Connectivity in Graphs and Digraphs

Maximizing vertex-, edge- and arc-connectivity with an emphasis on local connectivity properties

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Preface

Studying connectivity of graphs and digraphs has various applications in many areas. Whenever different agents or entities perform an individual or combined task, which requires those agents to communicate or exchange information or goods in some way, it is of interest that this communication or exchange works as smooth and effective as possible. These kind of scenarios can usually be modelled as graphs or digraphs, where the vertices represent the agents or entities in question, and the edges or arcs represent the connections between them. Common examples include computer networks, communication networks, electricity grids, social networks, transport infrastructures, supply chains or other logistic scenarios, to name a few. In all these settings it is important to have a reliable connection between the agents, where the throughput through the network in question should be as high as possible. To achieve this goal it is convenient to study the underlying graphs and digraphs of these networks and provide some mathematical measurements, which give information about both the reliability and throughput of a network.

In this thesis we deal with various parameters concerning the connectivity of graphs and digraphs, and present new results on these parameters themselves and how to maximize or optimize them in some sense. In particular, we take a closer look on local connectivity properties. For an overview on such connectivity parameters and recent results the reader may be referred to the surveys of Fàbrega and Fiol [30] and Hellwig and Volkmann [58]. We start in Chapter 1 by introducing the required terminology and notation as well as the definitions of the considered connectivity parameters. Furthermore, we give a short insight on how these parameters relate to the reliability of networks, and discuss the complexity of their computation. The main part of this thesis is then divided into two parts, the first part dealing with graphs and the second one dealing with digraphs.

Part I of this thesis includes Chapters 2 and 3 and covers some important connectivity parameters for graphs. We discuss results on the vertex- and edge-connectivity in Chapter 2 and 3, respectively, where we emphasize on local connectivity properties. The results of Chapter 2 and Section 3.1 mostly include conditions on the minimum degree for graphs to be maximally or maximally local (edge-)connected. Section 3.2 deals with the more refined parameter of restricted edge-connectivity, where we give some sufficient degree conditions for optimality in triangle-free and p-partite graphs. In Section 3.3 we then introduce the new parameter of local restricted edge-connectivity as a generalization of the concept of restricted edge-connectivity presented in Section 3.2.

In Part II, namely Chapters 4–7, we study connectivity parameters in digraphs. In Chapter 4 we discuss the maximum local connectivity in bipartite tournaments as well as the connectivity in local tournaments. Chapter 5 introduces a concept of restricted arc-connectivity for digraphs due to Volkmann [108], where we give some sufficient degree and regularity criteria for optimality in tournaments and bipartite tournaments. Chapter 6 deals with the special problem of decycling bipartite tournaments. The problem is to find the minimum number of arcs, whose deletion ensures the acyclicity of an arbitrary m-by-n bipartite tournament. In Chapter 7 we discuss a special flow problem in networks, where we want to find maximum flows with especially beneficial local flow properties. Therefore, we introduce the new concept of maximum local flows and use it to define a unique perfect flow in an arbitrary network.

Finally, in Chapter 8 we give a summary of the main contributions of this thesis and present some open problems related to the results of this work.

Aachen, December 2012 Andreas Holtkamp

Contents

Chapter 1

Introduction

1.1 Terminology and notation

Throughout this work we consider finite graphs and digraphs without loops and multiple edges or arcs. The *vertex set* and *edge set* (*arc set*) of a *graph* G (*digraph* D) are denoted by $V(G)$ ($V(D)$) and $E(G)$ ($A(D)$), respectively. An *edge* $e = uv \in E(G)$ is an unordered pair of vertices $u, v \in V(G)$, whereas an *arc* $e = uv \in A(D)$ is an ordered pair of vertices $u, v \in V(D)$. If D is a digraph, we refer to the *converse digraph* $D^{-1} = (V(D), \{yx \colon xy \in A(D)\})$.

For a vertex $v \in V(G)$, the *open neighbourhood* $N_G(v) = N(v)$ is the set of all vertices adjacent to v, and $N[v] = N_G[v] = N(v) \cup \{v\}$ is the *closed neighbourhood* of v. For $X, Y \subseteq V(G)$ we define $(X, Y) = \{xy \in E(G) \colon x \in X,\, y \in Y\}$ and $[X, Y] = |(X, Y)|$. We denote $N[X] = N_G[X] = \bigcup_{v \in X} N_G[v]$, and $G[X]$ is the graph induced by X, which is also called *induced subgraph* of G. By $G - X$ we refer to the graph induced by $V(G) \setminus X$, and by $G - S$ we refer to the subgraph where the edges $S \subseteq E(G)$ are removed from G. The numbers $n = n(G) = |V(G)|$, $m = m(G) = |E(G)|$ and $d(v) = d_G(v) = |N(v)|$ are called the *order*, the *size* of G and the *degree* of v, respectively. The *minimum degree* of a graph G is denoted by $\delta = \delta(G) = \min\{d(v) \colon v \in V(G)\}$, and the *maximum degree* by $\Delta = \Delta(G) = \max\{d(v) \colon v \in V(G)\}$.

For a vertex $v \in V(D)$ we denote by $N^+(v)$ and $N^-(v)$ the set of *out-* and *in-neighbours* of v, respectively. For a vertex subset $X \subseteq V(D)$ the notation $N^+(X)$ refers to the vertex set $\bigcup_{v \in X} N^+(v) \setminus X$, and $N^-(X)$ accordingly. The *out-degree* $d^+(v) = d^+_D(v) = |N^+(v)|$ of a vertex v is the number of out-neighbours of v in D, and analogously, $d^-(v) = d^-_D(v) = |N^-(v)|$ denotes the *in-degree*. $\delta^+ = \delta^+(D) = \min\{d^+(v) \colon v \in V(D)\}$ denotes the *minimum out-degree* of D, and $\delta^- = \delta^-(D) = \min\{d^-(v) : v \in V(D)\}$ the *minimum in-degree*. Also, we refer to the *minimum degree* $\delta = \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$. The *(global) irregularity* $i_g(D)$ of a digraph D is defined as $i_g(D) = \max\{\max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\} : x, y \in V(D)\}$. In case $i_g(D) = 0$ we call D a *regular* digraph, and in case $i_g(D) = 1$ an *almost regular* digraph.

For two vertex subsets $X, Y \subseteq V(D)$ we denote by $\overline{X}$ the vertex set $V(D) \setminus X$. We define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$ and denote $[X, Y] = |(X, Y)|$. We write $\omega^+(X)$ instead of $(X, \overline{X})$, and $\omega^-(X)$ for $(\overline{X}, X)$. $D[X]$ is the digraph induced by X (also called *induced subdigraph*), and $D - X$ denotes the digraph induced by the vertex set $\overline{X}$. For an arc $xy \in A(D)$ we refer to the digraph $D - xy$, where the arc xy is removed from D, and define $D - S$ for $S \subseteq A(D)$ analogously. Let $x_1, x_2, \ldots, x_n$ be the vertices of D and $D_1, D_2, \ldots, D_n$ digraphs, then $H = D[D_1, D_2, \ldots, D_n]$ is defined by $V(H) = \bigcup_{i=1}^{n} V(D_i)$ and $A(H) = (\bigcup_{i=1}^{n} A(D_i)) \cup \{y_i y_j : y_i \in V(D_i),\ y_j \in V(D_j),\ x_i x_j \in A(D)\}$. If $xy \in A(D)$, then we write $x \to y$ and say x *dominates* y. If every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, we abbreviate e. g. $\{x\} \to Y$ to $x \to Y$.

If $\{u_1, u_2, \ldots, u_p\} \subseteq V(G)$ $(\subseteq V(D))$ with $\{u_1u_2, u_2u_3, \ldots, u_{p-1}p, u_pu_1\} \subseteq E(G)$ $(\subseteq A(D))$, then we call $C_p = u_1u_2 \ldots u_pu_1$ a *cycle* (*directed cycle*) of *length* p for $p \geq 3$ $(p \geq 2)$. Cycles of length p are also called *p-cycles*. For two distinct vertices $u, v \in V(G)$ $(u, v \in V(D))$ we refer to a *path* (*directed path*) $P = u_0u_1 \ldots u_{m-1}u_m$ from u to v of *length* $m \geq 1$, if $u_0 = u$, $u_m = v$, $u_i \neq u_j$ for all $i, j \in \{0, 1, \ldots, m\}$ with $i \neq j$ and $\{u_0u_1, u_1u_2, \ldots, u_{m-1}u_m\} \subseteq E(G)$ $(\subseteq A(D))$. A path from u to v is also called *u-v-path*, and a path of length m is called *m-path*. If there exists a (directed) path from u to v, then the length of a shortest such path is called *distance* from u to v. When considering digraphs, all cycles and paths throughout this work are considered to be directed.

We denote by K_n a *complete graph (or digraph)* on n vertices, i. e. a graph (digraph) with n vertices and all possible edges (arcs). An induced complete subgraph K_p in a graph G is called a *clique* or *p-clique*. The *clique-number* $\omega(G)$ of a graph G is the maximum order over all cliques in G. A vertex set $X \subseteq V(G)$ is called *independent* if its induced subgraph contains no edges. We call a graph G *p-colorable* or *p-partite* if its vertex set can be partitioned into p independent sets $U_1, U_2, \ldots, U_p$ with $p \geq 2$, where U_i are called the *partite sets* of G for $1 \leq i \leq p$. If a graph G is q-colorable, but not $(q-1)$-colorable, then q is called the *chromatic number of* G, denoted $\chi(G)$.

Graphs containing no C_3 are called *triangle-free*, and digraphs without any cycles are called *acyclic*. The graph obtained from a K_4 by removing an arbitrary edge is called *diamond*, and the graph consisting of $p+2$ vertices, where two connected vertices have exactly p common neighbours and no further edges is called *p-diamond* (cf. Figure 2.1 on page 15). We note that by this definition a 2-diamond is equal to the diamond. A graph is called *diamond-free* (*p-diamond-free*) if it contains no diamond (p-diamond) as a (not necessarily induced) subgraph. In other words, a p-diamond-free graph is a graph, where all two distinct vertices u and v with $uv \in E(G)$ have at most $p-1$ common neighbours.

The *underlying graph* of a digraph D is the graph with vertex set $V(D)$ and edge set $\{uv\colon uv \in A(D) \text{ or } vu \in A(D)\}$ without multiple edges. An *orientation* of a graph G is a digraph, where all edges $uv \in E(G)$ are replaced by either the arc uv or vu. Orientations of graphs are also called *oriented graphs*. Orientations of complete graphs are called *tournaments* and denoted by $T = T_n$, where n is the number of vertices of T. A 2-partite graph with partite sets U and V is also called *bipartite*, where U and V are called *bipartite sets*. A p-partite graph is called *complete* if it contains all possible edges between its p partite sets. A *complete bipartite graph* $K_{m,n}$ is a complete 2-partite graph, where m and n are the cardinalities of the bipartite sets. Graphs isomorphic to $K_{1,n-1}$ for $n \geq 2$ are also called *stars*. An orientation of a complete bipartite graph is called *bipartite tournament*. A digraph which has at least one arc between every pair of distinct vertices is called *semicomplete digraph*. If for every vertex $x \in V(D)$ of a digraph D the set of out-neighbours as well as the set of in-neighbours each induce a semicomplete digraph (tournament), then D is called a *locally semicomplete digraph* (*local tournament*).

1.2 Concepts of connectivity

For detailed information on former results and the history of connectivity we refer the reader to the surveys by Fàbrega and Fiol [30] and Hellwig and Volkmann [58]. In this section we introduce the concepts and definitions of connectivity outlined in this work.

We call a graph G *connected* if there is a path from any vertex to any other vertex in G. A digraph is called *connected* if its underlying graph is, and *strongly connected* or *strong* if there is a path from any vertex to any other vertex. A maximal connected induced subgraph of a graph is called *component*, and a maximal strong induced subdigraph of a digraph is called *strong component*. A vertex v of a connected graph G is called a *cut-vertex* of G if its removal divides G into at least two components. If a graph does not contain any cut-vertices, then it is called *2-connected*. A vertex subset $S \subset V(G)$ of a connected graph G is called *separating set* if $G - S$ consists of at least two components, and a vertex set $S \subset V(D)$ of a strong digraph D is called *separating set*, if $D - S$ is not strong. A *minimal* *separating set* is minimal with respect to inclusion, and a *minimum separating set* is one of minimal cardinality. If there exists a path between two vertices u and v in a graph, we say a vertex set $S \subset V(G)$ *separates u and v* if both are in different components of $G - S$.

1.2.1 Vertex-connectivity in graphs

The *connectivity (number)* $\kappa(G)$ of graph G is the smallest number of vertices whose deletion disconnects the graph or produces the trivial graph (the latter only applying to complete graphs). The *local connectivity (number)* $\kappa(u, v) = \kappa_G(u, v)$ between two distinct vertices u and v of a graph G, is the maximum number of internally disjoint u-v-paths in G. It is a well-known consequence of Menger's theorem [75] that

$$\kappa(G) = \min\{\kappa_G(u, v) \colon u, v \in V(G)\}. \tag{1.1}$$

Furthermore, the number of internally disjoint u-v-paths in G equals the minimum cardinality of a vertex subset S separating u and v in G in case $uv \notin E(G)$, i. e. $|S| = \kappa(u, v)$ and u and v are in different components of $G - S$. For $uv \in E(G)$ there exists a vertex subset S separating u and v in $G - uv$ with $|S| = \kappa_{G-uv}(u, v) = \kappa_G(u, v) - 1$. Of course, removing the neighbours of a vertex of minimum degree always disconnects a graph or produces the trivial graph, which has already been known by Whitney [119] in 1932. Also, the number of internally disjoint u-v-paths in a graph is trivially limited by the number of neighbours of u and v. Hence, we have $\kappa(G) \le \delta(G)$ and $\kappa(u, v) \le \min\{d(u), d(v)\}$.

We call a graph G *maximally connected* when $\kappa(G) = \delta(G)$ and *maximally local connected* when $\kappa(u, v) = \min\{d(u), d(v)\}$ for all pairs u and v of distinct vertices in G.

1.2.2 Edge-connectivity

An *edge-cut* in a connected graph G is a subset $S \subseteq E(G)$ such that $G - S$ is disconnected. The *edge-connectivity (number)* $\lambda(G)$ of a graph G is the smallest number of edges whose deletion disconnects the graph. The *local edge-connectivity (number)* $\lambda_G(u, v) = \lambda(u, v)$ between two distinct vertices u and v of a graph G, is the maximum number of edge-disjoint u-v-paths in G. It is a well-known consequence of Menger's theorem [75] that

$$\lambda(G) = \min\{\lambda_G(u, v) \colon u, v \in V(G)\}. \tag{1.2}$$

Of course, it is always possible to disconnect a graph by deleting all edges adjacent to a vertex of minimum degree leading to Whitney's inequality [119]

$$\kappa(G) \leq \lambda(G) \leq \delta(G),$$

which has been proven in 1932. In a similar way it is possible to disconnect two arbitrary vertices u and v, leading to $\lambda(u, v) \leq \min\{d(u), d(v)\}$. We call a graph G *maximally edge-connected* when $\lambda(G) = \delta(G)$ and *maximally local edge-connected* when $\lambda(u, v) = \min\{d(u), d(v)\}$ for all pairs u and v of distinct vertices in G.

1.2.3 Restricted edge-connectivity

The k-restricted edge-connectivity we consider here is due to Fàbrega and Fiol [29]. An edge-cut S is called a *k-restricted edge-cut* if every component of $G - S$ has at least k vertices. Assuming that G has k-restricted edge-cuts, the *k-restricted edge-connectivity (number)* of G, denoted by $\lambda_k(G)$, is defined as the minimum cardinality over all k-restricted edge-cuts of G, i. e.

$$\lambda_k(G) = \min\{|S| \colon S \subset E(G) \text{ is a } k\text{-restricted edge-cut}\}.$$

A connected graph G is called *k-restricted edge-connected* if $\lambda_k(G)$ exists. A k-restricted edge-cut $(X, \overline{X})$ is called a *minimum k-restricted edge-cut* if $[X, \overline{X}] = \lambda_k(G)$. It is clear that for any minimum k-restricted edge-cut $(X, \overline{X})$, the graph $G - (X, \overline{X})$ has exactly two connected components. If $(X, \overline{X})$ is a minimum k-restricted edge-cut, then X is called a *k-fragment* of G. Let

$$r_k(G) = \min\{|X| : X \text{ is a } k\text{-fragment of } G\}.$$

Obviously, $k \leq r_k(G) \leq |V(G)|/2$. A k-fragment X is called a *k-atom* of G if $|X| = r_k(G)$.

Following [13], [74], and [84], we define the *minimum k-edge-degree*, by

$$\xi_k(G) = \min\{[X, \overline{X}] \colon |X| = k \text{ and } G[X] \text{ is connected}\}.$$

A k-restricted edge-connected graph G with $\lambda_k(G) \leq \xi_k(G)$ is said to be *optimally k-restricted edge-connected* (for short *λ_k-optimal*) if $\lambda_k(G) = \xi_k(G)$. $\lambda_1(G) = \lambda(G)$ and $\lambda_2(G) = \lambda'(G)$ correspond to the *edge-connectivity* and *restricted edge-connectivity*, respectively, and accordingly $\xi_1(G) = \delta(G)$ and $\xi_2(G) = \xi(G)$ are also known as the minimum (vertex) degree and the *minimum edge degree*. In particular, the notion of λ_1-optimality coincides with maximum edge-connectivity. A graph G is called *super-λ_k* if every minimum k-restricted edge-cut isolates a connected subgraph of order k.

1.2.4 Local restricted edge-connectivity

The local restricted edge-connectivity considered here is a local variant of the restricted edge-connectivity presented above. We will discuss the legitimation of this new idea in Section 3.3. We define a graph G to be *local k-restricted edge-connected* if for every pair x and y of vertices of G there exists an edge-cut S such that each component of $G - S$ has order at least k, and x and y are in different components of $G - S$. In other words, for every pair x and y of vertices there exists a k-restricted edge-cut separating x and y. We denote the size of a minimum such edge-cut by $\lambda_k(x, y)$ and call it a *$minloc_k(x, y)$-cut*. We refer to $\lambda_k(x, y)$ as the *local k-restricted edge-connectivity (number) of x and y*. Obviously, if S is minimal, then $G - S$ has exactly two components. Note that $\lambda_1(x, y) = \lambda(x, y)$.

The value

$$\xi_k(x, y) = \min\{[X, \overline{X}] \colon |X| = k,\ G[X] \text{ connected},\ |\{x, y\} \cap X| = 1\}$$

denotes the minimum number of edges between a connected subgraph of order k that contains either x or y and the remaining graph, and is called the *minimum local k-edge degree of the vertices x and y*. Analogue to the concept of λ_k-optimality we introduce the corresponding definition for local k-restricted edge-connectivity. A local k-restricted edge-connected graph is called *optimally local k-restricted edge-connected* or *local λ_k-optimal* if

$$\lambda_k(x, y) = \xi_k(x, y)$$

for every pair x and y of vertices.

1.2.5 Vertex-connectivity in digraphs

The *local connectivity (number)* $\kappa_D(x, y) = \kappa(x, y)$ of two vertices x and y in a digraph D is the maximum number of internally disjoint x-y-paths in D, and the *connectivity (number)* of D can be defined as $\kappa(D) = \min\{\kappa(x, y) \colon x, y \in V(D)\}$. Clearly, $\kappa(x, y) \leq \min\{d^+(x),\ d^-(y)\}$ for all pairs x and y of vertices in D. We call a digraph D *maximally connected* when $\kappa(D) = \delta(D)$ and *maximally local connected* when

$$\kappa(x, y) = \min\{d^+(x),\ d^-(y)\}$$

for all pairs x and y of distinct vertices in D. By a well-known variant of Menger's theorem for digraphs due to Dirac [21] the local connectivity $\kappa(x, y)$ equals the minimum cardinality of a vertex subset S *separating x from y*, i. e. $|S| = \kappa(x, y)$ and there is no x-y-path in $D - S$ in case $xy \notin A(D)$, or $|S| = \kappa(x, y) - 1$ and there is no x-y-path in $D - S - xy$ in case $xy \in A(D)$.

1.2.6 Restricted arc-connectivity

We call a subset $S \subseteq A(D)$ of a strong digraph D *arc-cut*, if $D - S$ is not strong. According to Volkmann [108], a strong digraph D is called *restricted arc-connected*, if there exists an arc-cut S such that $D - S$ contains a non-trivial strong component D' and $D - V(D')$ contains at least one arc. An arc-cut with this property is called *restricted arc-cut*. For a restricted arc-connected digraph we define $\lambda'(D)$ as the size of a *minimum restricted arc-cut* of D, and call it the *restricted arc-connectivity (number)*.

In 2008, Wang and Lin [112] defined the *arc-degree* $\xi'(xy)$ for an arc xy of a digraph D with $yx \notin A(D)$ as

$$\begin{aligned}\xi'(xy) = \min\{&d^+(x) + d^+(y) - 1,\ d^+(x) + d^-(y) - 1,\\ &d^-(x) + d^+(y),\ d^-(x) + d^-(y) - 1\},\end{aligned}$$

and in case $yx \in A(D)$ as

$$\begin{aligned}\xi'(xy) = \min\{&d^+(x) + d^+(y) - 2,\ d^+(x) + d^-(y) - 1,\\ &d^-(x) + d^+(y) - 1,\ d^-(x) + d^-(y) - 2\}.\end{aligned}$$

Also, we call $\xi'(D) = \min\{\xi'(xy) \colon xy \in A(D)\}$ the *minimum arc-degree* of D. By this definition, these four degree sums correspond to the size of four different arc subsets, whose removal yields a digraph where neither x nor y can be on any cycle (cf. Figure 5.1 on page 92). Therefore, we denote by Ω_{xy} the set of arc subsets

$$\{\omega^+(\{x, y\}),\ \omega^+(x) \cup \omega^-(y),\ \omega^-(x) \cup \omega^+(y),\ \omega^-(\{x, y\})\}.$$

In analogy to the concept of restricted edge-connectivity in graphs Wang and Lin [112] also used the notion of λ'-*optimal* digraphs, i. e. digraphs fulfilling $\lambda'(D) = \xi'(D)$.

Further notations will be defined where needed.

1.3 Connectivity and reliability of networks

When studying the reliability of interconnection networks one usually assumes the possibility of nodes and connections to be failing. If failures arise the connectivity among the remaining nodes of the network should be as high as possible, and of course, it is favourable for the remaining network to remain connected and not split into several clusters. According to this, the connectivity and edge-connectivity are the minimum number of vertices and edges to be removed from a graph, respectively, before it splits into two or more components. Furthermore, according to Karl Menger's results from 1927 [75], these numbers also correspond to the maximum number of internally vertex- and edge-disjoint paths between two arbitrary vertices of a graph. Of course, it is always possible to disconnect a graph by deleting the vertices or edges adjacent to a vertex of minimum degree, leading to Whitney's inequality [119] $\kappa(G) \leq \lambda(G) \leq \delta(G)$ from 1932. Having this trivial upper bound, the motivation for studying sufficient criteria for graphs to be *maximally connected* and *maximally edge-connected* is obvious, and the same holds for the stronger *maximum local connectivity* and *maximum local edge-connectivity*.

Moreover, even graphs of the same connectivity or edge-connectivity may vary largely in the reliabilities of the underlying networks, when counting the number of different minimum (edge-)cuts or due to further structural properties. Therefore, the question for more refined measurements of the fault tolerance of networks appeared. However, there is an important difference between vertex- and edge-connectivity. When deleting vertices from a graph it is not possible to know for sure the number of components this graph might split into. But deleting a single edge from a graph can increase the number of components by at most one. To that effect, in 1983 Harary [49] proposed a quite general concept of *conditional edge-connectivity*. In a common model due to Moore and Shannon [78, 79] from 1956 one assumes that vertices never fail, but edges might fail independently with equal (small) probability $0 < p < 1$. Let G be a connected graph modelling a so called *Moore-Shannon model*, and for $1 \leq i \leq m = |E(G)|$ let S_i be the number of edge-cuts of size i in G. Then the *reliability of a network* can be determined as the probability for the underlying graph G to stay connected in case that edges fail, i. e.

$$R(G,p) = 1 - \sum_{i=1}^{m} S_i \cdot p^i \cdot (1-p)^{m-i}.$$

Provan and Ball [90] proved in 1983 that determining all coefficients S_i is NP-hard. Of course, $S_i = 0$ for all $i < \lambda(G)$. In [11] Bauer, Bauer, Suffel and Tindell determined $S_{\lambda(G)}$ and showed that among networks of equal order and size those with higher edge-connectivity and a smaller number of minimum edge-cuts model networks with higher reliability $R(G,p)$, if p is sufficiently small.

According to this, maximally edge-connected graphs model most reliable networks. To compare the reliability between two graphs of the same edge-connectivity, it is reasonable to study the number of minimum edge-cuts in both graphs. With respect to this, Esfahanian and Hakimi [26] first introduced and studied the *restricted edge-connectivity* in 1988, to exclude trivial edge-cuts from considerations. And eight years later Fàbrega and Fiol [29] presented a generalization of this concept, namely the *k-restricted edge-connectivity*. Among graphs of the same edge-connectivity $\lambda(G)$ and the same number of minimum edge-cuts, those with higher restricted edge-connectivity $\lambda_2(G)$ and fewer 2-restricted edge-cuts model more reliable networks (for p sufficiently small). And again, if those numbers coincide we can take a look at the 3-restricted edge-connectivity and so on. Generally speaking, λ_2-optimal graphs model more reliable networks than those who are not λ_2-optimal, and among two graphs of same order, size and edge-connectivity, which are both λ_i-optimal with same i-edge degrees for $1 \leq i < k$, but one being λ_k-optimal and the other one not, the first one will model a more reliable network than the latter one. Of course, similar arguments also hold for the stronger local restricted edge-connectivity. Furthermore, in this context the notion of super-λ_k graphs becomes interesting, since they only allow trivial k-restricted edge-cuts.

Analogously, maximally connected digraphs model more reliable directed networks. And similar to the above the concept of restricted arc-connectivity due to Volkmann [108] and Wang and Lin [112] excludes trivial arc-cuts from considerations, and therefore shall offer a more refined measurement of fault tolerance for directed interconnection networks.

1.4 Connectivity and complexity

Many connectivity based parameters can be calculated efficiently. In the following we give a short overview on the complexity of the determination of the different connectivity parameters discussed in this thesis. For an arbitrary graph $G = (V, E)$ or an arbitrary digraph $D = (V, A)$ we refer to $n = |V|$, and $m = |E|$ or $m = |A|$, respectively.

Connected graphs and strongly connected digraphs: By using depth-first search algorithms it is possible to decide whether a graph $G = (V, E)$ is connected or not with requirements linear in time and space, which leads to a complexity in $\mathcal{O}(n+m)$. The same holds for the strong connectivity of digraphs. In 1972, Tarjan [97] presented an algorithm to determine the strong components of a digraph in $\mathcal{O}(n + m)$. Furthermore, a result of Reingold [93] from 2008 indicates that the connectivity of undirected graphs is solvable with space requirements in $\mathcal{O}(\log n)$.

Connectivity number $\kappa(G)$: Similar to the algorithm mentioned in the last paragraph, in [97] Tarjan also presented a method for finding the 2-connected components of a graph in $\mathcal{O}(n + m)$, again using depth-first search. According to this, the complexity of the decision whether we have $\kappa(G) = 1$ or $\kappa(G) = 2$ is in $\mathcal{O}(n+m)$. According to a result of Henzinger, Rao and Gabow [59] the connectivity number $\kappa(G) = k$ can be determined with time consumption in $\mathcal{O}(\min\{k^3 + n,\, kn\}\, kn)$ for the undirected case, and in $\mathcal{O}(\min\{k^3 + n,\, kn\}\, m)$ for the directed case. A survey on the history of algorithms for determining the vertex- and edge-connectivity by Esfahanian can be found in [25].

Edge-connectivity number $\lambda(G)$: In 1991, Gabow [37] presented a sophisticated approach determining the edge connectivity number $\lambda(G) = k$ in time $\mathcal{O}(km \log(n^2/m))$ for both directed and undirected graphs using matroid theory. A simple method for computing $\lambda(G)$ is e. g. due to Stoer and Wagner [96] in 1997 with overall running time in $\mathcal{O}(nm + n^2 \log(n))$, which is based on a slightly more complicated approach by Nagamochi and Ibaraki [81] in 1992 with the same running time.

Local connectivity number $\kappa(u, v)$ and local edge-connectivity number $\lambda(u, v)$: The numbers $\kappa(u, v)$ and $\lambda(u, v)$ can be computed using maximum flow/ minimum cut algorithms. Determining maximum flows has been widely studied and can be done for example by using the algorithms of Ford and Fulkerson [32, 33] or Edmonds and Karp [22], which runs in $\mathcal{O}(nm^2)$. More sophisticated approachs e. g. are due to Dinic [20] ($\mathcal{O}(n^2m)$), or Goldberg and Tarjan [38] ($\mathcal{O}(nm \log(n^2/m))$).

k-restricted edge-connectivity number $\lambda_k(G)$: Esfahanian and Hakimi [26] showed that by solving at most $(m + \delta\Delta - 2)$ network flow problems it is possible to determine the restricted edge-connectivity number $\lambda_2(G) = \lambda'(G)$. For higher values of k the complexity of determining $\lambda_k(G)$ remains polynomial in time, but rises fast. Therefore, it is desirable to have conditions, which ensure large values of $\lambda_k(G)$ and can be computed rapidly. Such conditions are discussed in Section 3.2, where we give degree conditions to ensure λ_k-optimality.

Local k-restricted edge-connectivity number $\lambda_k(u, v)$: Like we discuss in Section 3.3, for fixed values of k the local k-restricted edge-connectivity number $\lambda_k(u, v)$ can be determined in polynomial time (cf. also Remark 1.1 in [50] from Hellwig, Rautenbach and Volkmann). A naive approach, however, would run in time at most $\mathcal{O}(k^3 n^{2k+2} m^2)$, e.g. following the trivial algorithm suggested in Section 3.3 on page 54. In Section 3.3 we present some degree conditions ensuring local λ_2-optimality, which can be checked in linear time.

Restricted arc-connectivity number $\lambda'(D)$: Volkmann [108] showed that the decision whether a digraph is restricted arc-connected can be made in time $\mathcal{O}(m(m + n))$. Computing the number $\lambda'(D)$ is more complex. According to this, in Chapter 5 we give some degree and regularity criteria to ensure λ'-optimality in tournaments and bipartite tournaments, which can be computed in linear time.

Part I

Connectivity in graphs

Chapter 2

Vertex-connectivity

In this chapter we discuss the (vertex-)connectivity of graphs and present some sufficient degree conditions for various classes of graphs to be maximally local connected. These results have been obtained in collaboration with Lutz Volkmann and have been subject to my diploma thesis in 2008. But since they have been the motivation and starting point of my research and fit very well in the scope of this work, we list the findings without proof.

According to the definitions given in Section 1.2.1 on page 4 the following observation is obvious.

Observation 2.1. *Every maximally local connected graph is maximally connected.*

Proof. Since G is maximally local connected, we have $\kappa(u,v) = \min\{d(u), d(v)\}$ for all pairs u and v of vertices in G. Thus (1.1) implies

$$\kappa(G) = \min_{u,v \in V(G)} \{\kappa(u,v)\} = \min_{u,v \in V(G)} \{\min\{d(u), d(v)\}\} = \delta(G).$$

□

Because of $\kappa(G) \leq \delta(G)$, there exists a special interest on graphs G with $\kappa(G) = \delta(G)$. Different authors have presented sufficient conditions for graphs to be maximally connected, as, for example Balbuena, Cera, Diánez, García-Vázquez and Marcote [3], Esfahanian [24], Fàbrega and Fiol [27, 28], Fiol [31], Hellwig and Volkmann [57], Soneoka, Nakada, Imase and Peyrat [95] and Topp and Volkmann [99]. However, closely related investigations for the local connectivity have received little attention until recently. In the next sections we will give such results, which generalize these ones in [16, 57, 99, 109].

2.1 Maximum (local) connectivity in graphs with bounded clique number

In 1993, Topp and Volkmann [99] gave a sufficient condition for p-partite graphs to be maximally connected.

Theorem 2.2 (Topp, Volkmann [99], 1993). *Let $p \geq 2$ be an integer. If G is a p-partite graph such that*

$$n(G) \leq \delta(G) \cdot \frac{2p-1}{2p-3},$$

then $\kappa(G) = \delta(G)$.

In 1941, Turán [100] presented an interesting upper bound on the size of a graph G with a given clique number $\omega(G) \leq p$.

Theorem 2.3 (Turán [100], 1941). *If G is a graph of order n such that $\omega(G) \leq p$, then*

$$2m(G) \leq \frac{p-1}{p} n^2.$$

As an application of Turán's theorem, Hellwig and Volkmann [57] obtained the following generalization of Theorem 2.2.

Theorem 2.4 (Hellwig, Volkmann [57], 2006). *Let $p \geq 2$ be an integer, and let G be a connected graph with clique number $\omega(G) \leq p$. If $n(G) \leq \delta(G)(2p-1)/(2p-3)$, then $\kappa(G) = \delta(G)$.*

In view of Observation 2.1, the next result is also an extension of Theorem 2.2.

Theorem 2.5 (Volkmann [109], 2008). *Let $p \geq 2$ be an integer, and let G be a p-partite graph. If $n(G) \leq \delta(G)(2p-1)/(2p-3)$, then G is maximally local connected.*

Using also Turán's theorem, we were able to present a new short proof of the following common generalization of Theorems 2.2, 2.4 and 2.5.

Theorem 2.6 (Holtkamp, Volkmann [66], 2010). *Let $p \geq 2$ be an integer, and let G be a graph with clique number $\omega(G) \leq p$. If*

$$n(G) \leq \delta(G) \cdot \frac{2p-1}{2p-3},$$

then G is maximally local connected.

In [99], the authors have shown that the condition $n \leq \delta(2p-1)/(2p-3)$ in Theorem 2.2 is best possible in the sense that for any positive integers p, δ, q, n with $p, q \geq 2$, $\delta = q(2p-3)$ and $n = \delta(2p-1)/(2p-3)+1 = q(2p-1)+1$ there exist p-partite graphs G of order n, minimum degree δ and connectivity $\kappa < \delta$. Thus Theorem 2.6 is also best possible in this sense.

2.2 Maximum (local) connectivity in diamond- and p-diamond-free graphs

In 2007, Dankelmann, Hellwig and Volkmann [16] gave a sufficient condition for connected diamond-free graphs to be maximally connected. In a diamond-free graph no two 3-cycles can be adjacent, i. e. sharing a common edge (cf. Figure 2.1). Thus, diamond-free graphs include bipartite graphs.

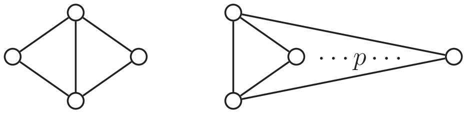

Figure 2.1: Left: A diamond. Right: A p-diamond.

Theorem 2.7 (Dankelmann, Hellwig, Volkmann [16], 2007)**.** *Let G be a connected diamond-free graph of order n and minimum degree $\delta \geq 3$. If $n \leq 3\delta$, then $\kappa(G) = \delta(G)$.*

The following family of examples will demonstrate that the condition $n \leq 3\delta$ in Theorem 2.7 does not guarantee that the graph is maximally local connected.

Example 2.8 (Holtkamp, Volkmann [65], 2009)**.** *Let $\delta \geq 3$ be an integer, and let G be a graph with vertex set $V(G) = \{u, u', v, v', w, w'\} \cup V_1 \cup V_2 \cup V_3$ such that $|V_1| = |V_2| = |V_3| = \delta - 2$ and edge set (cf. Figure 2.2)*

$$\begin{aligned} E(G) &= \{uu', uw, u'w, uw', vv', vw', v'w', vw\} \\ &\cup \{u'x \colon x \in V_1\} \cup \{w'x \colon x \in V_1\} \\ &\cup \{v'x \colon x \in V_2\} \cup \{wx \colon x \in V_2\} \\ &\cup \{ux \colon x \in V_3\} \cup \{vx \colon x \in V_3\} \\ &\cup \{xy \colon x \in (V_1 \cup V_2) \text{ and } y \in V_3\}. \end{aligned}$$

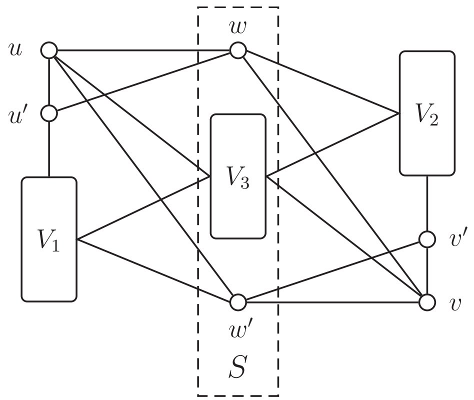

Figure 2.2: Graph from Example 2.8. The vertex set S separates u and v.

Obviously, $\delta(G) = \delta = d(u')$ and $n(G) = 3\delta(G)$. In addition, $d(u) = d(v) = \delta(G) + 1$, and the vertex set $S = V_3 \cup \{w, w'\}$ with $|S| = \delta(G)$ separates u and v. Consequently, G is not maximally local connected.

However, diamond-free graphs are maximally local connected when $n(G) \leq 3\delta(G) - 1$, and in addition, we have similar results for p-diamond-free graphs.

Theorem 2.9 (Holtkamp, Volkmann [65], 2009)**.** *Let $p \geq 2$ be an integer, and let G be a connected p-diamond-free graph. In addition, let $u, v \in V(G)$ two vertices of G and define $r = \min\{d_G(u), d_G(v)\} - \delta(G)$.*

(1) If $uv \notin E(G)$ and $n(G) \leq 3\delta(G) + r - 2p + 2$, then $\kappa_G(u, v) = \delta(G) + r$.

(2) If $uv \in E(G)$ and $n(G) \leq 3\delta(G) + r - 2p + 1$, then $\kappa_G(u, v) = \delta(G) + r$.

Theorem 2.10 (Holtkamp, Volkmann [65], 2009)**.** *Let G be a connected diamond-free graph with minimum degree $\delta(G) \geq 3$. If $n(G) \leq 3\delta(G) - 1$, then G is maximally local connected.*

Example 2.8 demonstrates that for each $\delta \geq 3$, there exists a diamond-free graph of order $n = 3\delta$ that is not maximally local connected. Thus Theorem 2.10 is best possible in this sense.

Combining the proof of Theorem 2.10 with Theorem 2.9, we obtain a generalization of Theorem 2.7 in the special case that $d_G(x) \notin \{\delta(G)+1, \delta(G)+2\}$ for each $x \in V(G)$.

Theorem 2.11 (Holtkamp, Volkmann [65], 2009). *Let G be a connected diamond-free graph with minimum degree $\delta(G) \geq 3$. If $n(G) \leq 3\delta(G)$ and $d_G(x) \notin \{\delta(G)+1, \delta(G)+2\}$ for each $x \in V(G)$, then G is maximally local connected.*

The graph G in Example 2.8 shows that Theorem 2.11 is not valid when G contains vertices of degree $\delta(G)+1$. If we connect in Example 2.8 the vertices u and v by a further edge, then the resulting graph G' demonstrates that Theorem 2.11 is also not valid when there exist vertices of degree $\delta(G')+2$. Therefore the given conditions in Theorem 2.11 are best possible too.

Theorem 2.12 (Holtkamp, Volkmann [65], 2009). *Let $p \geq 3$ be an integer, and let G be a connected p-diamond-free graph. If $n(G) \leq 3\delta(G) - 2p + 2$, then G is maximally local connected.*

Combining Theorem 2.12 with Observation 2.1, we obtain the following result immediately.

Corollary 2.13 (Holtkamp, Volkmann [65], 2009). *Let $p \geq 3$ be an integer, and let G be a connected p-diamond-free graph. If $n(G) \leq 3\delta(G) - 2p + 2$, then G is maximally connected.*

The next family of examples will demonstrate that the bounds given in Theorem 2.12 as well as in Corollary 2.13 are best possible for each $p \geq 3$.

Example 2.14 (Holtkamp, Volkmann [65], 2009). *Let G_3, G_4, G_5 and G_6 be the graphs depicted in Figure 2.3. Each G_i is an i-diamond-free graph with $\delta(G_i) = i$ and $n(G_i) = 3\delta(G_i) - 2i + 3 = i + 3$. In every case the removal of the vertex set S disconnects the graph. Since $|S| = i - 1$, the graphs G_i are not maximally connected and therefore not maximally local connected.*

Starting with these four graphs we are able to construct successively similar graphs G_p for all $p \geq 7$. Each G_p will be p-diamond-free with $\delta(G_p) = p$ and $n(G_p) = 3\delta(G_p) - 2p + 3 = p + 3$. A vertex set S with $|S| = p - 1$ will separate G_p, showing that neither of the graphs is maximally connected or maximally local connected. Given a graph G_i with the described properties, we can construct a graph G_{i+4} with the same qualities in the subsequently specified way. The existence of G_p for all $p \geq 7$ then follows by induction.

So let G_i be a graph with the properties mentioned above. We obtain the graph G_{i+4} by adding the four vertices u, u', v and v', the edges uu', vv' and all possible

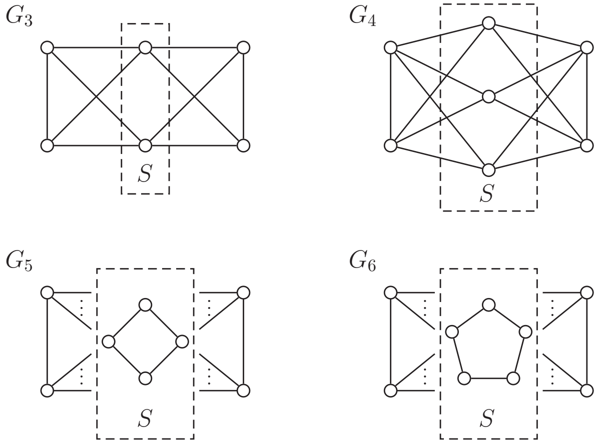

Figure 2.3: Four i-diamond-free graphs G_i of Example 2.14 with $n = 3\delta(G) - 2i + 3$. The vertex sets S separate the graphs with $|S| = \delta(G_i) - 1$.

edges between the four new vertices and the vertices of G_i, which means $\{xy\colon x \in \{u, u', v, v'\}$ and $y \in V(G_i)\}$.

2.3 Maximum (local) connectivity in $K_{2,p}$-free and C_4-free graphs

A graph is called $K_{2,p}$*-free* if it contains no $K_{2,p}$ as a (not necessarily induced) subgraph. Since a $K_{2,p}$-free graph is also p-diamond-free, the next corollary is an immediate consequence of Corollary 2.13.

Corollary 2.15 (Holtkamp, Volkmann [67], 2011)**.** *Let $p \geq 3$ be an integer, and let G be a connected $K_{2,p}$-free graph. If $n(G) \leq 3\delta(G)-2p+2$, then G is maximally local connected.*

The next result is a direct consequence of Corollary 2.15 and Observation 2.1.

Corollary 2.16 (Holtkamp, Volkmann [67], 2011)**.** *Let $p \geq 3$ be an integer, and let G be a connected $K_{2,p}$-free graph. If $n(G) \leq 3\delta(G)-2p+2$, then G is maximally connected.*

The following examples will demonstrate that the condition $n(G) \leq 3\delta(G) - 2p + 2$ in Corollaries 2.15 and 2.16 is best possible for $p = 3$ and $p \geq 5$.

Example 2.17 (Holtkamp, Volkmann [67], 2011)**.** *The connected graph in Figure 2.4 is $K_{2,3}$-free with minimum degree $\delta = 4$ and order $n = 3\delta - 6 + 3 = 9$. The vertex set S with $|S| = 3$ disconnects the graph, and therefore it is neither maximally connected nor maximally local connected. Thus the condition $n(G) \leq 3\delta(G) - 2p + 2$ in Corollaries 2.15 and 2.16 are best possible for $p = 3$.*

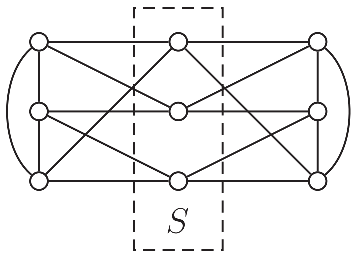

Figure 2.4: $K_{2,3}$-free graph of Example 2.17 with $\delta = 4$ and $n = 3\delta - 3 = 9$ vertices which is not maximally (local) connected.

Let G_3, G_4, G_5 and G_6 be the graphs depicted in Figure 2.5. Each G_p is a connected $K_{2,p}$-free graph with $\delta(G_p) = p$ and $n(G_p) = 3\delta(G_p) - 2p + 3 = p + 3$. The graphs G_5 and G_6 are not maximally connected and therefore not maximally local connected, since the removal of the vertex set S with $|S| = \delta(G_p) - 1 = p - 1$ disconnects the graphs. So Corollaries 2.15 and 2.16 are best possible for $p = 5$ and $p = 6$.

Starting with the four graphs G_3, G_4, G_5 and G_6 we are able to construct successively similar graphs G_p for all $p \geq 7$. Each G_p will be connected and $K_{2,p}$-free with $\delta(G_p) = p$ and $n(G_p) = 3\delta(G_p) - 2p + 3 = p + 3$. A vertex set S with $|S| = p - 1$ will separate G_p, showing that neither of the graphs is maximally connected or maximally local connected.

Given a graph G_p with the described properties, we can construct a graph G_{p+4} with the same qualities in the subsequently specified way. For G_{p+4} not to be maximally (local) connected the maximally (local) connectivity of G_p is irrelevant (e.g. G_3 and G_4 are maximally (local) connected). The existence of G_p for all $p \geq 7$ then follows by induction.

So let G_p be a graph with the properties mentioned above. We obtain the graph

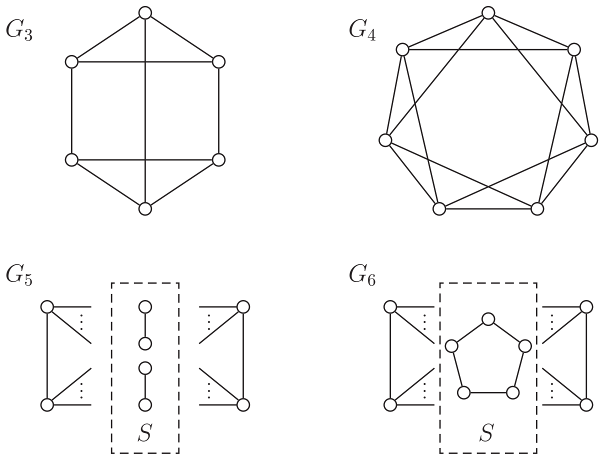

Figure 2.5: $K_{2,p}$-free graphs G_p ($p \in \{3, 4, 5, 6\}$) of Example 2.17 with $\delta(G_p) = p$ and $n = 3\delta(G_p) - 2p + 3 = \delta(G_p) + 3 = p + 3$. The graphs G_5 and G_6 are not maximally (local) connected, G_3 and G_4 are.

G_{p+4} by adding four new vertices u, u', v and v', the edges uu' and vv' as well as all possible edges between the four new vertices and the vertices of G_p that means $\{xy | x \in \{u, u', v, v'\}$ and $y \in V(G_p)\}$.

Next we will present an improved condition on maximally local connectivity for $K_{2,4}$-free graphs. For the proof we used Theorem 2.9.

Theorem 2.18 (Holtkamp, Volkmann [67], 2011)**.** *Let G be a connected $K_{2,4}$-free graph with minimum degree $\delta(G) \geq 3$. If $n(G) \leq 3\delta(G) - 5$, then G is maximally local connected.*

Combining Theorem 2.18 with Observation 2.1, we obtain the next result immediately.

Corollary 2.19 (Holtkamp, Volkmann [67], 2011)**.** *Let G be a connected $K_{2,4}$-free graph with minimum degree $\delta \geq 3$. If $n(G) \leq 3\delta(G) - 5$, then G is maximally connected.*

The example in Figure 2.6 demonstrates that the bound given in Theorem 2.18 as well as in Corollary 2.19 is best possible, at least for $\delta = 4$.

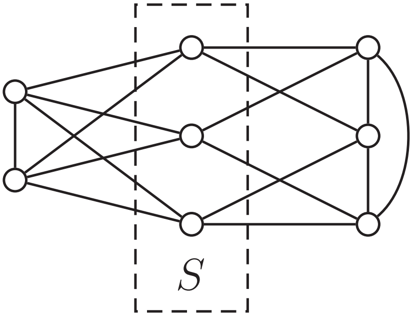

Figure 2.6: $K_{2,4}$-free graph with $\delta = 4$ and $n = 3\delta - 4 = 8$ vertices which is not maximally (local) connected.

In 2007, Dankelmann, Hellwig and Volkmann [16] presented the following sufficient condition for C_4-free graphs to be maximally connected. A graph is called *C_4-free* if it contains no 4-cycle.

Theorem 2.20 (Dankelmann, Hellwig and Volkmann [16], 2007)**.** *Let G be a connected C_4-free graph of order n and minimum degree $\delta \geq 2$. If*

$$n \leq \begin{cases} 2\delta^2 - 3\delta + 2 & \text{if } \delta \text{ is even,} \\ 2\delta^2 - 3\delta + 4 & \text{if } \delta \text{ is odd,} \end{cases}$$

then G is maximally connected.

Using Theorem 2.19, we proved a similar result for C_4-free graphs to be maximally local connected.

Theorem 2.21 (Holtkamp, Volkmann [67], 2011)**.** *Let G be a connected C_4-free graph of order n, minimum degree $\delta \geq 3$, $u, v \in V(G)$ and $r = \min\{d(u), d(v)\} - \delta$. If*

$$n \leq \begin{cases} 2\delta^2 - 5\delta + 6 - r & \text{if } uv \notin E(G), \\ 2\delta^2 - 5\delta + 7 - r & \text{if } uv \in E(G), \end{cases}$$

then $\kappa(u, v) = \delta + r$.

Theorem 2.22 (Holtkamp, Volkmann [67], 2011)**.** *Let G be a connected C_4-free graph of order n and minimum degree $\delta \geq 3$. If $n \leq 2\delta^2 - 6\delta + 10 - 5/\delta$, then G is maximally local connected.*

Chapter 3

Edge-connectivity

While in Chapter 2 we discussed the maximum (vertex-)connectivity and maximal local (vertex-)connectivity of several types of graphs in terms of their minimum degree, in this chapter we will make similar investigations for the edge-connectivity. We introduced the basic definitions of maximally edge-connected and maximally local edge-connected graphs in Section 1.2.2 on page 4. In Section 3.1 we present a new result for diamond-free graphs, which generalizes some results obtained by Volkmann [103] and Fricke, Oellermann and Swart [35] on bipartite graphs. In Sections 3.2 and 3.3 we will then discuss the more refined parameters of restricted edge-connectivity and local restricted edge-connectivity, respectively.

According to the definitions given in Section 1.2.2 the following observation is immediate.

Observation 3.1. *Every maximally local edge-connected graph is maximally edge-connected.*

Proof. In a maximally local edge-connected graph G we have $\lambda(u, v) = \min\{d(u), d(v)\}$ for all pairs u and v of vertices in G. Thus (1.2) implies

$$\lambda(G) = \min_{u,v \in V(G)} \{\lambda(u, v)\} = \min_{u,v \in V(G)} \{\min\{d(u), d(v)\}\} = \delta(G).$$

□

Because of $\lambda(G) \leq \delta(G)$, there is a special interest in graphs G with $\lambda(G) = \delta(G)$. Different authors have presented sufficient conditions for a graph to be maximally edge-connected, as, for example Dankelmann and Volkmann [17, 18, 19], Fàbrega and Fiol [28], Fiol [31], Hellwig and Volkmann [51, 54], Lin, Miller and Rodger [70], Moriarty and Christopher [80], Volkmann [104, 110], and Wang, Xu and Wang

[111]. However, closely related investigations for the local edge-connectivity have received little attention until recently. Fricke, Oellermann and Swart [35] studied the local edge-connectivity of p-partite graphs and graphs with bounded diameter. Hellwig and Volkmann [52] and Volkmann [105] gave sufficient conditions for the maximally local edge-connectivity of p-partite graphs and graphs with bounded clique number.

3.1 Maximum (local) edge-connectivity in diamond-free graphs

In a diamond-free graph two connected vertices can not have more than one common neighbour. Also, the notion of a diamond-free graph is equivalent to a graph without adjacent 3-cycles, i. e. a graph where no two 3-cycles share a common edge. Thus every bipartite graph is also diamond-free. We repeat Figure 2.1 to illustrate the structure of diamonds.

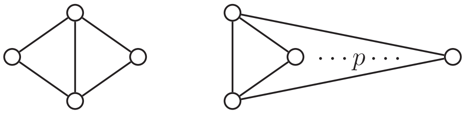

Figure 2.1: Left: A diamond. Right: A p-diamond.

In [103], Volkmann proved that bipartite graphs G with $n(G) \leq 4\delta(G) - 1$ are maximally edge-connected. Fricke, Oellermann and Swart [35] showed that this condition even guarantees the maximally local edge-connectivity of G. By looking at diamond-free graphs Dankelmann, Hellwig and Volkmann [16] were able to generalize a similar result on the maximally local (vertex-)connectivity of bipartite graphs. We will now give a generalization of the results of Volkmann and Fricke, Oellermann and Swart by proving that it is sufficient for G to be diamond-free with $n(G) \leq 4\delta(G) - 1$ to imply maximally local edge-connectivity. For a vertex $u \in V(G)$ we denote $E(u) = \{uv \in E(G) \colon v \in V(G)\}$, i. e. the set of all edges incident with u.

Theorem 3.2 (Holtkamp [60], 2011). *Let G be a diamond-free graph with $\delta(G) \geq 3$. If $n(G) \leq 4\delta(G) - 1$, then G is maximally local edge-connected.*

Proof. Assume G is not maximally local edge-connected. Therefore, we have two vertices $u, v \in V(G)$ with $r = \min\{d(u), d(v)\} - \delta \geq 0$ and an edge set S separating

u and v with $|S| \leq \delta + r - 1$. Let U be the component of $G - S$ with $u \in V(U)$. Since $n \leq 4\delta - 1$ and by symmetry of u and v, without loss of generality, we may assume

$$n(U) \leq 2\delta - 1. \tag{3.1}$$

Furthermore, since $d(u) \geq \delta + r > |S|$ the vertex u must have at least one neighbour in $V(U)$ and, in addition, at least for one neighbour $u' \in V(U)$ of u, we have $E(u') \cap S = \emptyset$ (i. e. none of the edges incident with u' is in S). We distinguish two cases:

Case 1. u and u' have a common neighbour in $V(U)$. Let $u'' \in N(u) \cap N(u') \cap V(U)$. Since G is diamond-free, u, u' and u'' can have no further common neighbours (pairwise). Let $W = (N(u) \cap V(U)) \setminus \{u', u''\}, W' = (N(u') \cap V(U))) \setminus \{u, u''\}$ and $W'' = (N(u'') \cap V(U)) \setminus \{u, u'\}$. Since G is diamond-free, $W \cap W' = \emptyset, W \cap W'' = \emptyset$ and $W' \cap W'' = \emptyset$. By $T = E(u) \cap S$ we refer to the edges of S incident with u, and let $T' = E(u') \cap S$ and $T'' = E(u'') \cap S$, respectively. Since no edge incident with u' is in S, we have

$$|W'| \geq \delta - 2. \tag{3.2}$$

Together with (3.1) this leads to

$$2\delta - 1 \geq n(U) \geq |W| + |W'| + |W''| + 3 \geq |W| + |W''| + \delta + 1.$$

Hence we have

$$|W| + |W''| \leq \delta - 2. \tag{3.3}$$

Obviously, it is $|T| + |W| \geq \delta + r - 2$ and $|T''| + |W''| \geq \delta - 2$. Thus, we deduce

$$2\delta + r - 4 \leq |T| + |T''| + |W| + |W''| \overset{(3.3)}{\leq} |T| + |T''| + \delta - 2,$$

which implies

$$|T| + |T''| \geq \delta + r - 2. \tag{3.4}$$

We now take a closer look on the vertices in W'. Assume there is a vertex $w \in W'$ with $E(w) \cap S = \emptyset$. Since G is diamond-free, w cannot be adjacent to u or u'', and have at most one neighbour in W'. Therefore, it is

$$\begin{aligned} 2\delta - 1 \overset{(3.1)}{\geq} n(U) &\geq |N(w) \setminus (W' \cup \{u'\})| + |W'| + |\{u, u', u''\}| \\ &\overset{(3.2)}{\geq} \delta - 2 + \delta - 2 + 3 = 2\delta - 1. \end{aligned}$$

So w must have exactly one neighbour $w' \in W'$ which cannot have further neighbours in U, and, of course, $\delta \geq 4$. Since G is diamond-free, w' is only adjacent to

w and u', but can not have neighbours in $(N(w) \setminus \{u', w'\}) \cup \{u, u''\} \cup (W' \setminus \{w\})$. Thus, w' must have at least $\delta - 2$ incident edges in S, i. e. $|E(w') \cap S| \geq \delta - 2$. Hence, every vertex in W' is either incident with at least one edge in S, or has exactly one neighbour in W' with at least 2 incident edges in S, and this neighbour cannot have further neighbours in W'. As a consequence, with $\delta \geq 4$ and $|T'| = |E(W') \cap S| = |\{E(x)|x \in W'\} \cap S|$ we obtain

$$|T'| \geq |W'| \geq \delta - 2. \tag{3.5}$$

By combining (3.5) with (3.4), we now deduce

$$|S| \geq |T| + |T'| + |T''| \geq \delta + r - 2 + \delta - 2 = \delta + r + (\delta - 4),$$

which is a contradiction to $|S| \leq \delta + r - 1$ for $\delta \geq 4$. In case $\delta = 3$ this deduction shows that all edges in S are incident with either u, u'' or w, where $w \in W'$, and since $|S| = r + 2$, $|T| + |T''| = r + 1$ and $|T'| = |W'| = 1$, the vertex w must have exactly one incident edge in S and one more neighbour $x \in V(U)$ besides u'. Since G is diamond-free, x can now only be adjacent to at most one of the vertices u, u' and u''. Hence, x must either have one more neighbour in U leading to $n(U) \geq 6 = 2\delta$, or an edge of S must be incident with x, a contradiction on the size of U or S.

Case 2. u and u' have no common neighbour in $V(U)$. Again, we define $W = (N(u) \cap V(U)) \setminus \{u'\}$ and $W' = (N(u') \cap V(U)) \setminus \{u\}$. Now $W \cap W' = \emptyset$ and

$$|W'| \geq \delta - 1. \tag{3.6}$$

Let $T = E(u) \cap S$. Since $|W| \geq \delta + r - 1 - |T|$, we conclude

$$2\delta - 1 \overset{(3.1)}{\geq} n(U) \geq |W| + |W'| + 2 \geq \delta + r - 1 - |T| + \delta - 1 + 2 = 2\delta + r - |T|$$

and, therefore

$$|T| \geq r + 1. \tag{3.7}$$

(3.6) and (3.7) together with $|S| \leq \delta + r - 1$ lead to the conclusion that there must be a vertex $u'' \in W'$ such that no edge in S is incident with u''. Now u'' can have at most one neighbour in W', hence we have $|W''| \geq \delta - 2$ where $W'' = N(u'') \setminus (W' \cup \{u, u'\})$, which leads us to

$$2\delta - 1 \overset{(3.1)}{\geq} n(U) \geq |W'| + |W''| + 2 \geq \delta - 1 + \delta - 2 + 2 = 2\delta - 1.$$

We conclude that u'' must have a neighbour $w' \in W'$, and since G is diamond-free, w' must be incident with at least $\delta - 2$ edges in S, i. e. $|T'| \geq \delta - 2$ where $T' = E(w') \cap S$. Furthermore, we must have $W \subseteq W''$ and

$$\delta + r - 1 \geq |S| \geq |T| + |T'| \overset{(3.7)}{\geq} r + 1 + \delta - 2 = \delta + r - 1.$$

Then it follows that $|T| = r + 1$ and thus $|W| = \delta - 2$ and $W = W''$. Now, an arbitrary vertex $w \in W(= W'')$ cannot be adjacent to u' or w', and no edge in S is incident with w. Furthermore, w cannot have a neighbour in W, otherwise we would have a diamond in U together with u and u''. Thus, $W \cap (N(w) \setminus \{u, u''\}) = \emptyset$, leading us to

$$2\delta - 1 \overset{(3.1)}{\geq} n(U) \geq |W| + |N(w) \setminus \{u, u''\}| + |\{u, u', u'', w'\}| \geq \delta - 2 + \delta - 2 + 4 = 2\delta,$$

a contradiction. □

As a direct consequence of Theorem 3.2 we obtain the following results.

Corollary 3.3 (Holtkamp [60], 2011)**.** *Let G be a diamond-free graph with $\delta(G) \geq 3$. If $n(G) \leq 4\delta(G) - 1$, then G is maximally edge-connected.*

Corollary 3.4 (Volkmann [103], 1988)**.** *Let G be a bipartite graph with $\delta(G) \geq 3$. If $n(G) \leq 4\delta(G) - 1$, then G is maximally edge-connected.*

Corollary 3.5 (Fricke, Oellermann, Swart [35], 2000)**.** *Let G be a bipartite graph with $\delta(G) \geq 3$. If $n(G) \leq 4\delta(G) - 1$, then G is maximally local edge-connected.*

To see that Theorem 3.2 and Corollary 3.3 are sharp in the sense that for every integer p there exists a diamond-free graph G with $\delta(G) = p$ and $n(G) = 4\delta(G)$, which is not maximally edge-connected and, therefore not maximally local edge-connected, we consider the following example.

Example 3.6 (Holtkamp [60], 2011)**.** *Let G be the graph obtained from two complete bipartite graphs $K_{p,p}$ ($p \geq 2$) by adding one arbitrary edge between them.*

Of course, G from Example 3.6 is diamond-free with edge-connectivity $\lambda(G) = 1$, while $\delta(G) = p$ and $n(G) = 4\delta(G)$. Therefore, G is not maximally edge-connected and not maximally local edge-connected.

To see that Theorem 3.2 does not hold for $\delta(G) = 2$, we consider the following graph.

Example 3.7 (Holtkamp [60], 2011)**.** *Let G be the graph obtained from two 3-cycles by adding one arbitrary edge between them like depicted in Figure 3.1.*

The graph G from Example 3.7 has $\delta(G) = 2$, but $\lambda(G) = 1$. Thus, G is not maximally edge-connected and not maximally local edge-connected, but we have $n(G) = 6 \leq 7 = 4\delta(G) - 1$.

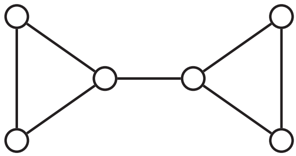

Figure 3.1: Graph showing that Theorem 3.2 does not hold for $\delta(G) = 2$.

3.2 Restricted edge-connectivity

In the last section we studied the conventional edge-connectivity of graphs. But like already pointed out in Section 1.3 there is a special interest in more refined measurements for the fault tolerance of networks. We illustrated the concept of restricted edge-connectivity, first introduced and studied by Esfahanian and Hakimi [26] in 1988. In general, graphs with larger edge-connectivity model more reliable networks. But even among graphs of the same edge-connectivity there are large differences in the reliabilities of the underlying networks due to further structural properties. Therefore, in Section 1.2.3 on page 5 we gave the basic definitions of k-restricted edge-connectivity and presented the concept of λ_k-optimality, which helps to study the edge-connectivity of graphs in a deeper sense. In Sections 3.2.1, 3.2.2 and 3.2.3 we will now discuss the λ_2-, λ_3- and λ_k-optimality of triangle-free graphs, respectively. These results have been obtained in collaboration with Luis Pedro Montejano and Dirk Meierling. Finally, in Section 3.2.4 we present similar results for p-partite graphs, which generalize some of the results on λ_k-optimality given in Section 3.2.3. These are the results of further studies with Dirk Meierling.

The k-restricted edge-connectivity we consider in this work is due to Fàbrega and Fiol [29], and has been widely studied since (e. g. [7, 13, 114, 122, 123, 126, 127]). The main idea of this concept is to exclude trivial edge-cuts from considerations. Esfahanian and Hakimi [26] showed that all graphs G except *stars* (i. e. graphs isomorphic to $K_{1,n-1}$ for $n \geq 2$) allow restricted edge-cuts and proved an upper bound for the restricted edge-connectivity number of a graph.

Theorem 3.8 (Esfahanian, Hakimi [26], 1988)**.** *Every connected graph G of order $n \geq 4$, except a star $K_{1,n-1}$, is 2-restricted edge-connected and satisfies*

$$\lambda(G) \leq \lambda_2(G) \leq \xi(G).$$

Together with a result from Bonsma, Ueffing and Volkmann [13] on λ_3 we have $\lambda_k(G) \leq \xi_k(G)$ for $1 \leq k \leq 3$ and all graphs G aside from a class of exceptions for $k = 3$ determined in [13], which we will discuss in Section 3.2.2. Also in [13] the authors give a number of examples, which show that $\lambda_k(G) \leq \xi_k(G)$ is not true in general for $k \geq 4$.

3.2.1 λ_2-optimality in triangle-free graphs

In this section we present a lower bound on the cardinality of 2-fragments in triangle-free graphs that are not λ_2-optimal in terms of ξ_2. The starting point of this work have been some recent results of Yuan and Liu [122], who gave the following sufficient condition for triangle-free graphs to be λ_2-optimal.

Theorem 3.9 (Yuan, Liu [122], 2010)**.** *Let G be a connected triangle-free graph of order $n \geq 4$. If $d(u) + d(v) \geq 2\left\lfloor \frac{n+2}{4} \right\rfloor + 1$ for each pair u, v of vertices at distance 2, then G is λ_2-optimal.*

In the same work the authors provided a sufficient criterion for a k-restricted edge-connected graph G with $\lambda_k(G) \leq \xi_k(G)$ to be λ_k-optimal.

Lemma 3.10 (Yuan, Liu [122], 2010)**.** *Let G be a k-restricted edge-connected graph with $\lambda_k(G) \leq \xi_k(G)$, and let U be a k-fragment of G. If there is a connected subgraph H of order k in $G[U]$ such that*

$$[V(H), U \setminus V(H)] \leq [U \setminus V(H), \overline{U}],$$

then G is λ_k-optimal.

If a 2-restricted edge-connected graph is not λ_2-optimal, we have the following tight lower bound on the cardinality of the 2-fragments of its 2-restricted edge-cuts.

Theorem 3.11 (Holtkamp, Meierling, Montejano [63], 2012)**.** *Let G be a 2-restricted edge-connected and triangle-free graph with minimum degree $\delta \geq 1$. If G is not λ_2-optimal, then $r_2(G) \geq \max\{3, \frac{1}{\delta}((\delta-1)\xi_2(G)+2\delta+1)\}$.*

Proof. Note that $\lambda_2(G) \leq \xi_2(G)$ and since G is not λ_2-optimal, we have $\lambda_2(G) < \xi_2(G)$. Let U be a 2-atom of G. If $|U| = 2$, then $\xi_2(G) \leq [U, \overline{U}] = \lambda_2(G)$, which is a contradiction. So $r_2(G) = |U| \geq 3$. For $\delta = 1$ we have $r_2(G) \geq 3 = \frac{1}{\delta}((\delta-1)\xi_2(G)+2\delta+1)$. Therefore, we assume $\delta \geq 2$.

Let xy be an edge with $x, y \in U$ such that $[\{x,y\}, \overline{U}]$ is minimal among all edges in U. Let $X = (N(x) \cap U) \setminus \{y\}$ and $Y = (N(y) \cap U) \setminus \{x\}$. Then $X \cup Y \neq \emptyset$ and, since G is triangle-free, we have $X \cap Y = \emptyset$. The choice of xy implies that

$$[v, \overline{U}] \geq [y, \overline{U}] \tag{3.8}$$

and $[w, \overline{U}] \geq [x, \overline{U}]$ for all vertices $v \in X$ and $w \in Y$.

If $[x, \overline{U}] \geq 1$ and $[y, \overline{U}] \geq 1$, then

$$[\{x,y\}, U \setminus \{x,y\}] = |X| + |Y| \leq [X \cup Y, \overline{U}] \leq [U \setminus \{x,y\}, \overline{U}]$$

and thus, G is λ_2-optimal by Lemma 3.10, a contradiction. So assume, without loss of generality, that $[x, \overline{U}] = 0$.
Note that

$$\begin{aligned}\xi_2(G) &\leq [\{x,y\}, V(G) \setminus \{x,y\}] \\ &= [\{x,y\}, U \setminus \{x,y\}] + [\{x,y\}, \overline{U}] \leq r_2(G) - 2 + [\{x,y\}, \overline{U}]\end{aligned}$$

and thus,

$$r_2(G) \geq \xi_2(G) + 2 - [\{x,y\}, \overline{U}].$$

Moreover, we have

$$\xi_2(G) \geq 2(\delta - 1). \tag{3.9}$$

Thus, for $[y, \overline{U}] = 0$ we obtain

$$\begin{aligned} r_2(G) &\geq \xi_2(G) + 2 - [\{x,y\}, \overline{U}] = \xi_2(G) + 2 \\ &= \frac{1}{\delta}(\delta\xi_2(G) - \xi_2(G) + \xi_2(G) + 2\delta) \\ &\overset{(3.9)}{\geq} \frac{1}{\delta}((\delta-1)\xi_2(G) + 2\delta - 2 + 2\delta) \\ &\overset{(\delta\geq 2)}{\geq} \frac{1}{\delta}((\delta-1)\xi_2(G) + 2\delta + 2), \end{aligned}$$

and we are done. Analogously, in case $[y,\overline{U}]=1$ we have

$$r_2(G) \geq \frac{1}{\delta}\left((\delta-1)\xi_2(G)+2\delta-2+\delta\right). \tag{3.10}$$

If G does not fulfill the conclusion of this theorem, then (3.10) implies $\delta=2$, $\xi_2(G)=2\delta-2$ and $r_2(G)=\xi_2(G)+1=3$. Since $d(x)\geq\delta\geq 2$, it follows that U induces the path zxy in G with $[z,\overline{U}]\geq 1$. Therefore, $[\{x,y\},U\setminus\{x,y\}]=[\{x,y\},z]=1\leq[z,\overline{U}]=[U\setminus\{x,y\},\overline{U}]$ and thus, G is λ_2-optimal by Lemma 3.10, a contradiction.

Hence, we may assume that $[y,\overline{U}]\geq 2$. If $[y,\overline{U}]\cdot|X|\geq|X|+|Y|$, then

$$[\{x,y\},U\setminus\{x,y\}]=|X|+|Y|\leq[y,\overline{U}]\cdot|X|\overset{(3.8)}{\leq}[X,\overline{U}]\leq[U\setminus\{x,y\},\overline{U}]$$

and G is λ_2-optimal by Lemma 3.10, a contradiction. So assume that $[y,\overline{U}]\cdot|X|\leq|X|+|Y|-1$. Since $|Y|\leq|U|-|X|-2$ and $|X|\geq\delta-1$,

$$[y,\overline{U}]\leq\frac{|X|+|Y|-1}{|X|}\leq\frac{|U|-3}{|X|}\leq\frac{|U|-3}{\delta-1}.$$

With the use of this inequality we deduce

$$\begin{aligned}\xi_2(G)&\leq[\{x,y\},V(G)\setminus\{x,y\}]\leq|U|-2+[y,\overline{U}]\\&\leq|U|-2+\frac{|U|-3}{\delta-1}=\frac{\delta}{\delta-1}|U|-\frac{2\delta+1}{\delta-1}.\end{aligned}$$

Since $|U|=r_2(G)$, we conclude that

$$r_2(G)\geq\frac{\delta-1}{\delta}\xi_2(G)+\frac{2\delta+1}{\delta}$$

and the proof is completed. □

The following result of Ueffing and Volkmann [101] is a direct consequence of Theorem 3.11.

Corollary 3.12 (Ueffing, Volkmann [101], 2003)**.** *Let G be a 2-restricted edge-connected and triangle-free graph with minimum degree $\delta\geq 2$. If G is not λ_2-optimal, then*

$$r_2(G)\geq\begin{cases}2\delta-1 & \text{if } \delta\geq 3,\\ 4 & \text{if } \delta=2.\end{cases}$$

Proof. Since $\xi_2(G) \geq 2(\delta - 1)$ it follows from Theorem 3.11 that

$$r_2(G) \geq \frac{1}{\delta}\left((\delta-1)\xi_2(G) + 2\delta + 1\right) \geq \frac{1}{\delta}\left(2(\delta-1)^2 + 2\delta + 1\right) = 2\delta - 2 + \frac{3}{\delta}.$$

Because $r_2(G)$ is an integer, we conclude that $r_2(G) \geq 2\delta - 1$ for $\delta \geq 3$ and $r_2(G) \geq 4$ for $\delta = 2$. □

The graphs defined in the following example show that the bound in Theorem 3.11 is tight.

Example 3.13 (Holtkamp, Meierling, Montejano [63], 2012)**.** *For $\delta \geq 2$ and $s \geq \max\{\delta - 1, 2\}$ let H_1 and H_2 be copies of the complete bipartite graphs $K_{\delta,(\delta-1)s+2}$ and $K_{s+1,\delta s}$, respectively. Join H_1 and H_2 by all possible edges between their partition sets of size δ and $s+1$ (see Figure 3.2).*

The resulting graph G of Example 3.13 is bipartite and has minimum degree δ. Furthermore, it fulfills $\lambda_2(G) = (s+1)\delta$, $\xi_2(G) = (s+1)\delta + 1$ and $r_2(G) = (\delta-1)s + 2 + \delta$. In particular, G is not λ_2-optimal. Moreover,

$$\frac{\delta-1}{\delta}\xi_2(G) + \frac{2\delta+1}{\delta} = (\delta-1)(s+1) + 3 = r_2(G)$$

which shows that the bound in Theorem 3.11 is tight.

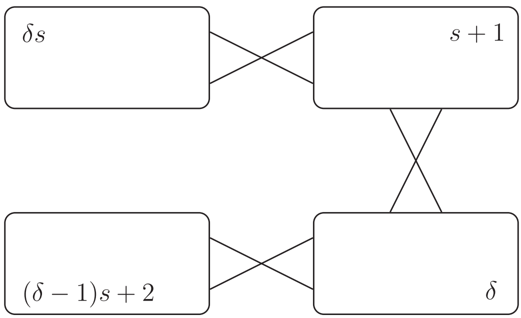

Figure 3.2: Graphs showing the tightness of Theorem 3.11.

3.2.2 λ_3-optimality in triangle-free graphs

We will now present a sufficient condition for triangle-free graphs, which fulfill an Ore-type degree condition for non-adjacent vertices, to be λ_3-optimal. Such a result has been supposed by Ou in [82].

Conjecture 3.14 (Ou [82]). *Let G be a connected triangle-free graph of order $n \geq 6$. If $d(u) + d(v) \geq \frac{n}{2} + 2$ for each pair u, v of non-adjacent vertices, then G is λ_3-optimal.*

However, it turns out that this bound is slightly inaccurate as we will see in Theorem 3.17. Moreover, we present examples that show the tightness of this result.

In 2002, Bonsma, Ueffing and Volkmann [13] characterized the graphs that are not 3-restricted edge-connected.

Theorem 3.15 (Bonsma, Ueffing, Volkmann [13], 2002). *A connected graph G is 3-restricted edge-connected if and only if $n \geq 6$ and G is not isomorphic to the net N or to any graph of the family F in Figure 3.3.*

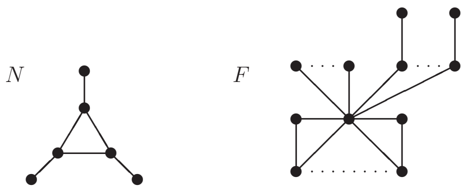

Figure 3.3: All graphs that are not 3-restricted edge-connected.

Note that N as well as every graph in F contains a triangle. The same authors showed that the following inequality is true.

Theorem 3.16 (Bonsma, Ueffing, Volkmann [13], 2002). *If G is a 3-restricted edge-connected graph, then $\lambda_3(G) \leq \xi_3(G)$.*

Now we are able to prove the main result of this section.

Theorem 3.17 (Holtkamp, Meierling, Montejano [63], 2012). *Let G be a connected triangle-free graph of order $n \geq 6$. If $d(u) + d(v) \geq 2\left\lfloor \frac{n}{4} \right\rfloor + 3$ for each pair u, v of non-adjacent vertices, then G is λ_3-optimal.*

Proof. Since G is not isomorphic to the net N and does not belong to the graph class F depicted in Figure 3.3, it follows from Theorem 3.15 that G is 3-restricted edge-connected. Thus, Theorem 3.16 yields $\lambda_3(G) \leq \xi_3(G)$.

Let $(U, \overline{U})$ be a minimum 3-restricted edge-cut of G with $4 \leq |U| \leq |\overline{U}|$. This implies $|U| \leq \frac{n}{2}$. Let H be a connected subgraph of order 3 of $G[U]$ such that $[V(H), V(G) \backslash V(H)]$ is minimal among all connected subgraphs of order 3 of $G[U]$. Note that H, as well as any other connected subgraph of G of order 3, is a path on 3 vertices, since G is triangle-free. Let $H = xyz$. Since our goal is to apply Lemma 3.10, we have to show that

$$[V(H), U \setminus V(H)] \leq [U \setminus V(H), \overline{U}].$$

Since G is triangle-free, x and y as well as y and z do not have common neighbours. Hence we may partition $N(H) \setminus (\overline{U} \cup V(H))$ as follows:

$$\begin{aligned}
X_0 &= \{v \in N(x) \setminus (\overline{U} \cup V(H) \cup N(z))\colon N(v) \cap \overline{U} = \emptyset\},\\
X_1 &= \{v \in N(x) \setminus (\overline{U} \cup V(H) \cup N(z))\colon |N(v) \cap \overline{U}| \geq 1\},\\
Y_0 &= \{v \in N(y) \setminus (\overline{U} \cup V(H))\colon N(v) \cap \overline{U} = \emptyset\},\\
Y_1 &= \{v \in N(y) \setminus (\overline{U} \cup V(H))\colon |N(v) \cap \overline{U}| \geq 1\},\\
Z_0 &= \{v \in N(z) \setminus (\overline{U} \cup V(H) \cup N(x))\colon N(v) \cap \overline{U} = \emptyset\},\\
Z_1 &= \{v \in N(z) \setminus (\overline{U} \cup V(H) \cup N(x))\colon |N(v) \cap \overline{U}| \geq 1\},\\
W_0 &= \{v \in (N(x) \cap N(z)) \setminus (\overline{U} \cup V(H))\colon N(v) \cap \overline{U} = \emptyset\},\\
W_1 &= \{v \in (N(x) \cap N(z)) \setminus (\overline{U} \cup V(H))\colon |N(v) \cap \overline{U}| = 1\},\\
W_2 &= \{v \in (N(x) \cap N(z)) \setminus (\overline{U} \cup V(H))\colon |N(v) \cap \overline{U}| \geq 2\}.
\end{aligned}$$

Claim 1. We have $d(v) \geq \left\lfloor \frac{n}{4} \right\rfloor + 2$ for all $v \in N(H)$.

Suppose that $v \in X_0 \cup X_1$. Based on the choice of H, we conclude that $d(v) \geq d(z)$. If $d(z) \geq \left\lfloor \frac{n}{4} \right\rfloor + 2$, we are done. Otherwise $d(z) \leq \left\lfloor \frac{n}{4} \right\rfloor + 1$. Since v and z are not adjacent, it follows that

$$d(v) \geq 2\left\lfloor \frac{n}{4} \right\rfloor + 3 - d(z) \geq \left\lfloor \frac{n}{4} \right\rfloor + 2.$$

We can analogously show that Claim 1 is true if $v \in Z_0 \cup Z_1$ or $v \in W_0 \cup W_1 \cup W_2$.

Suppose that $v \in Y_0 \cup Y_1$. Based on the choice of H, we conclude that $d(v) \geq \max\{d(x), d(z)\}$. Since x and z are not adjacent, it follows that

$$d(v) \geq \max\{d(x), d(z)\} \geq \left\lfloor \frac{n}{4} \right\rfloor + 2.$$

So Claim 1 is proved.

Claim 2. $X_0 \cup Y_0 \cup Z_0 \cup W_0 \cup W_1$ is an independent vertex set.

Suppose that $u, v \in X_0 \cup Y_0 \cup Z_0 \cup W_0 \cup W_1$ such that u and v are adjacent. Since G is triangle-free, their respective neighbourhoods are disjoint. Furthermore, at most one of them is in W_1. It follows that

$$|U| \geq d(u) + d(v) - [\{u, v\}, \overline{U}] \geq d(u) + d(v) - 1 \overset{\text{(Claim 1)}}{\geq} 2\left(\left\lfloor \frac{n}{4} \right\rfloor + 2\right) - 1 > \frac{n}{2},$$

a contradiction. So Claim 2 is proved.

Now we distinguish two cases depending on the number of vertices in X_0, Y_0, Z_0, W_0 and W_1.

Case 1. Suppose that $X_0 = Y_0 = Z_0 = W_0 = W_1 = \emptyset$. Then

$$\begin{aligned}
[V(H), U \setminus V(H)] &= |X_1| + |Y_1| + |Z_1| + 2|W_2| \\
&\leq \sum_{v \in X_1 \cup Y_1 \cup Z_1 \cup W_2} |N(v) \cap \overline{U}| \\
&\leq [U \setminus V(H), \overline{U}].
\end{aligned}$$

Hence G is λ_3-optimal by Lemma 3.10.

Case 2. Suppose that $A = X_0 \cup Y_0 \cup Z_0 \cup W_0 \cup W_1 \neq \emptyset$. Let $B_1 = N(A) \cap N(V(H)) \cap U$, $B_2 = (U \cap N(A)) \setminus N[V(H)]$ and $B = B_1 \cup B_2$. Note that A is an independent set by Claim 2. Furthermore, A, B_1 and B_2 are disjoint subsets of U. Let a be an arbitrary vertex in A. Note that a has at least

$$|N(a) \cap B| \geq \left\lfloor \frac{n}{4} \right\rfloor + 2 - [a, V(H)] - [a, \overline{U}] \geq \left\lfloor \frac{n}{4} \right\rfloor - [a, \overline{U}] \geq \left\lfloor \frac{n}{4} \right\rfloor - 1 \qquad (3.11)$$

neighbours in B.

If $b \in N(a) \cap B_1$, then

$$2\left(\left\lfloor \frac{n}{4} \right\rfloor + 2\right) \leq d(a) + d(b) \leq |U| + [\{a, b\}, \overline{U}].$$

Since $|U| \leq \frac{n}{2}$, it follows that $[\{a, b\}, \overline{U}] \geq 3$. Since G is triangle-free, $a \in W_0 \cup W_1$ implies $b \notin W_2$ and therefore $[V(H), \{a, b\}] \leq 3$. All in all we conclude that

$$[V(H), \{a, b\}] \leq 3 \leq [\{a, b\}, \overline{U}].$$

If $b, b' \in N(a) \cap B_2$, then b and b' are not adjacent. It follows that

$$\begin{aligned} 2\left\lfloor \frac{n}{4} \right\rfloor + 3 &\le d(b) + d(b') \\ &\le 2(|U| - |N(a) \cap B| - |V(H)|) + [\{b, b'\}, \overline{U}] \\ &\le n - 2\left(\left\lfloor \frac{n}{4} \right\rfloor - 1\right) - 6 + [\{b, b'\}, \overline{U}] \\ &= n - 2\left\lfloor \frac{n}{4} \right\rfloor - 4 + [\{b, b'\}, \overline{U}] \end{aligned}$$

and thus,

$$[\{b, b'\}, \overline{U}] \ge 4\left\lfloor \frac{n}{4} \right\rfloor - n + 7 \ge 4.$$

Therefore, there exists at most one vertex $b \in N(a) \cap B_2$ with $[b, \overline{U}] \le 1$. Let $B' = B_1 \cup B_2'$ with $B_2' = \{b \in B_2 \colon [b, \overline{U}] \ge 2\}$. Then

$$[V(H), \{a, b\}] \le 2 \le [\{a, b\}, \overline{U}]$$

for every $b \in N(a) \cap B_2'$.

If there exists a matching M of size $|A|$ connecting vertices of A and B', then

$$\begin{aligned} [U \setminus V(H), \overline{U}] &\ge \sum_{ab \in M} [\{a, b\}, \overline{U}] + \sum_{v \in N(H) \setminus V(M)} [v, \overline{U}] \\ &\ge \sum_{ab \in M} [V(H), \{a, b\}] + \sum_{v \in N(H) \setminus V(M)} [V(H), v] \\ &= [V(H), U \setminus V(H)] \end{aligned}$$

and thus, G is λ_3-optimal by Lemma 3.10. We distinguish two cases to show the existence of such a matching.

Subcase 2.1. Suppose that $A \setminus W_1 \neq \emptyset$. For any vertex $a \in A \setminus W_1$, (3.11) implies that $|B| \ge |N(a) \cap B| \ge \left\lfloor \frac{n}{4} \right\rfloor$. Thus

$$|A| \le |U| - |B| - |V(H)| \le \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor - 3.$$

Moreover,

$$|N(a) \cap B'| \ge |N(a) \cap B| - 1 \ge \left\lfloor \frac{n}{4} \right\rfloor - 1$$

and thus $|N(a) \cap B'| \ge |A|$ for every vertex $a \in A$. For $\emptyset \neq S \subseteq A$ we now have

$$|N(S) \cap B'| \ge |A| \ge |S|,$$

and therefore according to König [68] and Hall [48] there is a maximum matching M of size $|A|$ between A and B'.

Subcase 2.2. Suppose that $A = W_1$. Then $|B| \geq |N(a) \cap B| \geq \lfloor \frac{n}{4} \rfloor - 1$ for every vertex $a \in A$. Furthermore,

$$|A| \leq |U| - |B| - |V(H)| \leq \left\lfloor \frac{n}{2} \right\rfloor - \left(\left\lfloor \frac{n}{4} \right\rfloor - 1 \right) - 3 = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor - 2$$

and

$$|N(a) \cap B'| \geq |N(a) \cap B| - 1 \geq \left\lfloor \frac{n}{4} \right\rfloor - 2.$$

If $n = 4r + s$ with $0 \leq s \leq 3$, it follows that

$$|A| \leq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor - 2 < r \quad \text{and} \quad |N(a) \cap B'| \geq \left\lfloor \frac{n}{4} \right\rfloor - 2 = r - 2.$$

Hence $|N(a) \cap B'| \geq |A| - 1$ and $|N(a) \cap B| \geq |A|$ for every vertex $a \in A$. It follows that there exists a matching M of size $|A| - 1$ connecting vertices of A and B'. If M can be extended to a matching of size $|A|$ connecting vertices of A and B', then we are done. So assume that M cannot be extended in this way. Let a denote the single vertex in $a \in A \setminus V(M)$.

If $|A| \geq 2$, then $|N(a) \cap B'| \geq |A| - 1 \geq 1$. Moreover, a has at most $|A| - 1$ neighbours in B', since M cannot be extended. Furthermore, $|N(a) \cap B| \geq |A|$ implies that a has another neighbour in $B_2 \setminus B'$. Let b and b' denote these neighbours such that $b \in B_2 \setminus B'$ and $b' \in B'$ with $a'b' \in M$. Note that b and b' are not adjacent and $b' \notin W_2$, since $a \in W_1$, so b' can have at most one neighbour in $V(H)$. Then

$$\begin{aligned}
2 \left\lfloor \frac{n}{4} \right\rfloor + 3 &\leq d(b) + d(b') \\
&\leq 2(|U| - |N(a) \cap B| - |V(H)|) + 1 + [\{b, b'\}, \overline{U}] \\
&\leq n - 2 \left\lfloor \frac{n}{4} \right\rfloor - 3 + [\{b, b'\}, \overline{U}]
\end{aligned}$$

and thus, $[\{b, b'\}, \overline{U}] \geq 3$. It follows that

$$[\{a, a', b, b'\}, \overline{U}] \geq 5 \geq [V(H), \{a, a', b, b'\}].$$

If $|A| = 1$, then $A = \{a\}$ and $B' = \emptyset$. Note that $|B| \geq |A|$ and thus, $B_2 = B \neq \emptyset$. Let $b \in B_2$ be an arbitrary vertex. Then $[b, \overline{U}] \geq 2$ and thus,

$$[\{a, b\}, \overline{U}] \geq 2 \geq [V(H), \{a, b\}].$$

In both cases, let $M' = M \cup \{ab\}$. Then

$$\begin{aligned}
[U \setminus V(H), \overline{U}] &\geq \sum_{vw \in M'} [\{v,w\}, \overline{U}] + \sum_{v \in N(H) \setminus V(M')} [v, \overline{U}] \\
&\geq \sum_{vw \in M'} [V(H), \{v,w\}] + \sum_{v \in N(H) \setminus V(M')} [V(H), v] \\
&= [V(H), U \setminus V(H)]
\end{aligned}$$

and thus, G is λ_3-optimal by Lemma 3.10. This completes the proof of the theorem. □

The following examples show that the lower bound in the Ore-type condition of the theorem above is tight.

Example 3.18 (Holtkamp, Meierling, Montejano [63], 2012)**.** *Figure 3.4 shows the graphs defined in a) and b):*

a) Let H_1 be a copy of the complete bipartite graph $K_{r,r}$ and H_2 a copy of the complete bipartite graph $K_{r,s}$, where $s \in \{r, r+1\}$. Let U_i, V_i be the partition sets of H_i for $i = 1, 2$ such that $|V_2| = s$. If $r = 2$ and $s = 3$, let $x \in V_2$. Join x by two edges to V_1 and join $V_2 \setminus \{x\}$ by a perfect matching to U_1. Otherwise join U_1 and U_2 as well as V_1 and V_2 by a matching of size r. For $s = r + 1$ join the remaining vertex of V_2 to an arbitrary vertex of V_1.

b) Let H_1 be a copy of the complete bipartite graph $K_{r,r+1}$ and H_2 a copy of the complete bipartite graph $K_{s,r+1}$, where $s \in \{r, r+1\}$. Let U_i, V_i be the partition sets of H_i for $i = 1, 2$. Join V_1 and V_2 by a perfect matching.

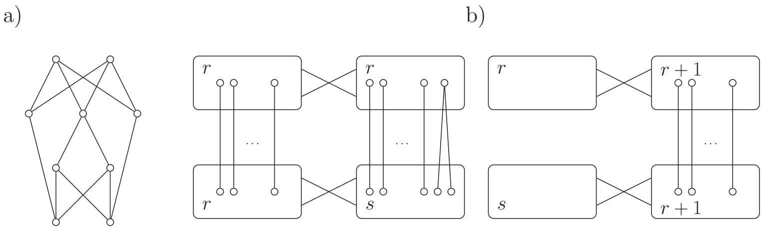

Figure 3.4: Graphs of Example 3.18 showing the tightness of Theorem 3.17.

For each $n = 4r + t$ in Example 3.18, where $r \geq 2$ and $0 \leq t \leq 3$, we have defined exactly one graph G_n. These graphs are triangle-free, have minimum degree $r + 1$, and fulfill $d(u) + d(v) \geq 2\left\lfloor \frac{n}{4} \right\rfloor + 2$ for each pair u, v of vertices. Furthermore, the edges between H_1 and H_2 form a 3-restricted edge-cut of size $< \xi_3(G_n) = 3r - 1$. Hence the graphs are not λ_3-optimal.

3.2.3 λ_k-optimality in triangle-free graphs

In this section we show that triangle-free graphs with high minimum degree are λ_k-optimal and super-λ_k. By definition, if G is super-λ_k, then G is λ_k-optimal. However, the converse is not true. For example, a cycle of length $n \geq 2k + 2$ is λ_k-optimal but not super-λ_k. Recent works on super k-restricted edge-connectivity can be found for example in Balbuena and García-Vázquez [4], and Wang, Lin and Li [113].

In 2005, Zhang and Yuan [126] proved that, except for the class of flowers, graphs with minimum degree greater than or equal to $k-1$ are k-restricted edge-connected, where a connected graph G with $|V(G)| \geq 2k$ is called a *flower* if it contains a cut vertex u such that every component of $G - u$ has order at most $k - 1$. Moreover, for the same class of graphs they showed that $\lambda_k(G) \leq \xi_k(G)$.

Theorem 3.19 (Zhang, Yuan [126], 2005)**.** *Let G be a connected graph not isomorphic to a flower and k a positive integer with $k \leq \delta(G)+1$. Then G is k-restricted edge-connected and $\lambda_k(G) \leq \xi_k(G)$.*

Furthermore, for graphs that are not λ_k-optimal, in 2007 the same authors gave a lower bound on the order of their k-fragments.

Theorem 3.20 (Zhang, Yuan [127], 2007)**.** *Let G be a k-restricted edge-connected graph with minimum degree δ. If $\lambda_k(G) < \xi_k(G)$, then $r_k(G) \geq \max\{k+1, \delta - k + 1\}$.*

The first result of this section, namely Theorem 3.21, will present a new lower bound on the order of these k-fragments, which is a generalization of the earlier result from Ueffing and Volkmann [101] in Corollary 3.12 for the case $k = 2$.

Theorem 3.21 (Holtkamp, Meierling, Montejano [63], 2012)**.** *Let G be a k-restricted edge-connected and triangle-free graph with $\lambda_k(G) \leq \xi_k(G)$ and minimum degree δ. If G is not λ_k-optimal, then $r_k(G) \geq \max\{k + 1,\ 2\delta - k + 1\}$.*

Since Theorem 3.21 is a special case of Theorem 3.33 presented in the next section, we omit the proof here. The following example shows that the bound given in Theorem 3.21 is tight.

Example 3.22 (Holtkamp, Meierling, Montejano [63], 2012)**.** *For $\delta \geq 2$ and $s > \frac{k^2}{2} - \frac{1}{2}$ when k is odd and $s > \frac{k^2}{2}$ when k is even, let H_1, H_2 be two copies of the complete bipartite graph $K_{s-k+1,s}$. Let U_i, V_i be the partition sets of H_i for $i = 1, 2$, then we join V_1 and V_2, i. e. the independence sets of cardinality s, by $k-1$ perfect matchings.*

The resulting graph G from Example 3.22 is bipartite and s-regular. Furthermore, it fulfills $\lambda_k(G) \leq ks - s$, $\xi_k(G) \geq ks - \frac{k^2}{2} + \frac{1}{2}$ if k is odd, and $\xi_k(G) \geq ks - \frac{k^2}{2}$ if k is even. Therefore, G fulfills $\lambda_k(G) < \xi_k(G)$. Moreover, $r_k(G) = 2s - k + 1$, which shows that the bound in Theorem 3.21 is tight.

In 2009, Yuan, Liu and Wang [123] showed the λ_k-optimality for bipartite graphs with high minimum degree.

Theorem 3.23 (Yuan, Liu & Wang [123], 2009)**.** *Let G be a bipartite graph of order $n \geq 2k$. If $\delta(G) \geq \frac{n+2k}{4}$, then G is λ_k-optimal.*

With the use of Theorem 3.21 it is now very easy to prove the following generalization of this result.

Corollary 3.24 (Holtkamp, Meierling, Montejano [63], 2012)**.** *Let G be a connected and triangle-free graph of order $n \geq 2k$. If*

$$\delta(G) \geq \frac{1}{2}\left(\left\lfloor \frac{n}{2} \right\rfloor + k\right),$$

then G is λ_k-optimal.

Proof. Since $k \leq \delta(G) + 1$, by Theorem 3.19 it follows that $\lambda_k(G) \leq \xi_k(G)$. Therefore, as $r_k(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$, the inequality is an equality by Theorem 3.21. □

As a special case of Theorem 3.38 in the next section, we can give a lower bound on the order of the k-fragments of a λ_k-optimal graph that are larger than k.

Theorem 3.25 (Holtkamp, Meierling, Montejano [63], 2012)**.** *Let G be a λ_k-optimal and triangle-free graph. If U is a k-fragment of G with $|U| \geq k+1$, then $|U| \geq 2\delta(G) - k$.*

As a consequence of this result, we obtain Corollary 3.27, which is a generalization of this earlier result from Yuan, Liu and Wang [123] in 2009.

Theorem 3.26 (Yuan, Liu, Wang [123], 2009)**.** *Let G be a bipartite graph of order $n \geq 2k$. If $\delta(G) \geq \frac{n+2k+3}{4}$, then G is super-λ_k.*

Corollary 3.27 (Holtkamp, Meierling, Montejano [63], 2012)**.** *Let G be a triangle-free graph of order $n \geq 2k$. If*

$$\delta(G) \geq \frac{1}{2}\left(\left\lfloor \frac{n}{2} \right\rfloor + k + 1\right),$$

then G is super-λ_k.

Proof. By Corollary 3.24, G is λ_k-optimal. Suppose on the contrary that G is not super-λ_k. Then there exists a k-fragment U such that $|U| \geq k+1$ and $|\overline{U}| \geq k+1$. We may suppose that $|U| \leq |\overline{U}|$ which means $|U| \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Therefore, combining the fact that $\delta(G) \geq \frac{1}{2}(\left\lfloor \frac{n}{2} \right\rfloor + k + 1)$ with Theorem 3.25, it follows that $|U| \geq 2\delta(G) - k \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$, contradicting $|U| \leq \left\lfloor \frac{n}{2} \right\rfloor$. □

The following upper bound for ξ_k in regular graphs is trivial. (Take a tree of order k in G and count the outgoing edges.)

Observation 3.28. *If G is a δ-regular graph, then $\xi_k(G) \leq k\delta - 2(k-1)$.*

Together with Corollary 3.27 this observation gives a lower bound on the order of k-fragments in terms of ξ_k for regular and triangle-free graphs that are not λ_k-optimal.

Corollary 3.29 (Holtkamp, Meierling, Montejano [63], 2012)**.** *Let G be a k-restricted edge-connected, δ-regular and triangle-free graph with $\lambda_k(G) \leq \xi_k(G)$. If G is not λ_k-optimal, then $r_k(G) \geq \frac{2}{k}(\xi_k(G) - 1) + 5 - k$.*

Proof. Since $\xi_k(G) \leq k\delta - 2(k-1)$, it follows that $\frac{\xi_k(G)+2k-2}{k} \leq \delta$. Then, by Theorem 3.21, we have

$$r_k(G) \geq 2\delta - k + 1 \geq 2\left(\frac{\xi_k(G) + 2k - 2}{k}\right) - k + 1 = \frac{2}{k}(\xi_k(G) - 2) + 5 - k.$$

□

3.2.4 λ_k-optimality in p-partite graphs

We call a p-partite graph G *balanced* if the cardinality of its largest and smallest partite sets differ by at most 1. The number of edges in the complete p-partite balanced graph on n vertices is denoted by $t_p(n)$. In order to prove the main result of this section we will use the following variation of Turán's Theorem 2.3.

Lemma 3.30 (Holtkamp, Meierling [61]). *Let G be a connected graph of order n with clique number ω and W a connected subgraph of G of order k with the maximum number of edges among all connected subgraphs of G of order k. If*

$$|E(G)| > t_\omega(n) + |E(W)| - t_\omega(k),$$

then

(a) there exists a vertex $v \in V(G) \setminus V(W)$ such that

$$|N(v) \cap V(W)| \geq \left\lceil \frac{(\omega - 1)k + 1}{\omega} \right\rceil;$$

(b) for every vertex $w \in N(v) \cap V(W)$ it is $|N(w) \cap V(W)| \geq |N(v) \cap V(W)| - 1$;

(c) for every vertex $w \in V(W) \setminus N(v)$ it is $|N(w) \cap V(W)| \geq |N(v) \cap V(W)|$.

Proof. For $k = 1$ it is $|E(G)| \leq t_\omega(n) = t_\omega(n) + |E(W)| - t_\omega(k)$ by Theorem 2.3. So let $k \geq 2$ and suppose that $|E(G)| > t_\omega(n) + |E(W)| - t_\omega(k)$. Among all graphs of order n that do not fulfill the desired inequality, let G be chosen such that $|E(G)|$ is maximal.

Let G^* be a balanced ω-partite graph of order n with partite sets $V_1, V_2, \ldots, V_\omega$ fulfilling $\lceil n/\omega \rceil \geq |V_1| \geq |V_2| \geq \cdots \geq |V_\omega| \geq \lfloor n/\omega \rfloor$. Furthermore, let W^* be an induced balanced ω-partite subgraph of G^* of order k with partite sets $W_i \subset V_i$ fulfilling $\lceil k/\omega \rceil \geq |W_1| \geq |W_2| \geq \cdots \geq |W_\omega| \geq \lfloor k/\omega \rfloor$. As the $K_{\omega+1}$-free graph with the maximum number of edges is a balanced ω-partite graph by Theorem 2.3, we define the edges of G^* such that $|E(G^*)| = |E(G)|$ and $|E(W^*)|$ is minimum. From this construction it follows that

$$|E(G)| = |E(G^*)| \leq t_\omega(n) + |E(W^*)| - t_\omega(k).$$

If $|E(W^*)| \leq |E(W)|$, then $|E(G)| \leq t_\omega(n) + |E(W)| - t_\omega(k)$ holds, a contradiction. Thus, let

$$|E(W)| < |E(W^*)|. \tag{3.12}$$

Hence, $|E(W^*)| \geq 1$ and $|E(G^* - W^*)| = t_\omega(n - k)$. Furthermore, by Theorem 2.3 we have

$$|E(G - W)| \leq |E(G^* - W^*)| \tag{3.13}$$

as $G^* - W^*$ is a complete balanced ω-partite graph. Since

$$\begin{aligned}&|E(G^* - W^*)| + [V(G^* - W^*), V(W^*)] + |E(W^*)|\\ =&|E(G - W)| + [V(G - W), V(W)] + |E(W)|,\end{aligned}$$

it follows from (3.12) and (3.13) that

$$[V(G-W),V(W)] \geq [V(G^*-W^*),V(W^*)]+1 \geq \left\lceil \frac{\omega-1}{\omega}k(n-k)\right\rceil +1.$$

Hence, there exists at least one vertex $v \in V(G)\setminus V(W)$ such that

$$|N(v)\cap V(W)| \geq \left\lceil \frac{(\omega-1)k+1}{\omega}\right\rceil .$$

This proves (a).

Before we turn our attention to (b) and (c), we shall show that for every $w \in W$, the graph W' induced by the vertex set $(V(W)-\{w\})\cup\{v\}$ is connected. Assume that W' is not connected. Note that the vertex set $(N(v)\cap W)\cup\{v\}$ belongs to one component of W'. Hence, there exists a component C of W' whose vertex set is contained in $V(W)\setminus N(v)$. Since C is either a single vertex or has a spanning tree with a leaf that is not a cut-vertex of W, there exists a vertex $u \in V(C)$ such that $W-u$ is connected. Consequently, the graph W'' induced by the vertex set $(V(W)-\{u\})\cup\{v\}$ is also connected. Since $|N(u)\cap V(W)| \leq |V(C)| < |N(v)\cap V(W)|$ and u and v are not adjacent, it follows that W'' is a connected subgraph of G of order k with $|E(W'')| > |E(W)|$, a contradiction.

Now we are able to prove (b) and (c). Assume that the inequality is not valid for a vertex w. Then the graph W' induced by $(V(W)-\{w\})\cup\{v\}$ is a connected subgraph of G of order k with $|E(W')| > |E(W)|$. This contradiction completes the proof of the lemma. □

Lemma 3.31 (Holtkamp, Meierling [61])**.** *Let G be a connected graph of order n with clique number ω and W a connected subgraph of G of order k with the maximum number of edges among all connected subgraphs of G of order k. If ω divides k or if ω divides $k+1$, then*

$$|E(G)| \leq t_\omega(n)+|E(W)|-t_\omega(k).$$

Proof. Assume that $|E(G)| > t_\omega(n)+|E(W)|-t_\omega(k)$. With $t = |N(v)\cap V(W)|$ it follows by Lemma 3.30 (b) and (c) that

$$|E(W)| \geq t(t-1)+(k-t)t = (k-1)t.$$

If ω divides k, then $t \geq \frac{\omega-1}{\omega}k+1$ by Lemma 3.30 (a). Hence,

$$|E(W)| \geq (k-1)\left(\frac{\omega-1}{\omega}k+1\right) \geq \frac{\omega-1}{\omega}k^2$$

and thus, W is a complete balanced ω-partite graph of order k.

If ω divides $k+1$, then $t \geq \frac{\omega-1}{\omega}(k+1)$ by Lemma 3.30 (a). Hence,

$$|E(W)| \geq \frac{\omega-1}{\omega}(k^2-1) \geq \left\lfloor \frac{\omega-1}{\omega}k^2 \right\rfloor$$

and again, W is a complete balanced ω-partite graph of order k.

Therefore, it is $|E(W)| = t_\omega(k)$ in both cases and the lemma holds by Theorem 2.3. □

Lemma 3.32 (Holtkamp, Meierling [61]). *Let G be a connected graph of order n with chromatic number χ. If W is a connected subgraph of G of order k with the maximum number of edges among all connected subgraphs of G of order k, then*

$$|E(G)| \leq t_\chi(n) + |E(W)| - t_\chi(k).$$

Proof. Assume that $|E(G)| > t_\chi(n) + |E(W)| - t_\chi(k)$.

Firstly, we shall show that if H is a subgraph of G with $|V(H)| = k$ and $|E(H)| \geq |E(W)|$, then H is connected. Assume that H is not connected. With $t = |N(v) \cap V(W)|$ it follows by Lemma 3.30 that

$$\begin{aligned} 2|E(W)| = \sum_{w \in V(W)} |N(w) \cap V(W)| &\geq t(t-1) + (k-t)t = t(k-1) \\ &> \frac{\chi-1}{\chi}(k-1)^2 \geq 2t_\chi(k-1) \geq 2|E(H)|, \end{aligned}$$

a contradiction.

Secondly, we shall show that G is a complete balanced χ-partite graph. Recall that

$$|E(G)| \geq t_\chi(n) + |E(W)| - t_\chi(k) + 1,$$

where W is a connected subgraph of G of order k and the maximum number of edges. Furthermore,

$$|E(W)| = \max\{|E(H)| \colon H \subset G, |V(H)| = k\},$$

since we have shown in the first part of the proof that every subgraph of G that has at least as many edges as W is connected.

Obviously there exists a partition of $V(G)$ into χ independent sets. Assume that G is not complete χ-partite. Let e be an edge between two partite sets that is missing in G. Then $G' = G \cup \{e\}$ fulfills

$$|E(G')| = |E(G)| + 1 \geq t_\chi(n) + |E(W)| - t_\chi(k) + 2 \geq t_\chi(n) + |E(W')| - t_\chi(k) + 1,$$

where W' is a connected subgraph of G' of order k with the maximum number of edges, a contradiction to the maximality of G.

Assume that G is not balanced. Without loss of generality, let V_1 and V_χ be the largest and the smallest partite set of G, respectively, with $|V_1| = |V_\chi| + r$ for an integer $r \geq 2$. By removing one vertex from V_1 and adding it to V_χ we obtain a graph G'' with

$$\begin{aligned} |E(G'')| &= |E(G)| + (r-1) \\ &\geq t_\chi(n) + |E(W)| - t_\chi(k) + r \\ &\geq t_\chi(n) + |E(W'')| - (r-1) - t_\chi(k) + r \\ &= t_\chi(n) + |E(W'')| - t_\chi(k) + 1, \end{aligned}$$

where W'' is a connected subgraph of G'' of order k with the maximum number of edges, a contradiction to the maximality of G.

Since G is a complete balanced χ-partite graph, W is also complete balanced χ-partite. This obvious contradiction completes the proof of the lemma. □

We can now present the main result of this section.

Theorem 3.33 (Holtkamp, Meierling [61]). *Let G be a k-restricted edge-connected graph with minimum degree δ, clique number ω and chromatic number χ. If $\lambda_k(G) < \xi_k(G)$, then $r_k(G) \geq k+1$ and*

$$(\chi - 1)(r_k(G) + k) \geq \chi\delta - \chi + 2\sqrt{\chi}.$$

Furthermore, if p divides k or if p divides $k+1$, where $p = \omega$ or $p = \chi$, then

$$(p-1)(r_k(G) + k) \geq p\delta + 1.$$

Proof. Let $\lambda_k = \lambda_k(G)$, $\xi_k = \xi_k(G)$ and $r_k = r_k(G)$. Let U be a λ_k-atom of G. If $|U| = k$, then $\xi_k \leq [U, \overline{U}] = \lambda_k$, a contradiction. So assume that $r_k = |U| \geq k+1$. Let W be a connected subgraph of $G[U]$ of order k with the maximum number of edges among all connected subgraphs of $G[U]$ of order k. Let $p = \omega$ or $p = \chi$. By Lemma 3.32, we have

$$|E(G[U])| \leq t_p(r_k) + |E(W)| - t_p(k).$$

As $\lambda_k = [U, \overline{U}] = \sum_{v \in U} d(v) - 2|E(G[U])|$, it follows that

$$\lambda_k \geq \sum_{v \in U} d(v) - 2\left(t_p(r_k) + |E(W)| - t_p(k)\right).$$

Since
$$\xi_k \le [V(W), \overline{V(W)}] = \sum_{v\in W} d(v) - 2|E(W)|$$
and $\lambda_k \le \xi_k - 1$, we have
$$\delta(r_k - k) \le \sum_{v\in U\setminus V(W)} d(v) \le 2t_p(r_k) - 2t_p(k) - 1. \tag{3.14}$$
With (3.14) and $s = (p-1)r_k^2/p - 2t_p(r_k)$ and $t = (p-1)k^2/p - 2t_p(k)$ it follows that
$$2t_p(r_k) - 2t_p(k) = \frac{p-1}{p}\left(r_k^2 - k^2\right) + t - s. \tag{3.15}$$
Let $k = ap + b$ with integers $a \ge 0$ and $0 \le b \le p-1$. Then
$$2t_p(k) = b(a+1)(k-(a+1)) + (p-b)a(k-a) = k(k-a) - ab - b,$$
whereas
$$\frac{p-1}{p}k^2 = \frac{p-1}{p}(ap+b)^2 = k(k-a) - ab - \frac{b^2}{p}.$$
Likewise, if $r_k = cp + d$ with integers $c \ge 0$ and $0 \le d \le p-1$, then
$$2t_p(r_k) = r_k(r_k - c) - cd - d \quad\text{and}\quad \frac{p-1}{p}r_k^2 = r_k(r_k-c) - cd - \frac{d^2}{p}.$$
It follows that
$$t - s = b - d - \frac{b^2 - d^2}{p}. \tag{3.16}$$
Using (3.14), (3.15) and (3.16), we conclude that
$$\delta(r_k - k) \le \frac{p-1}{p}\left(r_k^2 - k^2\right) + b - d - \frac{b^2-d^2}{p} - 1. \tag{3.17}$$
If $b = 0$, i. e. if p divides k, we conclude that
$$\delta(r_k - k) \le \frac{p-1}{p}\left(r_k^2 - k^2\right) - 1.$$
Since δ, r_k and k are integers, it follows that
$$(p-1)(r_k(G) + k) \ge p\delta + 1.$$

So assume that $b \geq 1$. Then

$$\begin{aligned}
\delta &\leq \frac{p-1}{p}(r_k+k) + \frac{b-d-\frac{b^2-d^2}{p}-1}{r_k-k} \\
&= \frac{p-1}{p}(r_k+k) + \underbrace{\frac{b-d-\frac{b^2-d^2}{p}-1}{(c-a)p+d-b}}_{=f(b,d)}. \qquad (3.18)
\end{aligned}$$

We now take a closer look on the function $f(b,d)$.

Case 1. Let $d > b$. From the definition of b and d we have $c-a \geq 0$ and $1 \leq b < d \leq p-1$. Note that the maximum of f is positive. Hence,

$$\max f(b,d) \leq \max g(b,d),$$

where

$$g(b,d) = -1 + \frac{b+d}{p} + \frac{1}{b-d}.$$

Firstly, we look for a maximum of g in the interior of the set

$$N = \{(b,d) \in \mathbb{R}^2 \colon b \geq 1, b+1 \leq d \leq p-1\}.$$

Since

$$\frac{\partial}{\partial d} g(b,d) = \frac{1}{p} + \frac{1}{(b-d)^2} > 0$$

for all points $(b,d) \in N$, the maximum of g on N has to be on the boundary of N. If $b=1$, then $g(1,d) = -1 + \frac{d+1}{p} - \frac{1}{d-1} < 0$. If $b = d-1$, then $g(d-1,d) = -2 + \frac{2d-1}{p} < 0$. Finally, for $d = p-1$ we have $g(b,p-1) = -1 + \frac{b+p-1}{p} - \frac{1}{p-1-b} = g_3(b)$ with

$$g_3'(b) = \frac{1}{p} + \frac{1}{(p-1-b)^2} = 0 \iff b = p-1 \pm \sqrt{p}.$$

Since $(p-1+\sqrt{p}, d) \notin N$ and $g_3''(p-1-\sqrt{p}) < 0$, the function g_3 has a maximum in $b = p-1-\sqrt{p}$ with value

$$g_3(p-1-\sqrt{p}) = 1 - \frac{2}{p} - \frac{2}{\sqrt{p}} < 1 - \frac{2}{\sqrt{p}}.$$

Hence

$$\max f(b,d) \leq \max g(b,d) \leq g_3(p-1-\sqrt{p}) = 1 - \frac{2}{p} - \frac{2}{\sqrt{p}} < 1 - \frac{2}{\sqrt{p}}. \qquad (3.19)$$

Case 2. Let $d \le b$. Since $r_k > k$ we have $c - a \ge 1$. Furthermore, $1 \le b \le p-1$ and $0 \le d \le b$. Note again that the maximum of f is positive. Hence,

$$\max f(b,d) \le \max g(b,d),$$

where

$$g(b,d) = \frac{b - d - \frac{b^2-d^2}{p} - 1}{p+d-b}.$$

Firstly, we look for a maximum of g in the interior of the set $M = \{(b,d) \in \mathbb{R}^2 \colon 1 \le b \le p-1, 0 \le d \le b\}$. It is

$$\begin{aligned}\frac{\partial}{\partial b} g(b,d) &= \frac{1}{(p+d-b)^2}\left(b - d - \frac{b^2-d^2}{p} - 1\right) + \frac{1}{p+d-b}\left(1 - \frac{2b}{p}\right)\\ &= \frac{1}{(p+d-b)^2}\left(b - d - \frac{b^2-d^2}{p} - 1 + (p+d-b)\left(1 - \frac{2b}{p}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial d} g(b,d) &= \frac{-1}{(p+d-b)^2}\left(b - d - \frac{b^2-d^2}{p} - 1\right) + \frac{1}{p+d-b}\left(-1 + \frac{2d}{p}\right)\\ &= \frac{1}{(p+d-b)^2}\left(-b + d + \frac{b^2-d^2}{p} + 1 + (p+d-b)\left(-1 + \frac{2d}{p}\right)\right).\end{aligned}$$

Furthermore, we obtain

$$\frac{\partial}{\partial b} g(b,d) = 0 \iff \frac{1}{p}(b-d)^2 - 1 + p - 2b = 0 \tag{3.20}$$

$$\frac{\partial}{\partial d} g(b,d) = 0 \iff \frac{1}{p}(b-d)^2 + 1 - p + 2d = 0. \tag{3.21}$$

Subtracting (3.20) from (3.21) we get

$$2 - 2p + 2d + 2b = 0 \iff b = p - d - 1.$$

Insertion of $b = p - d - 1$ into (3.20) leads to

$$\frac{1}{p}(p - 2d - 1)^2 - 1 + p - 2(p - d - 1) = -\frac{1}{p}(p - 2d - 1)(2d + 1) = 0.$$

Since $d \ge 0$, we conclude that $d = (p-1)/2$. Together with $b = p - d - 1$ we obtain $b = d = (p-1)/2$. Since $g((p-1)/2, (p-1)/2) = -1/p$, it is sufficient to look at the boundary of M for a maximum of g.

Now we look for a maximum of g on the boundary of the triangle M. For $b = d$ we have $g(b, d) = -1/p$. If $d = 0$, we obtain

$$g(b, 0) = \frac{1}{p-b}\left(b - \frac{b^2}{p} - 1\right) = \frac{b\left(1 - \frac{b}{p}\right)}{p\left(1 - \frac{b}{p}\right)} - \frac{1}{p-b} = \frac{b}{p} + \frac{1}{b-p} = h_1(b).$$

The function h_1 takes its maximum in $b = p - \sqrt{p}$ with value

$$h_1(p - \sqrt{p}) = 1 - \frac{2}{\sqrt{p}},$$

since $p \geq 2$. If $b = p - 1$, then

$$g(p-1, d) = \frac{1}{d+1}\left(p - 2 - d - \frac{(p-1)^2 - d^2}{p}\right) = -1 + \frac{d-1}{p} + \frac{1}{d+1} = h_2(d).$$

Since $h_2''(d) = \frac{2}{(d+1)^3} > 0$, the function h_2 does not have a maximum in $(0, p-1)$. Finally, on the corners of M we have $g(0, 0) = g(p-1, p-1) = g(p-1, 0) = -1/p$. Putting all this together, we obtain

$$\max f(b, d) \leq \max g(b, d) \leq 1 - \frac{2}{\sqrt{p}} \tag{3.22}$$

and

$$\max f(p-1, d) \leq \max g(p-1, d) \leq -\frac{1}{p} \tag{3.23}$$

in the special case that $b = p - 1$, i. e. that p divides $k + 1$. Together with (3.18) and (3.19) this leads to

$$\delta \leq \frac{p-1}{p}(r_k + k) + 1 - \frac{2}{\sqrt{p}} \iff (p-1)(r_k + k) \geq p\delta - p + 2\sqrt{p}$$

and

$$\delta \leq \frac{p-1}{p}(r_k + k) - \frac{1}{p} \iff (p-1)(r_k + k) \geq p\delta + 1$$

in the case that p divides $k + 1$. This completes the proof of the theorem. $\square$

It is easy to see that Theorem 3.21 is a direct consequence of Theorem 3.33. The following examples show the sharpness of Theorem 3.33. The first example deals with the case $\chi = \omega = 2$.

Example 3.34 (Holtkamp, Meierling [61]). *For $\delta \geq 2$ and $s > \frac{k^2}{2} - \frac{1}{2}$ when k is odd and $s > \frac{k^2}{2}$ when k is even, let H_1, H_2 be two copies of the complete bipartite graph $K_{s-k+1,s}$. Let U_i, V_i be the partition sets of H_i for $i = 1, 2$, then we join V_1 and V_2, i. e. the independent sets of cardinality s, by $k-1$ perfect matchings.*

The resulting graph G of Example 3.34 is bipartite and s-regular. Furthermore, it fulfills $\lambda_k(G) \leq ks - s$, $\xi_k(G) \geq ks - \frac{k^2}{2} + \frac{1}{2}$ if k is odd, and $\xi_k(G) \geq ks - \frac{k^2}{2}$ if k is even. Therefore, G fulfills $\lambda_k(G) < \xi_k(G)$. Moreover, $r_k(G) = 2s - k + 1$, which shows that the bound in Theorem 3.33 and Theorem 3.20 is tight.

The next example shows the sharpness of the bound in Theorem 3.33 in the case $\chi = \omega = p \geq 3$ and p divides k.

Example 3.35 (Holtkamp, Meierling [61]). *For $p \geq 3$ let $k = p - 1$. We consider the complete p-partite graph with partition sets $V_i = \{x_i, y_i\}$ for $1 \leq i \leq p$, and remove the edges $x_i y_{i+1}$ for $1 \leq i \leq p-1$, as well as the edge $x_p y_1$.*

The resulting graph G of Example 3.35 has minimum degree $\delta = 2p-3$, chromatic number $\chi = p$ and clique number $\omega = p$. Hence, $\xi_k = k(\delta - (k-1)) = (p-1)^2$. Moreover, $\lambda_k \leq p(p-2) < \xi_k$ (take $G[\{x_1, x_2, \ldots, x_p\}]$), which means G is not λ_k-optimal. Furthermore, $r_k = p = \frac{p}{p-1}\delta - k + \frac{1}{p-1}$.

In the case that neither ω nor χ divides k, we do not know whether the bound in Theorem 3.33 is sharp.

With Theorem 3.33 and Theorem 3.19 we can now give a lower bound on the minimum degree for a p-partite graph to be λ_k-optimal.

Corollary 3.36 (Holtkamp, Meierling [61]). *Let G be a connected graph of order $n \geq 2k$ with chromatic number χ. If*

$$\delta(G) > \frac{\chi - 1}{\chi}\left(\left\lfloor \frac{n}{2} \right\rfloor + k\right) + \frac{\chi - 2\sqrt{\chi}}{\chi},$$

then G is λ_k-optimal.

Corollary 3.37 (Holtkamp, Meierling [61]). *Let G be a connected graph of order $n \geq 2k$ with chromatic number χ and clique number ω. Furthermore, if p divides k or if p divides $k+1$, where $p = \omega$ or $p = \chi$, and*

$$\delta(G) > \frac{p-1}{p}\left(\left\lfloor \frac{n}{2} \right\rfloor + k\right) - \frac{1}{p},$$

then G is λ_k-optimal.

Proof of Corollaries 3.36 and 3.37. Since $k \leq \delta(G)+1$, by Theorem 3.19 it follows that $\lambda_k(G) \leq \xi_k(G)$. Therefore, as $r_k(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$, assuming G it not λ_k-optimal, we have a contradiction to Theorem 3.33. □

Similar to the proof of Theorem 3.33, we can give a lower bound on the order of the k-fragments of a λ_k-optimal graph that are larger than k.

Theorem 3.38 (Holtkamp, Meierling [61]). *Let G be a k-restricted edge-connected graph with minimum degree δ, clique number ω and chromatic number χ. If $\lambda_k(G) = \xi_k(G)$ and $r_k(G) \geq k+1$, then*

$$(\chi - 1)(r_k(G) + k) \geq \chi\delta - \chi + 2\sqrt{\chi} - 1.$$

Furthermore, if p divides k or if p divides $k+1$, where $p = \omega$ or $p = \chi$, then

$$(p-1)(r_k(G) + k) \geq p\delta.$$

Proof. The proof is analogue to the proof of Theorem 3.33, but starting with $\lambda_k(G) = \xi_k(G)$ instead of $\lambda_k(G) < \xi_k(G)$. Therefore, instead of (3.14) we obtain

$$\delta(G)(r_k - k) \leq 2t_p(r_k) - 2t_p(k),$$

leading to

$$p\delta(G) \leq (p-1)(r_k + k) + p - 2\sqrt{p} + 1$$

and

$$p\delta(G) \leq (p-1)(r_k + k),$$

respectively. □

As a consequence of this result, we obtain the following conditions for a λ_k-optimal graph to be super-λ_k.

Corollary 3.39 (Holtkamp, Meierling [61]). *Let G be a connected graph of order $n \geq 2k$ with chromatic number χ. If*

$$\delta(G) > \frac{\chi - 1}{\chi}\left(\left\lfloor \frac{n}{2} \right\rfloor + k\right) + \frac{\chi - 2\sqrt{\chi} + 1}{\chi},$$

then G is super-λ_k.

Corollary 3.40 (Holtkamp, Meierling [61]). *Let G be a connected graph of order $n \geq 2k$ with chromatic number χ and clique number ω. Furthermore, if p divides k or if p divides $k+1$, where $p = \omega$ or $p = \chi$, and*

$$\delta(G) > \frac{p-1}{p}\left(\left\lfloor \frac{n}{2} \right\rfloor + k\right),$$

then G is super-λ_k.

Proof of Corollaries 3.39 and 3.40. By Corollary 3.36 or 3.37, G is λ_k-optimal. Assume G is not super-λ_k. Thus, there exists a k-fragment U such that $|U| \geq k+1$ and $|\overline{U}| \geq k+1$. We may assume $|U| \leq |\overline{U}|$, which means $|U| \leq \left\lfloor \frac{n}{2} \right\rfloor$. But combining the lower bound on $\delta(G)$ with Theorem 3.38 we obtain $|U| > \left\lfloor \frac{n}{2} \right\rfloor$, a contradiction. □

3.3 Local restricted edge-connectivity

When dealing with the concept of restricted edge-connectivity, and considering the local extensions for maximally connected and maximally edge-connected graphs illustrated in Chapter 2 and Sections 3.1 and 3.2, the question for local properties of k-restricted edge-connected graphs arises. In this section we legitimate a local generalization of the k-restricted edge-connectivity defined in Section 1.2.4 on page 6. We obtain some first results concerning this local k-restricted edge-connectivity in Section 3.3.1 and deal with the cases $k = 2$ and $k = 3$ in particular. Then, in Section 3.3.2 we turn our attention to a generalization of the concept of λ_k-optimality, namely the concept of local λ_k-optimality of a graph, in line with the concepts of local restricted edge-connectivity, and obtain some results on local λ_2-optimality. The considerations of this chapter have been made in collaboration with Dirk Meierling.

As we have seen in Chapter 2 and Section 3.1 many conditions ensuring maximum vertex- and edge-connectivity in a graph already imply stronger local properties, namely the maximum local vertex- and maximum local edge-connectivity, respectively. Now the concept of restricted edge-connectivity excludes trivial edge-cuts from considerations, like seen in Section 1.3 or Chapter 3.2, and therefore offers a more refined measurement for the fault tolerance of interconnection networks. However, no attempt has been made to explore local extensions of this concept. Yet, in the spirit of this work it is only natural to determine whether the conditions assuring λ_k-optimality already imply some stronger local optimality-criteria. The following considerations show some similarity to the results of Hellwig, Rautenbach and Volkmann [50] on *p-q-restricted edge-connectivity*, i. e. edge-cuts leaving two components of order at least p and q, respectively. However, we consider a more general case in the sense that we ensure the existence of k-k-restricted edge-cuts separating two arbitrary vertices of a graph.

If a graph G is local k-restricted edge-connected (cf. Section 1.2.4), then $\lambda_k(G) = \min\{\lambda_k(x, y) \colon\ x \neq y\}$. Hence, the concept of local restricted edge-connectivity includes the notion of local edge-connectivity as well as the notion of restricted edge-connectivity. From the definitions of k-restricted edge-connectivity and its local variant we deduce our first observations.

Observation 3.41. *Every local k-restricted edge-connected graph is k-restricted edge-connected.*

Observation 3.42. *Every local $(k+1)$-restricted edge-connected graph G is local k-restricted edge-connected and fulfills*

$$\lambda_k(G) \leq \lambda_{k+1}(G).$$

In the last chapter we presented Theorem 3.8 from Esfahanian and Hakimi [26] and Theorem 3.15 from Bonsma, Ueffing and Volkmann [13], which characterized all 2- and 3-restricted edge-connected graphs fulfilling also $\lambda_2(G) \leq \xi_2(G)$ and $\lambda_3(G) \leq \xi_3(G)$, respectively. In the next sections we discuss similar results in terms of local restricted edge-connectivity. Therefore, we defined the minimum local k-edge degree in Section 1.2.4 on page 6 as local variant of the minimum k-edge degree $\xi_k(G)$. The existence of $\xi_k(x,y)$ for every pair of distinct vertices x and y in a local k-restricted edge-connected graph is a basic fact and will be addressed in Lemma 3.45. Note that $\xi_2(x,y) = \min\{d(e)\colon e$ incident to either x or $y\}$. Furthermore, the k-edge degree $\xi_k(G) = \min\{\xi_k(x,y)\colon x \neq y\}$ of G can be expressed in terms of $\xi_k(x,y)$. We make the following observation.

Observation 3.43. *Every local λ_k-optimal graph with $\lambda_k(G) \leq \xi_k(G)$ is λ_k-optimal.*

Proof. By Observation 3.41, G is k-restricted edge-connected. So let S be a minimum k-restricted edge-cut and $u, v \in V(G)$ be vertices from different components of $G - S$. Thus, we have

$$\xi_k(G) = \min\{\xi_k(x,y) : x \neq y\} \leq \xi_k(u,v) = \lambda_k(u,v) = \lambda_k(G) \leq \xi_k(G),$$

where the last inequality holds by the assumption. □

3.3.1 Local k-restricted edge-connectivity

The following lemma compares the local k-restricted edge-connectivity $\lambda_k(x,y)$ with the number of edges between a vertex subset and its complement, both of order at least k and containing exactly one of x and y.

Lemma 3.44 (Holtkamp, Meierling [62])**.** *Let G be a local k-restricted edge-connected graph and x and y two vertices of G. If X is a subset of $V(G)$ such that $G[X]$ has a component of order at least k containing x and $G[\overline{X}]$ has a component of order at least k containing y, then $[X, \overline{X}] \geq \lambda_k(x,y)$.*

Proof. Firstly, assume that $G[X]$ is connected. Let $B \subset \overline{X}$ be a maximal subset containing y such that $G[B]$ is connected. Since G is connected and there are no edges between B and $\overline{X} \setminus B$, the graph $G[\overline{B}]$ is also connected. Hence, $(B, \overline{B})$ is a k-restricted edge-cut separating x and y. It follows that $\lambda_k(x, y) \le [B, \overline{B}] \le [X, \overline{X}]$.

If $G[X]$ is not connected, then let $A \subset X$ be a maximal subset containing x such that $G[A]$ is connected. We conclude by the first case that $(A, \overline{A})$ is a k-restricted edge-cut separating x and y. Hence, $[A, \overline{A}] \ge \lambda_k(x, y)$. Since there are no edges between A and $X \setminus A$, it follows that $[X, \overline{X}] \ge [A, \overline{A}] \ge \lambda_k(x, y)$. $\square$

The next lemma states a basic but not trivial fact, which is helpful for subsequent results and implies the existence of $\xi_k(x, y)$ in local k-restricted edge-connected graphs.

Lemma 3.45 (Holtkamp, Meierling [62]). *A connected graph G is local k-restricted edge-connected if and only if for every pair x and y of vertices there exist two disjoint sets $X = \{x, x_1, x_2, \ldots, x_{k-1}\}$ and $Y = \{y, y_1, y_2, \ldots, y_{k-1}\}$ such that $G[X]$ and $G[Y]$ are connected.*

Proof. Firstly, let G be local k-restricted edge-connected and x and y two vertices of G. By definition there exists a minimum edge-cut S such that $G - S$ has exactly two components each of order at least k such that one of the components contains x and the other contains y. So the sets X and Y exist.

Now assume that for every pair x and y of vertices of G there exist two disjoint sets $X = \{x, x_1, x_2, \ldots, x_{k-1}\}$ and $Y = \{y, y_1, y_2, \ldots, y_{k-1}\}$ such that $G[X]$ and $G[Y]$ are connected. Let q be the maximum number of edge-disjoint paths connecting X and Y such that the internal vertices are contained in $V(G) \setminus (X \cup Y)$. If we remove a minimal set S of edges from these q paths such that $G - S$ is disconnected, then $G - S$ consists of two components. Clearly, one of them contains X and the other one Y. Hence, G is local k-restricted edge-connected. $\square$

Lemma 3.45 leads to the following observation about the computational complexity of determining the local k-restricted edge-connectivity number of a graph. For fixed k and a pair x and y of vertices in a graph G, the value $\lambda_k(x, y)$ can be computed in polynomial time by contracting all possible choices X and Y of disjoint vertex sets of cardinalities k with $x \in X$ and $y \in Y$ that induce connected subgraphs of G and determining minimum sets of edges that separate the two vertices created by the contractions which can clearly be done using max-flow algorithms (cf. Remark 1.1 in [50] from Hellwig, Rautenbach and Volkmann).

The next result presents a first classification of graphs that are not local k-restricted edge-connected.

Theorem 3.46 (Holtkamp, Meierling [62]). *Let G be a connected graph of order at least $2k$. If G has a cut-vertex that isolates a component of order at most k, then G is not local k-restricted edge-connected.*

Proof. Let v be a cut-vertex that isolates a component H of order at most k. If x and y are vertices of H, then every minimum edge-cut S that separates x and y has the property that $G - S$ has exactly two components. One of the components contains v and one of $\{x, y\}$ which means that the other component has at most $|V(H)| - 1 \leq k - 1$ vertices. Hence, G is not local k-restricted edge-connected. □

Note that for every positive integer k there exist local k-restricted edge-connected graphs containing a cut-vertex that isolates a component of order $k+1$ as the next example shows.

Example 3.47 (Holtkamp, Meierling [62]). *Let k be a positive integer. Join a complete graph K_s on $s \geq k+2$ vertices with a complete graph K_{k+2} by identifying a vertex of K_s to an arbitrary vertex of K_{k+2}. The resulting graph G is local k-restricted edge-connected and has exactly one cut-vertex v with the property that $G - v$ has two components of order $s - 1$ and $k + 1$, respectively.*

The following result provides a helpful tool in finding k-restricted edge-cuts separating fixed pairs of vertices.

Lemma 3.48 (Frank [34], 1976). *If G is a 2-connected graph and x and y are a pair of distinct vertices, then the vertex set of G can be partitioned into two subsets X and Y of arbitrary order such that $x \in X$, $y \in Y$ and both $G[X]$ and $G[Y]$ are connected.*

A generalization of the vertex partitioning problem presented in Lemma 3.48 has been proven by Lovász [71] and Györi [47]. The above considerations directly imply the following result.

Corollary 3.49 (Holtkamp, Meierling [62]). *Every 2-connected graph of order at least $2k$ is local k-restricted edge-connected.*

Lemma 3.48 allows us to obtain more detailed sufficient conditions for the existence of $\text{minloc}_k(x, y)$-cuts.

Theorem 3.50 (Holtkamp, Meierling [62]). *Let G be a connected graph of order $n \geq 2k$ and x and y be two vertices of G.*

(1) *If* $\kappa(x, y) \geq 2$ *and no cut-vertex* w *leaves* x *and* y *in a common component of order at most* $2k-2$ *in case* $w \neq x, y$*, or leaves* x *or* y *in a component of order at most* $k-1$ *in case* $w = y$ *or* $w = x$*, respectively, then there exists a* k*-restricted edge-cut separating* x *and* y.

(2) *If* $\kappa(x, y) = 1$ *and there exists a cut-vertex that leaves one of* x *and* y *in a component of order* s *with* $k \leq s \leq n-k$*, then there exists a* k*-restricted edge-cut separating* x *and* y.

Proof. If $\kappa(x, y) \geq 2$, then let $W \subset V(G)$ with $x, y \in W$ be the maximal vertex set such that $G[W]$ is 2-connected. According to Lemma 3.48 W can be partitioned into vertex sets X and Y with $x \in X$ and $y \in Y$ of arbitrary order such that $G[X]$ and $G[Y]$ are connected. Furthermore, for each such partitioning X and Y of W the edge set (X, Y) is an edge-cut of G separating x and y. Assume to the contrary that (X, Y) is not a k-restricted edge-cut for each such partitioning. Note that $G - (X, Y)$ has exactly two components, and let X' and Y' with $X \subseteq X'$ and $Y \subseteq Y'$ denote the according vertex sets. We choose a partitioning X, Y of W with the properties from above such that $m = \min\{|X'|, |Y'|\}$ is maximal. Since $G - (X, Y)$ is not a k-restricted edge-cut we have $m \leq k-1$. Let $Z \subseteq Y \setminus \{y\}$ be the vertices of $Y \setminus \{y\}$ that have a neighbour in X. Since $\kappa(x, y) \geq 2$ and y is not a cut-vertex of G we have $Z \neq \emptyset$. If a vertex $z \in Z$ is not a cut-vertex of $G[Y']$, then $X \cup \{z\}, Y \setminus \{z\}$ is a partitioning of W with the properties from above and $\min\{|X' \cup \{z\}|, |Y' \setminus \{z\}|\} = m+1$, contradicting the choice of X, Y. Thus, every vertex $z \in Z$ is a cut-vertex of Y', and x and y must be in the same component C of $G - \{z\}$ with $|V(C)| \geq 2k-1$. Choosing $z' \in Z$ such that the component D of $G[Y']$ with $y \in V(D)$ has the maximum number of vertices implies $|V(D)| \geq k$, where $X \cup \{z'\}, Y \setminus \{z'\}$ is a partitioning of W with $\min\{|X' \cup \{z\}|, |Y' \setminus \{z\}|\} \geq m+1$, a contradiction.

So assume that $\kappa(x, y) = 1$ and let v be a cut-vertex with the property that x and y are in distinct components H_1 and H_2 of $G - v$ such that H_1 is of order s with $k \leq s \leq n-k$. It follows that $(V(H_1), \overline{V(H_1)})$ is a k-restricted edge-cut separating x and y. □

The following examples show that the bound in Theorem 3.50 is sharp.

Example 3.51 (Holtkamp, Meierling [62])**.** *Let* k *be a positive integer.*

(1) *Let* x *be a vertex of a complete graph* K_s *on* $s \geq 2k-1$ *vertices. Let* G *be the graph obtained from* K_s *by adding the two paths* $u_1 u_2 \cdots u_{k-1} x$ *and* $v_1 v_2 \cdots v_{k-1} x$ *and the edge* $u_1 v_1$*. The graph* $G - x$ *has two components, one of order* $s-1$ *and the other of order* $2k-2$*, but* G *is not local* k*-restricted edge-connected, since it contains no* k*-restricted edge-cut separating* u_1 *and* v_1.

(2) Let x be a vertex of a complete graph K_t on $t \geq k+1$ vertices and y a vertex of a complete graph K_{k-2}. Let G be the graph obtained from K_t and K_{k-2} by adding a vertex v adjacent to x and y. The graph $G - v$ has two components, one of order $k+1 \leq t = n-k+1$ and the other of order $k-2$, but G is not local k-restricted edge-connected, since it contains no k-restricted edge-cut separating x and y.

Using Theorem 3.50, we conclude that if every cut-vertex of a graph leaves only large components, then the graph in question is local k-restricted edge-connected.

Corollary 3.52 (Holtkamp, Meierling [62]). *Let G be a connected graph of order at least $2k$. If G has no cut-vertex that isolates a component of order at most $2k-2$, then G is local k-restricted edge-connected.*

Corollary 3.53 (Holtkamp, Meierling [62]). *Let $k \neq 2$ be a positive integer and G a connected graph of order at least $2k$. If $\delta(G) \geq 2k-2$, then G is local k-restricted edge-connected.*

Proof. For $k = 1$ the proposition is trivially true. Removing a cut-vertex leaves components of order at least $2k-2$. Either such a component is of order at least $2k-1$, or it is isomorphic to a complete graph K_{2k-2}. In the former case the result follows from Theorem 3.50 (1), or we have $\kappa(x, y) = 1$ where x and y are adjacent or separated by a cut-vertex, which together with $\delta(G) \geq 2k-2 \geq k+1$ leads to the designated result by Theorem 3.50 (2). In the latter case considering $2k-2 \geq k+1 \geq 4$ for $k \geq 3$ it is easy to see the local k-restricted edge-connectivity. □

We now take a look at the case $k = 2$ that was left out in Corollary 3.53. Theorem 3.46 and Corollary 3.52 directly imply the following results.

Corollary 3.54 (Holtkamp, Meierling [62]). *A connected graph of order at least 4 is not local 2-restricted edge-connected if and only if it satifies at least one of the following conditions (see Figure 3.5).*

(1) It contains a vertex of degree 1.

(2) It contains two adjacent vertices of degree 2 that have a common neighbour.

Corollary 3.55 (Holtkamp, Meierling [62]). *Every graph of minimum degree at least 3 is local 2-restricted edge-connected.*

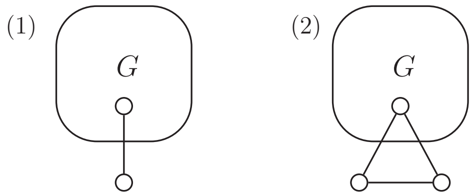

Figure 3.5: Graphs that are not local 2-restricted edge-connected contain at least one of these two structures.

For $k \geq 3$, Theorem 3.46 and Corollaries 3.52 and 3.53 do not cover all graphs of order at least $2k$. In the next result we characterize the class of graphs that are not local 3-restricted edge-connected. A *paw* is a graph on 4 vertices that is isomorphic to a star $K_{1,3}$ on 4 vertices plus an edge (and therefore contains exactly two vertices of degree 2).

Theorem 3.56 (Holtkamp, Meierling [62])**.** *A connected graph of order at least* 6 *is not local* 3*-restricted edge-connected if and only if it satifies at least one of the following conditions (see Figure 3.6).*

(1) It contains a cut-vertex that isolates a component of order at most 3*.*

(2) It contains a cut-vertex v that isolates a component of order 4 *such that at least two of its vertices are not adjacent to v.*

(3) It contains a cut-vertex v that isolates a paw such that one of its vertices of degree 2 *is not adjacent to v.*

(4) It contains a cut-vertex that isolates a path of order 4*.*

Proof. Suppose first that G satisfies at least one of (1)–(4). If G satisfies (1), then it is not local 3-restricted edge-connected by Theorem 3.46. Now let v be a cut-vertex of G that isolates a component C of order 4 and assume that G is local 3-restricted edge-connected. Note that every $\lambda_3(x, y)$-cut S that separates two vertices x and y of C has the property that one component H of $G - S$ contains exactly three vertices, namely one of x and y, and the other two vertices of C. The situation is depicted in Figure 3.6. If G satisfies (2), then let x and y be two vertices of C that are not adjacent to v; if G satisfies (3), then let x be the vertex of degree 3 in the paw and y be the vertex of degree 2 in the paw that is not adjacent to v; and if G satisfies (4), then let $wxyz$ be the P_4 isolated by v. In each case it is not possible to separate the vertices x and y into components of order at least 3. Hence G is not local 3-restricted edge-connected.

Suppose now that G satisfies none of (1)–(4), and let x and y be two arbitrary vertices of G.

Suppose first that $\kappa(x, y) \geq 2$. If no cut-vertex leaves x and y in a common component of order at most 4, then by Theorem 3.50 there exists a 3-restricted edge-cut separating x and y. Since G does not satisfy (1), assume that v is a cut-vertex of G that leaves x and y in a common component C of order 4. Since G does not satisfy (2), at least three vertices of C are adjacent to v. We may assume, without loss of generality, that x is adjacent to v. If $G[V(C) \setminus \{x\}]$ is connected, then $(V(C) \setminus \{x\}, \overline{V(C) \setminus \{x\}})$ is a 3-restricted edge-cut separating x and y. So assume that x is a cut-vertex of C. Since G does not satisfy (1), every leaf of C is adjacent to v. If $C - x$ has y as an isolated vertex, then y is adjacent to v. Hence, $(V(C) \setminus \{y\}, \overline{V(C) \setminus \{y\}})$ is a 3-restricted edge-cut separating x and y. So $C - x$ has two components, namely a 2-path yz and an isolated vertex w. Note that w is adjacent to v. Since C is not a P_4, both y and z are adjacent to x. Since G satisfies neither (2) nor (3), the vertex y is adjacent to v and thus, $(V(C) \setminus \{y\}, \overline{V(C) \setminus \{y\}})$ is a 3-restricted edge-cut separating x and y.

Suppose now that $\kappa(x, y) = 1$. If there exists a cut-vertex that leaves one of x and y in a component of order s with $3 \leq s \leq n - 3$, then by Theorem 3.50 there exists a 3-restricted edge-cut separating x and y. So assume that no cut-vertex that separates x and y has the above property. Since G does not satisfy (1), the two components containing x and y both have order at least $n - 2$ by Theorem 3.50. It follows that $n \geq 2n - 3$, a contradiction. □

An immediate consequence of Theorem 3.56 is the following.

Corollary 3.57 (Holtkamp, Meierling [62]). *Every graph with and least 6 vertices and minimum degree at least 4 is local 3-restricted edge-connected.*

For $k \geq 4$, we are not able to characterize the class of graphs that are not local k-restricted edge-connected, but the next result generalizes Corollaries 3.55 and 3.57 to arbitrary values of k.

Theorem 3.58 (Holtkamp, Meierling [62]). *Every connected graph G of order at least $2k$ and minimum degree at least $k + 1$ is local k-restricted edge-connected.*

Proof. The proof is by induction on k. Let x and y be two arbitary vertices of G. We shall show that x and y can be separated by a k-restricted edge-cut. For $k = 2$ and $k = 3$ the proposition holds due to Corollaries 3.55 and 3.57. So let $k \geq 4$.

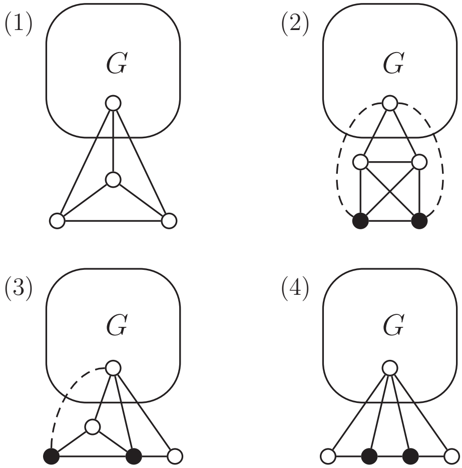

Figure 3.6: Graphs that are not local 3-restricted edge-connected contain at least one of these 4 structures. The vertices that cannot be separated by a 3-restricted edge-cut are colored black. Dashed lines indicate that the corresponding edges are missing.

If $\kappa(x, y) = 1$, then there is either a cut-vertex v of G such that x and y are in different components of $G - v$, or x and y are adjacent. Since $\delta(G) \geq k + 1$, in both cases x and y can be separated by a k-restricted edge-cut by Theorem 3.50 (2).

If $\kappa(x, y) \geq 2$ and no cut-vertex leaves x and y in a common component of order at most $2k - 2$, then there exists a k-restricted edge-cut separating x and y by Theorem 3.50 (1). So assume that G has at least one cut-vertex that isolates a component of order at most $2k - 2$ that contains both x and y. Among all such cut-vertices let v be chosen such that it isolates a component of minimum order containing both x and y. Denote this component by H and note that H is 2-connected due to this choice. Furthermore, note that $G - H$ contains at least $k + 2$ vertices. Let $w \notin \{x, y\}$ be an arbitrary vertex of H. Note that $\delta(G - w) \geq k$. By the induction hypothesis, it follows that there exists an edge-cut $S = (X, \overline{X})$ of $G' = G - w$ such that $x \in X$, $y \in \overline{X}$, $|X|, |\overline{X}| \geq k - 1$ and both $G'[X]$ and $G'[\overline{X}]$ are connected. Since G' contains at least $2k - 1$ vertices, at least one of X and $\overline{X}$ contains at least k vertices. Moreover, w has neighbours in at least one of X and

$\overline{X}$.

If $|X|, |\overline{X}| \geq k$ or if $|X| = k-1$ and w has neighbours in X, then it is easy to see that x and y can be separated by a k-restricted edge-cut of G.

So assume that $|X| = k-1$, $|\overline{X}| \geq k$ and $N(w) \cap X = \emptyset$. Note that the component of $G' - S$ that contains v also contains all vertices of $G - H$, since v is also a cut-vertex of G'. It follows that $v \in \overline{X}$, since $|X| = k-1$ and $|V(G-H)| \geq k+2$. Hence, $|\overline{X} \cap V(H)| = |V(H)| - |X| - 1 \leq k-2$. Therefore

$$k+1 \leq \delta(G) \leq d(w) \leq |N(w) \cap V(H)| + 1 \leq |\overline{X} \cap V(H)| + 1 \leq k-1,$$

a contradiction. This completes the proof of this theorem. □

If we join a complete graph K_s on $s \geq 2k-1$ with a complete graph K_{k+1} by identifying two arbitrary vertices, we obtain a graph that is not local k-restricted edge-connected with minimum degree $\delta = k$. This shows that the condition in Theorem 3.58 is sharp.

3.3.2 Local λ_2-optimality

We now turn our attention to an upper bound for $\lambda_2(x,y)$. In the same way that $\lambda' = \lambda_2$ is bounded from above by $\xi = \xi_2$ (see Theorem 3.8), $\lambda_2(x,y)$ is bounded from above by $\xi_2(x,y)$.

Theorem 3.59 (Holtkamp, Meierling [62]). *Every local 2-restricted edge-connected graph satisfies*

$$\lambda_2(x,y) \leq \xi_2(x,y)$$

for every pair x and y of vertices.

Proof. Without loss of generality, let $e = xv$ with $v \neq y$ be an edge with $d(e) = \xi_2(x,y)$. Let $S = (\{x,v\}, V(G) \setminus \{x,v\})$ and note that $|S| = d(e)$.

If y belongs to a non-trivial component of $G - S$, then there exists a set $S' \subset S$ such that $G - S'$ has exactly two components of order at least 2. Furthermore, one of the components contains x and the other contains y. Hence $\lambda_2(x,y) \leq |S'| \leq |S| = d(e)$.

So assume that y is isolated in $G - S$. By Corollary 3.54 (1) the vertex y has degree at least 2 and hence $N(y) = \{x,v\}$. Moreover, by Corollary 3.54 (2) the vertex x has a neighbour different from v and y and thus, $d(x) \geq 3$. It follows that $d(yv) = d(v) < d(x) + d(v) - 2 = d(xv)$, a contradiction to our assumption. □

The next example shows a class of graphs for which the inequality $\lambda_2(x,y) \le d(xy)$ is not valid. This explains why the edge xy is not considered in the minimum on the right-hand-side of the inequality in Theorem 3.59.

Example 3.60 (Holtkamp, Meierling [62]). *Let G be a graph that consists of the 4-path $wxyz$ and the complete graph K_{2s} on $2s \ge 4$ vertices with vertex set $\{y_1, y_2, \ldots, y_{2s}\}$ together with the edges wy_i and zy_{s+i} for $i = 1, 2, \ldots, s$. The graph G has exactly two minimum local 2-restricted edge-cuts separating x and y, namely $\{xy, zy_{s+1}, zy_{s+2}, \ldots, zy_{2s}\}$ and $\{xy, wy_1, wy_2, \ldots, wy_s\}$, both of size $s+1$. Therefore $d(xy) = 2 < s+1 = \lambda_2(x,y)$.*

Due to Observation 3.43 and Theorem 3.8 every local λ_2-optimal graph is λ_2-optimal. To illustrate the difference between λ_2-optimality and local λ_2-optimality we consider the graph class defined in the following example.

Example 3.61 (Holtkamp, Meierling [62]). *Let $C = u_1u_2 \cdots u_m u_1$ and $C' = v_1v_2 \cdots v_n v_1$ be two cycles of length $m, n \ge 3$. Let G be the graph that consists of C and C' joined by the edges u_1v_1 and u_2v_2. Then $\lambda_2(G) = \xi_2(G) = \lambda_2(u_i, v_j) = 2$ for every choice of i and j, but $\xi_2(u_1, v_1) = 3$. Hence, G is λ_2-optimal, but not local λ_2-optimal.*

In the next theorem we take a closer look at the connection between $\lambda_2(x,y)$ and $\xi_2(x,y)$.

Theorem 3.62 (Holtkamp, Meierling [62]). *Let G be a graph of order at least 4. If x and y are two vertices of G that can be separated by a 2-restricted edge-cut, then*

- *either $\lambda_2(x,y) < \xi_2(x,y)$,*
- *or $\lambda_2(x,y) = \xi_2(x,y)$ and there exists a 2-restricted edge-cut S separating x and y such that at least one component of $G - S$ has exactly two vertices.*

Proof. By Theorem 3.59, we have $\lambda_2(x,y) \le \xi_2(x,y)$. Without loss of generality, let $xv \in E(G)$ with $\lambda_2(x,y) = \xi_2(x,y) = d(xv)$. If y is not isolated in $G - (\{v,x\}, \overline{\{v,x\}})$, then $G - (\{v,x\}, \overline{\{v,x\}})$ has exactly two components, since otherwise there would be a smaller $\text{minloc}_2(x,y)$-cut. Thus, $(\{v,x\}, \overline{\{v,x\}})$ is a $\text{minloc}_2(x,y)$-cut isolating xv.

So let y be isolated in $G - (\{v,x\}, \overline{\{v,x\}})$. Therefore, $N(y) \subset \{v,x\}$ and, since x and y can be separated by a 2-restricted edge-cut, y and v are adjacent and x has a neighbour different from v and y. But then $d(xv) \ge 3 > 2 \ge d(yv)$, a contradiction. □

Using Theorem 2.3 from Turán, we obtain the following lower bound on the order of the components left by a minimum local k-restricted edge-cut of a graph that is not local λ_k-optimal.

Theorem 3.63 (Holtkamp, Meierling [62]). *Let G be a local 2-restricted edge-connected graph with minimum degree δ and clique number ω containing a pair x and y of vertices with*

$$\lambda_2(x,y) < \xi_2(x,y).$$

If S is a minloc$_2(x,y)$-cut, then both components of $G-S$ contain at least

- $\max\{3, 2\delta-1\}$ *vertices when* $\omega = 2$,
- $\max\{3, \delta, \lceil \frac{\omega}{\omega-1}\delta - \frac{3\omega-6}{\omega-1} \rceil\}$ *vertices when* $\omega \geq 3$.

Proof. Let X be the component of $G-S$ that contains x. Since $|X| = 2$ or $|Y| = 2$ would imply $\lambda_2(x,y) = \xi_2(x,y)$, we assume, without loss of generality, that $3 \leq |X| \leq |Y|$ and thus, $|X| \leq \lfloor n/2 \rfloor$. Let $v \in X$ with $d(xv) \leq d(xw)$ for every edge xw with $w \in X$. Hence,

$$\sum_{a \in X} d(a) - 2|E(X)| = |S| = \lambda_2(x,y) \leq \xi_2(x,y) - 1 \leq d(xv) - 1 = d(x) + d(v) - 3.$$

Using Theorem 2.3, it follows that

$$\delta(|X|-2) \leq \sum_{a \in X \setminus \{x,v\}} d(a) \leq 2|E(X)| - 3 \leq \frac{\omega-1}{\omega}|X|^2 - 3 \tag{3.24}$$

and thus,

$$\delta(|X|-2) \leq \frac{\omega-1}{\omega}\left(((|X|+2)(|X|-2)+4\right) - 3.$$

The last inequality is equivalent to

$$|X| \geq \frac{\omega}{\omega-1}\delta - \frac{4}{|X|-2} + \frac{3\omega}{(|X|-2)(\omega-1)} - 2. \tag{3.25}$$

Using $2|E(X)| \leq |X|(|X|-1)$ in the last step of inequality (3.24), we directly obtain $|X| \geq \delta$.

To derive better bounds, we take a closer look at the function

$$f(z) = -\frac{4}{z-2} + \frac{3\omega}{(z-2)(\omega-1)} = \frac{1}{z-2}\left(\frac{3\omega}{\omega-1} - 4\right),$$

where $3 \le z \le \lfloor n/2 \rfloor$ and $\omega \ge 2$. Note that $f \equiv 0$ for $\omega = 4$. Since $g(z) = 1/(z-2)$ is monotonically decreasing, the function f is monotonically decreasing for $\omega \le 3$ and monotonically increasing for $\omega \ge 5$. Hence, if $\omega \le 3$, then f is minimal in $z = \lfloor n/2 \rfloor$ and if $\omega \ge 5$, then f is minimal in $z = 3$.

If $\omega = 2$, then (3.25) leads to

$$|X| \ge 2\delta + \frac{2}{\lfloor n/2 \rfloor - 2} - 2$$

and thus, $|X| \ge 2\delta - 1$.

If $\omega = 3$, then (3.25) yields

$$|X| \ge \frac{3}{2}\delta + \frac{1}{2}\frac{1}{|X| - 2} - 2.$$

For even δ this leads to $|X| \ge \frac{3}{2}\delta - 1$, and for odd δ we conclude that $|X| \ge \frac{3}{2}\delta - \frac{3}{2} = \frac{3}{2}(\delta - 1)$.

Finally, if $\omega \ge 4$, then again by (3.25) we obtain

$$|X| \ge \frac{\omega}{\omega - 1}\delta + \frac{3\omega}{\omega - 1} - 6 = \frac{\omega}{\omega - 1}\delta - \frac{3\omega - 6}{\omega - 1}$$

which completes the proof. □

The following results are immediate by Theorem 3.63.

Corollary 3.64 (Holtkamp, Meierling [62]). *Let G be a connected graph of minimum degree δ containing a pair x and y of vertices that can be separated by a 2-restricted edge-cut. If*

$$\lambda_2(x, y) < \xi_2(x, y)$$

and S is a minloc$_2(x, y)$-cut, then both components of $G - S$ contain at least $\max\{3, \delta\}$ *vertices.*

Corollary 3.65 (Holtkamp, Meierling [62]). *Every graph of minimum degree $\delta > \lfloor n/2 \rfloor$ is local λ_2-optimal.*

Note that, using Theorem 3.63, we can derive better lower bounds for the minimum degree of a graph in terms of the clique number.

Corollary 3.66 (Holtkamp, Meierling [62]). *A connected graph G is local λ_2-optimal if it satisfies one of the following conditions.*

(1) Its clique number is 2 *and its minimum degree is greater than* $\frac{1}{2}\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)$.

(2) Its clique number ω *is at least* 3 *and its minimum degree is greater than* $\frac{\omega-1}{\omega}\left(\left\lfloor\frac{n}{2}\right\rfloor+6\right)-3$.

Our last examples show that the bounds in Theorem 3.63 are almost sharp.

Example 3.67 (Holtkamp, Meierling [62])**.** *Let* $s,\omega \geq 2$ *be two integers. Let* H *be the complete* ω*-partite graph with partite sets* $V_1, V_2, \ldots, V_\omega$ *such that* $|V_i| = s$ *for* $i = 1, 2, \ldots, \omega-1$ *and* $|V_\omega| = s-1$ *and let* H' *be a copy of* H *with partite sets* $V_1', V_2', \ldots, V_\omega'$. *We define* G *to be the graph that consists of* H *and* H' *plus a perfect matching between* $V(H) \setminus V_\omega$ *and* $V(H') \setminus V_\omega'$.

The graph G from Example 3.67 is $(\omega-1)s$-regular and thus, $\delta(G) = \delta = (\omega-1)s$ and $\xi_2(x,y) = 2(\omega-1)s-2$ for every pair x and y of vertices. Furthermore, if $x \in V(H)$ and $y \in V(H')$, then $(V(H), V(H'))$ is a 2-restricted edge-cut separating x and y. Hence, $\lambda_2(x,y) \leq [V(H), V(H')] = (\omega-1)s < \xi_2(x,y)$. Moreover, both components of $G-(V(H), V(H'))$ have exactly $\omega s-1$ vertices and $\frac{\omega}{\omega-1}\delta+\frac{3\omega}{\omega-1}-6 = \omega s-3+\frac{3}{\omega-1}$. This shows that for $\omega = 2$ and $\omega = 3$, the bound in Theorem 3.63 is sharp and for $\omega \geq 4$, the bound in Theorem 3.63 can be improved by at most 1.

Example 3.68 (Holtkamp, Meierling [62])**.** *Let* $s,\omega \geq 2$ *be two integers. Let* H *be the complete graph* $K_{\omega-1}$ *with vertices* $v_1, v_2, \ldots, v_{\omega-1}$ *and* H' *the complete* ω*-partite graph with partite sets* $V_1, V_2, \ldots, V_\omega$ *such that* $|V_i| = s$ *for* $i = 1, 2, \ldots, \omega$. *We define* G *to be the graph that consists of* H *and* H' *plus the edges of a matching between* $V(H)$ *and* $V(H') \setminus V_\omega$ *that covers all vertices of* H.

In the graph G from Example 3.68 every vertex $v_i \in V(G)$ has degree $\delta(G) = \delta = \omega-1$. If $x \in V(H)$ and $y \in V(H')$, then $(V(H), V(H'))$ is a 2-restricted edge-cut separating x and y. Hence, $\lambda_2(x,y) \leq [V(H), V(H')] = \omega-1 < 2(\omega-1)-2 = 2\delta-2 = \xi_2(x,y)$. Moreover, the smaller one of the two components of $G-(V(H), V(H'))$ has exactly $\omega-1 = \delta$ vertices. This shows that for $\omega \geq 4$, the bound in Theorem 3.63 is sharp.

Part II

Connectivity in digraphs

Chapter 4

Vertex-connectivity

In this chapter we discuss the (vertex-)connectivity in digraphs defined in Section 1.2.5 on page 6. Analogue to the similar concept for graphs illustrated earlier in Chapter 2, we now study sufficient criteria for digraphs to be maximally connected and maximally local connected. However, when it comes to directed graphs in general, the problem of finding manageable sufficient conditions for maximum (local) connectivity becomes essentially more complex, if not untraceable. Thus, it is common to focus on simple structured digraphs, as for example bipartite digraphs, highly regular digraphs, or tournaments. In Section 4.1 we give such a result for the maximum local connectivity of regular and almost regular bipartite tournaments. We will then discuss the connectivity of local tournaments in Section 4.2, where we present a generalization of a result on tournaments by Carsten Thomassen [98] from 1980. The investigations of this chapter have been undertaken in close collaboration with Yubao Guo and Sebastian Milz.

By the definitions given in Section 1.2.5 the following is obvious.

Observation 4.1. *Every maximally local connected digraph is maximally connected.*

The maximal connectivity of bipartite graphs and digraphs has been studied in Chapter 3 and also by various authors, for example, Balbuena and Carmona [2], Fàbrega and Fiol [28], Hellwig and Volkmann [56], Topp and Volkmann [99], and Volkmann [106]. In particular, the following result is closely related to the topic of this chapter.

Theorem 4.2 (Balbuena, Carmona [2], 2001)**.** *Let D be a bipartite digraph. If*

$$d_D^+(u) + d_D^-(v) \geq \frac{|V(D)| + \delta(D)}{2}$$

for all pairs of vertices u, v with $d_D(u, v) \geq 3$, then $\kappa(D) = \delta(D)$.

Theorem 3.5 in [106] claims that all bipartite oriented graphs D with $n \geq 4$ and $\delta \geq 1$ have $\kappa(D) \geq (8\delta - n)/3$. The author then uses this bound in Corollary 3.6 to obtain the maximal connectivity of all bipartite oriented graphs with $\delta \geq 1$ and $n \leq 5\delta + 2$, including regular and almost regular bipartite tournaments. However, by considering an arbitrary r-regular bipartite tournament T with $r \geq 1$ we have

$$\kappa(T) \geq \frac{8r - 4r}{3} = \frac{4}{3}r > r,$$

a contradiction. This means the lower bound given by Theorem 3.5 in [106] does not hold.

Furthermore, Yeo [121] studied the connectivity of multipartite tournaments.

Theorem 4.3 (Yeo [121], 1998)**.** *If D is a multipartite tournament, then*

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - \alpha(D) - 2i_l(D)}{3} \right\rceil,$$

where $\alpha(D)$ refers to the independence number of D, and $i_l(D)$ is the local irregularity.

Volkmann [107] presented examples that show the sharpness of this bound for regular p-partite tournaments with $p = 3k + 1$ and $k \geq 1$. Regular and almost regular bipartite tournaments, however, are maximally connected, and besides a small exceptional family even maximally local connected.

4.1 Maximum (local) connectivity in regular and almost regular bipartite tournaments

In this section we prove that all regular and almost regular bipartite tournaments are maximally connected. Also, we characterize all almost regular bipartite tournaments which are not maximally local connected. Therefore, we consider the almost regular bipartite tournaments from the following example.

Example 4.4 (Guo, Holtkamp, Milz [45])**.** *For an integer $s \geq 1$ we define the bipartite tournament $T_1(s)$, which has bipartite sets $U = U_1 \dot\cup U_2$ and $W = W_1 \dot\cup W_2$, where $|U_1| = |U_2| = s$ and $|W_1| = |W_2| = s + 1$, and the arc set such that $U_1 \to W_1 \to U_2 \to W_2 \to U_1$.*

Similarly, we define $T_2(s)$ in the same way, but with $|U_1| = s+1$ instead of $|U_1| = s$.

It is easy to verify that $\delta(T_1(s)) = \delta(T_2(s)) = s$, and $T_1(s+1)$ and $T_2(s)$ $(s \geq 1)$ are almost regular bipartite tournaments which are maximally connected, but not maximally local connected, since the removal of U_2 separates two arbitrary vertices $x, y \in U_1$ with $d^+(x) = d^-(y) = s+1$ and $|U_2| = s$. Furthermore, we recognize that $T_1(1)$ is maximally local connected.

In the following, we study the local connectivity between two vertices x and y from the same bipartite set (see Theorem 4.5), and then take a look at x and y from different bipartite sets (see Theorem 4.6).

Theorem 4.5 (Guo, Holtkamp, Milz [45]). *Let T be a bipartite tournament with $i_g(T) \leq 1$ and x, y two vertices from the same bipartite set. Then*

$$\kappa(x, y) = \min\{d^+(x), d^-(y)\},$$

except T is isomorphic to $T_1(s+1)$ or $T_2(s)$ with $s \geq 1$ from Example 4.4.

Proof. Let U and W be the bipartite sets of T, and, without loss of generality, let $x, y \in U$. Obviously, in T we have $2\delta \leq |U|, |W| \leq 2\delta + 2$. For $a \in \{0, 1\}$ we have $\min\{d^+(x), d^-(y)\} = \delta + a$. Suppose to the contrary that $\kappa(x, y) < \min\{d^+(x), d^-(y)\}$, then there exists a vertex set $S \subseteq V(T)\backslash\{x, y\}$ separating x from y in T with $|S| \leq \delta + a - 1$. In the following, we denote $X^* = X\backslash S$ for $X \subseteq V(T)$. We distinguish two cases.

Case 1. $|N^+(x) \cap N^-(y)| = t \geq 1$. We have

$$1 \leq t \leq \delta + a - 1.$$

Of course, $N^+(x) \cap N^-(y) \subseteq S$. We define

$$W_1 = N^+(x) \cap N^+(y), \quad W_2 = N^-(x) \cap N^-(y) \quad \text{and} \quad W_3 = N^-(x) \cap N^+(y).$$

We have $\delta + a - t \leq |W_1|, |W_2| \leq \delta + 1 - t$, and W_1^* as well as W_2^* are nonempty. Furthermore, we define the vertex sets

$$U_1 = N^+(W_1^*) \cap N^-(W_2^*), \quad U_2 = N^+(W_1^*) \setminus U_1 \quad \text{and} \quad U_3 = N^-(W_2^*) \setminus U_1.$$

We notice that $U_i \subseteq U$ and $W_i \subseteq W$ for $i = 1, 2, 3$. Furthermore, U_i and also W_j are pairwise disjoint for all $1 \leq i, j \leq 3$ (see Figure 4.1).

In the further argumentation of the proof we repeatedly take advantage of a certain symmetry in the definition of these vertex sets. According to this, we note that by considering the converse digraph $T^{-1} = (V(T), \{yx\colon xy \in A(T)\})$ we can adopt

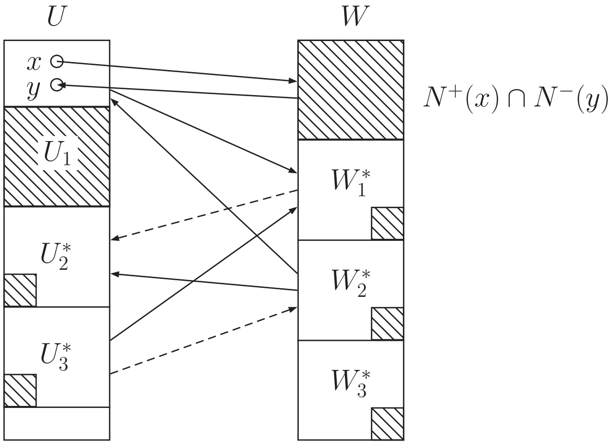

Figure 4.1: Structure of bipartite tournament T from Case 1 in the proof of Theorem 4.5. Shaded regions belong to the vertex set S. Continuous arcs depict a dominance relation between the vertex subsets in question. For two vertex subsets $A, B \subseteq V(T)$ a dashed arc from A to B indicates that for all vertices $a \in A$ and $b \in B$ we have $N^+(a) \cap B \neq \emptyset$ and $N^-(b) \cap A \neq \emptyset$.

the properties of W_1 in an analogue way for W_2, and vice versa. Also, we have the same symmetry between the vertex sets U_2 and U_3.

Let $r_i = |U_i \cap S|$ and $q_i = |W_i \cap S|$ for $i = 1, 2, 3$, and let $r = r_1 + r_2 + r_3$ and $q = q_1 + q_2 + q_3$. We have

$$t + r + q \leq |S| \leq \delta + a - 1. \tag{4.1}$$

It must be $U_1 \subseteq S$, since otherwise there would be an x-y-path of length 4 in $T - S$. Thus, $r_1 = |U_1|$ and $|U_2|, |U_3| \geq \delta - r_1$. By (4.1) we deduce

$$|U_i^*| \geq \delta - r_1 - r_i \geq t + q - a + 1 \geq t \geq 1 \quad \text{for} \quad i = 2, 3. \tag{4.2}$$

If there exists an arc u_2w_2 with $u_2 \in U_2^*$ and $w_2 \in W_2^*$, then $x \to W_1^* \to u_2 \to w_2 \to y$ and $T - S$ contains an x-y-path. Thus, $W_2^* \to U_2^*$ and therefore $\delta + 1 \geq d^+(w_2) \geq 2 + |U_2^*|$ holds for every vertex $w_2 \in W_2^*$. By the symmetry of W_1 and W_2, and U_2 and U_3 mentioned above, we also have $U_3^* \to W_1^*$ and $\delta + 1 \geq d^-(w_1) \geq 2 + |U_3^*|$

holds for all $w_1 \in W_1^*$, leading to

$$|U_2^*|, |U_3^*| \le \delta - 1. \tag{4.3}$$

Together with (4.2) we conclude

$$\delta - r_1 - r_i \le |U_i^*| \le \delta - 1 \quad \text{for} \quad i = 2, 3,$$

and we have

$$r_1 + r_i \ge 1 \quad \text{for} \quad i = 2, 3.$$

Since every vertex in U_2^* must have at least $\delta - t$ positive neighbours in $W \backslash (N^+(x) \cap N^-(y))$, we have the following inequality:

$$\begin{aligned} |U_2^*|(\delta - t) &\le [U_2^*,\, W \backslash (N^+(x) \cap N^-(y))] \\ &= [U_2^*, W_1^*] + [U_2^*, W_1 \backslash W_1^*] + [U_2^*, W_2] + [U_2^*, W_3^*] + [U_2^*, W_3 \backslash W_3^*]. \end{aligned} \tag{4.4}$$

We will use (4.4) to obtain a lower bound on $[U_2^*, W_3^*]$. Firstly, we observe that $[U_2^*, W_1^*] + [W_1^*, U_2^*] = |U_2^*| \cdot |W_1^*|$, and since every vertex in W_1^* has at least $\delta - r_1 - r_2$ positive neighbours in U_2^* we have

$$[U_2^*, W_1^*] \le |U_2^*| \cdot |W_1^*| - |W_1^*|(\delta - r_1 - r_2). \tag{4.5}$$

Secondly, since $W_2^* \to U_2^*$ we have

$$[U_2^*, W_2] = [U_2^*, W_2 \backslash W_2^*] \le |U_2^*| \cdot q_2. \tag{4.6}$$

Thirdly, we see that

$$[U_2^*, W_i \backslash W_i^*] \le |U_2^*| \cdot q_i \quad \text{for} \quad i = 1, 3. \tag{4.7}$$

Using (4.5), (4.6) and (4.7) in (4.4) we conclude

$$[U_2^*, W_3^*] \ge |W_1^*|(\delta - r_1 - r_2) + |U_2^*|(\delta - t - q - |W_1^*|).$$

Taking into account that $\delta + a - t - q_1 \le |W_1^*| \le \delta + 1 - t - q_1$ we further deduce

$$\begin{aligned} [U_2^*, W_3^*] &\ge (\delta + a - t - q_1)(\delta - r_1 - r_2) + |U_2^*|(\delta - t - q - (\delta + 1 - t - q_1)) \\ &= (\delta + a - t)(\delta - r_1 - r_2) - q_1(\delta - r_1 - r_2) - |U_2^*|(q_2 + q_3 + 1). \end{aligned}$$

Because of $|U_2^*| \geq \delta - r_1 - r_2$ this leads to

$$[U_2^*, W_3^*] \geq (\delta + a - t)(\delta - r_1 - r_2) - |U_2^*|(q+1).$$

Considering $q+1 \geq 1$ and $|U_2^*| \leq \delta - 1$ from (4.3), and using (4.1), we deduce

$$\begin{aligned}[U_2^*, W_3^*] &\geq (\delta + a - t)(\delta - r_1 - r_2) - (\delta - 1)(q+1) \\ &\geq (q + r + 1)(\delta - r_1 - r_2) - (\delta - 1)(q+1) \\ &= (q+1)(1 - r_1 - r_2) + r(\delta - r_1 - r_2) \\ &\geq (q+1)(1 - r_1 - r_2) + r(t + r_3 + q - a + 1) \\ &= (r_3 + 1)(q+1) + (t - a + r_3)r \qquad (4.8) \\ &\geq 1.\end{aligned}$$

This shows that W_3^* is nonempty and there exists at least one arc from U_2^* to W_3^*. Let $w_3 \in W_3^*$ and $u_2 \in U_2^*$ such that $u_2w_3 \in A(T)$. If there exists an arc $w_3u_3 \in A(T)$ with $u_3 \in U_3^*$, then $x \to W_1^* \to u_2 \to w_3 \to u_3 \to W_2^* \to y$ and $T - S$ contains an x-y-path. Thus, we have $U_3^* \to w_3$, and by definition of W_3 we have $yw_3 \in A(T)$.

Now assume w_3 has at least $r+1$ negative neighbours in U_2^*. Let $b = 1$ in case $|U| = 2\delta + 2$, and $b = 0$ otherwise. Then we have

$$\begin{aligned}\delta + b \leq d^+(w_3) &\leq |U| - |\{y\}| - |U_3^*| - (r+1) \\ &\leq 2\delta + 1 + b - 1 - (\delta - r_1 - r_3) - r - 1 \\ &= \delta + b - 1 - r_2 \\ &\leq \delta + b - 1,\end{aligned}$$

a contradiction. Following this conclusion, every vertex in W_3^* can have at most r negative neighbours in U_2^*, leading to

$$\begin{aligned}|N^+(U_2^*) \cap W_3^*| &\geq \frac{[U_2^*, W_3^*]}{r} \overset{(4.8)}{\geq} \frac{(r_3+1)(q+1) + (t - a + r_3)r}{r} \\ &= t - a + r_3 + \frac{(r_3+1)(q+1)}{r} > t - a + r_3,\end{aligned}$$

and thus,

$$|N^+(U_2^*) \cap W_3^*| \geq t - a + r_3 + 1. \qquad (4.9)$$

By the symmetry of U_2 and U_3 mentioned earlier, we also obtain

$$|N^-(U_3^*) \cap W_3^*| \geq t - a + r_2 + 1. \qquad (4.10)$$

We note that
$$N^+(U_2^*) \cap N^-(U_3^*) \cap W_3^* = \emptyset, \tag{4.11}$$
since otherwise there would be an x-y-path of length 6 in $T - S$.

For the cardinality of W we have
$$t + |W_1| + |W_2| + |W_3| = |W| \leq 2\delta + 1 + a.$$
Since $\delta + a - t \leq |W_1|, |W_2|$ we deduce
$$|W_3| \leq 2\delta + 1 + a - 2(\delta + a - t) - t = t - a + 1.$$
Together with (4.9), (4.10) and (4.11) this leads to
$$\begin{aligned} t - a + 1 \geq |W_3| &\geq |N^+(U_2^*) \cap W_3^*| + |N^-(U_3^*) \cap W_3^*| \\ &\geq t - a + r_3 + 1 + t - a + r_2 + 1 \\ &= 2(t - a + 1) + r_2 + r_3, \end{aligned}$$
a contradiction, since $t - a + 1 \geq 1$.

Case 2. $N^+(x) \cap N^-(y) = \emptyset$. We define the vertex subsets W_i and U_i as well as the integers r_i, q_i, r, q for $i = 1, 2, 3$ like before (see Figure 4.2). We observe that $|W_1|, |W_2| \geq \delta + a$, and thus $|W_1^*|, |W_2^*| \geq 1$ and $|W_3| \leq 1$. Analogue to Case 1 we also have $|U_2^*| \geq \delta - r_1 - r_2$ and $|U_3^*| \geq \delta - r_1 - r_3$. We distinguish two subcases.

Subcase 2.1. $r_1 = |U_1| \leq \delta - 1$. At first we show that $q \geq a$. Assume to the contrary that $a = 1$ and $q = 0$. Then $W_1^* = W_1$, $W_2^* = W_2$, and
$$\begin{aligned} |U_2^*| + |U_3^*| &\geq \delta - r_1 - r_2 + \delta - r_1 - r_3 \\ &= \delta - r_1 + \delta - r \geq \delta - r_1 \\ &\geq 1. \end{aligned}$$
Therefore, we have $|U_2^*| \geq 1$ or $|U_3^*| \geq 1$. From $a = 1$ we deduce $|W| = 2\delta + 2$. Assume $|U_3^*| \geq 1$ and $u_3 \in U_3^*$. By definition of U_3, u_3 must have a positive neighbour $w_2 \in W_2^*$. Furthermore, u_3 must have $\delta + 1$ negative neighbours in W. Since w_2 already is a positive neighbour of u_3, there must be a vertex $w_1 \in W_1^*$ with $w_1 u_3 \in A(T - S)$. Thus, we have an x-y-path in $T - S$, a contradiction. Therefore, we have $U_3^* = \emptyset$. By the symmetry of U_2 and U_3, an analogue deduction also shows $U_2^* = \emptyset$. It follows
$$q \geq a. \tag{4.12}$$
By (4.1) we conclude
$$|U_i^*| \geq \delta - r_1 - r_i \geq 1 + q - a \geq 1 \;\text{ for }\; i = 2, 3.$$

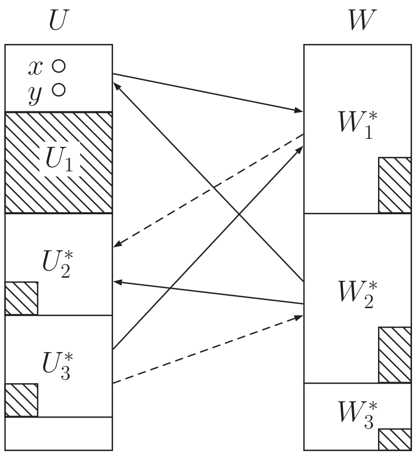

Figure 4.2: Structure of bipartite tournament T from Case 2 in the proof of Theorem 4.5. Shaded regions belong to the vertex set S. Continuous arcs depict a dominance relation between the vertex subsets in question. For two vertex subsets $A, B \subseteq V(T)$ a dashed arc from A to B indicates that for all vertices $a \in A$ and $b \in B$ we have $N^+(a) \cap B \neq \emptyset$ and $N^-(b) \cap A \neq \emptyset$.

Now assume there exists a vertex $u_2 \in U_2^*$ with $q_2 - a + 2$ negative neighbours in W_1^*. We further note that $W_2^* \to U_2^*$ implies $|N^-(u_2) \cap W_2^*| = |W_2^*|$, leading to

$$\begin{aligned}\delta + 1 \geq d^-(u_2) &\geq |N^-(u_2) \cap W_1^*| + |N^-(u_2) \cap W_2^*| \\ &\geq q_2 - a + 2 + |W_2^*| \\ &\geq q_2 - a + 2 + \delta + a - q_2 \\ &= \delta + 2,\end{aligned}$$

a contradiction. Thus, every vertex of U_2^* has at most $q_2 - a + 1$ negative neighbours in W_1^*. We can adopt inequality (4.5) from Case 1, and deduce

$$|U_2^*|(q_2 - a + 1) \geq [W_1^*, U_2^*] = |U_2^*| \cdot |W_1^*| - [U_2^*, W_1^*] \overset{(4.5)}{\geq} |W_1^*|(\delta - r_1 - r_2).$$

Together with $|W_1^*| \geq \delta + a - q_1$, $|U_2^*| \leq \delta - 1$ from (4.3), and $q_2 - a + 1 \leq \delta - q_1 - r$

from (4.1) we deduce

$$\begin{aligned}
0 &\leq |U_2^*|(q_2 - a + 1) - |W_1^*|(\delta - r_1 - r_2) \\
&\leq (\delta - 1)(q_2 - a + 1) - (\delta + a - q_1)(\delta - r_1 - r_2) \\
&\leq \delta(\delta - q_1 - r) - (q_2 - a + 1) - (\delta + a - q_1)(\delta - r_1 - r_2) \\
&= -\delta r_3 - a(\underbrace{\delta - r_1 - r_2 - 1}_{\geq q-a \overset{(4.12)}{\geq} 0}) - q_1(r_1 + r_2) - q_2 - 1 < 0,
\end{aligned}$$

a contradiction.

Subcase 2.2. $r_1 = |U_1| \geq \delta$. Because of $r_1 \leq |S| \leq \delta + a - 1$ we see that $|U_1| = \delta$, $a = 1$, and $\delta \leq |S|$ with $S = U_1$, i. e. $\kappa(x, y) \geq \delta$. We only need to consider the case $\kappa(x, y) = \delta$. Furthermore, we have $|W_1| = |W_2| = \delta + 1 = \min\{d^+(x), d^-(y)\}$, and thus $|W| = 2\delta + 2$. Let $R = U\backslash(\{x, y\} \cup U_1)$ and $v \in R$ be an arbitrary vertex from R.

If $v \in N^+(W_1)$, then there exists a vertex $w_1 \in W_1$ with $w_1 v \in A(T)$ and $W_2 \to v$, since otherwise we have an x-y-path in $T - S$. But now, we have

$$\delta + 1 \geq d^-(v) \geq 1 + |W_2| = \delta + 2,$$

a contradiction. Thus, we have $v \to W_1$, and by symmetry of W_1 and W_2 also $W_2 \to v$. Since $v \in R$ has been chosen arbitrarily, we have $W_2 \to R \to W_1$. This implies $(R \cup \{x, y\}) \to W_1 \to U_1 \to W_2 \to (R \cup \{x, y\})$ and for $s = \delta$ T is isomorphic to the bipartite tournament $T_1(s)$ in case $|U| = 2\delta$, and isomorphic to $T_2(s)$ in case $|U| = 2\delta + 1$ (see Example 4.4). □

In its main ideas, the proof of the following result is similar to the one above.

Theorem 4.6 (Guo, Holtkamp, Milz [45]). *Let T be a bipartite tournament with $i_g(T) \leq 1$, and x, y two vertices from different bipartite sets. Then*

$$\kappa(x, y) = \min\{d^+(x), d^-(y)\}.$$

Proof. Let U, W be the bipartite sets of G with $x \in U$ and $y \in W$, and let $a \in \{0, 1\}$ such that $\min\{d^+(x), d^-(y)\} = \delta + a$. We distinguish two cases.

Case 1. $yx \in A(T)$. Assume to the contrary that there exists a vertex set S separating x from y in T with $|S| \leq \delta + a - 1$. For a vertex set $X \subseteq V(T)$ we denote $X^* = X\backslash S$. We define

$$W_1 = N^+(x) \setminus \{y\} = N^+(x) \quad \text{and} \quad U_1 = N^-(y) \setminus \{x\} = N^-(y).$$

Obviously, we have $\delta + a \leq |U_1|, |W_1| \leq \delta + 1$, showing that U_1^* and W_1^* are nonempty. We define the vertex sets

$$W_2 = N^-(U_1^*) \setminus W_1, \qquad U_2 = N^+(W_1^*) \setminus U_1,$$
$$W_3 = N^+(U_2^*) \setminus (W_1 \cup W_2), \quad \text{and} \quad U_3 = N^-(W_2^*) \setminus (U_1 \cup U_2).$$

Figure 4.3 shows the structure of T. Like already seen in the proof of Theorem 4.5, in the further argumentation we repeatedly take advantage of a certain symmetry in the definition of these vertex sets. According to this, we note that by considering the converse digraph $T^{-1} = (V(T), \{yx : xy \in A(T)\})$ we can adopt the properties of W_1 in an analogue way for U_1, and vice versa. Also, we have the same symmetry between the vertex sets W_2 and U_2, as well as W_3 and U_3.

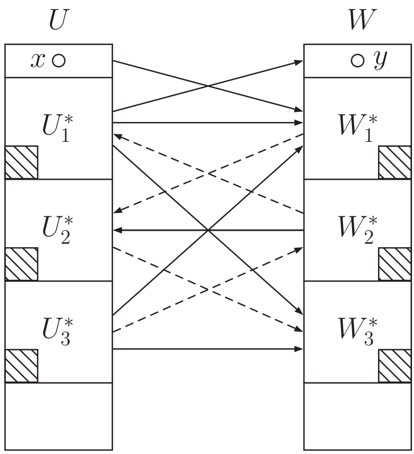

Figure 4.3: Structure of bipartite tournament T from the proof of Theorem 4.6. Shaded regions belong to the vertex set S. Continuous arcs depict a dominance relation between the vertex subsets in question. For two vertex subsets $A, B \subseteq V(T)$ a dashed arc from A to B indicates that for all vertices $a \in A$ and $b \in B$ we have $N^+(a) \cap B \neq \emptyset$ and $N^-(b) \cap A \neq \emptyset$.

Let $r_i = |U_i \cap S|$ and $q_i = |W_i \cap S|$ for $i = 1, 2, 3$, and $r = r_1 + r_2 + r_3$ and $q = q_1 + q_2 + q_3$. We have

$$r + q \leq |S| \leq \delta + a - 1. \tag{4.13}$$

Furthermore, by the definition of r_1 and q_1 we have

$$\delta + a - r_1 \le |U_1^*| \le \delta + 1 - r_1,$$

and

$$\delta + a - q_1 \le |W_1^*| \le \delta + 1 - q_1.$$

We have $U_1^* \to W_1^*$, since otherwise there would be an x-y-path in $T - S$. Because of $x \to W_1^*$ and $U_1^* \to y$ we deduce $|U_1^*|, |W_1^*| \le \delta$. It follows

$$q_1 = |W_1 \backslash W_1^*| \ge \delta + a - \delta = a$$

and

$$r_1 = |U_1 \backslash U_1^*| \ge \delta + a - \delta = a.$$

Furthermore, every vertex in W_1^* must have at least $\delta - r_1$ positive neighbours in $U \backslash U_1$, leading to $|U_2| \ge \delta - r_1$. Analogously, we have $|W_2| \ge \delta - q_1$. Using this, we can see that U_2^* and W_2^* are nonempty, since

$$|U_2^*| \ge \delta - r_1 - r_2 \overset{(4.13)}{\ge} r_3 + q + 1 - a \ge r_3 + q_2 + q_3 + a + 1 - a \ge 1$$

and

$$|W_2^*| \ge \delta - q_1 - q_2 \ge q_3 + r_2 + r_3 + a + 1 - a \ge 1.$$

Of course, the existence of an arc from U_2^* to W_2^* would imply an x-y-path in $T - S$ of length 5, a contradiction. Thus, we have $W_2^* \to U_2^*$. By definition of W_2 a vertex $w_2 \in W_2^*$ must have a positive neighbour in U_1^*, and also $w_2 \to x$. This leads to

$$\delta + 1 \ge d^+(w_2) \ge |U_2^*| + 1 + |\{x\}| \ge |U_2^*| + 2,$$

and therefore,

$$|U_2^*| \le \delta - 1.$$

Analogously we have

$$|W_2^*| \le \delta - 1.$$

In the following we show that W_3^* and by the symmetry mentioned above also U_3^* are nonempty. We have

$$\begin{aligned} |U_2^*| \cdot \delta &\le [U_2^*, W] \\ &= [U_2^*, W_1^*] + [U_2^*, W_1 \backslash W_1^*] + [U_2^*, W_2] + [U_2^*, W_3^*] + [U_2^*, W_3 \backslash W_3^*]. \end{aligned} \tag{4.14}$$

Since every vertex in W_1^* has at least $\delta - r_1 - r_2$ positive neighbours in U_2^* we conclude

$$\begin{aligned} [U_2^*, W_1^*] &= |U_2^*| \cdot |W_1^*| - [W_1^*, U_2^*] \\ &\le |U_2^*| \cdot |W_1^*| - |W_1^*|(\delta - r_1 - r_2). \end{aligned} \tag{4.15}$$

Furthermore, we have

$$[U_2^*, W_i \backslash W_i^*] \leq q_i \cdot |U_2^*| \quad \text{for} \quad i = 1, 3, \tag{4.16}$$

and since $W_2^* \to U_2^*$ we deduce

$$[U_2^*, W_2] = [U_2^*, W_2 \backslash W_2^*] \leq q_2 \cdot |U_2^*|. \tag{4.17}$$

Using the inequalities (4.15), (4.16), and (4.17) in (4.14), and taking into account that $\delta + a - q_1 \leq |W_1^*| \leq \delta + 1 - q_1$ as well as $|U_2^*| \leq \delta - 1$ we conclude

$$\begin{aligned}
[U_2^*, W_3^*] &\geq |W_1^*|(\delta - r_1 - r_2) + (\delta - q - |W_1^*|)|U_2^*| \\
&\geq (\delta + a - q_1)(\delta - r_1 - r_2) - (q_2 + q_3 + 1)|U_2^*| \\
&\geq (\delta + a - q_1)(\delta - r_1 - r_2) - (q_2 + q_3 + 1)(\delta - 1) \qquad (4.18) \\
&= \delta^2 - \delta(\underbrace{r_1 + r_2 + q + 1 - a}_{\overset{(4.13)}{\leq} \delta - r_3}) + (\underbrace{q_1 - a}_{\geq 0})(r_1 + r_2) + q_2 + q_3 + 1 \\
&\geq \delta r_3 + (q_1 - a)(r_1 + r_2) + q_2 + q_3 + 1 \qquad (4.19) \\
&\geq 1.
\end{aligned}$$

By symmetry we also have $[U_3^*, W_2^*] \geq 1$, thus, W_3^* and U_3^* are nonempty. Since there is a path from any vertex of $U_1^* \cup U_3^*$ to y in $T - S$, and a path from x to any vertex of $W_1^* \cup W_3^*$ in $T - S$, it follows $(U_1^* \cup U_3^*) \to (W_1^* \cup W_3^*)$.

Now $U_1^* \to (\{y\} \cup W_1^* \cup W_3^*)$, and for an arbitrary vertex $u_1 \in U_1^*$ we have

$$\delta + 1 \geq d^+(u_1) \geq |\{y\}| + |W_1^*| + |W_3^*| \geq 1 + \delta + a - q_1 + |W_3^*|,$$

leading to

$$|W_3^*| \leq q_1 - a \leq \delta - 1. \tag{4.20}$$

Since any vertex of W_3^* is dominated by at least 1 vertex of U_1^* we deduce

$$\begin{aligned}
|W_3^*| &\geq \frac{[U_2^*, W_3^*]}{\delta} \\
&\overset{(4.18)}{\geq} \frac{(\delta + a - q_1)(\delta - r_1 - r_2)}{\delta} - \frac{\delta - 1}{\delta}(q_2 + q_3 + 1) \\
&> \frac{\delta^2 - \delta(r_1 + r_2 + q_1 - a) + (q_1 - a)(r_1 + r_2)}{\delta} - q_2 - q_3 - 1 \\
&\geq \delta - q - r_1 - r_2 + a - 1.
\end{aligned}$$

Considering that the inequality above is strict we arrive at

$$|W_3^*| \geq \delta - q - r_1 - r_2 + a \overset{(4.13)}{\geq} r_3 + 1.$$

By an analogue deduction due to the symmetry of W_3^* and U_3^* we also obtain

$$|U_3^*| \geq \delta - r - q_1 - q_2 + a \geq q_3 + 1.$$

For an arbitrary $w_3 \in W_3^*$ we further have

$$|U_1^*| + |U_2^* \cap N^-(w_3)| + |U_3^*| \leq d^-(w_3) \leq \delta + 1.$$

Together with the lower bounds for U_1^* and U_3^* this leads to

$$\begin{aligned} 1 \leq |U_2^* \cap N^-(w_3)| &\leq \delta + 1 - |U_1^*| - |U_3^*| \\ &\leq \delta + 1 - \delta - a + r_1 - q_3 - 1 = r_1 - q_3 - a. \end{aligned}$$

Hence, every vertex in W_3^* has at most $r_1 - q_3 - a$ negative neighbours in U_2^*. By using (4.20) we now deduce

$$q_1 - a \geq |W_3^*| \geq \frac{[U_2^*, W_3^*]}{r_1 - q_3 - a},$$

which implies

$$(q_1 - a)(r_1 - q_3 - a) \geq [U_2^*, W_3^*] \overset{(4.19)}{\geq} \delta r_3 + (q_1 - a)(r_1 + r_2) + q_2 + q_3 + 1.$$

This finally leads to the contradiction

$$0 < \delta r_3 + q_2 + q_3 + 1 \leq (q_1 - a)(-r_2 - q_3 - a) \leq 0.$$

Case 2. $xy \in A(T)$. Assume to the contrary that there exists a vertex set S separating x from y in $T - xy$ with $|S| \leq \delta + a - 2$. We adopt the notations from Case 1 and have

$$r + q \leq |S| \leq \delta + a - 2. \tag{4.21}$$

Furthermore, we have $\delta + a - q_1 - 1 \leq |W_1^*| \leq \delta - q_1$ and $\delta + a - r_1 - 1 \leq |U_1^*| \leq \delta - r_1$. Obviously,

$$q_1, r_1 \geq a - 1.$$

It is easy to see that the inequalities (4.14), (4.15), (4.16) and (4.17) still remain valid for the case considered here. Analogue to (4.18) we deduce

$$\begin{aligned} [U_2^*, W_3^*] &\geq |W_1^*|(\delta - r_1 - r_2) + (\delta - q - |W_1^*|)|U_2^*| \\ &\geq |W_1^*|(\delta - r_1 - r_2) - (q_2 + q_3)|U_2^*| \\ &\geq |W_1^*|(\delta - r_1 - r_2) + |U_2^*| - (q_2 + q_3 + 1)|U_2^*| \\ &\geq |W_1^*|(\delta - r_1 - r_2) + (\delta - r_1 - r_2) - (q_2 + q_3 + 1)|U_2^*| \\ &\geq (|W_1^*| + 1)(\delta - r_1 - r_2) - (q_2 + q_3 + 1)(\delta - 1) \\ &\geq (\delta + a - q_1)(\delta - r_1 - r_2) - (q_2 + q_3 + 1)(\delta - 1). \end{aligned} \tag{4.22}$$

Taking a closer look at this last inequality we obtain

$$\begin{aligned}
[U_2^*, W_3^*] &\geq \delta^2 - \delta(r_1 + r_2 + q + 1 - a) + (q_1 - a)(r_1 + r_2) + q_2 + q_3 + 1 \\
&\overset{(4.21)}{\geq} \delta^2 - \delta(\delta - r_3 - 1) + (q_1 - a)(r_1 + r_2) + q_2 + q_3 + 1 \\
&= \delta r_3 + \delta + (q_1 - a)(r_1 + r_2) + q_2 + q_3 + 1 \\
&\overset{(4.21)}{\geq} \delta r_3 + r + q - a + 2 + (q_1 - a)(r_1 + r_2) + q_2 + q_3 + 1 \\
&= (\delta + 1) r_3 + \underbrace{(q_1 - a + 1)}_{\geq 0}(r_1 + r_2 + 1) + 2(q_2 + q_3 + 1) \qquad (4.23) \\
&\geq 2.
\end{aligned}$$

This shows that W_3^* and by symmetry also U_3^* are nonempty with $[U_2^*, W_3^*] \geq 2$ and $[U_3^*, W_2^*] \geq 2$. Since there is no x-y-path in $(T - xy) - S$, we have $(U_1^* \cup U_3^*) \to (W_1^* \cup W_3^*)$.

Furthermore, for any vertex $u_1 \in U_1^*$ we have

$$\delta + 1 \geq d^+(u_1) \geq |\{y\}| + |W_1^*| + |W_3^*| \geq 1 + \delta + a - 1 - q_1 + |W_3^*|,$$

such that

$$|W_3^*| \leq q_1 - a + 1 \leq \delta - 1. \qquad (4.24)$$

Since any vertex of W_3^* is dominated by at least 1 vertex of U_1^*, we deduce

$$\begin{aligned}
|W_3^*| &\geq \frac{[U_2^*, W_3^*]}{\delta} \\
&\overset{(4.22)}{\geq} \frac{(\delta + a - q_1)(\delta - r_1 - r_2)}{\delta} - \frac{\delta - 1}{\delta}(q_2 + q_3 + 1) \\
&> \frac{\delta^2 - \delta(r_1 + r_2 + q_1 - a + 1) + \delta + (q_1 - a)(r_1 + r_2)}{\delta} - q_2 - q_3 - 1 \\
&\geq \frac{\delta^2 - \delta(r_1 + r_2 + q_1 - a + 1) + (q_1 - a + 1)(r_1 + r_2)}{\delta} - q_2 - q_3 - 1 \\
&\geq \delta - q - r_1 - r_2 + a - 2.
\end{aligned}$$

Since the inequality above is strict, this implies

$$|W_3^*| \geq \delta - q - r_1 - r_2 + a - 1 \overset{(4.21)}{\geq} r_3 + 1.$$

Analogously we also obtain

$$|U_3^*| \geq \delta - r - q_1 - q_2 + a - 1 \overset{(4.21)}{\geq} q_3 + 1.$$

Any vertex $w_3 \in W_3^*$ fulfills

$$\begin{aligned} 1 \leq |U_2^* \cap N^-(w_3)| &\leq \delta + 1 - |U_1^*| - |U_3^*| \\ &\leq \delta + 1 - \delta - a + 1 + r_1 - q_3 - 1 \\ &= r_1 - q_3 - a + 1. \end{aligned}$$

This shows that every vertex of W_3^* has at most $r_1 - q_3 - a + 1$ negative neighbours in U_2^*. Therefore, we have

$$q_1 - a + 1 \geq |W_3^*| \geq \frac{[U_2^*, W_3^*]}{r_1 - q_3 - a + 1},$$

leading to

$$\begin{aligned} (q_1 - a + 1)(r_1 - q_3 - a + 1) &\geq [U_2^*, W_3^*] \\ &\overset{(4.23)}{\geq} (\delta + 1)r_3 + (q_1 - a + 1)(r_1 + r_2 + 1) + 2(q_2 + q_3 + 1). \end{aligned}$$

Finally, we arrive at the contradiction

$$0 < (\delta + 1)r_3 + 2(q_2 + q_3 + 1) \leq (q_1 - a + 1)(-r_2 - q_3 - a) \leq 0,$$

and the proof is complete. □

As an immediate consequence of Theorem 4.5 and Theorem 4.6 we obtain the following two corollaries.

Corollary 4.7 (Guo, Holtkamp, Milz [45])**.** *Regular and almost regular bipartite tournaments are maximally connected.*

Corollary 4.8 (Guo, Holtkamp, Milz [45])**.** *Almost regular bipartite tournaments that are not isomorphic to either $T_1(s+1)$ or $T_2(s)$ with $s \geq 1$ from Example 4.4 are maximally local connected.*

The following example shows that the upper bound for the irregularity in Theorem 4.5 and Theorem 4.6 is best possible.

Example 4.9. *Let B be the bipartite tournament depicted in Figure 4.4 consisting of two cycles C_a and C_b both of length 4, such that all arcs between C_a and C_b are directed from $V(C_a)$ to $V(C_b)$.*

Obviously, $i_g(B) = 2$ and $\delta(B) = 1$. Since B is not strong it is not maximally connected, and thus, not maximally local connected.

It is easy to construct bipartite tournaments similar to Example 4.9 with arbitrary irregularity and higher minimum degree, which are not maximally connected. However, B is the only bipartite tournament with irregularity two which is not maximally connected. The proof of this can be found in Milz [76].

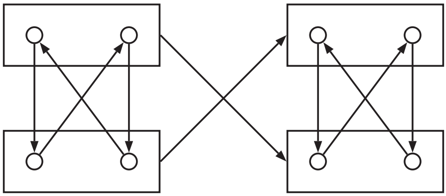

Figure 4.4: Bipartite tournament B from Example 4.9.

4.2 Lower bound on the connectivity of local tournaments

Thomassen [98] studied the connectivity of tournaments according to their irregularity.

Theorem 4.10 (Thomassen [98], 1980)**.** *If T is a tournament with $i_g(T) \leq k$, then*

$$\kappa(T) \geq \left\lceil \frac{|V(T)| - 2k}{3} \right\rceil . \tag{4.25}$$

He also characterized the tournaments for which (4.25) holds with equality. Lichiardopol [69] presented a generalization of this result for oriented graphs.

Theorem 4.11 (Lichiardopol [69], 2008)**.** *If T is an oriented graph, then*

$$\kappa(T) \geq \left\lceil \frac{2\delta^+(T) + 2\delta^-(T) + 2 - n(T)}{3} \right\rceil . \tag{4.26}$$

It is also shown that (4.26) implies (4.25) for tournaments and that for tournaments with $\delta^+ \neq \delta^-$ Theorem 4.11 is an improvement of Theorem 4.10. In this section we will prove two lower bounds on the connectivity of two classes of local tournaments. One of them implies Lichiardopol's bound for tournaments. Although local tournaments are oriented graphs, our bound gives a better approximation for the connectivity of local tournaments.

Every tournament is also a local tournament, and every local tournament is also a locally semicomplete digraph. The structure of these digraphs has been studied by Bang-Jensen [8] and Guo and Volkmann [46]. A collection of their results and proofs can be found in [10]. The following results will be helpful for the proof of our main result in the next section.

Lemma 4.12 (Bang-Jensen [8], 1990). *Let D be a strong locally semicomplete digraph and let S be a minimal separating set of D. Then $D - S$ is connected.*

According to this property, it is helpful to study the structure of connected locally semicomplete digraphs that are not strong. Since local tournaments are locally semicomplete digraphs, the following results hold for local tournaments as well.

Theorem 4.13 (Bang-Jensen [8], 1990). *Let D be a connected locally semicomplete digraph that is not strong. Then the following holds for D.*

1. *If A and B are distinct strong components of D with at least one arc between them, then either $A \to B$ or $B \to A$.*
2. *If A and B are strong components of D such that $A \to B$, then A and B are semicomplete digraphs.*
3. *The strong components of D can be ordered in a unique way $D_1, D_2, \dots, D_p$ such that there are no arcs from D_j to D_i for $j > i$, and D_i dominates D_{i+1} for $i = 1, 2, \dots, p-1$.*

For a digraph D fulfilling the condition of Theorem 4.13 the unique ordering of its strong components is called the *a*cyclic ordering of the strong components of D.

Theorem 4.14 (Guo, Volkmann [46], 1994). *Let D be a connected locally semicomplete digraph that is not strong and let $D_1, D_2, \dots, D_p$ be the acyclic ordering of the strong components of D. Then D can be decomposed into $r \geq 2$ induced subdigraphs $D'_1, D'_2, \dots, D'_r$ which satisfy the following properties.*

1. *$D'_1 = D_p$ and D'_i consists of some strong components of D and is semicomplete for $i \geq 2$.*
2. *D'_{i+1} dominates the initial component of D'_i and there exists no arc from D'_i to D'_{i+1} for $i = 1, \dots, r-1$.*
3. *If $r \geq 3$, then there is no arc between D'_i and D'_j for i, j satisfying $|i - j| \geq 2$.*

The unique sequence $D'_1, D'_2, \dots, D'_r$ is called the semicomplete decomposition *of D.*

Finally, the next lemma determines the structure of locally semicomplete digraphs that are not semicomplete.

Lemma 4.15 (Bang-Jensen, Guo, Gutin, Volkmann [9], 1997)**.** *If a strong locally semicomplete digraph D is not semicomplete, then there exists a minimal separating set S such that $D - S$ is not semicomplete. Furthermore, if $D_1, D_2, \ldots, D_p$ is the acyclic ordering of the strong components of $D - S$ and $D'_1, D'_2, \ldots, D'_r$ is the semicomplete decomposition of $D - S$, then $r \geq 3$, $D[S]$ is semicomplete and we have $D_p \to S \to D_1$.*

The inequality (4.25) in Theorem 4.10 gives a lower bound on the connectivity of tournaments with respect to their order $|V(T)|$ and irregularity $i_g(T)$. In Theorem 4.11 Lichiardopol uses the minimum out-degree, the minimum in-degree and the order of an oriented graph for a lower bound on its connectivity. Considering local tournaments, the number of vertices becomes an unsuitable parameter for a lower bound on the connectivity. A simple example for this is the oriented cycle C_n with n vertices, which is a local tournament with connectivity $\kappa(C_n) = 1$ for every $n \geq 3$. However, a more meaningful parameter for local tournaments seems to be the minimum degree. We note that if a digraph D is not strong, we have $\kappa(D) = 0$. The following result gives a lower bound on the connectivity of strong local tournaments.

Theorem 4.16 (Guo, Holtkamp, Milz [44])**.** *Let D be a strong local tournament with $i_g(D) \leq k$. If there exists a minimum separating set S such that $D - S$ is a tournament, then*

$$\kappa(D) \geq \left\lceil \frac{2 \cdot \max\{\delta^+, \delta^-\} + 1 - k}{3} \right\rceil,$$

else

$$\kappa(D) \geq \left\lceil \frac{2 \cdot \max\{\delta^+, \delta^-\} + 2|\delta^+ - \delta^-| + 1 - 2k}{3} \right\rceil.$$

Proof. Let $S \subset V(D)$ be a minimum separating set and $s = |S|$, thus, $\kappa(D) = s$ and S is also a minimal separating set. If possible, we choose S such that $D - S$ is a tournament. Since D is a strong local tournament, according to Lemma 4.12 the digraph $D - S$ is connected but not strong. Thus, $D - S$ has the structure as described in Theorem 4.13. Let $D_1, D_2, \ldots, D_p$ be the acyclic ordering of the strong components of $D - S$ for some $p \geq 2$. Denote $n_i = |V(D_i)|$ for all $i = 1, 2, \ldots, p$. Theorem 4.13 implies that D_p is a tournament. Thus there is a vertex $x^* \in V(D_p)$ dominating at most $\lceil (n_p - 1)/2 \rceil$ vertices in D_p. By the acyclic ordering of the strong components of $D - S$ a vertex in $V(D_p)$ can only dominate vertices of D_p or S, leading to

$$\delta^+ \leq |N_D^+(x^*)| \leq d_{D_p}^+(x^*) + s \leq \frac{n_p - 1}{2} + s$$

which implies

$$n_p \geq 2(\delta^+ - s) + 1. \tag{4.27}$$

Similarly, every vertex in D_1 can only be dominated by vertices in D_1 or S. Therefore, an analogue deduction for a vertex of D_1 leads to

$$n_1 \geq 2(\delta^- - s) + 1. \tag{4.28}$$

We consider two cases, whether $D - S$ is a tournament or not.

Case 1. If $D - S$ is a tournament, then every vertex in D_1 dominates all vertices of $V(D_p)$. We have $x^* \in V(D_1)$ with at least $\lceil (n_1 - 1)/2 \rceil$ positive neighbours in D_1. Considering the vertices dominated by x^* in $D - S$ together with $i_g(D) \leq k$ we have

$$d^+_{D_1}(x^*) + n_p \leq \delta + k.$$

This implies

$$n_p \leq \delta + k - d^+_{D_1}(x^*) \leq \delta + k - \frac{n_1 - 1}{2},$$

and with the use of (4.27) and (4.28) we deduce

$$2(\delta^+ - s) + 1 \leq n_p \leq \delta + k - (\delta^- - s).$$

This implies

$$s \geq \frac{2\delta^+ + \delta^- - \delta + 1 - k}{3}.$$

Considering a vertex $y^* \in V(D_p)$ with at least $\lceil (n_p - 1)/2 \rceil$ negative neighbours in D_p an analogue deduction leads to

$$s \geq \frac{2\delta^- + \delta^+ - \delta + 1 - k}{3}.$$

Altogether we have

$$\begin{aligned} s &\geq \max\left\{ \frac{2\delta^+ + \delta^- - \delta + 1 - k}{3}, \frac{2\delta^- + \delta^+ - \delta + 1 - k}{3} \right\} \\ &= \frac{\max\{\delta^+, \delta^-\} + \delta^+ + \delta^- - \delta + 1 - k}{3}. \end{aligned}$$

Taking into account that $\delta^+ + \delta^- - \delta = \max\{\delta^+, \delta^-\}$ we have

$$s \geq \frac{2 \cdot \max\{\delta^+, \delta^-\} + 1 - k}{3}.$$

Case 2. If $D - S$ is not a tournament, then by Lemma 4.15 we have $D_p \to S$. Therefore,

$$[V(D_p), S] = n_p \cdot s.$$

Also, $D[S]$ is a tournament and thus $|A(D[S])| = \frac{1}{2}s(s-1)$. For $w \in S$ we have

$$\delta + k \geq d^-_{D-(S\setminus\{w\})}(w) + d^-_{D[S]}(w).$$

For the number of arcs from D_p to S we now deduce

$$\begin{aligned} n_p \cdot s = [V(D_p), S] &\leq \sum_{w \in S} d^-_{D-(S\setminus\{w\})}(w) \\ &\leq \sum_{w \in S} \Big(\delta + k - d^-_{D[S]}(w)\Big) \\ &= s(\delta + k) - |A(D[S])| \\ &= s(\delta + k) - \frac{1}{2}s(s-1), \end{aligned}$$

which implies

$$0 \leq s(\delta + k - n_p) - \frac{1}{2}s(s-1) = s\left(\delta + k - n_p - \frac{s}{2} + \frac{1}{2}\right).$$

Since $s \geq 1$, we have

$$n_p \leq \delta + k - \frac{s}{2} + \frac{1}{2}. \tag{4.29}$$

Combining (4.29) with (4.27) yields

$$2(\delta^+ - s) + 1 \leq n_p \leq \delta + k + \frac{1}{2} - \frac{s}{2},$$

and thus

$$s \geq \frac{2(2\delta^+ - \delta) + 1 - 2k}{3}.$$

By Lemma 4.15 we also have $S \to D_1$ and in analogy to (4.29) we deduce

$$n_1 \leq \delta + k - \frac{s}{2} + \frac{1}{2}.$$

In combination with (4.28) we conclude

$$2(\delta^- - s) + 1 \leq n_1 \leq \delta + k + \frac{1}{2} - \frac{s}{2},$$

which implies

$$s \geq \frac{2(2\delta^- - \delta) + 1 - 2k}{3}.$$

Altogether we have

$$\begin{aligned} s &\geq \max\left\{\frac{2(2\delta^+ - \delta) + 1 - 2k}{3}, \frac{2(2\delta^- - \delta) + 1 - 2k}{3}\right\} \\ &= \frac{4 \cdot \max\{\delta^+, \delta^-\} - 2\delta + 1 - 2k}{3}. \end{aligned}$$

Since $\max\{\delta^+, \delta^-\} - \delta = |\delta^+ - \delta^-|$, we arrive at

$$s \geq \frac{2 \cdot \max\{\delta^+, \delta^-\} + 2|\delta^+ - \delta^-| + 1 - 2k}{3}.$$

□

Since Theorem 4.16 uses the irregularity instead of the order of a digraph, it is not included in Theorem 4.11. To see this, we recognize e.g. that for C_n with $n \geq 6$ our lower bound implies $\kappa(C_n) \geq 1$, while the inequality (4.26) becomes trivial.

However, when considering tournaments the bounds of Theorem 4.16 and Theorem 4.11 coincide.

Corollary 4.17 (Guo, Holtkamp, Milz [44]). *Theorems 4.16 and 4.11 imply the same lower bound on the connectivity of tournaments.*

Proof. Let T be a tournament with irregularity $i(T) = k$, then T has exactly $n = 2\delta(T) + 1 + k$ vertices. We have

$$\begin{aligned} \left\lceil \frac{2\delta^+(T) + 2\delta^-(T) + 2 - n}{3} \right\rceil &= \left\lceil \frac{2\delta^+(T) + 2\delta^-(T) + 1 - 2\delta(T) - k}{3} \right\rceil \\ &= \left\lceil \frac{2\max\{\delta^+(T), \delta^-(T)\} + 1 - k}{3} \right\rceil. \end{aligned}$$

Therefore, our bound and the one given by Lichiardopol coincide for tournaments. □

In [69] it has been shown that Theorem 4.10 follows from Theorem 4.11. By Corollary 4.17 we see that Theorem 4.10 also follows from Theorem 4.16. If T is a tournament with $i_g(T) \leq k$, then $|V(T)| \leq 2\delta + k + 1$, and if S is a minimum separating set, then $T - S$ is a tournament as well. By Theorem 4.16 we obtain

$$\begin{aligned}\kappa(T) &\geq \left\lceil \frac{2\max\{\delta^+, \delta^-\} + 1 - k}{3} \right\rceil = \left\lceil \frac{2\delta + 2|\delta^+ - \delta^-| + 1 - k}{3} \right\rceil \\ &\geq \left\lceil \frac{|V(T)| - 2k + 2|\delta^+ - \delta^-|}{3} \right\rceil \geq \left\lceil \frac{|V(T)| - 2k}{3} \right\rceil.\end{aligned}$$

In case $\delta^+ \neq \delta^-$ we have an improvement of the lower bound in (4.25) by Thomassen.

For tournaments T the examples given by Thomassen [98] also show the sharpness of Theorem 4.16 for the case that T has a minimum separating set S such that $T - S$ is a tournament. In the other case, the following examples confirm that the second bound presented in Theorem 4.16 is best possible as well.

Example 4.18 (Guo, Holtkamp, Milz [44])**.** *Let T_0 be a single vertex and T_r an r-regular tournament for $r \geq 1$. C_{2i} denotes the directed cycle of length $2i$. For integers $l, k \geq 0$ and $i \geq 2$ we define the digraph H by*

$$H = C_{2i}[T_l, T_{l+k}, T_l, T_{l+k}, \ldots, T_l, T_{l+k}].$$

Obviously, H is a strong local tournament, where every subdigraph of the form T_r has $2r + 1$ vertices. Therefore, we have $\delta^+(H) = \delta^-(H) = \delta(H) = 3l + k + 1$ and $i_g(H) = k$. The vertex set of every subdigraph T_l is a separating set, thus, according to Theorem 4.16 we have

$$2l + 1 = |V(T_l)| \geq \kappa(H) \geq \frac{2\delta(H) - 2k + 1}{3} = 2l + 1.$$

Finally, we notice that a result similar to Theorem 4.16 cannot be obtained for locally semicomplete digraphs. The following examples show that the gap between the minimum degree and the connectivity of localyl semicomplete digraphs can be arbitrarily large.

Example 4.19 (Guo, Holtkamp, Milz [44])**.** *Let K_r be the complete digraph on $r \geq 1$ vertices. For integers $l, k \geq 1$ and $i \geq 2$ we define the digraph F by*

$$F = C_{2i}[K_l, K_{l+k}, K_l, K_{l+k}, \ldots, K_l, K_{l+k}].$$

According to this definition F is a strong locally semicomplete digraph, which is $(2l + k - 1)$-regular. The vertex set of every subdigraph K_l is a separating set, thus, it is easy to see that $\kappa(F) = l$.

Chapter 5

Restricted arc-connectivity

In this chapter we discuss a concept of restricted arc-connectivity, namely $\lambda'(D)$, proposed by Lutz Volkmann [108] in 2007. Only one year later Wang, Lin and Li [112] added the notion of the arc-degree $\xi'(D)$. We introduced the related definitions in Section 1.2.6 on page 7. In Sections 5.1 and 5.2 we will now study the restricted arc-connectivity and the λ'-optimality of tournaments, respectively, where the λ'-optimality is defined in the style of λ_2-optimality in graphs, i. e. $\lambda'(D) = \xi'(D)$. We will then proceed in the same way for bipartite tournaments in Sections 5.3 and 5.4. The results of this chapter have been obtained together with Steffen Grüter, Yubao Guo and Eduard Ulmer.

Like presented in Section 1.2.6 the arc degree $\xi'(xy)$ of an arc $xy \in A(D)$ of a digraph D is defined in case $yx \notin A(D)$ as

$$\begin{aligned}\xi'(xy) = \min\{&d^+(x) + d^+(y) - 1,\ d^+(x) + d^-(y) - 1,\\ &d^-(x) + d^+(y),\ d^-(x) + d^-(y) - 1\},\end{aligned}$$

and in case $yx \in A(D)$ as

$$\begin{aligned}\xi'(xy) = \min\{&d^+(x) + d^+(y) - 2,\ d^+(x) + d^-(y) - 1,\\ &d^-(x) + d^+(y) - 1,\ d^-(x) + d^-(y) - 2\}.\end{aligned}$$

The four degree sums given in this definition correspond to the size of four different arc subsets, whose removal yields a digraph where neither x nor y can be on any cycle (cf. the set Ω_{xy} in Figure 5.1). According to this property the arc-degree is closely related to the edge-degree, which has proven very useful for the investigation of restricted edge-cuts in graphs, e. g. Esfahanian and Hakimi [26]. Likewise similar results for graphs, Wang, Lin and Li presented the following theorem.

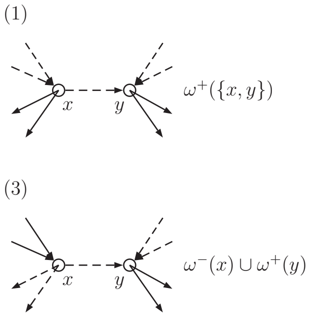

Figure 5.1: The four arc subsets of Ω_{xy}. Continuous arcs depict the arc subsets specified on the right hand side of pictures (1)–(4).

Theorem 5.1 (Wang, Lin, Li [112], 2008)**.** *Let D be a strong digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$, and xy be an arbitrary arc of D. Then every arc subset $S \in \Omega_{xy}$ is a restricted arc-cut, thus, D is restricted arc-connected. Furthermore, we have $\lambda'(D) \leq \xi'(D)$.*

Further results on restricted arc-connectivity can be found for example in Balbuena and García-Vázquez [5], Chen, Liu and Meng [14], and Guo and Guo [43].

5.1 Restricted arc-connectivity in tournaments

In this section we study the restricted arc-connectivity of tournaments. Note that a restricted arc-connected tournament has at least 5 vertices. Also, considering the arc-degree $\xi'(xy)$ of an arc $xy \in A(T)$ of a tournament T, we always have $yx \notin A(T)$. T is said to be *vertex pancyclic*, if every vertex is contained in a cycle of length i for all $3 \leq i \leq n$. Due to Moon [77] we have the following property for strong tournaments.

Theorem 5.2 (Moon [77], 1966)**.** *Let T be a strong tournament with $n \geq 3$. Then T is vertex pancyclic.*

This result already implies the existence of restricted arc-cuts for all strong tournaments with $n \geq 5$, which has been mentioned earlier by Meierling, Volkmann and Winzen [72], who studied the restricted arc-connectivity of generalizations of tournaments.

Observation 5.3. *Let T be a strong tournament with $n \geq 5$. Then T is restricted arc-connected.*

Proof. By Theorem 5.2, T contains a 3-cycle C_3. Therefore, $[V(C_3), \overline{V(C_3)}]$ is a restricted arc-cut of T. □

Let T_n be a transitive tournament on n vertices with its unique ordering $x_1 x_2 \dots x_n$ such that $x_i \to x_j$ for all $1 \leq i < j \leq n$. From T_n we obtain a strong tournament D_n by replacing the arc $x_1 x_n$ with the arc $x_n x_1$. We call the strong tournament D_n a *quasi-transitive strong tournament*, and note that for $n \geq 4$ we still have the unique ordering $x_1 x_2 \dots x_n$.

Lemma 5.4 (Grüter, Guo, Holtkamp, Ulmer [42])**.** *Let T be a quasi-transitive strong tournament with $n \geq 5$. Then $\lambda'(T) = 2$ and $\xi'(T) = 1$.*

Proof. Let $x_1 x_2 \dots x_n$ be the unique ordering of T with $x_n \to x_1$ and $x_i \to x_j$ for all other $1 \leq i < j \leq n$. Then we have $1 \leq \xi'(T) \leq \xi'(x_n x_1) \leq d^+(x_n) + d^-(x_1) - 1 = 1 + 1 - 1 = 1$.

By Observation 5.3, T is restricted arc-connected. At first we notice that $T - x_n x_1$ is acyclic, thus, $x_n x_1$ can not be contained in any minimum restricted arc-cut. Also, every minimum restricted arc-cut of T must contain at least one arc of the hamiltonian cycle $x_1 x_2 \dots x_n x_1$. Removing an arbitrary arc $x_i x_{i+1}$ $(1 \leq i \leq n-1)$ from the hamiltonian cycle still leaves the $(n-1)$-cycle $x_1 x_2 \dots x_i x_{i+2} \dots x_n x_1$ in case $1 \leq i \leq n-2$, or the $(n-1)$-cycle $x_1 x_2 \dots x_{n-2} x_n x_1$ if $i = n-1$. Thus, every minimum restricted arc-cut must contain at least 2 arcs. Since $\{x_1 x_2, x_1 x_3\}$ is a minimum restricted arc-cut of size 2, we have $\lambda'(T) = 2$. □

However, it turns out that with the exception of quasi-transitive strong tournaments we have the following upper bound on $\lambda'(T)$ in a tournament T.

Theorem 5.5 (Grüter, Guo, Holtkamp, Ulmer [42])**.** *Let T be a strong tournament with $n \geq 5$ vertices. If T is not a quasi-transitive strong tournament, then*

$$\lambda'(T) \leq \xi'(T).$$

Proof. By Observation 5.3, T is restricted arc-connected, and by Theorem 5.1 we only need to consider the case $\delta^+(T) \leq 2$ and $\delta^-(T) \leq 2$. Let $u, v \in V(T)$ with $d^+(u) \leq 2$ and $d^-(v) \leq 2$. Considering the arc uv or vu directly implies

$1 \leq \xi'(T) \leq 4$. Now let xy be an arc of T with $\xi'(xy) = \xi'(T)$. If $T - \{x, y\}$ is not transitive, then it clearly contains a cycle and has a non-trivial strong component. Thus, we have a restricted arc-cut $S \in \Omega_{xy}$ with $\lambda'(T) \leq |S| = \xi'(xy) = \xi'(T)$. So we assume from now on that $T - \{x, y\}$ is a transitive tournament with the unique ordering $x_1 x_2 \ldots x_{n-2}$. We consider four cases.

Case 1. $\xi'(T) = 1$. Let $d^+(x) = 1$ and $z \in V(T)$ be a positive neighbour of y. Since $x \to y$, we have $z \to x$. Now T is a quasi-transitive strong tournament $x_1 x_2 \ldots x_{n-2} xy$ in case $d^+(y) = 1$, and $yx_1 \ldots x_{n-2}x$ if $d^-(y) = 1$.

Let now $d^-(x) = 1$ and $z \in V(T)$ be the unique negative neighbour of x. By $\xi'(T) = 1$ we have $d^-(y) = 1$ and therefore $y \to z$. Again, $xyx_1 \ldots x_{n-2}$ with $z = x_{n-2}$ is a quasi-transitive strong tournament.

Case 2. $\xi'(T) = 2$. We consider two subcases.

Case 2.1. $d^+(x) = 1$. This directly implies $d^+(y) \leq 2$ or $d^-(y) \leq 2$, and $x_i \to x$ for all $1 \leq i \leq n-2$. Because of $\xi'(x_{n-2}x) \geq 2$ we have $x_{n-2}y \in A(T)$, and according to $\xi'(x_1 x_2) \geq 2$ we have $yx_1, yx_2 \in A(T)$. It follows that either $\{x_3, x_4, \ldots, x_{n-3}\} \to y$ or $y \to \{x_3, x_4, \ldots, x_{n-3}\}$. In the former case we have the minimum restricted arc-cut $\{xy, x_{n-2}y\} \in \Omega_{x_{n-2}x}$ leaving the cycle $yx_1x_2 \ldots x_{n-3}y$ for $n \geq 6$. In the latter case it is $\{yx_1, yx_2\} \in \Omega_{x_1x_2}$ a minimum restricted arc-cut leaving the $(n-2)$-cycle $yx_3 \ldots x_{n-2}xy$ for $n \geq 6$. In case $n = 5$ we have the minimum restricted arc-cut $\{yx_1, xy\} \in \Omega_{x_1x}$ leaving the 3-cycle yx_2x_3y.

Case 2.2. $d^+(x) = 2$. It is either $d^+(y) = 1$ or $d^-(y) = 1$. The first case implies $d^+(x_{n-2}) = 2$ leading to $x_{n-2}x, x_{n-2}y \in A(T)$. Furthermore, according to $2 \leq d^-(x_1) + d^-(x_2) - 1$ and $d^+(y) = 1$ we have a positive neighbour x_j of x with $j \in \{1, 2\}$. Thus, for $n \geq 6$ we have the minimum restricted arc-cut $\omega^+(\{x_{n-2}, y\}) \in \Omega_{x_{n-2}y}$ of size 2, leaving the cycle $xx_jx_{j+1} \ldots x_{n-3}x$. If $n = 5$ we either have the 3-cycle xx_1x_2x and the minimum restricted arc-cut $\omega^+(\{x_3, y\}) \in \Omega_{x_3y}$ of size 2, or $d^-(x_1) = 1$ leading to the contradiction $\xi'(T) \leq d^+(y) + d^-(x_1) - 1 = 1 + 1 - 1 = 1$.

So assume $d^-(y) = 1$. We have $y \to \{x_1, x_2, \ldots, x_{n-2}\}$ and $x_{n-2}x \in A(T)$. According to $2 \leq \xi'(x_{n-3}x_{n-2})$, it is $x_{n-3}x \in A(T)$. Furthermore, because of $d^+(x) = 2$ the vertex x must have another negative neighbour $x_j \in \{x_1, x_2, \ldots x_{n-4}\}$ for $n \geq 6$. Therefore, we have the minimum restricted arc-cut $\omega^+(\{x_{n-3}, x_{n-2}\}) \in \Omega_{x_{n-3}x_{n-2}}$ of size 2, leaving the cycle $xyx_1 \ldots x_jx$. In case $n = 5$ we have $\xi'(yx_3) = 2$ with the 3-cycle xx_1x_2x, i. e. $\lambda'(T) = 2$.

Case 3. $\xi'(T) = 3$. Because of $\xi'(x_1x_2) \geq 3$ there are at least 3 arcs from $\{x, y\}$ to $\{x_1, x_2\}$, as well as $\xi'(x_{n-3}x_{n-2}) \geq 3$ implies at least 3 arcs from $\{x_{n-3}, x_{n-2}\}$ to $\{x, y\}$. Thus, $d^+(x) \geq 2$ and $d^-(y) \geq 2$. Also, $d^+(y) = 1$ would imply $x_{n-3}y, x_{n-2}y \in A(T)$, leading to the contradiction $d^+(x_{n-2}) + d^+(y) - 1 = 2$. Furthermore, $d^-(x) = 1$ leads to the contradiction $d^-(x)+d^-(x_1)-1 \leq 1+2-1 = 2$. Therefore, we have $d^-(x) \geq 2$ and $d^+(y) \geq 2$.

By the definition of the arc-degree $\xi'(xy)$ we are in one of the three cases $d^+(x) = d^+(y) = 2$, $d^+(x) = d^-(y) = 2$, or $d^-(x) = d^-(y) = 2$. Taking this into account, it is easy to verify that $\{x, y\} \to x_1$ and $x_{n-2} \to \{x, y\}$. Since x must have a negative neighbour x_j with $2 \leq j \leq n-3$, we have $\xi'(x_{n-2}y) = 3$, and therefore a minimum restricted arc-cut in $\Omega_{x_{n-2}y}$ leaving the cycle $xx_1 \dots x_jx$.

Case 4. $\xi'(T) = 4$. Because of $\delta^+(T) \leq 2$ and $\delta^-(T) \leq 2$ we may assume that $d^-(x) = 2$ and $d^+(y) = 2$. Since $T - \{x, y\}$ is transitive with the ordering $x_1x_2 \dots x_{n-2}$, we have $d^-(x_1) \leq 2$, and therefore the contradiction $d^-(x)+d^-(x_1)-1 \leq 3$.

□

5.2 λ'-optimality in tournaments

Since $\lambda'(D) \leq \xi'(D)$ in general holds for many digraphs D, there is a special interest in digraphs with $\lambda'(D) = \xi'(D)$, i. e. λ'-optimal digraphs. The following result of Wang, Lin and Li [112] gives a helpful tool for studying λ'-optimality.

Theorem 5.6 (Wang, Lin, Li [112], 2008)**.** *Let D be a restricted arc-connected digraph with $\lambda'(D) \leq \xi'(D)$. If D has no minimum restricted arc-cut of the form $\omega^+(X)$ for some $X \subseteq V(D)$, then D is λ'-optimal.*

Using this we are now able to prove the main result of this section.

Theorem 5.7 (Grüter, Guo, Holtkamp, Ulmer [42])**.** *Let T be a restricted arc-connected tournament with $n \geq 5$. If $\delta(T) \geq \frac{n+1}{4}$, then T is λ'-optimal.*

Proof. By Theorem 5.5 and $\delta(T) \geq 2$, we have $\lambda'(T) \leq \xi'(T)$. So assume to the contrary that $\lambda'(T) \leq \xi'(T) - 1$. According to Theorem 5.6 we see that T has a minimum restricted arc-cut of the form $\omega^+(X)$ for some $X \subseteq V(T)$. At first we prove

$$|X| > 2 \text{ and } |\overline{X}| > 2. \tag{5.1}$$

Let $X = \{x\}$ and $\omega^+(\{x\})$ be a minimum restricted arc-cut. It is clear that $T - \omega^+(\{x\})$ is not strong. Let $D_1, D_2, \dots, D_t$ with $t \geq 2$ be the unique ordering of the strong components of $T - \omega^+(\{x\})$, such that D_i dominates D_j for $i < j$. Because of $d^+_{T-\omega^+(\{x\})}(x) = 0$ it is $V(D_t) = \{x\}$. Since $\omega^+(\{x\})$ is a minimum restricted arc-cut, $T - \{x\}$ cannot be strong, thus, $t \geq 3$. If D_1 is trivial with

$V(D_1) = \{y\}$ we obtain $\delta(T) = d^-(y) = 1 < (n+1)/4$, a contradiction. Thus, D_1 is not trivial and with $X' = \{x\} \cup V(D_{t-1})$ the arc set $\omega^+(X')$ is a minimum restricted arc-cut with $|X'| \geq 2$ and $|\overline{X'}| \geq 2$. Analogously, for $|\overline{X}| = 1$ we obtain a minimum restricted arc-cut $\omega^+(X'')$ with $|X''| \geq 2$ and $|\overline{X''}| \geq 2$.

So assume $X = \{x, y\}$ or $\overline{X} = \{x, y\}$ with $xy \in A(T)$. Since $\omega^+(X)$ or $\omega^-(\overline{X})$ is a minimum restricted arc-cut, we deduce

$$\lambda'(T) = \omega^+(X) = d^+(x) + d^+(y) - 1 \geq \xi'(xy) \geq \xi'(T),$$

or

$$\lambda'(T) = \omega^-(\overline{X}) = d^-(x) + d^-(y) - 1 \geq \xi'(xy) \geq \xi'(T),$$

a contradiction to the assumption. Therefore, (5.1) holds.

Now we define $I = \{x \in V(T)\colon \min\{d^+(x), d^-(x)\} = \delta(T)\}$. We consider the following 3 cases.

Case 1. $|I| = 1$. For some integer $\theta \geq 1$ we have $d^+(z) \geq \delta + \theta$ for all $z \in X \setminus I$. It is $\xi'(T) \leq 2\delta + \theta$, and we deduce

$$\begin{aligned}
\lambda'(T) &= \omega^+(X) = \sum_{z \in X} d^+(z) - \binom{|X|}{2} \\
&\geq 1 \cdot \delta + (|X| - 1) \cdot (\delta + \theta) - \frac{1}{2}|X|(|X| - 1) + \xi'(T) - 2\delta - \theta \\
&= \xi'(T) - \frac{1}{2}\underbrace{\left(|X|^2 - (2\delta + 2\theta + 1)|X| + 4\delta + 4\theta\right)}_{= f(|X|)}.
\end{aligned}$$

In the following we will show that $f(|X|) < 2$, which implies $\lambda'(T) \geq \xi'(T) - f(|X|)/2 > \xi'(T) - 1$, a contradiction. Analogously, since $\lambda'(T) = \omega^-(\overline{X})$, the same conclusion holds for $f(|\overline{X}|) < 2$. So we may assume, without loss of generality, that $|X| \leq n/2$. To complete the proof we now take a closer look on the parabola $f(x)$ and show $f(x) < 2$ for $x \in [3, n/2]$. It is easy to check that

$$\begin{aligned}
f(x) - 2 &= x^2 - (2\delta + 2\theta + 1)x + 4\delta + 4\theta - 2 \\
&= \left(x - \delta - \theta - \frac{1}{2}\right)^2 - \left(\delta + \theta - \frac{3}{2}\right)^2 \\
&= (x - 2\delta - 2\theta + 1)\,(x - 2)\,.
\end{aligned}$$

For $x \geq 3$ it is $(x-2) > 0$. Furthermore, since $\delta \geq (n+1)/4$ we have $x - 2\delta - 2\theta + 1 \leq n/2 - 2((n+1)/4) - 2 + 1 < 0$. Thus, $f(x) - 2 < 0$ and Case 1 is complete.

Case 2. $|I| = 2$. It follows that $\xi'(T) \leq 2\delta$. Furthermore,

$$\begin{aligned}
\lambda'(T) &= \omega^+(X) = \sum_{z\in X} d^+(z) - \binom{|X|}{2} \\
&\geq 2\cdot\delta + (|X|-2)\cdot(\delta+1) - \frac{1}{2}|X|(|X|-1) + \xi'(T) - 2\delta \\
&= \xi'(T) - \frac{1}{2}\underbrace{\left(|X|^2 - (2\delta+3)|X| + 4\delta + 4\right)}_{=g(|X|)}.
\end{aligned}$$

Similar to the deduction above we have $g(x) - 2 = (x - 2\delta - 1)(x-2) < 0$ for $x \in [3, n/2]$, since $x - 2\delta - 1 \leq n/2 - 2((n+1)/4) - 1 = -3/2 < 0$. Again, this leads to the contradiction $\lambda'(T) \geq \xi'(T) - g(|X|)/2 > \xi'(T) - 1$.

Case 3. $|I| = t \geq 3$. It is easy to see that $\xi'(T) = 2\delta - 1$. In the following we may also assume $t \leq |X|$, since we only count the positive degrees of vertices in X. Therefore, we have

$$\begin{aligned}
\lambda'(T) &= \omega^+(X) = \sum_{z\in X} d^+(z) - \binom{|X|}{2} \\
&\geq t\cdot\delta + (|X|-t)\cdot(\delta+1) - \frac{1}{2}|X|(|X|-1) + \xi'(T) - (2\delta-1) \\
&= \xi'(T) - \frac{1}{2}\underbrace{\left(|X|^2 - (2\delta+3)|X| + 2t + 4\delta - 2\right)}_{=h(|X|)}.
\end{aligned}$$

Again, we show $h(x) < 2$ for $x \in [3, n/2]$. We have

$$\begin{aligned}
h(x) - 2 &= x^2 - (2\delta+3)x + 2t + 4\delta - 4 \\
&= (x - 2\delta - 1)(x-2) + 2t - 6 \\
&\leq (x - 2\delta - 1)(t-2) + 2(t-3),
\end{aligned}$$

since $(x - 2\delta - 1) \leq 0$. For $x \leq (n-1)/2$ it is $(x - 2\delta - 1) \leq -2$, and therefore $h(x) - 2 \leq -2(t-2) + 2(t-3) = -2 < 0$. In the remaining case we have $x = n/2$, and thus $|X| = |\overline{X}| = n/2$. Now n is an even number, and therefore we must have $\delta \geq (n+2)/4$. This again leads to $(x - 2\delta - 1) \leq -2$, and therefore $h(x) - 2 < 0$. From the deduction before we once more conclude $\lambda'(T) \geq \xi'(T) - h(|X|)/2 > \xi'(T) - 1$, a contradiction, and the proof is complete. □

To see the sharpness of Theorem 5.7 we give the following example of tournaments T_δ with minimum degree $\delta = n/4$, which are not λ'-optimal.

Example 5.8 (Grüter, Guo, Holtkamp, Ulmer [42]). *Let TT_A and TT_B be two transitive tournaments with the vertex sets $V(TT_A) = \{a_1, a_2, \dots, a_t\}$ and $V(TT_B) = \{b_1, b_2, \dots, b_t\}$ $(t \geq 2)$, and with arc sets $A(TT_A) = \{a_i \to a_j\colon\ i < j\}$ and $A(TT_B) = \{b_i \to b_j\colon\ i < j\}$, respectively. We define the tournament T with vertex set $V(T) = V(TT_A) \cup V(TT_B)$ and arc set $A(T) = A(TT_A) \cup A(TT_B) \cup \{a_i \to b_j\colon\ i \geq j\} \cup \{b_j \to a_i\colon\ i < j\}$. Now let T' be a copy of T with vertex sets $V(T') = \{a'_1, a'_2, \dots, a'_t\} \cup \{b'_1, b'_2, \dots, b'_t\}$ and arcs accordingly. Finally, we define the tournament T_t to be the conjunction of T and T' plus the arcs $\{b_i \to a'_i\colon\ i = 1, 2, \dots, t\}$ and all other arcs directed from $V(T')$ to $V(T)$.*

By this definition T_t has $n = 4t$ vertices and minimum degree $\delta(T_t) = d^+(a_1) = d^+(b_1) = t = \frac{1}{4}n$. We have $\xi'(T_t) = \xi'(a_1 b_1) = 2t-1$. For $X = \{a_1, \dots, a_t, b_1, \dots, b_t\}$ the arc set $\omega^+(X)$ is a minimum restricted arc-cut of size t. Since $\lambda'(T_t) \leq t < 2t - 1 = \xi'(T_t)$, T_t is not λ'-optimal for all $t \geq 2$.

To complete our considerations concerning tournaments we present some direct consequences of Theorem 5.7. We note that in a tournament the (global) irregularity equals $i_g(T) = n - 2\delta(T) - 1$. In case $i_g(T) = 0$ the tournament T is regular, and in case $i_g(T) = 1$ it is almost regular. We have the following corollaries.

Corollary 5.9 (Grüter, Guo, Holtkamp, Ulmer [42]). *Let T be a restricted arc-connected tournament. If $i_g(T) \leq \frac{n-3}{2}$, then T is λ'-optimal.*

Proof. We have $i_g(T) = (n - 1 - \delta) - \delta = n - 2\delta - 1 \leq \frac{n-3}{2}$ leading to $\delta \geq \frac{n+1}{4}$, and the result follows directly from Theorem 5.7. □

Corollary 5.10 (Grüter, Guo, Holtkamp, Ulmer [42]). *Regular tournaments with $n \geq 5$ are λ'-optimal.*

Corollary 5.11 (Grüter, Guo, Holtkamp, Ulmer [42]). *Almost regular tournaments with $n \geq 5$ are λ'-optimal.*

5.3 Restricted arc-connectivity in bipartite tournaments

In this section we study the restricted arc-connectivity of bipartite tournaments. The following basic observation gives a helpful tool for later considerations.

Observation 5.12 (Grüter, Guo, Holtkamp [41]). *Let D be a strong digraph and $uv \in A(D)$. If $D - \{u, v\}$ contains a (non-trivial) cycle, then D is restricted arc-connected and $\lambda'(D) \leq \xi'(uv)$.*

Proof. Since $D-\{u,v\}$ contains a cycle, every element of Ω_{uv} is a restricted arc-cut of D, and thus $\lambda'(D) \leq \xi'(uv)$. □

We notice that cycles, strong components, restricted arc-cuts and the arc-degree are invariant due to conversion, i. e. $\lambda'(D) = \lambda'(D^{-1})$ and $\xi'_D(xy) = \xi'_{D^{-1}}(yx)$ for $xy \in A(D)$. First of all, we discuss the existence of restricted arc-cuts in bipartite tournaments and make the following observation.

Lemma 5.13 (Grüter, Guo, Holtkamp [41])**.** *Let $T = (U \dot\cup V, A)$ be a strong bipartite tournament. Then T is restricted arc-connected if and only if $|U|, |V| \geq 3$.*

Proof. Let S be a restricted arc-cut of T. A non-trivial strong component D_1 of $T - S$ must contain a cycle of length at least 4, with at least 2 vertices from each bipartite set. Since $T - V(D_1)$ contains an arc, each bipartite set must at least contain 2 vertices. Thus, $|U|, |V| \geq 3$ is a necessary condition for a bipartite tournament to be restricted arc-connected.

To see that this condition is sufficient, we note that every strong bipartite tournament contains a 4-cycle. □

We have seen that all strong bipartite tournaments $T = (U\dot\cup V, A)$ with $|U|, |V| \geq 3$ are restricted arc-connected. We will now turn our attention to an upper bound on the size of restricted arc-cuts in T, and prove that $\lambda'(T) \leq \xi'(T)$ holds for all strong bipartite tournaments except the members of the following family of almost acyclic bipartite tournaments, where $\lambda'(T) = 2 > 1 = \xi'(T)$. It is easy to see that an acyclic bipartite tournament $T = (U \dot\cup V, A)$ has a unique partitioning of the bipartite sets U and V, i. e. $U = U_1 \dot\cup U_2 \dot\cup \ldots \dot\cup U_r$ and $V = V_1 \dot\cup V_2 \dot\cup \ldots \dot\cup V_r$, such that (without loss of generality) $U_1 \to V_1 \to \cdots \to U_r \to V_r$ for $r \geq 1$, and $U_i \to V_r$, $U_i \to V_{j-1}$ and $V_i \to U_j$ for all $1 \leq i < j \leq r$. Note that $V_r = \emptyset$ is possible in case $r \geq 2$. We call this unique partitioning the *acyclic ordering* of T, like depicted in Figure 5.2.

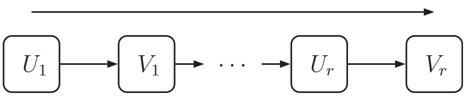

Figure 5.2: Acyclic ordering of acyclic bipartite tournaments with bipartite sets $U = U_1 \dot\cup U_2 \dot\cup \ldots \dot\cup U_r$ and $V = V_1 \dot\cup V_2 \dot\cup \ldots \dot\cup V_r$ for $r \geq 1$. $V_r = \emptyset$ is possible in case $r \geq 2$. All arcs are directed from left to right.

Example 5.14 (Grüter, Guo, Holtkamp [41]). *Let $T' = (U \,\dot\cup\, V, A')$ be an acyclic bipartite tournament with $|U|, |V| \geq 3$, and the acyclic ordering $U_1 \to V_1 \to U_2 \to V_2 \to \cdots \to U_r \to V_r$ with $r \geq 2$, where $U = U_1 \,\dot\cup\, U_2 \,\dot\cup \ldots \dot\cup\, U_r$ and $V = V_1 \,\dot\cup\, V_2 \,\dot\cup \ldots \dot\cup\, V_r$ are disjoint unions and $U_i \to V_r$, $U_i \to V_{j-1}$ and $V_i \to U_j$ for all $1 \leq i < j \leq r$. Furthermore, let $U_1 = \{u\}$, $V_r = \{v\}$ and $|V_1|, |U_r| \geq 2$. From T' we obtain the strong bipartite tournament $T = (U \,\dot\cup\, V, A)$ by deleting the arc uv and adding the arc vu instead. We call T an* almost acyclic bipartite tournament *and refer to $\mathcal{T}$ as the family of all digraphs isomorphic to an almost acyclic bipartite tournament. We notice that the acyclic ordering of T' can be transferred to T, and we call this ordering the* almost acyclic ordering *of T (see Figure 5.3).*

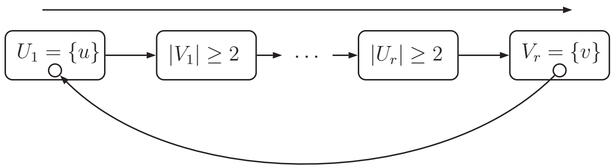

Figure 5.3: Almost acyclic ordering of a member of family $\mathcal{T}$ with bipartite sets $U = U_1 \,\dot\cup\, U_2 \,\dot\cup \ldots \dot\cup\, U_r$ and $V = V_1 \,\dot\cup\, V_2 \,\dot\cup \ldots \dot\cup\, V_r$ for $r \geq 2$. All arcs except vu are directed from left to right.

By this definition, for all $T \in \mathcal{T}$ we have $\xi'(T) = \xi'(vu) = 1$. Furthermore, since vu is on every cycle of T, it cannot be an element of any restricted arc-cut. On the other hand, removing an arbitrary arc from $A(T) \setminus \{vu\}$ still leaves a strong tournament. Thus, we have $\lambda'(T) \geq 2$ and for $x \in V_1$ and $y \in U_r$ we have $\xi'(xy) = 2$ with a restricted arc-cut $S = \omega^-(x) \cup \omega^+(y) = \{ux, yv\}$ of size 2, i. e. $\lambda'(T) = 2$.

With the help of Observation 5.12 we now prove that the members of $\mathcal{T}$ are the only strong bipartite tournaments with $\lambda'(T) > \xi'(T)$. For a vertex $x \in V(G)$ we define the abbreviation $\delta(x) = \min\{d^+(x), d^-(x)\}$.

Theorem 5.15 (Grüter, Guo, Holtkamp [41]). *Let $T = (U \,\dot\cup\, V, A)$ be a strong biparite tournament such that $|U|, |V| \geq 3$. If T is not a member of the family $\mathcal{T}$ from Example 5.14, then T is restricted arc-connected and $\lambda'(T) \leq \xi'(T)$.*

Proof. By Lemma 5.13 T is restricted arc-connected. Let $xy \in A(T)$ be an arc with minimum arc-degree $\xi'(xy) = \xi'(T)$. We only need to consider the case that

$T - \{x, y\}$ is acyclic, and, without loss of generality, with the acyclic ordering $U_1 \to V_1 \to \cdots \to U_r \to V_r$ and $r \geq 1$. In the following we assume that $x \in U$ and $y \in V$, and discuss the case $y \in U$ and $x \in V$ afterwards.

Since T is strong, we have $y \to U_1$. If $r = 1$, then it must be $x \to y \to U_1 \to V_1 \to x$ with $|U_1|, |V_1| \geq 2$. This implies $T \in \mathcal{T}$. Hence, let now $r \geq 2$ and we consider two cases.

Case 1. $V_r = \emptyset$. Since T is strong, we have $U_r \to y$. Considering an arbitrary vertex $u_1 \in U_1$ we have $d^-(u_1) = 1$ and therefore

$$\delta(y) = \delta(y) + d^-(u_1) - 1 = \xi'(yu_1) \geq \xi'(xy) \geq \delta(x) + \delta(y) - 1,$$

leading to $\delta(x) \leq 1$ and hence $\delta(x) = 1$.

By considering an arbitrary vertex $v_1 \in V_1$ we either have $v_1x \in A(T)$ or $xv_1 \in A(T)$. In the former case we deduce

$$\xi'(T) \leq \xi'(v_1x) \leq d^-(v_1) + \delta(x) = |U_1| + 1.$$

In the latter case x dominates the vertices v_1 and y, leading to

$$\xi'(T) \leq \xi'(xv_1) \leq d^-(x) + d^-(v_1) - 1 = |U_1| + 1.$$

Analogously, considering an arbitrary vertex $v_{r-1} \in V_{r-1}$ we either have

$$\xi'(v_{r-1}x) \leq d^+(v_{r-1}) + \delta(x) - 1 = |U_r| + 1$$

or

$$\xi'(xv_{r-1}) \leq d^-(x) + d^+(v_{r-1}) = |U_r| + 1,$$

i.e. $\xi'(T) \leq |U_r| + 1$.

Together with $\xi'(T) = \xi'(xy) \geq \delta(x) + \delta(y) - 1 = \delta(y)$ this leads to $\delta(y) \leq |U_1| + 1$ and $\delta(y) \leq |U_r| + 1$, and considering $(U_r \cup \{x\}) \to y \to U_1$ we arrive at

$$\min\{|U_1|, |U_r| + 1\} \leq \delta(y) \leq \min\{|U_1| + 1, |U_r| + 1\}. \tag{5.2}$$

We consider two subcases.

Case 1.1. $|U_1| \leq |U_r|$. From (5.2) we have $|U_1| \leq \delta(y) \leq |U_1| + 1$ and therefore

$$|U_1| + 1 \geq \xi'(xy) \geq \delta(x) + \delta(y) - 1 \geq |U_1|.$$

Since $T - \{x, v_1\}$ contains a cycle, by Observation 5.12 we have $\lambda'(T) \leq |U_1| + 1$. In case $\xi'(T) = |U_1| + 1$ this implies $\lambda'(T) \leq \xi'(T)$. For $\xi'(T) = |U_1|$ we have $d^+(y) = |U_1|$ and $d^+(x) = 1$.

If $|V_1| = 1$ with $V_1 = \{v_1\}$, then $r \geq 3$ and for $u_2 \in U_2$ we have $\xi'(v_1u_2) = d^-(v_1) + d^-(u_2) - 1 = |U_1|$, where $T - \{v_1, u_2\}$ contains a cycle, thus, we have $\lambda'(T) \leq \xi'(v_1u_2) = |U_1|$ by Observation 5.12.

So let now $|V_1| \geq 2$. If $|U_1| = 1$ with $U_1 = \{u_1\}$, then T is an almost acyclic bipartite tournament with the almost acyclic ordering $u_1 \to V_1 \to \cdots \to V_{r-1} \to (U_r \cup \{x\}) \to y$, i. e. $T \in \mathcal{T}$. For $|U_1| \geq 2$ and arbitrary vertices $u_1 \in U_1, v_1 \in V_1$ we have $\xi'(u_1v_1) = d^-(u_1) + d^-(v_1) - 1 = |U_1|$, where $T - \{u_1, v_1\}$ contains a cycle, thus, $\lambda'(T) \leq \xi'(u_1v_1) = |U_1|$ by Observation 5.12.

Case 1.2. $|U_1| \geq |U_r| + 1$. From (5.2) we have $\delta(y) = |U_r| + 1$ and either $\xi'(v_{r-1}x) \leq |U_r| + 1$ or $\xi'(xv_{r-1}) \leq |U_r| + 1$. Since $|U_r| + 1 = \delta(x) + \delta(y) - 1 \leq \xi'(xy) = \xi'(T)$ and $T - \{x, v_{r-1}\}$ contains a cycle, we have $\lambda'(T) \leq |U_r| + 1 \leq \xi'(T)$ by Observation 5.12.

Case 2. $V_r \neq \emptyset$. Since T is strong, we have $V_r \to x$. Considering an arbitrary vertex $v_r \in V_r$ we have

$$\delta(x) = d^+(v_r) + \delta(x) - 1 = \xi'(v_rx) \geq \xi'(xy) \geq \delta(x) + \delta(y) - 1,$$

and therefore $\delta(y) \leq 1$.

In case $d^+(y) = 1$ we have $U_1 = \{u_1\}$. Let v_1 be an arbitrary vertex $v_1 \in V_1$. If $xv_1 \in A(T)$, then $T - \{y, u_1\}$ contains a cycle and $\xi'(yu_1) = d^+(y) + d^-(u_1) - 1 = 1$, thus, $\lambda'(T) = 1$ by Observation 5.12. If $xv_1 \notin A(T)$, then we have $d^-(v_1) = 1$ and $\xi'(u_1v_1) = 1 = \xi'(xy)$, which implies $\delta(x) = 1$. Since at least v_1 and the vertices of V_r dominate x, we have $d^+(x) = 1$. For an arbitrary vertex $v_r \in V_r$ we now have $\xi'(v_rx) = d^+(v_r) + d^+(x) - 1 = 1$ and $T - \{v_r, x\}$ contains a cycle, thus, $\lambda'(T) = 1$ by Observation 5.12.

So let now $d^-(y) = 1$. For a vertex $u_1 \in U_1$ we have $\xi'(yu_1) = d^-(y) + d^-(u_1) - 1 = 1$, which implies $\delta(x) = 1$. In case $d^-(x) = 1$ we have $x \to V_1$ and there is a cycle in $T - \{y, u_1\}$, thus, $\lambda'(T) = 1$ by Observation 5.12. Let now $d^+(x) = 1$. For $|U_1|, |V_r| \geq 2$ we have an almost acyclic bipartite tournament with the almost acyclic ordering $y \to U_1 \to V_1 \to \dots U_r \to V_r \to x$, thus, $T \in \mathcal{T}$. If $U_1 = \{u_1\}$ we have $\xi'(u_1v_1) = d^-(u_1) + d^-(v_1) - 1 = 1$ for an arbitrary vertex $v_1 \in V_1$, where $T - \{u_1, v_1\}$ contains a cycle. Analogously, if $V_r = \{v_r\}$ we have $\xi'(u_rv_r) = d^+(u_r) + d^+(v_r) - 1 = 1$ for an arbitrary vertex $u_r \in U_r$ and $T - \{u_r, v_r\}$ contains a cycle. In both cases applying Observation 5.12 finishes the proof.

Now we consider the case that $x \in V$ and $y \in U$. If $V_r = \emptyset$, then it is not hard to see that for every vertex $u_r \in U_r$ the arc u_rx is also of minimum degree $\xi'(u_rx) = \xi'(T)$. Either there is an arc u_rx such that $T - \{x, u_r\}$ contains a cycle and we are done by Observation 5.12, or $T - \{x, u_r\}$ is acyclic and we are in the situation of Case 1 with $y' = x \in V$ and $x' = u_r \in U$.

In case $V_r \neq \emptyset$, we consider the converse digraph T^{-1} with $yx \in A(T^{-1})$. We notice that cycles, strong components, restricted arc-cuts and the arc-degree are invariant due to conversion. Thus, we are in the situation of Case 2 from above with the roles of U and V as well as x and y interchanged, which completes the proof. □

5.4 λ'-optimality in bipartite tournaments

For studying the λ'-optimality of bipartite tournaments we make use of the following result of Balbuena, García-Vázquez, Hansberg and Montejano [6].

Lemma 5.16 (Balbuena, García-Vázquez, Hansberg, Montejano [6]). *Let D be a restricted arc-connected digraph such that $\lambda'(D) < \xi'(D)$ and let S be a minimum restricted arc-cut. Then the vertex set $V(D)$ can be partitioned into two subsets, X, $\overline{X}$ such that $S = \omega^+(X) = \omega^-(\overline{X})$ and both induced subdigraphs $D[X]$ and $D[\overline{X}]$ of D contain an arc.*

Combining Example 5.14 and Theorem 5.15 we have $\lambda'(T) \leq \xi'(T)+1$ for all strong bipartite tournaments. In fact, Example 5.14 gives us a characterization of all strong bipartite tournaments with $\lambda'(T) = \xi'(T) + 1$, namely the family $\mathcal{T}$. Using Lemma 5.16 we now give a sufficient condition on the minimum degree for bipartite tournaments T to be optimally restricted arc-connected, i. e. $\lambda'(T) = \xi'(T)$.

Theorem 5.17 (Grüter, Guo, Holtkamp [41]). *If T is a restricted arc-connected bipartite tournament with $\delta(T) \geq (n+3)/8$, then T is λ'-optimal.*

Proof. In a restricted arc-connected bipartite tournament we have $n \geq 6$ and therefore $\delta(T) \geq 2$ by assumption. Thus, T cannot be a member of the family $\mathcal{T}$, and we have $\lambda'(T) \leq \xi'(T)$ by Theorem 5.15. Assuming T is not λ'-optimal, then, by Lemma 5.16, T has a minimum restricted arc-cut of the form $\omega^+(X)$ for some $X \subseteq V(T)$ with $|X|, |\overline{X}| \geq 2$.

We notice that $\omega^+_{T^{-1}}(\overline{X})$ is a minimum restricted arc-cut in T^{-1} containing the same arcs (but conversed) as $\omega^+_T(X)$. Furthermore, it is $\lambda'(T) = \lambda'(T^{-1})$ and $\xi'(T) = \xi'(T^{-1})$. Either $|X| \leq n/2$, or $|\overline{X}| \leq n/2$ and we consider the converse digraph T^{-1} instead of T. Hence, we may assume $|X| \leq n/2$.

Let U and V be the partite sets of T with $3 \leq |U| \leq |V|$. Furthermore, let $X_U = X \cap U$ and $X_V = X \cap V$, thus, $|X| = |X_U| + |X_V|$ and by a well-known result of Turán [100] we have $|X_U| \cdot |X_V| = |A(T[X])| \leq \frac{1}{4}|X|^2$. We define $\delta_U =$

$\min\{d^+(u), d^-(u) : u \in U\}$, and δ_V respectively. We have

$$\begin{aligned}\omega^+(X) &= |[X_U, \overline{X}]| + |[X_V, \overline{X}]| \geq \sum_{u \in X_U} d^+(u) + \sum_{v \in X_V} d^+(v) - |X_U| \cdot |X_V| \\ &\geq \sum_{u \in X_U} \delta_U + \sum_{v \in X_V} \delta_V - |X_U| \cdot |X_V| \geq |X_U| \cdot \delta_U + |X_V| \cdot \delta_V - |X_U| \cdot |X_V|.\end{aligned}$$

Of course, we either have $\xi'(T) = \delta_U + \delta_V - 1$ or $\xi'(T) = \delta_U + \delta_V$ according to the definition of the minimum arc-degree. In the former case, we obtain the following lower bound on $\lambda'(T)$:

$$\begin{aligned}\lambda'(T) &\geq \omega^+(X) \geq \omega^+(X) + \xi'(T) - (\delta_U + \delta_V - 1) \\ &> \xi'(T) + |X_U| \cdot \delta_U + |X_V| \cdot \delta_V - |X_U| \cdot |X_V| - (\delta_U + \delta_V)\end{aligned}$$

In the latter case we either have $\xi'(T) = \delta_U^- + \delta_V^+$ or $\xi'(T) = \delta_U^+ + \delta_V^-$, leading to $\delta_U^+ > \delta_U$ or $\delta_V^+ > \delta_V$. Again, this implies

$$\begin{aligned}\lambda'(T) &\geq \omega^+(X) \geq \omega^+(X) + \xi'(T) - (\delta_U + \delta_V) \\ &> \xi'(T) + |X_U| \cdot \delta_U + |X_V| \cdot \delta_V - |X_U| \cdot |X_V| - (\delta_U + \delta_V).\end{aligned}$$

We consider the following two cases.

Case 1. $\delta(T) \geq (n+4)/8$. From the two strict inequalities above we deduce

$$\begin{aligned}\lambda'(T) &> \xi'(T) - |X_U| \cdot |X_V| + (|X_U| - 1) \cdot \delta_U + (|X_V| - 1) \cdot \delta_V \\ &\geq \xi'(T) - \frac{1}{4}|X|^2 + (|X| - 2) \cdot \frac{n+4}{8} \\ &= \xi'(T) - \frac{1}{4} \underbrace{\left(|X|^2 - |X| \cdot \frac{n+4}{2} + n + 4\right)}_{=f(|X|)}.\end{aligned}$$

Since

$$f(|X|) - 4 = |X|^2 - \frac{n+4}{2}|X| + n + 4 - 4 = \underbrace{\left(|X| - \frac{n}{2}\right)}_{\leq 0} \underbrace{(|X| - 2)}_{\geq 0} \leq 0,$$

we have $f(|X|) \leq 4$, and it is easy to see that $\lambda'(T) \geq \xi'(T)$. Together with $\lambda'(T) \leq \xi'(T)$ we obtain the λ'-optimality.

Case 2. $\delta(T) = (n+3)/8$. Similar to Case 1 we obtain

$$\begin{aligned}\lambda'(T) &> \xi'(T) - |X_U| \cdot |X_V| + (|X_U| - 1) \cdot \delta_U + (|X_V| - 1) \cdot \delta_V \\ &\geq \xi'(T) - \frac{1}{4}|X|^2 + (|X| - 2) \cdot \frac{n+3}{8} \\ &= \xi'(T) - \frac{1}{4} \underbrace{\left(|X|^2 - |X| \cdot \frac{n+3}{2} + n + 3 \right)}_{=g(|X|)} .\end{aligned}$$

We conclude

$$g(|X|) - 4 = |X|^2 - \frac{n+3}{2}|X| + n + 3 - 4 = \left(|X| - \frac{n-1}{2} \right) \underbrace{(|X| - 2)}_{\geq 0} .$$

In case $|X| \leq (n-1)/2$ this implies $g(|X|) \leq 4$, and it follows $\lambda'(T) \geq \xi'(T)$. Together with $\lambda'(T) \leq \xi'(T)$ we have λ'-optimality.

In the remaining case we have $|X| = n/2 = |\overline{X}|$. Thus, n must be an even number. On the other hand, since $\delta(T) = (n+3)/8$ is an integer, $n+3$ must also be an even number, a contradiction, and Case 2 is proved. □

To see the sharpness of Theorem 5.17 we give the following examples of bipartite tournaments with minimum degree $\delta = (n+2)/8$, which are not λ'-optimal.

Example 5.18 (Grüter, Guo, Holtkamp [41])**.** *For an integer $t \geq 2$ we define the strong bipartite tournament T_t with partite sets $U = A_1 \dot\cup B_2 \dot\cup A_3 \dot\cup B_4$ and $V = B_1 \dot\cup A_2 \dot\cup B_3 \dot\cup A_4$ with $|A_1| = |B_1| = t-1$ and $|A_i| = |B_i| = t$ for $2 \leq i \leq 4$, and the arc set such that $A_1 \to A_2 \to A_3 \to A_4 \to A_1$, as well as $B_1 \to B_2 \to B_3 \to B_4 \to B_1$, and $(A_1 \cup A_3) \to (B_1 \cup B_3)$. Also, the vertex set $A_2 \cup A_4$ dominates the vertex set $B_2 \cup B_4$, except for the arcs $\{b_i a_i \colon 1 \leq i \leq t\}$, where $A_2 = \{a_1, a_2, \ldots, a_t\}$ and $B_4 = \{b_1, b_2, \ldots, b_t\}$.*

By this definition we have $n = 8t - 2$, $\delta(T_t) = t = (n+2)/8$ and $\xi'(T_t) = 2t - 1$. Furthermore, T_t has the restricted arc-cut $S = \{b_i a_i \colon 1 \leq i \leq t\}$, which implies $\lambda'(T_t) = t$. Therefore, T_t is not λ'-optimal.

Chapter 6

Decycling bipartite tournaments by deleting arcs

Somehow related to the studies of connectivity is the analysis of cycles in a digraph. As for example many algorithms run very fast on digraphs with few or none cycles, the question on the structure and number of cycles in a digraph becomes very interesting. In this chapter we discuss how far away bipartite tournaments are from being acyclic. Therefore, in Section 6.1 we define the decycling index $\nabla'(D)$ of a digraph D and the maximum decycling index $\overline{\overline{\nabla}}'(m,n)$ of m-by-n bipartite tournaments, which have been introduced by Beineke and Vandell [12, 102]. In Section 6.2 we then discuss 5-by-6-, 6-by-6-, 5-by-7-, 6-by-7- and 5-by-8 bipartite tournaments in particular. The results of this chapter have been obtained in collaboration with Lutz Volkmann.

6.1 Maximum decycling index of bipartite tournaments

The *decycling index* of a digraph D, denoted by $\nabla'(D)$, is the minimum number of arcs whose removal yields an acyclic digraph. By $\overline{\overline{\nabla}}'(G)$ we denote the *maximum decycling index* among all orientations of a graph G. Therefore, $\overline{\overline{\nabla}}'(m,n) = \overline{\overline{\nabla}}'(K_{m,n})$ is the maximum decycling index of all m-by-n bipartite tournaments.

The decycling index was studied for tournaments by Reid [92] and for bipartite tournaments by Vandell [102]. Vandell determined the numbers $\overline{\overline{\nabla}}'(2,n), \overline{\overline{\nabla}}'(3,n)$, and $\overline{\overline{\nabla}}'(4,n)$ for all positive integers n, as well as $\overline{\overline{\nabla}}'(5,5)$. In this chapter we present $\overline{\nabla}'(5,6)$, $\overline{\nabla}'(6,6)$, and $\overline{\nabla}'(5,7)$, as well as results on $\overline{\nabla}'(6,7)$ and $\overline{\nabla}'(5,8)$. Firstly, we find some lower and upper bounds for the mentioned cases, which lead to only two possible values for the maximum arc decycling number in each case. Secondly, we prove further degree conditions for the tournaments with high decycling index. Then, in [64] we used a computer program to decide between the two possible values, where the degree conditions helped to reduce the runtime of the algorithm. In this chapter we restrict ourselves on presenting the results obtained in [64]. For more details on the computer program we refer the reader to this article. The following results of Vandell [102] are useful for our investigations.

Lemma 6.1 (Vandell [102], 2010). *For all positive integers m, s, and t,*

$$\overline{\nabla}'(m, s+t) \geq \overline{\nabla}'(m, s) + \overline{\nabla}'(m, t).$$

Theorem 6.2 (Vandell [102], 2010). $\overline{\nabla}'(2, n) = \lfloor \frac{n}{2} \rfloor$.

Theorem 6.3 (Vandell [102], 2010). $\overline{\nabla}'(3, n) = \lfloor \frac{2n}{3} \rfloor$.

Lemma 6.4 (Vandell [102], 2010). $7 \leq \overline{\nabla}'(4,6) \leq \overline{\nabla}'(4,7)$.

Theorem 6.5 (Vandell [102], 2010). *For $n \geq 2$,*

$$\overline{\nabla}'(4, n) = \begin{cases} \lfloor \frac{7n}{6} \rfloor - 1 & if\ n \equiv 1,\ 3 \pmod 6, \\ \lfloor \frac{7n}{6} \rfloor & if\ n \equiv 0,\ 2,\ 4,\ 5 \pmod 6. \end{cases}$$

Theorem 6.6 (Vandell [102], 2010). $\overline{\nabla}'(5,5) = 6$.

Either deleting all arcs going into or out of an arbitrary vertex v, leaves a digraph without any cycle through v. Therefore, the following lemma is immediate.

Lemma 6.7 (Holtkamp, Volkmann [64], 2012). *Let T be an m-by-n bipartite tournament with partite sets X, Y with $|X| = m \geq 2$, $|Y| = n \geq 1$, and $v \in X$. Then*

$$\nabla'(T) \leq \min\{d^+(v), d^-(v)\} + \overline{\nabla}'(m-1, n).$$

This directly leads to the following result.

Corollary 6.8 (Holtkamp, Volkmann [64], 2012). *For all positive integers i, m and n with $m - i \geq 1$ we have*

$$\overline{\nabla}'(m, n) \leq \overline{\nabla}'(m-i, n) + i \cdot \left\lfloor \frac{n}{2} \right\rfloor.$$

Furthermore, by combining Theorem 6.3 and Corollary 6.8 we obtain a trivial upper bound for the general case.

Corollary 6.9 (Holtkamp, Volkmann [64], 2012). *For all positive integers m and n with $m \geq 3$ and $n \geq 2$ we have*

$$\overline{\nabla}'(m,n) \leq \left\lfloor \frac{2n}{3} \right\rfloor + (m-3)\left\lfloor \frac{n}{2} \right\rfloor.$$

To obtain a trivial lower bound for the general case, we can decompose one partite set of an m-by-n bipartite tournament into disjoint subsets of order 2 and one subset of order 3. Thus, using Lemma 6.1, Theorem 6.2 and Theorem 6.3 we obtain:

Corollary 6.10 (Holtkamp, Volkmann [64], 2012). *For all positive integers m and n with $3 \leq m \leq n$ we have*

$$\overline{\nabla}'(m,n) \geq \left\lfloor \frac{2n}{3} \right\rfloor + \left(\left\lfloor \frac{m-3}{2} \right\rfloor\right) \cdot \left\lfloor \frac{n}{2} \right\rfloor.$$

6.2 Determining $\overline{\nabla}'(5,6)$, $\overline{\nabla}'(6,6)$, $\overline{\nabla}'(5,7)$, $\overline{\nabla}'(6,7)$ and $\overline{\nabla}'(5,8)$

According to Lemma 6.4, we have $\overline{\nabla}'(4,6) \geq 7$, and so we deduce from Lemma 6.1 our first proposition.

Proposition 6.11 (Holtkamp, Volkmann [64], 2012). $\overline{\nabla}'(5,6) \geq 7$.

Proposition 6.12 (Holtkamp, Volkmann [64], 2012). $\overline{\nabla}'(5,6) \leq 8$.

Proof. Let $X = \{x_1, x_2, \ldots, x_5\}$ and $Y = \{y_1, y_2, \ldots, y_6\}$ be the partite sets of a 5-by-6 bipartite tournament T. Then $\min\{d^-(y_1), d^+(y_1)\} \leq 2$. By Lemma 6.7 and Theorem 6.6 we obtain

$$\overline{\nabla}'(5,6) \leq \overline{\nabla}'(5,5) + 2 = 8.$$

□

The proof of Proposition 6.12 shows the next corollary immediately.

Corollary 6.13 (Holtkamp, Volkmann [64], 2012). *Let $X = \{x_1, x_2, \ldots, x_5\}$ and $Y = \{y_1, y_2, \ldots, y_6\}$ be the partite sets of a bipartite tournament T. If $d^-(y_i) \leq 1$ or $d^+(y_i) \leq 1$ for any $i \in \{1, 2, \ldots, 6\}$, then $\nabla'(T) \leq 7$.*

Using Corollary 6.13, Proposition 6.11 and a computer program described in [64], we obtained the next result.

Theorem 6.14 (Holtkamp, Volkmann [64], 2012)**.** $\overline{\nabla}'(5,6) = 7$.

Proposition 6.15 (Holtkamp, Volkmann [64], 2012)**.** $\overline{\nabla}'(6,6) \geq 10$.

Proof. In view of Lemma 6.1 and Theorems 6.2 and 6.5, we obtain

$$\begin{aligned}\overline{\nabla}'(6,6) &= \overline{\nabla}'(6,4+2) \\ &\geq \overline{\nabla}'(6,4) + \overline{\nabla}'(6,2) \\ &= 7 + 3 = 10.\end{aligned}$$

□

Theorem 6.16 (Holtkamp, Volkmann [64], 2012)**.** $\overline{\nabla}'(6,6) = 10$.

Proof. Let $X = \{x_1, x_2, \ldots, x_6\}$ and $Y = \{y_1, y_2, \ldots, y_6\}$ be the partite sets of a bipartite tournament T. Then there exists a vertex x_i, say x_1, such that $d^-(x_1) \leq 3$. By Lemma 6.7 and Theorem 6.14 we obtain

$$\overline{\nabla}'(6,6) \leq \overline{\nabla}'(5,6) + 3 = 10.$$

Now Proposition 6.15 leads to the desired result. □

Theorem 6.16 confirms a conjecture of Vandell [102].

Proposition 6.17 (Holtkamp, Volkmann [64], 2012)**.** $\overline{\nabla}'(5,7) \geq 8$.

Proof. In view of Lemma 6.1 and Theorems 6.2 and 6.6, we obtain

$$\begin{aligned}\overline{\nabla}'(5,7) &= \overline{\nabla}'(5,5+2) \\ &\geq \overline{\nabla}'(5,5) + \overline{\nabla}'(5,2) \\ &= 6 + 2 = 8.\end{aligned}$$

□

Proposition 6.18 (Holtkamp, Volkmann [64], 2012)**.** $\overline{\nabla}'(5,7) \leq 9$.

Proof. Let $X = \{x_1, x_2, \ldots, x_5\}$ and $Y = \{y_1, y_2, \ldots, y_7\}$ be the partite sets of a bipartite tournament T. Then $\min\{d^-(y_1), d^+(y_1)\} \leq 2$. By Lemma 6.7 and Theorem 6.14 we obtain

$$\overline{\nabla}'(5,7) \leq \overline{\nabla}'(5,6) + 2 = 9.$$

□

	y_1	y_2	y_3	y_4	y_5	y_6	y_7
x_1	-	-	$\oplus$	-	-	$\oplus$	$\oplus$
x_2	-	$\oplus$	-	-	$\oplus$	-	$\oplus$
x_3	+	-	$\oplus$	+	-	-	-
x_4	+	+	-	$\ominus$	-	+	-
x_5	-	-	-	+	$\oplus$	-	-

Figure 6.1: Tournament T_1 from the proof of Theorem 6.19. A +/- entry in line i and row j in the table specifies an arc $x_i \to y_j$ / $x_i \leftarrow y_j$. Removing the 9 circled arcs decycles T_1.

Theorem 6.19 (Holtkamp, Volkmann [64], 2012)**.** $\overline{\nabla}'(5,7) = 9$.

Proof. The computer program in [64] shows that $\nabla'(T_1) = 9$ for the 5-by-7 bipartite tournament T_1 in Figures 6.1 and 6.2. Thus, Proposition 6.18 leads to

$$\overline{\nabla}'(5,7) = 9.$$

□

There are many further 5-by-7 bipartite tournaments T with the property $\nabla'(T) = 9$. In the next proposition we give some sufficient conditions for 5-by-7 bipartite tournaments T such that $\nabla'(T) \leq 8$.

Proposition 6.20 (Holtkamp, Volkmann [64], 2012)**.** *Let $X = \{x_1, x_2, \dots, x_5\}$ and $Y = \{y_1, y_2, \dots, y_7\}$ be the partite sets of a bipartite tournament T.*

If $d^-(y_i) \leq 1$ or $d^+(y_i) \leq 1$ for any $i \in \{1, 2, \dots, 7\}$, then $\nabla'(T) \leq 8$.

If $d^-(x_i) \leq 1$ or $d^+(x_i) \leq 1$ for any $i \in \{1, 2, \dots, 5\}$, then $\nabla'(T) \leq 8$.

If there are two distinct indices $i, j \in \{1, 2, \dots, 5\}$ such that $d^-(x_i) \leq 2$ or $d^+(x_i) \leq 2$ and $d^-(x_j) \leq 2$ or $d^+(x_j) \leq 2$, then $\nabla'(T) \leq 8$.

Proof. If $d^-(y_i) \leq 1$ or $d^+(y_i) \leq 1$ for any $i \in \{1, 2, \dots, 7\}$, then Lemma 6.7 and Theorem 6.14 lead to $\nabla'(T) \leq \overline{\nabla}'(5,6) + 1 = 8$.

If $d^-(x_i) \leq 1$ or $d^+(x_i) \leq 1$ for any $i \in \{1, 2, \dots, 5\}$, then Lemma 6.7 and Theorem 6.5 imply that $\nabla'(T) \leq \overline{\nabla}'(4,7) + 1 = 8$.

Assume that there are two distinct indices $i, j \in \{1, 2, \dots, 5\}$ such that $d^-(x_i) \leq 2$ or $d^+(x_i) \leq 2$ and $d^-(x_j) \leq 2$ or $d^+(x_j) \leq 2$. By Lemma 6.7 and Theorem 6.3 we obtain $\nabla'(T) \leq \overline{\nabla}'(3,7) + 4 = 8$. □

Proposition 6.21 (Holtkamp, Volkmann [64], 2012)**.** $\overline{\nabla}'(6,7) \geq 11$.

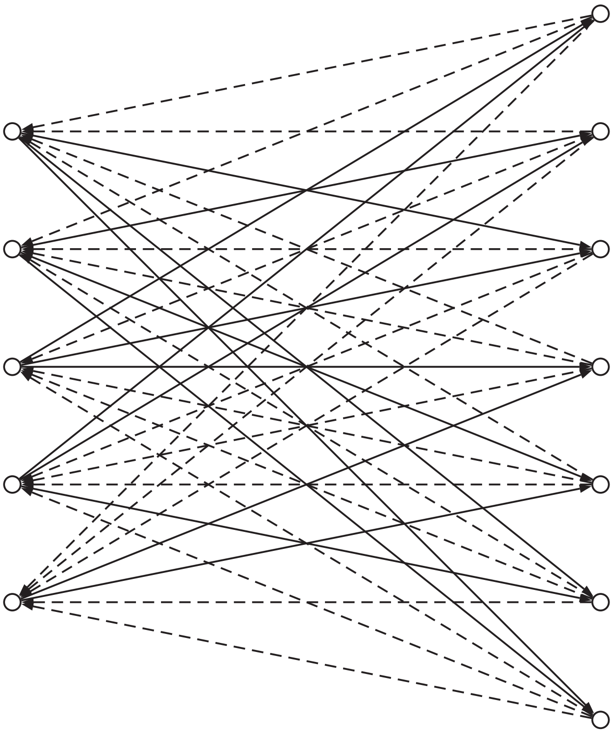

Figure 6.2: Drawing of tournament T_1. Arcs from left to right (right to left) are depicted as continuous (dashed) lines.

Proof. In view of Lemma 6.1 and Theorems 6.3 and 6.5, we obtain

$$\begin{aligned}\overline{\nabla}'(6,7) &= \overline{\nabla}'(6,4+3)\\ &\geq \overline{\nabla}'(6,4)+\overline{\nabla}'(6,3)\\ &= 7+4=11.\end{aligned}$$

$\square$

Proposition 6.22 (Holtkamp, Volkmann [64], 2012)**.** $\overline{\nabla}'(6,7) \leq 12$.

Proof. Let $X = \{x_1, x_2, \ldots, x_6\}$ and $Y = \{y_1, y_2, \ldots, y_7\}$ be the partite sets of a bipartite tournament T. Then $\min\{d^-(x_1), d^+(x_1)\} \leq 3$. By Lemma 6.7 and Proposition 6.18 we obtain

$$\overline{\nabla}'(6,7) \leq \overline{\nabla}'(5,7) + 3 \leq 9 + 3 = 12.$$

□

The proof of Proposition 6.22 and Lemma 6.7 lead immediately to the next propositions.

Proposition 6.23 (Holtkamp, Volkmann [64], 2012)**.** *Let $X = \{x_1, x_2, \ldots, x_6\}$ and $Y = \{y_1, y_2, \ldots, y_7\}$ be the partite sets of a bipartite tournament T. If $d^-(x_i) \leq 2$ or $d^+(x_i) \leq 2$ for any $i \in \{1, 2, \ldots, 6\}$, then $\nabla'(T) \leq 11$.*

Proposition 6.24 (Holtkamp, Volkmann [64], 2012)**.** *Let $X = \{x_1, x_2, \ldots, x_6\}$ and $Y = \{y_1, y_2, \ldots, y_7\}$ be the partite sets of a bipartite tournament T. If $d^-(y_i) \leq 1$ or $d^+(y_i) \leq 1$ for any $i \in \{1, 2, \ldots, 7\}$, then $\nabla'(T) \leq 11$.*

Proof. Assume, without loss of generality, $d^-(y_1) \leq 1$. Deleting the arc going into y_1 leaves a digraph without any cycle through y_1. According to Theorem 6.16, we obtain $\nabla'(T) \leq \overline{\nabla}'(6,6) + 1 = 11$. □

Proposition 6.25 (Holtkamp, Volkmann [64], 2012)**.** $\overline{\nabla}'(5,8) \geq 10$.

Proof. In view of Lemma 6.1 and Theorem 6.5, we obtain

$$\begin{aligned}\overline{\nabla}'(5,8) &= \overline{\nabla}'(5,4+4) \\ &\geq \overline{\nabla}'(5,4) + \overline{\nabla}'(5,4) \\ &= 5 + 5 = 10.\end{aligned}$$

□

Proposition 6.26 (Holtkamp, Volkmann [64], 2012)**.** $\overline{\nabla}'(5,8) \leq 11$.

Proof. Let $X = \{x_1, x_2, \ldots, x_5\}$ and $Y = \{y_1, y_2, \ldots, y_8\}$ be the partite sets of a bipartite tournament T. Then $\min\{d^-(y_1), d^+(y_1)\} \leq 2$. By Lemma 6.7 and Proposition 6.18, we obtain

$$\overline{\nabla}'(5,8) \leq \overline{\nabla}'(5,7) + 2 \leq 9 + 2 = 11.$$

□

The proof of Proposition 6.26 and Lemma 6.7 immediately lead to the next proposition.

Proposition 6.27 (Holtkamp, Volkmann [64], 2012)**.** *Let* $X = \{x_1, x_2, \ldots, x_5\}$ *and* $Y = \{y_1, y_2, \ldots, y_8\}$ *be the partite sets of a bipartite tournament* T*. If* $d^-(y_i) \leq 1$ *or* $d^+(y_i) \leq 1$ *for any* $i \in \{1, 2, \ldots, 8\}$*, then* $\nabla'(T) \leq 10$*.*

Chapter 7

Local connectivity properties in networks

In this chapter we present an approach how to use local connectivity properties to support local flow optimization decisions in a network flow problem. We start by introducing the model of the underlying application on which this scenario is based on in Section 7.1, and present our approach in Section 7.2. Basically, motivated by the results of the previous chapters, instead of just looking for any maximum flow we demand stronger local properties and derive distribution rates from this *maximum local flow*. Interestingly, this interpretation of K. Menger's ideas [75] in a more general local way leads to the definition of a unique *perfect flow* for any network, presented in Section 7.3. Finally, we discuss how to determine maximum local flows in Section 7.4. The work on this project has been performed in collaboration with Michael Herty and Ute Ziegler. The theoretical studies of maximum local flows and perfect flows emerged in various discussions with Yubao Guo.

7.1 Network flow optimization scenario

We consider a model for a supply network published by Armbruster, Degond and Ringhofer [1], Göttlich and Herty [40], Göttlich [39] and Fuegenschuh, Göttlich, Herty, Klar and Martin [36]. The supply network is modeled by a connected digraph $N = (V, A)$ called *network*, where every vertex has at least one incoming and one outgoing arc, except for the two exceptional vertices v_{in} with $d^-(v_{in}) = 0$ and v_{out} with $d^+(v_{out}) = 0$ called the *source* (or *inflow vertex*) and *target* (or *outflow vertex*), respectively. Moreover, for every vertex $v \in V(N)$ there exists a

v_{in}-v-path and a v-v_{out}-path in N. Furthermore, every arc $uv \in A(N)$ is equipped with *capacity* $cap(uv) \in \mathbb{N}$ modelling the maximum capacity of the associated processor in the supply chain.

This model is now used to simulate the (continuous) flow of particles through the supply network over time, where at every moment the flow of particles at the arcs is given by its density and velocity. Also every processor is featured with a buffer to store particles, in case it is already working at full capacity. The authors of [1, 36, 39, 40] approached this continuous optimization problem by solving a linear mixed integer problem and using the solution to make a decision on the distribution rates R_{uv} for every arc $uv \in A(N)$, where the flow arriving at a vertex is partitioned among its outgoing arcs according to these rates. These distribution rates can be changed over time and are the only way to control the flow through the supply network. Since the structure of the supply network is fixed, the idea now is to provide further information for the decision process at every vertex based on local connectivity properties of the underlying digraph. Therefore, in the following section we calculate *cut values* for every arc $uv \in A(N)$ and derive recommendable distribution rates R_{uv} from them.

7.2 Local connectivity properties in maximum flows

Let now $cap(uv) \in \mathbb{R}$ for all $uv \in A(N)$ of network N. We call a function $f : A(N) \to \mathbb{R}$ *flow* in N if $0 \leq f(uv) \leq cap(uv)$ for every $uv \in A(N)$, and

$$\sum_{x \in N^-(v)} f(xv) = \sum_{y \in N^+(v)} f(vy)$$

holds for every vertex $v \in V(N) \setminus \{v_{in}, v_{out}\}$. By $val(f) = \sum_{x \in N^+(v_{in})} f(v_{in}x)$ we denote the *value of* f, and a *maximum flow* is a flow f with $val(f) = \max\{val(g)\colon g \text{ is a flow in N}\}$. By a *cut* in N we refer to an arc-cut S separating source and target, such that there is no v_{in}-v_{out}-path in $N - S$. We denote the *capacity of a cut* S by $cap(S) = \sum_{uv \in S} cap(uv)$, and refer to a *minimum cut* if $cap(S) = \min\{cap(T)\colon T \text{ is a cut in } N\}$. We note that a minimum cut is always of the form $S = (X, \overline{X})$ with $X \subset V(N)$, where $v_{in} \in X$ and $v_{out} \in \overline{X}$. Due to the results of Elias, Feinstein and Shannon [23] and Ford, Fulkerson [32] we have the well-known *Max-Flow-Min-Cut Theorem*, which states that the value of maximum flows equals the capacity of minimum cuts in a network.

If the flow of particles through a network N runs at full capacity, then the corresponding flow f in N is a maximum flow. In this case, of course, every arc

$uv \in A(N)$ which is part of some minimum cut in N must also run at full capacity in f, i.e. $f(uv) = cap(uv)$. But the values of f on arcs that are not part of a minimum cut in N are not fixed in general, as maximum flows in networks must not be unique. In fact, networks can allow for a large variety of maximum flows, and each of those may differ essentially in the local flow values at some given vertex, which could result in some inappropriate behaviour according to local connectivity properties. In order to derive reasonable distribution rates for arbitrary vertices/arcs, the first thing we assure is that no arc should have a larger flow value than suggested by the portion of his importance in the connectivity structure of the network. To this end, for an arbitrary arc $uv \in A(N)$ of network N we call a cut S in N a *local cut of the arc* uv, if $S \setminus \{uv\}$ is *not* a cut in N (i.e. uv is an essential part of S). We note that not every arc must have local cuts in a network (cf. Figure 7.1).

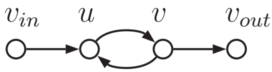

Figure 7.1: A network where the arc vu is not in any local cut.

More precisely, we make the following observation.

Observation 7.1. *Let $N = (V, A)$ be a network with source $v_{in} \in V(N)$ and target $v_{out} \in V(N)$. An arc $uv \in A(N)$ is part of some local cut if and only if there exists a v_{in}-v_{out}-path in N using uv.*

Proof. If $uv \in A(N)$ is part of a local cut S, then by definition there exists a v_{in}-v_{out}-path in $N - (S \setminus \{uv\})$, of course using the arc uv, since there are no v_{in}-v_{out}-paths in $N - S$.

Let now $P = v_0v_1...v_iv_{i+1}...v_{m-1}v_m$ be a v_{in}-v_{out}-path with $m \geq 1$, $v_0 = v_{in}$, $v_m = v_{out}$, $u = v_i$ and $v = v_{i+1}$ for some $i \in \{0, 1, \ldots, m-1\}$. Then

$$(\{v_0, v_1, \ldots, v_i\}, \overline{\{v_0, v_1, \ldots, v_i\}})$$

is a local cut of uv. □

By *minimum local cut* of an arc $uv \in A(N)$, denoted by S_{uv}, we refer to a local cut of uv of minimum capacity, which does not need to be unique in N. We note that a minimum local cut S_{uv} is always of the form $(X, \overline{X})$ with $X \subset V(N)$, where $\{v_{in}, u\} \subseteq X$ and $\{v_{out}, v\} \subseteq \overline{X}$. In case uv is part of some minimum cut in N, we note that S_{uv} is also a minimum cut in N. For a given network N with maximum flow f we now obtain the *transformed network* N' or *transformation of* N by changing the capacities of all arcs according to the following rule. If an arc $uv \in A(N)$ admits no local cut in N, then let $cap_{N'}(uv) = 0$ (as the arc plays no important role in any cut), and for every arc $uv \in A(N)$ with an arbitrary minimum local cut S_{uv} let

$$cap_{N'}(uv) = \frac{cap_N(uv) \cdot val_N(f)}{cap_N(S_{uv})}.$$

For instance, consider a network N with maximum flow f of value 100, and an arc $uv \in A(N)$ with capacitiy $cap_N(uv) = 50$, which is part of a minimum local cut S_{uv} of capacity $cap_N(S_{uv}) = 200$. Since the arc uv locally processes an amount of 50 in a minimum local cut of 200, it is only reasonable for uv not to exceed the portion of $50/200 = 1/4$ of the maximum flow f in N, as this quotient corresponds to the local importance of the arc uv. Thus, the capacity of uv should not exceed $cap_{N'} = (1/4) \cdot 100 = 25$. We note that according to this transformation rule we have $0 \leq cap_{N'}(uv) \leq cap_N(uv)$ for every arc $uv \in A(N)$.

We will now show that for any network N the corresponding network N' allows a maximum flow of the same value.

Theorem 7.2. *Let N be a network with maximum flow f, and N' be the transformation of N. Then there exists a maximum flow f' in N' with $val_{N'}(f') = val_N(f)$.*

Proof. Since the capacities of the arcs in N' are lower or equal to those in N, the inquality $val_{N'}(f') \leq val_N(f)$ is obvious.

Let now S be a minimum cut in N'. Without loss of generality let every arc $uv \in S$ with $cap_{N'}(uv) = 0$ also have $cap_N(uv) = 0$, since otherwise $S \setminus \{uv\}$ is also a cut in N and therefore also a minimum cut in N'. By S^* we denote the subset of S with all arcs of capacity 0 in both N and N' removed, i. e. $S^* = S \setminus \{uv \in S \colon cap_N(uv) = cap_{N'}(uv) = 0\}$. We have

$$val_{N'}(f') = cap_{N'}(S) = \sum_{uv \in S^*} cap_{N'}(uv) = \sum_{uv \in S^*} \frac{cap_N(uv) \cdot val_N(f)}{cap_N(S_{uv})},$$

where S_{uv} denotes a minimum local cut in N for every arc $uv \in S^*$. Of course, S is a cut in N. Furthermore, for all $uv \in S^*$ the set $S \setminus \{uv\}$ can not be a cut in N, since in this case $S \setminus \{uv\}$ would also be a cut in N', contradicting the minimality of S. Thus, S is a local cut of uv in N for every arc $uv \in S^*$ with $cap_N(S) \geq cap_N(S_{uv})$. We deduce

$$\sum_{uv \in S^*} \frac{cap_N(uv) \cdot val_N(f)}{cap_N(S_{uv})} \geq val_N(f) \cdot \sum_{uv \in S^*} \frac{cap_N(uv)}{cap_N(S)}$$

$$= \frac{val_N(f)}{cap_N(S)} \cdot \underbrace{\sum_{uv \in S} cap_N(uv)}_{=cap_N(S)} = val_N(f),$$

and therefore $val_{N'}(f') = val_N(f)$. □

According to Theorem 7.2, every network N allows a maximum flow keeping the stronger local properties mentioned above. We call such a maximum flow f in N, which is also a maximum flow in N', a *maximum local flow in* N. Figure 7.2 shows a simple network N and its transformation N'.

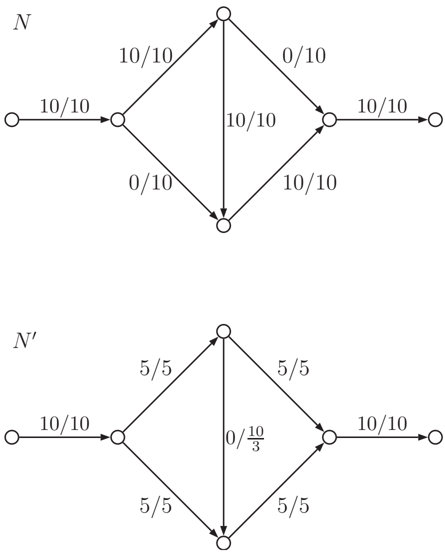

Figure 7.2: A network N and its transformation N'. The numbers on the arcs represent the values of a maximum flow and the capacities, respectively. The maximum flow in N' is a maximum local flow in N. Also, in this case the maximum flow in N' is unique, and therefore also a perfect flow in N (cf. Section 7.3).

In the application mentioned above we computed a maximum local flow and proposed distribution rates according to its flow values, which were then integrated into the decision process. All capacities and flow values there have been restricted to integer values. Determining maximum flows in those networks has been widely studied and can be done for example by using the algorithms of Ford and Fulkerson [32, 33] or Edmonds and Karp [22], which runs in $\mathcal{O}(nm^2)$. For large networks one might also consider more sophisticated approachs as for example Dinic [20] ($\mathcal{O}(n^2m)$), Goldberg and Tarjan [38] ($\mathcal{O}(nm\log(n^2/m))$) or others. The main problem here, however, is computing the new capacities when transforming a network N to N', which we will discuss in Section 7.4. But first, we want to extend the theoretical implications of the presented approach in the next section.

7.3 Perfect flow in networks

As seen in the last section, when transforming the capacities of network N to obtain N', such that the capacity of no arc in N' can exceed its appropriate portion as measured by its contribution to a minimum local cut in N, the new capacities fulfill $0 \leq cap_{N'}(uv) \leq cap_N(uv)$. Like already established in Theorem 7.2 maximum flows in N' are also maximum flows in N, but hold stronger local properties. Of course, if this is true for network N, then the same should hold for network N' as well. Thus, repeating this procedure and deriving network N'' from network N' will further improve the local properties of the resulting maximum flow in N''. By $N^{(k)}$ we denote the k-th iteration of this process, where $f^{(k)}$ denotes a maximum flow in $N^{(k)}$. Like seen in the last section, we note that $f^{(k)}$ is also a maximum flow and maximum local flow in N. The capacities of every arc $uv \in A(N)$ form a monotonically decreasing sequence due to $0 \leq cap_{N^{(k)}}(uv) \leq cap_{N^{(k-1)}}(uv) \leq \ldots \leq cap_N(uv)$. Since the capacities are also bounded below by zero, these sequences must converge for large k. According to the transformation rule, for an arc $uv \in A(N)$ which is part of a minimum cut in $N^{(k)}$ we have $cap_{N^{(l)}}(uv) = cap_{N^{(k)}}(uv)$ for all $l \geq k$, and all arcs $uv \in A(N)$ not being part of a minimum cut in $N^{(k)}$ fulfill $cap_{N^{(k+1)}}(uv) < cap_{N^{(k)}}(uv)$. So let $\overline{N}$ and $\overline{f}$ be the boundary values of this iteration, i. e.

$$\overline{N} = \lim_{k\to\infty} N^{(k)} \quad \text{and} \quad \overline{f} = \lim_{k\to\infty} f^{(k)}.$$

Every arc of $\overline{N}$ must be part of some minimum cut, thus, we have $val_{\overline{f}}(uv) = cap_{\overline{N}}(uv)$ for every arc $uv \in A(N)$. We call the unique flow $\overline{f}$ the *perfect flow in* N. Unfortunately, even in simple networks the number of iterations necessary to obtain $\overline{N}$ can be infinite. For instance, let uv be the arc from top to bottom in network N from Figure 7.2. It is easy to see that its capacity is given by the recursive formula

$$cap_{N^{(k)}}(uv) = 10 \cdot \left(1 - \frac{10}{10 + cap_{N^{(k-1)}}(uv)}\right) \text{ for } k \geq 2,$$

which defines a monotonically decreasing sequence with starting value $cap_{N'}(uv) = 10/3$, which converges to 0.

7.4 Determining maximum local flows

In order to determine a maximum local flow for a given network N we have to calculate its transformation N'. Hence, for every arc $uv \in A(N)$ we have to decide whether or not it belongs to some local cut (e. g. with the help of Observation 7.1), and if so, determine the capacity of a minimum local cut S_{uv}. It is easy to see that such a minimum local cut is always of the form $(X, \overline{X})$ for some (not necessarily unique) $X \subseteq V(N)$, where $u, v_{in} \in X$ as well as $v, v_{out} \in \overline{X}$, and there exists a v_{in}-u-path in X and a v-v_{out}-path in $\overline{X}$ by the definition of a local cut. Altogether, the problem of determining $cap(S_{uv})$ shows some similarity to the Minimum Multiway Cut or Multiterminal Cut problem with 3 terminals, which has been proven to be NP-hard by Dahlhaus et al. [15]. However, so far we have not been able to prove or disprove the following conjecture.

Conjecture 7.3. *Let N be a network with $uv \in A(N)$. Then finding a minimum local cut S_{uv} is NP-hard.*

Given a network N, the following naive approach with exponential running time can be used to compute its transformation N'.

Algorithm 7.4.

Input: *A network N.*

Step 1: *For all $e \in A(N)$ initialize values* $a_e = \begin{cases} 0, & \textit{if } cap_N(e) = 0, \\ \infty, & \textit{if } cap_N(e) \neq 0. \end{cases}$
Set $N' = N$. Determine a maximum flow f in N.

Step 2: **for** *every subset $X \subseteq V(N) \setminus \{v_{in}, v_{out}\}$* **do**
  for *every arc $e \in (v_{in} \cup X, \overline{v_{in} \cup X})$ with $a_e \neq 0$* **do**
    if $(cap_N((v_{in} \cup X, \overline{v_{in} \cup X})) \leq a_e$
    and $(v_{in} \cup X, \overline{v_{in} \cup X})$ *is a local cut of e*)
    then $a_e = cap_N((v_{in} \cup X, \overline{v_{in} \cup X}))$.

Step 3: **for** *every arc $e \in A(N)$ with $a_e \neq 0$* **do**
  if $a_e = \infty$ **then** *set $cap_{N'}(e) = 0$*
  else *set $cap_{N'}(e) = (cap_N(e) \cdot val_N(f))/a_e$.*

Output: N'.

Having the transformation N', standard techniques for determining a maximum flow in N' provide a maximum local flow in N.

Chapter 8

Conclusion

In this thesis we studied various connectivity parameters in graphs and digraphs with an emphasis on local connectivity properties. In the following we will summarize our main contributions in Section 8.1 and give an outlook on some interesting open problems for the future in Section 8.2.

8.1 Summary

In Part I of this thesis including Chapters 2 and 3 we studied some important connectivity parameters for graphs such as the vertex-, edge- and restricted edge-connectivity. We presented new results for all of these parameters to ensure maximality/optimality in various classes of graphs.

In particular, in Chapter 2 we gave degree conditions to obtain maximum connectivity and maximum local connectivity in graphs with bounded clique number, as well as in p-partite graphs, diamond-free graphs, p-diamond-free graphs, $K_{2,p}$-free graphs and C_4-free graphs. Our results generalized some earlier ones given in [16, 57, 99, 109]. Also, we presented examples showing the sharpness of the conditions in our results.

In the beginning of Chapter 3 we proved a new result for the maximum local edge-connectivity of diamond-free graphs, including bipartite graphs. Our result is a generalization of earlier results for the maximum edge-connectivity and maximum local edge-connectivity of bipartite graphs presented in [35, 103]. Furthermore, we gave examples to see that the degree condition in this result is tight.
In Section 3.2 we presented the concept of restricted edge-connectivity first introduced and studied in [26], and the more general case of k-restricted edge-

connectivity introduced in [29]. Firstly, we gave a new and tight lower bound on the cardinality of the 2-fragments in triangle-free, 2-restricted edge-connected and not λ_2-optimal graphs, which includes an earlier result from [101]. Secondly, we proved a conjecture given in [82] on the λ_3-optimality of connected, triangle-free graphs. Examples showed the tightness of this Ore-type degree condition. Thirdly, we gave new results for the λ_k-optimality in triangle-free graphs, which generalize some of the results in [101, 123]. Finally, we proved the λ_k-optimality of p-partite graphs fulfilling a certain degree condition. Again, examples showed the sharpness of this condition.
In Section 3.3 we proposed a new concept of local k-restricted edge-connectivity as a local generalization of the k-restricted edge-connectivity. We gave a legitimation for the introduction of this new parameter and studied sufficient criteria for graphs to be local k-restricted edge-connected. We then turned our attention to the general concept of local λ_k-optimality, where we studied the cases $k = 2$ and $k = 3$ in particular.

In Part II of this thesis, namely Chapters 4–7, we discussed connectivity parameters for digraphs.

In Chapter 4 we studied the vertex-connectivity and proved that regular and almost regular bipartite tournaments are maximally local connected, except for a small family of almost regular bipartite tournaments, which are maximally connected, but not maximally local connected. Furthermore, we gave a new lower bound for the vertex-connectivity of local tournaments, which generalizes an earlier result in [98] on tournaments. Examples given in [98] also show the sharpness of our result. Also, we gave an example showing that a similar result can not be obtained for the more general class of locally semicomplete digraphs.

In Chapter 5 we discussed a concept of restricted arc-connectivity proposed in [108]. Our results of this chapter cover the restricted arc-connectivity and λ'-optimality in tournaments and bipartite tournaments. We showed that essentially all strong tournaments and bipartite tournaments are restricted arc-connected. Furthermore, we characterized the small exceptional families of tournaments and bipartite tournaments T for which the inequality $\lambda'(T) \leq \xi'(T)$ does not hold, where $\xi'(T)$ is the arc-degree proposed in [112]. Also, we gave degree conditions and regularity criteria ensuring λ'-optimality in tournaments and bipartite tournaments.

In Chapter 6 we discussed the decycling of bipartite tournaments. We proved a conjecture given in [102] on the maximum decycling index of 6-by-6 bipartite tournaments using a computer program. Altogether, this computer program helped to determine the numbers $\overline{\nabla}'(5,6) = 7$, $\overline{\nabla}'(6,6) = 10$ and $\overline{\nabla}'(5,7) = 9$.

Finally, in Chapter 7 we studied local connectivity properties in a network flow problem. We proposed the new concept of maximum local flows as a special case of maximum flows holding stronger local properties, and legitimated this new approach on the basis of an application scenario. In fact, we interpreted K. Menger's ideas [75] in a more general local way. We proved that every network admits such a maximum local flow and used our idea to define a unique perfect flow in a network.

8.2 Outlook and open problems

The vertex- and edge-connectivity of graphs from Chapter 2 and Section 3.1 have been widely studied in the literature. However, the restricted edge-connectivity discussed in Section 3.2 is a very promising field for studying the reliability of network structures, and still offers a lot of open problems. While the 2-restricted edge-connectivity has already drawn some attention over the last 24 years (e. g. [7, 26, 53, 55, 73, 83, 84, 94, 101, 115, 116, 120]), the 3-restricted edge-connectivity has only been addressed by a few authors within the last 10 years (e. g. [13, 74, 82, 84, 114, 118, 124]). In the near future various classes of graphs shall be investigated for the 3-restricted edge-connectivity, with the question of finding sufficient criteria ensuring λ_3-optimality. Of course, studying the k-restricted edge-connectivity for larger values of k, i. e. $k \geq 4$, is also desirable. So far, a number of publications cover mostly very basic properties and special cases there (e. g. [61, 63, 74, 84, 85, 86, 87, 88, 89, 91, 117, 125, 126, 127]). One main problem here, however, is that for $k \geq 4$ the equality $\lambda_k(G) \leq \xi_k(G)$ is no longer true in general [13].

In this thesis, however, the main focus has been on local connectivity properties. According to this, the new concept of local k-restricted edge-connectivity presented in Section 3.3 offers many possibilities for future research. For instance, since a local λ_2-optimal graph is also λ_2-optimal by Observation 3.43, we can now take a look at various conditions implying λ_2-optimality, and study if these conditions already imply local λ_2-optimality provided that G is local 2-restricted edge-connected. First, we state this problem in its general form.

Problem 8.1 (Holtkamp, Meierling [62]). *Let $\mathcal{P}$ be a property that guarantees a graph G to be λ_2-optimal. Provided that G is local 2-restricted edge-connected, does $\mathcal{P}$ imply that G is* local *λ_2-optimal?*

For example, Wang and Li [115] showed that a 2-restricted edge-connected graph of order n with the property that $d(u)+d(v) \geq n+1$ for all pairs u, v of non-adjacent vertices is λ_2-optimal. Hence, one specific problem is the following.

Problem 8.2 (Holtkamp, Meierling [62]). *Let G be a local 2-restricted edge-connected graph with $d(u)+d(v) \geq n+1$ for all pairs u, v of non-adjacent vertices. Is G local λ_2-optimal?*

From Part II of this thesis a promising field for future research surely offers the quite new concept of restricted arc-connectivity of digraphs due to [108, 112] investigated in Chapter 5. Besides tournaments and bipartite tournaments one might study the restricted arc-connectivity and λ'-optimality of multipartite tournaments or other simply structured classes of digraphs.

Another group of open problems evolves from Chapter 7. First of all, we repeat the following conjecture.

Conjecture 7.3. Let N be a network with $uv \in A(N)$. Then finding a minimum local cut S_{uv} is NP-hard.

If Conjecture 7.3 holds, then it is very interesting to study the complexity of the maximum local flow problem in acyclic or planar networks, networks with bounded cut-width, or other special classes of networks. Also, a polynomial-time approximation scheme for maximum local flows and perhaps even perfect flows in arbitrary networks could be developed.

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