Competition for Order Flow as a Coordination Game [*]

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Competition for Order Flow as a Coordination Game

ABSTRACT

Competition for order flow can be characterized as a coordination game with multiple equilibria. Analyzing competition between dealer markets and a crossing network, we show that the crossing network is more stable for lower traders' disutilities from unexecuted orders. By introducing private information, we prove existence of a unique equilibrium with market consolidation. Assets with low volatility and large volumes are traded on crossing networks, others on dealer markets. Efficiency requires more assets to be traded on crossing networks. If traders' disutilities differ sufficiently, a unique equilibrium with market fragmentation exists. Low disutility traders use the crossing network while high disutility traders use the dealer market. The crossing network's market share is inefficiently small.

1 Introduction

Intermarket competition for order flow between trading venues is growing significantly. Key forces behind the increase in competition between markets and different market structures are the globalization and deregulation of financial markets as well as further advances in the automation of trading. Recent innovations in trading technologies and a reduction in communication costs have strengthened the popularity of alternative trading systems (ATSs), such as crossing networks (CNs) and electronic communication networks (ECNs). ATSs offer market participants the possibility to meet directly without the intervention of an intermediary. In the U.S., regulation by the Securities and Exchange Commission (SEC) led to a proliferation of ATSs, in particular it allows ECNs to register as exchanges.[1] This intensifies competition between existing markets and electronic trading venues even further. In Europe, CNs such as PositUK and E–Crossnet have gained market share, and an increasingly number of other initiatives is currently being considered by the industry, i.e. European brokers start to organize their own CNs to save transaction costs. As there is no special regulatory status for ATSs in Europe, these CNs are not surveyed by authorities.

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[1] See Securities and Exchange Commission (1998); SEC Rule 17a gives ECNs the right to register as stock exchanges; upon approval, ECNs would be self-regulated stock exchanges.

ATSs attract order flow away from existing exchanges by offering lower commission prices and the longest after-hours trading. ATSs, thus, increase intermarket competition for order flow. However, one of the major concerns with respect to multi-market trading is the question of liquidity. In order to provide a liquid platform, new trading venues must attract sufficient order flow to ensure a high probability of order execution.

This paper examines intermarket competition for order flow between traditional stock exchanges, i.e. dealer markets, and an electronic matching market. Trading at established dealer markets guarantees immediate order execution at bid and ask prices quoted by market makers. Trading at the electronic market is less expensive, as traders do not have to pay for an intermediary's services but only a small commission. However, the execution of an order submitted is uncertain. The probability of execution depends on the number of orders submitted. As more traders direct their orders to the electronic matching market, the probability of any particular order being matched against a counter order increases, raising the probability of execution for all submitted orders and improving liquidity. This, in turn, attracts even more traders to submit their orders to the electronic market.

Intermarket competition can be understood as a coordination game among traders. If and only if many traders coordinate to trade on the electronic market the probability of order execution is high and expected payoff from trading at this market exceeds the payoff from trading at dealer markets. Coordination failure would result in the immediate failure of the new market due to a lack of liquidity. This raises the question under which circumstances and for which sort of assets or commodities electronic matching markets can co–exist with dealer markets or even replace them. Related to these are the questions which parameters influence the traders' decision where to trade.

Theoretical research that addresses the question whether markets with different market structures and trading mechanisms can co–exist is rather limited. Most existing market microstructure models analyze different trading mechanisms in isolation. There exists, however, some theoretical research that analyzes liquidity–based competition for order flow in models of multi–market trading that are close to our view of intermarket competition as a coordination game. Pagano (1989), Gehrig (1993), and most recently Hendershott and Mendelson (2000) analyze the interaction of markets with a focus on the traders' choice of trading venue. Market liquidity arises endogenously as a result of the traders' decision where to trade.[2]

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[2]Glosten (1994) and Parlour and Seppi (1998) also address the question of the ability of markets to co-exist. Glosten (1994) examines an idealized electronic limit order book and shows that it does not invite competition from other markets while other markets do. Based on Glosten (1994), Parlour and Seppi (1998) present a model of competition for order flow between different pairings of pure limit order markets and hybrid specialist/limit order markets. These models jointly describe

Pagano (1989) examines competition between two centralized markets and between a centralized market and direct search for a trading partner. He focuses on the role of heterogeneous traders' beliefs about the actions of other traders and their impact on markets' performance. There is no intermediary but liquidity arises as a function of scale. Depending on the transaction cost differential between markets, multiple rational expectation equilibria arise. If markets have identical transaction costs, the equilibrium in which both markets exist is unstable and trade will rather concentrate on one of them. If markets differ in transaction costs or the search mechanism there may be either fragmentation or consolidation of trading, depending on the traders' initial expectations about other traders' decisions where to trade. When there is fragmentation, smaller traders go to the less expensive but illiquid market and larger traders to the more expensive but liquid market.[3]

Considering competition between a centralized market with an intermediary that offers guaranteed execution and a decentralized search market where heterogeneous liquidity traders meet randomly and negotiate prices, Gehrig (1993) analyzes how the intermediaries pricing behavior is affected by the existence and efficiency of the search market and the bargaining process in the search market. He shows that there is an equilibrium in which traders with large gains from trade choose to trade with the monopolistic intermediary while traders with low gains from trade enter the search market. Here, both markets co–exist and order flow is fragmented. However, there are multiple equilibria depending on the traders' initial beliefs about other traders' choice of venue.

Hendershott and Mendelson (2000) study the interaction between a competitive dealer market and a passive crossing network and analyze the impact of the introduction of a CN on traders and the dealer market. While trading at the dealer market is certain and expensive, trading at the CN is cheaper but uncertain and entails waiting costs. In a rather complex model with different types of heterogeneous liquidity and informed traders, they show that, in general, low liquidity preference traders use the CN exclusively and traders with medium net gain use the CN opportunistically, while high liquidity preference traders go to the dealer market.

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liquidity demand and supply by assuming different types of traders who trade via different order types, i.e. liquidity suppliers are limit order traders and intermediaries, liquidity is demanded by market order submitters. Viswanathan and Wang (1998) analyze the traders' choice between a limit-order book, a dealership market and a hybrid market structure of the two when traders differ in size and risk aversion.

[3]Similar to Pagano (1989), Chowdhry and Nanda (1991) analyze how the ability of traders to choose where to trade affects functioning and liquidity of markets in the presence of informational asymmetries and liquidity traders in each market, who are not allowed to switch at another market. They show that the market with the largest number of liquidity traders, who cannot move between markets, attracts liquidity and informed traders, resulting in a concentration of trading in this market.

These models suggest that markets with different trading costs and market structures may co–exist. They have one feature in common: there exist multiple equilibria in each model resulting from a coordination problem among traders. Whether order flow is consolidated on one market or fragmented depends on the initial beliefs of traders about other traders' behavior. Given fragmentation of order flow, a sudden change in the initial beliefs could result in traders switching markets and trade concentrating on one single market. Multiple equilibria models do not allow comparative statics and analyses of the impact of alternative policies with respect to multi–market trading.

Our paper presents an approach to examine the intermarket competition for order flow that allows to solve the coordination problem and remove indeterminacy. First, we develop a model of liquidity–based competition between a pure dealer market and a CN. Traders' choices of market venues depend on transaction costs, probabilities of order execution and expected losses from unexecuted orders. If the disutility of unexecuted orders θ is the same for all traders and common knowledge among them, multiple equilibria exist. We analyze and compare efficiency and stability of these equilibria and calculate critical market shares that are necessary for a CN to drive out dealer markets. Using the geometric distribution for the number of traders, we show that there are three equilibria for most interesting values of θ with market shares for the CN of zero, one, and some $\tilde{\alpha}(\theta)$ strictly in between. The latter equilibrium is highly unstable, so that we would not expect a CN to co–exist with dealer markets.

Adopting an approach of Carlsson and van Damme (1993a,b) and Morris and Shin (1998) who proved that small payoff uncertainty can remove indeterminacy of equilibria in coordination games, we introduce noisy private information about the value of trading. This results in traders being uncertain about the other traders' decisions where to trade. Instead of worsening the multiplicity problem, as one would expect at first sight, this uncertainty leads to an additional equilibrium restriction that results in a considerable reduction in the number of equilibria and, under certain conditions, in a unique equilibrium.

If the noise in private information on θ is sufficiently small, there exists a unique equilibrium with a threshold signal x^* up to which agents place orders at the CN. Agents who receive signals above x^* trade at the dealer market. Threshold x^* rises with rising bid–ask–spread at the dealer market, with rising market thickness and with falling trading costs at the CN. Surprisingly, x^* rises the more precise the private information becomes. However, this effect is too small to be exploited by any specific information policy. The equilibrium is inefficient as each agent's decision to use the CN has a positive externality on other users. The equilibrium switching signal is lower than the efficient switching signal. If the variance of private

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information approaches zero, almost all assets will be traded either at the CN or at a dealer market. Hence, we would not expect both market venues to co–exist.

Finally, we allow disutilities of unexecuted orders to differ across individuals. In each equilibrium there is a threshold θ^* such that all traders with lower disutilities submit orders to the CN, while traders with higher disutilities trade at the dealer market. There is a unique equilibrium, if disutilities have a uniform distribution and spread far enough to have some traders for whom going to the dealer market is a dominant strategy and a sufficient number of traders for whom going to the CN is a dominant strategy. In this case dealer markets and a CN co–exist with market shares being determined by the distribution of disutilities. If the equilibrium is unique, the market share of the CN is inefficiently small.

The major achievement of our work is the removal of the multiplicity of equilibria in the analysis of intermarket competition. We prove existence of a unique equilibrium if traders have private information about θ or if private values θ^i are sufficiently different. In contrast to models with multiple equilibria, our analysis provides a definitive answer to the question whether markets can co–exist and order flow is fragmented or whether trading concentrates on a single market. If traders have the same disutility from unexecuted orders, order flow concentrates on one market. While existing models cannot predict whether and on which market trade consolidates, our model shows that assets with low price volatility and large turnovers are traded at a CN, while assets with high volatility or small volumesare traded at dealer markets. If disutilities differ sufficiently across individuals, both markets co–exist and order flow is fragmented. Traders with low disutilities use the CN and traders with higher disutilities go to the dealer market. Our results are in line with existing literature; when there is fragmentation of order flow, traders cluster together according to their typical characteristics, i.e. their order size as in Pagano (1989) or their liquidity preference as in Gehrig (1993) and Hendershott and Mendelson (2000). While our model is closely related to the latter as we also analyze competition between a CN and dealer markets, we differ from it in that we derive our results by assuming one single type of traders like Pagano (1989) and Gehrig (1993).

In the next chapter, we introduce the basic market structure underlying our analysis. Chapter 3 deals with the game with common knowledge of values of trade, chapter 4 analyses private information and chapter 5 lays out the game with private disutilities of unexecuted orders. Chapter 6 gives conclusions and an outlook on future research.

2 Traders and Markets

There are exogenously given numbers of potential buyers N_b and sellers N_s that are independently and identically distributed with $\text{prob}(N = 0) > 0$. In particular, we assume a geometric distribution with $\text{E}(N) = \lambda$. Furthermore, we assume that all agents have the same probability of being selected.

The geometric distribution follows from the idea that there is an infinite set of agents out of which potential buyers and sellers are selected randomly. With probability γ a first agent is selected as buyer. A second buyer is selected with probability γ if and only if another buyer has been selected already. Thus, the probability of having at least n buyers is γ^n. The total number of buyers N_b has a geometric distribution with an expected number of $\lambda = \frac{\gamma}{1-\gamma}$ and $\text{prob}(N) = (1 - \gamma)\,\gamma^N = \frac{\lambda^N}{(1+\lambda)^{N+1}}$. The same procedure is applied to select sellers.

Agents do not know their position in the selection process. For any buyer the number of *additional* buyers has the same geometric distribution with expected value λ. It is a basic property of the geometric distribution that the conditional probability of having $k + N$ buyers, given that there are k buyers already, is the same as the unconditional probability for N buyers. This property makes calculations easier and allows to interpret potential buyers and sellers as small and without market power.

Each trader can decide to buy [sell] one unit of the one and only asset either at market A or at market B. In the following sequel we distinguish two cases: either agents decide for one of the two markets without knowing whether they will enter this market as a buyer or seller, or agents are allowed to condition their choice on the market side. Traders are risk neutral and maximize expected payoff.

Market A is a dealer market (DM) where traders trade with market makers who set bid and ask prices at which they are willing to buy or sell the asset. We assume that bid and ask prices do not depend on the volume of trade. In particular, market makers quote prices even if there are no traders at the DM. We normalize the mid–point of bid and ask price to zero, so that traders can buy the asset at price t_A and sell at $-t_A$, where t_A is half of the bid–ask–spread and sometimes referred to as the DM's transaction fee.

Market B is an electronic crossing network (CN) which offers purely transactional services without any intervention by an intermediary and without price discovery. Orders can be submitted to the CN as market orders and are executed at the mid–point between bid and ask price observed at the DM, i.e. zero. If an order is executed at the CN, the trader pays a small fee $t_B < t_A$.

There may be an imbalance of orders on the two sides of market B, in which case the excess side is rationed stochastically. In this case, one runs the risk of an order not being executed. Orders on the excess side are randomly selected to match orders on the short side. The probabilities with which orders are executed are determined by the numbers of buyers and sellers who place their orders at the CN, n_b and n_s. The probability of a buy order at the CN to be executed is

$$\pi_b = \min\{1, n_s/n_b\}. \tag{1}$$

The probability of a sell order to be executed is

$$\pi_s = \min\{1, n_b/n_s\}. \tag{2}$$

Unexecuted orders might be passed on to another trading system and executed there at some later point in time possibly at different costs. The loss in expected payoff from orders that are not executed instantaneously may depend on the costs of passing orders to other trading venues, on expected price differentials, on the length of the time interval until the order can be executed elsewhere, on the traders' impatience or urgency to trade, and on the asset's price volatility. We assume a reduced form one–shot game, where unexecuted orders leave the trader with some disutility $\theta \in [\check{\theta}, \hat{\theta}]$. Parameter θ is also referred to as value of trade, but it should not be confused with the expected payoff from carrying out a trade at all. θ essentially is the difference in payoffs between trading now and having the choice to trade next period.

The payoff for a trader at the DM is $\theta - t_A$ with certainty. At the CN the expected payoff is $\mathrm{E}((\theta - t_B)\,\pi_b)$ for a potential buyer and $\mathrm{E}((\theta - t_B)\,\pi_s)$ for a potential seller. For simplicity of the exposition,[4] we assume $\check{\theta} > t_B$.

The model and its parameters are assumed to be common knowledge. For the disutility of unexecuted orders θ we consider three cases: First, we assume that all traders face the same disutility θ if an order remains unexecuted. In the next chapter, we assume in addition that θ is common knowledge. We see that this assumption leads to multiple equilibria for a wide range of values. We then introduce small noise in the observation of θ, so that agents only have private information about this variable. One possible interpretation of the noise in observation may be that traders cannot observe the realization of the underlying value θ but a noisy signal. Traders are uncertain about the signals observed by others, but know that they are in some surrounding of θ. Another interpretation may be that all traders receive

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[4]If $\theta < t_B$, agents would prefer not to trade. This requires a third strategy "no trade" to be considered, besides A and B. We verified that this would not alter our results.

the same message with regard to θ but interpret this message differently. As a result, traders' expected values of trading are clustered round θ.[5] We shall see that uncertainty with private information creates an additional restriction that leads to a unique equilibrium, if the variance of private information is sufficiently small. Then, there exists a threshold signal x^* such that traders with smaller signals trade at the CN while traders with higher signals go to the DM.

In a third part, we argue that the value of trade may differ across individuals and analyze equilibria of a game with private values of trade θ^i. The dispersion in private values of trade may result from disparities in, for example, endowments or (time) preferences across traders. In general, θ^i may be influenced by a trader's liquidity preference, her risk aversion, idiosyncratic beliefs or inside information. There is a unique equilibrium if and only if private values are spread over a sufficiently wide range. The unique equilibrium is associated with a critical value θ^*, such that all agents with smaller values place their orders at the CN, while agents with higher values go to the DM.

3 Common Knowledge Game

The set of players in this game is $[0, 1]$. A random process selects subsets of buyers $\mathbb{N}_b$ and sellers $\mathbb{N}_s$, whose finite sizes N_b and N_s are independently and identically distributed. As explained before, we assume a geometric distribution with $\mathrm{E}(N_b) = \mathrm{E}(N_s) = \lambda > 0$.

Assume that θ is common knowledge. An individual strategy is a function

$$a^i : [\check{\theta}, \hat{\theta}] \times \{b, s\} \to \{0, 1\}. \tag{3}$$

$a^i(\theta, b) = 1$ means that agent i goes to market B if she is a buyer and the value of trade is θ. If she is a seller, she goes to B iff $a^i(\theta, s) = 1$. We allow for strategies to depend on whether a player is selected as a buyer or a seller. Strategies may not, however, depend on the sets $\mathbb{N}_b$ and $\mathbb{N}_s$, as we assume that these sets are unknown to traders when they choose a market. Even if θ is common knowledge and strategies are mutually known, each trader faces some uncertainty about successful execution of an order placed at market B, because the total number of buyers and sellers, N_b and N_s, are unknown.

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[5]For an illustration, one may think of θ being an indicator of exogenously given price volatility of the traded asset. A high value of θ reflects high volatility and, because prices may change quickly, a high urgency to trade with individual levels of urgency differing around θ. A low value of θ reflects a low volatility and, thus, a low demand for immediacy.

Given a strategy combination $a = (a^i)_{i \in [0,1]}$ and disutility θ, the proportions of agents who submit orders to the CN if selected as buyers or sellers, respectively, are

$$\alpha_b(\theta, a) = \int_0^1 a^i(\theta, b) \, \mathrm{d}i \qquad \text{and} \qquad \alpha_s(\theta, a) = \int_0^1 a^i(\theta, s) \, \mathrm{d}i \tag{4}$$

Lemma 1 *Suppose a fraction α_b of all traders goes to market B if selected as buyers and a fraction α_s of all traders goes to market B if selected as sellers. The probability with which a buy order is executed is given by*

$$\Pi(\alpha_b, \alpha_s) = \frac{\alpha_s}{\alpha_b} \ln \left(1 + \frac{\alpha_b \lambda}{1 + \alpha_s \lambda} \right).$$

The probability of execution of a sell–order is $\Pi(\alpha_s, \alpha_b)$, accordingly.

Proof see Appendix. Lemma 1 is a generalization of a result by Hendershott and Mendelson (2000, Proposition 3, p. 2081).

The expected payoff for a buyer with strategy a^i, provided that others play strategy combination a is given by

$$U_b^i(\theta, a) = a^i(\theta, b) \, (\theta - t_B) \, \Pi(\alpha_b(\theta, a), \alpha_s(\theta, a)) + (1 - a^i(\theta, b)) \, (\theta - t_A). \tag{5}$$

Accordingly, the expected payoff for a seller is

$$U_s^i(\theta, a) = a^i(\theta, s) \, (\theta - t_B) \, \Pi(\alpha_s(\theta, a), \alpha_b(\theta, a)) + (1 - a^i(\theta, s)) \, (\theta - t_A) \tag{6}$$

and the expected payoff of agent i is

$$U^i(\theta, a) = \begin{cases} U_b^i(\theta, a) & \text{if} \quad i \in \mathbb{N}_b \\ U_s^i(\theta, a) & \text{if} \quad i \in \mathbb{N}_s. \end{cases} \tag{7}$$

Definition 1 *A Nash equilibrium of the game with common knowledge of θ is a strategy combination a^* with*[6]

$$U^i(\theta, a^*) \geq U^i(\theta, \tilde{a}^i, a^{*-i}) \qquad \forall \theta, \quad \forall \tilde{a}^i, \quad \forall i.$$

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[6]By $\tilde{a}^i, a^{*-i}$ we denote a strategy combination a^*, where the strategy of player i has been replaced by $\tilde{a}^i$.

In equilibrium a trader decides for B if expected gains at B exceed those at A. She decides for A if it is the other way round. The expected payoff of going to B instead of A is

$$\tilde{U}_b(\theta, a) = (\theta - t_B)\,\Pi(\alpha_b, \alpha_s) - (\theta - t_A) \tag{8}$$

for a buyer and for a seller accordingly

$$\tilde{U}_s(\theta, a) = (\theta - t_B)\,\Pi(\alpha_s, \alpha_b) - (\theta - t_A). \tag{9}$$

Using (5) to (9) it is straightforward that a strategy combination a^* is a Nash equilibrium if and only if

$$a^{*i}(\theta, b) = \begin{cases} 1 & \text{if} \quad \tilde{U}_b(\theta, a^*) > 0 \\ 0 & \text{if} \quad \tilde{U}_b(\theta, a^*) < 0 \end{cases} \tag{10}$$

and

$$a^{*i}(\theta, s) = \begin{cases} 1 & \text{if} \quad \tilde{U}_s(\theta, a^*) > 0 \\ 0 & \text{if} \quad \tilde{U}_s(\theta, a^*) < 0 \end{cases} \tag{11}$$

To analyze equilibria, we first show that the same proportions of buyers and sellers submit orders to the CN.

Lemma 2 *If a^* is a Nash equilibrium of the common knowledge game, then*

$$\alpha_b(\theta, a^*) = \alpha_s(\theta, a^*) = \alpha(\theta, a^*) \qquad \textit{for all } \theta.$$

Proof Let a^* be a Nash equilibrium. Suppose $\alpha_b(\theta, a^*) > \alpha_s(\theta, a^*)$ for some θ. Then $\Pi(\alpha_b, \alpha_s) < \Pi(\alpha_s, \alpha_b)$ and $\tilde{U}_b(\theta, a^*) < \tilde{U}_s(\theta, a^*)$. On the other hand, $\alpha_b(\theta, a^*) > \alpha_s(\theta, a^*)$ implies $\alpha_b(\theta, a^*) > 0$ and $\alpha_s(\theta, a^*) < 1$ and therefore $\tilde{U}_b(\theta, a^*) \geq 0$ and $\tilde{U}_s(\theta, a^*) \leq 0$. This is a contradiction to the inequality above. Therefore, $\alpha_b(\theta, a^*) = \alpha_s(\theta, a^*) \quad \forall \theta$. QED

Associated with each Nash equilibrium a^* is a market share for the CN of $\alpha(\theta, a^*)$ The probability of order execution is

$$\tilde{\pi}(\alpha) = \ln\left(1 + \frac{\alpha\,\lambda}{1 + \alpha\,\lambda}\right). \tag{12}$$

$\tilde{\pi}$ is strictly increasing in α up to

$$\bar{\pi} = \tilde{\pi}(1) = \ln\left(1 + \frac{\lambda}{1 + \lambda}\right) < 0.7. \tag{13}$$

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Proposition 1 *A strategy combination a^* is a Nash equilibrium of the game with common knowledge if and only if*

$$\alpha_b(\theta, a^*) = \alpha_s(\theta, a^*) = 1 \qquad \text{for} \quad \theta \in (t_B, t_A),$$

$$\alpha_b(\theta, a^*) = \alpha_s(\theta, a^*) \in \{0, \tilde{\alpha}(\theta), 1\} \quad \text{for} \quad \theta \in [t_A, \theta_0],$$

$$\alpha_b(\theta, a^*) = \alpha_s(\theta, a^*) = 0 \qquad \text{for} \quad \theta > \theta_0,$$

where $\theta_0 = \frac{t_A - \bar{\pi} t_B}{1 - \bar{\pi}}$ *and* $\tilde{\alpha}(\theta) = \tilde{\pi}^{-1}\left(\frac{\theta - t_A}{\theta - t_B}\right) = \frac{1}{\lambda} \frac{\exp\left(\frac{\theta - t_A}{\theta - t_B}\right) - 1}{2 - \exp\left(\frac{\theta - t_A}{\theta - t_B}\right)}.$

Proof see Appendix. Nash equilibria are illustrated in Figure 1.

. Insert Figure 1 here

Figure 1 *Nash equilibria of the common knowledge game. For $t_B = 1$, $t_A = 2$ and $\lambda = 15$, $\theta_0 = 3.953$.*

Equilibrium strategies are individually optimal at each value of trade θ, given that all other agents play the strategies of the same equilibrium.

If $\theta < t_A$, agents would loose from trading at market A, but for $\theta \geq \check{\theta} > t_B$, they profit from trading at market B. Here, $a^{*i}(\theta, b) = a^{*i}(\theta, s) = 1$ is a dominant strategy and the only equilibrium.

For $\theta = t_A$ there are two equilibria: in equilibrium A, all agents go to the DM where they receive nothing. This is a Nash equilibrium because a single trader cannot gain from switching to market B. Without a trading partner her order would not be executed. But any coalition with at least one trader on each side of the market could improve upon their payoffs by switching to B.

In equilibrium B, all agents go to the CN where they expect positive gains from trade. Equilibrium B is a strong equilibrium[7] where any coalition would loose by switching to market A.

If θ exceeds t_A, agents would still prefer to trade at market B since it is cheaper. However, there is uncertainty about the execution of an order at B; at market A gains are lower but certain. If all traders go to market A, a single trader who switches to B will face probability zero of her order to be executed and hence loose her gains from trade. Therefore, it is an equilibrium if all traders go to A.

If all agents place orders at market B, the probability of successful order execution is high. Agents get the reward from the transaction and save transaction costs with high probability. At market A it is guaranteed that they get a small reward $\theta - t_A$. If this reward is sufficiently small, it is more than compensated by the higher reward at B with positive probability. Thus, there is a B–equilibrium for some $\theta > t_A$.

From equation (12) we know that the probability of order execution at the CN rises in its market share. However, even if all agents place their orders at market B, the probability of successful execution is bounded below 1. The probability that there is a lack of trading partners is $\text{prob}(N_b > N_s) > 0$. There is a positive probability to be rationed. Hence, there is a $\theta_0 > t_A$ such that for $\theta > \theta_0$ it pays to go to market A, even if all other agents go to B. That is to say, for larger values of trade going to A is the only equilibrium.

For smaller trading values $\theta \in [t_A, \theta_0]$ going to B is an equilibrium, because expected payoff at market B is higher than at A if all traders go to B. θ_0 is determined by the equality of expected payoffs at both markets for the maximal probability of order execution $\bar{\pi}$. Expected payoff on market B is $(\theta - t_B)\bar{\pi}$. A trader going to market A gets a payoff $\theta - t_A$. This does not exceed the expected payoff on market B iff $\theta \leq \theta_0$.

Market shares of zero and one for either of the two markets are equilibria for $\theta \in [t_A, \theta_0]$. However, the relative stability of these equilibria changes in θ.

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[7] A strategy combination a^* is a strong equilibrium if

$$U^i(\theta, a^*) \geq U^i\left(\theta, (\tilde{a}^i)_{i \in K}, (a^{*i})_{i \notin K}\right) \qquad \forall \theta \quad \forall \tilde{a} \quad \forall K \subseteq \mathbb{N}.$$

With rising θ the A–equilibrium becomes stronger as coalitions of growing size are needed to improve their payoffs by switching to B and to create an execution probability that outweighs certainty of the lower gain at A. For $\theta \geq \theta_0$, the A–equilibrium is strong and cannot be improved upon by any coalition.

The B–equilibrium is always strong up to θ_0. However, there is a relative strength of the B–equilibrium that declines with θ above t_A: smaller coalitions could be attracted to market A. They would loose relative to the B–equilibrium but after their change in strategy, market size at B would be too small to re–attract single traders as the expected payoff at A would now exceed that at B. The triggering coalition would require a compensation for the lower payoffs at A.

Strength of A and relative weakness of B are logically the same. The size of the marginal coalition necessary to raise the expected payoff at B over the gain at A and induce agents to switch from an A–equilibrium to B is equal to the one that must remain at B to prevent A from being more attractive than B after the drain.

The strength of the A–equilibrium is more direct than that of B. Any CN that is successful in coordinating a sufficient proportion of strategies can intrude a DM–monopoly. To intrude a CN–monopoly, dealer(s) would have to compensate all traders by fees lower than t_A until the CN is so drained out that lower transaction costs do not compensate for execution risk and the DM is more attractive even with fee t_A.

In addition to the pure A– and B–equilibrium, there is a mixed equilibrium at which both markets co–exist with a market share for the CN of $\tilde{\alpha}(\theta)$ that decreases from one to zero as θ rises from t_A to θ_0. In this equilibrium, the size of the market share of B just generates an execution probability for which expected payoffs at A and B equal each other, so that no agent wants to switch. If expected payoffs are the same on both markets, traders on market B cannot gain from switching to A. On the other hand, a single trader, say buyer, who switches from A to B, would reduce $\mathrm{E}(\pi_b)$ and, thus, would expect smaller gains from trade at market B.

Mixed equilibrium market share $\tilde{\alpha}(\theta)$ is given by

$$\tilde{U}_b(\theta, \tilde{\alpha}) = 0 \quad \Leftrightarrow \quad \tilde{\pi}(\tilde{\alpha})\,(\theta - t_B) = (\theta - t_A) \quad \Leftrightarrow \quad \tilde{\alpha}(\theta) = \tilde{\pi}^{-1}\left(\frac{\theta - t_A}{\theta - t_B}\right). \quad (14)$$

The mixed equilibrium is very weak. Expected payoffs are the same on both markets and any coalition with at least one trader on each side of the market can improve their payoffs by switching from A to B. But instability of the mixed equilibrium is asymmetric. Agents who go to the CN in a mixed equilibrium have no incentive to

form a deviating coalition, as they cannot gain by switching to the DM. Once a CN has a market share of $\tilde{\alpha}(\theta)$, one should expect that it takes over the whole market.

Market share $\tilde{\alpha}(\theta)$ is also the minimal size of a coalition that is needed to induce agents to switch from an A–equilibrium to market B and establish a CN that offers higher expected payoffs than the DM. As expected, critical mass $\tilde{\alpha}$ rises in θ and t_B and falls in t_A, as these changes reduce the cost advantage of $t_A - t_B$ relative to the value of trading θ. A rise in the expected number of traders λ, i.e. rising "thickness" of the market, increases probability of order execution and makes the CN more attractive. This lowers $\tilde{\alpha}(\theta)$.

With respect to the situation where existing DMs face upcoming electronic CNs, we suggest the following interpretation: the lower the value of immediate trade θ, the easier is it to intrude the market with a CN. Intrusion is possible only if $\theta \leq \theta_0$. For $\theta > \theta_0$ agents trade at the DM exclusively. While it is increasingly easy for an intruding CN to win the market when θ goes down, it becomes more and more costly for market makers to re–attract order flow. To convince traders to return to the DM, they must be compensated for higher bid–ask spreads. This, in turn, is costly for market makers. It pays for the DM to protect itself against attempts to establish a CN–monopoly. A CN may fail to enter the market if its market share stays below the critical value $\tilde{\alpha}(\theta)$.

For $\theta \geq \theta_0$, the A–equilibrium is efficient. The B–equilibrium is efficient if $\theta \leq \theta_0$. It operates at lower costs and for $\theta < \theta_0$ this cost advantage is more valuable to agents than the execution risk. Risk at market B is minimized when all agents go there.

Equilibrium market shares are the same if we restrict agents to strategies for which the market choice can only depend on θ, but not on the market side. In other words, multiplicity of equilibria does not depend on the consideration of asymmetric strategies.

For trading values $\theta \in [t_A, \theta_0]$ multiple equilibria exist. Although these equilibria differ in strength, it is not possible to predict at which market trade consolidates. With multiple Nash equilibria, it is even possible that different agents play strategies belonging to different equilibria, so that played strategies do not form a Nash equilibrium at all. This opens another way for co–existing DM and CN, besides mixed equilibria. Mixed equilibria are extremely unstable and strategy combinations that are no Nash equilibrium will not survive in the long run either. We would therefore not expect to observe co–existence of DMs and a CN when θ is the same for all agents and common knowledge. The existence of multiple equilibria is consistent with other models when considering the ability of markets to co–exist. For examples, see Pagano (1989), Gehrig (1993), Parlour and Seppi (1998), Hendershott

and Mendelson (2000), as well as related work examining markets with positive externalities as for example, Katz and Shapiro (1985).

4   Private Information Game

Suppose that traders do not know a trade's exact value θ. Suppose, they each get a private signal x^i, but are uncertain about θ. When there is uncertainty about θ, there is also uncertainty about the signals of other agents. Even if strategies are known, the actual behavior of other agents is uncertain to each trader. This uncertainty creates an additional restriction for equilibria that can be used to eliminate strategies that are equilibria under common knowledge of θ.

Given θ, signals x^i are independently and identically distributed. We assume that $\mathrm{E}(\theta \,|\, x^i)$ rises with rising x^i. For means of exposition, we assume that x^i has a uniform distribution in $[\theta - \epsilon, \theta + \epsilon]$ and θ has a uniform distribution in $[\check{\theta}, \hat{\theta}]$, so that for $\check{\theta} + \epsilon < x^i < \hat{\theta} - \epsilon$ the posterior distribution of θ conditional on x^i is uniform in an ϵ–surrounding of x^i. Furthermore, we assume that $\check{\theta} + \epsilon < t_A$, so there are signals below t_A for which $\mathrm{E}(\theta \,|\, x^i) = x^i$. Here, agent i's expected profit from executed trade is x^i minus transaction costs.

An individual strategy is a function $a^i : \mathbb{R} \to \{0, 1\}$. $a^i(x^i) = 1$ [0] means that agent i goes to market B [A] if her signal is x^i.

We think of traders choosing the market irrespective of their wish to buy or sell the asset. They decide on the market depending on their signal before they are selected as buyers or sellers. This must be taken into account for any interpretation of equilibria. The market that we consider has the same people trading on both sides. This may be a suitable assumption for many asset markets, but not for all. It is most certainly not appropriate for product markets, where buyers and sellers are firms of different branches, for retail markets or markets with participants who exercise market power.

Denote the conditional density of signal x^i for given disutility θ by $f(x^i \,|\, \theta)$. The proportion of players who go to the CN if selected as buyers or sellers is

$$\alpha(\theta, a) = \int_{-\infty}^{\infty} \int_0^1 f(x^i \,|\, \theta)\, a^i(x^i) \, \mathrm{d}i \, \mathrm{d}x^i. \tag{15}$$

The probability of order execution at the CN when the value of trade is θ and when agents play a strategy combination a is

$$\pi(\theta, a) = \tilde{\pi}(\alpha(\theta, a)). \tag{16}$$

The expected payoff for agent i going to market B instead of A is

$$\tilde{U}(x^i, a) = \mathrm{E}((\theta - t_B)\,\pi(\theta, a) - \theta + t_A \mid x^i). \tag{17}$$

4.1 Dominated Strategies

To start analysis of this game, we need some regions in which going to either market is a dominated strategy. It is easy to conceive that there are signals so bad that the expected value of trade is smaller than t_A and other signals so good that expected gains from trade exceed θ_0. As this depends on the probability space, we formally assume the existence of signals $\underline{x}^0, \bar{x}^0$ in the interior of the signal space, for which

$$E(\theta \mid \underline{x}^0) = t_A \qquad \text{and} \qquad E(\theta \mid \bar{x}^0) = \theta_0, \tag{18}$$

so that for all strategies a, $\tilde{U}(x^i, a)$ is positive for some $x^i < \underline{x}^0$ and negative for $x^i > \bar{x}^0$. Given the uniform distribution of values of trade and signals as described above, $\underline{x}^0 = t_A$ and $\bar{x}^0 = \theta_0$.

Using (17), we find

$$\tilde{U}(x^i, a) > 0 \quad \forall a \qquad \Leftrightarrow \qquad t_A - \mathrm{E}(\theta \mid x^i) > 0 \qquad \Leftrightarrow \qquad x^i < \underline{x}^0.$$

Thus, for an agent who gets signal $x^i < \underline{x}^0$, it is a dominant strategy to go to market B. The intuitive reason for this is that the trader expects a positive reward at B with some probability that depends on the strategies of other agents. Since $\check{\theta} > t_B$ the agent cannot loose money at B, while the expected reward at A is negative.

Using (17) again, we find

$$\tilde{U}(x^i, a) < 0 \quad \forall a \qquad \Leftrightarrow \qquad t_A - \bar{\pi}\, t_B - (1 - \bar{\pi})\,\mathrm{E}(\theta \mid x^i) < 0 \qquad \Leftrightarrow \qquad x^i > \bar{x}^0.$$

Thus, for an agent who gets signal $x^i > \bar{x}^0$, it is a dominant strategy to go to market A. The strategy profile that would give a buyer the highest incentive to go to B is given when all traders go to B. The probability of success in this case is $\bar{\pi} < 1$, as described in the previous section. An agent who gets signal $\bar{x}^0$ and has the optimistic belief $\bar{\pi}$ is indifferent between the two markets. For higher signals, the expected value of trade is so big that certainty of execution at A outweighs the lower costs at B even for the highest possible execution probability at that market.

If agent i gets a signal $x^i \in (\underline{x}^0, \bar{x}^0)$, her payoff from going to B instead of A may be positive or negative, depending on the strategies of other players. There is no dominant strategy for these intermediate signals.

Traders are rational and do not play a dominated strategy. Hence, all agents who get signals below $\underline{x}^0$ go to market B and agents who get signals above $\bar{x}^0$ go to market A. As we assume that traders know that others are rational, each agent concludes that the others will not play a dominated strategy. This gives an additional insight that can be used to eliminate even more strategies.

4.2 Iterated Elimination of Dominated Strategies

As rationality is common knowledge an agent will not play a strategy that is dominated if she considers only those strategies of other players that have not been eliminated yet. Starting with $k = 0$, at step $k+1$ agents consider only the strategies that assign B to signals below $\underline{x}^k$ and A to signals above $\bar{x}^k$.

In this game, market choices are strategic complements. The more agents that decide for market B the higher is the incentive for each agent to go to the same market. After eliminating dominated strategies, the best [worst] thing that can happen to a potential trader on market B is that all other potential traders who have signals in $[\underline{x}^k, \bar{x}^k]$ go to market B [A]. This maximizes [minimizes] the probability of execution of an order at market B. So, the best [worst] strategy combination that an agent at step $k+1$ must consider is given when all other agents play strategy $I_{\bar{x}^k}$ [$I_{\underline{x}^k}$]. Define a strategy I_y by

$$I_y(x^i) = \begin{cases} 1 & \text{if } x^i \le y \\ 0 & \text{if } x^i > y. \end{cases} \tag{19}$$

In other words, an agent playing this strategy goes to market B if and only if her signal is not bigger than y.

It is a dominant strategy to go to market B whenever the lowest expected return there exceeds the certain return at A, i.e. $\tilde{U}(x^i, I_{\underline{x}^k}) > 0$. This is the case when $x^i < \underline{x}^{k+1}$, defined by

$$\underline{x}^{k+1} = \inf\{x \mid \tilde{U}(x, I_{\underline{x}^k}) = 0\}. \tag{20}$$

If the highest expected return from B is lower than the gain at A, it is a dominant strategy to go to market A. This happens for $x^i > \bar{x}^{k+1}$, defined by

$$\bar{x}^{k+1} = \sup\{x \,|\, \tilde{U}(x, I_{\bar{x}^k}) = 0\}. \tag{21}$$

$\tilde{U}(x, I_k)$ rises in k. Since $\underline{x}^k < \bar{x}^k$, we have $\underline{x}^{k+1} < \bar{x}^{k+1}$. If rationality is common knowledge, players go to market B if they get signals below $\underline{x}^k$, and they go to A if their signals exceed $\bar{x}^k$ for any k.

Common knowledge of rationality takes this procedure to the limits, where a trader with signal x^i will always go to market B if $x^i < \underline{x}^\infty = \lim_{k\to\infty} \underline{x}^k$ and always go to market A if $x^i > \bar{x}^\infty = \lim_{k\to\infty} \bar{x}^k$. Sequences $\underline{x}^k$ and $\bar{x}^k$ are monotone and bounded, so that limit points exist and are given by

$$\begin{aligned}
\underline{x}^\infty &= \inf\{x \,|\, \tilde{U}(x, I_x) = 0\} \\
\bar{x}^\infty &= \sup\{x \,|\, \tilde{U}(x, I_x) = 0\},
\end{aligned} \tag{22}$$

where

$$\tilde{U}(x, I_x) = \mathrm{E}((\theta - t_B)\,\tilde{\pi}(F(x|\theta)) - \theta + t_A \,|\, x) \tag{23}$$

and $F(x|\theta)$ is he cumulative density of signal x given value of trade θ. $\underline{x}^\infty$ and $\bar{x}^\infty$ characterize the set of rationalizable strategies.

Proposition 2 *A rationalizable strategy of the game with private information about θ is a strategy a^* with*

$$\begin{aligned}
a^*(x^i) &= 1 &&\textit{for}\quad x^i < \underline{x}^\infty \\
a^*(x^i) &\in \{0, 1\} &&\textit{for}\quad x^i \in [\underline{x}^\infty, \bar{x}^\infty] \\
a^*(x^i) &= 0 &&\textit{for}\quad x^i > \bar{x}^\infty.
\end{aligned}$$

Since we have strategic complements, we know from Milgrom and Roberts (1990) that the range of rationalizable strategies is limited by Nash equilibria. Indeed, it is easy to see that $I_{\underline{x}^\infty}$ and $I_{\bar{x}^\infty}$ are Nash equilibria of the private information game. Since all Nash equilibria are rationalizable, there is no Nash equilibrium where agents go to market A for signals below $\underline{x}^\infty$ or to B for signals above $\bar{x}^\infty$.

Limit points $\underline{x}^\infty$ and $\bar{x}^\infty$ are the smallest and the biggest solution of equation $\tilde{U}(x, I_x) = 0$. In general, this equation may have several solutions, so that we are

left with multiple equilibria, although we could clearly reduce the set of disutilities with unpredictable outcomes in comparison to the game with common knowledge. However, if $\tilde{U}(x, I_x)$ is monotone in x, there would be only one solution: a unique equilibrium. Signals enter this function in two ways. The partial derivative of $\tilde{U}$ with respect to x is negative. An increase in x increases expected return at B at a marginal rate that equals execution probability $\pi < 0.7$. Expected returns at A are certain and, therefore, rise at a marginal rate of 1. If execution probability is not affected, an increase in x lowers the expected payoff of going to B instead of A at a rate $1 - \pi$. On the other hand, an increase in the switching point up to which traders go to market B may change the probabilities of order execution in a way that depends on the assumed probability distributions. This effect may be positive and could even exceed the negative partial derivative. Hence, the net effect depends on the probability distributions, as does multiplicity of rationalizable strategies. For uniform distribution of values of trade and signals the latter effect vanishes, because agents always attribute the same probability to other signals being higher or lower than their own.

Theorem 1 *For uniform distribution of values of trade and signals, there is a signal x^*, such that any rationalizable strategy assigns market B to signals below x^* and market A to signals above x^*. x^* is the unique solution to*

$$\tilde{U}(x^*, I_{x^*}) = \int_0^1 (x^* + \epsilon - 2\,\epsilon\,\alpha - t_B)\,\tilde{\pi}(\alpha)\,\mathrm{d}\alpha - x^* + t_A = 0. \tag{24}$$

Proof see Appendix. Given uniform distribution, there is a critical signal x^*, such that agents with lower signals use the CN, while agents with higher signals go to the DM.

The intuitive explanation of the existence of a unique equilibrium under uncertainty about θ is the following. The noise of the signal eliminates common knowledge about θ. A trader observing a private signal knows neither the true value of θ, nor does she know which signals other traders have obtained. However, traders are known to be rational and to take certain actions at certain information sets, i.e. dominant strategies exist for signals $x < \underline{x}^0$ and $x > \bar{x}^0$, respectively. The knowledge that traders do not employ dominant strategies results in a unique best response for traders who, based on the observations of their private signals, believe that other traders have observed signals $x < \underline{x}^0$ or $x > \bar{x}^0$, respectively, and play a dominant strategy. The unique best response is to follow the same action as it offers a higher expected payoff. These traders' best responses, in turn, influence the responses of other traders, who believe that some traders believe that others have observed signals $x < \underline{x}^0$ or $x > \bar{x}^0$, respectively, and so on. If this infection argument

results in a unique action profile, as it does for uniform distribution of θ and signals, a unique equilibrium is obtained. Clearly, higher order beliefs and the existence of dominant strategies are the key factors in determining a unique equilibrium.

In the common knowledge game, payoffs are certain in equilibrium. In the private information game, the actual behavior of traders and payoffs is uncertain due to the lack of common knowledge. Traders need to take into account potential losses from trading at the CN instead of trading at the DM in case the order submitted to the CN remains unexecuted. They weigh expected gains from trading at the CN determined by the probability of order execution against certain gains from trading at the DM. At the equilibrium switching signal x^*, expected gains at both markets are equal. This is an additional equilibrium restriction that enables us to eliminate strategies that are equilibria in the common knowledge game.

A trader i who receives signal x^* attaches equal probability to all values of θ within $[x^* - \epsilon, x^* + \epsilon]$. If all traders who receive signals lower than x^* choose the CN, the proportion of traders at the CN is $\alpha(\theta, I_{x^*}) = \frac{x^* - \theta + \epsilon}{2\epsilon} \in [0, 1]$ and the execution probability is $\pi^* = E(\pi(\theta, I_{x^*}) \mid x^*)$. For trader i the expected gain at the CN $\int_{x^* - \epsilon}^{x^* + \epsilon} (\theta - t_B) \, \pi(\theta, I_{x^*}) \, d\theta$ just compensates $x^* - t_A$ (see Figure 2).

As the proof of Theorem 1 shows, with uniform distribution of values of trade and signals, the probability of order execution at signal x^* does not change with rising ϵ. It is given by

$$\pi^* = E(\pi(\theta, I_{x^*}) \mid x^*) = \int_0^1 \tilde{\pi}(\alpha) \, d\alpha. \tag{25}$$

Corollary 1 *Given uniform distribution of values of trade and signals, as ϵ goes to zero, the critical signal x^* approaches*

$$x_0^* = \frac{t_A - \pi^* t_B}{1 - \pi^*}.$$

Proof As ϵ approaches zero, (24) shows that

$$\tilde{U}(x^*, I_{x^*}) \to (x^* - t_B) \, \pi^* - x^* + t_A = 0.$$

Solving for x^* gives the equation in Corollary 1. QED

When ϵ approaches zero, uncertainty about the value of trade vanishes. But, as Morris and Shin (1997, 1998) have argued, with positive ϵ there is never common

knowledge that the value of trade is contained in any strict subset $\Theta \subset [\check{\theta}, \hat{\theta}]$. For $\epsilon > 0$ there is always uncertainty about higher beliefs. The lack of common knowledge, even for arbitrarily small ϵ is a fundamental difference to the previous game, and it is for this reason that reducing ϵ does not lead to an approximation of the Nash equilibria in the common knowledge game.

. Insert Figure 2 here

Figure 2 *Nash equilibrium of the private information game. Agents switch markets at signal x^*, where the expected payoff from trading at the CN equals the certain*

payoff from trading at the DM, i.e. the areas A and B are of equal size. For $t_B = 1$,
$t_A = 2$, $\lambda = 15$ and $\epsilon = 0.1$, $\theta_0 = 3.953$, $x^* = 3.434$ and $x_0^* = 3.445$.

If the distribution of θ is not uniform, there may be multiple equilibria. However, Morris and Shin (2000) have shown that for a quite general class of symmetric coordination games with binary choices equilibria approach a single strategy profile that does not depend on higher moments of the probability distribution as the variance of private information approaches zero.[8]

It is easy to check that our game satisfies the conditions required for this result. It follows that for any continuous probability distribution of values of trade and signals, as the variance of private information approaches zero, all agents with signals below x_0^* place orders at the CN, while agents with signals above x_0^* go to the DM.

Uniqueness of the equilibrium allows us to do some comparative statics on (24):

Corollary 2 *Given uniform distribution of values of trade and signals, the critical signal x^* rises with rising t_A or λ and with falling t_B for any $\epsilon > 0$. A rise in ϵ lowers x^*.*

Proof see Appendix.

The higher x^*, the larger the unconditional expected market share of the CN is. In our leading example, this share is $\text{prob}(x^i < x^*) = \frac{x^* - \check{\theta}}{\hat{\theta} - \check{\theta}}$. However, it cannot be concluded that a CN should take efforts to reduce ϵ in order to increase its market share, because the effect of ϵ on x^* is very small.

Corollary 3 *Given uniform distribution of values of trade and signals, for all $\epsilon > 0$,*

$$x_0^* - \frac{\pi^*}{1 - \pi^*} \epsilon < x^* < x_0^*.$$

Proof see Appendix.

This shows that precision of private information has no big impact on x^*. Since $\frac{\pi^*}{1-\pi^*} < \frac{7}{3}$, a reduction in ϵ raises x^* by a magnitude in the order of the change in ϵ. So even for positive ϵ, the equilibrium threshold is close to x_0^*. This is especially important as x_0^* is much easier to calculate than x^*.

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[8]More generally, Frankel, Morris and Pauzner (2000) have shown that a large class of global games with strategic complementarities has equilibria converging towards a single strategy profile as the noise in private information approaches zero.

Given θ, the market share of the CN is $F(x^* \mid \theta)$, where F is the cumulative distribution of signals. If $\theta < x^* - \epsilon$, all traders get signals below x^* and choose to trade at the CN. If $\theta > x^* + \epsilon$, all traders get signals above x^* and trade at the DM. For $\theta \in [x^* - \epsilon, x^* + \epsilon]$ the market share of the CN is $\frac{x^* - \theta + \epsilon}{2\epsilon}$. As ϵ approaches zero, the CN's market share approaches 1 [0] for $\theta < [>] x_0^*$.

Both markets co–exist if $\theta \in [x^* - \epsilon, x^* + \epsilon]$. However, this is an event with probability $\frac{2\epsilon}{\bar{\theta} - \underline{\theta}}$ that is very small and approaches zero for $\epsilon \to 0$. Hence, we would expect to observe co–existence of both market forms only for very few assets.

In comparison to the results of the common knowledge game, we see that uncertainty about the value of trade destabilizes a DM monopoly for low values of trade. Uncertainty makes it easier to coordinate on a CN, because traders believe that others might believe that some traders go to the CN anyway. Accordingly, there is a positive execution probability that is sufficient to attract some traders who know that the value of trade is above t_A.

For high values of θ up to θ_0, a CN could win the market provided that θ is common knowledge. Although this is a strong equilibrium, the CN would loose the market again as soon as the value of trade becomes uncertain. If a DM offers new services that create uncertainty in the observation of θ, it can hinder traders to coordinate on the CN and, thus, prevent intrusion or can win back the market at low costs. In fact, this is much cheaper than winning back market shares under common knowledge, where some traders must be compensated for the initial loss they face when switching from the strong B–equilibrium in the common knowledge game.

In the common knowledge game, the B–equilibrium is efficient whenever $\theta < \theta_0$. In the private information game, efficient strategy combinations coordinate trade at the CN up to a signal k^* that is close to θ_0. The efficient switching signal k^* maximizes $\mathrm{E}(U^i(I_k))$ over k.

Proposition 3 *Given uniform distribution of values of trade and signals, the efficient strategy combination in the private information game is I_{k^*}, where*

$$k^* = \theta_0 - \epsilon \, \frac{\bar{\pi} - 2 \int_0^1 \alpha \, \tilde{\pi}(\alpha) \, \mathrm{d}\alpha}{1 - \bar{\pi}} < \theta_0. \tag{26}$$

Proof see Appendix. Proposition 3 shows that the efficient switching signal in the private information game is smaller than θ_0 up to which it is efficient to place orders at market B in the common knowledge game. However, the deviation from θ_0 is smaller than $\frac{\bar{\pi}}{1 - \bar{\pi}} \epsilon < \frac{7}{3} \epsilon$ and disappears for $\epsilon \to 0$, while x^* converges to $x_0^* < \theta_0$.

Theorem 2 *Given uniform distribution of values of trade and signals, the unique equilibrium switching signal x^* is smaller than the efficient switching signal k^*.*

Proof see Appendix. For uniform distribution, the equilibrium switching signal x^* is smaller than k^* and hence, the equilibrium is inefficient. Reason are network externalities that arise from strategic complementarities. Agents should use the CN at signals at which, in equilibrium, they do not use it, because the decision to go to the CN increases expected payoff also for other users of the CN. The externality is not accounted for by individual decisions. This can be used to argue that CNs and other electronic market places need public support to overcome inefficiencies.

5 Private Value Game

In this chapter we assume that values of trade differ across individuals. Each agent i has a private value of trade $\theta^i \geq t_B$. The distribution of private values is defined by a density function $f(\theta^i)$. To simplify exposition, we identify agents with their private values, i.e. $i = \theta^i$.

Each trader knows her private value of trade and distribution f. However, she does not know how many traders are selected as buyers and sellers and what their private values are. Traders for both market sides are selected according to the random process described in the previous chapters. After selection, a trader decides on which market she places her order. A strategy is a function $a^i : \{b, s\} \rightarrow \{0, 1\}$, where $a^i(b) = 1$ means that trader i goes to the CN if she is selected as buyer.

Let $\alpha_b(a)$ $[\alpha_s(a)]$ be the expected proportion of traders who go to market B if they are selected as buyers [sellers]. For any given strategy combination a, the probability of execution is $\pi_b(a) = \Pi(\alpha_b(a), \alpha_s(a))$ for a buy order and $\pi_s(a) = \Pi(\alpha_s(a), \alpha_b(a))$ for a sell order.

Execution probabilities do not depend on any value of trade θ in contrast to the previous game. Here, there is a given distribution of private values θ^i that are affixed to traders, so that for any strategy combination expected probabilities of order execution only depend on the proportions of buyers and sellers going to market B. In particular, execution probabilities are the same for all traders and do not depend on θ^i.

Given strategy combination a, the expected payoff for trader i is

$$U(\theta^i, a) = \begin{cases} a^i(b)\,(\theta^i - t_B)\,\pi_b(a) + (1 - a^i(b))\,(\theta^i - t_A) & \text{for} \quad i \in \mathbb{N}_b \\ a^i(s)\,(\theta^i - t_B)\,\pi_s(a) + (1 - a^i(s))\,(\theta^i - t_A) & \text{for} \quad i \in \mathbb{N}_s. \end{cases} \tag{27}$$

Definition 2 *A Nash equilibrium of the game with private values of trade is a strategy combination a^* with*

$$U(\theta^i, a^*) \geq U(\theta^i, \tilde{a}^i, a^{*-i}) \qquad \forall \tilde{a}^i \qquad \forall i.$$

To characterize equilibria of this game, we start by defining excess utilities for going to market B instead of A,

$$\tilde{U}_b(\theta^i, a) = (\theta^i - t_B)\, \pi_b(a) - (\theta^i - t_A), \tag{28}$$

$$\tilde{U}_s(\theta^i, a) = (\theta^i - t_B)\, \pi_s(a) - (\theta^i - t_A). \tag{29}$$

Since probabilities of order execution are bounded below 1, $\tilde{U}_b$ and $\tilde{U}_s$ are strictly decreasing in θ^i for any strategy combination a. Using this property, we can show that in each equilibrium there is a value $\theta^* < \theta_0$ such that all agents with private values below θ^* place their orders at the CN, while agents with private values above θ^* trade at the DM.

Lemma 3 *If a^* is a Nash equilibrium of the private value game, there is a unique $\theta^*(a^*) \in [t_A, \theta_0]$, such that*

$$a^{*i}(b) = a^{*i}(s) = \begin{cases} 1 & if \quad \theta^i < \theta^*(a^*) \\ 0 & if \quad \theta^i > \theta^*(a^*) \end{cases}$$

Proof see Appendix.

The associated proportion of agents who place orders at the CN is given by $\alpha^* = F(\theta^*)$, where F is the cumulative distribution of private values. The probability of order execution is $\tilde{\pi}(\alpha^*)$, and excess utility from going to B instead of A is

$$\hat{U}(\theta^*) = (\theta^* - t_B)\, \tilde{\pi}(F(\theta^*)) - (\theta^* - t_A). \tag{30}$$

In equilibrium $\hat{U}(\theta^*) = 0$, and any strategy combination, where agents go to B iff their private values are smaller than θ^*, is a Nash equilibrium if $\hat{U}(\theta^*) = 0$.

Theorem 3 *If private values have a uniform distribution in $[t_B, \hat{\theta}]$ and $\hat{\theta} > \theta_0$ and $\frac{t_A - t_B}{\hat{\theta} - t_B}\, \lambda \geq \frac{1}{2}$, the private value game has a unique Nash equilibrium.*

Proof see Appendix.

For a unique equilibrium, there must be a sufficient mass of agents with private values below t_A. Theorem 3 requires that the expected number of these agents is at least 1/2. This guarantees a minimal probability of order execution $\tilde{\pi}(\frac{t_A - t_B}{\hat{\theta} - t_B}) \geq$ $\ln(4/3) \approx 0.288$. Given this probability at the lower end of $[t_A, \theta_0]$, the increase in π that is associated with a rising threshold value θ is too weak to compensate for the increasing disadvantage of B stemming from uncertainty of gain θ.

If disutilities of unexecuted orders differ sufficiently between traders to have some traders for whom going to the DM is a dominant strategy and a sufficient mass of traders for whom going to the CN is a dominant strategy, then there is a unique equilibrium with a threshold θ^* such that all traders with lower disutilities place orders at the CN while traders with higher values go to the DM. Under these conditions, both market types co–exist. The market share of the CN rises with rising t_A and falling t_B. Figure 3 illustrates a unique Nash equilibrium.

Insert Figure 3 here

Figure 3 *Nash equilibrium of the private value game. An intersection of the two curves at θ^* represents a Nash equilibrium if all agents with private values below θ^* go to B and agents with private values above θ^* go to A. For $t_B = 1$, $t_A = 2$, $\lambda = 15$ and $\hat{\theta} = 10$, $\theta_0 = 3.953$ and $\theta^* = 3.433$.*

If the distribution of private values is such that $\theta^i \in (t_A, \theta_0)$ with probability one, there are at least three Nash equilibria with properties comparable to the equilibria

in the common knowledge game: either trade concentrates on one of the two markets or both markets co–exist and order flow is fragmented, with trader i going to market A [B] if $\theta^i > [<] \theta^*$, where $\hat{U}(\theta^*) = 0$.

If there are three equilibria, the 'mixed' equilibrium is weak. A coalition of agents with positive mass and private values slightly higher than θ^* can improve by switching from A to B. The A-equilibrium is robust against deviations of small coalitions and the B-equilibrium is strong. If there are multiple equilibria, the one with the largest market share for the CN Pareto–dominates the others. If $\theta^i \in [t_A, \theta_0)$ for all i, only the B–equilibrium is efficient.

If there is a unique equilibrium, for example under conditions of Theorem 3, it corresponds to the 'mixed' equilibrium in the common knowledge game and is strong. No coalition can improve by changing strategies. Efficiency depends on whether private values are assigned to agents randomly or whether these values are inherent properties of agents' preferences. In the latter case, and if utility is not transferable across agents, any allocation different from equilibrium reduces expected payoff for some agents. If execution risks can be hedged, payoff is transferable and the efficient threshold θ^{**} maximizes the sum of individual payoffs, weighted with respective probabilities of the agents' participation as given by density function f. The same holds if private values are randomly assigned to agents. Here, f is the density function of the private value for each agent. Again, the efficient threshold θ^{**} maximizes expected payoff with respect to distribution f. Define

$$\theta^{**} = \arg \max_k \mathrm{E}(U(\theta^i, a_k^i)), \qquad \text{where} \ \ a_k^i = 1 \ \ \text{iff} \ \ \theta^i \geq k. \tag{31}$$

Theorem 4 *If the private value game has a unique equilibrium, the associated threshold θ^* is smaller than θ^{**}.*

Proof see Appendix.

In the private information game, we saw that assets are traded at a DM that would be more efficiently traded at a CN. Here, we get a similar result: The market share of the CN is inefficiently small. Traders with private values of trade between θ^* and θ^{**} give their orders to the DM. If they would go to the CN instead, the overall gains of all traders induced by higher liquidity of the CN exceed the losses that those traders must expect at the CN whose values are closest to θ^{**}.

Results of this game are related to Gehrig (1993), who also showed that traders with a low liquidity preference choose direct trading instead of trading with an intermediary. Gehrig's (1993) model differs from ours in that he does not consider

a CN, but traders must search for partners by themselves and matching occurs with rather low probability. Thus, execution probability is bounded above far from unity. In Gehrig's model, a whole continuum of traders is active on both sides of the market. Hence, there is no uncertainty about market size. Introducing a CN in Gehrig's model would guarantee multiple equilibria, and execution probability at the CN would either be zero or one. Uncertainty in Gehrig (1993) stems from the search process, while in our model uncertainty is due to the random selection of active traders. In both models, the probability of order execution at the direct market is limited — a crucial feature for uniqueness of the critical value θ^* that divides customers of the two markets. In equilibrium, uncertainty of order execution is a result of the market mechanism at the direct market. CNs and other E–business platforms, as we start to observe them, increase this probability and attractiveness of the direct market tremendously. A striking example is the comparison between garage sales and Internet auctions. Improvements in the mechanisms for direct marketing increase their market share accordingly.

Our results are also related to those of Herrendorf, Valentinyi and Waldmann (2000) who study multiplicity and indeterminacy in two–sector models with sector–specific labor and positive externalities. In their model, individuals differ in productivity and choose the sector in which they work. Herrendorf, Valentinyi and Waldmann show that enough heterogeneity in agents' sector–specific productivity can ensure uniqueness of the chosen stationary state as it prevents sufficiently many agents from changing their choice in reaction to a change in beliefs about the production of the sectors. This is in line with our result that a sufficient mass of traders with private values below t_A ensures a unique equilibrium.

6   Conclusion and Outlook

The proliferation of ATSs such as CNs considerably increases intermarket competition for order flow. While increased competition may be welcomed from the perspective of market and price efficiency, the enhanced choice of trading venues fragments the order flow and reduces liquidity which is key to the functioning of financial markets. In this paper we presented three models building on the idea that liquidity–based competition for order flow between DMs and a CN may be understood as a coordination game among traders. We addressed the question whether and under which circumstances electronic matching markets can co–exist with DMs or replace them. While in existing models of intermarket competition multiple equilibria arise and, therefore, answers to these questions are not definitive, we have shown that under certain conditions a unique equilibrium exists that allows clear answers.

The models presented differ in the assumptions about traders' disutility from unexecuted orders or values of immediate trade θ. The first game, in which θ is assumed to be the same for all traders and common knowledge, exhibits multiple equilibria for most interesting values of trade $\theta \in [t_A, \theta_0]$. Here, traders do not only face the problem of coordinating on the strong equilibrium but coordinating on a Nash equilibrium at all. Strategy combinations that are no Nash equilibria are not expected to survive in the long run and mixed equilibria are extremely unstable. Therefore, we would not expect to observe a CN co–existing with DMs. We found that the lower the value of trade θ, the easier it is for a CN to intrude the market, with intrusion being possible for values of θ up to θ_0. For $\theta < \theta_0$, coordination at the CN is efficient because the cost advantage from trading at the CN is more valuable to agents than the risk of non–execution which is minimized if all agents go to the CN.

If traders have noisy private information about θ, the set of assets for which multiple equilibria exist is reduced considerably. We proved existence of a unique equilibrium in the private information game if θ and signals have uniform distribution. For low values of θ, uncertainty about the value of trade makes it easier to coordinate on the CN than in the common knowledge game and, thus, destabilizes a DM monopoly. For high values of θ, agents trade at the DM. While in the common knowledge game for high values of θ up to θ_0 trading at the CN is a strong equilibrium, in the private information game uncertainty about θ can prevent intrusion of the CN or even allow the dealer to win back the market at low cost. Since the probability of dual trading is very small and approaches zero for small variance of private information, we would expect to observe both markets co–existing for only a few assets.

In the private value game, values of trade differ across traders. If there are some traders with private values above θ_0 and a sufficient mass of traders with private values below t_A ensuring a minimum probability of order execution at the CN, a unique equilibrium exists such that traders with values of trade below a certain threshold go to the CN and others go to the DM. Under these conditions, both markets co–exist.

In the private information game, existence of a unique equilibrium requires that *potential* values of θ are distributed widely, such that there exist extreme potential values for which going to either market is a dominant strategy. Posterior individual expectations about θ may be arbitrarily close to each other. In the private value game, however, existence of a unique equilibrium requires that *actual* private values θ^i are distributed widely.

The features of the private value game can be combined with private information if there is uncertainty about some payoff–relevant variables, aside from θ, such as the distribution of private values or λ. As we know by now, payoff–uncertainty is the driving force behind the reduction in the number of equilibria in coordination

games. However, it remains to be shown which conditions are needed to yield unique results for such an extended model.

The games presented above provide a broad platform to study intermarket competition for order flow between DMs and a CN. With respect to further research, we can think of a number of extensions. One may be to make the half–spread t_A endogenous by explicitly considering the DM's market structure and the dealer's cost structure. This would move the critical signal in one or the other direction. If a dealer quotes her prices based on her expectations about the number of traders, market microstructure theory suggests that the spread must widen if the expected number of traders at the DM decreases to ensure coverage of the dealer's costs. In a dynamic, multiple–period model, a strategic dealer may lower prices for some periods in order to re–attract traders. The CN may also set t_B strategically, even with negative values, i.e. it offers little presents or bonus points to attract traders and gain market share in the beginning. Thus, the introduction of a CN may be seen as the entry of a competitor into a market that leads to a limitation of existing dealer(s)' room to quote prices.

Other possible extensions could analyze alternative market structures, for example hybrid markets or automatic routing of unexecuted orders from CN to DM. We believe that this would basically influence the parameters of the model, transaction costs at the DM, market thickness and disutilities of unexecuted orders. A hybrid structure of the DM, in general, decreases the transaction costs differential which lowers the critical signal and, thus, reduces the possibility for a CN to enter the market. Automatic routing lowers the disutility from unexecuted orders θ and makes it easier for the CN to enter the market.

An important extension of our model would be the implementation of an endogenous price discovery process as it exists at another type of ATSs; namely ECNs. In this case, the price at which orders are executed is not taken from a primary exchange but is derived endogenously, depending on the preferences of the market participants. For example, an ECN with periodic trading may set prices that maximize turnover. As a result, only a few orders would be rationed, the probability of order execution would be close to one and thus, it would be easier to intrude the market. Alternatively, in a continuous-trading model, buy and sell limit orders are submitted to the ECN and matched. Unexecuted buy and sell limit orders are stored in the order book and matched against new incoming limit orders. This would require a dynamic model that discounts the expected utility from order execution in later periods.

Appendix

Proof of Lemma 1

Suppose the probability of a buyer [seller] to get a signal leading her to go to market B is α_b [α_s]. Then, the additional number of buyers k has a geometric distribution with $E(k) = \alpha_b \lambda$. For a buyer on market B the probability of having k additional buyers on this market is

$$p_b(k) = \frac{1}{1 + \alpha_b \lambda} \left(\frac{\alpha_b \lambda}{1 + \alpha_b \lambda} \right)^k.$$

The probability of having r sellers is

$$p_s(r) = \frac{1}{1 + \alpha_s \lambda} \left(\frac{\alpha_s \lambda}{1 + \alpha_s \lambda} \right)^r.$$

The probability of execution of a buyer's order, given that there are k additional buyers and r sellers, is

$$\pi_b(k, r) = \begin{cases} \frac{r}{k+1} & \text{if} \quad r \le k \\ 1 & \text{if} \quad r > k. \end{cases}$$

The conditional probability of order execution, given that there are k additional buyers, is

$$E(\pi_b \mid k) = \sum_{r=0}^{k} \frac{r}{k+1} \, p_s(r) + 1 \cdot p_s(r > k) = \sum_{r=0}^{k} \frac{r}{k+1} \, p_s(r) + 1 - \sum_{r=0}^{k} p_s(r)$$

$$= 1 - \sum_{r=0}^{k} \left(1 - \frac{r}{k+1} \right) p_s(r) = 1 - \frac{1}{1 + \alpha_s \lambda} \left[\sum_{r=0}^{k} q_s^r - \frac{1}{k+1} \sum_{r=0}^{k} r \, q_s^r \right],$$

where $q_s := \frac{\alpha_s \lambda}{1 + \alpha_s \lambda}$. Using

$$1 - q_s = \frac{1}{1 + \alpha_s \lambda}, \qquad \sum_{r=0}^{k} q^r = \frac{1 - q^{k+1}}{1 - q} \quad \text{and} \quad \sum_{r=0}^{k} r \, q^r = \frac{q \left[1 - q^k \left(k + 1 - q \, k \right) \right]}{(1 - q)^2}$$

we find that

$$\mathrm{E}(\pi_b \mid k) = q_s^{k+1} + \frac{\alpha_s \lambda}{k+1} \left[1 - q_s^k \left(k + 1 - q_s \, k \right) \right]$$

$$= q_s^{k+1} + \alpha_s \, \lambda \left[\frac{1 - q_s^{k+1}}{k+1} - q_s^k + q_s^{k+1} \right] = q_s^{k+1} \left(1 + \alpha_s \, \lambda \right) + \alpha_s \, \lambda \left[\frac{1 - q_s^{k+1}}{k+1} - q_s^k \right].$$

The probability of order execution is

$$\mathrm{E}(\pi_b) = \sum_{k=0}^{\infty} \mathrm{E}(\pi_b \mid k) \, p_b(k)$$

$$= \sum_{k=0}^{\infty} \left[q_s^{k+1} \left(1 + \alpha_s \, \lambda \right) + \alpha_s \, \lambda \left(\frac{1 - q_s^{k+1}}{k+1} - q_s^k \right) \right] \frac{1}{1 + \alpha_b \, \lambda} \, q_b^k$$

where $q_b := \frac{\alpha_b \lambda}{1 + \alpha_b \lambda}$. This equals

$$\frac{1 + \alpha_s \, \lambda}{1 + \alpha_b \, \lambda} \sum_{k=0}^{\infty} q_s^{k+1} \, q_b^k + \frac{\alpha_s \, \lambda}{1 + \alpha_b \, \lambda} \sum_{k=0}^{\infty} \left(\frac{q_b^k}{k+1} - \frac{q_s^{k+1} \, q_b^k}{k+1} - q_s^k \, q_b^k \right)$$

$$= \frac{1 + \alpha_s \, \lambda}{1 + \alpha_b \, \lambda} \, q_s \sum_{k=0}^{\infty} q_s^k \, q_b^k + \frac{\alpha_s \, \lambda}{1 + \alpha_b \, \lambda} \left[\frac{1}{q_b} \sum_{k=1}^{\infty} \frac{q_b^k - q_s^k \, q_b^k}{k} - \sum_{k=0}^{\infty} q_s^k \, q_b^k \right] = \frac{\alpha_s}{\alpha_b} \sum_{k=1}^{\infty} \frac{q_b^k - q_s^k \, q_b^k}{k}.$$

Using

$$\sum_{k=1}^{\infty} q^k / k = - \ln(1 - q)$$

we find that

$$\mathrm{E}(\pi_b) = \frac{\alpha_s}{\alpha_b} \left[\ln(1 - q_b \, q_s) - \ln(1 - q_b) \right] = \frac{\alpha_s}{\alpha_b} \ln \frac{1 + (\alpha_s + \alpha_b) \, \lambda}{1 + \alpha_s \, \lambda} = \frac{\alpha_s}{\alpha_b} \ln \left(1 + \frac{\alpha_b \, \lambda}{1 + \alpha_s \, \lambda} \right).$$

Execution probability for sell orders is calculated accordingly by changing subscripts b and s.
\hfill QED

Proof of Proposition 1

A strategy combination a^* is a Nash equilibrium iff either

$$\tilde{U}(\theta, \alpha(\theta, a^*)) > 0 \quad \wedge \quad a^{*i}(\theta, b) = a^{*i}(\theta, s) = 1 \tag{32}$$

$$\text{or} \quad \tilde{U}(\theta, \alpha(\theta, a^*)) < 0 \quad \wedge \quad a^{*i}(\theta, b) = a^{*i}(\theta, s) = 0 \tag{33}$$

$$\text{or} \quad \tilde{U}(\theta, \alpha(\theta, a^*)) = 0 \tag{34}$$

If $t_B < \theta < t_A$, then $\tilde{U}(\theta, \alpha) > 0$ for all α. This excludes (33) and (34), while (32) holds. So, in equilibrium $\alpha(\theta, a^*) = 1$.

If $\theta = t_A$, then $\tilde{U}(\theta, \alpha) \geq 0$ for all α. This excludes (33). (32) holds, and (34) is equivalent to $\tilde{\pi}(\alpha) = 0 \Leftrightarrow \alpha(\theta, a^*) = 0$. There are two equilibria with market shares of zero and one for the CN.

If $\theta > t_A$, (33) holds for all $\theta > t_A$. (32) requires $(\theta - t_B)\tilde{\pi}(1) > \theta - t_A$, which is equivalent to $\theta < \theta_0$. (34) $\Leftrightarrow (\theta - t_B)\tilde{\pi}(\alpha) = \theta - t_A \Leftrightarrow \tilde{\pi}(\alpha) = \frac{\theta - t_A}{\theta - t_B}$. $\tilde{\pi}$ is a continuous and increasing function in α reaching from zero to $\bar{\pi}$. Hence, there is a unique solution $\tilde{\alpha}(\theta) = \tilde{\pi}^{-1}\left(\frac{\theta - t_A}{\theta - t_B}\right) \leq 1$ for all $\theta \leq \theta_0$. For $\theta > \theta_0$, there is no solution to (34) with $\alpha \in [0, 1]$. $\hspace{2cm}$ QED

Proof of Theorem 1

For uniform distribution of values of trade and signals and geometric distribution of market size,

$$\pi(\theta, I_x) = \begin{cases} \bar{\pi} & \text{if} \quad \theta < x - \epsilon \\ \ln\left(1 + \frac{(x - \theta + \epsilon)\lambda}{2\epsilon + (x - \theta + \epsilon)\lambda}\right) & \text{if} \quad x - \epsilon \leq \theta \leq x + \epsilon \\ 0 & \text{if} \quad \theta > x + \epsilon. \end{cases}$$

and

$$\tilde{U}(x, I_x) = \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} (\theta - t_B)\pi(\theta, I_x)\,\mathrm{d}\theta - x + t_A.$$

Probability $\pi(\theta, I_x)$ depends on the difference between x and θ and the integral is evaluated around x. Substituting α for $\frac{x - \theta + \epsilon}{2\epsilon}$, we find

$$\tilde{U}(x, I_x) = \int_0^1 (x + \epsilon - 2\epsilon\alpha - t_B)\tilde{\pi}(\alpha)\,\mathrm{d}\alpha - x + t_A$$

and

$$\frac{\tilde{U}}{x} = \int_0^1 \tilde{\pi}(\alpha)\,\mathrm{d}\alpha - 1 < 0.$$

As $\tilde{U}(x, I_x)$ is strictly decreasing in x, there is a unique x^* with $\tilde{U}(x^*, I_{x^*}) = 0$. QED

Proof of Corollary 2

From (24), the derivatives of x^* w.r.t. t_A, t_B, and λ are obvious. Differentiating (24) yields

$$\frac{\mathrm{d}x^*}{\mathrm{d}\epsilon} = \frac{\int_0^1 (1 - 2\,\alpha)\,\tilde{\pi}(\alpha)\,\mathrm{d}\alpha}{\int_0^1 (1 - \tilde{\pi}(\alpha))\,\mathrm{d}\alpha}.$$

As $\tilde{\pi}(\alpha) < 1$ for all $\alpha \in [0, 1]$, the denominator is positive. Splitting the numerator up, substituting y for $1 - \alpha$ in the second integral and rearranging terms gives

$$\int_0^1 (1 - 2\,\alpha)\,\tilde{\pi}(\alpha)\mathrm{d}\alpha = \int_0^{1/2} (1 - 2\,\alpha)\,\tilde{\pi}(\alpha)\mathrm{d}\alpha - \int_0^{1/2} (1 - 2\,y)\,\tilde{\pi}(1 - y)\,\mathrm{d}y$$

$$= \int_0^{1/2} (1 - 2\,\alpha)\,[\tilde{\pi}(\alpha) - \tilde{\pi}(1 - \alpha)]\,\mathrm{d}\alpha$$

$$= \int_0^{1/2} (1 - 2\,\alpha)\,\ln\frac{1 + \lambda + 2\,\alpha\,(1 - \alpha)\,\lambda^2 + \alpha\,\lambda}{1 + \lambda + 2\,\alpha\,(1 - \alpha)\,\lambda^2 + (1 - \alpha)\,\lambda}\,\mathrm{d}\alpha < 0$$

For $\alpha \in [0, 1/2]$ the numerator is smaller than the denominator. Hence, logarithm and integral are negative. This proofs that x^* falls with rising ϵ. QED

Proof of Corollary 3

Corollaries 1 and 2 imply $x^* < x_0^*$. From Theorem 1 we know that $\tilde{U}(x^*, I_{x^*}) = 0$.

$$\tilde{U}(x^*, I_{x^*}) = (x^* - \epsilon - t_B)\,\pi^* + 2\,\epsilon \int_0^1 (1 - \alpha)\,\tilde{\pi}(\alpha)\,\mathrm{d}\alpha - x^* + t_A = 0.$$

As the integral is positive,

$$(x^* - \epsilon - t_B)\,\pi^* - x^* + t_A < 0 \quad \Leftrightarrow \quad x^* > x_0^* - \epsilon\,\frac{\pi^*}{1 - \pi^*}.$$

<div align="right">QED</div>

Proof of Proposition 3

The efficient switching point k^* maximizes $\mathrm{E}(U^i(I_k))$ over k.

$$\mathrm{E}(U^i(I_k)) = \frac{1}{\hat\theta - \check\theta} \int_{\check\theta}^{\hat\theta} F(k|\theta)\,(\theta - t_B)\,\tilde\pi(F(k|\theta)) + (1 - F(k|\theta))\,(\theta - t_A)\,\mathrm{d}\theta.$$

$$= \frac{1}{\hat\theta - \check\theta} \left[\int_{\check\theta}^{k-\epsilon} (\theta - t_B)\,\bar\pi\,\mathrm{d}\theta + \int_{k+\epsilon}^{\hat\theta} (\theta - t_A)\,\mathrm{d}\theta \right.$$

$$\left. + \int_{k-\epsilon}^{k+\epsilon} \frac{k - \theta + \epsilon}{2\,\epsilon}\,(\theta - t_B)\,\tilde\pi\left(\frac{k - \theta + \epsilon}{2\,\epsilon}\right) + \frac{\epsilon - k + \theta}{2\,\epsilon}\,(\theta - t_A)\,\mathrm{d}\theta \right]$$

$$= \frac{1}{\hat\theta - \check\theta} \left[\int_{\check\theta}^{k-\epsilon} (\theta - t_B)\,\bar\pi\,\mathrm{d}\theta + \int_{k+\epsilon}^{\hat\theta} (\theta - t_A)\,\mathrm{d}\theta \right.$$

$$\left. + 2\,\epsilon \int_0^1 \alpha\,(k + \epsilon - 2\,\epsilon\,\alpha - t_B)\,\tilde\pi(\alpha) + (1 - \alpha)\,(k + \epsilon - 2\,\epsilon\,\alpha - t_A)\,\mathrm{d}\alpha, \right]$$

where we substituted α for $\frac{k-\theta+\epsilon}{2\,\epsilon}$. If there is an interior optimum k^*, the derivative $\mathrm{dE}(U^i(I_k))/\mathrm{d}k$ equals zero at k^*.

$$\frac{\mathrm{dE}(U^i(I_k))}{\mathrm{d}k} = \frac{1}{\hat\theta - \check\theta} \left[(k - \epsilon - t_B)\,\bar\pi - k - \epsilon + t_A + 2\,\epsilon \int_0^1 \alpha\,\tilde\pi(\alpha) + (1 - \alpha)\,\mathrm{d}\alpha \right].$$

Setting the derivative to zero and solving for k gives (26). QED

Proof of Theorem 2

Given uniform distribution of values of trade and signals, the proof of Theorem 1 shows that $\tilde U(x, I_x)$ is strictly decreasing in x. At the equilibrium switching signal

<div align="center">35</div>

$\tilde{U}(x^*, I_{x^*}) = 0$. Hence, $x^* < k^*$ is equivalent to $\tilde{U}(k^*, I_{k^*}) < 0$. Using (24), (25) and (26) we find

$$\tilde{U}(k^*, I_{k^*}) = \int_0^1 \left[\theta_0 - \epsilon \frac{\bar{\pi} - 2 \int_0^1 \alpha\, \tilde{\pi}(\alpha)\, \mathrm{d}\alpha}{1 - \bar{\pi}} + \epsilon(1 - 2\,\alpha) - t_B \right] \tilde{\pi}(\alpha)\, \mathrm{d}\alpha$$

$$-\theta_0 + \epsilon \frac{\bar{\pi} - 2 \int_0^1 \alpha\, \tilde{\pi}(\alpha)\, \mathrm{d}\alpha}{1 - \bar{\pi}} + t_A$$

$$= \theta_0\, (\pi^* - 1) - \pi^*\, t_B + t_A + \frac{\epsilon\, \Delta}{1 - \bar{\pi}}$$

where $\Delta = \pi^*\, (1 - 2\,\bar{\pi}) + \bar{\pi} - 2\, (2 - \pi^* - \bar{\pi}) \int_0^1 \alpha\, \tilde{\pi}(\alpha)\, \mathrm{d}\alpha$. Henceforth, $x^* < k^*$ is equivalent to

$$\frac{\epsilon\, \Delta}{1 - \bar{\pi}} < \theta_0\, (1 - \pi^*) - t_A + \pi^*\, t_B$$

$$\Leftrightarrow \quad \epsilon\, \Delta < (t_A - \bar{\pi}\, t_B)\, (1 - \pi^*) - (1 - \bar{\pi})\, (t_A - \pi^*\, t_B) = (t_A - t_B)\, (\bar{\pi} - \pi^*)$$

Given our assumption that $\epsilon < t_A - t_B$, a sufficient condition for this is

$$\Delta < \bar{\pi} - \pi^*$$

$$\Leftrightarrow \quad (1 - \bar{\pi})\, \pi^* < (2 - \pi^* - \bar{\pi}) \int_0^1 \alpha\, \tilde{\pi}(\alpha)\, \mathrm{d}\alpha.$$

$$\Leftarrow \quad 2\, (1 - \bar{\pi}) < 2 - \pi^* - \bar{\pi} \quad \wedge \quad \frac{\pi^*}{2} < \int_0^1 \alpha\, \tilde{\pi}(\alpha)\, \mathrm{d}\alpha.$$

$$\Leftrightarrow \quad -\bar{\pi} < -\pi^* \quad \wedge \quad \int_0^1 \tilde{\pi}(\alpha)\, \mathrm{d}\alpha < 2 \int_0^1 \alpha\, \tilde{\pi}(\alpha)\, \mathrm{d}\alpha.$$

$$\Leftrightarrow \quad \bar{\pi} > \pi^* \quad \wedge \quad \int_0^1 (1 - 2\,\alpha)\, \tilde{\pi}(\alpha)\, \mathrm{d}\alpha < 0.$$

These inequalities follow from monotonicity of $\tilde{\pi}$. The second has been proved above in Corollary 2. $\hfill$ QED

Proof of Lemma 3

Let a^* be a Nash equilibrium of the private value game. $a^{i*}(b) = 1[0]$ if $\tilde{U}_b(\theta^i, a^*) > [<]\, 0$ and $a^{i*}(s) = 1[0]$ if $\tilde{U}_s(\theta^i, a^*) > [<]\, 0$. $\tilde{U}_b(\theta^i, a)$ and $\tilde{U}_s(\theta^i, a)$ are positive for $\theta^i < t_A$ and negative for $\theta^i > \theta_0$ for all a. As $\hat{U}_b$ and $\tilde{U}_s$ are strictly decreasing in θ^i, there are a $\theta_b^*(a^*)$ and a $\theta_s^*(a^*)$, such that $a^{i*}(b) = 1[0]$ if $\theta^i < [>]\, \theta_b^*(a^*)$ and $a^{i*}(s) = 1[0]$ if $\theta^i < [>]\, \theta_s^*(a^*)$.

Suppose $\theta_b^*(a) < \theta_s^*(a)$. Then $\alpha_b(a) < \alpha_s(a)$, and $\pi_b(a) > \pi_s(a)$. Then $\tilde{U}_b^i(a) > \tilde{U}_s^i(a)$ for all i. This implies $\theta_b^*(a) > \theta_s^*(a)$ which is a contradiction to the inequality above. Hence, $\theta_b^*(a) = \theta_s^*(a)$. $\hfill$ QED

Proof of Theorem 3

Since $\hat{U}(\theta)$ is continuous and positive for $\theta < t_A$ and negative for $\theta > \theta_0$, it is sufficient to show that in equilibrium $\hat{U}$ is strictly decreasing.

$$\frac{d\hat{U}(\theta)}{d\theta} = \tilde{\pi}(F(\theta)) - 1 + (\theta - t_B)\, \frac{\partial \tilde{\pi}}{\partial \alpha}\, f(\theta). \tag{35}$$

For geometric distribution of the number of buyers and sellers and uniform distribution of private values,

$$\frac{d\hat{U}(\theta)}{d\theta} = \tilde{\pi}(\alpha) - 1 + \frac{\theta - t_B}{\hat{\theta} - t_B}\, \frac{\lambda}{(1 + 2\,\alpha\,\lambda)\,(1 + \alpha\,\lambda)}.$$

$\frac{\theta - t_B}{\hat{\theta} - t_B} = \alpha$ and in equilibrium $\tilde{\pi}(\alpha) = \frac{\theta - t_A}{\theta - t_B}$.

$$\left.\frac{d\hat{U}(\theta)}{d\theta}\right|_{\theta^*} = \frac{\theta - t_A}{\theta - t_B} - 1 + \frac{\alpha\,\lambda}{1 + 3\,\alpha\,\lambda + 2\,\alpha^2\,\lambda^2} = -\frac{t_A - t_B}{\theta - t_B} + \frac{\alpha\,\lambda}{1 + 3\,\alpha\,\lambda + 2\,\alpha^2\,\lambda^2}.$$

$$\left.\frac{d\hat{U}(\theta)}{d\theta}\right|_{\theta^*} < 0 \quad \Leftrightarrow \quad \frac{t_A - t_B}{\hat{\theta} - t_B} > \frac{\theta - t_B}{\hat{\theta} - t_B}\, \frac{\alpha\,\lambda}{1 + 3\,\alpha\,\lambda + 2\,\alpha^2\,\lambda^2}$$

$$\Leftrightarrow \quad (1 + 3\,\alpha\,\lambda + 2\,\alpha^2\,\lambda^2)\, \frac{t_A - t_B}{\hat{\theta} - t_B} > \alpha^2\,\lambda$$

$$\Leftarrow \quad 2\,\alpha^2\,\lambda^2\,\frac{t_A - t_B}{\hat{\theta} - t_B} \geq \alpha^2\,\lambda \quad \Leftrightarrow \quad \frac{t_A - t_B}{\hat{\theta} - t_B}\,\lambda \geq \frac{1}{2}.$$

<div align="right">QED</div>

Proof of Theorem 4

The efficient threshold θ^{**} maximizes (over k)

$$\mathrm{E}(U(\theta^i, a_k)) = \int_{t_B}^{k} f(\theta)\,(\theta - t_B)\,\tilde{\pi}(F(k))\,\mathrm{d}\theta + \int_{k}^{\hat{\theta}} f(\theta)\,(\theta - t_A)\,\mathrm{d}\theta.$$

The first order condition implies

$$(\theta^{**} - t_B)\,\tilde{\pi}(F(\theta^{**})) + \int_{\check{\theta}}^{\theta^{**}} f(\theta)\,(\theta - t_B)\,\tilde{\pi}'(\cdot)\,\mathrm{d}\theta - \theta^{**} + t_A = 0. \tag{36}$$

In equilibrium $\hat{U}(\theta^*) = 0$. If there is a unique equilibrium θ^*, the derivative of $\hat{U}$ is negative at equilibrium. Therefore, $\theta^* < \theta^{**}$ if and only if $\hat{U}(\theta^{**}) < 0$. (30) and (36) imply

$$\hat{U}(\theta^{**}) = (\theta^{**} - t_B)\,\tilde{\pi}(F(\theta^{**})) - \theta^{**} + t_A = -\int_{\check{\theta}}^{\theta^{**}} f(\theta)\,(\theta - t_B)\,\tilde{\pi}'(\cdot)\,\mathrm{d}\theta < 0.$$

<div align="right">QED</div>

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