Coherent Sheaves on Calabi-Yau manifolds, Picard-Fuchs equations and potential functions

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Kathrin Natalie Peternell

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Dekan der Fakultät für Mathematik und Physik
Prof. Dr. Gregor Herten

Erstgutachter
Priv.-Doz. Dr. Emanuel Scheidegger

Zweitgutachter
Prof. Sheldon Katz, PhD

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Chapter 1

Introduction

Calabi-Yau manifolds and the general setting. Calabi-Yau manifolds are simply-connected compact projective manifolds X with trivial canonical bundle $K_X = \Omega_X^n$, where $n = \dim X$, and the additional property that

$$H^0\left(X, \Omega_X^j\right) = 0 \text{ for } 2 \leqslant j \leqslant n-1.$$

By Yau's solution of the Calabi conjecture, X carries a Kähler metric with vanishing Ricci curvature.

Calabi-Yau manifolds are not only central objects in complex geometry, but also play an important role in string theory. In fact, much of the mathematical theory of Calabi-Yau manifolds has its origin in physics, e.g. mirror symmetry. Calabi-Yau manifolds themselves have a special property in deformation theory: their first-order deformations are unobstructed, although the obstructions live in a space $H^2(X, T_X)$ which is not 0. So their local moduli are smooth of dimension $\dim H^1(X, T_X)$.

Motivated by the study of D-branes in string theory, physicists started to study deformations of geometric objects on Calabi-Yau manifolds. For each joint moduli problem, e.g. the moduli problem of X together with a coherent sheaf on it, one has to specify the notion of a family of geometric objects. For example, a simultaneous deformation of a complex manifold X and a coherent sheaf $\mathcal{F}$ on X consists of a deformation $\mathcal{X} \to S$ of X, where $\mathcal{X}$ and S are complex spaces, and a coherent sheaf $\tilde{\mathcal{F}}$ on $\mathcal{X}$, flat over S. These types of deformations are subject of this thesis.

Picard-Fuchs equations and potential functions. Given a Calabi-Yau 3-fold X and a geometric object, say A, we are interested in the joint "local moduli space" $\mathcal{M} = \mathcal{M}(X, A)$ of A and X. This local moduli space is realized as a subspace of the Zariski tangent space $\mathcal{T} \cong \mathbb{C}^N$ with $N := \dim \mathcal{T}_{(X,A)}\mathcal{M}$ of $\mathcal{M}$ at A, which is the space of first-order deformations.

Thus, given a first-order deformation, one tries to extend it order by order. However in general there are obstructions to do this, and to understand how $\mathcal{M}$ sits in $\mathcal{T}$ is the same as to understand the obstructions. So $\mathcal{M}$ parametrizes – up to convergence – the deformations that are unobstructed to each order.

The hope is then to have a so-called holomorphic potential function $f : U \to \mathbb{C}$ defined on an open set $U \subset \mathcal{T}$ such that the critical locus of f is $\mathcal{M}$.

More specifically, consider the space of first-order deformations $\mathcal{T}(X, A)$ of X and A, and the space of first-order deformations $\mathcal{T}(X) \cong H^1(X, T_X)$ of X, leading to a forgetful morphism $\mathcal{T}(X, A) \to \mathcal{T}(X)$. Then there should be open neighbourhoods $U \subset \mathcal{T}(X, A)$ and $V \subset \mathcal{T}(X)$ of 0 and a holomorphic function $f : U \to \mathbb{C}$ such that

$$\left\{d_{U/V} f = 0\right\} = \mathcal{M}(X, A) \cap U. \tag{1.0.0.1}$$

The technical tool to treat infinitesimal deformations are functors of Artin rings. These functors associate to an Artin ring the space of deformations of the given object over the spectrum of the Artin ring. In this setting, obstructions are cohomology classes that vanish if a deformation over the spectrum of an Artin ring can be extended to a deformation over the spectrum of a larger Artin ring. For geometric deformation problems, usually the space of first-order deformations is given by the first cohomology group (or Ext-group) of a certain coherent sheaf. The space of obstructions is contained in the second cohomology of the same coherent sheaf. In most cases, the obstructions of a deformation problem are difficult to compute. Therefore it is useful to have a potential function that gives information about the obstructions.

Physicists propose that in some situations such a potential function should exist as a solution of a certain differential equation associated with the deformation problem, the Picard-Fuchs equation. In the special case of a family of Calabi-Yau manifolds, the Picard-Fuchs equation is satisfied by all periods induced by the variation of the complex structure that is related to the given deformation problem. Furthermore, the potential function should be a generating function of the obstructions. We explain this in more detail below.

In several cases potential functions are known to exist, although not all expected properties are established yet. For example, sometimes just the existence of a holomorphic potential function is known, but it is still unknown whether it is a solution of a differential equation with specific properties.

The existence problem for potential functions is therefore closely related to the task of constructing Picard-Fuchs equations associated with deformation problems. These Picard-Fuchs equations are differential equations arising from certain Gauß-Manin connections. We explain this in the simplest possible situation.

Let $\mathcal{X} \to T$ be a deformation of a Calabi-Yau n-fold over a complex manifold T, i.e., a family of Calabi-Yau manifolds, and let Ω be a family of holomorphic n-forms on $\mathcal{X}$. Then $R^n\pi_*\mathbb{Z}$ is a local system and the holomorphic vector bundle associated with this local system carries a canonical connection, the Gauß-Manin connection. Now we apply repeatedly the Gauß-Manin connection to obtain sections

$$\nabla^0 [\Omega], \nabla^1 [\Omega], \ldots, \nabla^m [\Omega] \in H^0 (T, R^n\pi_*\mathbb{Z} \otimes \mathcal{O}_T).$$

As the n-th cohomology of the fibres is finite-dimensional, these classes will be linearly dependent if m is large enough. The resulting equation is called the Picard-Fuchs equation associated with the family $\mathcal{X}$.

For families of Calabi-Yau manifolds that are special complete intersections in (weighted) projective spaces the Picard-Fuchs equation can be calculated explicitly using the so-called Griffiths-Dwork method. This method was first introduced by Griffiths in [Gri69] for Calabi-Yau hypersurfaces in a projective space. Using a residue map it is possible to represent classes in the $(n-1)$-th cohomology $H^{n-1}(X, \mathbb{C})$ of a Calabi-Yau hypersurface X in $\mathbb{P}^n$ by rational forms on $\mathbb{P}^n$ with poles along X. The Griffiths-Dwork method uses a correspondence between the Hodge filtration on $H^{n-1}(X, \mathbb{C})$ and a filtration given by the pole order on a complex of rational forms. On the level of these rational forms one constructs the requested linear combination.

The case of complete intersections of codimension 2 was first established by Libgober and Teitelbaum in dimension 5 in [LT93]. Strictly speaking, the Picard-Fuchs equation is calculated for the mirror or a finite quotient of the Calabi-Yau manifold.

We will now explain this in case of a quintic $X \subset \mathbb{P}^4$. Then the Hodge numbers are easily computed as

$$h^{1,1}(X) = 1 \text{ and } h^{1,2}(X) = 101.$$

This creates a problem, since the Picard-Fuchs equation is calculated in $H^3(X, \mathbb{C})$, which has dimension 204. Therefore the number m appearing in the Griffiths-Dwork procedure will be large. To avoid this difficulty, one considers special families of quintics X admitting a certain action of a finite group G, and the Picard-Fuchs equation can be computed using G-invariant cohomology. This can be reinterpreted as working on the mirror $\check{X}$ instead of X. On $\check{X}$ the Hodge numbers are interchanged so that

$$h^{1,1}\left(\check{X}\right) = 101 \text{ and } h^{1,2}\left(\check{X}\right) = 1.$$

Consequently $\dim H^3\left(\check{X},\mathbb{C}\right) = 4$. Alternatively, one may argue on X/G. In this context,

$$h^{1,2}(X/G) = h^{1,2}(X)^G = 1 \text{ and } \dim H^3(X/G,\mathbb{C}) = \dim H^3(X,\mathbb{C})^G = 4.$$

The general task, as proposed by examples from physics, is to study Picard-Fuchs equations and potential functions for pairs (X, A) as at the beginning of our discussion, where A is now a coherent sheaf $\mathcal{F}$. This might be hopeless for a general coherent sheaf; in particular there seems to be no Hodge theory and therefore no Gauß-Manin connection available. Instead of a general coherent sheaf, one might choose for instance vector bundles of rank two or coherent sheaves $\iota_* \mathcal{O}_Y$ for a subspace Y in X with inclusion $\iota : Y \hookrightarrow X$. This leads to the discussion of Picard-Fuchs equations for pairs (X, D) consisting of a Calabi-Yau 3-fold X and a smooth divisor D. The Zariski tangent space to the first-order deformations $\operatorname{Def}(X, D)$ is given by

$$\operatorname{Def}(X,D) \cong H^1(X, T_X(-\log D)),$$

which is dual to $H^2\left(X, \Omega_X^1(\log D)\right)$. So the deformations of the pair (X, D) are governed by the logarithmic complex $\Omega_X^\bullet(\log D)$.

The connection to Hodge theory is therefore given by the mixed Hodge structure on the hypercohomology of the logarithmic complex, which computes the relative de Rham cohomology $H^3(X, D, \mathbb{C})$. In fact, by Deligne's theorem we obtain

$$H^3(X,D,\mathbb{C})^\vee \cong \mathbb{H}^3(X, T_X^\bullet(-\log D)) \cong \bigoplus_{p+q=3} H^p(X, T^q(-\log D)).$$

A family of deformations of a pair (X, D) thus gives a Gauß-Manin connection and hence a Picard-Fuchs equation. The Griffiths-Dwork method was first used by Jockers and Soroush in [JS09a] and [JS09b] for calculating Picard-Fuchs equations and candidates of potential functions for pairs. The mathematical theory for the Griffiths-Dwork reduction of pairs was carried out by Li, Lian and Yau in [LLY12].

There are also methods from toric geometry to obtain Picard-Fuchs equations for pairs (X, D), introduced by Lerche, Mayr and Warner in [LMW02a] and [LMW02b]. These authors derived a candidate for a potential function for deformations of non-compact toric Calabi-Yau 3-folds X with toric divisors satisfying additional symmetry assumptions arising from $\mathcal{N} = 1$-special geometry. They used a mixed Hodge structure on the relative cohomology of pairs.

Subsequently, in [AHMM09], [AHJ$^+$10], [AHJ$^+$11] and [AHJ$^+$12] Alim, Hecht, Mayr and Mertens transferred the ideas of Lerche, Mayr and Warner to compact Calabi-Yau manifolds and derived systems of Picard-Fuchs operators. In this context, Morrison and Walcher discovered in [MW09] a different differential equation which is solved by Hodge-theoretic normal functions. These normal functions are obtained by integrating a holomorphic 3-form over a certain cycle.

It has to be mentioned that for deformations even of complexes of coherent sheaves on a *fixed* Calabi-Yau 3-fold Brav, Bussi and Joyce showed in [BBBBJ15] that there is a holomorphic function whose critical locus gives the unobstructed deformations. However it is not known that this function is the solution of a differential equation and there is in general no explicit construction of the function.

Chern-Simons functional. Another situation is known in which a potential function was found by completely different methods. To explain this, we first recall the real Chern-Simons functional on the space of connections of a fixed real 3-manifold. The critical locus of the real Chern-Simons functional is nothing but the subspace of flat connections.

In analogy to the real situation, Thomas [Tho00] developed a holomorphic Chern-Simons functional. Since in this setting it will be apparent how generating functions of the obstructions come into the picture, we will give a very detailed exposition.

To begin with, let $E \to X$ be a complex differentiable vector bundle on a fixed Calabi-Yau 3-fold X, and let $\mathcal{A}$ be the space of $\overline{\partial}$-operators on E. The holomorphic structures on $E \to X$ correspond to those $\overline{\partial}$-operators $\overline{\partial}_E$ on E satisfying $\overline{\partial}_E \circ \overline{\partial}_E = 0$, i.e., to those $\overline{\partial}_E$ that are integrable. We assume that $E \to X$ is a holomorphic vector bundle and fix a holomorphic structure $\overline{\partial}_0$ on E. Then $\mathcal{A}$ is an affine space with associated vector space $\mathcal{A}^{0,1}(\mathrm{End}(E))$. As usual $\mathcal{A}^{0,1}(\mathrm{End}(E))$ denotes the sheaf of $(0,1)$-forms with values in $\mathrm{End}(E)$.

Then the holomorphic Chern-Simons functional for E will be a holomorphic function on $\mathcal{A}$, given as follows

$$\mathrm{CS}_{\overline{\partial}_0}\left(\overline{\partial}_0 + a\right) := \frac{1}{4\pi^2}\int_X \mathrm{tr}\left(\frac{1}{2}\overline{\partial}_0 a \wedge a + \frac{1}{3} a \wedge a \wedge a\right) \wedge \Omega \qquad (1.0.0.2)$$

for each $a \in \mathcal{A}^{0,1}(\mathrm{End}(E))$. Here Ω denotes a holomorphic non-vanishing 3-form on X and $\mathrm{tr} : \mathcal{E}nd(E) \to \mathcal{O}_X$ the trace map.

The functional $\mathrm{CS} = \mathrm{CS}_{\overline{\partial}_0}$ descends to a functional on $\mathcal{A}/\mathcal{G}_E$, where $\mathcal{G}_E$ denotes the gauge group of E, the space of complex linear $\mathcal{C}^\infty$-automorphisms of the vector bundle E. Then Thomas proves

$$\mathrm{Crit}(\mathrm{CS}) = \mathcal{A}^{\mathrm{int}},$$

where $\mathcal{A}^{\mathrm{int}}$ denotes the space of integrable $\overline{\partial}$-operators on $E \to X$. Thus the critical points of the Chern-Simons functional are the holomorphic structures on $E \to X$.

To continue, we observe that $a \in \mathrm{Crit}\left(\mathrm{CS}_{\overline{\partial}_0}\right)$ is equivalent to saying that a satisfies the Maurer-Cartan equation

$$\overline{\partial}_0 a + a \wedge a = 0.$$

We assume now additionally that E is simple, i.e. all holomorphic endomorphisms of E are just multiples of the identity. This assumption is satisfied in the important case that E is semistable for some ample polarization. Under this assumption even more is true: There exists an open neighbourhood U of $\overline{\partial}_0$ in $\mathcal{A}/\mathcal{G}_E$ such that the critical locus of $\mathrm{CS}_{\overline{\partial}_0}$ is a local moduli space of E, and U can be identified with an open neighbourhood V of 0 in $\mathrm{Ext}^1(E,E) \cong \mathrm{Def}(E)$, the space of first-order deformations of E.

At the level of tangent spaces we have

$$H^1\left(\mathcal{A}^{0,1}(\mathrm{End}(E)), \overline{\partial}_{\mathrm{End}E}\right) = H^1(X, \mathrm{End}E) = \mathrm{Ext}^1(E,E) \cong \mathrm{Def}(E).$$

The obstruction of extending a bundle infinitesimally from order n to $n+1$ can be described by so-called Massey products

$$r_n\left(a^{\otimes n}\right) \in \mathrm{Ext}^2(E,E),$$

where $a \in A := \oplus_n \Gamma\left(\mathcal{A}^{0,n}(\mathcal{H}om(E,E))\right)$ and $r_n : A^{\otimes n} \to A$ are morphisms subject to further restrictions – so-called product relations – making A into an $\mathcal{A}_\infty$-algebra. Then

$$H^1(A) = \mathrm{Ext}^1(E,E) \text{ and } H^2(A) = \mathrm{Ext}^2(E,E).$$

The $\mathcal{A}_\infty$-structure on A also induces an $\mathcal{A}_\infty$-structure on $H^*(X, \mathrm{End}(E))$.

Fix a Ricci-flat metric on X and a metric on E such that every class $a \in \mathrm{Ext}^1(E,E)$ is represented by a unique $\overline{\partial}_0$-harmonic form α. Then we define a potential function $\widehat{\mathrm{CS}}_{\overline{\partial}_0}$ on $\mathrm{Ext}^1(E,E)$ by setting

$$\widehat{\mathrm{CS}}_{\overline{\partial}_0}([\alpha]) = \sum_{n \geqslant 1} \frac{(-1)^{\frac{n(n+1)}{2}}}{n+1} \int_X \Omega \wedge \mathrm{tr}\left(a \wedge r_n\left(a^{\otimes n}\right)\right). \tag{1.0.0.3}$$

As $r_1(a) = \overline{\partial}_0 a$ and $r_2(a,a) = a \wedge a$, the functional $\widehat{\mathrm{CS}}_{\overline{\partial}_0}$ can be seen as an extension of the holomorphic Chern-Simons functional $\mathrm{CS}_{\overline{\partial}_0}$ by adding Massey products of higher order.

Formula (1.0.0.3) uses the non-degenerate pairing between the space of first-order deformations of $E \to X$ and the space that contains the obstructions, see Thomas in [Tho97], namely

$$\mathrm{Ext}^1(E,E) \times \mathrm{Ext}^2(E,E) \to \mathrm{Ext}^3(\mathcal{O}_X, K_X) \overset{\wedge\Omega}{\to} H^{3,3}(X) \overset{\int_X}{\to} \mathbb{C}.$$

Here the first map is given by Serre-Duality using $K_X \cong \mathcal{O}_X$. We emphasize that in the case of vector bundles, Serre-duality provides a non-degenerate pairing between the space of first-order deformations and obstructions. This is no longer the case for other deformation problems, e.g. for general coherent sheaves.

In this context the Maurer-Cartan equation generalizes to

$$\sum_{n\geqslant 1} r_n\left(a^{\otimes n}\right) = 0.$$

In [Laz01] and [Jia17] it is suggested that the critical loci of $\widehat{\mathrm{CS}}_{\bar{\partial}_0}$ and $\mathrm{CS}_{\bar{\partial}_0}$ agree up to $\mathcal{A}^{\infty}$-relations that should not affect the deformation theory. Nevertheless, the functional $\widehat{\mathrm{CS}}_{\bar{\partial}_0}$ carries richer information.

For other deformation problems, e.g. for pairs $(X, \mathcal{F})$, where $\mathcal{F}$ is a coherent sheaf which is not locally free, the pairing between the space of first-order deformations and their obstructions is missing. However, it is possible to look at the generalized Maurer-Cartan equation as generating function of the obstructions. The critical locus of this map gives the unobstructed first-order deformations.

In [MW09] Morrison and Walcher suggest a generalization of the Chern-Simons functional to arbitrary objects $B \in D^b(X)$ in the derived category of coherent sheaves on X. If the topological Chern class $c_2^{\mathrm{top}}(B) \in H^4(X, \mathbb{Z})$ of B vanishes, then the algebraic Chern class $c_2^{\mathrm{alg}}(B) \in \mathrm{CH}^2(\mathrm{X})$ in the Chow group of algebraic cycles of codimension 2 modulo rational equivalence yields a normal function $\nu_B = \nu_{c_2^{\mathrm{alg}}(B)} \in J^3(X)$, which is a holomorphic section of the Griffiths intermediate Jacobian fibration associated with the variation of Hodge structure satisfying certain properties. Morrison and Walcher study two algebraic Chern-classes $C_{+/-} := c_2^{\mathrm{alg}}\left(E_{+/-}\right) \in \mathrm{CH}^2(\mathrm{X})$ of two holomorphic vector bundles $E_{+/-}$ on a fixed quintic 3-fold X such that the homology class $[C_+ - C_-] = 0 \in H_2(X)$ vanishes. They derive a differential equation that is satisfied by the normal function which is defined as the period of a holomorphic 3-form on X over a 3-chain Γ with $\partial\Gamma = C_+ - C_-$. The differential equation they obtain coincides with the Picard-Fuchs equation for the quintic Calabi-Yau hypersurface. The normal function coincides with a potential function W restricted to the critical locus.

Potential function and Noether-Lefschetz locus. A similar situation is studied by Voisin in the appendix of [Cle05]. On a Calabi-Yau 3-fold X, let $\iota : D \hookrightarrow X$ be a smooth very ample divisor in X. We consider a class $\lambda \in H^2_{van}(D, \mathbb{Z}) \cap H^{1,1}(D)$, where

$$H^2_{van}(D, \mathbb{Z}) = \left\{ a \in H^2(D, \mathbb{Z}) \middle| \, \iota_*(a) = 0 \right\}.$$

Then the first-order deformations of (X, D) are unobstructed. Given a deformation $(\mathcal{X}, \mathcal{D}) = (X_s, D_s)_{s\in S}$ of (X, D) over a complex manifold S, there is a unique smooth family of cycles $\Lambda_S := (\lambda_s)_{s \in S}$ extending λ such that $\lambda_s \in H^2_{van}(D_s, \mathbb{Z})$ for each $s \in S$. However λ_s will in general no longer be of type $(1,1)$.

We consider the local moduli space R for (X, D). Then basically - up to a choice of 3-forms – Voisin constructs a holomorphic potential function

$$\phi_{NL} : R \to \mathbb{C}$$

such that the critical locus of ϕ_{NL} is the subspace of R where λ_s is of type $(1,1)$. This subspace is also called the Noether-Lefschetz locus. In this thesis, we interprete ϕ_{NL} as a potential function for a deformation problem.

In the same vein, in [Cle05] Clemens obtains a potential function for pairs (X, C) consisting of a Calabi-Yau 3-fold X and a smooth curve C.

Outline of the thesis. We explain now the content of the thesis in detail. After presenting some preliminaries in Chapter 2, we collect in Chapter 3 those basics in deformation theory that will be used later in this thesis. In particular, we formulate carefully all relevant deformation problems. Furthermore, we elaborate the proof of a well-known theorem (see e.g. [DF89]), stating that each deformation of the projective space $\mathbb{P}(E)$ of a holomorphic vector bundle E over a compact n-dimensional complex manifold X satisfying $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ is isomorphic to the projective fibre space of a locally free sheaf on a deformation of X. As a consequence it is possible to describe the deformations of X together with a vector bundle E in terms of the projective bundle $\mathbb{P}(E) \to X$. This will be used intensively in Chapter 6.

Chapter 4 recalls foundational notions from Hodge theory: pure and mixed Hodge structures and their variations. Then we introduce the notion of residues, which will form a basic tool in the Griffiths-Dwork theory. The connection between residues and mixed Hodge structures is provided by the hypercohomology on the complex of logarithmic differential forms. We also give a brief account for the de Rham cohomology of pairs (X, Z) consisting of a submanifold Z or a smooth divisor. For effective computations in specific examples, we will further need to consider residues for certain singular divisors as well as invariant cohomology in the presence of a finite group.

Chapter 5 offers a detailed discussion of two deformation problems and their relation. As already mentioned, physicists came up with examples of deformation problems for pairs $(X, \mathcal{F})$, where $\mathcal{F}$ is a special coherent sheaf on a Calabi-Yau 3-fold X. A first approach is the study of simultaneous deformations of X together with the direct image sheaf $\mathcal{F} = \iota_* \mathcal{O}_D$ of the trivial sheaf on a subspace given as a hypersurface D of X. We will see that situation is copnnected to the mixed Hodge structure on a cohomology group related to the deformation theory for pairs (X, D). This is the starting point of this thesis.

First we study the connection between the two deformation problems discussed last, even in a more general context.

To be specific, we consider a compact complex manifold X and a compact submanifold Z. In practise, X will be a Calabi-Yau 3-fold and Z a divisor. If $\iota : Z \hookrightarrow X$ denotes the inclusion map, then $\iota_* \mathcal{O}_Z$ is a coherent sheaf. So we relate the deformations of (X, Z) and $(X, \iota_* \mathcal{O}_Z)$:

1.0.1 Theorem. *(Theorem 5.1.1 in Chapter 5) Let Z be a closed submanifold of codimension d of a compact complex manifold X and $\iota : Z \hookrightarrow X$ be the inclusion. Then there is a natural linear isomorphism of simultaneous first-order deformations*

$$\mathrm{Def}\,(X, \iota_* \mathcal{O}_Z) \cong \mathrm{Def}\,(X, Z) \oplus H^1\,(Z, \mathcal{O}_Z)\,. \tag{1.0.1.1}$$

As a preparation, we observe that a simultaneous deformation of X and the structure sheaf $\iota_* \mathcal{O}_Z$ yields a coherent sheaf $\mathcal{F}$ on a deformation of X that is a locally free sheaf of rank 1 on $\mathrm{Supp}\,(\mathcal{F})$. This leads to the following exact sequence which is the basis of the proof of Theorem 1.0.1.

$$0 \to \mathrm{Def}\,(X, Z) \xrightarrow{\zeta} \mathrm{Def}\,(X, \iota_* \mathcal{O}_Z) \xrightarrow{\xi} H^1\,(Z, \mathcal{O}_Z) \to 0. \tag{1.0.1.2}$$

To establish the linearity of the maps ζ and ξ we observe that they extend to natural transformations of the corresponding deformation functors.

Concerning obstructions, we show

1.0.2 Theorem. *(Theorem 5.1.2 in Chapter 5) Assume that $H^1\,(Z, \mathcal{O}_Z) = 0$ or $H^2\,(Z, \mathcal{O}_Z) = 0$, e.g. Z is an ample divisor in a Calabi-Yau 3-fold or a smooth curve in a compact manifold. Then*

$$\mathrm{Obs}\,(X, \iota_* \mathcal{O}_Z) = \mathrm{Obs}\,(X, Z) \subset H^2\,(X, T_X \langle -Z \rangle)\,.$$

Furthermore, we replace the trivial line bundle $\mathcal{O}_Z \in \mathrm{Pic}\,(Z)$ by an arbitrary line bundle $L \in \mathrm{Pic}\,(Z)$ and examine to which extent Theorem 1.0.1 holds. This is relevant to obtain a Noether-Lefschetz-type theorem. In detail we show

1.0.3 Theorem. *(Theorem 5.4.2 in Chapter 5) Let $L \in \mathrm{Pic}\,(Z)$. Then there is a canonical morphism of vector spaces*

$$\Theta : \mathrm{Def}\,(X, \iota_* L) \to \mathrm{Def}\,(X, \iota_* \mathcal{O}_Z)\,.$$

1. *The map Θ is injective if and only if $H^1\,(Z, \mathcal{O}_Z) = 0$.*

2. *Θ is surjective if and only if, for each first-order deformation $(\mathcal{X}, \mathcal{Z})$ of (X, Z), there is a preimage of L under the restriction map $\mathrm{Pic}\,(\mathcal{Z}) \to \mathrm{Pic}\,(Z)$.*

3. If $H^2(Z, \mathcal{O}_Z) = 0$, then Θ is surjective.

We then derive the existence of a potential function for deformations of a pair $(X, \iota_* \mathcal{O}_D(C))$, where X is a Calabi-Yau 3-fold, D a very ample smooth divisor on X with inclusion $\iota : D \hookrightarrow X$ and C a divisor on D whose class is of type $(1,1)$ and vanishes in X. Let ω be holomorphic non-vanishing 3-form on X.

1.0.4 Theorem. *There are local open sets $\tilde{W}$ and $\tilde{Z}$ in the spaces of first-order deformations of the deformation problems $(X, \iota_* \mathcal{O}_D(C), \omega)$ and (X, ω) and a holomorphic map*

$$\psi_{NL} : \tilde{W} \to \mathbb{C}$$

such that the following property is satisfied:

$$\mathcal{M}_{\tilde{W}}(X, \iota_* \mathcal{O}_D(C), \omega) = \left\{ \tilde{w} \in \tilde{W} \middle| d_{\tilde{W}|\tilde{Z}} \psi_{NL}(\tilde{w}) = 0 \right\},$$

where $\mathcal{M}_{\tilde{W}}(X, \iota_* \mathcal{O}_D(C), \omega) \subset \tilde{W}$ denotes the space of unobstructed deformations of $(X, \iota_* \mathcal{O}_D(C), \omega)$ inside $\tilde{W}$ and $d_{\tilde{W}|\tilde{Z}}$ is the relative differential with respect to the projection $\tilde{W} \to \tilde{Z}$.

This will be a consequence of a theorem of Voisin, addressed above. Thus the Noether-Lefschetz locus is the critical locus of a potential function for the deformation problem of the pair $(X, \iota_* \mathcal{O}_D(C), \omega)$. In [AHJ$^+$11] for several examples it is shown that there exists a system of differential equations satisfying certain properties.

1.0.5 Corollary. *In the situation of the examples considered in [AHJ$^+$11] the function ψ_{NL} satisfies a system of Picard-Fuchs operators.*

Thus, for the deformation problem $(X, \iota_* \mathcal{O}_D(C), \omega)$ there exists a holomorphic potential function as a solution of a differential equation satisfying the property that its critical locus gives the unobstructed deformations. However, a representation as a generating function of the obstructions is not established yet.

We are now turning to the case of pairs (X, C), where C is a smooth curve in the Calabi-Yau 3-fold X, having in mind to possibly set up a Picard-Fuchs equation.

Chapter 6 studies special curves C, namely those curves arising as the zero-locus of a section in a vector bundle of rank 2 on X.

1.0.6 Theorem. *(Theorem 6.1.1 in Chapter 6) Let X be a Calabi-Yau 3-fold, $E \to X$ be a holomorphic vector bundle of rank 2 on X and $[s] \in$*

$\mathbb{P}\left(H^0(X,E)\right)$ *be the class of a holomorphic section* $s \in H^0(X,E)$ *such that* $C := \{s = 0\}$ *is a smooth connected curve.*

Then the space of first-order deformations $\operatorname{Def}(X,E,[s])$ *of the pair* $(X,E,[s])$ *and the space of their obstructions* $\operatorname{Obs}(X,E,[s])$ *satisfy the following properties:*
There is a locally free sheaf Q *of rank* 5 *on* X *such that*

$$\begin{aligned} \operatorname{Def}(X,E,[s]) &\cong \operatorname{Ext}^1(Q,\mathcal{O}_X), \\ \operatorname{Obs}(X,E,[s]) &\subseteq \operatorname{Ext}^2(Q,\mathcal{O}_X). \end{aligned}$$

In order to prove Theorem 1.0.6 we introduce the projectivised bundle $\mathbb{P}(E)$. Inside $\mathbb{P}(E)$ we consider the divisor $D := \mathbb{P}(\mathcal{J}_C \otimes \det E)$ which turns out to be the blow-up of X along C. The deformation theory of $(X,E,[s])$ coincides with the deformation theory of $(\mathbb{P}(E), D)$. Since

$$\operatorname{Def}(\mathbb{P}(E), D) \cong H^1\left(\mathbb{P}(E), T_{\mathbb{P}(E)}(-\log D)\right),$$

we define $Q := \pi_*\left(T_{\mathbb{P}(E)}(-\log D)\right)^\vee$, which turns out to be a locally free sheaf of rank 5 on X. The sheaf Q carries much more information, subsumed by

1.0.7 Theorem. *(Theorem 6.4.1 in Chapter 6) There are exact sequences*

$$0 \to \Omega^1_X \to Q \to E \otimes \mathcal{J}_C \to 0 \tag{1.0.7.1}$$

and

$$0 \to E^\vee \to Q^\vee \to T_X\langle -C\rangle \to 0, \tag{1.0.7.2}$$

where $T_X\langle -C\rangle$ *is a certain subsheaf of* T_X *controlling the deformations of the pair* (X,C)*.*

Next we obtain a relation between deformations of pairs (X,E) and triples $(X,E,[s])$.

1.0.8 Theorem. *(Theorem 6.4.2 in Chapter 6) We assume the setting of Theorem 1.0.7. The logarithmic tangent sequence*

$$0 \to T_{\mathbb{P}}(-\log D) \to T_{\mathbb{P}} \to \iota_*\mathcal{N}_{D|\mathbb{P}} \to 0$$

induces a sequence

$$0 \to Q^\vee \to \pi_*(T_{\mathbb{P}}) \to \mathcal{J}_C \otimes \det E \to 0, \tag{1.0.8.1}$$

which in cohomology gives

$$H^0(X, \mathcal{J}_C \otimes \det E) \to H^1(X, Q^\vee) \to H^1(X, \pi_* T_{\mathbb{P}}). \tag{1.0.8.2}$$

This sequence can be interpreted as the natural sequence of first-order deformations

$$\operatorname{Def}([s]) \to \operatorname{Def}(X,E,[s]) \to \operatorname{Def}(X,E).$$

We now discuss the case of a splitting bundle. Suppose that X is a Calabi-Yau 3-fold and D_1, D_2 are ample divisors on X meeting transversally in a smooth curve C and set $E := \mathcal{O}_X(D_1) \oplus \mathcal{O}_X(D_2)$.

1.0.9 Corollary. *(Corollary 6.3.7, Chapter 6) If $D_1 - D_2$ is ample, we have isomorphisms of spaces of first-order deformations:*

$$\operatorname{Def}(X, D_1, D_2) \cong \operatorname{Def}(X, E, [s]) \cong \operatorname{Def}(X, C)$$

Finally we consider the deformation problem for $(X, E, [s])$, where E is not necessarily a splitting bundle. We derive the existence of a holomorphic function whose critical locus contains the unobstructed deformations provided $H^1(X, \det E^\vee) = H^2(X, \det E^\vee) = 0$, e.g., if $\det E$ is ample. This is based on the existence of a potential function for the deformations of the pair (X, C) constructed by Clemens in [Cle05]. If $C \cong \mathbb{P}^1$, this function is a potential function in the strict sense. For details see Corollary 6.5.5 in Chapter 6.

Since we only know the existence of the potential function and not an explicit form, we study Picard-Fuchs equations in the remaining chapters, hoping that one will finally find potential functions as solutions of differential equations. Accessing a well-advanced theory of solutions for differential equations, one would obtain the explicit form.

In *Chapter 7* we turn to Picard-Fuchs equations and describe the Griffiths-Dwork reduction first for hypersurfaces in a projective space and then for complete intersections of codimension 2 in a projective space. We extend Libgober and Teitelbaum's theory to any dimension, fill gaps in their proof and show carefully that the Hodge filtration of the forms on the complement of the Calabi-Yau manifold is isomorphic to the filtration by pole order of rational forms with poles along the Calabi-Yau manifold. The precise formulation is as follows:

1.0.10 Theorem. *(Corollary 7.3.10 in Chapter 7) Let Y_1, Y_2 be hypersurfaces in $\mathbb{P}^n$ which intersect transversally. Let $d_1 := \deg(Y_1) = d_1$ and $d_2 := \deg(Y_2) = d_2$. Then for each $p = 0, \ldots, n$ there exists a map Ψ_p such that*

$$\Psi_p : H^0\left(\mathbb{P}^n, \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} \mathcal{O}_{\mathbb{P}^n}(p_1 Y_1 + p_2 Y_2 + K_{\mathbb{P}^n})\right) \to F^{n-p}H^{n-2}\left(V^\lambda, \mathbb{C}\right).$$

If n is odd, then Ψ_p is surjective. If n is even, then $\operatorname{im}(\Psi_p)$ *has codimension 1 in $F^{n-p}H^{n-2}\left(V^\lambda, \mathbb{C}\right)$ for $p \geqslant \frac{n}{2} + 1$.*

We describe in detail the Griffiths-Dwork algorithm for explicitly computing the Picard-Fuchs equation for a Calabi-Yau manifold that appears as a complete intersection of codimension 2. A detailed example is worked out in a Singular programme, presented in an appendix.

Chapter 8 treats pairs (X, D) consisting of a Calabi-Yau 3-fold X and a smooth divisor D. In [JS09a] Jockers and Soroush set up a Picard-Fuchs equation using a Griffiths-Dwork algorithm for several hypersurfaces X in weighted projective space, in particular for X being a quintic and D a certain divisor of degree 4.

First we define a residue map for pairs on a suitable complex of logarithmic forms and also for the cohomology of rational forms. The residues are then elements in the relative cohomology $H^{n-1}(X, D)$. Using the mixed Hodge structure which was introduced by Deligne on the hypercohomology group $\mathbb{H}^{n-1}(\Omega_X^\bullet(\log D))$ of the complex of logarithmic differential forms, we compare the Hodge filtration and the pole-order filtration and set up a basis for the relative cohomology $H^{n-1}(X, D, \mathbb{C})$.

To compute the Gauß-Manin connection, we will use the work of Li, Lian and Yau. Their theory will be presented in detail, calculations will be carried out, constructions will be made precise and at the same time extended to triples in Chapter 9.

Next the Griffiths-Dwork reduction is set up and we present an example where X is a quintic 3-fold with symmetries and D is a special divisor of degree 4 cut out by a hypersurface of degree 4. An important point here is that this hypersurface as well as the divisor D are singular. This requires detailed explanations, e.g. residues have to be defined carefully. This difficulty has appearently never been discussed in the connection with the Griffiths-Dwork reduction. We address this point in detail and explain why the methods presented so far still work. In particular we will describe the structure of the surface D and the hypersurface H of degree 4 as well as their singularities. Furthermore, a Hodge theory for D and H has to be set up. This, together with the paper of Li, Lian and Yau, provides the mathematical foundation of the work of Jockers and Soroush. A Singular programme for computing Picard-Fuchs operators in this situation is presented in the appendix.

The *final chapter* is connected with the question whether there is a Picard-Fuchs equation for the pair (X, C) consisting of a Calabi-Yau 3-fold X and a smooth curve C in X. We first observe that $H^3(X, C, \mathbb{C})$ is not a good candidate for the local system underlying the Gauß-Manin connection. Therefore we consider a complete intersection curve $C = D_1 \cap D_2$ where D_i are smooth divisors in X meeting transversally. We define a cohomology group $H^3(X, D_1, D_2, \mathbb{C})$ as a de Rham cohomology of triples of forms $(\alpha, \beta_1, \beta_2) \in \mathcal{A}_X^3 \oplus \mathcal{A}_{D_1}^2 \oplus \mathcal{A}_{D_2}^2$ with differential

$$d(\alpha, \beta_1, \beta_2) := \left(d\alpha, \alpha|_{D_1} - d\beta_1, \alpha|_{D_2} - d\beta_2\right)$$

as well as a homology group $H_3(X, D_1, D_2)$ and set up a perfect pairing

$$H^3(X, D_1, D_2, \mathbb{C}) \times H_3(X, D_1, D_2) \to \mathbb{C},$$

which is needed for defining periods via integration. Then we extend the theory of Li, Lian and Yau to local systems $H^3(X, D_1, D_2, \mathbb{C})$ when (X, D_1, D_2) varies. After that we set up the general formalism for a Griffiths-Dwork algorithm for triples and discuss an example. One might hope that the Picard-Fuchs equation associated to (X, D_1, D_2) is solved by the potential function for (X, C) constructed by Clemens in [Cle05] as mentioned above.

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Chapter 2

Preliminaries

In this short chapter we collect some general definitions and set up notions.

2.1 Notation

1. In this thesis we work entirely in the category of complex spaces in the sense of Grauert-Remmert [GR77].

2. If $\mathcal{S}$ is a sheaf and $\alpha \in \mathcal{S}(U)$ for an open set U, which we do not want to specify further, we simply write $\alpha \in \mathcal{S}$.

3. Given a complex manifold, Ω_X^p will denote the sheaf of holomorphic $p-$forms. If $n = \dim X$, then we denote by $\omega_X = \Omega_X$ the dualizing sheaf of X.

4. A Calabi-Yau manifold X is a projective connected simply connected complex manifold with trivial canonical bundle K_X such that additionally
$$H^0(X, \Omega_X^q) = 0$$
for $2 \leqslant q \leqslant \dim X - 1$. Often, the complex dimension of X will be 3, then the assumption that X is simply connected with trivial canonical bundles automatically yields $H^{1,0}(X) = H^{0,1}(X) = H^{2,0}(X) = H^{0,2}(X) = 0$.

2.2 Singularities

At a very few places singular spaces have to be considered. The basic definitions and facts are collected below. We refer e.g. to [KM98].

2.2.1 Definition. *Let X be a normal complex space.*

1. *X is said to have rational singularities if there is a resolution of singularities $\pi : \widehat{X} \to X$ such that $R^j \pi_* \left(\mathcal{O}_{\widehat{X}}\right) = 0$ for $j > 0$.*

2. *Assume* $\dim X = 2$. *A singular rational point $x_0 \in X$ is said to be a du Val singularity or rational double point if there exists a resolution $\pi : \hat{X} \to X$ such that $K_{\hat{X}} \cdot C = 0$ for all curves C contracted by π.*

3. *The canonical Weil divisor K_X is said to be $\mathbb{Q}$-Cartier if some multiple mK_X is a Cartier divisor. In terms of sheaves, $\mathcal{O}_X(K_X)$ is the reflexive sheaf $\iota_* \left(\omega_{X_{\mathrm{reg}}}\right)$, and the condition is that $\left(\mathcal{O}_X(K_X)^{\otimes m}\right)^{\vee\vee}$ is locally free for some m.*

4. *X is said to have only canonical singularities if K_X is $\mathbb{Q}$-Cartier and if there is a resolution of singularities $\pi : \widehat{X} \to X$ such that*
$$K_{\widehat{X}} = \pi^* (K_X) + \sum_i \lambda_i E_i$$
where the E_i are the π-exceptional divisors and $\lambda_i \geqslant 0$.

5. *X is said to be Gorenstein if K_X is Cartier and if X is Cohen-Macaulay, i.e., all local rings $\mathcal{O}_{X,x}$ are Cohen-Macaulay.*

6. *X is said to have only quotient singularities if every point $x \in X$ has a neighborhood of the form U/G with $U \subset \mathbb{C}^n$ and $G \subset \mathrm{Gl}(\mathrm{n}, \mathbb{C})$ a finite group.*

We will use the following facts, for which we again refer to [KM98].

2.2.2 Remark. Let X be a normal complex space.

1. If X has only rational singularities, then X is Cohen-Macaulay.

2. If X is a divisor or a complete intersection in a complex manifold, then X is Gorenstein.

3. In dimension 2, rational Gorenstein singularities are the same as du Val singularites, often also called ADE singularities.

4. Quotient singularities are rational.

2.3 Projective fibre spaces

In order to study the deformation theory of a coherent sheaf it might be useful to look at the projective fibre spaces of the sheaf. Let $\mathcal{F}$ be a coherent sheaf on a complex manifold X. Then to $\mathcal{F}$ is associated a projective fibre space
$$\mathbb{P}(\mathcal{F}) \xrightarrow{\pi} X.$$
Its basic properties are collected below, we refer to [BS76] or [Har77] (in the algebraic category).

1. π is a surjective map of complex spaces and $\pi^{-1}(x) \cong \mathbb{P}^{r-1}$, where $r = \dim_{\mathbb{C}} \mathcal{F}_x / m_x \mathcal{F}_x$.
2. There is a locally free sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ on $\mathbb{P}(\mathcal{F})$ of rank 1 such that
$$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)\big|_{\pi^{-1}(x)} \cong \mathcal{O}_{\mathbb{P}^{r-1}}(1)$$
for all $x \in X$ such that
$$\pi_*\left(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(k)\right) \cong S^k(\mathcal{F})$$
for all k and
$$H^q\left(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(k) \otimes \pi^*(\mathcal{G})\right) \simeq H^q\left(X, S^k(\mathcal{F}) \otimes \mathcal{G}\right)$$
for each coherent sheaf $\mathcal{G}$ on X and all $q \geqslant 0$.

We will also need the following property: Let $\mathcal{E}, \mathcal{F}$ be coherent sheaves and
$$\mathcal{E} \to \mathcal{F} \to 0$$
be a surjective morphism. Then there is an injective map
$$\iota : \mathbb{P}(\mathcal{F}) \hookrightarrow \mathbb{P}(\mathcal{E})$$
such that
$$\iota^*\left(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\right) \cong \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1).$$

2.3.1 Remark.

- One should notice that the complex space $\mathbb{P}(\mathcal{F})$ is not necessarily irreducible.
- If $\mathcal{F}$ is locally-free of rank r, then $\mathbb{P}(\mathcal{F})$ is a $\mathbb{P}^{r-1}$–bundle.

Chapter 3

Deformation Theory for geometric objects

In this chapter we refer the definitions and important properties of all deformation problems appearing in this thesis. All of the material can be found in [Ser06] or [Har10].

3.1 Functors of Artin rings

The technical tool to treat infinitesimal deformations are functors of Artin rings.

3.1.1 Definition.

1. *Let $\mathcal{A}$ be the category of Artin rings, i.e the category of complex, local Artinian $\mathbb{C}$-algebras. A functor of Artin rings is a covariant functor*

$$F : \mathcal{A} \to (\text{sets})$$

from $\mathcal{A}$ to the category of sets.

2. *Let $F : \mathcal{A} \to (\text{sets})$ be a functor of Artin rings.*

   a. *The functor F fulfils property (H_0) if $F(\mathbb{C})$ consists of one point.*

   b. *The functor F satisfies property (H_ϵ) if the following condition holds: Let*

$$\begin{tikzcd} A' \arrow[dr] & & A'' \arrow[dl] \\ & A & \end{tikzcd}$$

   be a diagram in $\mathcal{A}$. We consider the natural map

$$\alpha : F\left(A' \times_A A''\right) \to F\left(A'\right) \times_{F(A)} F\left(A''\right)$$

induced by the commutative diagram

$$\begin{tikzcd} F\left(A' \times_A A''\right) \arrow[r] \arrow[d] & F\left(A''\right) \arrow[d] \\ F\left(A'\right) \arrow[r] & F\left(A\right). \end{tikzcd}$$

Then α is bijective if $A = \mathbb{C}$ and $A'' = \mathbb{C}[t]/t^2$.

3. *If the functor F satisfies both properties (H_0) and (H_ϵ), then we call F a deformation functor.*

In this context the following result is important.

3.1.2 Theorem. *([Ser06], Lemma 2.2.1) Let F be a deformation functor. Then $F\left(\mathrm{Spec}\left(\mathbb{C}[t]/t^2\right)\right)$ has the structure of a $\mathbb{C}$-vector space.*

Examples for functors of Artin rings are functors of infinitesimal deformations of a geometric object.

3.1.3 Definition. *With a geometric object $\mathcal{X}$ we associate a functor of Artin rings* $\mathrm{Def}(\mathcal{X}) : \mathcal{A} \to (\mathrm{sets})$ *by defining*

$$\begin{aligned} \mathrm{Def}_A(\mathcal{X}) \quad := \quad & \{\textit{isomorphism classes of deformations of the object } \mathcal{X} \\ & \quad \textit{over the Artin ring } \mathrm{Spec}(A)\}. \end{aligned}$$

We also write $\mathrm{Def}_{\mathrm{Spec}(A)}(\mathcal{X}) = \mathrm{Def}_A(\mathcal{X})$ *and* $\mathrm{Def}(\mathcal{X}) = \mathrm{Def}_{\mathrm{Spec}(\mathbb{C}[t]/t^2)}(\mathcal{X})$. *A* first-order deformation *of $\mathcal{X}$ is an isomorphism class of deformations of $\mathcal{X}$ over* $\mathrm{Spec}\left(\mathbb{C}[t]/t^2\right)$.

For each deformation problem under consideration we will give a precise definition of the notion of a deformation and of the notion of isomorphy. *All functors we consider in the following will be deformation functors in the sense of Definition 3.1.1.*

Since the definiton of an obstruction space is technically a little complicated, we simply refer to Definition 2.2.9 in [Ser06] and confine ourselves with the following.

3.1.4 Definition. *Let F be a deformation functor of Artin rings, describing the deformations of a geometric object X. Let ζ be a deformation of X over* $\mathrm{Spec}(A)$. *Then the obstruction map for ζ is a map*

$$o_\zeta : \{\ \textit{small extensions } e \textit{ of } A\} \to V$$

to a vector space V, called the obstruction space, with the following property: if $\tilde{A}$ is the Artin ring given by e, then ζ lifts to $\operatorname{Spec}\left(\tilde{A}\right)$ *if and only if* $o_\zeta(e) = 0$.
The deformation ζ is unobstructed if $o_\zeta(e) = 0$ for each small extension e of A.
The geometric object X is called unobstructed if every infinitesimal deformation ζ of X is unobstructed.

3.2 Deformation theory of a compact complex manifold

We first treat the case of a compact complex manifold.

3.2.1 Definition. *A* deformation *of X over a (connected) complex space S consists of a complex space $\mathcal{X}$ and a proper flat surjective morphism $\pi : \mathcal{X} \to S$ together with an isomorphism $\mathcal{X}|_{s_0} \cong X$ for a point $s_0 \in S$.*

Two deformations $\mathcal{X}$ and $\mathcal{X}'$ of X over the same complex space S are isomorphic if there is a morphism of complex spaces $\phi : \mathcal{X} \to \mathcal{X}'$ such that the following diagram is commutative

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{X}' \arrow[dl] \\ & S. & \end{tikzcd}$$

We obtain a deformation functor by setting

$$\begin{aligned} F(A) := \operatorname{Def}_A(X) \quad := \quad & \{\text{ isomorphism classes of deformations of } X \\ & \text{over } \operatorname{Spec}(A)\}, \end{aligned}$$

where A is an Artin ring.

The following theorem is classical.

3.2.2 Theorem. *(e.g. [Har10], p. 38) The space of first-order deformations* $\operatorname{Def}(X)$ *of X and the space of obstructions* $\operatorname{Obs}(X)$ *of X satisfy*

$$\begin{aligned} \operatorname{Def}(X) &\cong H^1(X, T_X), \\ \operatorname{Obs}(X) &\subseteq H^2(X, T_X). \end{aligned}$$

3.2.3 Remark. If X is a Calabi-Yau manifold, then by a theorem of Tian-Todorov $\operatorname{Obs}(X) = 0$, although $H^2(X, T_X) \neq 0$ (see e.g. [GHJ03], Theorem 14.10).

3.3 Deformation theory of a submanifold in a fixed complex manifold

Next we consider pairs of compact complex manifolds. More precisely, let Z be a closed complex submanifold of the compact complex manifold X and let $\iota : Z \hookrightarrow X$ be the inclusion map.

3.3.1 Definition. *([Ser06], p. 161) A* deformation of Z in X *over a complex space S consists of a cartesian diagram*

$$\begin{tikzcd} \mathcal{Z} \arrow[rr, "J"] \arrow[dr, "\pi\circ J"'] & & X \times S \arrow[dl, "\pi"] \\ & S, & \end{tikzcd}$$

where $\mathcal{Z} \subset X \times S$ is a closed subscheme of $X \times S$ with inclusion map J and $\pi \circ J$ is a flat morphism. Furthermore, in the following diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\iota"] & \mathcal{Z} \arrow[d, "J"] \\ X \arrow[r] \arrow[d] & X \times S \arrow[d, "\pi"] \\ \mathrm{Spec}\,(\mathbb{C}) \arrow[r, "s"] & S, \end{tikzcd}$$

the pullback of the right column by s has to be isomorphic to the left column.

Two deformations $\mathcal{Z}$ and $\mathcal{Z}'$ of Z in X over the same complex space S are isomorphic if

$$\begin{tikzcd} \mathcal{Z} \arrow[rr, "J"] \arrow[dr] & & X \times S \arrow[dl] \\ & S & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{Z}' \arrow[rr, "J'"] \arrow[dr] & & X \times S \arrow[dl] \\ & S & \end{tikzcd}$$

is a pair of isomorphisms of deformations

$$\alpha : \mathcal{Z} \to \mathcal{Z}', \beta : \mathcal{X} \to \mathcal{X}'$$

which makes the following diagram commutative:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "J"] \arrow[d, "\alpha"'] & \mathcal{X} \arrow[d, "\beta"] \\ \mathcal{Z}' \arrow[r, "J'"] & \mathcal{X}'. \end{tikzcd}$$

3.3.2 Theorem. *The space of first-order deformations* $\mathrm{Def}_X(Z)$ *of* Z *in* X *and the space of obstructions* $\mathrm{Obs}_X(Z)$ *satisfy*

$$\begin{aligned}\mathrm{Def}_X(Z) &\cong H^0\left(Z, \mathcal{N}_{Z|X}\right),\\ \mathrm{Obs}_X(Z) &\subseteq H^1\left(Z, \mathcal{N}_{Z|X}\right).\end{aligned}$$

3.3.3 Remark. If X is a Calabi-Yau manifold and Z an ample divisor, then $\mathrm{Obs}_X(Z) = 0$, since $H^1\left(Z, \mathcal{N}_{Z|X}\right) = 0$.

3.4 Deformation theory of a coherent sheaf on a fixed complex manifold

Let $\mathcal{F}_0$ be a coherent sheaf on the compact complex manifold X.

3.4.1 Definition. *([Har10], p. 14, in the algebraic case) A deformation of* $\mathcal{F}_0$ *over a complex space* S *consists of a coherent sheaf* $\mathcal{F}$ *on* $X \times S$ *and a morphism of* $\mathcal{O}_X$*-modules* $\lambda : \mathcal{F} \to \mathcal{F}_0$ *which induces an isomorphism* $\mathcal{F}|_{X \times \{s_0\}} \cong \mathcal{F}_0$.

Two deformations $\lambda : \mathcal{F} \to \mathcal{F}_0$ *and* $\lambda' : \mathcal{F}' \to \mathcal{F}$ *of* $\mathcal{F}_0$ *over the same complex space* S *are isomorphic if there is a isomorphism of sheaves* $\phi : \mathcal{F} \to \mathcal{F}'$ *compatible with* λ *and* λ'.

In this context we have

3.4.2 Theorem. *The space of first-order deformations* $\mathrm{Def}(\mathcal{F}_0)$ *of* $\mathcal{F}_0$ *on* X *and the space of infinitesimal obstructions* $\mathrm{Obs}(\mathcal{F}_0)$ *of* $\mathcal{F}_0$ *satisfy*

$$\begin{aligned}\mathrm{Def}(\mathcal{F}_0) &\cong \mathrm{Ext}^1_X(\mathcal{F}_0, \mathcal{F}_0),\\ \mathrm{Obs}(\mathcal{F}_0) &\subseteq \mathrm{Ext}^2_X(\mathcal{F}_0, \mathcal{F}_0).\end{aligned}$$

3.5 Deformation theory of a pair consisting of a smooth divisor in a complex manifold

Let $\iota : Z \hookrightarrow X$ be an embedding of a compact, possibly reducible, hypersurface or of a compact complex submanifold into a smooth compact complex manifold. Following [Ser06] we state

3.5.1 Definition. *A* deformation of j parametrized by a complex space S *is a cartesian diagram*

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "j"] & \mathcal{Z} \arrow[d, "J"] \\ X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\psi"] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "s"] & S, \end{tikzcd}$$

where ψ and $\psi \circ J$ are flat and the pullback of the right column by s is isomorphic to the left column. We denote the deformations of the pair (Z, X) by $\mathcal{Z} \overset{J}{\hookrightarrow} \mathcal{X}$ or $(\mathcal{X}, \mathcal{Z})$. An isomorphism between two deformations of j over the same complex space S

$$\begin{tikzcd} \mathcal{Z} \arrow[rr, "J"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & S & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{Z}' \arrow[rr, "J'"] \arrow[dr] & & \mathcal{X}' \arrow[dl] \\ & S & \end{tikzcd}$$

is a pair of isomorphisms of deformations

$$\alpha : \mathcal{Z} \to \mathcal{Z}', \beta : \mathcal{X} \to \mathcal{X}'$$

which makes the following diagram commutative:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "J"] \arrow[d, "\alpha"'] & \mathcal{X} \arrow[d, "\beta"] \\ \mathcal{Z}' \arrow[r, "J'"] & \mathcal{X}'. \end{tikzcd}$$

3.5.2 Remark. Note that J is again a closed embedding; see [Ser06], 3.4.4.

3.5.3 Notation. *We denote the set of isomorphism classes of infinitesimal deformations of the embedding $j : Z \to X$ over $S = \operatorname{Spec}(A)$ by $\operatorname{Def}_A(j : Z \hookrightarrow X)$, or $\operatorname{Def}_A(Z, X)$ if j is fixed.*

In order to describe manifolds together with a subspace we need the following definition; for details on logarithmic bundles see Chapter 4.

3.5.4 Definition. *Let X be a complex manifold.*

1. *Given a divisor $D \subset X$ with simple normal crossings, we set*

$$T_X(-\log D) := \Omega^1_X(\log D)^\vee .$$

2. *Let $Z \subset X$ be a submanifold and $\pi : \widehat{X} \to X$ the blow-up of $Z \subset X$ with exceptional divisor E; then we define*

$$T_X\langle -Z\rangle := \pi_*\left(T_{\widehat{X}}(-\log E)\right).$$

In this terminology we state

3.5.5 Theorem. *([Ser06], Prop. 3.4.17)*

1. *Let X be a compact complex manifold and D a divisor on X with simple normal crossings. Then the spaces of first-order deformations* $\mathrm{Def}\,(X,D)$ *and obstructions* $\mathrm{Obs}\,(X,D)$ *of the pair (X,D) are given by*

$$\begin{aligned}\mathrm{Def}\,(X,D) &\cong H^1\left(X,T_X\left(-\log D\right)\right)\\ \mathrm{Obs}\,(X,D) &\subseteq H^2\left(X,T_X\left(-\log D\right)\right).\end{aligned}$$

2. *Let $Z\subset X$ be a compact complex submanifold. Then the spaces of first-order deformations* $\mathrm{Def}\,(X,Z)$ *and obstructions* $\mathrm{Obs}\,(X,Z)$ *of the pair (X,Z) are given by*

$$\begin{aligned}\mathrm{Def}\,(X,Z) &\cong H^1\left(X,T_X\left\langle -Z\right\rangle\right)\\ \mathrm{Obs}\,(X,Z) &\subseteq H^2\left(X,T_X\left\langle -Z\right\rangle\right).\end{aligned}$$

3.5.6 Remark. Let X be a Calabi-Yau manifold and D an ample divisor on X with simple normal crossings; then $\mathrm{Obs}\,(X,D)=0$. In fact, the logarithmic tangent sequence

$$0\to T_X\left(-\log D\right)\to T_X\to \iota_*\mathcal{N}_{D|X}\to 0$$

induces in cohomology

$$0=H^1\left(D,\mathcal{N}_{D|X}\right)\to H^2\left(X,T_X\left(-\log D\right)\right)\to H^2\left(X,T_X\right)\to H^2\left(D,\mathcal{N}_{D|X}\right).$$

The obstructions $\mathrm{Obs}\,(X,D)\subset H^2\left(X,T_X\left(-\log D\right)\right)$ are mapped to the obstructions of deforming X in $H^2\left(X,T_X\right)$, which vanish by the theorem by Tian-Todorov. Since D is ample, $H^1\left(D,\mathcal{N}_{D|X}\right)=0$. Thus the map $H^2\left(X,T_X\left(-\log D\right)\right)\to H^2\left(X,T_X\right)$ is injective and $\mathrm{Obs}\,(X,D)=0$.

3.6 Simultaneous deformations of a coherent sheaf and its complex base manifold

Let X be a compact complex manifold and $\mathcal{F}_0$ be a coherent sheaf on X_0.

3.6.1 Definition. *([Har10], p. 53, [Ser06] p. 146)*
A simultaneous deformation of X and $\mathcal{F}_0$ *is a pair $(\mathcal{X},\mathcal{F})$ consisting of a deformation $\mathcal{X}$ of X over S and a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$, which is flat over S, together with a map $\mathcal{F}\to\mathcal{F}_0$ such that the induced map $\mathcal{F}\otimes_{\mathcal{O}_\mathcal{X}}\mathcal{O}_{X_0}\to\mathcal{F}_0$ is an isomorphism. Two deformations $(\mathcal{X},\mathcal{F})$ and $(\mathcal{X},\mathcal{F})$ over S are isomorphic if there is an isomorphism of deformations $f:\mathcal{X}\to\mathcal{X}'$ and an isomorphism $\mathcal{F}\to f^*\mathcal{F}'$.*

3.6.2 Notation. *We denote the space of isomorphism classes of simultaneous first-order deformations of* $\mathcal{F}_0$ *and* X *over* $\operatorname{Spec}\left(\mathbb{C}[t]/\left(t^2\right)\right)$ *by* $\operatorname{Def}(X, \mathcal{F}_0)$ *and the space of obstructions of extending infinitesimal deformations by* $\operatorname{Obs}(X, \mathcal{F}_0)$.

The general result concerning first-order deformations and obstructions is stated in the derived category:

3.6.3 Theorem. *([HT10], [Li]) Let* X *be a smooth projective variety and* $E \in D(X)$ *a perfect complex, e.g. given by a single coherent sheaf. Let* $G^\bullet$ *be the mapping cone of the Atiyah class* $R\mathcal{H}om(E, E)[-1] \to X$. *Then*

$$\begin{aligned} \operatorname{Def}(X, E) &\cong \operatorname{Ext}^1_X(G^\bullet, \mathcal{O}_X), \\ \operatorname{Obs}(X, E) &\subset \operatorname{Ext}^2_X(G^\bullet, \mathcal{O}_X). \end{aligned}$$

If the sheaf E is locally free, the situation is much easier.

3.6.4 Corollary. *Let* X *be a compact complex manifold and* E *a locally free sheaf on* X. *Let*

$$0 \to \operatorname{End}(E) \to D(E) \to T_X \to 0 \tag{3.6.4.1}$$

be the Atiyah sequence, where $D(E)$ *denotes the sheaf of differential operators of order* $\leqslant 1$ *with diagonal symbol.*

$$\begin{aligned} \operatorname{Def}(X, E) &\cong H^1(X, D(E)), \\ \operatorname{Obs}(X, E) &\subseteq H^2(X, D(E)). \end{aligned}$$

3.6.5 Remark. The extension class defining the exact sequence 3.6.4.1 is the Atiyah class $\operatorname{At}(E)$ of E.

3.7 Deformations of projective bundles

Deformations of the pair (X, E), where X is a compact complex manifold and E a holomorphic vector bundle on X, are closely related to the deformations of the projective bundle $\mathbb{P}(E)$, as we shall see now.

The following proposition is in principle well-known, see [DF89], p. 202, and [Kod63] for a version over manifolds. In lack of a proper reference, we give the proof.

3.7.1 Proposition. *Let* E *be a locally free sheaf of rank* r *over the compact* n*-dimensional complex manifold* X. *Suppose*

$$H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0.$$

Let $\mathcal{Y} \to S$ *be a deformation of* $\mathbb{P}(E)$ *over the spectrum of an Artin ring or the germ of a complex space* S. *Then there exists a deformation* $\tau : \mathcal{X} \to S$ *of* X *and a locally free sheaf* $\mathcal{E}$ *over* $\mathcal{X}$ *such that*

$$\mathcal{Y} \cong \mathbb{P}(\mathcal{E}).$$

Proof of Proposition 3.7.1: Let $\tau_0 : \mathbb{P}(E) \to X$ be the projection of the projective fibre space of E. Let $\mathcal{H}_0$ be the subspace of the Hilbert scheme $\mathcal{H}ilb(\mathbb{P}(E))$ of $\mathbb{P}(E)$ consisting of all fibres of τ_0. Then at each fibre F of τ_0 the space $\mathcal{H}ilb(\mathbb{P}(E))$ is smooth of dimension n according to [Ser06], Prop. 4.4.7, as the Zariski tangent space at F satisfies

$$T_F\mathcal{H}ilb(\mathbb{P}(E)) \cong H^0\left(F, \mathcal{N}_{F|\mathbb{P}(E)}\right) = H^0\left(F, \mathcal{O}_F^n\right) \cong \mathbb{C}^n$$

and furthermore $H^1\left(F, \mathcal{N}_{F|\mathbb{P}(E)}\right) = 0$.

Then $\mathcal{H}_0 \cong X$ is an irreducible component of $\mathcal{H}ilb(\mathbb{P}(E))$, since all fibres of τ_0 already form an n-dimensional family parametrized by X.

Let $\mathcal{H}ilb(\mathcal{Y}/S)$ be the relative Hilbert scheme of $\mathcal{Y} \to S$. According to [Ser06], Prop. 4.4.7, it is relatively smooth of dimension n over S at any point $[F] \in \mathcal{H}_0$. Let $\mathcal{H}$ be the irreducible component of $\mathcal{H}ilb(\mathcal{Y}/S)$ containing all fibres $[F]$ of τ_0. Then $\mathcal{H}_0$ is the central fibre of $\mathcal{H} \to S$. Hence the projection $\mathcal{H} \to S$ has n-dimensional smooth fibres, i.e. it is a submersion. In particular it is flat.

Let

$$q : U \to \mathcal{H}$$

be the universal family of $\mathcal{H}$, i.e., set-theoretically

$$U = \{(x, F) |\, x \in F\} \subset \mathcal{Y} \times \mathcal{H}.$$

Notice also that q is a locally trivial $\mathbb{P}^{r-1}$-bundle, because of the local rigidity of the projective space. Let $p : U \to \mathcal{Y}$ be the projection. Furthermore let

$$q_0 : U_0 := \{(x, F) |\, x \in F\} \subset \mathbb{P}(E) \times \mathcal{H}_0 \to \mathcal{H}_0$$

be the universal family of $\mathcal{H}_0$ and $p_0 : U_0 \to \mathbb{P}(E)$ the projection to $\mathbb{P}(E)$. Obviously p_0 is an isomorphism.

Then the restriction of the maps p and q to the central fibre in S yields the maps p_0 and q_0.

Over the reduced point $0 \in S$, we obtain the family

$$q_0 : U_0 \to \mathcal{H}_0$$

with projection $p_0 : U_0 \to \mathbb{P}(E)$.

We define

$$\mathcal{X} := \mathcal{H},$$

hence $\mathcal{X} \to S$ is a flat family.

We prove that p is an isomorphism. This follows easily from the fact that p_0 is an isomorphism and from the fact that locally both spaces are of the form $V \times S$, where V is a small open set in either $\mathcal{H}_0$ or $\mathbb{P}(E)$. Let $\tau := q \circ p^{-1} : \mathcal{Y} \to \mathcal{X}$ be the induced map.

It remains to construct $\mathcal{E}$ such that $\mathcal{Y} \cong \mathbb{P}(\mathcal{E})$. It is sufficient (actually equivalent) to construct a line bundle $\mathcal{L} \in \operatorname{Pic}(\mathcal{Y})$ such that $\mathcal{L}|_{\tau^{-1}(x)} = \mathcal{O}_{\mathbb{P}^{r-1}}(1)$ for each $x \in \mathcal{X}$.

Once $\mathcal{L}$ has been found, we set

$$\mathcal{E} := \tau_*(\mathcal{L}).$$

Let

$$\mathcal{L}_0 := \mathcal{O}_{\mathbb{P}(E)}(1).$$

We aim to extend $\mathcal{L}_0$ to a line bundle on $\mathcal{Y}$.

Since $\mathcal{Y} \to S$ is flat and $H^2\left(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}\right) = 0$, there is a surjective map

$$R^2\pi_*\mathcal{O}_{\mathcal{Y}}\big|_{\{0\}} \to H^2\left(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}\right) = 0,$$

where $0 \in S$ denotes the central point in S. By semicontinuity (see e.g. [Har77], III, 12.11) this map is an isomorphism on S, hence $R^2\pi_*\mathcal{O}_{\mathcal{Y}} = 0$. Using this vanishing and $H^i\left(S, R^j\pi_*\mathcal{O}_{\mathcal{Y}}\right) = 0$ for $i > 0$, as S is Stein, the Leray spectral sequence for $\pi : \mathcal{Y} \to S$ yields $H^2(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = 0$.

The exponential sequences for $\mathcal{Y}$ and $\mathbb{P}(E)$, the vanishing

$$\begin{aligned} & H^2\left(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}\right) \cong H^2(X, \mathcal{O}_X) = 0, \\ & H^1\left(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}\right) \cong H^1(X, \mathcal{O}_X) = 0 \\ \text{and} \qquad & H^2(\mathcal{Y}, \mathbb{Z}) \cong H^2(\mathbb{P}(E), \mathbb{Z}) \end{aligned}$$

show that there is a line bundle $\mathcal{L} \in \operatorname{Pic}(\mathcal{Y})$ such that $\mathcal{L}|_{\mathbb{P}(E)} \cong \mathcal{L}_0$.

Since $\mathcal{L}_0|_{\tau^{-1}(x)} = \mathcal{O}_{\mathbb{P}^{r-1}}(1)$ for $x \in X$, we obtain that $\mathcal{L}|_{\tau^{-1}(x)} = \mathcal{O}_{\mathbb{P}^{r-1}}(1)$ for each $x \in X$. □

3.7.2 Remark. If we omit in Proposition 3.7.1 the assumption $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, then there exists a deformation $\mathcal{X} \to S$ of X such that $\mathcal{Y}$ is a $\mathbb{P}^{r-1}$-bundle over $\mathcal{X}$.

3.7.3 Corollary. *Let E be a locally free sheaf of rank r over the compact n-dimensional complex manifold X. Suppose*

$$H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0.$$

Then there is an isomorphism of functors

$$\operatorname{Def}_\bullet(X, E) \to \operatorname{Def}_\bullet(X, \mathbb{P}(E)).$$

3.7.4 Remark. Let X be a compact complex manifold and E be a holomorphic vector bundle on X. We consider the relative tangent bundle sequence

$$0 \to T_{\mathbb{P}(E)|X} \to T_{\mathbb{P}(E)} \to \pi^*T_X \to 0.$$

Taking π_* and using the relative Euler sequence, we obtain the exact sequence

$$0 \to E \otimes E^\vee/\mathcal{O}_X \to \pi_*T_{\mathbb{P}(E)} \to T_X \to 0.$$

A comparison with Corollary 3.6.4 shows $D(E) = \pi_*T_{\mathbb{P}(E)} \oplus \mathcal{O}_X$.

3.8 Simultaneous deformations of a holomorphic vector bundle and a section

Let X be a compact complex manifold and $E \to X$ a holomorphic vector bundle of rank r on X. Let $s \in H^0(X, E)$, $s \not\equiv 0$, be a holomorphic section of E and $[s] \in \mathbb{P}\left(H^0(X, E)^\vee\right)$ the class of s in the projective space associated to the vector space $H^0(X, E)$.

3.8.1 Definition. *A simultaneous deformation of X, E and $[s]$ over a complex space S consists of a simultaneous deformation*

$$\mathcal{E} \xrightarrow{\pi} \mathcal{X} \to S$$

of E and X over S (Definition 3.6.1) and a class $[\tilde{s}] \in \mathbb{P}\left(H^0(\mathcal{X}, \mathcal{E})^\vee\right)$ of a holomorphic section $\tilde{s} \in H^0(\mathcal{X}, \mathcal{E})$ such that $\left[\tilde{s}|_{\{s_0\}}\right] \cong [s] \in \mathbb{P}\left(H^0(X, E)^\vee\right)$.

Two deformations $(\mathcal{X}, \mathcal{E}, [s])$ and $(\mathcal{X}', \mathcal{E}', [s'])$ over $\operatorname{Spec}(A)$ *are isomorphic if there is an isomorphism $\phi : \mathcal{X} \to \mathcal{X}'$ over* $\operatorname{Spec}(A)$ *and an isomorphism of locally free sheaves $\mu : \phi^*\mathcal{E} \to \mathcal{E}'$ such that $\mu(\phi^*[s]) = [s']$.*

We denote the set of isomorphism classes of simultaneous deformations of X, E and $[s]$ over $\operatorname{Spec}(A)$ *by* $\operatorname{Def}_A(X, E, [s])$ *and the obstructions to extend a first-order deformation by* $\operatorname{Obs}(X, E, [s])$.

3.8.2 Remark. We obtain a deformation functor which will be denoted by $\operatorname{Def}_\bullet(X, E, [s])$.

The space of first-order deformations and the space of obstructions will be studied in Chapter 6.

For further reference we state the following.

3.8.3 Lemma. *Let S be the spectrum of an Artin ring or the germ of a complex space containing* 0. *Let U be an open set in $\mathbb{C}^n$ and $\mathcal{D}$ be a divisor in $U \times S$ such that $\mathcal{D}_0 = \mathcal{D} \cap (U \times \{0\})$ is a smooth divisor. Then $\mathcal{D} \to S$ is a submersion.*

Proof of Lemma 3.8.3: As the question is local, we may assume $\mathcal{D} = \{f = 0\}$ with $f \in \mathcal{O}_{U \times S}(U \times S)$.

1. We suppose first that S is a complex manifold. We may assume that $f|_{\mathcal{D}_0 \cap U}$ is regular, hence it is regular on $U \times \{s\}$ for all $s \in S$. Hence $\mathcal{D}$ is smooth and $\mathcal{D} \to S$ is a submersion.

2. In general we embed S into an open set $W \subset \mathbb{C}^N$. Since the question is local in U, we may assume that f lifts to $\tilde{f} \in \mathcal{O}_{U \times W}(U \times W)$. Therefore $\mathcal{D}$ lifts to a divisor $\tilde{\mathcal{D}} := \left\{\tilde{f} = 0\right\}$. By (1) the map $\tilde{\mathcal{D}} \to W$ is a submersion, hence $\mathcal{D} \to S$ is a submersion by base change, compare [Har77], $III, 10.1(b)$.
$\square$

Chapter 4

Variation of Hodge structures on the cohomology of special complex manifolds

In this section we briefly review what we will need from Hodge theory. As general references we use [PS08], [Voi02], [Voi03] and [EV92].

4.1 Variation of pure Hodge structures

4.1.1 Definition. *([PS08], p. 17) Let $H_{\mathbb{R}}$ be an $\mathbb{R}$-vector space of finite dimension over $\mathbb{R}$. Let $H := H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $H_{\mathbb{R}}$.*

A pure Hodge structure of weight *n on H is given by a lattice $H_{\mathbb{Z}} \subset H_{\mathbb{C}}$ together with a direct sum decomposition*

$$H = \bigoplus_{p+q=n} H^{p,q}$$

with $H^{q,p} = \overline{H^{p,q}}$. The numbers $h^{p,q} := \dim_{\mathbb{C}} H^{p,q}$ are called the Hodge numbers *of the Hodge decomposition.*

4.1.2 Definition. *([PS08], p. 241) Let S be a complex manifold. A* variation of (pure) Hodge structure *on S consists of the following data:*

1. *a local system $\mathbb{V}_{\mathbb{Z}}$ of finitely generated abelian groups on S,*
2. *a finite decreasing filtration $\{\mathcal{F}^p\}$ of the holomorphic vector bundle $\mathcal{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_S$ by holomorphic subbundles (the* Hodge filtration*).*

These data are subject to the following conditions:

1. *for each* $s \in S$ *the filtration* $\{\mathcal{F}^p(s)\}$ *of* $\mathbb{V}(s) \cong \mathbb{V}_{\mathbb{Z},s} \otimes_{\mathbb{Z}} \mathbb{C}$ *defines a Hodge structure of weight* k *on the finitely generated abelian group* $\mathbb{V}_{\mathbb{Z},s}$*,*

2. *the connection* $\nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_S} \Omega^1_S$ *whose sheaf of horizontal sections is* $\mathbb{V}_{\mathbb{C}}$ *satisfies the* Griffiths' transversality condition
$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_S.$$

4.2 Logarithmic differentials and residues

The notion of logarithmic differential forms will be crucial in all that follows. For details we refer to [EV92] and [PS08].

4.2.1 The complex of logarithmic differential forms

Let X be a complex manifold of dimension n and D a divisor in X with simple normal crossings, let $\iota : D \hookrightarrow X$ be the inclusion. Furthermore, we define $U := X \backslash D$ and $j : U \hookrightarrow X$ to be the inclusion.

4.2.1 Definition. *Let* $\Omega^\bullet_X(\log D)$ *be the smallest subcomplex of sheaves containing* $\Omega^\bullet_X$*, which is stable under the exterior product such that for each local section* f *of* $j_*\mathcal{O}_U$ *that is meromorphic along* D*, the quotient* $\frac{df}{f}$ *is a local section of* $\Omega^1_X(\log D)$*.*

A section of $j_*\Omega^p_U$ *is said to have a logarithmic pole along* D *if it is a section of* $\Omega^p_X(\log D)$*.*

The following is easy to prove.

4.2.2 Theorem.

1. *A section* α *of* $j_*\Omega^p_U$ *has a logarithmic pole along* D *if and only if* α *and* $d\alpha$ *have at most simple poles along* D*.*

2. *The sheaf* $\Omega^1_X(\log D)$ *is locally free and* $\Omega^p_X(\log D) = \bigwedge^p \Omega^1_X(\log D)$*. If* $\{z_1, \ldots, z_n\}$ *are local holomorphic coordinates of* X *in a neighbourhood* U *of* $p = (0, \ldots, 0) \in X$ *such that* $X \cap U = \{z_1 \cdot \ldots \cdot z_k = 0\}$*, then a local base of* $\Omega^1_X(\log D)$ *consists of* $\left(\frac{dz_i}{z_i}\right)_{1 \leqslant i \leqslant k}$ *and* $(dz_j)_{k \leqslant j \leqslant n}$*.*

4.2.3 Theorem. *([PS08], Prop. 4.3, p. 91) The inclusion of complexes*
$$\Omega^\bullet_X(\log D) \hookrightarrow j_*\Omega^\bullet_U$$
is a quasi-isomorphism and induces an isomorphism
$$H^k(U, \mathbb{C}) = \mathbb{H}^k(X, \Omega^\bullet_X(\log D)),$$
where $\mathbb{H}^k(X, \Omega^\bullet_X(\log D))$ *denotes the hypercohomology of the logarithmic complex* $\Omega^\bullet_X(\log D)$*.*

4.2.4 Theorem. *(See e.g. [Voi02], p. 198/199) If the divisor Y with simple normal crossings has k irreducible components, then the logarithmic de Rham complex $\Omega^\bullet_X(\log Y)$ is exact in degree $\geqslant k+1$.*

Proof of Theorem 4.2.4: As the statement is local, we can assume that X is an n-dimensional polydisc, i.e.

$$X = D_1 \times \ldots \times D_n,$$

where $D_i := \{ z_i \in \mathbb{C} | \, |z_i| < r_i \}$ with $r_i \in \mathbb{R}$ for $1 \leqslant i \leqslant n$ denotes an open disc. Furthermore we can assume

$$Y = \{ (z_1, \ldots, z_n) \in D_1 \times \ldots \times D_n | \, z_1 \cdots z_k = 0 \},$$

Thus locally

$$U := X \backslash Y = D_1^* \times \cdots \times D_k^* \times D_{k+1} \times \cdots \times D_n$$

retracts onto the product of circles $\left(S^1\right)^k$. Using the Künneth formula, we get $H^r\left(\left(S^1\right)^k, \mathbb{C}\right) = 0$ for each $r \geqslant k+1$. □

4.2.2 The residue map

We now introduce the notion of residues, following [PS08], p. 93. We will use only the case of one or two components.

4.2.5 Notation. *Let X be a complex manifold and $D = \sum_{i=1}^N D_i$ be a divisor with simple normal crossings. We set:*

$$\begin{aligned} D_I &:= D_{i_1} \cap \ldots \cap D_{i_m} \text{ for } I = \{i_1, \ldots, i_m\} \\ D(I) &:= \sum_{j \notin I} D_I \cap D_j. \end{aligned}$$

Furthermore, let $a_I : D_I \hookrightarrow X$ be the inclusion and

$$\begin{aligned} D(0) &:= X \\ D(m) &:= \coprod_{|I|=m} D_I, \text{ for } m = 1, \ldots, N. \end{aligned}$$

Let $a_m = \coprod_{|I|=m} a_I : D(m) \to X$ be the inclusion.

Let $1 \leqslant m \leqslant N$ and $I = \{i_1, \ldots, i_m\}$ be an index set of cardinality m. For $p \in D_I$ we choose holomorphic coordinates in a neighbourhood around $p = (0, \ldots, 0)$ such that $D_{i_j} = \{z_j = 0\}$ for each $j = 1, \ldots, m$ and the other components are given by $\{z_j = 0\}$ for $j = m+1, \ldots, N$. Then every local section ω of $\Omega^p_X(\log D)$ can be written as

$$\omega = \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_m}{z_m} \wedge \eta + \eta', \tag{4.2.5.1}$$

where η has at most poles along the components D_j with $j \notin I$ and η' is not divisible by the form $\frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_m}{z_m}$.

4.2.6 Theorem. *There is a well-defined global map, the* residue map, *on the complex of logarithmic differential forms, which is given as follows:*

$$\mathrm{res}_I : \Omega_X^\bullet (\log D) \to \Omega_{D_I}^\bullet (\log (D(I))) [-m], \tag{4.2.6.1}$$

by setting locally $\mathrm{res}_I (\omega) := \eta|_{D_I}$, *where η is given by 4.2.5.1.*

4.2.7 Remark. If D is a smooth divisor, then we can locally find holomorphic coordinates $\{z_1, \ldots, z_n\}$ on X such that $D = \{z_1 = 0\}$. Then the residue map 4.2.6.1 has the form

$$\mathrm{res} : \Omega_X^\bullet (\log D) \to \Omega_D^\bullet [-1], \omega = \frac{dz_1}{z_1} \wedge \eta + \eta' \mapsto \eta|_D, \tag{4.2.7.1}$$

where η and η' do not contain $\frac{dz_1}{z_1}$.

4.2.8 Example. The residue maps lead to various sequences which we write down explicitly for the case of one or two divisors. These are the only relevant situations for us. Let X be a complex manifold.

1. Let $D \subset X$ be a smooth irreducible divisor. Then there are sequences
$$0 \to \Omega_X^k \to \Omega_X^k (\log D) \to \Omega_D^{k-1} \to 0. \tag{4.2.8.1}$$
The dual sequence reads
$$0 \to T_X (-\log D) \to T_X \to \mathcal{N}_{D|X} \to 0. \tag{4.2.8.2}$$
2. Let D_1 and D_2 be smooth irreducible divisors such that $D_1 \cup D_2$ has simple normal crossings. Then there is an exact sequence
$$0 \to \Omega_X^k (\log D_2) \to \Omega_X^k (\log (D_1 \cup D_2)) \to \Omega_{D_1}^{k-1} (\log (D_1 \cap D_2)) \to 0. \tag{4.2.8.3}$$

In Chapter 8 we will be in a slightly more general situation where one of the two divisors in Example 4.2.8 is mildly singular. In the following we will make the necessary preparations. In order not to have trouble with coherence problems, we assume from now on that X is projective.

4.2.9 Definition. *Let X be a projective manifold and D a reduced divisor on X. Let $X_0 := X \backslash \mathrm{Sing}(D)$, i.e. $D \cap X_0$ is a divisor with simple normal crossings, and $\iota : X_0 \hookrightarrow X$ be the inclusion. Then we set*

$$\tilde{\Omega}_X^q (\log D) := \iota_* \left(\Omega_{X_0}^q \log (D \cap X_0) \right).$$

4.2.10 Proposition. *Let X be a projective manifold and D a normal divisor. For any $k \geqslant 1$ there are exact residue sequences*

$$0 \to \Omega_X^k \to \tilde{\Omega}_X^k (\log D) \xrightarrow{\psi} \tilde{\Omega}_D^{k-1}$$

extending the classical residue sequence 4.2.8.1. The map ψ is surjective outside $\mathrm{Sing}(D)$.

Proof of Proposition 4.2.10: We set $X_0 := X \backslash \mathrm{Sing}\,(D)$ and $D_0 := D \cap X_0$ and consider the usual residue sequence

$$0 \to \Omega^k_{X_0} \xrightarrow{\phi_0} \Omega^k_{X_0}\,(\log D_0) \xrightarrow{\psi_0} \Omega^{k-1}_{D_0} \to 0.$$

The map ϕ_0 extends to $\phi : \Omega^k_X \to \tilde{\Omega}^k_X\,(\log D)$. Let $Q := \mathrm{coker}\,(\phi)$. Obviously Q is supported on D and $Q|_{D_0} \cong \Omega^{k-1}_{D_0}$; thus Q is torsion-free as a sheaf on D. Hence $Q \subset \tilde{\Omega}^{k-1}_D$ and we obtain a map $\psi : \tilde{\Omega}^k_X\,(\log D) \to \tilde{\Omega}^{k-1}_D$, extending ψ_0. Obviously, ϕ is injective and $\mathrm{im}\,(\phi) = \ker\,(\psi)$. Thus we obtain a sequence

$$0 \to \Omega^k_X \xrightarrow{\phi} \tilde{\Omega}^k_X\,(\log D) \xrightarrow{\psi} \tilde{\Omega}^{k-1}_D.$$

□

4.2.11 Proposition. *In the setting of Proposition 4.2.10 the sequence*

$$0 \to \Omega^1_X \to \tilde{\Omega}^1_X\,(\log D) \xrightarrow{\psi} \mathcal{O}_D \to 0$$

is exact.

Proof of Proposition 4.2.11: It remains to be shown that ψ is surjective. Let $x_0 \in D$ be a singular point and h be a local holomorphic function on D defined in an open neighbourhood of x_0. We lift h to a holomorphic function $\tilde{h}$ locally on X and write $D = \{f = 0\}$ near x_0. Define $\omega := \frac{df}{f} \wedge \tilde{h}$ locally on X, then ω is a local section in $\tilde{\Omega}^1_X\,(\log D)$ and $\psi\,(\omega) = h$. □

4.2.12 Remark. If $n = \dim X$ in the setting of Proposition 4.2.10, then the sequence

$$0 \to \Omega^n_X \to \tilde{\Omega}^n_X\,(\log D) \xrightarrow{\psi} \tilde{\Omega}^{n-1}_D \to 0$$

is exact. In fact, $\tilde{\Omega}^n_X\,(\log D) = \mathcal{O}_X\,(K_X + D)$ and $\tilde{\Omega}^{n-1}_D = \mathcal{O}_D\,(K_D)$; so the assertion follows from the adjunction formula.

Now we prove that ψ is surjective in special cases; we only treat the situation we are interested in.

4.2.13 Proposition. *Let X be a 3-dimensional projective manifold and $D \subset X$ a normal surface with rational singularities. Then all sequences*

$$0 \to \Omega^k_X \to \tilde{\Omega}^k_X\,(\log D) \xrightarrow{\psi} \tilde{\Omega}^{k-1}_D \to 0$$

are exact.

Proof of Proposition 4.2.13: It remains to treat the case $k = 2$. Let $\pi : \widehat{X} \to X$ be an embedded resolution of singularities of D. So the strict transform $\widehat{D}$ of D in $\widehat{X}$ is smooth, and we obtain the residue sequence

$$0 \to \Omega^2_{\widehat{X}} \to \Omega^2_{\widehat{X}}\left(\log \widehat{D}\right) \to \Omega^1_{\widehat{D}} \to 0. \tag{4.2.13.1}$$

Since $\pi_*\Omega^2_{\widehat{X}} \cong \Omega^2_X$, since $\pi_*\Omega^1_{\widehat{D}} \cong \tilde{\Omega}^1_D$ (D has quotient singularities, then use Remark 4.2.42) and since $R^1\pi_*\Omega^2_{\widehat{X}} = 0$ (π consists only of blow-ups of points), we obtain the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2_X \arrow[r] \arrow[d, "\cong"] & \pi_*\left(\Omega^2_{\widehat{X}}\left(\log \widehat{D}\right)\right) \arrow[r] \arrow[d, "\alpha"] & \tilde{\Omega}^1_D \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \Omega^2_X \arrow[r] & \tilde{\Omega}^2_X(\log D) \arrow[r] & Q \arrow[r] & 0.
\end{tikzcd}$$

Since $Q \subset \tilde{\Omega}^1_D$, and since β is generically an isomorphism, β is an isomorphism everywhere; so is α. This proves the claim. $\square$

4.2.14 Remark.

1. The proof of Proposition 4.2.13 works in any dimension and any degree provided
$$\pi_*\Omega^q_{\widehat{D}} \cong \Omega^q_D$$
for all q and that the map
$$R^1\pi_*\Omega^q_{\widehat{X}} \to R^1\pi_*\Omega^q_{\widehat{X}}\left(\log \widehat{D}\right)$$
induced by Sequence 4.2.13.1 is injective.

   The first condition is satisfied if D has canonical singularities, see [GKKP11].

2. The proof also shows the following: Let $\pi : \widehat{X} \to X$ be an embedded resolution for the normal divisor $D \subset X$. We assume that D has only quotient or canonical singularities and that the injectivity assumption in 1. holds. Let $\widehat{D}$ be the strict transform of D. Then
$$\pi_*\Omega^k_{\widehat{X}}\left(\log \widehat{D}\right) \cong \tilde{\Omega}^k_X(\log D).$$

4.2.15 Proposition. *Let X be a projective manifold, D_1 a smooth divisor, D_2 a normal divisor such that $D = D_1 \cap D_2$ is normal and D_1 and D_2 meet transversally outside* Sing(D). *Then we have exact sequences*

$$0 \to \tilde{\Omega}^k_X(\log D_2) \to \tilde{\Omega}^k_X(\log(D_1 \cup D_2)) \xrightarrow{\psi} \tilde{\Omega}^{k-1}_{D_1}(\log D).$$

If $k = 1$, then $\tilde{\Omega}^{k-1}_{D_1}(\log D) = \mathcal{O}_{D_1}$ and ψ is surjective.

Proof of Proposition 4.2.15: The proof is the same as for Proposition 4.2.11 and Proposition 4.2.10 and therefore is omitted. □

4.2.16 Proposition. *Let X be a 4-dimensional projective manifold, $D_1 \subset X$ be a smooth divisor and $D_2 \subset X$ be a normal divisor with only canonical singularities. We assume furthermore*

1. *$D = D_1 \cap D_2$ is a normal divisor with rational singularities;*
2. *D_1 and D_2 meet transversally outside the singular locus $\operatorname{Sing}(D)$;*
3. *$D_1 \cap \operatorname{Sing}(D_2)$ is contained in the locus of D_2 where D_2 has the local product structure $U \times S$ with U an open set in $\mathbb{C}$ and S a local surface with a rational double point.*

Then we have exact sequences

$$0 \to \tilde{\Omega}_X^k (\log D_2) \to \tilde{\Omega}_X^k (\log (D_1 \cup D_2)) \xrightarrow{\kappa} \tilde{\Omega}_{D_1}^{k-1} (\log D) \to 0$$

for each $k = 1, \ldots, 4$.

Proof of Proposition 4.2.16: We only need to treat the cases $k = 2, 3$. Using Proposition 4.2.15 it suffices to show that κ is surjective. By our assumption, it suffices to show surjectivity of κ at $p \in \operatorname{Sing}(D)$.

Case 1: k=2 Since $\tilde{\Omega}_X^2 (\log (D_1 \cup D_2))$ and $\tilde{\Omega}_{D_1}^1 (\log D)$ are reflexive, it suffices to show that

$$\kappa_V : \tilde{\Omega}_X^2 (\log (D_1 \cup D_2)) (V \backslash p) \to \tilde{\Omega}_{D_1}^1 (\log D) \left(\tilde{V} \backslash p\right)$$

is surjective for V an arbitrary small Stein neighbourhood of p in X and $\tilde{V} = V \cap D_1$. Clearly, κ_V is surjective if

$$H^1 \left(V \backslash p, \tilde{\Omega}_X^2 (\log (D_2))\right) = 0. \tag{4.2.16.1}$$

For Equation 4.2.16.1 we use the residue sequence

$$0 \to \Omega_X^2 \to \tilde{\Omega}_X^2 (\log D_2) \to \tilde{\Omega}_{D_2}^1 \to 0 \tag{4.2.16.2}$$

on V. For Sequence 4.2.16.2, we use the local product structure of D_2: locally $D_2 = U \times S \subset U \times Y = V$ with a smooth 3-fold Y. Then Sequence 4.2.16.2 has the form

$$0 \to \Omega_{U \times Y}^2 \to \tilde{\Omega}_{U \times Y}^2 (\log (U \times S)) \to \tilde{\Omega}_{U \times S}^1 \to 0. \tag{4.2.16.3}$$

Sequence 4.2.16.2 follows from the exact sequences

$$0 \to \Omega_Y^k \to \tilde{\Omega}_Y^k (\log S) \to \tilde{\Omega}_S^{k-1} \to 0, \tag{4.2.16.4}$$

see 4.2.13, because we have the following isomorphisms:

$$\begin{aligned}
\Omega^2_{U\times Y} &\cong \left(\mathrm{pr}_1^*\Omega^1_U \otimes \mathrm{pr}_2^*\Omega^1_Y\right) \oplus \mathrm{pr}_2^*\Omega^2_Y, \\
\tilde{\Omega}^2_{U\times Y}\left(\log\left(U\times S\right)\right) &\cong \left(\mathrm{pr}_1^*\Omega^1_U \otimes \mathrm{pr}_2^*\tilde{\Omega}^1_Y\left(\log S\right)\right) \oplus \mathrm{pr}_2^*\tilde{\Omega}^2_Y\left(\log S\right), \\
\tilde{\Omega}^1_{U\times S} &\cong \mathrm{pr}_1^*\Omega^1_U \oplus \mathrm{pr}_2^*\tilde{\Omega}^1_S.
\end{aligned}$$

The middle isomorphism is at first only valid on $U\times Y \setminus \mathrm{Sing}\left(D_2\right)$, then due to reflexivity also on $U\times Y$. These isomorphisms provide a decomposition of Sequence 4.2.16.3 in two sequences related to the exact sequences 4.2.16.4 for $k=1,2$. Then the exactness of Sequence 4.2.16.2 follows.

By 4.2.16.2, the equality 4.2.16.1 comes down to

$$H^1\left(V\backslash p, \Omega^2_X\right) = 0 \tag{4.2.16.5}$$

and

$$H^1\left(V'\backslash p, \tilde{\Omega}^1_{D_2}\right) = 0, \tag{4.2.16.6}$$

where $V' := V\cap D_2$ is Stein again.

To show Equation 4.2.16.5, observe that the Riemann extension theorem (for locally free sheaves, see [Sch61] and [Sch64]) gives

$$H^1\left(V,\Omega^2_V\right) \xrightarrow{\cong} H^1\left(V\backslash p,\, \Omega^2_V\big|_{V\backslash p}\right).$$

But $H^1\left(V,\Omega^2_V\right)=0$, since V is Stein.

To show Equation 4.2.16.6, we claim that

$$H^1\left(V',\, \tilde{\Omega}^1_{D_2}\Big|_{V'}\right) \to H^1\left(V'\backslash p,\, \tilde{\Omega}^1_{D_2}\Big|_{V'}\right) \tag{4.2.16.7}$$

is an isomorphism. To see this, we need $\mathrm{prof}\left(\tilde{\Omega}^1_{\mathrm{D_2,p}}\right) \geqslant 3$ (see [BS76], II.3.10). Here prof denotes the homological codimension of $\tilde{\Omega}^1_{D_2,p}$ as an $\mathcal{O}_{D_2,p}$-module. Since

$$H^1\left(V',\, \tilde{\Omega}^1_{D_2}\Big|_{V'}\right) \cong H^1\left(V', \mathrm{pr}_1^*\Omega^1_U\right) \oplus H^1\left(V', \mathrm{pr}_2^*\tilde{\Omega}^1_S\right)$$

and $\mathrm{pr}_1^*\Omega^1_U \cong \mathcal{O}_{U\times S}$, furthermore $U\times S$ is Cohen-Macaulay, it suffices to show that $\mathrm{prof}\left(\mathrm{pr}_2^*\tilde{\Omega}^1_{\mathrm{S}}\right) \geqslant 3$. Since $V' = U\times S$, this is equivalent to $\mathrm{prof}_{\mathcal{O}_{\mathrm{S,p'}}}\left(\tilde{\Omega}^1_{\mathrm{S,p'}}\right) \geqslant 2$, where $p=(0,p')$. Indeed, let $\left(f_{1,p'}, f_{2,p'}\right)$ be a regular sequence for $\tilde{\Omega}^1_{S,p'}$ and t a holomorphic coordinate of U. We write t for pr_1^*t. Then $\left(t_p, \left(\mathrm{pr}_2^*f_1\right)_p, \left(\mathrm{pr}_2^*f_2\right)_p\right)$ is a regular sequence for $\left(\mathrm{pr}_2^*\tilde{\Omega}^1_S\right)_p$. This follows as the quotient sheaf $\left(\mathrm{pr}_2^*\tilde{\Omega}^1_S\right)_p / \left(t\cdot \mathrm{pr}_2^*\tilde{\Omega}^1_S\right)_p$ has support $\{0\}\times S$ and equals $\tilde{\Omega}^1_S$.

We obtain $\operatorname{prof}_{\mathcal{O}_{\mathrm{S,p}'}}\left(\tilde{\Omega}^1_{\mathrm{S,p}'}\right) \geqslant 2$ by e.g. [BS76], II.3.15, as $\tilde{\Omega}^1_S$ is a reflexive sheaf.

Then we conclude with Theorem B.

Case 2: k=3 This is done in exactly the same way. □

We comment on the assumption 3 in Proposition 4.2.16:

4.2.17 Remark. Since D_2 has canonical Gorenstein singularities, there exists a finite set $M \subset \operatorname{Sing}(D_2)$ such that at $p \in \operatorname{Sing}(D) \backslash M$, the variety D_2 has locally the form $U \times S$ as assumed in Proposition 4.2.16, see [Rei87].

There is an alternative way to define residues by using tube maps in case X is compact:

4.2.18 Theorem. *([Gri69], Prop. 3.5) Let $V \subset \mathbb{P}^n$ be a smooth hypersurface. There is a $\mathbb{Z}$-linear map, the tube map,*

$$\tau : H_{n-1}(V, \mathbb{Z}) \to H_n(\mathbb{P}^n \backslash V, \mathbb{Z})$$

such that $\tau(\gamma)$ is given geometrically by taking a tube over γ. The map τ is always surjective and injective if n is even.

The proof actually can be adapted to show the following:

4.2.19 Corollary. *Let X be a projective manifold and $D \subset X$ a smooth divisor. Then Theorem 4.2.18 can be adapted to this situation such that we get a tube map*

$$\tau : H_{n-1}(D, \mathbb{Z}) \to H_n(X \backslash D, \mathbb{Z})$$

satisfying the properties stated in Theorem 4.2.18.

4.2.20 Theorem. *For $\omega \in H^0\left(X, \Omega^p_X(\log D)\right)$ the residue can be computed as an integral for each $\gamma \in C_p(D)$ by*

$$\int_\gamma \operatorname{res}^p_{D|X}(\omega) := \int_{\tau(\gamma)} \omega.$$

This leads to a generalization of the notion of the residue.

4.2.21 Definition. *For each rational k-form $\eta \in H^0\left(X, \Omega^p_X(D)\right)$ on X with poles along D we define a residue by*

$$\int_\gamma \operatorname{res}^p_{D|X}(\eta) := \int_{\tau(\gamma)} \eta$$

for each cycle $\gamma \in C_k(D)$.

We now turn to the cohomological level.

4.2.22 Definition. *The residue map 4.2.31.1 on the complex of logarithmic differential forms induces a residue map in cohomology*

$$\mathrm{Res}^k_{X|\mathbb{P}^n} : H^k\left(X \backslash D, \mathbb{C}\right) \to H^{k-1}\left(D, \mathbb{C}\right) \qquad (4.2.22.1)$$

for each k.

The following alternative definition of the residue map in cohomology is important:

4.2.23 Theorem. *([Voi03], Chapter 6.1.1)*

1. *Let $U := X\backslash D$. The residue map 4.2.22.1 can also be defined as composition*

   $$\mathrm{Res}^k_{D|X} : H^k\left(U, \mathbb{C}\right) \to H^{k+1}\left(X, U, \mathbb{C}\right) \cong H^{k+1}\left(T, \partial T, \mathbb{C}\right) \cong H^{k-1}\left(D, \mathbb{C}\right),$$

   where T is a tubular neighbourhood of D in X.

2. *The residue map is part of the long exact sequence of relative cohomology of the pair (X, U), i.e.,*

   $$\ldots \to H^k\left(X, \mathbb{Z}\right) \to H^k\left(U, \mathbb{Z}\right) \stackrel{\mathrm{Res}^k_{D|X}}{\to} H^{k-1}\left(D, \mathbb{Z}\right) \stackrel{l_*}{\to} H^{k+1}\left(X, \mathbb{Z}\right) \to \ldots,$$

   where l_* denotes the Gysin morphism.

To fix notations, we recall

4.2.24 Definition.

1. $$H^k\left(X, \mathbb{Q}\right)_{\mathrm{prim}} := \ker\left(L : H^k\left(X, \mathbb{R}\right) \to H^{2n-k+2}\left(X, \mathbb{R}\right)\right)$$

2. $$H^k\left(D, \mathbb{C}\right)_{\mathrm{van}} := \ker\left(l_* : H^k\left(D, \mathbb{Q}\right) \to H^{k+2}\left(X, \mathbb{Q}\right)\right)$$

4.2.25 Theorem. *([Voi03], 2.3.3) If D is ample, then there is an isomorphism*

$$H^n\left(X, \mathbb{Q}\right)_{\mathrm{prim}} = H^n\left(X, \mathbb{Q}\right)/l_* H^{n-2}\left(D, \mathbb{Q}\right)$$

and an exact sequence

$$0 \to H^n\left(X, \mathbb{Q}\right)_{\mathrm{prim}} \stackrel{j^*}{\to} H^n\left(U, \mathbb{Q}\right) \stackrel{\mathrm{Res}^n_{D|X}}{\to} H^{n-1}\left(D, \mathbb{Q}\right)_{\mathrm{van}} \to 0. \qquad (4.2.25.1)$$

4.2.3 The notion of mixed Hodge structures and variations of mixed Hodge structures

Mixed Hodge structures occur our context in the following way:

4.2.26 Definition. *(See e.g. [PS08], p. 62) Let $H_{\mathbb{Z}}$ be a finitely generated $\mathbb{Z}$-module and $H := H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification.*

A mixed Hodge structure *of $H_{\mathbb{R}}$ on a finite dimensional $\mathbb{Z}$-module $H_{\mathbb{Z}}$ consists of:*

1. *the* Hodge filtration*, i.e. a decreasing filtration*

$$H \supset \ldots \supset F^{p+1} \supset F^p \supset F^{p-1} \supset \ldots \supset 0,$$

2. *the* weight filtration*, an increasing filtration*

$$0 \subset \ldots \subset W_{m-1} \subset W_m \subset W_{m-1} \subset \ldots \subset H$$

such that the two filtrations verify the condition:

for each m the Hodge filtration induces a pure Hodge structure of weight m on the m-th graded element $\mathrm{Gr}_m^W := W_m/W_{m-1}$ of the weight filtration.

4.2.27 Remark. The general element of the induced filtration is

$$F^p \mathrm{Gr}_m^W = (W_m \cap F^p)/W_{m-1}.$$

4.2.28 Definition. *(See e.g. [PS08], p.362) Let S be a complex manifold. A* variation of mixed Hodge structure *on S consists of the following data:*

1. *a local system $H_{\mathbb{Z}}$ of finitely generated abelian groups on S,*

2. *the* Hodge filtration*, i.e. a finite decreasing filtration $\{\mathcal{F}\}$ of the holomorphic vector bundle $\mathcal{H} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_S$ by holomorphic subbundles,*

3. *the* weight filtration*, i.e. a finite increasing filtration $\{W_m\}$ of the local system $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ by local subsystems.*

These data are subject to the following conditions:

1. *for each $s \in S$ the filtrations $\{\mathcal{F}^p(s)\}$ and $\{W_m\}$ of $H(s) \cong H_{\mathbb{Z},s} \otimes_{\mathbb{Z}} \mathbb{C}$ define a mixed $\mathbb{Q}$-Hodge structure on the $\mathbb{Q}$-vector space $H_{\mathbb{Q},s}$,*

2. *the connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{O}_S} \Omega_S^1$ whose sheaf of horizontal sections is $H_{\mathbb{C}}$ satisfies the* Griffiths' transversality condition

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega_S^1.$$

4.2.4 The mixed Hodge structure on the hypercohomology of the complex of logarithmic differential forms

Let U be a complex algebraic manifold and X a good compactification, i.e. $D := X \backslash U$ is a divisor with simple normal crossings.

4.2.29 Theorem. *(See e.g. [PS08], Thm. 4.2, p. 90) The following two filtrations put a mixed Hodge structure on $H^k(U, \mathbb{C})$:*

1. *The decreasing Hodge filtration on $H^k(U, \mathbb{C})$ is induced by the decreasing trivial filtration $F^\bullet \Omega_X^\bullet(\log D)$ on $\Omega_X^\bullet(\log D)$, i.e.,*
$$F^p H^k(U, \mathbb{C}) := \operatorname{im}\left(\mathbb{H}^k(X, F^p \Omega_X^\bullet(\log D)) \to H^k(U, \mathbb{C})\right),$$
where
$$F^p \Omega_X^\bullet(\log D) :\equiv 0 \to \ldots \to 0 \to \Omega_X^p(\log D) \to \Omega_X^{p+1}(\log D) \to \ldots .$$

2. *The increasing weight filtration on $H^k(U, \mathbb{C})$ is induced by the increasing weight filtration $W_\bullet \Omega_X^\bullet(\log D)$ on $\Omega_X^\bullet(\log D)$, i.e.,*
$$W_m H^k(U, \mathbb{C}) := \operatorname{im}\left(\mathbb{H}^k(X, W_{m-k} \Omega_X^\bullet(\log D)) \to H^k(U, \mathbb{C})\right),$$
where
$$W_m \Omega_X^p(\log D) := \begin{cases} 0 & \text{for } m < 0 \\ \Omega_X^p(\log D) & \text{for } m \geqslant p \\ \Omega_X^{p-m} \wedge \Omega_X^m(\log D) & \text{if } 0 \leqslant m \leqslant p. \end{cases}$$

4.2.30 Theorem. *([PS08], p. 93) The residue map 4.2.6.1 restricts to a surjective map*
$$\operatorname{res}_I : W_m \Omega_X^\bullet(\log D) \to \Omega_{D_I}^\bullet[-m]$$
and induces an isomorphism of complexes
$$\operatorname{res}_m = \bigoplus_{|I|=m} \operatorname{res}_I : \operatorname{Gr}_m^W \Omega_X^\bullet(\log D) \xrightarrow{\cong} a_{m*} \Omega_{D(m)}^\bullet[-m].$$

4.2.31 Remark. If the divisor D is smooth, we can locally find holomorphic coordinates $\{z_1, \ldots, z_n\}$ on X such that $D = \{z_1 = 0\}$. Then we get the residue map
$$\operatorname{res} : \Omega_X^\bullet(\log D) \to \Omega_D^\bullet[-1], \omega = \frac{dz_1}{z_1} \wedge \eta + \eta' \mapsto \eta|_D, \tag{4.2.31.1}$$
where η and η' do not contain $\frac{dz_1}{z_1}$. It induces a surjective map
$$\operatorname{res} : W_1 \Omega_X^\bullet(\log D) \to \Omega_D^\bullet[-1]$$
and an isomorphism
$$\operatorname{res} : \operatorname{Gr}_1^W \Omega_X^\bullet(\log D) = \Omega_X^{\bullet-1} \wedge \Omega_X^1(\log D) \xrightarrow{\cong} a_{1*} \Omega_D^\bullet[-1].$$

4.2.32 Remark. ([PS08], [Voi03])

1. The spectral sequence associated to the Hodge filtration of $H^k(U,\mathbb{C})$ degenerates at E_1, furthermore the maps
$$\mathbb{H}^k\left(X, F^p\Omega_X^\bullet(\log D)\right) \to H^k(U,\mathbb{C})$$
are injective. Therefore
$$\begin{aligned} F^pH^k(U,\mathbb{C}) &= \operatorname{im}\left(\mathbb{H}^k\left(X, F^p\Omega_X^\bullet(\log D)\right) \to H^k(U,\mathbb{C})\right) \\ &\cong \mathbb{H}^k\left(X, F^p\Omega_X^\bullet(\log D)\right) \cong \bigoplus_{\substack{r+s=k \\ s\geqslant p}} H^r\left(X, \Omega_X^s(\log D)\right) \end{aligned}$$
and
$$\mathbb{H}^k\left(X, \Omega_X^\bullet(\log D)\right) \cong \bigoplus_{r+s=k} H^r\left(X, \Omega_X^s(\log D)\right).$$

2. If the divisor D is smooth, i.e. it has one smooth irreducible component, then the weight filtration on $\Omega_X^\bullet(\log D)$ consists of two steps:
$$W_0\Omega_X^\bullet(\log D) \equiv \Omega_X^\bullet$$
and
$$W_1\Omega_X^\bullet(\log D) \equiv \Omega_X^\bullet(\log D).$$
Therefore
$$W_3H^3(U,\mathbb{C}) \subset W_4H^3(U,\mathbb{C}) = H^3(U,\mathbb{C}).$$

4.2.5 Relative de Rham-Cohomology

Later we will need a de Rham theory for pairs; the relevant definitions are found below.

Let X be an n-dimensional compact complex manifold, $D \overset{\iota}{\hookrightarrow} X$ a smooth hypersurface.

4.2.33 Definition. *(See e.g. [BT82], p.78) The* relative cohomology *of X and D is the cohomology of the complex*
$$\mathcal{A}^\bullet_{(X,D)} := \left(\mathcal{A}_X^k \oplus \mathcal{A}_D^{k-1}\right)_{k\in\mathbb{N}}$$
with the differential
$$\tilde{d}(\alpha,\beta) := (d\alpha, \iota^*\alpha - d\beta)$$
for $\alpha \in \mathcal{A}_X^k, \beta \in \mathcal{A}_D^{k-1}$, i.e.,
$$H^k(X,D,\mathbb{C}) := \frac{\left\{(\alpha,\beta) \in \mathcal{A}_X^k \oplus \mathcal{A}_D^{k-1} \,\middle|\, \tilde{d}(\alpha,\beta) = 0\right\}}{\tilde{d}\left(\mathcal{A}_X^{k-1} \oplus \mathcal{A}_D^{k-2}\right)}$$
for each $k \in \mathbb{N}$.

4.2.34 Remark.
We get a short exact sequence of complexes

$$0 \to \mathcal{A}_D^{\bullet-1} \to \mathcal{A}_{(X,D)}^{\bullet} \to \mathcal{A}_X^{\bullet} \to 0$$

which yields a long exact sequence in cohomology

$$\begin{aligned}&\cdots \to H^{k-1}(X,\mathbb{C}) \to H^{k-1}(D,\mathbb{C}) \to H^k(X,D,\mathbb{C}) \to H^k(X,\mathbb{C}) \to \\ &\to H^k(D,\mathbb{C}) \to H^{k+1}(X,D,\mathbb{C}) \to \cdots .\end{aligned}$$

Therefore we get a decomposition of the relative cohomology

$$\begin{aligned}H^k(X,D,\mathbb{C}) &\cong \ker\left(H^k(X,\mathbb{C}) \to H^k(D,\mathbb{C})\right)\\ &\oplus \operatorname{coker}\left(H^{k-1}(X,\mathbb{C}) \to H^{k-1}(D,\mathbb{C})\right).\end{aligned}$$

We denote

$$H_{var}^{n-1}(D) := \operatorname{coker}\left(H^{n-1}(X,\mathbb{C}) \to H^{n-1}(D,\mathbb{C})\right).$$

4.2.35 Definition.

1. *We define the relative singular chain complex $C_\bullet(X,D)$ by setting*

$$C_k(X,D) := C_k(X)/C_k(D).$$

Here $C_k(X)$ denotes the space of k-cycles in X (with complex coefficients), $C_k(D)$ is analogously defined. We obtain homology groups $H_k(X,D)$.

2. *There is a duality pairing between the relative cohomology $H^k(X,D,\mathbb{C})$ and the relative homology $H_k(X,D,\mathbb{C})$ for each k, which is given by*

$$\begin{aligned}\Pi: \quad & H_k(X,D,\mathbb{C}) \times H^k(X,D,\mathbb{C}) \to \mathbb{C},\\ & ([\gamma],[(\eta_1,\eta_2)]) \mapsto \int_\gamma \eta_1 - \int_{\partial(\gamma)} \eta_2.\end{aligned}$$

Then we have the following classical theorem:

4.2.36 Theorem. *The pairing in Definition 4.2.35 is non-degenerate.*

4.2.37 Theorem. *There is an isomorphism*

$$H^{2n-k}(X,D,\mathbb{C}) \cong H^k(X\backslash D,\mathbb{C}) \cong \mathbb{H}^k(\Omega_X^\bullet(\log(D))).$$

Proof of Theorem 4.2.37: By Lefschetz duality we have an isomorphism

$$H_k(X\backslash D, \mathbb{Z}) \cong H^{2n-k}(X, D, \mathbb{Z}).$$

for each $0 \leqslant k \leqslant 2n$. Using the universal coefficient theorem for $X\backslash D$ we get

$$H^k(X\backslash D, \mathbb{Z}) \cong H^{2n-k}(X, D, \mathbb{Z}).$$

□

4.2.38 Remark. If we define $H^k(X, D, \mathbb{C}) := \operatorname{Hom}(H_k(X, D), \mathbb{C})$, then Theorem 4.2.37 remains true for divisors with simple normal crossings.

4.2.6 Hodge theory on quotients by finite groups

In this chapter we consider a normal projective variety X with a finite group G acting on X. Then the quotient X/G is again a normal projective variety. Let $p : X \to X/G$ be the quotient map. This will be important in Chapter 8, where X is a quintic.

The group G acts on the cohomology $H^q(X, \mathbb{C})$. More generally, if $\mathcal{F}$ is a G-sheaf (i.e., $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module and $\rho^*(\mathcal{F}) \cong \mathcal{F}$ for each $\rho \in G$), then G acts on $H^q(X, \mathcal{F})$.

We define the sheaf $p_*(\mathcal{F})^G$ by

$$p_*(\mathcal{F})^G(U) := \mathcal{F}\left(p^{-1}(U)\right)^G$$

for any open set $U \subset X/G$, where $\mathcal{F}\left(p^{-1}(U)\right)^G$ denotes the G-invariant part of $\mathcal{F}\left(p^{-1}(U)\right)$.

4.2.39 Definition. *Let $H^q(X, \mathbb{C})^G$ be the G-invariant part of $H^q(X, \mathbb{C})$ and $H^q(X, \mathcal{F})^G$ be the G-invariant part of $H^q(X, \mathcal{F})$.*

4.2.40 Proposition. *Given $q \geqslant 0$, then there are isomorphisms*

1. $H^q(X, \mathbb{C})^G \cong H^q(X/G, \mathbb{C})$,
2. $H^q(X, \mathcal{F})^G \cong H^q\left(X/G, p_*(\mathcal{F})^G\right)$ *and*
3. $H^q(X, \mathcal{O}_X)^G \cong H^q\left(X/G, \mathcal{O}_{X/G}\right)$.

Proof of Proposition 4.2.40: For 1, we refer to [Bre72], Theorem 2.4 and Theorem 7.2. For 2, we refer to a manuscript of J. Kollár [Kol]. 3 is a special case of 2, since $p_*(\mathcal{O}_X)^G = \mathcal{O}_{X/G}$, see also [GKKP11], Lemma 4.2. □

In the following we consider normal projective varieties with only quotient singularities, i.e., Y is locally of the form U/G, where $U \subset \mathbb{C}^n$ is an open ball and $G \subset \mathrm{Gl}(\mathrm{n}, \mathbb{C})$ a small finite group. Following [Ste77], see also [PS08], Section 2.5 (where Y is called a V-manifold), we define the Hodge groups $H^{p,q}(Y)$ of Y.

4.2.41 Definition. *Let Y be a normal projective variety.*

1. *Let $\iota : Y_{reg} \hookrightarrow Y$ be the regular part of Y. We define $\tilde{\Omega}^q_Y := \iota_* \left(\Omega^q_{Y_{reg}}\right)$ (in accordance with the log case treated before).*
2. *We set $H^{p,q}(Y) := H^q\left(Y, \tilde{\Omega}^p_Y\right)$.*

4.2.42 Remark. [Ste77] We suppose that Y has only quotient singularities.

1. If $\pi : \tilde{Y} \to Y$ is a desingularisation, then $\tilde{\Omega}^q_Y = \pi_* \left(\Omega^q_{\tilde{Y}}\right)$.
2. There is a Frölicher-type spectral sequence with E_1-term
$$E_1^{p,q} = H^{p,q}(Y)$$
converging to $H^{p+q}(Y, \mathbb{C})$, which degenerates at E_1, so that
$$H^r(Y, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(Y).$$
3. These remarks also apply if Y is a normal projective toric variety, [Dan78], Theorem 12.5, instead assuming quotient singularities. For the analogue of 1 see [GKKP11], even in greater generality.

4.2.43 Theorem. *Let X be a projective manifold.*

1. *Let $\dim X = 3$ and let $D \subset X$ be a normal divisor which has only quotient singularities or which is a toric variety. Then there is a spectral sequence $(E_r)_r$ with E_1-term*
$$E_1^{p,q} = H^q\left(X, \tilde{\Omega}^p_X(\log D)\right)$$
and the following properties:
   - *(E_r) converges to $\mathbb{H}^*\left(\tilde{\Omega}^\bullet_X(\log D)\right)$,*
   - *(E_r) degenerates at E_1,*
   - *$\mathbb{H}^r\left(\tilde{\Omega}^\bullet_X(\log D)\right) \cong H^r(X \backslash D, \mathbb{C})$ for all r.*
2. *We suppose that $\dim X = 4$. Let D_1 and D_2 be irreducible divisors, D_1 smooth and D_2 with only quotient singularities or a toric variety. Suppose additionally*

(i) $D = D_1 \cap D_2$ *is a normal divisor with rational singularities;*

(ii) D_1 *and* D_2 *meet transversally outside the singular locus* $\mathrm{Sing}\,(D)$*;*

(iii) $D_1 \cap \mathrm{Sing}\,(D_2)$ *is contained in the locus of* D_2 *where* D_2 *has the local product structure* $U \times S$ *with* U *an open set in* $\mathbb{C}$ *and* S *a local surface with a rational double point.*

Then there is a spectral sequence $(E_r)_r$ *with* E_1*-term*

$$E_1^{p,q} = H^q\left(X, \tilde{\Omega}_X^p\,(\log D_1 \cup D_2)\right)$$

and the following properties:

- (E_r) *converges to* $\mathbb{H}^*\left(\tilde{\Omega}_X^\bullet\,(\log D_1 \cup D_2)\right)$,
- (E_r) *degenerates at* E_1,
- $\mathbb{H}^r\left(\tilde{\Omega}_X^\bullet\,(\log D_1 \cup D_2)\right) \cong H^r\left(X \backslash (D_1 \cup D_2)\,, \mathbb{C}\right)$ *for all* r.

Proof of Theorem 4.2.43: The spectral sequence (E_r) is just the sequence of hypercohomology to the logarithmic complex in question. Then we consider the exact sequences of the complexes (see Proposition 4.2.16 and Proposition 4.2.13)

$$0 \to \Omega_X^\bullet \to \tilde{\Omega}_X^\bullet\,(D) \to \tilde{\Omega}_D^{\bullet-1} \to 0$$

and

$$0 \to \tilde{\Omega}_X^\bullet\,(\log D_2) \to \tilde{\Omega}_X^\bullet\,(\log\,(D_1 \cup D_2)) \to \tilde{\Omega}_{D_1}^{\bullet-1}\,(\log D) \to 0,$$

and apply Remark 4.2.42 to conclude that (E_r) at E_1. The last statements follows from the long exact sequence of hypercohomology attached to the above complexes. $\square$

Chapter 5

Comparison between two deformation problems associated with a closed submanifold in a compact complex manifold

In this chapter we consider pairs (X, Z), where X is a compact complex manifold and Z a compact complex submanifold in X, and compare the deformations of the pairs (X, Z) and $(X, \iota_* \mathcal{O}_Z)$, where ι denotes the inclusion of Z in X. The results of this chapter will be used later, when X is a Calabi-Yau 3-fold and Z is a smooth curve or a smooth divisor.

5.1 Statement of the main theorems

5.1.1 Theorem. *Let Z be a closed submanifold of codimension d of a compact complex manifold X and $\iota : Z \hookrightarrow X$ the inclusion. Then there is a natural linear isomorphism of simultaneous first-order deformations*

$$\operatorname{Def}(X, \iota_* \mathcal{O}_Z) \cong \operatorname{Def}(X, Z) \oplus H^1(Z, \mathcal{O}_Z). \tag{5.1.1.1}$$

Concerning obstructions, we restrict ourselves to the situation we are interested in:

5.1.2 Theorem. *We assume that $H^1(Z, \mathcal{O}_Z) = 0$ or $H^2(Z, \mathcal{O}_Z) = 0$, e.g. Z is an ample divisor in a Calabi-Yau 3-fold or a smooth curve in a compact manifold. Then*

$$\operatorname{Obs}(X, \iota_* \mathcal{O}_Z) = \operatorname{Obs}(X, Z) \subset H^2(X, T_X \langle -Z \rangle).$$

5.2 Preparations

We recall some definitions from the theory of coherent sheaves, see e.g. [GR84] or [Fis76].

5.2.1 Definition. *We denote by $\mathcal{F}$ a coherent sheaf on the complex manifold X.*

1. *Let* Ann $(\mathcal{F})$ *be the* annihilator *of $\mathcal{F}$, i.e.,*

$$\mathrm{Ann}\,(\mathcal{F})_x := \{\, f_x \in \mathcal{O}_{X,x} |\, f_x \cdot s_x = 0 \textit{ for each } s_x \in \mathcal{F}_x\}$$

for each $x \in X$.

2. *Let* Supp $(\mathcal{F})$ *denote the* support *of $\mathcal{F}$, i.e., the complex space*

$$\mathrm{Supp}\,(\mathcal{F}) := (\{\, p \in X |\, \mathcal{F}_p \neq 0\}\,, \mathcal{O}_X/\mathrm{Ann}\,(\mathcal{F}))\,.$$

So Supp $(\mathcal{F})$ is not only an analytic set, but carries a natural complex structure.

5.2.2 Lemma. *Let S be the spectrum of an Artin ring or the germ of a complex space. Let Z be a closed submanifold of codimension d of a compact complex manifold X and $\iota : Z \hookrightarrow X$ be the inclusion.*

Any deformation $(\mathcal{X}, \mathcal{F})$ of the pair $(X, \iota_* \mathcal{O}_Z)$ over S is a pair consisting of a deformation $\mathcal{X}$ of X over S and a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ which is a locally free sheaf of rank 1 *on* Supp $(\mathcal{F})$ *such that $\mathcal{F}|_X \cong \iota_* \mathcal{O}_Z$.*

Proof of Lemma 5.2.2: The restriction of $\mathcal{F}$ to its support Supp $(\mathcal{F})$ is a coherent sheaf whose restriction to the central fibre X is a locally free sheaf of rank 1 on Supp $(\mathcal{F}) \cap X = Z$, namely

$$\left(\mathcal{F}|_{\mathrm{Supp}(\mathcal{F})} \right)\Big|_X \cong \mathcal{O}_Z.$$

Since $\mathcal{X} \to S$ is a submersion, $\mathcal{X}$ is locally isomorphic to a product. So locally in $\mathcal{X}$ near $p \in$ Supp $(\mathcal{F})$, we can write

$$\mathcal{X} \cong U\,(p) \times S,$$

where $U\,(p)$ is a small neighbourhood of $p \in Z$. Moreover, we choose a suitable $\epsilon > 0$ and consider S as a subspace of $U_\epsilon\,(0)$, where $U_\epsilon\,(0) = \left\{ z \in \mathbb{C}^N \big|\, |z| < \epsilon \right\}$ for some $N > 0$. Hence $\mathcal{X}$ is locally a subspace of $U\,(p) \times U_\epsilon\,(0)$.

Possibly, after shrinking $U\,(p)$ and ϵ, we find a coherent sheaf $\tilde{\mathcal{F}}$ on $U\,(p) \times U_\epsilon\,(0)$ such that the restriction to $U\,(p) \times S$ is

$$\tilde{\mathcal{F}}\Big|_{U(p)\times S} = \mathcal{F}|_{U(p)\times S}\,.$$

We claim that $\tilde{\mathcal{F}}\Big|_{\mathrm{Supp}(\tilde{\mathcal{F}})}$ is locally free of rank 1 on $\mathrm{Supp}\left(\tilde{\mathcal{F}}\right)$, possibly after shrinking $U(p)$ and ϵ again. Since $\mathcal{F}|_Z = \mathcal{O}_Z$, we know

$$\dim_{\mathbb{C}} \tilde{\mathcal{F}}(x) = \dim_{\mathbb{C}} \tilde{\mathcal{F}}_x / m_x \tilde{\mathcal{F}}_x = \dim_{\mathbb{C}} \mathcal{F}(x) = 1$$

for all $x \in U(p) \times S$, where m_x is the maximal ideal at x.

The set

$$\left\{ x \in U(p) \times U_\epsilon(0) \mid \dim_{\mathbb{C}} \tilde{\mathcal{F}}_x / m_x \tilde{\mathcal{F}}_x > 1 \right\}$$

is analytic; in particular, it is closed (see e.g. [Fis76], p. 49). Hence after shrinking δ and ϵ again,

$$\dim_{\mathbb{C}} \tilde{\mathcal{F}}_x / m_x \tilde{\mathcal{F}}_x = 1$$

for each $x \in U(p) \times U_\epsilon(0)$. Hence the Nakayama lemma (see e.g. [AM69], p. 21) implies that $\tilde{\mathcal{F}}\Big|_{\mathrm{Supp}(\mathcal{F})}$ is locally free of rank 1 and so does $\mathcal{F}|_{U(p)\times S}$ near p for arbitrary $p \in \mathrm{Supp}(\mathcal{F})$. □

Since all computations are local, we can replace the trivial bundle $\mathcal{O}_Z$ in Lemma 5.2.2 by an arbitrary line bundle on Z and obtain also the following well-known lemma:

5.2.3 Lemma. *Let S be the spectrum of an Artin ring or the germ of a complex space. Let Z be a closed submanifold of codimension d of a compact complex manifold X and $\iota : Z \hookrightarrow X$, the inclusion. Let $L \in \mathrm{Pic}(Z)$ be a line bundle.*

Any deformation $(\mathcal{X}, \mathcal{F})$ of the pair $(X, \iota_* L)$ over S is a pair consisting of a deformation $\mathcal{X}$ of X over S and a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ which is a locally free sheaf of rank 1 on $\mathrm{Supp}(\mathcal{F})$ such that $\mathcal{F}|_X \cong \iota_* L$.

The method of the proof also shows the following classical lemma:

5.2.4 Lemma. *Let S be the spectrum of an Artin ring or the germ of a complex space. Let $\pi : \mathcal{X} \to S$ be a deformation of a compact complex manifold X over S. Let $\mathcal{E}$ be a coherent sheaf on $\mathcal{X}$ which is flat over S. Suppose that $\mathcal{E}|_X$ is locally free. Then $\mathcal{E}$ is locally free.*

Next we associate to an infinitesimal deformation of the pair (X, Z) an infinitesimal deformation of the pair $(X, \iota_* \mathcal{O}_Z)$ and vice versa.

5.2.5 Lemma. *Let Z be a closed submanifold of codimension d of a compact complex manifold X and $\iota : Z \hookrightarrow X$ be the inclusion. Let S be the spectrum of an Artin ring or the germ of a complex space.*

1. *We suppose that the closed subscheme* $\mathcal{Z} \overset{J}{\hookrightarrow} \mathcal{X}$ *is a deformation of the pair* (X, Z) *over* S*. Then by associating to* $\mathcal{Z} \overset{J}{\hookrightarrow} \mathcal{X}$ *the coherent sheaf*
$$\mathcal{F} := J_* \mathcal{O}_{\mathcal{Z}}$$
we get a deformation $(\mathcal{X}, \mathcal{F})$ *of the pair* $(X, \iota_* \mathcal{O}_Z)$ *over* S*.*

2. *Conversely we suppose that the pair* $(\mathcal{X}, \mathcal{F})$ *is a deformation of the pair* $(X, \iota_* \mathcal{O}_Z)$ *over* S*. Then we get an associated deformation of the pair* (X, Z) *over* S *by setting*
$$\mathcal{Z} := \mathrm{Supp}\,(\mathcal{F})\,.$$
Furthermore we define the map $\pi : \mathcal{Z} \to S$ *by setting* $\pi = p|_{\mathcal{Z}}$*, where* $p : \mathcal{X} \to S$ *is the projection.*

Proof of Lemma 5.2.5: (1.) $\mathcal{F} := J_* \mathcal{O}_{\mathcal{Z}}$ is a coherent sheaf on $\mathcal{X}$ which is flat over S. Indeed, the flatness of $\mathcal{F}$ over S follows immediately from the flatness of $\mathcal{Z}$ over S.

The inclusion $j_1 : Z \hookrightarrow \mathcal{Z}$ induces a morphism of sheaves
$$\mathcal{O}_{\mathcal{X}}/\mathcal{J}_{\mathcal{Z}} \to j_{1*}\left(\mathcal{O}_{\mathcal{X}}/\mathcal{J}_Z\right),$$
where $\mathcal{J}_{\mathcal{Z}}$ and $\mathcal{J}_Z$ denote the ideal sheaves for $\mathcal{Z}$ and Z in $\mathcal{X}$, respectively; i.e., we get a map of sheaves on $\mathcal{X}$
$$J_*\left(\mathcal{O}_{\mathcal{Z}}\right) \to \iota_*\left(\mathcal{O}_Z\right),$$
where ι also denotes the inclusion $Z \cong Z \times \mathrm{Spec}\,(\mathbb{C}) \hookrightarrow \mathcal{X}$. This yields an isomorphism
$$J_*\left(\mathcal{O}_{\mathcal{Z}}\right) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_X \cong J_* \mathcal{O}_{\mathcal{Z} \times_S \mathrm{Spec}(\mathbb{C})} \cong J_* j_{1*} \mathcal{O}_Z = \iota_* \mathcal{O}_Z.$$
Therefore $\mathcal{F}$ is an infinitesimal deformation of $\iota_* \mathcal{O}_Z$ over S.

(2.) We need to show that $\pi : \mathcal{Z} \to S$ is flat and that $\mathcal{Z} \times_S \mathrm{Spec}\,(\mathbb{C}) \cong Z$.

We note that $\mathcal{Z} \times_S \mathrm{Spec}\,(\mathbb{C})$ is the subspace of $X = \mathcal{X} \times_S \mathrm{Spec}\,(\mathbb{C})$ defined by $J := \mathrm{im}\,(\iota^*\left(\mathrm{Ann}\,(\mathcal{F})\right) \to \mathcal{O}_X)$, where $\iota : X \hookrightarrow \mathcal{X}$ is the inclusion.

Then
$$\begin{aligned} J_x &= \left\{ f_x \in \mathcal{O}_{X,x} \middle| \, f_x \cdot s_x = 0 \text{ for each } s_x \in \left(\mathcal{F}|_Z\right)_x \right\} \\ &= \left\{ f_x \in \mathcal{O}_{X,x} \middle| \, f_x \cdot s_x = 0 \text{ for each } s_x \in \left(\iota_* \mathcal{O}_Z\right)_x \right\} \\ &= J_{Z,x}. \end{aligned}$$

Furthermore we have to show that π is flat; i.e. for each $x \in \mathcal{Z}$ the local ring $\mathcal{O}_{\mathcal{Z},x}$ is flat over $\mathcal{O}_{S,\pi(x)}$.

Let $x \in \mathcal{Z}$. By assumption $\mathcal{F}$ is flat over S; i.e. for each $x \in \mathcal{X}$ the stalk $\mathcal{F}_x$ is flat over $\mathcal{O}_{S,p(x)}$. As $\mathcal{F}$ is a locally free sheaf of rank 1 on $\mathcal{Z}$ by Lemma 5.2.2, it is locally trivial, and thus $\mathcal{F}_x \cong \mathcal{O}_{\mathcal{Z},x}$ for each $x \in \mathcal{Z}$. Therefore the flatness of π follows. $\square$

5.3 Proof of Theorem 5.1.1

In this section we give the proof of Theorem 5.1.1. We want to establish an exact sequence of vector spaces

$$0 \to \mathrm{Def}\,(X, Z) \xrightarrow{\zeta} \mathrm{Def}\,(X, \iota_* \mathcal{O}_Z) \xrightarrow{\xi} H^1\,(Z, \mathcal{O}_Z) \to 0. \tag{5.3.0.1}$$

We first define the map

$$\zeta : \mathrm{Def}\,(X, Z) \to \mathrm{Def}\,(X, \iota_* \mathcal{O}_Z)$$

by associating to $(J : \mathcal{Z} \hookrightarrow \mathcal{X})$ the coherent sheaf $\mathcal{F}$ constructed in Lemma 5.2.5, (1). Second, in Step 1 below we are going to construct a map

$$\xi : \mathrm{Def}\,(X, \iota_* \mathcal{O}_Z) \to H^1\,(Z, \mathcal{O}_Z)\,, (\mathcal{X}, \mathcal{F}) \mapsto \xi\,(\mathcal{F})\,.$$

The resulting situation is then shown in the following diagram:

$$\begin{tikzcd}
\mathrm{Def}\,(X) \cong H^1\,(X, TX) \arrow[r, "\simeq"] & \mathrm{Def}\,(X) \cong H^1\,(X, TX) & \\
\mathrm{Def}\,(X, Z) \cong H^1\,(X, T_X\,(-\log Z)) \arrow[u] \arrow[r, "\zeta"] & \mathrm{Def}\,(X, \iota_* \mathcal{O}_Z) \arrow[u] \arrow[r, "\xi"] & H^1\,(Z, \mathcal{O}_Z) \\
\mathrm{Def}_X\,(Z) \cong H^0\,(Z, \mathcal{N}_{Z|X}) \arrow[u] & \mathrm{Def}_X\,(\iota_* \mathcal{O}_Z) \cong \mathrm{Ext}^1_X\,(\iota_* \mathcal{O}_Z, \iota_* \mathcal{O}_Z) \arrow[u] & H^1\,(Z, \mathcal{O}_Z) \arrow[u] \\
0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Step 1: Construction of the map ξ. We start with some preparations: Let $(\mathcal{X}, \mathcal{F})$ be a simultaneous first-order deformation of the pair $(X, \iota_* \mathcal{O}_Z)$. Let $\mathcal{Z} := \mathrm{Supp}\,(\mathcal{F})$ be the complex space defined in 5.2.1. According to Lemma 5.2.2 we know $\mathcal{F} \in \mathrm{Pic}\,(\mathcal{Z})$ with $\mathcal{F}|_Z = \mathcal{O}_Z \in \mathrm{Pic}\,(Z)$.

From Lemma 5.2.5, (2) we know that the map $\pi : \mathcal{Z} \subseteq \mathcal{X} \to S$ induced by the projection of $\mathcal{X}$ to S is a first-order deformation of Z in X. We observe that the square of the ideal sheaf J of Z in $\mathcal{Z}$ vanishes, i.e.

$$J \cong J/J^2 \cong \mathcal{N}^\vee_{Z|\mathcal{Z}}.$$

Furthermore we have

$$\mathcal{N}_{Z|\mathcal{Z}} \cong \pi^* \mathcal{N}_{\mathrm{Spec}(\mathbb{C})|\mathrm{Spec}(\mathbb{C}[t]/(t^2))} \cong \pi^* \mathcal{O}_{\mathrm{Spec}(\mathbb{C})} \cong \mathcal{O}_Z.$$

Thus, looking at the exponential sequences for Z and $\mathcal{Z}$ and the ideal sheaf sequence for Z in $\mathcal{Z}$, we get the following diagram:

$$\begin{tikzcd}
 & H^0(\mathcal{Z},\mathcal{O}_{\mathcal{Z}}) \arrow[d] & & \\
 & H^0(Z,\mathcal{O}_Z) \arrow[d] & & \\
 & H^1(\mathcal{Z},\mathcal{J}_{Z|\mathcal{Z}}) \cong H^1\left(Z,\mathcal{N}^\vee_{Z|\mathcal{Z}}\right) \cong H^1(Z,\mathcal{O}_Z) \arrow[d, "\beta"] & & \\
H^1(\mathcal{Z},\mathbb{Z}) \arrow[r, "\iota_{\mathcal{Z}}"] \arrow[d, "\mathrm{id}"', "\cong"] & H^1(\mathcal{Z},\mathcal{O}_{\mathcal{Z}}) \arrow[r, "\mu"] \arrow[d, "\alpha"] & \mathrm{Pic}(\mathcal{Z}) \arrow[r, "c_1"] \arrow[d, "\gamma"] & H^2(\mathcal{Z},\mathbb{Z}) \arrow[d, "\mathrm{id}"', "\cong"] \\
H^1(Z,\mathbb{Z}) \arrow[r, "\iota_Z"] & H^1(Z,\mathcal{O}_Z) \arrow[r, "\nu"] & \mathrm{Pic}(Z) \arrow[r, "c_1"] & H^2(Z,\mathbb{Z}) .
\end{tikzcd} \tag{5.3.0.2}$$

The map $H^0(\mathcal{Z},\mathcal{O}_{\mathcal{Z}}) \to H^0(Z,\mathcal{O}_Z)$ induced by the ideal sheaf sequence of $\mathcal{Z}$ in Z is surjective, and therefore $H^0(Z,\mathcal{O}_Z) \to H^1\left(Z,\mathcal{N}^\vee_{Z|\mathcal{Z}}\right)$ is the zero map. Hence the map $\beta : H^1\left(Z,\mathcal{N}^\vee_{Z|\mathcal{Z}}\right) \to H^1(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})$ is injective.

Furthermore we observe that the maps ι_Z and $\iota_{\mathcal{Z}}$ in the diagram are injective. To that end, we extend diagram 5.3.0.2 at the left side and get

$$\begin{tikzcd}
H^0(\mathcal{Z},\mathcal{O}_{\mathcal{Z}}) \arrow[r] \arrow[d] & H^0(\mathcal{Z},\mathcal{O}^*_{\mathcal{Z}}) \arrow[r] \arrow[d] & H^1(\mathcal{Z},\mathbb{Z}) \arrow[r, "\iota_{\mathcal{Z}}"] \arrow[d, "\mathrm{id}"', "\cong"] & H^1(\mathcal{Z},\mathcal{O}_{\mathcal{Z}}) \arrow[r, "\mu"] \arrow[d, "\alpha"] & \dots \\
H^0(Z,\mathcal{O}_Z) \arrow[r] & H^0(Z,\mathcal{O}^*_Z) \arrow[r] & H^1(Z,\mathbb{Z}) \arrow[r, "\iota_Z"] & H^1(Z,\mathcal{O}_Z) \arrow[r, "\nu"] & \dots .
\end{tikzcd} \tag{5.3.0.3}$$

As the map $H^0(Z,\mathcal{O}_Z) \to H^0(Z,\mathcal{O}^*_Z) \cong \mathbb{C}^*$ is surjective, the map ι_Z is injective. The commutativity of diagram 5.3.0.3 implies then that the map $H^0(\mathcal{Z},\mathcal{O}^*_{\mathcal{Z}}) \to H^1(\mathcal{Z},\mathbb{Z})$ is the zero map; thus $\iota_{\mathcal{Z}}$ is injective.

As $\mathcal{F} \in \mathrm{Pic}(\mathcal{Z})$ and $\gamma(\mathcal{F}) = \mathcal{O}_Z \in \mathrm{Pic}(Z)$ and furthermore $c_1(\mathcal{O}_Z) = 0 \in H^2(Z,\mathbb{Z})$, we know that

$$c_1(\gamma(\mathcal{F})) = 0 \in H^2(Z,\mathbb{Z}) \cong H^2(\mathcal{Z},\mathbb{Z}),$$

thus $c_1(\mathcal{F}) = 0 \in H^2(\mathcal{Z},\mathbb{Z})$. Therefore $\mathcal{F} = \mu\left(\tilde{\mathcal{F}}\right)$ for a class $\tilde{\mathcal{F}} \in H^1(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})$. Because of the commutativity of the second square of diagram 5.3.0.2 we have $\nu \circ \alpha\left(\tilde{\mathcal{F}}\right) = \mathcal{O}_Z \in \mathrm{Pic}(Z)$ and can conclude

$$\alpha\left(\tilde{\mathcal{F}}\right) = \iota_Z(\mathcal{F}')$$

for a class $\mathcal{F}' \in H^1(Z,\mathbb{Z})$. Under the isomorphism $\mathrm{id} : H^1(Z,\mathbb{Z}) \xrightarrow{\cong} H^1(\mathcal{Z},\mathbb{Z})$, the class $\mathcal{F}'$ can be viewed as a class $\tilde{\mathcal{F}}' \in H^1(\mathcal{Z},\mathbb{Z})$. As the first square of diagram 5.3.0.2 is commutative, we get $\alpha\left(\tilde{\mathcal{F}}\right) \cong \alpha\left(\iota_{\mathcal{Z}}\left(\tilde{\mathcal{F}}'\right)\right)$. As a consequence,

$$\alpha\left(\tilde{\mathcal{F}} - \iota_{\mathcal{Z}}\left(\tilde{\mathcal{F}}'\right)\right) = \alpha\left(\tilde{\mathcal{F}}\right) - \alpha\left(\tilde{\mathcal{F}}\right) = 0$$

and

$$\tilde{\mathcal{F}} - \iota_{\mathcal{Z}}\left(\tilde{\mathcal{F}}'\right) \in \operatorname{im}(\beta).$$

Let

$$\xi(\mathcal{F}) \in H^1(Z, \mathcal{O}_Z)$$

be the image under the isomorphism $H^1(Z, \mathcal{O}_Z) \cong H^1\left(Z, \mathcal{N}^\vee_{Z|\mathcal{Z}}\right)$ of the preimage of $\tilde{\mathcal{F}} - \iota_{\mathcal{Z}}\left(\tilde{\mathcal{F}}'\right) \in H^1(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ under the injective map β.

We have to prove that $\xi(\mathcal{F})$ is well defined, i.e. the definition is independent of the choices of $\tilde{\mathcal{F}}$. Let $\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2 \in H^1(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ be two classes satisfying

$$\mu\left(\tilde{\mathcal{F}}_1\right) = \mu\left(\tilde{\mathcal{F}}_2\right) = \mathcal{F} \in \operatorname{Pic}(\mathcal{Z}).$$

Let $a_1, a_2 \in H^1\left(Z, \mathcal{N}^\vee_{Z|\mathcal{Z}}\right)$ be those classes corresponding to the classes which are the preimages of $\tilde{\mathcal{F}}_1$ or $\tilde{\mathcal{F}}_2$, resp., under the map β. As $\mu \circ \beta(a_1 - a_2) = 0$, we have $\beta(a_1 - a_2) \in \operatorname{im}(\iota_{\mathcal{Z}})$. Therefore $\beta(a_1 - a_2) = \iota_{\mathcal{Z}}(\tilde{a})$ for a class $\tilde{a} \in H^1(\mathcal{Z}, \mathbb{Z})$. Let $a \in H^1(Z, \mathbb{Z})$ be the image of $\tilde{a}$ under the isomorphism $\mathrm{id} : H^1(\mathcal{Z}, \mathbb{Z}) \xrightarrow{\cong} H^1(Z, \mathbb{Z})$. Because of the commutativity of the diagram we get

$$0 = \alpha \circ \beta(a_1 - a_2) = \iota_Z(a).$$

This yields $a = 0$ and thus $\tilde{a} = 0$; hence $\beta(a_1 - a_2) = 0$ and, as β is injective, $a_1 = a_2$. Thus the construction of $\xi(\mathcal{F})$ is unique.

Step 2: Linearity of ζ. We first treat the linearity of ζ. Let F_1 be the deformation functor of (X, Z) and F_2 be the deformation functor of $(X, \iota_* \mathcal{O}_Z)$. Now we construct a morphism $f : F_1 \to F_2$. Let A be an Artin ring and $T = \operatorname{Spec}(A)$. Then $f(T)$ is the infinitesimal deformation $(\mathcal{X}, \mathcal{F})$ of $(X, \iota_* \mathcal{O}_Z)$ over T induced by the infinitesimal deformation $(\mathcal{X}, \mathcal{Z})$ of (X, Z) over T constructed in Lemma 5.2.5. Then $\zeta = f\left(\operatorname{Spec}\left(\mathbb{C}[t]/t^2\right)\right)$.

By [Ser06], p. 46, the map ζ is therefore linear. As they are deformation functors (see Chapter 3.1), they satisfy conditions H_0 and H_ϵ.

Step 3: Exactness. Now we show that the sequence (5.3.0.1) is exact. Obviously ζ is injective.

Moreover, ξ is surjective, as for each element $a \in H^1(Z, \mathcal{O}_Z)$ we can choose $\tilde{X} := X \times \operatorname{Spec}(\mathbb{C}[t]/(t^2))$ and $\mathcal{Z} := Z \times \operatorname{Spec}(\mathbb{C}[t]/(t^2))$. Then a corresponds to an element $\tilde{a} \in H^1\left(Z, \mathcal{N}^\vee_{Z|\mathcal{Z}}\right)$, and we can define

$$\mathcal{F} := \mu \circ \beta(\tilde{a}).$$

We get $\mathcal{F}|_X \cong \mathcal{O}_Z \in \operatorname{Pic}(Z)$ because of the commutativity of the diagram, and furthermore $\mathcal{F}$ is flat, as it is locally trivial on $\mathcal{Z}$. Thus $\mathcal{F} \in \operatorname{Def}_X(\iota_*\mathcal{O}_Z) \subset \operatorname{Def}(X, \iota_*\mathcal{O}_Z)$.

The composition $\xi \circ \zeta$ vanishes. In order to see this, let $J : \mathcal{Z} \hookrightarrow \mathcal{X}$ be a first-order deformation of the embedding $j : Z \hookrightarrow X$. Under the map ζ, the deformation J is mapped to the coherent sheaf $\mathcal{F} = J_*\mathcal{O}_{\mathcal{Z}}$ on the deformation space $\mathcal{X}$ of X. We choose $\tilde{\mathcal{F}} := 0 \in H^1(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$. Since β is injective, we can conclude that $x_{\mathcal{F}} = 0$; thus $\xi(\mathcal{F}) = 0$.

Next we show that $\xi^{-1}(0) \subseteq \operatorname{im}(\zeta)$, hence the sequence is exact as soon as we know that ξ is linear. Let the pair $(\mathcal{X}, \mathcal{F})$ be an infinitesimal deformation of the pair $(X, \iota_*\mathcal{O}_Z)$ with

$$\xi(\mathcal{F}, \mathcal{X}) = x_{\mathcal{F}} = 0 \in H^1(Z, \mathcal{O}_Z)$$

and $\tilde{x}_{\mathcal{F}} \in H^1\left(Z, \mathcal{N}^\vee_{Z|\mathcal{Z}}\right)$ be the image of $x_{\mathcal{F}}$ under the isomorphism

$$H^1(Z, \mathcal{O}_Z) \cong H^1\left(Z, \mathcal{N}^\vee_{Z|\mathcal{Z}}\right),$$

where $\mathcal{Z} := \operatorname{Supp}(\mathcal{F}) \subseteq \mathcal{X}$. Then

$$\mathcal{F} = \mu \circ \beta(\tilde{x}_{\mathcal{F}}) = \mu(0) = J_*\mathcal{O}_{\mathcal{Z}},$$

where $J : \mathcal{Z} \hookrightarrow \mathcal{X}$ is the inclusion. Therefore the first-order deformation $(\mathcal{X}, \mathcal{F})$ is the image of the first-order deformation $J : \mathcal{Z} \hookrightarrow \mathcal{X}$ of the inclusion $j : Z \hookrightarrow X$ under the map ζ. □

Step 4: Linearity of ξ. Let F_3 be the deformation functor of the sheaf $\mathcal{O}_Z$ on Z. We define a morphism of functors

$$g : F_3 \to F_2$$

by associating to the infinitesimal deformation $\mathcal{L}$ of $\mathcal{O}_Z$ the infinitesimal deformation of $(X, \iota_*\mathcal{O}_Z)$ consisting of $\mathcal{X} = X \times \operatorname{Spec}(\mathbb{C}[t]/t^n)$ and $\iota_*\mathcal{L}$. Let

$$\lambda := g\left(\operatorname{Spec}\left(\mathbb{C}[t]/t^2\right)\right) : \operatorname{Def}(\mathcal{O}_Z) \to \operatorname{Def}(X, \iota_*\mathcal{O}_Z).$$

Then λ is linear according to [Ser06], p. 46. By construction of ξ, we know $\xi \circ \lambda = \mathrm{id}$. Now ξ induces a map

$$\overline{\xi} : \operatorname{Def}(X, \iota_*\mathcal{O}_Z)/\operatorname{im}(\zeta) \to H^1(Z, \mathcal{O}_Z),$$

therefore it suffices to show that $\overline{\xi}$ is linear. Let

$$\tau : \mathrm{Def}\,(X, Z) \to \mathrm{Def}\,(X, \iota_* \mathcal{O}_Z) \to \mathrm{coker}\,(\lambda)$$

be the induced linear map. Since $\xi^{-1}(0) = \mathrm{im}\,(\zeta)$ and $\xi \circ \lambda = \mathrm{id}$, the map $\overline{\xi}$ is an isomorphism. Hence

$$\dim H^1(Z, \mathcal{O}_Z) = \dim \left(\mathrm{Def}\,(X, \iota_* \mathcal{O}_Z) / \mathrm{im}\,(\zeta)\right)$$

and $\overline{\lambda} : H^1(Z, \mathcal{Z}) \to \mathrm{Def}\,(X, \iota_* \mathcal{O}_Z) / \mathrm{im}\,(\zeta)$ is an isomorphism. Since $\overline{\xi} \circ \overline{\lambda} = \mathrm{id}$, the map $\overline{\xi} = \overline{\lambda}^{-1}$ is linear.

We have also seen that λ defines a splitting of Sequence 5.3.0.1. □

5.3.1 Remark. We summarize the situation of Theorem 5.1.1 in the following diagram:

$$\begin{tikzcd}
\mathrm{Def}\,(X) \cong H^1(X, TX) \arrow[r, "\simeq"] & \mathrm{Def}\,(X) \cong H^1(X, TX) & \\
\mathrm{Def}\,(X, Z) \cong H^1(X, T_X(-\log Z)) \arrow[u] \arrow[r, "\zeta"] & \mathrm{Def}\,(X, \iota_* \mathcal{O}_Z) \arrow[u] \arrow[r, "\xi"] & H^1(Z, \mathcal{O}_Z) \\
\mathrm{Def}_X(Z) \cong H^0(Z, \mathcal{N}_{Z|X}) \arrow[u] \arrow[r, "\alpha"] & \mathrm{Def}_X(\iota_* \mathcal{O}_Z) \cong \mathrm{Ext}^1_X(\iota_* \mathcal{O}_Z, \iota_* \mathcal{O}_Z) \arrow[u] \arrow[r, "\beta"] & H^1(Z, \mathcal{O}_Z) \arrow[u] \\
0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The maps α and β are defined analogously to the construction of ζ and ξ. It seems well-known that the lower exact sequence can also be constructed by applying $\mathrm{Ext}^1_X(\cdot, \iota_* \mathcal{O}_Z)$ to the ideal sheaf sequence

$$0 \to \mathcal{J}_Z \to \mathcal{O}_X \to \iota_* \mathcal{O}_Z \to 0,$$

compare [Tho00], Lemma 3.42.

In the next section we examine, to which extend it is important to take the trivial bundle instead of an arbitrary line bundle on Z in Theorem 5.1.1.

5.4 A generalization of Theorem 5.1.1

We recall the following situation already considered in the proof of Theorem 5.1.1.

5.4.1 Notation. *Given a first-order deformation $\mathcal{Z}$ of Z, we consider the following commutative diagram given by the exponential sequences*

$$
\begin{tikzcd}
 & & & H^2(Z, \mathcal{O}_Z) \cong H^2\left(Z, \mathcal{N}^\vee_{Z|\mathcal{Z}}\right) \arrow[d] & \\
H^1(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \arrow[r, "\delta"] \arrow[d, "\gamma"] & H^1\left(\mathcal{Z}, \mathcal{O}^*_{\mathcal{Z}}\right) \arrow[r] \arrow[d, "r"] & H^2(\mathcal{Z}, \mathbb{Z}) \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"', "\cong"] & H^2(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}) \arrow[r, "\mu"] \arrow[d, "\alpha"] & \dots \\
H^1(Z, \mathcal{O}_Z) \arrow[r, "\alpha"] \arrow[d, "\beta"] & H^1\left(Z, \mathcal{O}^*_Z\right) \arrow[r] & H^2(Z, \mathbb{Z}) \arrow[r, "\iota_Z"] & H^2(Z, \mathcal{O}_Z) \arrow[r, "\nu"] & \dots. \\
H^2(Z, \mathcal{O}_Z). & & & &
\end{tikzcd}
\tag{5.4.1.1}
$$

5.4.2 Theorem. *Let $L \in \mathrm{Pic}(Z)$. Then there is a canonical morphism of vector spaces*

$$\Theta : \mathrm{Def}(X, \iota_* L) \to \mathrm{Def}(X, \iota_* \mathcal{O}_Z).$$

1. *The map Θ is injective if and only if $H^1(Z, \mathcal{O}_Z) = 0$.*

2. *Θ is surjective if and only if, for each first-order deformation $(\mathcal{X}, \mathcal{Z})$ of (X, Z), there is a preimage of L under the restriction map $\mathrm{Pic}(\mathcal{Z}) \to \mathrm{Pic}(Z)$.*

3. *Θ is surjective if and only if the following holds: $\epsilon(c_1(L)) = 0$ and, choosing $\mathcal{L} \in \mathrm{Pic}(\mathcal{Z})$ so that $\mathcal{L}|_Z \otimes L^\vee = \alpha(\zeta)$ with $\zeta \in H^1(Z, \mathcal{O}_Z)$, then $\beta(\zeta) = 0$.*

4. *If $H^2(Z, \mathcal{O}_Z) = 0$, then Θ is surjective.*

Proof of Theorem 5.4.2: *(0.)* To construct the morphism Θ, let $(\mathcal{X}, \mathcal{F})$ be a first-order deformation of $(X, \iota_* L)$. By Lemma 5.2.3, there is a line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{Z})$ such that $\mathcal{F} = j_* \mathcal{L}$, where $\mathcal{Z} := \mathrm{Supp}(\mathcal{F})$ and $j : \mathcal{Z} \to \mathcal{X}$ is the inclusion.

We associate to $(\mathcal{X}, \mathcal{F})$ the pair $(\mathcal{X}, j_* \mathcal{O}_{\mathcal{Z}}) =: \Theta(\mathcal{X}, \mathcal{F})$, which is a first-order deformation of $(X, \iota_* \mathcal{O}_Z)$. We observe that $\iota_* \mathcal{O}_{\mathcal{Z}}$ is flat over S. In fact, this is a local question and locally $\mathcal{F} = \iota_* \mathcal{O}_{\mathcal{Z}}$.

Since this works for every infinitesimal deformation, we get a morphism between the corresponding deformation functors. Therefore Θ is linear.

(1.) We show that Θ is injective if and only if $H^1(Z, \mathcal{O}_Z) = 0$. We assume first that $H^1(Z, \mathcal{O}_Z) = 0$ and consider $(\mathcal{X}, \mathcal{F}) \in \mathrm{Def}(X, \iota_* L)$ such that $\Theta(\mathcal{X}, \mathcal{F}) = (X \times S, j_* \mathcal{O}_{Z \times S})$, where $S := \mathrm{Spec}\left(\mathbb{C}[t]/t^2\right)$ and $j : Z \times S \hookrightarrow X \times S$ is the inclusion. Then $\mathcal{F} = j_* \mathcal{L}$, where $\mathcal{L} \in \mathrm{Def}_Z(L) \cong H^1(Z, \mathcal{O}_Z) = 0$ is the trivial first-order deformation of $L \in \mathrm{Pic}(Z)$. Thus Θ is injective.

Conversely, if Θ is injective, then $H^1(Z, \mathcal{O}_Z) = 0$, since otherwise we should obtain a nontrivial deformation of the line bundle L, which yields a nontrivial deformation of $(\mathcal{X}, \mathcal{F})$ with trivial image under Θ.

Finally, *(2.)* is just a reformulation, *(4.)* is a consequence of *(2.)*, and *(3.)* is a diagram chase.

□

5.4.3 Remark.
In the situation of Theorem 5.4.2 we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Def}(X,Z) \arrow[r] & \mathrm{Def}(X,\iota_*\mathcal{O}_Z) \arrow[r] & \mathrm{Def}(\mathcal{O}_Z) \arrow[r] & 0 \\ & & \mathrm{Def}(X,\iota_*L)\,. \arrow[ul] \arrow[u, "\Theta"] \arrow[ur] & & \end{tikzcd} \tag{5.4.3.1}$$

In general it is not possible to define a map

$$\mathrm{Def}(X,Z) \to \mathrm{Def}(X,\iota_*L)\,.$$

In fact, if $\mathcal{Z}$ is a first-order deformation of Z, then $\mathrm{Pic}(\mathcal{Z}) \to \mathrm{Pic}(Z)$ is neither injective nor surjective in general.

A linear map $\mathrm{Def}(X,Z) \to \mathrm{Def}(X,\iota_*L)$ exists if and only if the restriction map $\mathrm{Pic}(\mathcal{Z}) \to \mathrm{Pic}(Z)$ is an isomorphism.

5.4.4 Remark. We consider a first-order deformation $(\mathcal{X}, \mathcal{Z})$ of (X, Z) and a line bundle L on Z. Then L extends to $\mathcal{Z}$ if and only if the following holds: We consider the composition

$$F : H^1(X, T_X) \times H^1\left(X, \Omega^1_X\right) \overset{\cup}{\to} H^2\left(X, T_X \otimes \Omega^1_X\right) \to H^2(X, T_X)$$

given by the cup product and the pairing $T_X \otimes \Omega^1_X \to \mathcal{O}_X$. Then

$$F(\xi, c_1(L)) = 0.$$

See [Ser06], 3.3.11, for details.

5.5 Proof of Theorem 5.1.2

In this subsection we prove:

5.5.1 Theorem. *We assume that* $H^1(Z, \mathcal{O}_Z) = 0$ *or* $H^2(Z, \mathcal{O}_Z) = 0$*; e.g., Z is an ample divisor in a Calabi-Yau* 3*-fold or a smooth curve in a compact complex manifold. Then*

$$\mathrm{Obs}(X, \iota_*\mathcal{O}_Z) = \mathrm{Obs}(X, Z) \subset H^2(X, T_X\langle -Z\rangle)\,.$$

Proof of Theorem 5.5.1: (1.) The assumption $H^1(Z, \mathcal{O}_Z) = 0$ implies immediately that $\mathrm{Def}(X, Z) \cong \mathrm{Def}(X, \iota_* \mathcal{O}_Z)$ and moreover that the deformations of the pairs (X, Z) and $(X, \iota_* \mathcal{O}_Z)$ over $\mathrm{Spec}(\mathbb{C}[t]/t^n)$ are equal, thus $\mathrm{Obs}(X, Z) \cong \mathrm{Obs}(X, \iota_* \mathcal{O}_Z)$.

(2.) By Lemma 5.2.5 it suffices to prove the following: given

$$(\mathcal{X}_n, \mathcal{F}_n) \in \mathrm{Def}_n(X, \iota_* \mathcal{O}_Z),$$

let $(\mathcal{X}_n, \mathcal{Z}_n)$ be the associated element in $\mathrm{Def}_n(X, Z)$. If $(\mathcal{X}_n, \mathcal{Z}_n)$ extends to an element in $\mathrm{Def}_{n+1}(X, Z)$, then $(\mathcal{X}_n, \mathcal{Z}_n)$ extends to an element in $\mathrm{Def}_{n+1}(X, \iota_* \mathcal{O}_Z)$.

Since $H^2(Z, \mathcal{O}_Z) = 0$, the deformations of line bundles on Z are unobstructed. Let $(\mathcal{X}_n, \mathcal{F}_n)$ be a deformation of $(X, \iota_* \mathcal{O}_Z)$ over $\mathrm{Spec}(\mathbb{C}[t]/t^n)$. We assume that there is a deformation $(\mathcal{X}_n, \mathcal{Z}_n)$ of (X, Z) over $\mathrm{Spec}(\mathbb{C}[t]/t^n)$ which can be extended to a deformation $(\mathcal{X}_{n+1}, \mathcal{Z}_{n+1})$ over $\mathrm{Spec}\left(\mathbb{C}[t]/t^{n+1}\right)$.

The ideal sheaf sequence

$$0 \to \mathcal{J}_{\mathcal{Z}_n|\mathcal{Z}_{n+1}} \to \mathcal{O}_{\mathcal{Z}_{n+1}} \to \mathcal{O}_{\mathcal{Z}_n} \to 0$$

yields, using

$$\mathcal{J}_{\mathcal{Z}_n|\mathcal{Z}_{n+1}} = \mathcal{J}^n_{Z|X} / \mathcal{J}^{n+1}_{Z|X} = \mathcal{O}_Z$$

and the assumption $H^2(Z, \mathcal{O}_Z) = 0$, the equality

$$H^2(\mathcal{Z}_n, \mathcal{O}_{\mathcal{Z}_n}) = H^2\left(\mathcal{Z}_{n+1}, \mathcal{O}_{\mathcal{Z}_{n+1}}\right).$$

Using the exponential sequences for $\mathcal{Z}_n$ and $\mathcal{Z}_{n+1}$, we can extend

$$\mathcal{F}_n|_{\mathrm{Supp}(\mathcal{F}_n)} = \mathcal{F}_n|_{\mathcal{Z}_n}$$

to a sheaf $\mathcal{G}_{n+1}|_{\mathcal{Z}_n} \in \mathrm{Pic}(\mathcal{Z}_{n+1})$ and set

$$\mathcal{F}_{n+1} = \iota_{n+1*}(\mathcal{G}_{n+1}),$$

where $\iota_{n+1} : \mathcal{Z}_n \hookrightarrow \mathcal{X}_{n+1}$ is the inclusion. Thus $(\mathcal{X}_n, \mathcal{F}_n)$ is extendable to $(n+1)$-th order.

Hence the theorem follows by applying and using the fact that the obstructions of (X, Z) are in $H^2(X, T_X\langle -Z\rangle)$ due to [Ser06], 3.4.17. $\square$

We conclude that for $H^1(Z, \mathcal{O}_Z) = 0$, e.g. Z is an ample divisor in a Calabi-Yau 3-fold, the deformation problems of (X, Z) and $(X, \iota_* \mathcal{O}_Z)$ coincide.

5.6 An example for a potential function: the Noether-Lefschetz locus

As an application of the previous results, we establish a potential function of a pair $(X, \iota_* \mathcal{O}_D(C))$, where $\iota : D \hookrightarrow X$ be a smooth very ample divisor in X such that C is a special divisor on D. This will be a consequence of a theorem by C. Voisin.

5.6.1 Setup. *Let X be a Calabi-Yau 3-fold; $\iota : D \hookrightarrow X$ be a smooth very ample divisor in X such that there is a class $\lambda \in H^2_{van}(D, \mathbb{Z}) \cap H^{1,1}(D)$, where*

$$H^2_{van}(D, \mathbb{Z}) = \left\{ a \in H^2_{van}(D, \mathbb{Z}) \middle| \, \iota_*(a) = 0 \right\}.$$

Then the first-order deformations of (X, D) are unobstructed according to Remark 3.5.6. For each deformation $(\mathcal{X}, \mathcal{D}) = (X_s, D_s)_{s \in S}$ of (X, D) over a complex space S we get a unique smooth family of cycles $\Lambda_S := (\lambda_s)_{s \in S}$ extending λ such that $\lambda_s \in H^2_{van}(D_s, \mathbb{Z})$ for each $s \in S$. However λ_s will in general no longer be of type $(1,1)$. We fix a holomorphic 3-form ω on X.

We will make use of the following theorem by C. Voisin, which is proven in the appendix to [Cle05].

5.6.2 Theorem. *([Cle05], Appendix) We assume the Setup 5.6.1. Then there exist*

- *an open neighbourhood $T \subset H^1(X, T_X)$ of $0 \in H^1(X, T_X)$ and*
- *an open neighbourhood $R \subset H^1(X, T_X(-\log D))$ of the point $0 \in H^1(X, T_X(-\log D))$ with projection $R \to T$,*
- *furthermore $\mathbb{C}^*$-bundles $\tilde{R} \to R$ and $\tilde{T} \to T$ such that $\tilde{R}$ and $\tilde{T}$ parametrize the deformations of (X, D, ω) and (X, ω),*
- *and a holomorphic map*

$$\phi_{NL} : \tilde{R} \to \mathbb{C}$$

such that the following property is satisfied:

The family of cycles $\Lambda_S = (\lambda_s)_{s \in S}$ induced by the deformation of (X, D, ω) corresponding to $\tilde{r} \in \tilde{R}$ stays of type $(1,1)$ if and only if

$$d_{\tilde{R}|\tilde{T}} \phi_{NL}(\tilde{r}) = 0.$$

Here $d_{\tilde{R}|\tilde{T}}$ is the relative differential with respect to the projection $\tilde{R} \to \tilde{T}$.

5.6.3 Definition. *The set*

$$\mathrm{NL}\left(X, D, \lambda, \omega\right) := \left\{ \tilde{r} \in \tilde{R} \middle| \, d_{\tilde{R}|\tilde{T}} \phi_{NL}\left(\tilde{r}\right) = 0 \right\}.$$

is called the Noether-Lefschetz locus.

We will now apply Voisin's theorem to the situation discussed before.
To be specific, we propose the following

5.6.4 Setup. *Let X be a Calabi-Yau 3-fold; $\iota : D \hookrightarrow X$ be a smooth very ample divisor in X. Let C be a divisor on D, such that*

$$c_1\left(\mathcal{O}_D\left(C\right)\right) \in H^2_{van}\left(D, \mathbb{Z}\right) \cap H^{1,1}\left(D\right).$$

Thus C is a divisor in D which is not effective. We fix a holomorphic 3-form ω on X.

We show that the Noether-Lefschetz locus is the critical locus of a potential function for the deformation problem of the pair $\left(X, \iota_* \mathcal{O}_D\left(C\right), \omega\right)$.

5.6.5 Theorem. *We assume the Setup 5.6.4. There are*

- *open neighbourhoods $Z \subset H^1\left(X, T_X\right)$ of $0 \in H^1\left(X, T_X\right)$ and*
- *$W \subset \mathrm{Def}\left(X, \iota_* \mathcal{O}_D\left(C\right)\right)$ of $0 \in \mathrm{Def}\left(X, \iota_* \mathcal{O}_D\left(C\right)\right)$,*
- *furthermore $\mathbb{C}^*$-bundles $\tilde{Z} \to Z$ and $\tilde{W} \to W$ such that $\tilde{Z}$ and $\tilde{W}$ parametrize the deformations of $\left(X, \omega\right)$ and $\left(X, \iota_* \mathcal{O}_D\left(C\right), \omega\right)$*
- *and a holomorphic map*

$$\psi_{NL} : \tilde{W} \to \mathbb{C}$$

such that the following property is satisfied:

$$\mathcal{M}_{\tilde{W}}\left(X, \iota_* \mathcal{O}_D\left(C\right), \omega\right) = \left\{ \tilde{w} \in \tilde{W} \middle| \, d_{\tilde{W}|\tilde{Z}} \psi_{NL}\left(\tilde{w}\right) = 0 \right\}$$

where $\mathcal{M}_{\tilde{W}}\left(X, \iota_* \mathcal{O}_D\left(C\right), \omega\right) \subset \tilde{W}$ denotes the space of unobstructed deformations of $\left(X, \iota_* \mathcal{O}_D\left(C\right), \omega\right)$ inside $\tilde{W}$ and $d_{\tilde{W}|\tilde{Z}}$ is the relative differential with respect to the projection $\tilde{W} \to \tilde{Z}$.

Proof of Theorem 5.6.5: We set $\lambda := c_1(\mathcal{O}_D(C))$ and aim to apply Theorem 5.6.2.
Step 1. We construct a map

$$\rho : \mathrm{Def}(X, \iota_* \mathcal{O}_D(C)) \to \mathrm{Def}(X, D) \cong H^1(X, T_X(-\log D)).$$

Let $(\mathcal{X}, \mathcal{F})$ be a first-order deformation of $(X, \iota_* \mathcal{O}_D(C))$. Let $\mathcal{D} := \mathrm{Supp}(\mathcal{F})$. According to Lemma 5.2.3 we know $\mathcal{F} = j_* \mathcal{L}$ with $\mathcal{L} \in \mathrm{Pic}(\mathcal{D})$ and the inclusion $j : \mathcal{D} \hookrightarrow \mathcal{X}$. Then we define

$$\rho(\mathcal{X}, \mathcal{F}) := (\mathcal{X}, \mathcal{D}),$$

which is a deformation of (X, D) according to Lemma 5.2.5. The same construction also applies to all infinitesimal deformations and to germs of complex spaces.

Therefore we obtain a map between the deformation functors of the pairs $(X, \iota_* \mathcal{O}_D(C))$ and (X, D). Hence ρ is linear by [Ser06], p. 46; see Chapter 3.1. Then ρ is injective, as $H^1(D, \mathcal{O}_D) = 0$.

In fact, if $(\mathcal{X}, \mathcal{F}) \in \mathrm{Def}(X, \iota_* \mathcal{O}_D(C))$ with $\rho(\mathcal{X}, \mathcal{F}) = (X \times S, D \times S) \in \mathrm{Def}(X, D)$ is the trivial deformation, then $\mathcal{X} = X \times S$ and $\mathcal{D} = D \times S$, and thus $\mathcal{F} = j_* \mathcal{L}$, where $\mathcal{L} \in \mathrm{Pic}(D \times S)$ is a first-order deformation of $\mathcal{O}_D(C)$, therefore trivial, since $H^1(D, \mathcal{O}_D) = 0$.

Furthermore ρ extends trivially to a map

$$\tilde{\rho} : \mathrm{Def}(X, \iota_* \mathcal{O}_D(C), \omega) \to \mathrm{Def}(X, D, \omega).$$

Step 2. Let $W := \tilde{\rho}^{-1}(R)$ and $Z := T$. Let $\tilde{W}$ and $\tilde{Z}$ be as in Theorem 5.6.2, and set

$$\psi_{NL} := \phi_{NL} \circ \tilde{\rho} : \tilde{W} \to \mathbb{C}.$$

Then ψ_{NL} is holomorphic, as ϕ_{NL} is holomorphic and $\tilde{\rho}$ is linear. Thus

$$\left\{ d_{\tilde{W}|\tilde{Z}} \psi_{NL} = 0 \right\} = \tilde{\rho}^{-1} \left\{ d_{\tilde{R}|\tilde{T}} \phi_{NL} = 0 \right\}.$$

In order to prove the theorem, we show that

$$\mathcal{M}_{\tilde{W}}(X, \iota_* \mathcal{O}_D(C), \omega) = \tilde{\rho}^{-1}(\mathrm{NL}(X, D, \lambda, \omega)).$$

a) Let $(\mathcal{X}, \mathcal{F})$ be an unobstructed deformation of $(X, \iota_* \mathcal{O}_D(C))$ over a contractible complex Stein space S.

Then $(\mathcal{X}, \mathcal{F})$ induces a deformation $(\mathcal{X}, \mathcal{D})$ of (X, D) over S as seen above. The family $\Lambda = (\lambda_s)_{s \in S}$ induced by the deformation $(\mathcal{X}, \mathcal{D})$ stays of type $(1,1)$ on each fibre, as $\lambda_s = c_1(\mathcal{L}_s)$ for each $s \in S$, where $\mathcal{L}_s \in \mathrm{Pic}(\mathcal{D}_s)$. Thus

$$\tilde{\rho}(\mathcal{X}, \mathcal{F}, \omega) \in \mathrm{NL}(X, D, \lambda, \omega).$$

b) Let $(\mathcal{X}, \mathcal{D}, \omega) \in \mathrm{NL}(X, D, \lambda, \omega)$. Then all members of the family $\Lambda = (\lambda_s)_{s \in S}$ stay of type $(1,1)$ along the deformation $(\mathcal{X}, \mathcal{D})$. Therefore there are line bundles $\mathcal{L}_s \in \operatorname{Pic}(\mathcal{D}_s)$, such that $\lambda_s = c_1(\mathcal{L}_s)$. We show that the line bundles $\mathcal{L}_s$ fit together to a line bundle on $\mathcal{L}$ on $\mathcal{D}$.

The exponential sequence for the complex space $\mathcal{D}$ yields the exact sequence

$$H^1(\mathcal{D}, \mathcal{O}^*_{\mathcal{D}}) \to H^2(\mathcal{D}, \mathbb{Z}) \xrightarrow{\tau} H^2(\mathcal{D}, \mathcal{O}_{\mathcal{D}}),$$

where $H^2(\mathcal{D}, \mathbb{Z}) \cong H^0\left(S, R^2\pi_*\mathbb{Z}\right)$ and $H^2(\mathcal{D}, \mathcal{O}_{\mathcal{D}}) \cong H^0\left(S, R^2\pi_*\mathcal{O}_{\mathcal{D}}\right)$ according to the Leray spectral sequence associated to the projection $\pi : \mathcal{D} \to S$. Let

$$\sigma : H^0\left(S, R^2\pi_*\mathbb{Z}\right) \to H^0\left(S, R^2\pi_*\mathcal{O}_{\mathcal{D}}\right)$$

be the map induced by τ. Then $\sigma(\Lambda) = 0$, since $\sigma(\lambda_s) \in \operatorname{Pic}(D_s)$ for each $s \in S$. So Λ induces a class $\alpha \in H^2(\mathcal{D}, \mathbb{Z})$ such that $\alpha|_{D_s} = \lambda_s$ and such that $\tau(\alpha) = 0$. Hence there exists $\mathcal{L} \in \operatorname{Pic}(\mathcal{D})$ such that $c_1(\mathcal{L}) = \alpha$.

Thus $(\mathcal{X}, \mathcal{L}, \omega)$ yields a deformation of $(X, \iota_*\mathcal{O}_D(C), \omega)$ over S and $\tilde{\rho}(\mathcal{X}, \mathcal{L}, \omega) = (\mathcal{X}, \mathcal{D}, \omega)$. □

5.6.6 Remark. In the situation of Setup 5.6.4 let $[\underline{\Gamma}] \in H_3(X, D)$ such that $[\partial\underline{\Gamma}] \in H_2(D)$ is Poincaré-dual to $c_1(\mathcal{O}_D(C))$. Let $[\underline{\omega}] = [(\omega, 0)] \in H^3(X, D, \mathbb{C})$. Then according to [Cle05] we can write

$$\psi_{NL} := \int_{\underline{\Gamma}} \underline{\omega}.$$

In [AHJ+11] Alim, Hecht, Jockers, Mayr, Mertens and Soroush give several examples which appear as hypersurfaces in weighted projective spaces. Using toric methods, they derive a generalized hypergeometric GKZ-system, i.e. a system of Picard-Fuchs equations associated with the deformation problem $(X, \iota_*\mathcal{O}_D(C))$, that is solved by the function ψ_{NL}.

Using the results from this paper we see that in addition to the result stated in Theorem 5.6.5 the function ψ_{NL} is a solution of a system of Picard-Fuchs equations satisfying certain properties.

5.6.7 Corollary. *In the situation of the examples considered in [AHJ+11] the function ψ_{NL} satisfies a system of Picard-Fuchs operators.*

Chapter 6

Simultaneous deformations of a holomorphic vector bundle and a section

In this chapter we consider Calabi-Yau 3-folds containing a curve C which is given by a section s in a holomorphic vector bundle E of rank 2. We aim to describe the first-order deformations and the obstructions for the triple $(X, E, [s])$. We construct a locally free sheaf Q of rank 5 such that $H^1(X, Q^\vee)$ describes the first-order deformations of the triple $(X, E, [s])$ and the obstructions are in $H^1(X, Q^\vee)$.

6.1 Situation and main theorem

Let X be a Calabi-Yau 3-fold, $E \to X$ a holomorphic vector bundle of rank 2 on X and $s \in H^0(X, E)$ a holomorphic section of E. We assume that $C := \{s = 0\}$ scheme-theoretically is a smooth connected curve in X, i.e. the ideal sheaf that is locally generated by s is the ideal sheaf of the complex manifold C. This situation is called the *Serre-correspondence* for holomorphic vector bundles of rank 2 and yields the exact *Koszul complex*

$$\begin{aligned} & 0 \to \det(E^\vee) \to E^\vee \to \mathcal{J}_C \to 0 \\ \Leftrightarrow \quad & 0 \to \mathcal{O}_X \xrightarrow{\cdot s} E \xrightarrow{\wedge s} \mathcal{J}_C \otimes \det(E) \to 0. \end{aligned}$$

As $\{s = 0\} = \{\lambda \cdot s = 0\}$ for each $\lambda \in \mathbb{C}^*$, we look at the class $[s] \in \mathbb{P}\left(H^0(X, E)\right)$ of s in the projective space of $H^0(X, E)$.

We recall from Section 3.8 that the deformations of the triple $(X, E, [s])$ form a deformation functor. The main result in this section is:

6.1.1 Theorem. *The space of simultaneous first-order deformations*

$$\mathrm{Def}(X, E, [s])$$

of the base manifold X, the vector bundle E and the class $[s]$ of the section s defined in 3.8.1 and the space of their obstructions $\mathrm{Obs}\,(X, E, [s])$ *satisfy the following properties:*

There is a locally free sheaf Q of rank 5 on X such that

1. $$\mathrm{Def}\,(X, E, [s]) \cong \mathrm{Ext}^1\,(Q, \mathcal{O}_X) \quad \text{and}$$

2. $$\mathrm{Obs}\,(X, E, [s]) \subseteq \mathrm{Ext}^2\,(Q, \mathcal{O}_X)\,.$$

The proof of Theorem 6.1.1 needs various preparations.

6.2 Proof of Theorem 6.1.1

We begin with various preparations.

6.2.1 Construction. We will see that the simultaneous deformations of E and X will be described in terms of the projective bundle

$$\pi : \mathbb{P} := \mathbb{P}\,(E) \to X$$

of E; see Section 2.3 for the theory of projective fibre spaces. All fibres of $\mathbb{P}$ are isomorphic to $\mathbb{P}^1$, and the dimension of the total space $\mathbb{P}$ as a complex manifold is 4. As the exact Koszul complex gives a surjective map $E \to \mathcal{J}_C \otimes \det\,(E)$, we get an injective map

$$\mathbb{P}\,(\mathcal{J}_C \otimes \det\,(E)) \hookrightarrow \mathbb{P}$$

of the associated projective fibre spaces. We observe that $\mathbb{P}\,(\mathcal{J}_C \otimes \det\,(E)) \to X$ is generically an isomorphism and has 1-dimensional fibres over C. Let

$$D := \mathbb{P}\,(\mathcal{J}_C) = \mathbb{P}\,(\mathcal{J}_C \otimes \det\,(E))$$

be the projective fibre space associated to $\mathcal{J}_C \otimes \det\,(E)$ and $\sigma : D \to X$ be the restriction of $\pi : \mathbb{P} \to X$ to $D \subset \mathbb{P}$.

6.2.2 Lemma. *The complex space D is a smooth divisor in $\mathbb{P}$. Moreover*

$$D = \{t = 0\}\,,$$

where $t \in H^0\,(\mathbb{P}, \mathcal{O}_{\mathbb{P}}\,(1))$ is the image of $\pi^*\,(s) \in H^0\,(\mathbb{P}, \pi^*\,(E))$ under the homomorphism

$$H^0\,(\mathbb{P}, \pi^*\,(E)) \to H^0\,(\mathbb{P}, \mathcal{O}_{\mathbb{P}}\,(1))$$

induced by the canonical surjection $\pi^*\,(E) \to \mathcal{O}_{\mathbb{P}}\,(1) \to 0$.

Furthermore, D is isomorphic to the blow-up of C in X.

Proof of Lemma 6.2.2: *Step 1.* First we show that $D = \{t = 0\}$.

We restrict X to $\tilde{X} := X \backslash C$ and define $\tilde{E} := E|_{\tilde{X}}$ with $\tilde{s} := s|_{\tilde{X}}$, furthermore $\tilde{\pi} : \tilde{\mathbb{P}} := \mathbb{P}\left(\tilde{E}\right) \to \tilde{X}$ and

$$\tilde{D} := D|_{\sigma^{-1}(\tilde{X})} = \mathbb{P}\left(\det \tilde{E}\right) \cong \tilde{X}.$$

Hence $\tilde{D}$ is a smooth divisor in $\tilde{\mathbb{P}}$. Furthermore we have

$$\tilde{\pi}_* \mathcal{O}_{\tilde{\mathbb{P}}}(1)\big|_{\tilde{D}} = \tilde{\pi}_* \mathcal{O}_{\mathbb{P}\left(\det \tilde{E}\right)}(1) \cong \det \tilde{E}. \tag{6.2.2.1}$$

Applying $\tilde{\pi}_*$ to the ideal sheaf sequence

$$0 \to \mathcal{J}_{\tilde{D}} \otimes \mathcal{O}_{\tilde{\mathbb{P}}}(1) \to \mathcal{O}_{\tilde{\mathbb{P}}}(1) \to \mathcal{O}_{\tilde{\mathbb{P}}}(1)\big|_{\tilde{D}} \to 0 \tag{6.2.2.2}$$

for the inclusion $\tilde{D} \subset \tilde{\mathbb{P}}$, we get

$$0 \to \tilde{\pi}_* \left(\mathcal{J}_{\tilde{D}} \otimes \mathcal{O}_{\tilde{\mathbb{P}}}(1)\right) \to \tilde{E} \to \det \tilde{E}. \tag{6.2.2.3}$$

The last map $\tilde{E} \to \det \tilde{E}$ is nothing but the surjective map $\kappa : \tilde{E} \to \det \tilde{E}$ which occurs in the Koszul complex

$$0 \to \mathcal{O}_{\tilde{X}} \to \tilde{E} \xrightarrow{\kappa} \det \tilde{E} \to 0, \tag{6.2.2.4}$$

as the ideal sheaf sequence 6.2.2.2 is defined using the inclusion $\tilde{D} \subset \tilde{\mathbb{P}}$, which is induced by the Koszul complex. Thus we get an exact sequence

$$0 \to \tilde{\pi}_* \left(\mathcal{J}_{\tilde{D}} \otimes \mathcal{O}_{\tilde{\mathbb{P}}}(1)\right) \to \tilde{E} \xrightarrow{\kappa} \det \tilde{E} \to 0. \tag{6.2.2.5}$$

Comparing 6.2.2.4 and 6.2.2.5 we obtain $\ker \kappa \cong \tilde{\pi}_* \left(\mathcal{J}_{\tilde{D}} \otimes \mathcal{O}_{\tilde{\mathbb{P}}}(1)\right) \cong \mathcal{O}_{\tilde{X}}$, thus

$$\mathcal{O}_{\tilde{\mathbb{P}}}\left(\tilde{D}\right) \cong \mathcal{O}_{\tilde{\mathbb{P}}}(1).$$

Hence $\tilde{D} = \{t' = 0\}$ for a section $t' \in H^0\left(\tilde{\mathbb{P}}, \mathcal{O}_{\tilde{\mathbb{P}}}(1)\right)$. Let $s' \in H^0\left(\tilde{X}, \tilde{E}\right)$ be the section of $\tilde{E}$ which is mapped to t' under the isomorphism

$$H^0\left(\tilde{X}, \tilde{E}\right) \cong H^0\left(\tilde{\mathbb{P}}, \mathcal{O}_{\tilde{\mathbb{P}}}(1)\right).$$

Then the inclusion map of the Koszul complex 6.2.2.4 is given by the multiplication with s' since $\tilde{D} = \{t' = 0\}$.

As the curve C has codimension 2 in X, by applying the Riemann extension theorem, we extend the holomorphic section s' to a section $s'' \in H^0(X, E)$ on X. The image of s'' under the isomorphism of sections defines an extension of $t' \in H^0\left(\tilde{\mathbb{P}}, \mathcal{O}_{\tilde{\mathbb{P}}}(1)\right)$ to a section $t'' \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ on $\mathbb{P}$.

Hence the restrictions $s|_{\tilde{X}}$ and $s''|_{\tilde{X}}$ of both sections s and s'' to $\tilde{X}$ define the same Koszul complex. They operate on $\mathcal{O}_{\tilde{X}}$ as multiplication with a holomorphic function, which can be extended to a function on X and thus is constant. So s and s'' differ by a constant complex number and therefore t and t'' do so. As $\tilde{D} \subseteq \{t'' = 0\}$ and $D|_{\sigma^{-1}(C)} \cong \mathbb{P}\left(\mathcal{J}_C/\mathcal{J}_C^2|_C\right) \subseteq \{t = 0\}$, we have $D \subseteq \{t = 0\}$ and since $\{t = 0\}$ is irreducible, $D = \{t = 0\}$.

Step 2. Next we show that $D \subset \mathbb{P}$ is smooth.

As this is a local problem on X, we may assume $X = \mathbb{C}^3, E = \mathcal{O}_X \oplus \mathcal{O}_X$ and $s = (z_1, z_2) \in H^0(X, \mathcal{O}_X \oplus \mathcal{O}_X)$, where z_1, z_2, z_3 are coordinates on $\mathbb{C}^3$, thus

$$C = \{z_1 = z_2 = 0\}.$$

Then $\mathbb{P} = \mathbb{P}^1 \times \mathbb{C}^3$ and the section $t \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ which is mapped to $s \in H^0(X, E)$ under the isomorphism $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \cong H^0(X, E)$, can be written in local coordinates

$$t = w_0 z_1 - w_1 z_2,$$

where $[w_0 : w_1]$ are homogeneous coordinates in $\mathbb{P}^1$. By computing the partial derivatives of t, we see that $D = \{t = 0\}$ is smooth.

Step 3. It remains to see that D is the blow-up of X along C.

Let $\varphi : \widehat{X} \to X$ be the blow-up of $C \subset X$. Then φ is a proper, holomorphic map, and the exceptional divisor

$$\varphi^{-1}(C) \cong \mathbb{P}\left(\mathcal{J}_C/\mathcal{J}_C^2|_C\right) \cong D|_{\sigma^{-1}(C)}$$

is a smooth hypersurface in $\widehat{X}$.

As $\sigma^{-1}(C) = D|_{\sigma^{-1}(C)} = \mathbb{P}\left(\mathcal{J}_C/\mathcal{J}_C^2 \otimes \det E\right) \cong \mathbb{P}\left(\mathcal{J}_C/\mathcal{J}_C^2\right)$ is a smooth hypersurface in D, we may apply the universal property of the blow-up which says that there is a unique map $\tau : D \to \widehat{X}$ such that the following diagram is commutative:

$$\begin{tikzcd} D \arrow[r, "\tau"] \arrow[dr, "\sigma"'] & \widehat{X} \arrow[d, "\varphi"] \\ & X. \end{tikzcd}$$

It remains to show that τ is biholomorphic. On $X \backslash C$ we have the diagram:

$$\begin{tikzcd} D\backslash\sigma^{-1}(C) \arrow[r, "\tau", "\cong"'] \arrow[dr, "\cong", "\sigma"'] & \widehat{X}\backslash\varphi^{-1}(C) \arrow[d, "\cong"', "\varphi"] \\ & X\backslash C, \end{tikzcd}$$

where

$$\varphi|_{\widehat{X}\backslash\varphi^{-1}(C)} : \widehat{X}\backslash\varphi^{-1}(C) \to X\backslash C$$

and

$$\sigma|_{D\backslash\sigma^{-1}(C)} : D\backslash\sigma^{-1}(C) \to X\backslash C$$

are biholomorphic. Thus

$$\tau|_{D\backslash\sigma^{-1}(C)} : D\backslash\sigma^{-1}(C) \to \widehat{X}\backslash\varphi^{-1}(C)$$

is also biholomorphic.

Over C we get the following diagram:

$$\begin{tikzcd} \mathbb{P}\left(\mathcal{J}_C/\mathcal{J}_C^2\right) \arrow[r, "\tau|_C"] \arrow[dr, "\sigma"'] & \mathbb{P}\left(\mathcal{J}_C/\mathcal{J}_C^2\right) \arrow[d, "\varphi"] \\ & C. \end{tikzcd}$$

τ is holomorphic, surjective and

$$\tau : \sigma^{-1}(x) \to \varphi^{-1}(x)$$

is finite for $x \in C$. Furthermore τ is birational, as it is biholomorphic outside an analytic set. Then from Zariski's Main Theorem it follows that the fibres of τ are connected, so τ is biholomorphic. Thus $D \cong \widehat{X}$. □

An immediate consequence of Lemma 6.2.2 is the following observation.

6.2.3 Corollary. *Using 3.5.5 we obtain*

$$\begin{aligned} \operatorname{Def}(\mathbb{P}, D) &\cong H^1(\mathbb{P}, T_{\mathbb{P}}(-\log D)), \\ \operatorname{Obs}(\mathbb{P}, D) &\subseteq H^2(\mathbb{P}, T_{\mathbb{P}}(-\log D)). \end{aligned}$$

6.2.4 Lemma. *The coherent sheaf $\pi_* T_{\mathbb{P}}(-\log D)$ on X is locally free of rank 5 and*

$$\begin{aligned} H^1(\mathbb{P}, T_{\mathbb{P}}(-\log D)) &\cong H^1(X, \pi_*(T_{\mathbb{P}}(-\log D))), \\ H^2(\mathbb{P}, T_{\mathbb{P}}(-\log D)) &\cong H^2(X, \pi_*(T_{\mathbb{P}}(-\log D))). \end{aligned}$$

Proof of Lemma 6.2.4: Once we know that

$$R^1\pi_*(T_{\mathbb{P}}(-\log D)) = 0, \tag{6.2.4.1}$$

we conclude the statement

$$H^1(\mathbb{P}, T_{\mathbb{P}}(-\log D)) = H^1(X, \pi_*(T_{\mathbb{P}}(-\log D))),$$

from the Leray spectral sequence. As the fibres of π have dimension 1,

$$R^2\pi_* T_{\mathbb{P}}(-\log D) = 0,$$

and the second statement follows by the Leray spectral sequnece, too.

Thus it remains to show that $R^1\pi_*(T_{\mathbb{P}}(-\log D)) = 0$.

Let $x \in X$ and $F := \pi^{-1}(x) \cong \mathbb{P}^1$ be a fibre of $\pi : \mathbb{P} \to X$.

Case 1**:** Let $x \in X \backslash C$, then F and D intersect in one point $p \in \mathbb{P}$. Restricting the residue sequence

$$0 \to T_{\mathbb{P}}\left(-\log D\right) \to T_{\mathbb{P}} \to \iota_* \mathcal{N}_{D|\mathbb{P}} \to 0,$$

where $\iota : D \hookrightarrow \mathbb{P}$ is the inclusion, to F, we obtain the exact sequence

$$T_{\mathbb{P}}\left(-\log D\right)|_F \to T_{\mathbb{P}}|_F \to \iota_* \mathcal{N}_{D|\mathbb{P}}\big|_F \to 0.$$

The first map is injective as the sheaf $\iota_* \mathcal{N}_{D|\mathbb{P}}\big|_F$ has support in p. So, outside the point p, we have

$$T_{\mathbb{P}}\left(-\log D\right)|_F = T_{\mathbb{P}}|_F,$$

and the map $T_{\mathbb{P}}\left(-\log D\right)|_F \to T_{\mathbb{P}}|_F$ is injective, since $T_{\mathbb{P}}\left(-\log D\right)|_F$ is torsion-free. Hence with $\mathcal{N}_{D|\mathbb{P}}\big|_F = \mathcal{O}_{\mathbb{P}}\left(D\right)|_{D \cap F} = \mathcal{O}_{\mathbb{P}}\left(D\right)|_p = \mathbb{C}_p$, we have the following exact sequence on $F \subset \mathbb{P}$:

$$0 \to T_{\mathbb{P}}\left(-\log D\right)|_F \to T_{\mathbb{P}}|_F \to \iota_* \mathbb{C}_p \to 0. \tag{6.2.4.2}$$

In order to compute $T_{\mathbb{P}}|_F$ we look at the tangent bundle sequence of $F \subset \mathbb{P}$:

$$0 \to T_F \to T_{\mathbb{P}}|_F \to \mathcal{N}_{F|\mathbb{P}} \to 0.$$

As $T_F \cong T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$ and $\mathcal{N}_{F|\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$, we conclude that the rank of $T_{\mathbb{P}}|_F$ is 4. Furthermore, as $\operatorname{Ext}^1_{\mathbb{P}^1}\left(\mathcal{O}^{\oplus 3}, \mathcal{O}_{\mathbb{P}^1}(2)\right) = \bigoplus_3 H^1\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)\right) = 0$, the sequence is split exact and thus

$$T_{\mathbb{P}}|_F \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}.$$

Returning to Sequence 6.2.4.2 we obtain

$$c_1\left(T_{\mathbb{P}}\left(-\log D\right)|_F\right) = 1.$$

As $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$ has 6 sections and $\mathbb{C}_p$ has 1 section, $T_{\mathbb{P}}\left(-\log D\right)|_F$ has 5 sections. Since there is an inclusion of $T_{\mathbb{P}}\left(-\log D\right)|_F = \bigoplus_{i=0}^4 \mathcal{O}\left(a_i\right)$ into $T_{\mathbb{P}}|_F \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$, there are two possibilities for $T_{\mathbb{P}}\left(-\log D\right)|_F$: either

$$\left(a_1, a_2, a_3, a_4\right) = \left(2, 0, 0, -1\right)$$

or

$$\left(a_1, a_2, a_3, a_4\right) = \left(1, 0, 0, 0\right).$$

Hence

$$H^1\left(F, T_{\mathbb{P}}\left(-\log D\right)|_F\right) = 0.$$

Case 2: Let $x \in C$, so $D \cap F = F$. Again, we restrict the residue sequence to F and obtain

$$T_{\mathbb{P}}(-\log D)|_F \xrightarrow{\lambda} T_{\mathbb{P}}|_F \to \iota_* \mathcal{N}_{D|\mathbb{P}}\big|_F \to 0,$$

which yields an exact sequence

$$0 \to \operatorname{im}(\lambda) \to T_{\mathbb{P}}|_F \xrightarrow{\mu} \iota_* \mathcal{N}_{D|\mathbb{P}}\big|_F \to 0.$$

First we determine $\operatorname{im}(\lambda)$. As $T_{\mathbb{P}}|_F \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$ and $\iota_* \mathcal{N}_{D|\mathbb{P}}\big|_F \cong \mathcal{O}_{\mathbb{P}^1}(1)|_F$, we conclude that $\operatorname{im}(\lambda)$ is locally free of rank 3, its first Chern class is 1 and it has 4 sections. The summand $\mathcal{O}_{\mathbb{P}^1}(2)$ of $T_{\mathbb{P}}|_F$ is mapped to 0 by μ; hence it has to be a summand of $\operatorname{im}(\lambda)$. So

$$\operatorname{im}(\lambda) = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)$$

with $\alpha + \beta = -1$ and $\alpha, \beta \leqslant 0$. Thus we get

$$\operatorname{im}(\lambda) = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

In order to determine $T_{\mathbb{P}}(-\log D)|_F$, we look at the exact sequence

$$0 \to \ker(\lambda) \to T_{\mathbb{P}}(-\log D)|_F \xrightarrow{\lambda} \operatorname{im}(\lambda) \to 0. \tag{6.2.4.3}$$

We recall that $c_1(T_{\mathbb{P}}(-\log D)|_F) = 1$. It follows that $\ker(\lambda)$ is locally free of rank 1 with $c_1(\ker(\lambda)) = 0$. Hence $\ker(\lambda) \cong \mathcal{O}_{\mathbb{P}^1}$. Therefore we get

$$H^1\left(\mathbb{P}^1, \ker(\lambda)\right) = 0 \text{ and } H^1\left(\mathbb{P}^1, \operatorname{im}(\lambda)\right) = 0.$$

The long exact sequence in cohomology associated to the short exact Sequence 6.2.4.3 yields

$$H^1(F, T_{\mathbb{P}}(-\log D)|_F) = 0.$$

Thus, in both cases $x \in C$ and $x \in X \backslash C$ we get

$$H^1(F, T_{\mathbb{P}}(-\log D)|_F) = 0$$

for $F = \pi^{-1}(x)$. Then, because of the constancy of the Euler characteristic $\chi(F, T_{\mathbb{P}}(-\log D)|_F)$,

$$\dim H^0(F, T_{\mathbb{P}}(-\log D)|_F)$$

is constant for all fibres F and by applying ([Har77], p. 288, III,12.9) the sheaf $\pi_* T_{\mathbb{P}}(-\log D)$ is locally free and its rank is

$$\dim\left(H^0(F, T_{\mathbb{P}}(-\log D)|_F)\right) = 5.$$

□

Next we identify the deformation problems of the triple $(X, E, [s])$ and the pair $(\mathbb{P}(E), D)$.

6.2.5 Theorem. *Let X be a Calabi-Yau 3-fold. An infinitesimal or local analytic deformation of the triple $(X, E, [s])$ induces in a natural way an infinitesimal or local analytic deformation of the pair $(\mathbb{P}, D)$.*

Conversely every infinitesimal or local analytic deformation of $(\mathbb{P}, D)$ is given by an infinitesimal or local analytic deformation of $(X, E, [s])$. The two constructions are inverse to each other.

This construction respects isomorphy of triples $(X, E, [s])$ and of pairs $(\mathbb{P}(E), D)$. Hence there is a linear isomorphism

$$\operatorname{Def}(X, E, [s]) \cong \operatorname{Def}(\mathbb{P}(E), D).$$

Proof of Theorem 6.2.5: *Step 1.* We start with a deformation of $(X, E, [s])$ over S, where S is the spectrum of an Artin ring or the germ of a complex space. So we have a deformation $\mathcal{X} \to S$ of X, a coherent sheaf $\mathcal{E}$ on $\mathcal{X}$, flat over S, such that $\mathcal{E}|_X \cong E$ and a section $\mu \in H^0(\mathcal{X}, \mathcal{E})$ with $\mu|_X = s$. Since $\mathcal{E}|_X \cong E$, the sheaf $\mathcal{E}$ is locally free by Lemma 5.2.4. We consider the projective bundle

$$\mathbb{P}(\mathcal{E}) \to \mathcal{X},$$

then $\mathbb{P}(\mathcal{E})$ is a deformation of $\mathbb{P}(E)$ over S. Let $\mathcal{C} \subset \mathcal{X}$ be the zero scheme of μ. Locally, μ is given by holomorphic functions $f, g \in \mathcal{O}_{\mathcal{X}}(U)$ for an open subset $U \subset \mathcal{X}$, and the ideal sheaf of $\mathcal{C}$ is generated by f and g. We obtain an exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \xrightarrow{\cdot\mu} \mathcal{E} \to \mathcal{J}_{\mathcal{C}} \otimes \det \mathcal{E} \to 0 \tag{6.2.5.1}$$

given by μ.

In fact, the section μ defines an exact sequence

$$\mathcal{O}_{\mathcal{X}} \xrightarrow{\cdot\mu} \mathcal{E} \to F \to 0,$$

where the cokernel F is a coherent sheaf of rank 1 on $\mathcal{X}$. Dualizing, we obtain

$$0 \to F^\vee \to \mathcal{E}^\vee \xrightarrow{\lambda} \mathcal{O}_{\mathcal{X}}.$$

We see that $\operatorname{im}(\lambda) = \mathcal{J}_{\mathcal{C}}$ and $\det \mathcal{E}^\vee \cong \det F^\vee \otimes \det \mathcal{J}_{\mathcal{C}} \cong \det F^\vee = F^\vee$. Therefore we get Sequence 6.2.5.1.

Now we define

$$\mathcal{D} := \mathbb{P}(\mathcal{J}_{\mathcal{C}} \otimes \det \mathcal{E}) \subset \mathbb{P}(\mathcal{E}).$$

Then

$$\mathcal{D}|_{\mathbb{P}(E)} = D,$$

since $\mathcal{D}|_{\mathbb{P}(E)} = \mathbb{P}(j^*(\mathcal{J}_{\mathcal{C}} \otimes \det \mathcal{E})) = \mathbb{P}(\mathcal{J}_C \otimes \det E)$, where $j : X \hookrightarrow \mathcal{X}$ is the inclusion. Therefore

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mathcal{D})\big|_{\mathbb{P}(E)} \cong \mathcal{O}_{\mathbb{P}(E)}(D),$$

hence $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mathcal{D})$ is locally free and $\mathcal{D}$ is a divisor in $\mathcal{X}$. Furthermore $\mathcal{D}$ is a submersion over S by Lemma 3.8.3, hence flat. In conclusion, $(\mathbb{P}(\mathcal{E}), \mathcal{D})$ is a deformation of $(\mathbb{P}(E), D)$ over S.

Step 2. Now let $(\mathcal{Y}, \mathcal{D})$ be a deformation of $(\mathbb{P}(E), D)$ over S. Let $\pi : \mathcal{Y} \to S$ be the projection. We will associate to $(\mathcal{Y}, \mathcal{D})$ a deformation of $(X, E, [s])$ over S.

By Proposition 3.7.1 there is a deformation $\tau : \mathcal{X} \to S$ of $X := \mathcal{X}_0$ and a locally free sheaf $\mathcal{E}$ on $\mathcal{X}$ such that $\mathcal{Y} \cong \mathbb{P}(\mathcal{E})$.

Let $\tilde{\pi} : \mathbb{P}(\mathcal{E}) \to \mathcal{X}$ be the projection. So we need to construct a section

$$\mu \in H^0(\mathcal{X}, \mathcal{E}),$$

such that $\mu|_{X_0} = s$.

Let $t \in H^0\left(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)\right)$ be the corresponding section such that $\tilde{\pi}_*(t) = s$. This means that t corresponds to s under the canonical isomorphism $H^0\left(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)\right) \cong H^0(X, E)$.

Therefore it suffices to construct a section $\tilde{\mu} \in H^0\left(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\right)$, such that $\tilde{\mu}|_{\mathbb{P}(E)} = t$. Then we set $\mu := \tilde{\pi}_*(\tilde{\mu})$ to be the image of $\tilde{\mu}$ under the canonical isomorphism $H^0\left(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\right) \cong H^0(\mathcal{X}, \mathcal{E})$.

Since $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mathcal{D})\big|_{\mathbb{P}(E)} \cong \mathcal{O}_{\mathbb{P}(E)}(D)$, we may write

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mathcal{D}) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \tilde{\pi}^*(L)$$

for a line bundle L on $\mathcal{X}$ and the restriction $L|_X = \mathcal{O}_X$ is trivial. Hence, $L = \mathcal{O}_{\mathcal{X}}$.

Then we let $\tilde{\mu}$ be the section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ defined by $\mathcal{D}$.

Step 3. Since $H^1(X, \mathcal{O}_X) = 0$, the Picard group $\operatorname{Pic}(X)$ is discrete, the deformations of E are the same as the deformations of $\mathbb{P}(E)$. Moreover, if $(\mathcal{X}, \mathcal{E}, [s]) \simeq (\mathcal{X}', \mathcal{E}', [s'])$, then $(\mathbb{P}(\mathcal{E}), \mathcal{D}) \simeq (\mathbb{P}(\mathcal{E}'), \mathcal{D}')$ and vice versa. Finally both constructions are inverse to each other up to isomorphy.

□

The same proof neglecting the divisor D representing the section s also shows

$$\operatorname{Def}(X, E) \cong \operatorname{Def}(\mathbb{P}(E)).$$

6.2.6 Definition. *We set* $Q := (\pi_* T_{\mathbb{P}}(-\log D))^\vee$.

Proof of Theorem 6.1.1: The proof is a combination of Corollary 6.2.3, Lemma 6.2.4 and Theorem 6.2.5. □

6.3 A special situation

In general $\mathrm{Def}\,(X, C)$ is much larger than $\mathrm{Def}\,(X, E, [s])$; we will comment on this in Theorem 6.5.1. Here we consider the case of complete intersections of ample divisors.

6.3.1 Remark. Let X be a Calabi-Yau 3-fold, $E \to X$ a holomorphic vector bundle of rank 2 with a section $s \in H^0(X, E)$. Let $C := \{s = 0\}$ be the zero set of s.

We assume that the set of simultaneous first-order deformations of X and the curve C is isomorphic to the set of simultaneous first-order deformations of X, E and $[s] \in \mathbb{P}\left(H^0(X, E)\right)$:

$$\mathrm{Def}\,(X, C) \cong \mathrm{Def}\,(X, E, [s])$$

Then

$$\mathrm{Def}\,(X, C) \cong \mathrm{Def}\,(X, E, [s]) \cong \mathrm{Def}\,(\mathbb{P}, D) \cong H^1(\mathbb{P}, T_{\mathbb{P}}(-\log D))$$

and there is a locally free sheaf Q of rank 5 on X such that

$$\mathrm{Def}\,(X, \mathcal{O}_C) \quad \cong \quad \mathrm{Ext}^1_X(Q, \mathcal{O}_X) \oplus H^1(C, \mathcal{O}_C).$$

Moreover for $i = 1, 2$ we have

$$H^i(X, Q^\vee) = H^i(\mathbb{P}, T_{\mathbb{P}}(-\log D)).$$

We give an example for a vector bundle $E \to X$ such that every infinitesimal deformation of C is induced by an infinitesimal deformation of the section s.

6.3.2 Example. Let $D_1, D_2 \subset X$ be smooth transversally intersecting divisors and write $D_i = \{s_i = 0\}, i = 1, 2$, for sections $s_i \in H^0(X, \mathcal{O}_X(D_i))$. We let $C := D_1 \cap D_2$ be the intersection of them. Then

$$E := \mathcal{O}_X(D_1) \oplus \mathcal{O}_X(D_2) \to X$$

is a holomorphic vector bundle of rank 2 and the curve C is the zero set of the section $s := (s_1, s_2) \in H^0(X, E)$. This is a very special case of the *Serre-construction* for holomorphic vector bundles of rank 2.

We write $L_i = \mathcal{O}_X(D_i)$ and assume that the line bundles $L_1, L_2, L_1 \otimes L_2^\vee$ are ample. Then E does not have any nontrivial first-order deformations, since by Kodaira vanishing

$$\begin{aligned} H^1(X, E^\vee \otimes E) &= H^1(X, \mathcal{O}_X) \oplus H^1(X, \mathcal{O}_X) \oplus H^1(X, L_1 \otimes L_2^\vee) \oplus \\ &\quad \oplus H^1(X, L_1^\vee \otimes L_2) \\ &= 0. \end{aligned}$$

As $\mathcal{N}_{C|X} \cong E|_C \cong L_1|_C \oplus L_2|_C$, we have

$$H^0\left(C, \mathcal{N}_{C|X}\right) = H^0\left(C, L_1|_C\right) \oplus H^0\left(C, L_2|_C\right).$$

The Koszul complex

$$\begin{aligned} & 0 \to L_1^\vee \otimes L_2^\vee \to L_1^\vee \oplus L_2^\vee \to \mathcal{J}_C \to 0 \\ \Leftrightarrow \quad & 0 \to L_2^\vee \to \mathcal{O}_X \oplus (L_2^\vee \otimes L_1) \to \mathcal{J}_C \otimes L_1 \to 0 \end{aligned}$$

yields $H^1(X, \mathcal{J}_C \otimes L_1) = 0$ and analogously $H^1(X, \mathcal{J}_C \otimes L_2) = 0$. The exact sequences

$$0 \to \mathcal{J}_C \otimes L_1 \to L_1 \to L_1|_C \to 0$$

and

$$0 \to \mathcal{J}_C \otimes L_2 \to L_2 \to L_2|_C \to 0$$

give surjective maps

$$H^0(X, L_1) \to H^0\left(C, L_1|_C\right) \to H^1(X, \mathcal{J}_C \otimes L_1) = 0$$

and

$$H^0(X, L_2) \to H^0\left(C, L_2|_C\right) \to H^1(X, \mathcal{J}_C \otimes L_2) = 0$$

and therefore a surjective map

$$H^0(X, E) \to H^0\left(C, \mathcal{N}_{C|X}\right) \to 0.$$

Thus every first-order deformation of C is induced by a first-order deformation of $s \in H^0(X, E)$, fixing X.

We consider a compact complex manifold X, containing smooth divisors D_1 and D_2. In order to continue, we need the notion of a deformation of a triple , which will be discussed in detail in Chapter 9.

6.3.3 Definition. *A* deformation of the triple (X, D_1, D_2) parametrized by a complex space S *consists of a deformation of two cartesian diagrams*

$$\begin{tikzcd} \mathcal{D}_1 \arrow[rr, "J_1"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & S & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathcal{D}_2 \arrow[rr, "J_2"] \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & S & \end{tikzcd}$$

such that each pair $(\mathcal{X}, \mathcal{D}_i)$ is a deformation of (X, D_i) over S in the sense of Definition 3.5.1.

An isomorphism between two deformations $(\mathcal{X}, \mathcal{D}_1, \mathcal{D}_2)$ and $(\mathcal{X}', \mathcal{D}_1', \mathcal{D}_2')$ of the triple (X, D_1, D_2) consists of a triple of isomorphisms $\alpha_i : \mathcal{D}_i \to \mathcal{D}_i'$ and $\beta : \mathcal{X} \to \mathcal{X}'$ such that each pair (α_i, β) is an isomorphism between the deformations $(\mathcal{X}, \mathcal{D}_i)$ and $(\mathcal{X}', \mathcal{D}_i')$.

6.3.4 Remark. If D_1 and D_2 are ample divisors, then the deformations of (X, D_1) and (X, D_2) are unobstructed, as seen above. Therefore the deformations of the triple (X, D_1, D_2) are unobstructed.

6.3.5 Theorem. *Let X be a Calabi-Yau 3-fold and L_1, L_2 be line bundles on X. Let $D_i \in |L_i|$ be smooth divisors, given by sections $s_i \in H^0(X, L_i)$ such that $\dim(D_1 \cap D_2) = 1$. We set $E := L_1 \oplus L_2$ and $s := s_1 \oplus s_2$. Let F be the deformation functor of (X, D_1, D_2) and G be the deformation functor of $(X, E, [s])$.*

1. *There is a natural transformation $\phi : F \to G$.*
2. *We suppose that $H^1(X, L_1 \otimes L_2^\vee) = H^1(X, L_1^\vee \otimes L_2) = 0$. Let A be an Artin ring. Then $\phi_A : F(A) \to G(A)$ is bijective, i.e., ϕ is an isomorphism of functors. In particular, there is a canonical linear isomorphism*
$$\mathrm{Def}(X, D_1, D_2) \cong \mathrm{Def}(X, E, [s]).$$

We prepare the proof by the following

6.3.6 Lemma. *Let $(\mathcal{X}, \mathcal{E})$ be an infinitesimal deformation of (X, E), where $E = L_1 \oplus L_2$ with L_1 and L_2 are line bundles on X, such that*
$$H^1(X, L_1 \otimes L_2^\vee) = H^1(X, L_1^\vee \otimes L_2) = 0.$$
Then there are line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathcal{X}$, such that $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$.

Proof of Lemma 6.3.6: We first prove the following

Claim: Let $\mathcal{X}$ be an infinitesimal deformation of a Calabi-Yau manifold X. Then the restriction map $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X)$ is an isomorphism.

Proof of the Claim: We obtain a commutative diagram
$$\begin{tikzcd}
\arrow[r] & H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \arrow[r] \arrow[d] & \mathrm{Pic}(\mathcal{X}) \arrow[r] \arrow[d] & H^2(\mathcal{X}, \mathbb{Z}) \arrow[r] \arrow[d] & H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \arrow[r] \arrow[d] & {} \\
\arrow[r] & H^1(X, \mathcal{O}_X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] & H^2(X, \mathbb{Z}) \arrow[r] & H^2(X, \mathcal{O}_X) \arrow[r] & {}
\end{tikzcd} \tag{6.3.6.1}$$
We observe that $H^q(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \cong H^q(X, \mathcal{O}_X)$ for $q = 1, 2$. To see this, let $\mathcal{J}$ be the ideal sheaf of $X \subset \mathcal{X}$. Then $\mathcal{J}^k/\mathcal{J}^{k+1}$ is a trivial sheaf on X (possibly zero). Hence $H^q\left(X_{k+1}, \mathcal{O}_{X_{k+1}}\right) \cong H^q(X_k, \mathcal{O}_{X_k})$ for the k-th infinitesimal neighbourhood X_k of $X \subset \mathcal{X}$. Since $H^q(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = H^q(X_k, \mathcal{O}_{X_k})$, we conclude the claim. Compare [Har10], Theorem 6.4.

By the claim the line bundles L_i extend uniquely to line bundles $\mathcal{L}_i$ on $\mathcal{X}$. Since

$$H^1\left(X, E^\vee \otimes (L_1 \oplus L_2)\right) = H^1\left(X, \mathcal{O}_X \oplus \mathcal{O}_X \oplus (L_1^\vee \otimes L_2) \oplus (L_1 \oplus L_2^\vee)\right) = 0,$$

we get by induction, using [Har10], Theorem 7.1, that $\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$ on all X_k, hence on $\mathcal{X}$. $\square$

Proof of Theorem 6.3.5: 1.) We establish a map

$$\Phi_A : \mathrm{Def}_A\left(X, D_1, D_2\right) \to \mathrm{Def}_A\left(X, E, [s]\right)$$

for any Artin ring A.

Let $(\mathcal{X}, \mathcal{D}_1, \mathcal{D}_2)$ be a deformation of (X, D_1, D_2) over $\mathrm{Spec}\,(A)$, given by liftings $\tilde{s}_i \in H^0\left(\mathcal{X}, \mathcal{O}_{\mathcal{X}}\left(\mathcal{D}_i\right)\right)$ of $s_i \in H^0\left(X, \mathcal{O}_X\left(D_i\right)\right)$. We set

$$\mathcal{E} := \mathcal{O}_{\mathcal{X}}\left(\mathcal{D}_1\right) \oplus \mathcal{O}_{\mathcal{X}}\left(\mathcal{D}_2\right)$$

and $\tilde{s} = \tilde{s_1} \oplus \tilde{s_2}$ and obtain $(\mathcal{X}, \mathcal{E}, [\tilde{s}]) \in \mathrm{Def}_A\left(X, E, [s]\right)$.

2.) Conversely, let $(\mathcal{X}, \mathcal{E}, [\tilde{s}]) \in \mathrm{Def}_A\left(X, E, [s]\right)$. Then $\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$ by Lemma 6.3.6 and $\tilde{s} = \tilde{s}_1 \oplus \tilde{s}_2$ with $\tilde{s}_i \in H^0\left(\mathcal{X}, \mathcal{L}_i\right)$. We define $\mathcal{D}_i := \{\tilde{s}_i = 0\}$. Then $\mathcal{D}_i$ is flat over $S := \mathrm{Spec}\,(A)$ since $\mathcal{O}_{\mathcal{X}}\left(\mathcal{D}_i\right)$ is locally free, therefore flat over S, and since $\mathcal{D}_i \cap X$ is a divisor in X.

3.) Clearly Φ_A respects the isomorphy, and both constructions are inverse to each other. Since $\Phi := \left(\Phi_A\right)_A$ is a natural transformation of deformation functors, $\Phi_{\mathrm{Spec}(\mathbb{C}[t]/t^2)}$ is linear. $\square$

This situation will be studied further in Chapter 9.

6.3.7 Corollary. *In the situation of Example 6.3.2 we suppose further $D_1 - D_2$ to be ample. Then we get isomorphisms of spaces of first-order deformations:*

$$\mathrm{Def}\left(X, D_1, D_2\right) \cong \mathrm{Def}\left(X, E, [s]\right) \cong \mathrm{Def}\left(X, C\right).$$

6.3.8 Corollary. *In the situation of Example 6.3.2 we suppose further $D_1 - D_2$ to be ample. Then the first-order deformations of (X, C) are unobstructed.*

Proof of Corollary 6.3.8: The first-order deformations of (X, D_1, D_2) are unobstructed (see Remark 6.3.4). $\square$

6.4 The structure of the sheaf Q

We continue to investigate the structure of the locally free sheaf Q.

6.4.1 Theorem. *There are exact sequences*

$$0 \to \Omega^1_X \to Q \to E \otimes \mathcal{J}_C \to 0 \qquad (6.4.1.1)$$

and

$$0 \to E^\vee \to Q^\vee \to T_X \langle -C \rangle \to 0. \qquad (6.4.1.2)$$

Proof of Theorem 6.4.1: As $R^1\pi_* (T_{\mathbb{P}} (-\log D)) = 0$ according to 6.2.4.1, the following sequence is exact

$$0 \to Q^\vee = \pi_* (T_{\mathbb{P}} (-\log D)) \to \pi_* T_{\mathbb{P}} \to \pi_* \iota_* \mathcal{N}_{D|\mathbb{P}} \to 0. \qquad (6.4.1.3)$$

Next we apply π_* to the exact sequence

$$0 \to T_{\mathbb{P}|X} \to T_{\mathbb{P}} \to \pi^* T_X \to 0$$

and get

$$0 \to \pi_* T_{\mathbb{P}|X} \to \pi_* T_{\mathbb{P}} \to T_X \to R^1\pi_* \left(T_{\mathbb{P}|X}\right) = 0. \qquad (6.4.1.4)$$

We notice that $R^1\pi_* \left(T_{\mathbb{P}|X}\right) = 0$, since $T_{\mathbb{P}|X}\big|_{\pi^{-1}(x)} = \mathcal{O}_{\mathbb{P}^1} (2)$. Applying π_* to the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}} (D) = \mathcal{O}_{\mathbb{P}} (1) \to \iota_* \mathcal{N}_{D|\mathbb{P}} \to 0,$$

we obtain

$$0 \to \mathcal{O}_X \to \pi_* \mathcal{O}_{\mathbb{P}} (1) \cong E \to \pi_* \iota_* \mathcal{N}_{D|\mathbb{P}} \to R^1\pi_* \mathcal{O}_{\mathbb{P}} = 0.$$

Therefore we have an isomorphism

$$\pi_* \iota_* \mathcal{N}_{D|\mathbb{P}} \cong E/\mathcal{O}_X.$$

The Koszul complex

$$0 \to \mathcal{O}_X \to E \to \mathcal{J}_C \otimes \det E \to 0$$

yields an isomorphism

$$\mathcal{J}_C \otimes \det E \cong E/\mathcal{O}_X,$$

where the inclusion $\mathcal{O}_X \to E$ is given by the section s. Thus

$$\pi_* \iota_* \mathcal{N}_{D|\mathbb{P}} \cong \mathcal{J}_C \otimes \det E. \qquad (6.4.1.5)$$

Furthermore, we apply π_* to the relative Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to \pi^* E^\vee \otimes \mathcal{O}_{\mathbb{P}}(1) \to T_{\mathbb{P}|X} \to 0$$

and get the exact sequence

$$0 \to \mathcal{O}_X \to E^\vee \otimes \pi_* \mathcal{O}_{\mathbb{P}}(1) \to \pi_* T_{\mathbb{P}|X} \to R^1\pi_*(\mathcal{O}_{\mathbb{P}}) = 0.$$

Therefore

$$\pi_* T_{\mathbb{P}|X} \cong (E \otimes E^\vee)/\mathcal{O}_X. \tag{6.4.1.6}$$

In summary we obtain the following commutative diagram consisting of the short exact Sequences 6.4.1.3 and 6.4.1.4:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker(\psi) \arrow[r, "\alpha"] \arrow[d] & \pi_* T_{\mathbb{P}|X} \arrow[r] \arrow[d] \arrow[dr, "\phi"] & \operatorname{coker}(\alpha) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_*(T_{\mathbb{P}}(-\log D)) \arrow[r] \arrow[d] \arrow[dr, "\psi"] & \pi_* T_{\mathbb{P}} \arrow[r] \arrow[d] & \pi_* \iota_* \mathcal{N}_{D|\mathbb{P}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{im}(\psi) \arrow[r] \arrow[d] & T_X \arrow[r] \arrow[d] & T_X/\operatorname{im}(\psi) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{6.4.1.7}$$

The map $\psi : Q^\vee \cong \pi_*(T_{\mathbb{P}}(-\log D)) \to T_X$ is defined to be the composition of the two maps $\pi_*(T_{\mathbb{P}}(-\log D)) \hookrightarrow \pi_* T_{\mathbb{P}}$ and $\pi_* T_{\mathbb{P}} \to T_X$. It is generically surjective if $\operatorname{rg}(\operatorname{im}(\psi)) = 3$.

Since by 6.4.1.6

$$\ker(\psi) \subseteq \pi_*\left(T_{\mathbb{P}|X}\right) \cong (E \otimes E^\vee)/\mathcal{O}_X$$

and $\operatorname{im}(\psi) \subseteq T_X$, we conclude $\operatorname{rg}(\ker(\psi)) \leqslant 3$ and $\operatorname{rg}(\operatorname{im}(\psi)) \leqslant 3$, thus $2 \leqslant \operatorname{rg}(\ker(\psi)) \leqslant 3$, as $\operatorname{rg}(Q) = 5$.

Claim 1. On $X \backslash C$ we have

1. $\ker(\psi)$ is locally free of rank 2,
2. $\operatorname{im}(\psi) = T_X$.

Proof of Claim 1:

We consider the map

$$\phi : \pi_* T_{\mathbb{P}|X} \cong (E \otimes E^\vee)/\mathcal{O}_X \to \pi_* \iota_* \mathcal{N}_{D|\mathbb{P}} \cong \mathcal{J}_C \otimes \det E,$$

defined in Diagram 6.4.1.7. We show that ϕ_x is surjective at any point in $x \in X \backslash C$. Since $\ker(\psi) = \ker(\phi)$, this proves the claim.

Let $\tilde{x} \in D$ with $\pi(\tilde{x}) = x$. Then $\tilde{x} \in \pi^{-1}(x) =: l$. We look at the linear map

$$\phi_x : \left(\pi_* T_{\mathbb{P}|X}\right)_x \cong H^0\left(l, T_{\mathbb{P}|X}\big|_l\right) \cong H^0\left(\mathbb{P}^1, T_{\mathbb{P}^1}\right) \to \left(\pi_* \iota_* \mathcal{N}_{D|\mathbb{P}}\right)_x .$$

Let $\widehat{\phi} : T_{\mathbb{P}|X} \to \iota_* \mathcal{N}_{D|\mathbb{P}}$ be the canonical composition $T_{\mathbb{P}|X} \to T_{\mathbb{P}} \to \iota_* \mathcal{N}_{D|\mathbb{P}}$. Then $\phi = \pi_* \widehat{\phi}$. Let $u \in \left(T_{\mathbb{P}|X}\right)_{\tilde{x}}, u \neq 0$, with

$$\widehat{\phi}_{\tilde{x}}(u) \neq 0 \in \left(\iota_* \mathcal{N}_{D|\mathbb{P}}\right)_{\tilde{x}} \cong \left(\pi|_D\right)_* \left(\iota_* \mathcal{N}_{D|\mathbb{P}}\right)_{\tilde{x}} \cong \left(\pi_* \iota_* \mathcal{N}_{D|\mathbb{P}}\right)_x .$$

Let $v \in \left(\pi_* T_{\mathbb{P}|X}\right)_x$ such that $\pi_*(u) = v$. Then

$$\phi_x(v) = \pi_* \widehat{\phi}_{\tilde{x}}(u) \neq 0 \in \left(\pi_* \iota_* \mathcal{N}_{D|\mathbb{P}}\right)_x ,$$

as $\pi|_D$ is an isomorphism. So for each $x \in X \backslash C$ we know $\phi_x \neq 0$. Thus we have shown that ϕ_x is surjective for each $x \in X \backslash C$; therefore ϕ is surjective. This finishes the proof of the claim.

Claim 2. There is an isomorphism

$$\pi_* \left(T_{\mathbb{P}|X} \otimes \mathcal{O}_D\right) \cong \mathcal{J}_C^2 \otimes \det E.$$

Proof of Claim 2: The exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(D) \cong \mathcal{O}_{\mathbb{P}}(1) \to \iota_* \mathcal{N}_{D|\mathbb{P}} \to 0$$

tensorized by $\pi^* E^\vee$, the relative Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to \pi^* E^\vee \otimes \mathcal{O}_{\mathbb{P}}(D) \to T_{\mathbb{P}|X} \to 0$$

(see e.g. [Har77], III, Ex. 8.4) and the ideal sheaf sequence of D in $\mathbb{P}$ yield the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{J}_{D|\mathbb{P}} \arrow[r] \arrow[d, "\delta"] & \mathcal{O}_{\mathbb{P}} \arrow[r] \arrow[d] \arrow[dr, "\mu"] & \iota_* \mathcal{O}_D \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \pi^* E^\vee \arrow[r] \arrow[d] & \pi^* E^\vee \otimes \mathcal{O}_{\mathbb{P}}(D) \arrow[r] \arrow[d] & \pi^* E^\vee \otimes \iota_* \mathcal{N}_{D|\mathbb{P}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{coker}(\delta) \arrow[r] \arrow[d] & T_{\mathbb{P}|X} \arrow[r] \arrow[d] & \operatorname{coker}(\beta) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \quad . \tag{6.4.1.8}$$

The maps β, δ and μ are defined as follows:

The map μ is defined as the composition of the maps $\mathcal{O}_{\mathbb{P}} \to \pi^* E^\vee \otimes \mathcal{O}_{\mathbb{P}}(D)$ and $\pi^* E^\vee \otimes \mathcal{O}_{\mathbb{P}}(D) \to \pi^* E^\vee \otimes \iota_* \mathcal{N}_{D|\mathbb{P}}$.

By restricting to a fibre F of π we see immediately that $\mu|_F \neq 0$; therefore μ is not zero.

Since $\mu \not\equiv 0$ and $\operatorname{Supp}\left(\pi^* E^\vee \otimes \iota_* \mathcal{N}_{D|\mathbb{P}}\right) = D$, it factorizes over $\iota_* \mathcal{O}_D$ such that the map

$$\beta : \iota_* \mathcal{O}_D \to \pi^* E^\vee \otimes \iota_* \mathcal{N}_{D|\mathbb{P}}$$

is defined. As a consequence, we define the map δ such that Diagram 6.4.1.8 is commutative. It follows immediately that

$$\operatorname{coker}(\delta) = T_{\mathbb{P}|X} \otimes \mathcal{O}_{\mathbb{P}}(-D) \text{ and } \operatorname{coker}(\beta) = T_{\mathbb{P}|X} \otimes \mathcal{O}_D.$$

As $\pi_*\left(\mathcal{J}_{D|\mathbb{P}}\right) = 0$ and

$$\begin{aligned} R^1\pi_*\left(\mathcal{J}_{D|\mathbb{P}}\right) &= R^1\pi_*\mathcal{O}_{\mathbb{P}} = R^1\pi_*\left(T_{\mathbb{P}|X} \otimes \mathcal{O}_{\mathbb{P}}(-D)\right) = R^1\pi_*\iota_*\mathcal{O}_D = \\ &= R^1\pi_*\pi^* E^\vee = 0, \end{aligned}$$

by applying π_* to Diagram 6.4.1.8 we get the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] \arrow[dr, "\mu"] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^\vee \arrow[r] \arrow[d] & E^\vee \otimes E \arrow[r] \arrow[d] & E^\vee \otimes \pi_*\iota_*\mathcal{N}_{D|\mathbb{P}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_*\left(T_{\mathbb{P}|X} \otimes \mathcal{O}_{\mathbb{P}}(-D)\right) \arrow[r] \arrow[d] & \left(E^\vee \otimes E\right)/\mathcal{O}_X \arrow[r] \arrow[d] & \pi_*\left(T_{\mathbb{P}|X} \otimes \mathcal{O}_D\right) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{6.4.1.9}$$

Using [Har77], III, Ex. 8.4, we get

$$T_{\mathbb{P}|X} = \det T_{\mathbb{P}|X} = -K_{\mathbb{P}|X} = \pi^* \det E^\vee \otimes \mathcal{O}_{\mathbb{P}}(2). \tag{6.4.1.10}$$

Hence

$$\begin{aligned} \pi_*\left(T_{\mathbb{P}|X} \otimes \mathcal{O}_D\right) &= \pi_*\left(\mathcal{O}_{\mathbb{P}(\mathcal{J}_C \otimes \det E)}(2) \otimes \pi^*\left(\det E^\vee\right)\right) = \\ &= S^2\left(\mathcal{J}_C \otimes \det E\right) \otimes \det E^\vee = \mathcal{J}_C^2 \otimes (\det E)^2 \otimes \det E^\vee = \mathcal{J}_C^2 \otimes \det E, \end{aligned}$$

proving Claim 2.

Claim 3. The cokernel of the map $\alpha : \ker(\psi) \to \pi_* T_{\mathbb{P}|X}$ in Diagram 6.4.1.7 satisfies
$$\operatorname{coker}(\alpha) \cong \mathcal{J}_C^2 \otimes \det E.$$

Proof of Claim 3: First we show that
$$\pi_* \left(T_{\mathbb{P}|X} \otimes \mathcal{O}_{\mathbb{P}}(-D)\right) \cong \ker(\psi).$$

We observe that
$$\operatorname{coker}(\delta) = T_{\mathbb{P}|X}(-D) \subset T_{\mathbb{P}}(-D) \subset T_{\mathbb{P}}(-\log D)$$

and therefore
$$\pi_* (\operatorname{coker}(\delta)) \subset \left(\pi_* (T_{\mathbb{P}}(-\log D)) \cap \pi_* T_{\mathbb{P}|X}\right),$$

thus
$$\pi_* (\operatorname{coker}(\delta)) \subset \ker(\psi).$$

Diagram 6.4.1.9 yields $\pi_* (\operatorname{coker}(\delta)) \cong E^\vee$, hence we have an inclusion $E^\vee \hookrightarrow \ker(\psi)$.

On $X \backslash C$, by Claim 1 we already have the following exact sequence
$$0 \to \ker(\psi) \to Q^\vee \to T_X \to 0,$$

which yields $\det \ker(\psi) = \det Q^\vee$ on X. Diagram 6.4.1.7 implies
$$\det(Q^\vee) = \det \pi_* T_{\mathbb{P}} \otimes \det E^\vee.$$

Furthermore
$$\det \pi_* T_{\mathbb{P}} = \det \pi_* T_{\mathbb{P}|X} = \det(E^\vee \otimes E) = \mathcal{O}_X.$$

Thus we know
$$\det \ker(\psi) = \det E^\vee.$$

Since $\operatorname{im}(\psi)$ is torsion-free and $\ker(\psi)$ is reflexive, the inclusion $E^\vee \hookrightarrow \ker(\psi)$ is an isomorphism.

Diagram 6.4.1.9 can be summarized by the following commutative diagram:

$$\begin{tikzcd}
 & & & \mathcal{O}_X \arrow[d] & \\
0 \arrow[r] & E^\vee \arrow[r, "\rho"] \arrow[d, "\zeta"] & E \otimes E^\vee \arrow[r, "\tau"] \arrow[d, "\mathrm{pr}"] & \mathcal{J}_C \otimes E \arrow[r] \arrow[d, "\nu"] & 0 \\
0 \arrow[r] & \ker(\psi) \arrow[r, "\alpha"] & (E \otimes E^\vee)/\mathcal{O}_X \arrow[r, "\phi"] & \mathcal{J}_C^2 \otimes \det E \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

This proves Claim 3.

Claim 4. We obtain the isomorphism

$$\mathrm{im}\,(\psi) \cong T_X \langle -C \rangle \,,$$

where $T_X \langle -C \rangle$ is the kernel of $T_X \to j_* \mathcal{N}_{C|X}$ and $j : C \hookrightarrow X$ is the inclusion.

Proof of Claim 4: Using $\mathrm{coker}\,(\alpha) = \mathcal{J}_C^2 \otimes \det E$ and $\pi_* \iota_* \mathcal{N}_{D|\mathbb{P}} \cong \mathcal{J}_C \otimes \det E$ Diagram 6.4.1.7 gives

$$T_X / \mathrm{im}\,(\psi) = \mathcal{J}_C / \mathcal{J}_C^2 \otimes \det E \cong \mathcal{N}_{C|X}^\vee \otimes \det \mathcal{N}_{C|X} \cong \mathcal{N}_{C|X}.$$

Since $T_X \to T_X / \mathrm{im}\,(\psi) = \mathcal{N}_{C|X}$ is the canonical map, we get the assertion of the claim $\mathrm{im}\,(\psi) = T_X \langle -C \rangle$.

Now the first row of Diagram 6.4.1.7 reads

$$0 \to E^\vee \to Q^\vee \to T_X \langle -C \rangle \to 0.$$

Dualizing the sequence we get

$$0 \to \Omega_X^1 \to Q \to E \to \mathcal{E}xt_X^1 \left(T_X \langle -C \rangle , \mathcal{O}_X \right) \to 0.$$

Now the proof is completed by proving the following claim:

Claim 5. The following equation holds

$$\mathcal{E}xt^1 \left(T_X \langle -C \rangle , \mathcal{O}_X \right) \cong E|_C \,.$$

Proof of Claim 5: Using the sequence

$$0 \to T_X \langle -C \rangle \to T_X \to j_* \mathcal{N}_{C|X} \to 0$$

we obtain

$$\begin{aligned} &\mathcal{E}xt^1 \left(T_X \langle -C \rangle , \mathcal{O}_X \right) \cong \mathcal{E}xt^2 \left(j_* \mathcal{N}_{C|X}, \mathcal{O}_X \right) \cong \mathcal{E}xt^2 \left(j_* \mathcal{O}_C \otimes E|_C \,, \mathcal{O}_X \right) \\ &\cong \mathcal{E}xt^2 \left(j_* \mathcal{O}_C, \mathcal{O}_X \right) \otimes E^\vee . \end{aligned}$$

By the local fundamental isomorphism (see e.g. [OSS11])

$$\mathcal{E}xt^2 \left(j_* \mathcal{O}_C, \mathcal{O}_X \right) \otimes E^\vee \cong \det \mathcal{N}_{C|X} \otimes E^\vee ,$$

we conclude the proof of the theorem. □

The following theorem provides a relation between the deformations of triples $(X, E, [s])$ with deformations of pairs (X, E).

6.4.2 Theorem. *We assume the setting of Theorem 6.4.1. The logarithmic tangent sequence*

$$0 \to T_{\mathbb{P}}(-\log D) \to T_{\mathbb{P}} \to \iota_* \mathcal{N}_{D|\mathbb{P}} \to 0$$

induces a sequence

$$0 \to Q^\vee \to \pi_*(T_{\mathbb{P}}) \to \mathcal{J}_C \otimes \det E \to 0, \tag{6.4.2.1}$$

which in cohomology gives

$$H^0(X, \mathcal{J}_C \otimes \det E) \to H^1(X, Q^\vee) \to H^1(X, \pi_* T_{\mathbb{P}}). \tag{6.4.2.2}$$

This sequence can be interpreted as the natural sequence of first-order deformations

$$\mathrm{Def}([s]) \to \mathrm{Def}(X, E, [s]) \to \mathrm{Def}(X, E).$$

Proof of Theorem 6.4.2: We already established Sequence 6.4.2.1 in 6.4.1.3 and 6.4.1.5. Taking cohomology of 6.4.2.1 yields 6.4.2.2.

Since $H^1(X, Q^\vee) \cong \mathrm{Def}(X, E, [s])$ and $H^1(X, \pi_* T_{\mathbb{P}}) \cong \mathrm{Def}(X, E)$, it remains to be shown that $H^0(X, \mathcal{J}_C \otimes \det E) \cong \mathrm{Def}([s])$. The first-order deformations of $[s]$ are given by $H^0(X, E)/H^0(X, \mathcal{O}_X)$ and

$$H^0(X, E)/H^0(X, \mathcal{O}_X) \cong H^0(X, \mathcal{J}_C \otimes \det E)$$

by the Koszul complex and $H^1(X, \mathcal{O}_X) = 0$. We omit the identifications of the maps. $\square$

6.5 A comparison theorem and a potential function

We now compare the first-order deformations of triples $(X, E, [s])$ and pairs (X, C) in the situation of Section 6.1.

6.5.1 Theorem. *Let X be a Calabi-Yau 3-fold, E be a holomorphic vector bundle of rank 2 on X and $s \in H^0(X, E)$ be a holomorphic section. We assume that $C := \{s = 0\}$ is a smooth curve in X and that*

$$H^1(X, \det E^\vee) = H^2(X, \det E^\vee) = 0.$$

Let

$$\zeta : \mathrm{Def}(X, E, [s]) \to \mathrm{Def}(X, C)$$

be the map which associates with a deformation $(\mathcal{X}, \mathcal{E}, [\tilde{s}]) \in \mathrm{Def}(X, E, [s])$ the deformation $(\mathcal{X}, \mathcal{C}) \in \mathrm{Def}(X, C)$, where $\mathcal{C} := \{\tilde{s} = 0\} \subset \mathcal{X}$.

Then the image of ζ consists exactly of those first-order deformations $(\mathcal{X}, \mathcal{C})$ for which there is a line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ extending $\det(E)$ such that $\mathcal{L}|_{\mathcal{C}} \cong K_{\mathcal{C}}$.

Proof of Theorem 6.5.1: In the sequel we will use without mentioning the following basic facts:

- The complex spaces $\mathcal{X}$ and $\mathcal{C}$ are Gorenstein; so the dualizing sheaves $K_{\mathcal{X}}$ and $K_{\mathcal{C}}$ are line bundles;
- $j^*(K_{\mathcal{X}}) \cong K_X \cong \mathcal{O}_X$ and $j^*(K_{\mathcal{C}}) \cong K_C$, where j denotes the inclusion of X in $\mathcal{X}$ and of C in $\mathcal{C}$;
- $j^*\left(\mathcal{J}_{\mathcal{C}/\mathcal{X}}\right) \cong \mathcal{J}_{C/X}$.

There are two inclusions to be proved. First we assume that $(\mathcal{X},\mathcal{C})$ is a first-order deformation of (X,C) coming from a first-order deformation $(\mathcal{X},\mathcal{E},[\tilde{s}])$ of $(X,E,[s])$. Then $\tilde{s}$ defines a Koszul sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \to \mathcal{E} \to \mathcal{J}_{\mathcal{C}} \otimes \det \mathcal{E} \to 0.$$

We set $\mathcal{L} := \det \mathcal{E}$ and obtain

$$\mathcal{L}|_{\mathcal{C}} = \left(\det\left(\mathcal{J}_{\mathcal{C}}/\mathcal{J}_{\mathcal{C}}^2\right)\right)^{\vee} \cong K_{\mathcal{C}}.$$

In the other direction, we consider a first-order deformation $(\mathcal{X},\mathcal{C})$ satisfying the property that there is a line bundle $\mathcal{L} \in \operatorname{Pic}(\mathcal{X})$ such that $\mathcal{L}|_{\mathcal{C}} \cong K_{\mathcal{C}}$. The section s induces the Koszul sequence

$$0 \to \mathcal{O}_X \to E \to \mathcal{J}_C \otimes L \to 0 \tag{6.5.1.1}$$

with $L = \det E$. We want to construct a Koszul sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \to \mathcal{E} \to \mathcal{J}_{\mathcal{C}} \otimes \det \mathcal{E} \to 0 \tag{6.5.1.2}$$

with $\mathcal{L} = \det \mathcal{E}$, whose restriction to X yields Sequence 6.5.1.1 up to isomorphism. Here $\mathcal{E}$ is a vector bundle of rank 2 on $\mathcal{X}$ with $j^*(\mathcal{E}) \cong E$ and $t \in H^0(\mathcal{X},\mathcal{E})$ a section extending s such that $\{t = 0\} = \mathcal{C}$ up to an automorphism of $\mathcal{E}$.

Sequence 6.5.1.2 is now given by the Serre-correspondence on $\mathcal{X}$, using our assumption that $K_{\mathcal{C}} = \mathcal{L}|_{\mathcal{C}}$. Since $\mathcal{X}$ is not smooth, some comments have to be made. Following the construction in [OSS11], the arguments go through provided we know the following.

1. $\mathcal{E}xt^k(\mathcal{O}_{\mathcal{C}},\mathcal{O}_{\mathcal{X}}) = 0$ for $k = 0, 1$;
2. $\mathcal{E}xt^2(\mathcal{O}_{\mathcal{C}},\mathcal{O}_{\mathcal{X}}) \cong \mathcal{H}om\left(\det\left(\mathcal{J}_{\mathcal{C}}/\mathcal{J}_{\mathcal{C}}^2\right), \mathcal{L}^{\vee}|_{\mathcal{C}}\right)$;
3. $H^2(\mathcal{X},\mathcal{L}^{\vee}) = 0$.

Assertion (1) is clear for $k = 0$, since $\mathcal{X}$ is Cohen-Macaulay. For $k = 1$, either we need to make a computation in local coordinates or we argue as follows. We choose $A \in \operatorname{Pic}(\mathcal{X})$ sufficiently ample on $\mathcal{X}$ such that $\mathcal{E}xt^1(\mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{X}) \otimes A$ is generated by global sections; then it suffices to show that

$$\mathcal{E}xt^1(\mathcal{O}_\mathcal{C} \otimes A^\vee, \mathcal{O}_\mathcal{X}) = \mathcal{E}xt^1(\mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{X}) \otimes A = 0,$$

and therefore we prove

$$H^0\left(\mathcal{X}, \mathcal{E}xt^1(\mathcal{O}_\mathcal{C} \otimes A^\vee, \mathcal{O}_\mathcal{X})\right) = 0.$$

Now we use the Grothendieck spectral sequence ("local to global Ext"),

$$E_2^{p,q} = H^p(\mathcal{X}, \mathcal{E}xt^q(\mathcal{O}_\mathcal{C} \otimes A^\vee, \mathcal{O}_\mathcal{X}))$$

converging to $\operatorname{Ext}^{p+q}(\mathcal{O}_\mathcal{C} \otimes A^\vee, \mathcal{O}_\mathcal{X})$. Since $\dim \mathcal{C} = 1$, we get $E_2^{2,0} = 0$. Therefore the vanishing follows, using Serre-duality and $\dim \mathcal{C} = 1$, from

$$\operatorname{Ext}^1(\mathcal{O}_\mathcal{C} \otimes A^\vee, \mathcal{O}_\mathcal{X}) \cong H^2(\mathcal{X}, \mathcal{O}_\mathcal{C} \otimes A^\vee) = 0.$$

Assertion (2) is the local fundamental isomorphism, see ([AK70], I.4.5).

Assertion (3) finally results from the assumption

$$H^2(X, L^\vee) = H^2(X, \det E^\vee) = 0$$

by tensoring the ideal sheaf sequence

$$0 \to \mathcal{J}_{X/\mathcal{X}} \cong j_*\mathcal{O}_X \to \mathcal{O}_\mathcal{X} \to j_*\mathcal{O}_X \to 0 \tag{6.5.1.3}$$

with $\mathcal{L}^\vee$ and using $\mathcal{L}|_X \cong L = \det E$ since E is constructed via Serre-construction.

So Sequence 6.5.1.2 is constructed, defining a section $t \in H^0(\mathcal{X}, \mathcal{E})$ such that $\{t = 0\} = \mathcal{C}$. We finally have to show that $t|_X = s$, at least up to an isomorphism of E. Tensoring the dual Koszul complex with E and taking cohomology, we obtain an exact sequence

$$H^0(X, E^\vee) \to H^0(X, E^\vee \otimes E) \to H^0(X, \mathcal{J}_C \otimes E) \to H^1(X, E^\vee).$$

Our assumption $H^1(X, \det E^\vee) = 0$ yields $H^1(X, E^\vee) = 0$, again by the dual Koszul sequence. Therefore, there exists $\lambda : E \to E$ such that $\lambda(s) = t|_X$. Then λ has to be an automorphism since both sections vanish in codimension 2 only. Indeed, $\det(\lambda) \in H^0(X, \det E^\vee \otimes \det E) = H^0(X, \mathcal{O}_X)$ is constant on X. Thus, if λ is not an automorphism, $\operatorname{rg}(\lambda) < 2$ and the zero-set of $\lambda(s)$ would be of codimension smaller than 2. Hence both Koszul sequences on X, defined by $t|_X$ and s, are isomorphic. $\square$

6.5.2 Remark.

1. By choosing the Koszul sequence 6.5.1.2 more carefully, it should be possible to get rid of the assumption $H^1(X, \det E^\vee) = 0$, which however is anyway used in the following.

2. If $\operatorname{Pic}(X) \cong \mathbb{Z}$, then the image of ζ consists exactly of those first-order deformations $(\mathcal{X}, \mathcal{C})$ for which there is a line bundle $\mathcal{L} \in \operatorname{Pic}(\mathcal{X})$ such that $\mathcal{L}|_{\mathcal{C}} \cong K_{\mathcal{C}}$.

The same methods also show the following

6.5.3 Proposition. *Let X be a Calabi-Yau 3-fold, $E \to X$ a vector bundle of rank 2 on X and $s \in H^0(X, E)$. Assume that $C = \{s = 0\}$ is a smooth curve and that $H^q(X, \det E^\vee) = 0$ for $q = 0, 1, 2$. Let F be the deformation functor of $(X, E, [s])$ and G the deformation functor of (X, C). For each $n \in \mathbb{N}$ let*

$$\tau_n : F(\operatorname{Spec}(\mathbb{C}[t]/t^n)) \to G(\operatorname{Spec}(\mathbb{C}[t]/t^n))$$

be the canonical map. Then the image of τ_n consists exactly of those n-th order deformations $(\mathcal{X}_n, \mathcal{C}_n)$ for which there is a line bundle $\mathcal{L}_n \in \operatorname{Pic}(\mathcal{X}_n)$ extending $\det(E)$ such that $\mathcal{L}_n|_{\mathcal{C}_n} \cong K_{\mathcal{C}_n}$.

We finally discuss potential functions. To explain the result, we simplify the situation a little and pretend that there is a potential function

$$\Phi_{CL} : \operatorname{Def}(X, C) \to \mathbb{C}$$

constructed in [Cle05] such that the critical locus of Φ_{CL} is just the set of points $r \in \operatorname{Def}(X, C)$ corresponding to unobstructed deformations. Then the corresponding first-order deformation $(\mathcal{X}_1, \mathcal{C}_1)$ of (X, C), coming from a first-order deformation $(\mathcal{X}_1, \mathcal{E}_1, [s_1])$ of $(X, E, [s])$, is unobstructed and therefore induces a formal unobstructed deformation $\left(\widehat{\mathcal{X}}, \widehat{\mathcal{E}}, [\widehat{s}]\right)$ of $(X, E, [s])$. However Proposition 6.5.3 says that this is the case exactly when there exists a line bundle $\widehat{\mathcal{L}} \in \operatorname{Pic}\left(\widehat{\mathcal{X}}\right)$ such that

$$\widehat{\mathcal{L}}\Big|_{\widehat{\mathcal{C}}} \cong K_{\widehat{\mathcal{C}}}.$$

As a preparation, we show

6.5.4 Proposition. *The map $\operatorname{Def}(X, E, [s]) \to \operatorname{Def}(X, C)$ is injective if and only if $H^1(X, \det E^\vee) = 0$.*

Proof of Proposition 6.5.4: We first assume that $H^1(X, \det E^\vee) = 0$. Then $H^1(X, E^\vee) = 0$ by the Koszul sequence and

$$H^0(X, \mathcal{J}_C) = H^1(X, \mathcal{J}_C) = 0.$$

Now we use the exact sequence

$$0 = H^1(X, E^\vee) \to H^1(X, Q^\vee) \to H^1(X, T_X\langle -C\rangle)$$

and conclude by Theorem 6.1.1.

In the other direction, we suppose that $\mathrm{Def}(X, E, [s]) \to \mathrm{Def}(X, C)$ is injective. Then we consider again the exact sequence

$$H^0(X, T_X\langle -C\rangle) \xrightarrow{\alpha} H^1(X, E^\vee) \to H^1(X, Q^\vee) \xrightarrow{\beta} H^1(X, T_X\langle -C\rangle).$$

By our assumption β is injective. Hence α is surjective. Since

$$H^0(X, T_X\langle -C\rangle) \subset H^0(X, T_X) = 0,$$

it follows that $H^1(X, E^\vee) = 0$. Therefore $H^1(X, \det E^\vee) = 0$. □

Using Proposition 6.5.3 we prove:

6.5.5 Corollary. *Let X be a Calabi-Yau 3-fold with holomorphic 3-form ω and E be a holomorphic vector bundle of rank 2 such that $H^1(X, \det E^\vee) = H^2(X, \det E^\vee) = 0$. Let $s \in H^0(X, E)$ be a section such that $C := \{s = 0\}$ is a smooth curve. Then there exist*

- *an open neighbourhood $T \subset H^1(X, T_X)$ of $0 \in H^1(X, T_X)$ and*
- *an open neighbourhood $R \subset H^1(X, Q^\vee)$ of the point $0 \in H^1(X, Q^\vee)$ with projection $R \to T$,*
- *furthermore $\mathbb{C}^*$-bundles $\tilde{R} \to R$ and $\tilde{T} \to T$ such that $\tilde{R}$ and $\tilde{T}$ parametrize the deformations of $(X, E, [s], \omega)$ and (X, ω),*
- *and a holomorphic map*

$$\phi_{CL} : \tilde{R} \to \mathbb{C}$$

such that the following property is satisfied. The point $\tilde{r} \in \tilde{R}$ defines an unobstructed deformation of $(X, E, [s], \omega)$ if and only if the following holds:

Let $(\mathcal{X}_1, \mathcal{E}_1, [s_1], \omega_1)$ be the first-order deformation of $(X, E, [s], \omega)$ given by $\tilde{r} \in \tilde{R}$ and $(\mathcal{X}_1, \mathcal{C}_1, \omega_1)$ the induced first-order deformation of (X, C, ω). Then

- *$d_{\tilde{R}|\tilde{T}}\phi_{CL}(\tilde{r}) = 0$;*

- *the induced formal deformation* $\left(\widehat{\mathcal{X}}, \widehat{\mathcal{C}}\right)$ *of* (X, C) *satisfies*

$$\left.\widehat{\mathcal{L}}\right|_{\widehat{\mathcal{C}}} \cong K_{\widehat{\mathcal{C}}}$$

  for a line bundle $\widehat{\mathcal{L}} \in \operatorname{Pic}\left(\widehat{\mathcal{X}}\right)$ *extending* $\det(E)$.

Here $d_{\tilde{R}|\tilde{T}}$ *denotes the relative differential with respect to the projection* $\tilde{R} \to \tilde{T}$.

Proof of Corollary 6.5.5: This follows using Proposition 6.5.3, Proposition 6.5.4 and [Cle05] in the same way as Theorem 5.6.5 is induced from Theorem 5.6.2. □

6.5.6 Remark. The second condition in Corollary 6.5.5 is satisfied if $H^1(C, \mathcal{O}_C) = 0$, i.e., if $C \cong \mathbb{P}^1$.

However it is not known so far that this potential function is a solution of a Picard-Fuchs equation.

Chapter 7

Picard-Fuchs Equations for Calabi-Yau manifolds

In this chapter we study Picard-Fuchs equations attached to families of Calabi-Yau manifolds embedded in a projective space. First we recall the construction of the Picard-Fuchs equation for a general family. Then we consider the classical case of hypersurfaces in projective space and explain the Griffiths-Dwork reduction. The heart of the chapter treats Calabi-Yau complete intersections of codimension 2 in projective space. We extend the methods of Libgober and Teitelbaum [LT93] from dimension 3 to any dimension and at the same time give rigorous proofs of some statements in [LT93].

7.1 Picard-Fuchs equation associated to a family of Calabi-Yau manifolds

Let $\pi : \mathcal{X} \to T$ be a proper family of Calabi-Yau n-folds over a connected complex manifold T, where π is a submersion such that the fibres $X_t := \pi^{-1}(t)$ of π in the complex manifold $\mathcal{X}$ are complex manifolds for each $t \in T$. Let $t_0 \in T$ be a distinguished point in T.

For each $t \in T$, we obtain a canonical (pure) Hodge structure of weight n on $H^n(X_t, \mathbb{C})$. According to the Ehresmann theorem, all fibres of π are diffeomorphic, and thus $H^n(X_s, \mathbb{C}) \cong H^n(X_t, \mathbb{C})$ for each $s, t \in T$. The local system $R^n\pi_*\mathbb{C}$ yields a variation of Hodge structure of the family π, and therefore there is a Gauß-Manin connection

$$\nabla : \Gamma\left(R^n\pi_*\mathbb{C} \otimes \mathcal{O}_T\right) \to \Gamma\left(R^n\pi_*\mathbb{C} \otimes \mathcal{O}_T\right) \otimes \Omega^1_T$$

on the associated holomorphic vector bundle $R^n\pi_*\mathbb{C} \otimes \mathcal{O}_T$. This connection satisfies Griffiths-transversality with respect to the Hodge filtration on $R^n\pi_*\mathbb{C}$ which is induced by the Hodge filtration of $H^n(X_{t_0}, \mathbb{C})$. Given a vector field $\frac{\partial}{\partial t}$ on T, we define

$$\nabla_{\frac{\partial}{\partial t}} : \Gamma\left(R^n\pi_*\mathbb{C} \otimes \mathcal{O}_T\right) \to \Gamma\left(R^n\pi_*\mathbb{C} \otimes \mathcal{O}_T\right),$$
$$\nabla_{\frac{\partial}{\partial t}}(s) := \nabla(s)\left(\frac{\partial}{\partial t}\right).$$

If it is clear which vector field is meant, we just write ∇ instead of $\nabla_{\frac{\partial}{\partial t}}$.

We shrink T to a small disc around t_0 such that the locally free sheaf $R^n\pi_*\left(\Omega^n_{\mathcal{X}|T}\right)$ is trivial on T. Here $\Omega^n_{\mathcal{X}|T}$ is the sheaf of relative n-forms, i.e., $\Omega^n_{\mathcal{X}|T} = \Lambda^n\left(\Omega^1_{\mathcal{X}|T}\right)$, where $\Omega^1_{\mathcal{X}|T}$ is the cokernel of the inclusion map $\pi^*\Omega^1_T \hookrightarrow \Omega^1_{\mathcal{X}}$. We have $R^n\pi_*\left(\Omega^n_{\mathcal{X}|T}\right) \cong \mathcal{O}_T$, as $H^n\left(X_t, \Omega^n_{X_t}\right) \cong \mathbb{C}$ for each $t \in T$. Thus there is a non-vanishing section

$$\Omega \in H^0\left(T, R^n\pi_*\left(\Omega^n_{\mathcal{X}|T}\right)\right),$$

which yields a family of holomorphic non-vanishing n-forms $(\Omega(t))_{t\in T}$ on $\mathcal{X}(t)$. We apply $\nabla_{\frac{\partial}{\partial t}}$ to the family $\Omega \in \Gamma\left(R^n\pi_*\mathbb{C} \otimes \mathcal{O}_T\right)$ of holomorphic n-forms arbitrarily many times. As $H^n\left(X_{t_0}, \mathbb{C}\right)$ is finite-dimensional, we get a linear dependence of

$$\nabla^0_{\frac{\partial}{\partial t}}[\Omega]\Big|_{t_0}, \nabla^1_{\frac{\partial}{\partial t}}[\Omega]\Big|_{t_0}, \ldots, \nabla^k_{\frac{\partial}{\partial t}}[\Omega]\Big|_{t_0} \in H^n\left(X_{t_0}, \mathbb{C}\right)$$

for $k > \dim_{\mathbb{C}} H^n\left(X_{t_0}, \mathbb{C}\right)$. Hence there are holomorphic functions $\lambda_j : T \to \mathbb{C}, j = 0, \ldots, k$ such that

$$\sum_{j=0}^{k} \lambda_j \cdot \nabla^j_{\frac{\partial}{\partial t}}[\Omega] = 0. \tag{7.1.0.1}$$

This linear dependence yields the *Picard-Fuchs equation* of the family π.

7.1.1 Definition. *For each n-cycle γ on X_{t_0} the well-defined holomorphic function $T \to \mathbb{C}$, $t \mapsto \int_\gamma [\Omega(t)]$, is called a* period *of the family of Calabi-Yau manifolds. Here $[\Omega(t)]$ denotes for each $t \in T$ the class of the non-vanishing holomorphic n-form $\Omega(t)$ in $H^n(X_t, \mathbb{C})$.*

7.1.2 Remark. The Picard-Fuchs equation of the family of Calabi-Yau manifolds π is satisfied by all period integrals $\int_\gamma [\Omega(t)], t \in T$, for each n-cycle γ on X_{t_0}, i.e.,

$$\sum_{j=0}^{k} \lambda_j \frac{\partial^j}{\partial t^j} \int_\gamma [\Omega(t)] = 0.$$

The following notation will be used in the whole chapter.

7.1.3 Notation. *Let $[x_0 : \ldots : x_n]$ be homogeneous coordinates of $\mathbb{P}^n$ and*

$$\Delta := \sum_{i=0}^{n} (-1)^i x_i \, dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n \in H^0\left(\mathbb{P}^n, K_{\mathbb{P}^n}(n+1)\right) \cong \mathbb{C}$$

be the canonical Euler-form on $\mathbb{P}^n$. Furthermore, let

$$S = \bigoplus_l S^l, S^l = H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(l)\right)$$

be the graded ring of homogeneous polynomials on $\mathbb{P}^n$.

In the following sections we present a method for explicitly calculating Picard-Fuchs equations for families of Calabi-Yau manifolds in projective varieties, called the *Picard-Fuchs reduction* or *Picard-Fuchs algorithm*.

The Griffiths-Dwork reduction is a method in order to explicitly calculate the Picard-Fuchs equation for Calabi-Yau n-folds. It was first introduced by Griffiths in [Gri69] for Calabi-Yau manifolds which are hypersurfaces in projective spaces. The method was extended to hypersurfaces in weighted projective spaces by Dolgachev in [Dol83] and to Calabi-Yau manifolds that are complete intersections in projective spaces by Libgober and Teitelbaum in [LT93].

7.2 The Griffiths-Dwork reduction for families of hypersurfaces in projective spaces

Let X be a Calabi-Yau hypersurface in $\mathbb{P}^n$, i.e. $X = \{f = 0\}$ for a homogeneous polynomial $f \in H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n+1)\right)$ of degree $n+1$ on $\mathbb{P}^n$.

7.2.1 Notation. *Let*

$$\mathcal{J}_f := \left\langle \frac{\partial f}{\partial x_i}, i = 0, \ldots, n \right\rangle$$

be the Jacobian ideal of f, i.e. the homogeneous ideal in the graded ring S, which is generated by the partial derivatives $\frac{\partial f}{\partial x_i}, i = 0, \ldots, n$, of f. The graduation of S induces a grading of the Jacobian ideal of f, namely

$$\mathcal{J}_f = \bigoplus_l \mathcal{J}_f^l,$$

where $\mathcal{J}_f^l$ is generated by the partial derivatives of f over S^l.

We are going to use the residue map

$$\mathrm{Res}^k_{X|\mathbb{P}^n} : H^k\left(\mathbb{P}^n \backslash X, \mathbb{Z}\right) \to H^{k-1}\left(X, \mathbb{Z}\right),$$

which was introduced in Definition 4.2.22. As X is an ample divisor in $\mathbb{P}^n$ and $H^k(\mathbb{P}^n, \mathbb{C})_{prim} = 0$ for all k, applying the short exact sequence of Theorem 4.2.25.1, the residue map yields an isomorphism

$$\begin{aligned} \mathrm{Res}^n_{X|\mathbb{P}^n} : H^n(\mathbb{P}^n \backslash X, \mathbb{Q}) &\overset{\cong}{\to} H^{n-1}(X, \mathbb{Q})_{van} \\ &:= \ker\left(l_* : H^{n-1}(X, \mathbb{Q}) \to H^n(\mathbb{P}^n, \mathbb{Q})\right), \end{aligned}$$

where l_* denotes the Gysin morphism. Furthermore,

$$H^{n-1}(X, \mathbb{C})_{prim} = H^{n-1}(X, \mathbb{C})_{van}.$$

First we need a representation of all rational forms on $\mathbb{P}^n$ in local coordinates.

7.2.2 Theorem. *([Gri69], Theorem 2.9) All rational $(n+1-l)$-forms on $\mathbb{P}^n$ can be written as*

$$\begin{aligned} \phi = \frac{1}{B} \sum_{j_1 < \ldots < j_l} (-1)^{j_1 + \ldots + j_l} & \left(\left(\sum_{k=1}^{l} (-1)^k x_{j_k} A_{j_1 \ldots \widehat{j_k} \ldots j_l} \right) \right. \\ & \left. dx_1 \wedge \ldots \wedge \widehat{dx_{j_1}} \wedge \ldots \wedge \widehat{dx_{j_l}} \wedge \ldots \wedge dx_{n+1} \right), \end{aligned}$$

where B and $A_{j_1 \ldots \widehat{j_k} \ldots j_l}$ are homogeneous polynomials on $\mathbb{P}^n$ such that

$$\deg B = \deg A_{j_1 \ldots \widehat{j_k} \ldots j_l} + (n+2-l).$$

7.2.3 Corollary. *([Gri69], Corollary 2.11) All rational n-forms on $\mathbb{P}^n$ can be written as*

$$\phi = \frac{P}{Q}\Delta$$

where P and Q are homogeneous polynomials on $\mathbb{P}^n$ with $\deg Q = \deg P + (n+1)$.

The following theorem shows that the residue map induces an isomorphism between the filtration by the order of the pole along X of rational forms on $\mathbb{P}^n$ with poles along X on the one hand and the Hodge filtration on X on the other hand.

7.2.4 Theorem. *([Voi03], 6.5, Chapter 6.1.3, 6.10, 6.11)*

1. *For each $p \in \mathbb{N}, 1 \leqslant p \leqslant n$, there is a surjective map*

$$\begin{aligned} \alpha_p : \quad & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(p(n+1) - n - 1)) \to F^{n-p+1} H^n(U, \mathbb{C}) \\ & \qquad \cong F^{n-p} H^{n-1}(X, \mathbb{C})_{van}, \\ & P \mapsto \mathrm{Res}^n_{X|\mathbb{P}^n}\left(\left[\frac{P}{f^p} \cdot \Delta\right]\right). \end{aligned}$$

2. Let

$$\begin{aligned}\overline{\alpha}_p : \quad & H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(p\left(n+1\right)-n-1\right)\right) \to F^{n-p}H^{n-1}\left(X,\mathbb{C}\right) \to \\ & \to F^{n-p}H^{n-1}\left(X,\mathbb{C}\right)/F^{n-p+1}H^{n-1}\left(X,\mathbb{C}\right) = H^{n-p,p-1}\left(X\right)\end{aligned}$$

be the composition of α_p *with the projection map. Then*

$$\ker\left(\overline{\alpha}_p\right) = \mathcal{J}_f^{p(n+1)-n-1}.$$

7.2.5 Corollary. *([Voi03], 6.12) The residue map induces a natural isomorphism*

$$R^{p(n+1)-n-1} \overset{\cong}{\to} H^{n-p,p-1}\left(X\right)_{prim},$$

where $R_f^l := S^l/\mathcal{J}_f^l$ *denotes the* l^{th} *component of the Jacobian ring* $R_f = S/\mathcal{J}_f$.

So far we did not need the Calabi-Yau property.

In the following we briefly describe the Griffiths-Dwork-Algorithm for Picard-Fuchs equations of Calabi-Yau hypersurfaces in projective spaces (see e.g. [CK99], Chapter 5.3 or [GHJ03], Chapter 18).

The idea is using the isomorphism of Corollary 7.2.5 in order to find a linear combination as in Equation 7.1.0.1.

Calculation of the Picard-Fuchs equation: Let $\pi : \mathcal{X} \to T$ be a deformation of a Calabi-Yau n-fold X, i.e., a flat proper family of Calabi-Yau n-folds over a connected complex manifold T.

Let $\Omega \in H^0\left(T, R^n\pi_*\left(\Omega^n_{\mathcal{X}|T}\right)\right)$ be a family of non-vanishing, holomorphic $(n-1)$-forms on $\mathcal{X}$ as in Section 7.1.

We assume that $X_t = \{f_t = 0\}$ with homogeneous polynomials

$$f_t \in H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(n+1\right)\right)$$

for each $t \in T$. Using the notation of Section 7.1 we aim to find a linear combination of $\nabla^m_{\frac{\partial}{\partial t}}[\Omega]$ in terms of $\nabla^0_{\frac{\partial}{\partial t}}[\Omega], \ldots, \nabla^{m-1}_{\frac{\partial}{\partial t}}[\Omega]$.

For each $t \in T$, using Theorem 7.2.4, we may write

$$[\Omega\left(t\right)] = \nabla^0_{\frac{\partial}{\partial t}}\left([\Omega]\right)\Big|_t = \left[\operatorname{res}^n_{X|\mathbb{P}^n}\left(\frac{1}{f_t}\cdot\Delta\right)\right] \in H^{n-1,0}\left(X_t\right).$$

As the Gauß-Manin connection ∇ is flat for holomorphic sections, we locally (in T) compute ∇ by taking partial derivatives of $\frac{1}{f_t}\cdot\Delta$ with respect to the parameter $t \in T$ due to the formula

$$\frac{\partial}{\partial t}\int_\gamma \Omega\left(t\right) = \int_\gamma \nabla_{\frac{\partial}{\partial t}}\left[\Omega\left(t\right)\right],$$

where $[\gamma] \in H_3(X_t)$; see [CK99], p. 75.

Then for each k we see that $\nabla^k(\Omega)\big|_t$ corresponds to the residue of a meromorphic form on $\mathbb{P}^n \backslash X$, which has a pole of order at most $k+1$ along X. In fact, let $k \in \mathbb{N}$. Taking partial derivatives of $\frac{1}{f_t} \cdot \Delta$ with respect to t, we find locally in T for the k-th application of the Gauß-Manin connection

$$\nabla^k_{\frac{\partial}{\partial t}}[\Omega](t) \quad = \quad \mathrm{Res}^n_{X|\mathbb{P}^n}\left[\frac{\partial^k}{\partial t^k}\left(\frac{1}{f_t} \cdot \Delta\right)\right] = \mathrm{Res}^n_{X|\mathbb{P}^n}\left[\sum_{i=2}^{k+1} \frac{g_{i,t}}{(f_t)^i} \cdot \Delta\right]$$

with suitable polynomials $g_{i,t} \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(i(n+1)-n-1))$ for all $i = 2, \ldots, k+1$.

Thus applying Theorem 7.2.4, we obtain for each $k = 0, \ldots, m-1$

$$\nabla^k_{\frac{\partial}{\partial t}}[\Omega](t) \in F^{n-(k+1)+1} H^{n-1}(X_t, \mathbb{C}) = F^{n-k} H^{n-1}(X_t, \mathbb{C}).$$

This result is consistent with Griffiths transversality. Especially for $k = m = n+1$, we get with $P_t \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m(n+1)-n-1))$

$$\nabla^m_{\frac{\partial}{\partial t}}[\Omega](t) = \mathrm{Res}^n_{X|\mathbb{P}^n}\left[\sum_{i=2}^{m} \frac{g_{i,t}}{(f_t)^i} \cdot \Delta + \frac{P_t}{(f_t)^{m+1}} \cdot \Delta\right] \in F^0 H^{n-1}(X_t, \mathbb{C}), \tag{7.2.5.1}$$

which must be a linear combination of the linearly independent classes $\nabla^0_{\frac{\partial}{\partial t}}[\Omega], \ldots, \nabla^{m-1}_{\frac{\partial}{\partial t}}[\Omega]$, as $m = \dim H^n(X_t, \mathbb{C})$.

To find this linear combination explicitly, we observe that every rational n-form on $\mathbb{P}^n$ with poles along X_t with numerator in the Jacobian ideal of f is cohomologous to a rational n-form whose pole order is reduced by 1:

7.2.6 Lemma. *For each $t \in T, l \in \mathbb{N}$ and homogeneous polynomials $g_j, j = 1, \ldots, n$, of degree $l(n+1)-n$, the following equality in cohomology holds:*

$$\frac{l \cdot \sum_{j=0}^n g_{j,t} \frac{\partial f_t}{\partial x_j}}{f_t^{l+1}} \cdot \Delta \cong \frac{\sum_{j=0}^n \frac{\partial g_{j,t}}{\partial x_j}}{f_t^l} \cdot \Delta, \tag{7.2.6.1}$$

i.e., the difference of these two forms is an exact rational $(n-1)$-form. Every cohomolgy relation of rational n-forms with poles along X_t has the form 7.2.6.1 for a choice of l and $g_{j,t}$, where $j = 1, \ldots, n$.

Proof of Lemma 7.2.6: For each $l \in \mathbb{N}$ and homogeneous polynomials $g_j, j = 1, \ldots, n$, of degree $l(n+1)-n$, we define the rational $(n-1)$-form

$$\begin{aligned} \phi_t \quad &:= \quad \frac{1}{f^l} \sum_{i<j} (-1)^{i+j} (x_i g_{j,t} - x_j g_{i,t}) \, dx_0 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n \\ &\in H^0\left(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(lX_t)\right). \end{aligned} \tag{7.2.6.2}$$

Differentiating ϕ, we get

$$d\phi_t = \frac{\left(l \sum_{j=0}^{n} g_{j,t} \frac{\partial f_t}{\partial x_j} - f_t \sum_{j=0}^{n} \frac{\partial g_{j,t}}{\partial x_j}\right) \cdot \Delta}{f_t^{l+1}},$$

which yields Equation 7.2.6.1. As every rational $(n-1)$-form on $\mathbb{P}^n$ with poles along X_t has the form 7.2.6.2, the second statement of the lemma follows. $\square$

Thus, as the class $\nabla^m_{\frac{\partial}{\partial t}}[\Omega](t)$ must be cohomologous to a linear combination of $\nabla^0_{\frac{\partial}{\partial t}}[\Omega], \ldots, \nabla^{m-1}_{\frac{\partial}{\partial t}}[\Omega]$, according to Lemma 7.2.6 the homogeneous polynomial P_t has to be in $\mathcal{J}_{f_t}^{m(n+1)-n-1}$. Using the relation 7.2.6.1 we replace the meromorphic form $\frac{P_t}{f_t^{m+1}} \cdot \Delta$ in 7.2.5.1 by a cohomologous form which has at most poles of order m along X_t.

In this way we find the linear combination for $\nabla^m_{\frac{\partial}{\partial t}}[\Omega](t)$ we were looking for.

7.3 The Griffiths-Dwork reduction for complete intersections of codimesion 2 in a projective space

In [LT93] the Griffiths-Dwork method for computing Picard-Fuchs equations is extended to families of Calabi-Yau manifolds that are complete intersections of codimesion 2 in $\mathbb{P}^5$.

In this section, we extend that procedure to complete intersections of codimsion 2 in projective spaces of any dimension.

7.3.1 Setup and computation of a residue map

Let $Q_1(\lambda)$ and $Q_2(\lambda)$ be two general homogeneous polynomials in $\mathbb{P}^n$ depending on a parameter $\lambda \in \mathbb{C}$, so that

$$V_i^\lambda := \{Q_i(\lambda) = 0\} \subset \mathbb{P}^n$$

is smooth and $\{Q_1(\lambda) = 0\} \cup \{Q_2(\lambda) = 0\}$ is a divisor with simple normal crossings in $\mathbb{P}^n$ and

$$V^\lambda := \{Q_1(\lambda) = Q_2(\lambda) = 0\} \subset \mathbb{P}^n$$

is a smooth Calabi-Yau $(n-2)$-fold. Thus, applying the adjunction formula, we have to assume that

$$\deg(Q_1) + \deg(Q_2) = n + 1.$$

In order to be able to describe cohomology classes on V^λ as rational forms on $\mathbb{P}^n$ with poles along the divisor $\{Q_1(\lambda)=0\} \cup \{Q_2(\lambda)=0\}$, we use a composition of two residue maps:

7.3.1 Definition. *For each $k \in \mathbb{N}, k \geqslant 1$, we define a residue map*

$$\operatorname{res}^{LT,k+1}_{V^\lambda|\mathbb{P}^n} : \Omega^{k+1}_{\mathbb{P}^n}\left(\log\left(V_1^\lambda \cup V_2^\lambda\right)\right) \to \Omega^{k-1}_{V^\lambda}$$

by

$$\operatorname{res}^{LT,k+1}_{V^\lambda|\mathbb{P}^n} := \operatorname{res}^{k}_{V^\lambda|V_1^\lambda} \circ \operatorname{res}^{k+1}_{V_1^\lambda|\mathbb{P}^n},$$

where

$$\operatorname{res}^{k+1}_{V_1^\lambda|\mathbb{P}^n} : \Omega^{k+1}_{\mathbb{P}^n}\left(\log\left(V_1^\lambda \cup V_2^\lambda\right)\right) \to \Omega^{k}_{V_1^\lambda}\left(\log\left(V^\lambda\right)\right)$$

and

$$\operatorname{res}^{k}_{V^\lambda|V_1^\lambda} : \Omega^{k}_{V_1^\lambda}\left(\log\left(V^\lambda\right)\right) \to \Omega^{k-1}_{V^\lambda}$$

are the residue maps defined in Chapter 4.2.2.

Let

$$\begin{aligned}\operatorname{Res}^{LT,k+1}_{V^\lambda|\mathbb{P}^n} : \quad & \mathbb{H}^{k+1}\left(\Omega^\bullet_{\mathbb{P}^n}\left(\log\left(V_1^\lambda \cup V_2^\lambda\right)\right)\right) \cong H^{k+1}\left(\mathbb{P}^n \backslash \left(V_1^\lambda \cup V_2^\lambda\right), \mathbb{C}\right) \\ & \to H^{k-1}\left(V^\lambda, \mathbb{C}\right)\end{aligned}$$

be the map induced in cohomology by $\operatorname{res}^{LT,k+1}_{V^\lambda|\mathbb{P}^n}$.

7.3.2 Remark.

1. Thus, the map $\operatorname{Res}^{LT,k+1}_{V^\lambda|\mathbb{P}^n}$ is the composition of

$$\operatorname{Res}^{k+1}_{V_1^\lambda|\mathbb{P}^n} : H^{k+1}\left(\mathbb{P}^n \backslash \left(V_1^\lambda \cup V_2^\lambda\right), \mathbb{C}\right) \to H^{k}\left(V_1^\lambda \backslash V^\lambda, \mathbb{C}\right)$$

   and

$$\operatorname{Res}^{k}_{V^\lambda|V_1^\lambda} : H^{k}\left(V_1^\lambda \backslash V^\lambda, \mathbb{C}\right) \to H^{k-1}\left(V^\lambda, \mathbb{C}\right).$$

2. The residue map $\operatorname{Res}^{LT,k+1}_{V^\lambda|\mathbb{P}^n}$ respects the Hodge filtration, i.e.

$$\operatorname{Res}^{LT,k+1}_{V^\lambda|\mathbb{P}^n}\left(F^i H^{k+1}\left(\mathbb{P}^n \backslash \left(V_1^\lambda \cup V_2^\lambda\right), \mathbb{C}\right)\right) \subset F^{i-2} H^{k-1}\left(V^\lambda, \mathbb{C}\right)$$

   for each $0 \leqslant i \leqslant k+1$ setting $F^j H^{k-1}\left(V^\lambda, \mathbb{C}\right) = H^{k-1}\left(V^\lambda, \mathbb{C}\right)$ for all $j < 0$ (see e.g. [Voi03], p. 159).

Similar to the Griffiths-Dwork reduction for hypersurfaces, we describe $(n-2)$-forms on V^λ via meromorphic forms of higher degree on the ambient space.

7.3.3 Theorem. *([LT93] for $n = 5$.) The residue map*

$$\mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n} : H^n\left(\mathbb{P}^n\backslash\left(V_1^\lambda \cup V_2^\lambda\right),\mathbb{C}\right) \to H^{n-2}\left(V^\lambda,\mathbb{C}\right)$$

induces for $n \in \mathbb{N}$ odd an isomorphism

$$\frac{H^n\left(\mathbb{P}^n\backslash\left(V_1^\lambda \cup V_2^\lambda\right),\mathbb{C}\right)}{\mathrm{im}\left(H^n\left(\mathbb{P}^n\backslash V_1^\lambda,\mathbb{C}\right)\oplus H^n\left(\mathbb{P}^n\backslash V_2^\lambda,\mathbb{C}\right)\right)} \cong H^{n-2}\left(V^\lambda,\mathbb{C}\right), \qquad (7.3.3.1)$$

and for $n \in \mathbb{N}$ even an isomorphism

$$\frac{H^n\left(\mathbb{P}^n\backslash\left(V_1^\lambda \cup V_2^\lambda\right),\mathbb{C}\right)}{\mathrm{im}\left(H^n\left(\mathbb{P}^n\backslash V_1^\lambda,\mathbb{C}\right)\oplus H^n\left(\mathbb{P}^n\backslash V_2^\lambda,\mathbb{C}\right)\right)} \oplus H^{n+2}\left(\mathbb{P}^n,\mathbb{C}\right) \cong H^{n-2}\left(V^\lambda,\mathbb{C}\right). \qquad (7.3.3.2)$$

Here, in both cases

$$\mathrm{im}\left(H^n\left(\mathbb{P}^n\backslash V_1^\lambda,\mathbb{C}\right)\oplus H^n\left(\mathbb{P}^n\backslash V_2^\lambda,\mathbb{C}\right)\right)$$

denotes the image of the map

$$H^n\left(\mathbb{P}^n\backslash V_1^\lambda,\mathbb{C}\right)\oplus H^n\left(\mathbb{P}^n\backslash V_2^\lambda,\mathbb{C}\right) \to H^n\left(\left(\mathbb{P}^n\backslash V_1^\lambda\right)\cap\left(\mathbb{P}^n\backslash V_2^\lambda\right),\mathbb{C}\right),$$

which appears in the Mayer-Vietoris sequence that is associated to $\left(\mathbb{P}^n\backslash V_1^\lambda\right)\cup\left(\mathbb{P}^n\backslash V_2^\lambda\right)$.

Proof of Theorem 7.3.3: In Step 1, we construct isomorphisms of the form 7.3.3.1 or 7.3.3.2, resp., which are not necessarily given as residues. In Step 2, we show that the residue map $\mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}$ provides such isomorphisms.

Step 1: We follow the arguments of [LT93]. Let $T\left(V^\lambda\right)$ be a tubular neighbourhood of $V^\lambda = V_1^\lambda \cap V_2^\lambda \subset \mathbb{P}^n$. Thus $\dim_{\mathbb{R}} T\left(V^\lambda\right) = 2n$. Applying the universal coefficient theorem, we choose an isomorphism

$$H^{n-2}\left(V^\lambda,\mathbb{C}\right) \cong H_{n-2}\left(V^\lambda,\mathbb{C}\right).$$

Using a retraction from $T\left(V^\lambda\right)$ onto V^λ and Poincaré-Lefschetz duality (see e.g. [Mas91], p. 379), we get

$$H_{n-2}\left(V^\lambda,\mathbb{C}\right) \cong H_{n-2}\left(T\left(V^\lambda\right),\mathbb{C}\right) \cong H^{n+2}\left(T\left(V^\lambda\right),\partial T\left(V^\lambda\right),\mathbb{C}\right).$$

Let $\tilde{T}$ be a neighbourhood around $T\left(V^\lambda\right)$ in $\mathbb{P}^n$ such that $\tilde{T}$ and $T\left(V^\lambda\right)$ are homotopy equivalent. Then, using the excision theorem for the pairs $\left(\mathbb{P}^n,\mathbb{P}^n\backslash T\left(V^\lambda\right)\right)$ and $\left(\tilde{T},\tilde{T}\backslash T\left(V^\lambda\right)\right)$, we obtain

$$\begin{aligned} H^{n+2}\left(\mathbb{P}^n,\mathbb{P}^n\backslash T\left(V^\lambda\right),\mathbb{C}\right) &\cong H^{n+2}\left(\tilde{T},\tilde{T}\backslash T\left(V^\lambda\right),\mathbb{C}\right) \cong \\ &\cong H^{n+2}\left(T\left(V^\lambda\right),\partial T\left(V^\lambda\right),\mathbb{C}\right). \end{aligned}$$

Thus

$$H^{n-2}(V_\lambda, \mathbb{C}) \cong H^{n+2}\left(\mathbb{P}^n, \mathbb{P}^n \backslash T\left(V^\lambda\right), \mathbb{C}\right) \cong H^{n+2}\left(\mathbb{P}^n, \mathbb{P}^n \backslash V^\lambda, \mathbb{C}\right).$$

7.3.4 Claim. *There are isomorphisms*

$$H^{n+2}\left(\mathbb{P}^n, \mathbb{P}^n \backslash V^\lambda, \mathbb{C}\right) \cong H^{n+1}\left(\mathbb{P}^n \backslash V^\lambda, \mathbb{C}\right)$$

if n is odd, or

$$H^{n+2}\left(\mathbb{P}^n, \mathbb{P}^n \backslash V^\lambda, \mathbb{C}\right) \cong H^{n+1}\left(\mathbb{P}^n \backslash V^\lambda, \mathbb{C}\right) \oplus H^{n+2}(\mathbb{P}^n, \mathbb{C})$$

if n is even, resp..

Proof of Claim 7.3.4: *Case 1: n odd.* We look at the exact sequence of the pair $\left(\mathbb{P}^n, \mathbb{P}^n \backslash V^\lambda\right)$:

$$\begin{aligned} \ldots \qquad &\to H^{n+1}(\mathbb{P}^n, \mathbb{C}) \to H^{n+1}\left(\mathbb{P}^n \backslash V^\lambda, \mathbb{C}\right) \to H^{n+2}\left(\mathbb{P}^n, \mathbb{P}^n \backslash V^\lambda, \mathbb{C}\right) \to \\ &\to H^{n+2}(\mathbb{P}^n, \mathbb{C}) \to H^{n+2}\left(\mathbb{P}^n \backslash V^\lambda, \mathbb{C}\right) \to \ldots . \end{aligned} \tag{7.3.4.1}$$

If n is odd, then $H^{n+2}(\mathbb{P}^n, \mathbb{C}) = 0$ and $H^{n+1}(\mathbb{P}^n, \mathbb{C}) \cong \mathbb{C}$. In order to see that the map

$$H^{n+1}(\mathbb{P}^n, \mathbb{C}) \to H^{n+1}\left(\mathbb{P}^n \backslash V^\lambda, \mathbb{C}\right)$$

is the zero map, we show equivalently that the dual map

$$H_{n+1}\left(\mathbb{P}^n \backslash V^\lambda, \mathbb{C}\right) \to H_{n+1}(\mathbb{P}^n, \mathbb{C})$$

is the zero map. As

$$H_{n+1}\left(\mathbb{P}^n \backslash V^\lambda, \mathbb{C}\right) \cong H^{n-1}\left(\mathbb{P}^n, V^\lambda, \mathbb{C}\right)$$

by Lefschetz-duality and $H_{n+1}(\mathbb{P}^n, \mathbb{C}) \cong H^{n-1}(\mathbb{P}^n, \mathbb{C})$ by Poincaré-duality, we can find the map we are looking for in the exact sequence of relative cohomology:

$$\ldots \to H^{n-1}\left(\mathbb{P}^n, V^\lambda, \mathbb{C}\right) \to H^{n-1}(\mathbb{P}^n, \mathbb{C}) \to H^{n-1}\left(V^\lambda, \mathbb{C}\right) \to \ldots .$$

The map

$$H^{n-1}(\mathbb{P}^n, \mathbb{C}) \cong \mathbb{C} \to H^{n-1}\left(V^\lambda, \mathbb{C}\right)$$

is injective, because it is not the zero map. In order to see this, let ω be a Kähler form associated to the Fubini-Study metric on $\mathbb{P}^n$. The image of $[\omega \wedge \ldots \wedge \omega] \in H^{n-1}(\mathbb{P}^n, \mathbb{C})$ under the map is $\left[\iota_{V^\lambda}^* \omega \wedge \ldots \wedge \iota_{V^\lambda}^* \omega\right] \neq 0 \in$

$H^{n-1}\left(V^{\lambda}, \mathbb{C}\right)$, where $\iota_{V^{\lambda}} : V^{\lambda} \hookrightarrow \mathbb{P}^n$ is the inclusion. This proves Claim 7.3.4 if n is odd.

Case 2: n even. If n is even, then $H^{n+1}(\mathbb{P}^n, \mathbb{C}) = 0$. Furthermore, as in the case of n being odd, the map

$$H^{n+2}(\mathbb{P}^n, \mathbb{C}) \to H^{n+2}\left(\mathbb{P}^n \backslash V^{\lambda}, \mathbb{C}\right)$$

in the exact sequence 7.3.4.1 vanishes. Therefore

$$H^{n+2}\left(\mathbb{P}^n, \mathbb{P}^n \backslash V^{\lambda}, \mathbb{C}\right) \cong H^{n+1}\left(\mathbb{P}^n \backslash V^{\lambda}, \mathbb{C}\right) \oplus H^{n+2}(\mathbb{P}^n, \mathbb{C}).$$

This proves Claim 7.3.4.

Finally we establish the isomorphism

$$H^{n+1}\left(\mathbb{P}^n \backslash V^{\lambda}, \mathbb{C}\right) \cong \frac{H^n\left(\mathbb{P}^n \backslash \left(V_1^{\lambda} \cup V_2^{\lambda}\right), \mathbb{C}\right)}{\operatorname{im}\left(H^n\left(\mathbb{P}^n \backslash V_1^{\lambda}, \mathbb{C}\right) \oplus H^n\left(\mathbb{P}^n \backslash V_2^{\lambda}, \mathbb{C}\right)\right)}.$$

We look at the Mayer-Vietoris sequence for the open sets $U := \mathbb{P}^n \backslash V_1^{\lambda}$ and $V := \mathbb{P}^n \backslash V_2^{\lambda}$. So we have $U \cup V = \mathbb{P}^n \backslash V^{\lambda}$ and $U \cap V = \mathbb{P}^n \backslash \left(V_1^{\lambda} \cup V_2^{\lambda}\right)$.

$$\begin{aligned} \ldots \quad & \to H^n\left(\mathbb{P}^n \backslash V^{\lambda}, \mathbb{C}\right) \to H^n\left(\mathbb{P}^n \backslash V_1^{\lambda}, \mathbb{C}\right) \oplus H^n\left(\mathbb{P}^n \backslash V_2^{\lambda}, \mathbb{C}\right) \to \\ & \to H^n\left(\mathbb{P}^n \backslash \left(V_1^{\lambda} \cup V_2^{\lambda}\right), \mathbb{C}\right) \to H^{n+1}\left(\mathbb{P}^n \backslash V^{\lambda}, \mathbb{C}\right) \to \\ & \to H^{n+1}\left(\mathbb{P}^n \backslash V_1^{\lambda}, \mathbb{C}\right) \oplus H^{n+1}\left(\mathbb{P}^n \backslash V_2^{\lambda}, \mathbb{C}\right) \to \ldots . \end{aligned}$$

In order to show that the map

$$H^{n+1}\left(\mathbb{P}^n \backslash V^{\lambda}, \mathbb{C}\right) \to H^{n+1}\left(\mathbb{P}^n \backslash V_1^{\lambda}, \mathbb{C}\right) \oplus H^{n+1}\left(\mathbb{P}^n \backslash V_2^{\lambda}, \mathbb{C}\right)$$

is the zero map, we apply Lefschetz- or Poincaré-duality, resp., to the dual maps and show that the maps

$$H^{n-1}\left(\mathbb{P}^n, V_i^{\lambda}, \mathbb{C}\right) \to H^{n-1}\left(\mathbb{P}^n, V^{\lambda}, \mathbb{C}\right)$$

for $i = 1, 2$ vanish. The relative cohomology $H^{n-1}\left(\mathbb{P}^n, V_i^{\lambda}, \mathbb{C}\right)$ for $i = 1, 2$ appears in the long exact sequence of the pair $\left(\mathbb{P}^n, V_i^{\lambda}\right)$:

$$\begin{aligned} \ldots \to H^{n-2}(\mathbb{P}^n, \mathbb{C}) \overset{\cong}{\to} H^{n-2}\left(V_i^{\lambda}, \mathbb{C}\right) \to H^{n-1}\left(\mathbb{P}^n, V_i^{\lambda}, \mathbb{C}\right) \to \\ \to H^{n-1}(\mathbb{P}^n, \mathbb{C}) \to H^{n-1}\left(V_i^{\lambda}, \mathbb{C}\right) \to \ldots . \end{aligned}$$

The Lefschetz hyperplane theorem yields an isomorphism $H^{n-2}(\mathbb{P}^n, \mathbb{C}) \cong H^{n-2}(V_i^\lambda, \mathbb{C})$. Furthermore the map

$$H^{n-1}(\mathbb{P}^n, \mathbb{C}) \to H^{n-1}\left(V_i^\lambda, \mathbb{C}\right)$$

is injective also owing to the Lefschetz hyperplane theorem; therefore, the map

$$H^{n-2}\left(V_i^\lambda, \mathbb{C}\right) \to H^{n-1}\left(\mathbb{P}^n, V_i^\lambda, \mathbb{C}\right)$$

is surjective.

If n is odd, then $H^{n-2}(\mathbb{P}^n, \mathbb{C}) = 0$; therefore $H^{n-2}(V_i^\lambda, \mathbb{C}) = 0$, and the map $H^{n-2}(V_i^\lambda, \mathbb{C}) \to H^{n-1}(\mathbb{P}^n, V_i^\lambda, \mathbb{C})$ is an isomorphism. Thus

$$H^{n-1}\left(\mathbb{P}^n, V_i^\lambda, \mathbb{C}\right) = 0,$$

and the map we were looking for is the zero map.

If n is even, $H^{n-1}(\mathbb{P}^n, \mathbb{C}) = 0$; therefore, the map

$$H^{n-2}\left(V_i^\lambda, \mathbb{C}\right) \to H^{n-1}\left(\mathbb{P}^n, V_i^\lambda, \mathbb{C}\right)$$

is surjective. As it is due to the isomorphism $H^{n-2}(\mathbb{P}^n, \mathbb{C}) \cong H^{n-2}(V_i^\lambda, \mathbb{C})$ the zero map, we know

$$H^{n-1}\left(\mathbb{P}^n, V_i^\lambda, \mathbb{C}\right) = 0$$

and the assertion follows, finishing Step 1.

Step 2: Let

$$\alpha : H^n\left(\mathbb{P}^n \backslash V_1^\lambda, \mathbb{C}\right) \oplus H^n\left(\mathbb{P}^n \backslash V_2^\lambda, \mathbb{C}\right) \to H^n\left(\mathbb{P}^n \backslash \left(V_1^\lambda \cup V_2^\lambda\right), \mathbb{C}\right)$$

be the map given by the Mayer-Vietoris sequence for the pair

$$\left(\mathbb{P}^n \backslash V_1^\lambda, \mathbb{P}^n \backslash V_2^\lambda\right).$$

At first we show

$$\ker\left(\mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\right) \supset \mathrm{im}\,(\alpha). \tag{7.3.4.2}$$

Let $[u] \in \mathrm{im}\,(\alpha)$. Then, using Grothendieck's Algebraic de Rham Theorem, we can write

$$[u] = \alpha\left(\left[v_1|_{\mathbb{P}^n \backslash \left(V_1^\lambda \cup V_2^\lambda\right)}, v_2|_{\mathbb{P}^n \backslash \left(V_1^\lambda \cup V_2^\lambda\right)}\right]\right)$$

with n-forms $v_i \in \Omega^n(\mathbb{P}^n \backslash V_i^\lambda)$ having poles along V_i^λ. Hence

$$\begin{aligned}\mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}([u]) &= \mathrm{Res}^{n-1}_{V^\lambda|V_1^\lambda} \circ \mathrm{Res}^n_{V_1^\lambda|\mathbb{P}^n}\left(\left[v_1|_{\mathbb{P}^n \backslash \left(V_1^\lambda \cup V_2^\lambda\right)}\right]\right) - \\ &\quad - \mathrm{Res}^{n-1}_{V^\lambda|V_1^\lambda} \circ \mathrm{Res}^n_{V_1^\lambda|\mathbb{P}^n}\left(\left[v_2|_{\mathbb{P}^n \backslash \left(V_1^\lambda \cup V_2^\lambda\right)}\right]\right) = 0,\end{aligned}$$

since v_2 does not have any poles along V_1^λ and $\mathrm{res}^n_{V_1^\lambda|\mathbb{P}^n}(v_1)$ does not have any poles along V_2^λ. This establishes 7.3.4.2.

Next we show that the map

$$\mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n} = \mathrm{Res}^{n-1}_{V^\lambda|V_1^\lambda} \circ \mathrm{Res}^n_{V_1^\lambda|\mathbb{P}^n}$$

is surjective.

$\mathrm{Res}^n_{V_1^\lambda|\mathbb{P}^n}$ induces a map between the direct summands of the mixed Hodge decompositions of $H^n\left(\mathbb{P}^n \backslash \left(V_1^\lambda \cup V_2^\lambda\right), \mathbb{C}\right)$ and $H^{n-1}\left(V_1^\lambda \backslash V^\lambda, \mathbb{C}\right)$ as shown in the following diagram:

$$\begin{tikzcd}
H^n\left(\mathbb{P}^n \backslash \left(V_1^\lambda \cup V_2^\lambda\right), \mathbb{C}\right) \arrow[r, "\cong"] \arrow[d, "\mathrm{Res}^n_{V_1^\lambda|\mathbb{P}^n}"] & \bigoplus_{p+q=n} H^q\left(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}\left(\log\left(V_1^\lambda \cup V_2^\lambda\right)\right)\right) \arrow[d] \\
H^{n-1}\left(V_1^\lambda \backslash V^\lambda, \mathbb{C}\right) \arrow[r, "\cong"] & \bigoplus_{p+q=n} H^q\left(V_1^\lambda, \Omega^{p-1}_{V_1^\lambda}\left(\log V^\lambda\right)\right) \arrow[d] \\
 & \bigoplus_{p+q=n} H^{q+1}\left(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}\left(\log V_2^\lambda\right)\right).
\end{tikzcd}$$

Each map

$$H^q\left(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}\left(\log\left(V_1^\lambda \cup V_2^\lambda\right)\right)\right) \to H^q\left(V_1^\lambda, \Omega^{p-1}_{V_1^\lambda}\left(\log V^\lambda\right)\right)$$

is part of the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \Omega^k_{\mathbb{P}^n}\left(\log V_2^\lambda\right) \to \Omega^k_{\mathbb{P}^n}\left(\log\left(V_1^\lambda \cup V_2^\lambda\right)\right) \to \Omega^{k-1}_{V_1^\lambda}\left(\log V^\lambda\right) \to 0$$

for $k \in \mathbb{N}$ (see e.g. [EV92], Prop. 2.3). Applying Serre duality and [EV92], Cor. 6.4, we get

$$H^{q+1}\left(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}\left(\log V_2^\lambda\right)\right) \cong H^{n-q-1}\left(\mathbb{P}^n, \Omega^{n-p}_{\mathbb{P}^n}\left(\log V_2^\lambda\right) - V_2^\lambda\right) = 0.$$

Therefore $\mathrm{Res}^n_{V_1^\lambda|\mathbb{P}^n}$ is surjective.

In the same way we get a diagram for the map $\mathrm{Res}^{n-1}_{V^\lambda|V_1^\lambda}$:

$$\begin{tikzcd}
H^{n-1}\left(V_1^\lambda \backslash V^\lambda, \mathbb{C}\right) \arrow[r, "\cong"] \arrow[d, "\mathrm{Res}^{n-1}_{V^\lambda|V_1^\lambda}"] & \bigoplus_{p+q=n-1} H^q\left(V_1^\lambda, \Omega^p_{V_1^\lambda}\left(\log V^\lambda\right)\right) \arrow[d] \\
H^{n-2}\left(V^\lambda, \mathbb{C}\right) \arrow[r, "\cong"] & \bigoplus_{p+q=n-1} H^q\left(V^\lambda, \Omega^{p-1}_{V^\lambda}\right) \arrow[d] \\
 & \bigoplus_{p+q=n-1} H^{q+1}\left(V_1^\lambda, \Omega^p_{V_1^\lambda}\right).
\end{tikzcd}$$

Again each map in the direct-sum decompositions is part of the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \Omega^p_{V_1^\lambda} \to \Omega^p_{V_1^\lambda}\left(\log V^\lambda\right) \to \Omega^{p-1}_{V^\lambda} \to 0. \qquad (7.3.4.3)$$

For the case that n is odd, we see that the map $\mathrm{Res}^{n-1}_{V^\lambda|V_1^\lambda}$ is surjective. In fact, using Poincaré-duality and the Lefschetz hyperplane theorem, we obtain

$$H^n\left(V_1^\lambda, \mathbb{C}\right) \cong H^{n-2}\left(V_1^\lambda, \mathbb{C}\right) \cong H^{n-2}\left(\mathbb{P}^n, \mathbb{C}\right) = 0;$$

therefore $H^{q+1}\left(V_1^\lambda, \Omega^p_{V_1^\lambda}\right) = 0$ for each p, q with $p + q = n - 1$. Together with the surjectivity of $\mathrm{Res}^n_{V_1^\lambda|\mathbb{P}^n}$ we know that, for n being odd, $\mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}$ is surjective. Furthermore, by Step 1 and 7.3.4.2 we have

$$\ker\left(\mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\right) = \mathrm{im}\left(H^n\left(\mathbb{P}^n\backslash V_1^\lambda, \mathbb{C}\right) \oplus H^n\left(\mathbb{P}^n\backslash V_2^\lambda, \mathbb{C}\right)\right).$$

This establishes Theorem 7.3.3 for the odd-n case.

If n is even, we similarly get

$$H^n\left(V_1^\lambda, \mathbb{C}\right) \cong H^{n-2}\left(\mathbb{P}^n, \mathbb{C}\right) \cong \mathbb{C};$$

therefore $H^{q+1}\left(V_1^\lambda, \Omega^p_{V_1^\lambda}\right) = 0$ for all $(p, q+1) \neq (m, m)$, and

$$H^m\left(V_1^\lambda, \Omega^m_{V_1^\lambda}\right) \cong \mathbb{C}.$$

Hence the only map between direct summands

$$H^q\left(V_1^\lambda, \Omega^p_{V_1^\lambda}\left(\log V^\lambda\right)\right) \to H^q\left(V^\lambda, \Omega^{p-1}_{V^\lambda}\right)$$

induced by $\mathrm{Res}^{n-1}_{V^\lambda|V_1^\lambda}$ which is not surjective is the map

$$\phi : H^{m-1}\left(V_1^\lambda, \Omega^m_{V_1^\lambda}\left(\log V^\lambda\right)\right) \to H^{m-1}\left(V^\lambda, \Omega^{m-1}_{V^\lambda}\right).$$

Applying [EV92], Cor. 6.4, gives

$$H^m\left(V_1^\lambda, \Omega^m_{V_1^\lambda}\left(\log V^\lambda\right)\right) \cong H^{n-1-m}\left(V_1^\lambda, \Omega^{n-1-m}_{V_1^\lambda}\left(\log V^\lambda\right) - V^\lambda\right) = 0.$$

Hence the long exact sequence in cohomology

$$\begin{aligned}\ldots \to H^{m-1}\left(V_1^\lambda, \Omega^m_{V_1^\lambda}\left(\log V^\lambda\right)\right) &\xrightarrow{\phi} H^{m-1}\left(V^\lambda, \Omega^{m-1}_{V^\lambda}\right) \to \\ \to H^m\left(V_1^\lambda, \Omega^m_{V_1^\lambda}\right) &\to H^m\left(V_1^\lambda, \Omega^m_{V_1^\lambda}\left(\log V^\lambda\right)\right) \to \ldots\end{aligned}$$

associated to 7.3.4.3 for $p = m$ yields $\operatorname{codim}(\operatorname{im}(\phi)) = 1$. By Step 1 and 7.3.4.2 we conclude for the case that n is even that

$$\ker\left(\mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\right) = \operatorname{im}\left(H^n\left(\mathbb{P}^n \backslash V_1^\lambda, \mathbb{C}\right) \oplus H^n\left(\mathbb{P}^n \backslash V_2^\lambda, \mathbb{C}\right)\right).$$

□

As a consequence of this theorem, we can represent each cohomology class of degree $(n-2)$ on V^λ as a rational form on $\mathbb{P}^n$ with poles along V_1^λ and V_2^λ.

7.3.5 Corollary. *([LT93] for $n = 5$.) If n is odd, each class*

$$a \in H^{n-2}\left(V^\lambda, \mathbb{C}\right)$$

corresponds, via the isomorphism of Theorem 7.3.3, to a class

$$b \in \frac{H^n\left(\mathbb{P}^n \backslash \left(V_1^\lambda \cup V_2^\lambda\right), \mathbb{C}\right)}{\operatorname{im}\left(H^n\left(\mathbb{P}^n \backslash V_1^\lambda, \mathbb{C}\right) \oplus H^n\left(\mathbb{P}^n \backslash V_2^\lambda, \mathbb{C}\right)\right)}.$$

If n is even, the same holds for $a \in W \subset H^{n-2}\left(V^\lambda, \mathbb{C}\right)$, where W is a linear subspace of codimension 1 in $H^{n-2}\left(V^\lambda, \mathbb{C}\right)$.

We give further information on the complementary space of the subspace W in Corollary 7.3.5.

7.3.6 Lemma. *We assume the setting of Theorem 7.3.3. Let*

$$\tau : H_{n-2}\left(V^\lambda, \mathbb{C}\right) \to H_{n-1}\left(V_1^\lambda \backslash V^\lambda, \mathbb{C}\right)$$

be the tube map. If n is even, then $\ker(\tau)$ is generated by the fundamental class of a linear section $\mathbb{P}^{\frac{n}{2}+1} \cap V^\lambda$ of V^λ.

Proof of Lemma 7.3.6: This is [Gri69], Prop. 3.5, for V_1^λ instead of $\mathbb{P}^n$. We may simply copy the proof, because $H_n\left(V_1^\lambda, \mathbb{C}\right) \cong \mathbb{C}$ according to the Lefschetz hyperplane theorem. □

7.3.7 Corollary. *Let $Z := \mathbb{P}^{\frac{n}{2}+1} \cap V^\lambda$ be a general linear section of V^λ. Then the cohomology class $[Z] \in H^{n-2}\left(V^\lambda, \mathbb{C}\right)$ of Z, i.e. the dual fundamental class, satisfies*

$$[Z] \notin \operatorname{im}\left(\mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\right).$$

Proof of Corollary 7.3.7: It suffices to show that

$$[Z] \notin \operatorname{im}\left(\operatorname{Res}^{n-1}_{V^\lambda|V_1^\lambda} : H^{n-1}\left(V_1\backslash V^\lambda, \mathbb{C}\right) \to H^{n-2}\left(V^\lambda, \mathbb{C}\right)\right).$$

But by definition of the residue, $\operatorname{Res}^{n-1}_{V^\lambda|V_1^\lambda} = \tau^*$, the dual of the tube map. Hence the claim follows from Lemma 7.3.6. □

7.3.8 Corollary. *We also have the following description of the image of the residue map:*

$$\begin{aligned}\operatorname{im}\left(\operatorname{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\right) &= H^{n-2}_{var}\left(V^\lambda\right) \\ &:= \operatorname{coker}\left(H^{n-2}\left(V_1^\lambda, \mathbb{C}\right) \overset{H^{n-2}(\iota)}{\to} H^{n-2}\left(V^\lambda, \mathbb{C}\right)\right) \subset H^{n-2}\left(V^\lambda, \mathbb{C}\right).\end{aligned}$$

Proof of Corollary 7.3.8: Let $[Z] \in H^{m-1,m-1}\left(V^\lambda\right)$ as in Corollary 7.3.7. In particular, $[Z] \in H^{n-2}(\iota)\left(H^{m-1,m-1}\left(V_1^\lambda\right)\right)$. Furthermore $[Z] \notin \operatorname{im}\left(\operatorname{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\right)$, and $\operatorname{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}$ has codimension 1 in $H^{n-2}\left(V_1^\lambda, \mathbb{C}\right)$. Therefore we know $\operatorname{im}\left(\operatorname{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\right) \subset H^{n-2}_{var}\left(V^\lambda\right)$. □

7.3.2 Pole order and Hodge filtration

Now we will show that under the isomorphism of Theorem 7.3.3 the Hodge filtration on $H^{n-2}\left(V^\lambda, \mathbb{C}\right)$ corresponds to the filtration given by the total pole order of rational forms with poles along V_1^λ and V_2^λ with classes in $H^n\left(\mathbb{P}^n\backslash\left(V_1^\lambda \cup V_2^\lambda\right), \mathbb{C}\right)$.
The following theorem is an extension of [Voi03], 6.5, Chapter 6.1.3, 6.10, 6.11, or [Gri69], Chapter 8, to divisors with simple normal crossings with two irreducible components.

7.3.9 Theorem. *Let X be a projective manifold of dimension n and $Y = Y_1 \cup Y_2$ be a divisor with simple normal crossings in X such that the following vanishing hypothesis is satisfied: For all positive integers p_1, p_2, i, j we assume*

$$H^i\left(X, \Omega^j_X(p_1Y_1 + p_2Y_2)\right) = 0. \tag{7.3.9.1}$$

Let $U := X\backslash Y$. For each integer $2 \leqslant p \leqslant n$ we consider the natural map

$$\begin{aligned}\Phi_p : H^0\left(X, \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} K_X(p_1Y_1 + p_2Y_2)\right) &\to H^n(U, \mathbb{C}), \\ \alpha &\mapsto [\alpha],\end{aligned}$$

which maps a section α, viewed as a sum of meromorphic n-forms on X which are holomophic on $U := X \backslash Y$ with poles of order p_i along $Y_i, i = 1, 2$, to the de Rham class $[\alpha]$ of $\alpha|_U$.

Then

$$\mathrm{im}\left(\Phi_p\right) = F^{n-p+2} H^n \left(U, \mathbb{C}\right).$$

Before giving the proof of Theorem 7.3.9, we state two consequences.

7.3.10 Corollary. *We assume $X = \mathbb{P}^n$ and $\deg(Y_1) = d_1, \deg(Y_2) = d_2$; then, according to Bott's theorem, the vanishing hypothesis is satisfied. Composing the map Φ_p of Theorem 7.3.9 with the residue map $\mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}$ yields a map*

$$\Psi_p : H^0 \left(\mathbb{P}^n, \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N}, \\ p_1+p_2=p}} \mathcal{O}_{\mathbb{P}^n}\left(p_1 Y_1 + p_2 Y_2 + K_{\mathbb{P}^n}\right) \right) \to F^{n-p} H^{n-2} \left(V^\lambda, \mathbb{C}\right).$$

If n is odd, then Ψ_p is surjective. If n is even, then $\mathrm{im}\left(\Psi_p\right)$ has codimension 1 in $F^{n-p} H^{n-2}\left(V^\lambda, \mathbb{C}\right)$ for $p \geqslant \frac{n}{2} + 1$.

Proof of Corollary 7.3.10: We have $\Psi_p = \mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n} \circ \Phi_p$. If n is odd, then Φ_p and $\mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n} \circ \Phi_p$ are surjective, and we conclude.

If n is even, then Ψ_p is surjective for $p < \frac{n}{2} + 1$. □

7.3.11 Corollary. *In the setting of Corollary 7.3.5, the class b is represented in de Rham cohomology by a form*

$$\eta = \sum_{i=1}^{n-1} \frac{P_i}{Q_1^i Q_2^{n-i}} \Delta, \tag{7.3.11.1}$$

where the $P_i \in H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(id_1 + (n-i)\, d_2 - n - 1\right)\right)$ are homogeneous polynomials of degree $id_1 + (n-i)\, d_2 - n - 1$.

Proof of Corollary 7.3.11.1: The proof is an application of Corollary 7.3.10 for $p = n$, using the fact that every rational n-form on $\mathbb{P}^n$ with poles along $Y_1 \cup Y_2$ is of the form $\frac{P}{Q}\Delta$ with homogeneous polynomials P and Q with $\deg(Q) + n + 1 = \deg(P)$ due to [Gri69], Cor. 2.11. Consequently, the map Ψ_p is given by

$$\sum_{i=1}^{p-1} P_i \mapsto \left[\sum_{i=1}^{n-1} \frac{P_i}{Q_1^i Q_2^{n-i}} \Delta \right].$$

□

7.3.12 Corollary. *We assume the setting of Corollary 7.3.10. Composing Ψ_p with the natural surjective map*

$$\kappa_p : \bigoplus_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(p_1 d_1 + p_2 d_2 - n - 1\right)\right) \to$$

$$\to H^0\left(\mathbb{P}^n, \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} \mathcal{O}_{\mathbb{P}^n}\left(p_1 Y_1 + p_2 Y_2 + K_{\mathbb{P}^n}\right)\right),$$

we obtain a map

$$\bigoplus_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(p_1 d_1 + p_2 d_2 - n - 1\right)\right) \to F^{n-p}H^{n-2}\left(V^\lambda, \mathbb{C}\right),$$

$$P = \oplus_{k=1}^{p-1} P_k \mapsto \sum_{k=1}^{p-1} \mathrm{Res}_{V^\lambda|\mathbb{P}^n}^{LT,n}\left(\left[\frac{P_k}{Q_1^k Q_2^{p-k}}\Delta\right]\right).$$

If n is odd, all classes in $F^{n-p}H^{n-2}\left(V^\lambda, \mathbb{C}\right)$ for $p = 2, \ldots, n$ are given as

$$\sum_{k=1}^{p-1} \mathrm{Res}_{V^\lambda|\mathbb{P}^n}^{LT,n}\left(\left[\frac{P_k}{Q_1^k Q_2^{p-k}}\Delta\right]\right)$$

for homogeneous polynomials $P_k \in H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(kd_1 + (p-k)\, d_2 - n - 1\right)\right)$. If n is even, this holds for all classes in the image of Ψ_p.

7.3.3 Proof of Theorem 7.3.9

For the proof of Theorem 7.3.9 we need some preparation. In particular we need to compute the cohomology $H^i\left(X, \sum_{(p_1,p_2)} \Omega_X^j\left(p_1 Y_1 + p_2 Y_2\right)\right)$.

7.3.13 Lemma. *Let $\mathcal{T}$ be a sheaf of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ and $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{T}$ sheaves of submodules. Then there is an exact sequence*

$$0 \to \mathcal{S}_1 \cap \mathcal{S}_2 \xrightarrow{j} \mathcal{S}_1 \oplus \mathcal{S}_2 \xrightarrow{\alpha} \mathcal{S}_1 + \mathcal{S}_2 \to 0, \tag{7.3.13.1}$$

where $j(x) = x \oplus (-x)$ and $\alpha(x \oplus y) = x + y$.

7.3.14 Lemma. *Let X be a projective manifold and $Y := Y_1 \cup Y_2$ a divisor with simple normal crossings. We assume $H^i\left(X, \Omega_X^j\left(p_1 Y_1 + p_2 Y_2\right)\right) = 0$ for each $i, j > 0$. Let $k > 0$ and $M_r := \{(k-1, 1), \ldots, (k-r, r)\}$. Then*

$$H^i\left(X, \sum_{(p_1,p_2)\in M_r} \Omega_X^j\left(p_1 Y_1 + p_2 Y_2\right)\right) = 0.$$

Proof of Lemma 7.3.14: We prove the lemma by induction over r.

For $r = 2$ we have $M_2 = \{(k-1,1), (k-2,2)\}$, and we obtain an exact sequence of type 7.3.13.1, namely

$$\begin{array}{rl} 0 & \to \Omega_X^j((k-2)Y_1 + Y_2) \to \\ & \to \Omega_X^j((k-1)Y_1 + Y_2) \oplus \Omega_X^j((k-2)Y_1 + 2Y_2) \to \\ & \to \Omega_X^j((k-1)Y_1 + Y_2) + \Omega_X^j((k-2)Y_1 + 2Y_2) \to 0. \end{array}$$

After applying Bott's vanishing theorem, the long exact sequence in cohomology yields

$$H^i\left(X, \sum_{(p_1,p_2)\in M_2} \Omega_X^j(p_1Y_1 + p_2Y_2)\right) = 0.$$

We assume $H^i\left(X, \sum_{(p_1,p_2)\in M_{r-1}} \Omega_X^j(p_1Y_1 + p_2Y_2)\right) = 0$ and consider the exact sequence

$$\begin{array}{rl} 0 & \to \Omega_X^j((k-r)Y_1 + (r-1)Y_2) \to \\ & \to \sum\limits_{(p_1,p_2)\in M_{r-1}} \Omega_X^j(p_1Y_1 + p_2Y_2) \oplus \Omega_X^j((k-r)Y_1 + rY_2) \to \\ & \to \sum\limits_{(p_1,p_2)\in M_r} \Omega_X^j(p_1Y_1 + p_2Y_2) \to 0. \end{array}$$

Now applying Bott's vanishing theorem and the induction hypothesis, we obtain a long exact sequence in cohomology we obtain

$$H^i\left(X, \sum_{(p_1,p_2)\in M_r} \Omega_X^j(p_1Y_1 + p_2Y_2)\right) = 0.$$

We proceed in this way until $r = k-1$. □

The same methods also show:

7.3.15 Lemma. *We assume the setting of Lemma 7.3.14. The natural map*

$$\begin{array}{rl} \kappa_p: & \bigoplus\limits_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} H^0(X, K_X(p_1Y_1 + p_2Y_2)) \to \\ & \to H^0\left(X, \sum\limits_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} K_X(p_1Y_1 + p_2Y_2)\right). \end{array}$$

is surjective.

7.3.16 Notation. *We denote by*

$$\left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} \Omega_X^k\left(p_1Y_1+p_2Y_2\right)\right)^c$$

the sheaf of closed differential k-forms in

$$\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} \Omega_X^k\left(p_1Y_1+p_2Y_2\right),$$

the sum being taken inside the sheaf of meromorphic k-foms.

We prepare the proof of Theorem 7.3.9 by

7.3.17 Lemma. *Let* $2\leqslant p\leqslant n$ *and*

$$\alpha\in\left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} \Omega_X^k\left(p_1Y_1+p_2Y_2\right)\right)^c.$$

1. *If* $p\geqslant 3$*, then, locally on an open subset* $V\subset X$*, we can write*

$$\alpha=d\beta+\gamma,$$

where

$$\begin{aligned}\beta &\in \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p-1}} \Omega_X^{k-1}\left(p_1Y_1+p_2Y_2\right)\\ \text{and } \gamma &\in \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p-1}} \Omega_X^k\left(p_1Y_1+p_2Y_2\right)\right)^c.\end{aligned}$$

2. *If* $p=2$*, then* α *is a logarithmic form.*

Proof of Lemma 7.3.17: Around a point of $Y_1\cap Y_2$ we choose local coordinates $\{z_1,\ldots,z_n\}$ such that $Y_1=\{z_1=0\}$ and $Y_2=\{z_2=0\}$. We write

$$\alpha=\alpha_1+\ldots+\alpha_{p-1} \text{ with } \alpha_j\in\Omega_X^k\left(jY_1+(p-j)\,Y_2\right) \text{ for } j=1,\ldots,p-1.$$

Then we can write

$$\alpha_j=\frac{dz_1\wedge\alpha_j'}{z_1^j}+\frac{\alpha_j''}{z_1^j},$$

where $\alpha_j' \in \Omega_X^{k-1}((p-j)Y_2)$ and $\alpha_j'' \in \Omega_X^{k}((p-j)Y_2)$ do not contain dz_1. Furthermore

$$\alpha_j' = \frac{dz_2 \wedge \beta_j'}{z_2^{p-j}} + \frac{\gamma_j'}{z_2^{p-j}},$$

where $\beta_j' \in \Omega_X^{k-2}$ and $\gamma_j' \in \Omega_X^{k-1}$ do not contain dz_1 and dz_2. Similarly we write

$$\alpha_j'' = \frac{dz_2 \wedge \beta_j''}{z_2^{p-j}} + \frac{\gamma_j''}{z_2^{p-j}},$$

where $\beta_j'' \in \Omega_X^{k-1}$ and $\gamma_j'' \in \Omega_X^{k}$ do not contain dz_1 and dz_2. Therefore

$$\alpha_j = \frac{dz_1 \wedge dz_2 \wedge \beta_j'}{z_1^j z_2^{p-j}} + \frac{dz_1 \wedge \gamma_j'}{z_1^j z_2^{p-j}} + \frac{dz_2 \wedge \beta_j''}{z_1^j z_2^{p-j}} + \frac{\gamma_j''}{z_1^j z_2^{p-j}}$$

and

$$\begin{aligned} \alpha \quad = \quad & dz_1 \wedge dz_2 \wedge \left(\sum_{j=1}^{p-1} \frac{\beta_j'}{z_1^j z_2^{p-j}} \right) + dz_1 \wedge \left(\sum_{j=1}^{p-1} \frac{\gamma_j'}{z_1^j z_2^{p-j}} \right) + \\ & + dz_2 \wedge \left(\sum_{j=1}^{p-1} \frac{\beta_j''}{z_1^j z_2^{p-j}} \right) + \sum_{j=1}^{p-1} \frac{\gamma_j''}{z_1^j z_2^{p-j}}. \end{aligned}$$

In order to calculate $d\alpha$ we use the following notation

$$\begin{aligned} d\beta_j' &:= dz_1 \wedge \beta_j^{'(1)} + dz_2 \wedge \beta_j^{'(2)} + \beta_j^{'(0)}, \\ d\gamma_j' &:= dz_1 \wedge \gamma_j^{'(1)} + dz_2 \wedge \gamma_j^{'(2)} + \gamma_j^{'(0)}, \\ d\beta_j'' &:= dz_1 \wedge \beta_j^{''(1)} + dz_2 \wedge \beta_j^{''(2)} + \beta_j^{''(0)}, \\ d\gamma_j'' &:= dz_1 \wedge \gamma_j^{''(1)} + dz_2 \wedge \gamma_j^{''(2)} + \gamma_j^{''(0)} \end{aligned}$$

for each $j = 1, \ldots, p-1$, where the differential forms

$$\begin{aligned} &\beta_j^{'(1)}, \beta_j^{'(2)} \in \Omega_X^{k-2}, \beta_j^{'(0)} \in \Omega_X^{k-1}, \gamma_j^{'(1)}, \gamma_j^{'(2)} \in \Omega_X^{k-1}, \gamma_j^{'(0)} \in \Omega_X^{k}, \\ &\beta_j^{''(1)}, \beta_j^{''(2)} \in \Omega_X^{k-1}, \beta_j^{''(0)} \in \Omega_X^{k}, \gamma_j^{''(1)}, \gamma_j^{''(2)} \in \Omega_X^{k}, \gamma_j^{''(0)} \in \Omega_X^{k+1} \end{aligned}$$

do not contain dz_1 and dz_2. We calculate

$$\begin{aligned} d\alpha &= dz_1 \wedge dz_2 \wedge \sum_{j=1}^{p-1} \left(\frac{\beta_j^{'(0)}}{z_1^j z_2^{p-j}} + \frac{(p-j)\,\gamma_j'}{z_1^j z_2^{p-j+1}} - \frac{\gamma_j^{'(2)}}{z_1^j z_2^{p-j}} - \frac{j\beta_j''}{z_1^{j+1} z_2^{p-j}} + \frac{\beta_j^{''(1)}}{z_1^j z_2^{p-j}} \right) \\ &\quad + dz_1 \wedge \sum_{j=1}^{p-1} \left(-\frac{\gamma_j^{'(0)}}{z_1^j z_2^{p-j}} - \frac{j\gamma_j''}{z_1^{j+1} z_2^{p-j}} + \frac{\gamma_j^{''(1)}}{z_1^j z_2^{p-j}} \right) \\ &\quad + dz_2 \wedge \sum_{j=1}^{p-1} \left(-\frac{\beta_j^{''(0)}}{z_1^j z_2^{p-j}} - \frac{(p-j)\,\gamma_j''}{z_1^j z_2^{p-j+1}} + \frac{\gamma_j^{''(2)}}{z_1^j z_2^{p-j}} \right) \\ &\quad + \sum_{j=1}^{p-1} \frac{\gamma_j^{''(0)}}{z_1^j z_2^{p-j}}. \end{aligned}$$

The assumption $d(\alpha) = 0$ yields four equations:

$$\sum_{j=1}^{p-1} \left(\frac{\beta_j^{'(0)}}{z_1^j z_2^{p-j}} + \frac{(p-j)\,\gamma_j'}{z_1^j z_2^{p-j+1}} - \frac{\gamma_j^{'(2)}}{z_1^j z_2^{p-j}} - \frac{j\beta_j''}{z_1^{j+1} z_2^{p-j}} + \frac{\beta_j^{''(1)}}{z_1^j z_2^{p-j}} \right) = 0, \quad (7.3.17.1)$$

$$\sum_{j=1}^{p-1} \left(-\frac{\gamma_j^{'(0)}}{z_1^j z_2^{p-j}} - \frac{j\gamma_j''}{z_1^{j+1} z_2^{p-j}} + \frac{\gamma_j^{''(1)}}{z_1^j z_2^{p-j}} \right) = 0, \quad (7.3.17.2)$$

$$\sum_{j=1}^{p-1} \left(-\frac{\beta_j^{''(0)}}{z_1^j z_2^{p-j}} - \frac{(p-j)\,\gamma_j''}{z_1^j z_2^{p-j+1}} + \frac{\gamma_j^{''(2)}}{z_1^j z_2^{p-j}} \right) = 0, \quad (7.3.17.3)$$

$$\sum_{j=1}^{p-1} \frac{\gamma_j^{''(0)}}{z_1^j z_2^{p-j}} = 0. \quad (7.3.17.4)$$

1. Now we assume $p \geqslant 3$ and aim to find a decomposition $\alpha = d\beta + \gamma$ such that the pole order of β and γ along Y is $p-1$. We can decompose α in the following way

$$\begin{aligned} \alpha &= dz_1 \wedge dz_2 \wedge \sum_{j=1}^{p-1} \frac{\beta_j'}{z_1^j z_2^{p-j}} + dz_1 \wedge \frac{\gamma_1'}{z_1 z_2^{p-1}} + dz_2 \wedge \frac{\beta_{p-1}''}{z_1^{p-1} z_2} + \\ &\quad + \omega_1 + \ldots + \omega_{p-2} + \sum_{j=1}^{p-1} \frac{\gamma_j''}{z_1^j z_2^{p-j}} \end{aligned} \quad (7.3.17.5)$$

with

$$\omega_j := dz_1 \wedge \frac{\gamma_{j+1}'}{z_1^{j+1} z_2^{p-(j+1)}} + dz_2 \wedge \frac{\beta_j''}{z_1^j z_2^{p-j}}$$

for $j = 1, \ldots, p-2$. We proceed to find the requested decomposition for every summand of α in 7.3.17.5.

For $p-j>1$ we calculate

$$dz_1 \wedge dz_2 \wedge \frac{\beta_j'}{z_1^j z_2^{p-j}} = d\left(\frac{1}{p-j-1}\frac{dz_1 \wedge \beta_j'}{z_1^j z_2^{p-j-1}}\right) + \frac{1}{p-j-1}\frac{dz_1 \wedge d\beta_j'}{z_1^j z_2^{p-j-1}},$$

and for $j>1$ we get

$$dz_1 \wedge dz_2 \wedge \frac{\beta_j'}{z_1^j z_2^{p-j}} = d\left(-\frac{1}{j-1}\frac{dz_2 \wedge \beta_j'}{z_1^{j-1} z_2^{p-j}}\right) - \frac{1}{j-1}\frac{dz_2 \wedge d\beta_j'}{z_1^{j-1} z_2^{p-j}}.$$

Thus for each $j=1,\ldots,p-1$ we can find a decomposition $dz_1 \wedge dz_2 \wedge \frac{\beta_j'}{z_1^j z_2^{p-j}} = d\beta + \gamma$, where the pole order of β and γ along Y is at most $p-1$.

In order to show that the summands $dz_1 \wedge \frac{\gamma_1'}{z_1 z_2^{p-1}}$ and $dz_2 \wedge \frac{\beta_{p-1}''}{z_1^{p-1} z_2}$ in the decomposition 7.3.17.5 of α have already poles of order $p-1$ along Y, we multiply equation 7.3.17.1 by $z_1^p z_2^p$ and get

$$\sum_{j=1}^{p-1}\left(\left(\beta_j^{\prime(0)} - \gamma_j^{\prime(2)} + \beta_j^{\prime\prime(1)}\right) z_1^{p-j} z_2^j + (p-j)\,\gamma_j' z_1^{p-j} z_2^{j-1} - j\beta_j'' z_1^{p-j-1} z_2^j\right) = 0. \tag{7.3.17.6}$$

This equation shows that

$$\beta_{p-1}'' = \tilde{\beta}_{p-1}'' z_1 \text{ for a form } \tilde{\beta}_{p-1}'' \in \Omega_X^{k-1}$$

and

$$\gamma_1' = \tilde{\gamma}_1' z_2 \text{ for a form } \tilde{\gamma}_1' \in \Omega_X^{k-1}.$$

So we replace the two k-forms by

$$dz_2 \wedge \frac{\tilde{\beta}_{p-1}''}{z_1^{p-2} z_2} \text{ and } dz_1 \wedge \frac{\tilde{\gamma}_1'}{z_1 z_2^{p-2}}.$$

Furthermore equation 7.3.17.6 yields that there are forms $\phi_j, \psi_j \in \Omega_X^{k-1}$ such that

$$(p-(j+1))\,\gamma_{j+1}' - j\beta_j'' = z_1\phi_j + z_2\psi_j$$

for each $j=1,\ldots,p-2$. For each $j=1,\ldots,p-2$ we calculate

$$\begin{aligned}
\omega_j &= d\left(\frac{1}{j}\frac{-\gamma_{j+1}'}{z_1^j z_2^{p-j-1}}\right) + dz_2 \wedge \frac{1}{j}\left(\frac{j\beta_j'' - (p-(j+1))\,\gamma_{j+1}'}{z_1^j z_2^{p-j}}\right) + \frac{1}{j}\frac{d\gamma_{j+1}'}{z_1^j z_2^{p-j-1}} \\
&= d\left(\frac{1}{j}\frac{-\gamma_{j+1}'}{z_1^j z_2^{p-j-1}}\right) - dz_2 \wedge \frac{1}{j}\left(\frac{\phi_j}{z_1^{j-1} z_2^{p-j}}\right) - dz_2 \wedge \frac{1}{j}\left(\frac{\psi_j}{z_1^j z_2^{p-j-1}}\right) + \\
&\quad + \frac{1}{j}\frac{d\gamma_{j+1}'}{z_1^j z_2^{p-j-1}}.
\end{aligned}$$

If $j > 1$, we are done; in the case that $j = 1$, the term $dz_2 \wedge \left(\frac{\phi_1}{z_2^{p-1}}\right)$ does not have the required pole order, but using the same methods as before we can replace it by

$$d\left(-\frac{1}{p-2}\frac{z_1\phi_1}{z_1 z_2^{p-2}}\right) + \frac{1}{p-2}\frac{z_1 d\phi_1}{z_1 z_2^{p-2}}.$$

So we have found the requested decomposition of each ω_j.

For the last summands of α, namely $\sum_{j=1}^{p-1} \frac{\gamma_j''}{z_1^j z_2^{p-j}}$, we conclude in a similar way as above using the equations 7.3.17.2 and 7.3.17.3 that there are forms $\tilde{\gamma}_1'', \mu_j, \nu_j, \tilde{\gamma}_{p-1}'' \in \Omega_X^k$ for $j = 2, \ldots, p-2$ such that

$$\gamma_1'' = \tilde{\gamma}_1'' z_2, \gamma_{p-1}'' = \tilde{\gamma}_{p-1}'' z_1 \text{ and } \gamma_j'' = z_1\mu_j + z_2\nu_j$$

for $j = 2, \ldots, p-2$.

We conclude that

$$\frac{\gamma_1''}{z_1 z_2^{p-1}} = \frac{\tilde{\gamma}_1''}{z_1 z_2^{p-2}}, \frac{\gamma_{p-1}''}{z_1^{p-1} z_2} = \frac{\tilde{\gamma}_{p-1}''}{z_1^{p-2} z_2}, \frac{\gamma_j''}{z_1^j z_2^{p-j}} = \frac{\mu_j}{z_1^{j-1} z_2^{p-j}} + \frac{\nu_j}{z_1^j z_2^{p-j-1}}$$

for $j = 2, \ldots, p-2$.

By adding all $(k-1)$-forms and all k-forms of the decompositions of the summands of α we get the statement of the lemma. This finishes the case $p \geqslant 3$.

2. If $p = 2$, then $\alpha \in \Omega_X^{k,c}(Y_1 + Y_2)$. In the same way as before we write

$$\alpha = dz_1 \wedge dz_2 \wedge \frac{\beta'}{z_1 z_2} + dz_1 \wedge \frac{\gamma'}{z_1 z_2} + dz_2 \wedge \frac{\beta''}{z_1 z_2} + \frac{\gamma''}{z_1 z_2}$$

where $\beta' \in \Omega_X^{k-2}, \gamma', \beta'' \in \Omega_X^{k-1}, \gamma'' \in \Omega_X^k$ do not contain dz_1 and dz_2. Using $d\alpha = 0$ we get $\gamma' = \tilde{\gamma}' z_2, \beta'' = \tilde{\beta}'' z_1$ and $\gamma'' = \tilde{\gamma}'' z_1 z_2$ for forms $\tilde{\gamma}', \tilde{\beta}'' \in \Omega_X^{k-1}, \tilde{\gamma}'' \in \Omega_X^k$.

Thus we see that α and $d\alpha$ have only simple poles along Y; therefore, α is a logarithmic form. □

7.3.18 Corollary. *Let*

$$\alpha \in \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} \Omega_X^k(p_1Y_1 + p_2Y_2)\right)^c$$

be as in Lemma 7.3.17 with $k \geqslant 3$. If $p_1 + p_2 \geqslant 2$, then we can locally write

$$\alpha = d\beta,$$

where

$$\beta \in \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p-1}} \Omega_X^{k-1}(p_1Y_1 + p_2Y_2).$$

Proof of Corollary 7.3.18: Applying Lemma 7.3.17 we write

$$\alpha = d\beta_1 + \gamma_1,$$

where

$$\beta_1 \in \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p-1}} \Omega_X^{k-1}(p_1Y_1+p_2Y_2) \text{ and } \gamma_1 \in \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p-1}} \Omega_X^{k}(p_1Y_1+p_2Y_2)\right)^c.$$

In the same way we write

$$\gamma_1 = d\beta_2 + \gamma_2,$$

where

$$\beta_2 \in \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p-2}} \Omega_X^{k-1}(p_1Y_1+p_2Y_2) \text{ and } \gamma_2 \in \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p-2}} \Omega_X^{k}(p_1Y_1+p_2Y_2)\right)^c.$$

Continuing in this way we obtain

$$\alpha = d(\beta_1 + \ldots + \beta_{p-2}) + \gamma_{p-2}$$

with

$$\beta := \beta_1 + \ldots + \beta_{p-2} \in \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p-1}} \Omega_X^{k-1}(p_1Y_1+p_2Y_2),$$

and $\gamma_{p-2} \in \left(\Omega_X^k(Y_1+Y_2)\right)^c$ is a logarithmic form. As according to 4.2.4 the logarithmic de Rham complex $\Omega_X^{\bullet}(\log Y)$ is exact in degree $k \geqslant 3$ for a divisor Y with simple normal crossings and two irreducible components, there is a form $\gamma \in \Omega_X^{k-1}(\log Y)$ such that $\gamma_{p-2} = d(\gamma)$. Thus the assertion is proven. $\square$

Proof of Theorem 7.3.9: We first show the following

Claim: There is a surjective map

$$\Lambda_p : H^0\left(X, \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} K_X(p_1Y_1+p_2Y_2)\right) \to H^{p-2}\left(X, \Omega_X^{n-(p-2),c}(\log Y)\right).$$

Proof of the Claim: By Lemma 7.3.14

$$H^i\left(X, \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} \Omega_X^j(p_1Y_1+p_2Y_2)\right) = 0,$$

for $i, j > 0$.

Let $2 \leqslant p \leqslant n$. For each $k \geqslant 3, q \geqslant 3$, Lemma 7.3.17 and Corollary 7.3.18 yield the following exact sequences

$$0 \to \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N}, \\ p_1+p_2=q-1}} \Omega_X^{k-1}\left(p_1Y_1 + p_2Y_2\right) \right)^c \to \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N}, \\ p_1+p_2=q-1}} \Omega_X^{k-1}\left(p_1Y_1 + p_2Y_2\right) \xrightarrow{d} \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N}, \\ p_1+p_2=q}} \Omega_X^{k}\left(p_1Y_1 + p_2Y_2\right) \right)^c \to 0. \tag{7.3.18.1}$$

Starting at

$$\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N}, \\ p_1+p_2=p}} K_X\left(p_1Y_1 + p_2Y_2\right) = \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N}, \\ p_1+p_2=p}} \Omega_X^{n}\left(p_1Y_1 + p_2Y_2\right) \right)^c$$

we use the following exact sequences, which are of the form 7.3.18.1:

$$0 \to \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N}, \\ p_1+p_2=p-1}} \Omega_X^{n-1}\left(p_1Y_1 + p_2Y_2\right) \right)^c \to \sum\nolimits_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N}, \\ p_1+p_2=p-1}} \Omega_X^{n-1}\left(p_1Y_1 + p_2Y_2\right) \xrightarrow{d} \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N}, \\ p_1+p_2=p}} \Omega_X^{n}\left(p_1Y_1 + p_2Y_2\right) \right)^c \to 0,$$

$$0 \to \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N}, \\ p_1+p_2=p-2}} \Omega_X^{n-2}\left(p_1Y_1 + p_2Y_2\right) \right)^c \to \sum\nolimits_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N}, \\ p_1+p_2=p-2}} \Omega_X^{n-2}\left(p_1Y_1 + p_2Y_2\right) \xrightarrow{d} \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N}, \\ p_1+p_2=p-1}} \Omega_X^{n-1}\left(p_1Y_1 + p_2Y_2\right) \right)^c \to 0,$$

$$\vdots$$

until we arrive at

$$0 \to \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=2}} \Omega_X^{n-(p-2)}\left(p_1Y_1+p_2Y_2\right) \right)^c \to \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=2}} \Omega_X^{n-(p-2)}\left(p_1Y_1+p_2Y_2\right) \xrightarrow{d} \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=3}} \Omega_X^{n-(p-3)}\left(p_1Y_1+p_2Y_2\right) \right)^c \to 0.$$

In the last sequence we have by Lemma 7.3.17,

$$\left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=2}} \Omega_X^{n-(p-2)}\left(p_1Y_1+p_2Y_2\right) \right)^c = \Omega_X^{n-(p-2),c}\left(Y_1+Y_2\right) = \Omega_X^{n-(p-2),c}\left(\log Y\right)$$

and similarly

$$\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=2}} \Omega_X^{n-(p-2)}\left(p_1Y_1+p_2Y_2\right) = \Omega_X^{n-(p-2)}\left(Y_1+Y_2\right).$$

The long exact sequences associated to these short exact sequences yield the following surjective maps using the vanishing hypotheses 7.3.9.1:

$$
\begin{aligned}
H^0\left(X, \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} K_X\left(p_1Y_1+p_2Y_2\right)\right) &\to H^1\left(X, \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p-1}} \Omega_X^{n-1}\left(p_1Y_1+p_2Y_2\right)\right)^c\right),\\
H^1\left(X, \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p-1}} \Omega_X^{n-1}\left(p_1Y_1+p_2Y_2\right)\right)^c\right) &\to H^2\left(X, \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p-2}} \Omega_X^{n-2}\left(p_1Y_1+p_2Y_2\right)\right)^c\right),\\
&\vdots
\end{aligned}
$$

until

$$
H^{p-3}\left(X, \left(\sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=3}} \Omega_X^{n-(p-3)}\left(p_1Y_1+p_2Y_2\right)\right)^c\right) \to H^{p-2}\left(X, \Omega_X^{n-(p-2),c}\left(\log Y\right)\right).
$$

This gives the surjective map Λ_p, finishing the proof of the claim.

We next establish an isomorphism

$$\mu_p : H^{p-2}\left(X, \Omega_X^{n-(p-2),c}(\log Y)\right) \to F^{n-(p-2)}H^n(U, \mathbb{C}).$$

In fact, this follows as $F^k\Omega_X^\bullet(\log Y)$ is a resolution of $\Omega_X^{k,c}(\log Y)$ in degree $k \geqslant 2$, because the logarithmic de Rham complex $\Omega_X^\bullet(\log Y)$ is exact in degree $k \geqslant 3$. Since $n - p + 2 \geqslant 2$, we obtain

$$H^{p-2}\left(X, \Omega_X^{n-(p-2),c}(\log Y)\right) = \mathbb{H}^n\left(F^{n-(p-2)}\Omega_X^\bullet(\log Y)\right).$$

As furthermore

$$F^{n-(p-2)}H^n(U, \mathbb{C}) = \operatorname{im}\left(\mathbb{H}^n\left(F^{n-(p-2)}\Omega_X^\bullet(\log Y)\right) \xrightarrow{f_p} \mathbb{H}^n\left(\Omega_X^\bullet(\log Y)\right)\right)$$

and since f_p is injective by the degeneracy at E_1 of the Frölicher spectral sequence, we get the requested isomorphism μ_p. Finally, one has to check that indeed $\mu_p \circ \Lambda_p = \Phi_p$; compare [Voi03].

One has to work through the connecting homomorphisms on the level of a suitable open covering for *Čech*-cohomology and has to use the canonical isomorphism given in [Voi02], Corollary 8.19. We omit the straightforward but tedious details. □

7.3.4 Griffiths-Dwork reduction for codim-2 complete intersections

In the following we describe the Griffiths-Dwork algorithm for complete intersections, which was introduced in [LT93]. For convenience we describe the method here again.

As before, let $Q_1^\lambda = Q_1$ and $Q_2^\lambda = Q_2$ be two homogeneous polynomials on $\mathbb{P}^n$ of degree d_1 and d_2 with the same properties as in Section 7.3.1, and $Y_i := \{Q_i = 0\}$; moreover

$$V^\lambda = \left\{Q_1^\lambda = 0\right\} \cap \left\{Q_2^\lambda = 0\right\}.$$

To set up the method we need a few definitions.

7.3.19 Definition.

1. *For $i = 1, 2$ let*

$$\mathcal{J}_i := \left\langle \frac{\partial Q_i}{\partial x_j}, j = 1, \ldots, n+1 \right\rangle$$

be the Jacobian ideal of Q_i in the graded ring of homogeneous polynomials $S = \bigoplus_l S^l$ in $(n+1)$ variables. Furthermore let J_i be the $1 \times (n+1)$ matrix

$$J_i := \left(\frac{\partial Q_i}{\partial x_1}, \ldots, \frac{\partial Q_i}{\partial x_{n+1}}\right).$$

2. *For each* $k > 2$ *we define a matrix* $K_k : S^{\oplus((n+1)(k-2)+2(k-1))} \to S^{\oplus(k-1)}$ *by*
$$K_k := \left(B_k, \quad I_{k-1} \cdot Q_1, \quad I_{k-1} \cdot Q_2\right),$$
where I_{k-1} *is the* $(k-1) \times (k-1)$*-identity matrix and* B_k *is the following* $(k-1) \times ((n+1)(k-2))$ *matrix*

$$B_k \quad := \quad \begin{pmatrix} (k-2)\,J_1 & 0 & 0 & \dots & 0 & 0 \\ J_2 & (k-3)\,J_1 & 0 & \dots & 0 & 0 \\ 0 & 2J_2 & (k-4)\,J_1 & \dots & 0 & 0 \\ 0 & 0 & 3J_2 & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2J_1 & 0 \\ 0 & 0 & 0 & \dots & (k-3)\,J_2 & J_1 \\ 0 & 0 & 0 & \dots & 0 & (k-2)\,J_2 \end{pmatrix}.$$

7.3.20 Notation. *We apply the matrix* K_p *to a subspace*
$$\widehat{S}_{p-1} \subset S^{\oplus((n+1)(p-2)+2(p-1))},$$
which is defined as follows:

Let
$$\begin{aligned} x_j = x_j^{p-1} &:= (p-1-j)\,d_1 + jd_2 - n \text{ for } j = 1, \dots, p-2 \text{ and} \\ y_j = y_j^{p-1} &:= (p-1-j)\,d_1 + jd_2 - n - 1 \text{ for } j = 1, \dots, p-1 \text{ and} \\ z_j = z_j^{p-1} &:= (p-j)\,d_1 + (j-1)\,d_2 - n - 1 \text{ for } j = 1, \dots, p-1. \end{aligned}$$
Then we define
$$\widehat{S}_{p-1} := \bigoplus_{j=1}^{p-2} \left((S^{x_j})^{\oplus(n+1)} \right) \oplus \bigoplus_{j=1}^{p-1} S^{y_j} \oplus \bigoplus_{j=1}^{p-1} S^{z_j}.$$
Furthermore we define
$$\check{S}_{p-1} := \bigoplus_{j=1}^{p-1} S^{y_j^p} = \bigoplus_{j=1}^{p-1} S^{(p-j)d_1 + jd_2 - n - 1}.$$

7.3.21 Remark. If we restrict the map $K_p : S^{\oplus((n+1)(p-2)+2(p-1))} \to S^{\oplus(p-1)}$ to $\widehat{S}_{p-1}$, we obtain a map $K_p : \widehat{S}_{p-1} \to \check{S}_{p-1}$.

7.3.22 Definition.

1. *For each* $k > 2$ *we define*

$$M_k^* := \operatorname{coker}(K_k) = S^{\oplus(k-1)}/\operatorname{im}(K_k)$$

and

$$M_k := \bigoplus_{l=1}^{k-1} M_k^*\left[(k-l)\,d_1 + ld_2 - n - 1\right] = \check{S}_{k-1}/\operatorname{im}(K_k)$$

to be the direct sum of the parts of M_k^* *which are homogeneous of degree* $((k-l)\,d_1 + ld_2 - n - 1)$ *for* $1 \leqslant l \leqslant k-1$*; furthermore* $M_2 := \mathbb{C}$.

2. *Using the description of classes in* $F^{n-p}H^{n-2}\left(V^\lambda, \mathbb{C}\right)$ *or* $\operatorname{im}(\Psi_p)$*, resp., in Theorem 7.3.12 we define an isomorphism*

$$\begin{aligned} \varrho_{p-1}: \quad & F^{n-p}H^{n-2}\left(V^\lambda, \mathbb{C}\right) \to \check{S}_{p-1}, \\ & \omega = \sum_{k=1}^{p-1} \operatorname{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\left(\left[\frac{P_k}{Q_1^k Q_2^{p-k}}\Delta\right]\right) \mapsto (P_{p-1}, \ldots, P_1) \end{aligned}$$

and a map

$$\begin{aligned} \tilde{\varrho}_{p-1}: \quad & \sum_{k=1}^{p-1} H^0\left(\mathbb{P}^n, K_{\mathbb{P}^n}\left(kY_1 + (p-k)\,Y_2\right)\right) \to \check{S}_{p-1}, \\ & \sum_{k=1}^{p-1} \frac{P_k}{Q_1^k Q_2^{p-k}}\Delta \mapsto (P_{p-1}, \ldots, P_1)\,. \end{aligned}$$

Furthermore, we define a surjective map

$$\begin{aligned} \overline{\varrho}_{p-1}: \quad & F^{n-p}H^{n-2}\left(V^\lambda, \mathbb{C}\right) \to M_p, \\ & \overline{\varrho}_{p-1} := \operatorname{pr}_{M_p} \circ \varrho_{p-1}, \end{aligned}$$

where

$$\operatorname{pr}_{M_p}: \check{S}_{p-1} \to M_p$$

denotes the projection onto M_p.

7.3.23 Lemma. *For* $k, l \in \mathbb{N}$ *and* A_i *homogeneous polynomials of degree* $kd_1 + ld_2$*, one obtains the relation in cohomology*

$$\frac{k\sum_{i=1}^{n+1} A_i \frac{\partial Q_1}{\partial x_i}}{Q_1^{k+1}Q_2^l}\Delta + \frac{l\sum_{i=1}^{n+1} A_i \frac{\partial Q_2}{\partial x_i}}{Q_1^k Q_2^{l+1}}\Delta \equiv \frac{\sum_{i=1}^{n+1} \frac{\partial A_i}{\partial x_i}}{Q_1^k Q_2^l}\Delta \text{ modulo exact forms.}$$

Proof of Lemma 7.3.23: We consider the rational $(n-1)$-form on $\mathbb{P}^n$

$$\phi := \sum_{\substack{k<l,\\ k,l=1}}^{n+1} \frac{x_k A_l - x_l A_k}{Q_1^k Q_2^l} dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge \widehat{dx_l} \wedge \ldots \wedge dx_{n+1}.$$

Then

$$d\phi = \frac{k \sum_{i=1}^{n+1} A_i \frac{\partial Q_1}{\partial x_i}}{Q_1^{k+1} Q_2^l}\Delta + \frac{l \sum_{i=1}^{n+1} A_i \frac{\partial Q_2}{\partial x_i}}{Q_1^k Q_2^{l+1}}\Delta - \frac{\sum_{i=1}^{n+1} \frac{\partial A_i}{\partial x_i}}{Q_1^k Q_2^l}\Delta.$$

□

7.3.24 Remark. Using the cohomology relation of Lemma 7.3.23 for each $p > 2$, we calculate for $A = \left(A_1, \ldots, A_{(n+1)(p-2)+2(p-1)}\right) \in \widehat{S}_{p-1}$ with

$$\begin{aligned}
&\tilde{\varrho}_{p-1}^{-1}\left(K_p\left(\begin{array}{c} A_1 \\ \vdots \\ A_{(n+1)(p-2)+2(p-1)} \end{array}\right)\right) = \\
&= \frac{\sum_{i=1}^{n+1} (p-2) A_i \frac{\partial Q_1}{\partial x_i}}{Q_1^{p-1} Q_2}\Delta + \frac{\sum_{i=1}^{n+1} A_i \frac{\partial Q_2}{\partial x_i}}{Q_1^{p-2} Q_2^2}\Delta + \\
&\quad + \frac{(p-3)\sum_{i=1}^{n+1} A_{i+n+1} \frac{\partial Q_1}{\partial x_i}}{Q_1^{p-2} Q_2^2}\Delta + \frac{2\sum_{i=1}^{n+1} A_{i+n+1} \frac{\partial Q_2}{\partial x_i}}{Q_1^{p-3} Q_2^3}\Delta + \ldots + \\
&\quad + \frac{\sum_{i=1}^{n+1} A_{i+(p-3)(n+1)} \frac{\partial Q_1}{\partial x_i}}{Q_1^2 Q_2^{p-2}}\Delta + \frac{(p-2)\sum_{i=1}^{n+1} A_{i+(p-3)(n+1)} \frac{\partial Q_2}{\partial x_i}}{Q_1 Q_2^{p-1}}\Delta + \\
&\quad + \frac{A_{(n+1)(p-2)+1} Q_1}{Q_1^{p-1} Q_2}\Delta + \ldots + \frac{A_{(n+1)(p-2)+p-2} Q_1}{Q_1^2 Q_2^{p-2}}\Delta + \\
&\quad + \frac{A_{(n+1)(p-2)+p+1} Q_2}{Q_1^{p-2} Q_2^2}\Delta + \ldots + \frac{A_{(n+1)(p-2)+2(p-1)} Q_2}{Q_1 Q_2^{p-1}}\Delta \\
&\equiv \frac{\sum_{i=1}^{n+1} \frac{\partial A_i}{\partial x_i}}{Q_1^{p-2} Q_2}\Delta + \frac{\sum_{i=1}^{n+1} \frac{\partial A_{i+n+1}}{\partial x_i}}{Q_1^{p-3} Q_2^2}\Delta + \frac{\sum_{i=1}^{n+1} \frac{\partial A_{i+2(n+1)}}{\partial x_i}}{Q_1^{p-4} Q_2^3}\Delta + \ldots + \\
&\quad + \frac{\sum_{i=1}^{n+1} \frac{\partial A_{i+(p-3)(n+1)}}{\partial x_i}}{Q_1 Q_2^{p-2}}\Delta + \frac{A_{(n+1)(p-2)+1}}{Q_1^{p-2} Q_2}\Delta + \ldots + \frac{A_{(n+1)(p-2)+p-2}}{Q_1 Q_2^{p-2}}\Delta + \\
&\quad + \frac{A_{(n+1)(p-2)+p+1}}{Q_1^{p-2} Q_2}\Delta + \ldots + \frac{A_{(n+1)(p-2)+2(p-1)}}{Q_1 Q_2^{p-2}}\Delta.
\end{aligned}$$

According to Corollary 7.3.5, we have omitted two summands with poles only along one of the hypersurfaces $Y_i, i = 1, 2$.

Then (in a slightly simplified notation)

$$K_pA \equiv \begin{pmatrix} \sum_{i=1}^{n+1} \frac{\partial A_i}{\partial x_i} + A_{(n+1)(p-2)+1} + A_{(n+1)(p-2)+p+1} \\ \sum_{i=1}^{n+1} \frac{\partial A_{i+n+1}}{\partial x_i} + A_{(n+1)(p-2)+2} + A_{(n+1)(p-2)+p+2} \\ \sum_{i=1}^{n+1} \frac{\partial A_{i+2(n+1)}}{\partial x_i} + A_{(n+1)(p-2)+3} + A_{(n+1)(p-2)+p+3} \\ \vdots \\ \sum_{i=1}^{n+1} \frac{\partial A_{i+(p-4)(n+1)}}{\partial x_i} + A_{(n+1)(p-2)+p-3} + A_{(n+1)(p-2)+2p-3} \\ \sum_{i=1}^{n+1} \frac{\partial A_{i+(p-3)(n+1)}}{\partial x_i} + A_{(n+1)(p-2)+p-2} + A_{(n+1)(p-2)+2(p-1)}, \end{pmatrix}$$
$$\in \bigoplus_{j=1}^{p-2} S^{y_j^{p-1}} = \check{S}_{p-2} \subset S^{\oplus(p-2)}.$$

7.3.25 Theorem. *For each $2 \leqslant p \leqslant n-1$ there is an exact sequence*

$$0 \to F^{n-p}H^{n-2}\left(V^\lambda, \mathbb{C}\right) \to F^{n-(p+1)}H^{n-2}\left(V^\lambda, \mathbb{C}\right) \overset{\overline{\varrho}_p}{\to} M_{p+1} \to 0. \tag{7.3.25.1}$$

If n is even, we have to replace $F^{n-r}H^{n-2}\left(V^\lambda, \mathbb{C}\right)$ by $\operatorname{im}(\Psi_r)$ for $r = p, p+1$.

Proof of Theorem 7.3.25: We just give the proof for n is odd. The exactness is shown in two steps:

1. Let $\omega \in F^{n-p}H^{n-2}\left(V^\lambda, \mathbb{C}\right)$. Then there are homogeneous polynomials $P_k \in H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(kd_1 + (p-k)\, d_2 - n - 1\right)\right), k = 1, \ldots, p-1$, such that

$$\begin{aligned} \omega &= \sum_{k=1}^{p-1} \operatorname{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\left(\left[\frac{P_k}{Q_1^k Q_2^{p-k}}\Delta\right]\right) = \sum_{k=1}^{p-1} \operatorname{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\left(\left[\frac{P_k Q_1}{Q_1^{k+1} Q_2^{p+1-(k+1)}}\Delta\right]\right) \\ &= \sum_{k=2}^{p} \operatorname{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\left(\left[\frac{P_{k-1} Q_1}{Q_1^{k} Q_2^{p+1-k}}\Delta\right]\right) \in F^{n-(p+1)}H^{n-2}\left(V^\lambda, \mathbb{C}\right). \end{aligned}$$

Then

$$\varrho_p(\omega) = \left(P_{p-1}Q_1, P_{p-2}Q_1, \ldots, P_1Q_1, 0\right).$$

Now we have to show that $\omega \in \ker\left(\overline{\varrho}_p\right)$, i.e. we have to find a tuple $A = \left(A_1, \ldots, A_{(n+1)(p-1)+2p}\right) \in \widehat{S}_p$ such that

$$\varrho_p(\omega) = K_{p+1}A.$$

In fact,

$$K_{p+1} \begin{pmatrix} A_1 \\ \vdots \\ A_{(n+1)(p-1)+2p} \end{pmatrix} =$$

$$= \begin{pmatrix} (p-1)\sum_{i=1}^{n+1} A_i \frac{\partial Q_1}{\partial x_i} \\ \sum_{i=1}^{n+1} A_i \frac{\partial Q_2}{\partial x_i} + (p-2)\sum_{i=1}^{n+1} A_{i+n+1}\frac{\partial Q_1}{\partial x_i} \\ 2\sum_{i=1}^{n+1} A_{i+n+1}\frac{\partial Q_2}{\partial x_i} + (p-3)\sum_{i=1}^{n+1} A_{i+2(n+1)}\frac{\partial Q_1}{\partial x_i} \\ \vdots \\ \sum_{i=1}^{n+1} A_{i+(p-2)(n+1)}\frac{\partial Q_1}{\partial x_i} + (p-2)\sum_{i=1}^{n+1} A_{i+(p-3)(n+1)}\frac{\partial Q_2}{\partial x_i} \\ (p-1)\sum_{i=1}^{n+1} A_{i+(p-2)(n+1)}\frac{\partial Q_2}{\partial x_i} \end{pmatrix}$$

$$+ \begin{pmatrix} A_{(n+1)(p-1)+1}Q_1 \\ A_{(n+1)(p-1)+2}Q_1 + A_{(n+1)(p-1)+p+2}Q_2 \\ A_{(n+1)(p-1)+3}Q_1 + A_{(n+1)(p-1)+p+3}Q_2 \\ \vdots \\ A_{(n+1)(p-1)+p-1}Q_1 + A_{(n+1)(p-1)+2p-1}Q_2 \\ A_{(n+1)(p-1)+2p}Q_2 \end{pmatrix} \in \check{S}^p.$$

The vector $A = (0, \ldots, 0, P_{p-1}, \ldots, P_1, 0, 0, \ldots, 0)$ solves the problem.

2. Let $\omega \in F^{n-(p+1)}H^{n-2}\left(V^\lambda, \mathbb{C}\right)$ such that $\overline{\varrho}_p(\omega) = 0$. Then there is an $A \in \widehat{S}_p$ such that $\varrho_p(\omega) = K_{p+1}A$. We have

$$K_{p+1}A \equiv \begin{pmatrix} \sum_{i=1}^{n+1} \frac{\partial A_i}{\partial x_i} + A_{(n+1)(p-1)+1} + A_{(n+1)(p-1)+p+2} \\ \sum_{i=1}^{n+1} \frac{\partial A_{i+n+1}}{\partial x_i} + A_{(n+1)(p-1)+2} + A_{(n+1)(p-1)+p+3} \\ \sum_{i=1}^{n+1} \frac{\partial A_{i+2(n+1)}}{\partial x_i} + A_{(n+1)(p-1)+3} + A_{(n+1)(p-1)+p+4} \\ \vdots \\ \sum_{i=1}^{n+1} \frac{\partial A_{i+(p-3)(n+1)}}{\partial x_i} + A_{(n+1)(p-1)+p-2} + A_{(n+1)(p-1)+2p-1} \\ \sum_{i=1}^{n+1} \frac{\partial A_{i+(p-2)(n+1)}}{\partial x_i} + A_{(n+1)(p-1)+p-1} + A_{(n+1)(p-1)+2p} \end{pmatrix}$$

$$\in \bigoplus_{j=1}^{p-1} S^{y_j^p} = \check{S}_{p-1} \subset S^{\oplus(p-1)}.$$

The resulting vector defines a vector $(P_{p-1}, \ldots, P_1)$ such that

$$\varrho_{p-1}^{-1}(P_{p-1}, \ldots, P_1) =: \omega' \in F^{n-p}H^{n-2}\left(V^\lambda, \mathbb{C}\right)$$

and ω' coincides with the given ω. This shows that Sequence 7.3.25.1 is exact. □

7.3.5 Description of the method by Libgober and Teitelbaum for calculating the Picard-Fuchs equation

In this section we present an extension of the Griffiths-Dwork method for calculating the Picard-Fuchs equation for a codim-2 complete intersection Calabi-Yau manifold, which was introduced by Libgober and Teitelbaum in [LT93].

We will proceed similarly to the case of Calabi-Yau hypersurfaces in Chapter 7.2.

We use the notation introduced in Chapter 7.3.1. Let $\mathcal{V} = \left(V^\lambda\right)_{\lambda\in T} \to T$ be a deformation of the Calabi-Yau $(n-2)$-fold $V^\lambda \subset \mathbb{P}^n$ with T a complex manifold. Let

$$\Omega \in H^0\left(T, R^{n-2}\pi_*\left(\Omega^{n-2}_{\mathcal{V}|T}\right)\right)$$

be a family of non-vanishing holomorphic $(n-2)$-forms on $\mathcal{V}$. Let $m := \dim_{\mathbb{C}} H^{n-2}\left(V^\lambda, \mathbb{C}\right)$. We assume that all Hodge groups $H^{p,q}\left(V^\lambda\right)$ are 1-dimensional, thus $m = n-1$. In practice, these Hodge-groups are not 1-dimensional. However, a finite group is operating on $\mathbb{P}^n$, leaving V^λ invariant, and the invariant Hodge-groups $H^{p,q}\left(V^\lambda\right)^G$ are 1-dimensional. Furthermore, all classes $\nabla^k_{\frac{\partial}{\partial\lambda}}[\Omega]$ are G-invariant.

Again, using the notation of Chapter 7.1, we aim to find a linear combination of $\nabla^m_{\frac{\partial}{\partial\lambda}}[\Omega]$ in terms of $\nabla^0_{\frac{\partial}{\partial\lambda}}[\Omega], \ldots, \nabla^{m-1}_{\frac{\partial}{\partial\lambda}}[\Omega]$.

According to Corollary 7.3.12 we get a family of holomorphic $(n-2)$-forms $\Omega(\lambda)$ by defining

$$[\Omega(\lambda)] = \nabla^0_{\frac{\partial}{\partial\lambda}}([\Omega])\Big|_\lambda = \mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\left[\left(\frac{1}{Q_1(\lambda)\, Q_2(\lambda)}\Delta\right)\right] \in H^{n-2,0}\left(V^\lambda\right).$$

Locally in T we compute ∇ by taking partial derivatives of the rational form $\frac{1}{Q_1(\lambda)Q_2(\lambda)}\Delta$ with respect to λ, i.e.,

$$\nabla^i_{\frac{\partial}{\partial\lambda}}[\Omega](\lambda) = \mathrm{Res}^{LT,n}_{V^\lambda|\mathbb{P}^n}\left[\frac{\partial^i}{\partial\lambda^i}\left(\frac{1}{Q_1(\lambda)\, Q_2(\lambda)}\Delta\right)\right].$$

It is common practice to use the differential operator $\Theta_\lambda := \lambda\frac{\partial}{\partial\lambda}$ instead of $\frac{\partial}{\partial\lambda}$. The aim is to determine the coefficients $s_{m-1}(\lambda), \ldots, s_0(\lambda) \in \mathbb{C}[\lambda]$ of the equation

$$\Theta^m_\lambda = s_{m-1}(\lambda)\cdot\Theta^{m-1}_\lambda + s_{m-2}(\lambda)\cdot\Theta^{m-2}_\lambda + \ldots + s_0(\lambda)\cdot\Theta^0_\lambda.$$

Locally in T we calculate

$$\Theta^k_\lambda\left(\frac{1}{Q_1(\lambda)\, Q_2(\lambda)}\Delta\right) = \sum_{r=2}^{k+2}\sum_{j=1}^{r-1}\frac{P^{(k)}_{r,j}}{Q_1^j Q_2^{r-j}}\Delta, \quad k = 1, \ldots, m,$$

for polynomials $P^{(k)}_{r,j} = P^{(k)}_{r,j}(\lambda) \in H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(jd_1 + (r-j)\,d_2 - n - 1\right)\right)$ depending on the parameter λ.

Step 1. We reduce the summands of $\Theta^m_\lambda\left(\frac{1}{Q_1Q_2}\Delta\right)$ with pole order $m+2$, namely $\sum_{j=1}^{m+1} \frac{P^{(m)}_{m+2,j}}{Q_1^j Q_2^{m+2-j}}\Delta$. As $\Theta^m_\lambda\left(\frac{1}{Q_1Q_2}\Delta\right)$ has to be cohomologous to a form with lower pole order in Q_1 and Q_2, there exists a vector $A^{(m)} = \left(A^{(m)}_1, \ldots, A^{(m)}_{(n+1)m+2(m+1)}\right) \in \widehat{S}_{m+1}$ such that

$$\tilde{\varrho}_{m+1}\left(\sum_{j=1}^{m+1} \frac{P^{(m)}_{m+2,j}}{Q_1^j Q_2^{m+2-j}}\Delta\right) = K_{m+2}A^{(m)}.$$

Then, according to Remark 7.3.24, we reduce $K_{m+2}A^{(m)}$ to an element $q^m = (q^m_1, \ldots, q^m_m) \in \check{S}_m$, i.e., $K_{m+2}A^{(m)} \equiv q^m$, by setting

$$\begin{aligned}
q^m_1 &:= \sum_{i=1}^{n+1} \frac{\partial A^{(m)}_i}{\partial x_i} + A^{(m)}_{(n+1)m+1} + A^{(m)}_{(n+1)m+m+3} \\
q^m_2 &:= \sum_{i=1}^{n+1} \frac{\partial A^{(m)}_{i+n+1}}{\partial x_i} + A^{(m)}_{(n+1)m+2} + A^{(m)}_{(n+1)m+m+4} \\
q^m_3 &:= \sum_{i=1}^{n+1} \frac{\partial A^{(m)}_{i+2(n+1)}}{\partial x_i} + A^{(m)}_{(n+1)m+3} + A^{(m)}_{(n+1)m+m+5} \\
&\vdots \\
q^m_{m-1} &:= \sum_{i=1}^{n+1} \frac{\partial A^{(m)}_{i+(m-2)(n+1)}}{\partial x_i} + A^{(m)}_{(n+1)m+m-1} + A^{(m)}_{(n+1)m+2m+1} \\
q^m_m &:= \sum_{i=1}^{n+1} \frac{\partial A^{(m)}_{i+(m-1)(n+1)}}{\partial x_i} + A^{(m)}_{(n+1)m+m} + A^{(m)}_{(n+1)m+2m+2}.
\end{aligned}$$

By this reduction process we have found a rational form which is cohomologous to $\nabla^m_{\frac{\partial}{\partial\lambda}}[\Omega]$ and whose pole order in Q_1 and Q_2 is at most $m+1$.

Step 2 until Step m.

We continue the reduction procedure in order to find the linear combination of the class $\nabla^m_{\frac{\partial}{\partial\lambda}}[\Omega]$ in terms of $\nabla^0_{\frac{\partial}{\partial\lambda}}[\Omega], \ldots, \nabla^{m-1}_{\frac{\partial}{\partial\lambda}}[\Omega]$ by using the matrices K_{p+2} for $p = m-1, \ldots, 1$.

Starting with an element $q^{p+1} \in \check{S}_{p+1}, p = m-1, \ldots, 1$, we use the matrix K_{p+2} to find an element $A^{(p)} \in \widehat{S}_{p+1}$ such that

$$q^{p+1} = K_{p+2}A^{(p)} + m_{p+2},$$

where $m_{p+2} \in M_{p+2}$. The element m_{p+2} appears now, as it might be possible that

$$q^{p+1} \notin \operatorname{im}\left(K_{p+2} : \widehat{S}_{p+1} \to \check{S}_{p+1}\right),$$

i.e., we cannot find a vector $A^{(p)} \in \widehat{S}_{p+1}$ with $q^{p+1} = K_{p+2}A^{(p)}$. According to Theorem 7.3.25, there could be an element $m_{p+2} \in M_{p+2} \cong H^{n-(p+2),p}\left(V^{\lambda}\right)$ such that $m_{p+2} \notin \operatorname{im}(K_{p+2})$.

Subsequently, we get q^p by reduction of $K_{p+2}A^{(p)}$. We repeat the reduction procedure successively for $p = m-2, \ldots, 1$.

After the last reduction, we obtain a vector $q^1 \in \check{S}_1$ that corresponds to the class $\Theta^0\left(\frac{1}{Q_1Q_2}\Delta\right)$.

Then, the classes m_i should yield the coefficients of the derivatives $\Theta^i\left(\frac{1}{Q_1Q_2}\Delta\right)$ in the Picard-Fuchs equation.

Computational Details: In the following we give a more detailed description of how to get the coefficients of the linear combination in practice. This procedure is implemented in a programme written in the Singular programming language in Appendix A.1.

Step 1' is exactly Step 1.

For all further steps we change the definition of the matrices K_{p+2} in the following way: For each $p = 1, \ldots, m-1$ we concatenate the $((p+1) \times 1)$-matrix $\tilde{\varrho}_{p+1}\left(\sum_{j=1}^{p+1} \frac{P^{(p)}_{p+2,j}}{Q_1^jQ_2^{p+2-j}}\Delta\right) \in S^{\oplus(p+1)}$ with the matrix $K_{p+2} : \widehat{S}_{p+1} \to \check{S}_{p+1}$; i.e. we define a map by

$$\tilde{K}_{p+2} : \widehat{S}'_{p+1} \to \check{S}_{p+1},\ \tilde{K}_{p+2} := \left(\tilde{\varrho}_{p+1}\left(\sum_{j=1}^{p+1} \frac{P^{(p)}_{p+2,j}}{Q_1^jQ_2^{p+2-j}}\Delta\right),\quad K_{p+2}\right),$$

where $\widehat{S}'_{p+1} := S^0 \oplus \widehat{S}_{p+1}$.

According to Griffiths transversality and $\dim H^{i,j}\left(V^{\lambda}\right) = 1$ for each $i + j = n-2$, we know that

$$\tilde{\varrho}_{p+1}\left(\sum_{j=1}^{p+1} \frac{P^{(p)}_{p+2,j}}{Q_1^jQ_2^{p+2-j}}\Delta\right)$$

generates $F^{n-(p+2)}H^{n-2}\left(V^{\lambda}, \mathbb{C}\right)/F^{n-(p+1)}H^{n-2}\left(V^{\lambda}, \mathbb{C}\right)$. Thus we can replace the classes $m_{p+2} \in M_{p+2}$ by adding the entry $\tilde{\varrho}_{p+1}\left(\sum_{j=1}^{p+1} \frac{P^{(p)}_{p+2,j}}{Q_1^jQ_2^{p+2-j}}\Delta\right)$ to the matrix K_{p+2} for each $p = 1, \ldots, m-1$.

Step 2'. Starting with $p = m-1$ we continue by finding a vector $A^{(p)} \in \widehat{S}'_{p+1}$ such that

$$q^{p+1} = \tilde{K}_{p+2}A^{(p)},$$

where $A^{(p)} = \left(A_0^{(p)}, \tilde{A}^{(p)}\right) \in \widehat{S}'_{p+1}$. Then in case $p = m-1$ we can write

$$q^m = A_0^{(m-1)} \cdot \tilde{\varrho}_m \left(\sum_{j=1}^{m} \frac{P_{m+1,j}^{(m-1)}}{Q_1^j Q_2^{m+1-j}} \Delta \right) + K_{m+1} \tilde{A}^{(m-1)}.$$

As in Step 1, we reduce $K_{m+1}\tilde{A}^{(m-1)}$ to an element $q^{m-1} \in \breve{S}_{m-1}$. We repeat the reduction procedure with q^{m-1} instead of q^m.

Step 3' until Step m'. We continue in this way until $p = 1$.

In Step 1', we got

$$\tilde{\varrho}_{m+1} \left(\sum_{j=1}^{m+1} \frac{P_{m+2,j}^{(m)}}{Q_1^j Q_2^{m+2-j}} \Delta \right) \equiv q^m.$$

Then, in Step 2',

$$q^m \equiv A_0^{(m-1)} \cdot \tilde{\varrho}_m \left(\sum_{j=1}^{m} \frac{P_{m+1,j}^{(m-1)}}{Q_1^j Q_2^{m+1-j}} \Delta \right) + q^{m-1},$$

and so on. We have to find the coefficients $s_p(\lambda) \in \mathbb{C}[\lambda]$ for $p = m-1, \ldots, 1$ by collecting the terms of the same pole order such that

$$s_{m-1}(\lambda) \cdot \tilde{\varrho}_m \left(\sum_{j=1}^{m} \frac{P_{m+1,j}^{(m-1)}}{Q_1^j Q_2^{m+1-j}} \Delta \right) = A_0^{(m-1)} \cdot \tilde{\varrho}_m \left(\sum_{j=1}^{m} \frac{P_{m+1,j}^{(m-1)}}{Q_1^j Q_2^{m+1-j}} \Delta \right) + \tilde{\varrho}_m \left(\sum_{j=1}^{m} \frac{P_{m+1,j}^{(m)}}{Q_1^j Q_2^{m+1-j}} \Delta \right)$$

and

$$s_{m-2}(\lambda) \cdot \tilde{\varrho}_{m-1} \left(\sum_{j=1}^{m-1} \frac{P_{m,j}^{(m-2)}}{Q_1^j Q_2^{m-j}} \Delta \right) = A_0^{(m-2)} \cdot \tilde{\varrho}_{m-1} \left(\sum_{j=1}^{m-1} \frac{P_{m,j}^{(m-2)}}{Q_1^j Q_2^{m-j}} \Delta \right) + \tilde{\varrho}_{m-1} \left(\sum_{j=1}^{m-1} \frac{P_{m,j}^{(m)}}{Q_1^j Q_2^{m-j}} \Delta \right) - s_{m-1}(\lambda) \cdot \tilde{\varrho}_{m-1} \left(\sum_{j=1}^{m-1} \frac{P_{m,j}^{(m-1)}}{Q_1^j Q_2^{m-j}} \Delta \right)$$

and so on. For general $p = m-1, \ldots, 1$ the coefficient $s_p(\lambda)$ has to satisfy

$$s_p(\lambda) \cdot \tilde{\varrho}_{p+1} \left(\sum_{j=1}^{p+1} \frac{P_{p+2,j}^{(p)}}{Q_1^j Q_2^{p+2-j}} \Delta \right) = A_0^{(p)} \cdot \tilde{\varrho}_{p+1} \left(\sum_{j=1}^{p+1} \frac{P_{p+2,j}^{(p)}}{Q_1^j Q_2^{p+2-j}} \Delta \right) + \tilde{\varrho}_{p+1} \left(\sum_{j=1}^{p+1} \frac{P_{p+2,j}^{(m)}}{Q_1^j Q_2^{p+2-j}} \Delta \right) - \sum_{i=1}^{m-p-1} s_{m-i}(\lambda) \cdot \tilde{\varrho}_{p+1} \left(\sum_{j=1}^{p+1} \frac{P_{p+2,j}^{(m-i)}}{Q_1^j Q_2^{p+2-j}} \Delta \right).$$

We explain an example carried out by Libgober and Teitelbaum.

7.3.26 Example. [LT93] Let V^λ be the complete intersection of two hypersurfaces in $\mathbb{P}^5$, given by the zero sets of the two homogeneous polynomials

$$Q_1 = x_1^3 + x_2^3 + x_3^3 - 3\lambda\ x_4x_5x_6 \in H^0\left(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)\right)$$

and

$$Q_2 = x_4^3 + x_5^3 + x_6^3 - 3\lambda\ x_1x_2x_3 \in H^0\left(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)\right).$$

Then $V^\lambda = V_1^\lambda \cap V_2^\lambda$ is a Calabi-Yau 3-fold for generic λ. According to [LT93] there is an action of a finite group $G \subset \mathrm{PGL}(5, \mathbb{C})$ that preserves $V_1^\lambda = \{Q_1 = 0\}$ and $V_2^\lambda = \{Q_2 = 0\}$ and all groups $H^{p,q}\left(V^\lambda\right)$ are 1-dimensional.

Then the Picard-Fuchs equation reads

$$\left(\Theta_\lambda^4 - \lambda\left(\Theta_\lambda + \frac{1}{3}\right)^2\left(\Theta_\lambda + \frac{2}{3}\right)^2\right) = 0.$$

A Singular-Programme can be found in the Appendix A.1.

Chapter 8

Picard-Fuchs operators for pairs

In this chapter the Griffiths-Dwork method for calculating the Picard-Fuchs equation is extended to pairs consisting of a Calabi-Yau manifold in a projective space and a smooth hypersurface in the Calabi-Yau manifold. The Griffiths-Dwork method was first transferred to pairs by Jockers and Soroush in [JS09a]. As the hypersurface appears as a complete intersection in the projective space, the method introduced by Libgober and Teitelbaum in [LT93] can be applied in this situation.

To be precise, we consider the following situation.

8.0.-25 Setup. *Let $X = \{P = 0\}$ be a Calabi-Yau hypersurface in a projective space $\mathbb{P}^n$, defined by a homogeneous polynomial $P \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n+1))$. Let $H = \{Q = 0\}$ be another smooth hypersurface in $\mathbb{P}^n$, defined by a homogeneous polynomial $Q \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ for some $k \in \mathbb{N}$. We assume that X and H intersect transversally; therefore $D = X \cap H$ is a smooth divisor in X.*

A central object will be the relative cohomology $H^{n-1}(X, D, \mathbb{C})$. We first define a residue map for the pair (X, D) at the level of forms. This will lead to a residue map on the relative cohomology. The residue map will be used in Section 8.2 in order to compare a Hodge filtration on $H^{n-1}(X, D, \mathbb{C})$ with a filtration by the pole order on the hypercohomology of a complex of rational forms. Then we describe the Griffiths-Dwork reduction using the work of Li, Lian and Yau [LLY12]. The theory of Li, Lian and Yau will be discussed and extended to triples in Chapter 9.

Finally we describe an example where X is a quintic. Here a major new difficulty arises, since a suitable D is no longer smooth.

8.1 Definition of residues for pairs

We recall the following

8.1.1 Notation. *Given a sheaf $\mathcal{S}$ and a section $\eta \in \mathcal{S}(U)$ for an open set U, we write shortly $\eta \in \mathcal{S}$ if we do not want to specify U explicitly.*

For convenience we introduce some notation before giving a definition for a residue map for pairs of logarithmic forms.

8.1.2 Notation. *For each form $\eta \in \Omega^k_{\mathbb{P}^n}(\log X)$ we define a form $\widehat{\eta} \in \Omega^{k+1}_{\mathbb{P}^n}(\log(X+H))$ by*

$$\widehat{\eta} := \eta \wedge \frac{dQ}{Q}.$$

Then

$$\mathrm{res}^{k+1}_{H|\mathbb{P}^n}(\widehat{\eta}) = \eta\big|_H \in \Omega^k_H(\log D).$$

8.1.3 Definition. *We define a complex*

$$\left((\Omega_{\mathbb{P}^n}(\log X) \oplus \Omega_{\mathbb{P}^n}(\log(X+H)))^\bullet, \tilde{d}\right)$$

with differential

$$\begin{aligned} \tilde{d}^k : \quad & \Omega^k_{\mathbb{P}^n}(\log X) \oplus \Omega^k_{\mathbb{P}^n}(\log(X+H)) \to \\ & \qquad\qquad \Omega^{k+1}_{\mathbb{P}^n}(\log X) \oplus \Omega^{k+1}_{\mathbb{P}^n}(\log(X+H)), \\ & \tilde{d}^k(\eta_1, \eta_2) := \left(d^k\eta_1, \widehat{\eta_1} - d^k\eta_2\right). \end{aligned}$$

Obviously

$$\tilde{d}^{k+1} \circ \tilde{d}^k \equiv 0$$

for each k.

8.1.4 Remark. There is an exact sequence of complexes

$$\begin{aligned} 0 \quad \to \quad & (\Omega^\bullet_{\mathbb{P}^n}(\log(X+H)), d) \to \left((\Omega_{\mathbb{P}^n}(\log X) \oplus \Omega_{\mathbb{P}^n}(\log(X+H)))^\bullet, \tilde{d}\right) \\ \to \quad & (\Omega^\bullet_{\mathbb{P}^n}(\log X), d) \to 0. \end{aligned}$$

8.1.5 Definition. *We define a residue map $\mathrm{res}^\bullet_{(X,D)|\mathbb{P}^n}$ for the pair (X, D) by*

$$\begin{aligned} \mathrm{res}^k_{(X,D)|\mathbb{P}^n} := \mathrm{res}^k_{X|\mathbb{P}^n} \oplus \mathrm{res}^{LT,k}_{D|\mathbb{P}^n} : \quad & \Omega^k_{\mathbb{P}^n}(\log X) \oplus \Omega^k_{\mathbb{P}^n}(\log(X+H)) \\ & \to \Omega^{k-1}_X \oplus \Omega^{k-2}_D. \end{aligned}$$

8.1.6 Remark.
The residue $\mathrm{res}^{\bullet}_{(X,D)|\mathbb{P}^n}$ maps exact forms of the complex

$$\left(\Omega^{\bullet}_{\mathbb{P}^n}(\log X) \oplus \Omega^{\bullet}_{\mathbb{P}^n}(\log(X+H))\right)$$

to exact forms of the complex $\Omega^{\bullet-1}_X \oplus \Omega^{\bullet-2}_D$. In fact, given

$$(\eta_1, \eta_2) \in \Omega^{k-1}_{\mathbb{P}^n}(\log X) \oplus \Omega^{k-1}_{\mathbb{P}^n}(\log(X+H)),$$

then

$$d\left(\mathrm{res}^{k-1}_{X|\mathbb{P}^n}(\eta_1), \mathrm{res}^{LT,k-1}_{D|\mathbb{P}^n}(\eta_2)\right) = \mathrm{res}^{k}_{(X,D)|\mathbb{P}^n}\left(\tilde{d}(\eta_1,\eta_2)\right).$$

This last equality follows immediately from

$$\mathrm{res}^{k-1}_{X|\mathbb{P}^n}(\eta_1)\Big|_D = \mathrm{res}^{LT,k}_{D|\mathbb{P}^n}(\widehat{\eta_1}),$$

which is a consequence of

$$\mathrm{res}^{LT,k}_{D|\mathbb{P}^n}(\widehat{\eta_1}) = \mathrm{res}^{k-1}_{D|H} \circ \mathrm{res}^{k}_{H|\mathbb{P}^n}(\widehat{\eta_1}) = \mathrm{res}^{k-1}_{D|H}(\eta_1|_H) = \mathrm{res}^{k-1}_{X|\mathbb{P}^n}(\eta_1|_H).$$

8.1.7 Definition. *According to Remark 8.1.6, for each $k \geqslant 2$ the residue map $\mathrm{res}^{k}_{(X,D)|\mathbb{P}^n}$ defined in Definition 8.1.5 descends to a map which we also denote by*

$$\mathrm{res}^{k}_{(X,D)|\mathbb{P}^n} : \left(\left(\Omega^{k}_{\mathbb{P}^n}(\log X) \oplus \Omega^{k}_{\mathbb{P}^n}(\log(X+H))\right), \tilde{d}\right) \to \left(\Omega^{k-1}_X \oplus \Omega^{k-2}_D, d\right).$$

8.1.8 Definition. *The map $\mathrm{res}^{k}_{(X,D)|\mathbb{P}^n}$ induces a map in cohomology*

$$\mathrm{Res}^{k}_{(X,D)|\mathbb{P}^n} : \mathbb{H}^k\left(\Omega^{\bullet}_{\mathbb{P}^n}(\log X) \oplus \Omega^{\bullet}_{\mathbb{P}^n}(\log(X+H)), \tilde{d}\right) \to H^{k-1}(X, D, \mathbb{C}).$$

8.1.9 Remark. We obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{H}^k\left(\Omega^{\bullet}_{\mathbb{P}^n}(\log(X+H))\right) \arrow[r] \arrow[d, "\mathrm{Res}^{LT,k}_{D|\mathbb{P}^n}"] & \mathbb{H}^k\left(\Omega^{\bullet}_{\mathbb{P}^n}(\log X) \oplus \Omega^{\bullet}_{\mathbb{P}^n}(\log(X+H))\right) \arrow[r] \arrow[d, "\mathrm{Res}^{k}_{(X,D)|\mathbb{P}^n}"] & \mathbb{H}^k\left(\Omega^{\bullet}_{\mathbb{P}^n}(\log X)\right) \arrow[d, "\mathrm{Res}^{k}_{X|\mathbb{P}^n}"] \\
H^{k-2}(D,\mathbb{C}) \arrow[r] & H^{k-1}(X,D,\mathbb{C}) \arrow[r] & H^{k-1}(X,\mathbb{C}).
\end{tikzcd} \tag{8.1.9.1}$$

As we aim to work with rational forms on $\mathbb{P}^n$ with poles along the hypersurfaces X and H, we transfer the definition of $\mathrm{Res}^{\bullet}_{(X,D)|\mathbb{P}^n}$ to the cohomology of rational forms. Therefore we define a complex of global rational forms whose cohomology coincides with the hypercohomology of the complex of logarithmic forms defined in Definition 8.1.3.

8.1.10 Notation. *Let*

$$\Omega^k_{\mathbb{P}^n}(*X) = \sum_{m\geqslant 0} \Omega^k_{\mathbb{P}^n}(mX)$$

be the sheaf of meromorphic k-forms which are holomorphic outside X.

8.1.11 Definition. *Let $\left(\mathcal{K}^\bullet, \tilde{d}^\bullet\right)$ be the complex defined by*

$$\mathcal{K}^k := H^0\left(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(*X) \oplus \Omega^k_{\mathbb{P}^n}(*X + *H)\right)$$

with differential

$$\tilde{d}^k : \mathcal{K}^k \to \mathcal{K}^{k+1}, \tilde{d}^k(\eta_1, \eta_2) := \left(d^k\eta_1, \eta_1 \wedge \frac{dQ}{Q} - d^k\eta_2\right).$$

The following remark is in order.

8.1.12 Remark.

1. Let $(\eta_1, \eta_2) \in H^0\left(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}(mX) \oplus \Omega^k_{\mathbb{P}^n}(sX + rH)\right)$, then

   $$\tilde{d}^k(\eta_1, \eta_2) \in H^0\left(\mathbb{P}^n, \Omega^k_{\mathbb{P}^n}((m+1)X) \oplus \Omega^k_{\mathbb{P}^n}((s+1)X + (r+1)H)\right),$$

   thus $\tilde{d}^\bullet$ is well-defined.

2. Obviously $\tilde{d}^k \circ \tilde{d}^{k-1} = 0$ for each $k \in \mathbb{N}$.

We now compare the complex of global logarithmic pairs with the complex $\mathcal{K}^\bullet$ just defined.

8.1.13 Theorem. *The complexes $\left(\mathcal{K}^\bullet, \tilde{d}\right)$ and*

$$\left(H^0\left(\mathbb{P}^n, \left(\Omega_{\mathbb{P}^n}(\log X) \oplus \Omega_{\mathbb{P}^n}(\log(X+H))\right)^\bullet\right), \tilde{d}^\bullet\right)$$

are quasiisomorphic; therefore

$$H^q\left(\mathcal{K}^\bullet, \tilde{d}\right) \cong \mathbb{H}^q\left(\Omega^\bullet_{\mathbb{P}^n}(\log X) \oplus \Omega^\bullet_{\mathbb{P}^n}(\log(X+H)), \tilde{d}\right)$$

for each $q \in \mathbb{N}$.

Proof of Theorem 8.1.13: We use the exact sequences of complexes

$$\begin{aligned}0 &\to H^0\left(\mathbb{P}^n, \Omega^\bullet_{\mathbb{P}^n}\left(\log\left(X+H\right)\right)\right) \to \\ &\to H^0\left(\mathbb{P}^n, \Omega^\bullet_{\mathbb{P}^n}\left(\log X\right) \oplus \Omega^\bullet_{\mathbb{P}^n}\left(\log\left(X+H\right)\right)\right) \to H^0\left(\mathbb{P}^n, \Omega^\bullet_{\mathbb{P}^n}\left(\log X\right)\right) \\ &\to \ldots\end{aligned}$$

and

$$0 \to H^0\left(\mathbb{P}^n, \Omega^\bullet_{\mathbb{P}^n}\left(*X + *H\right)\right) \to \mathcal{K}^\bullet \to H^0\left(\mathbb{P}^n, \Omega^\bullet_{\mathbb{P}^n}\left(*X\right)\right) \to \ldots.$$

Now we use the quasiisomorphisms

$$\Omega^\bullet_{\mathbb{P}^n}\left(\log X\right) \overset{\cong}{\to} \iota_*\Omega^\bullet_{\mathbb{P}^n\setminus X}$$

and

$$\Omega^\bullet_{\mathbb{P}^n}\left(\log\left(X+H\right)\right) \overset{\cong}{\to} j_*\Omega^\bullet_{\mathbb{P}^n\setminus(X\cup H)},$$

where $\iota : X \hookrightarrow \mathbb{P}^n$ and $j : X \cup H \hookrightarrow \mathbb{P}^n$ denote the inclusion maps. Hence we obtain quasiisomorphisms

$$H^0\left(\mathbb{P}^n, \Omega^\bullet_{\mathbb{P}^n}\left(\log\left(X+H\right)\right)\right) \to H^0\left(\mathbb{P}^n, \Omega^\bullet_{\mathbb{P}^n}\left(*X + *H\right)\right)$$

and

$$H^0\left(\mathbb{P}^n, \Omega^\bullet_{\mathbb{P}^n}\left(\log X\right)\right) \to H^0\left(\mathbb{P}^n, \Omega^\bullet_{\mathbb{P}^n}\left(*X\right)\right)$$

and by the 5-Lemma a quasiisomorphism

$$H^0\left(\mathbb{P}^n, \Omega^\bullet_{\mathbb{P}^n}\left(\log X\right) \oplus \Omega^\bullet_{\mathbb{P}^n}\left(\log\left(X+H\right)\right)\right) \to \mathcal{K}^\bullet.$$

□

We next define a residue map on the cohomology of the complex $\left(\mathcal{K}^\bullet, \tilde{d}\right)$ of global rational forms. As we are going to see that this residue map coincides with the residue map in Definition 8.1.8, we use the same notation.

8.1.14 Definition. *The residue map for classes of rational forms is defined by*

$$\begin{aligned}\mathrm{Res}^q_{(X,D)|\mathbb{P}^n} : H^q\left(\mathcal{K}^\bullet, \tilde{d}\right) &\to H^{q-1}\left(X, D, \mathbb{C}\right) \cong \mathrm{Hom}\left(H_{q-1}\left(X, D\right), \mathbb{C}\right), \\ [(\eta_1, \eta_2)] &\mapsto \Big(\mathrm{Res}^q_{(X,D)|\mathbb{P}^n}[(\eta_1, \eta_2)] : H_{q-1}\left(X, D\right) \to \mathbb{C}, \\ &\qquad [\gamma] \mapsto \int_{\tau(\gamma)} \eta_1 - \int_{\tau'(\partial\gamma)} \eta_2\Big).\end{aligned}$$

It is easily checked that the map $\mathrm{Res}^q_{(X,D)|\mathbb{P}^n}$ is well-defined.

8.1.15 Corollary. *The residue map*

$$\mathrm{Res}^k_{(X,D)|\mathbb{P}^n} : \mathbb{H}^k\left(\Omega^\bullet_{\mathbb{P}^n}\left(\log X\right) \oplus \Omega^\bullet_{\mathbb{P}^n}\left(\log\left(X+H\right)\right), \tilde{d}\right) \to H^{k-1}\left(X, D, \mathbb{C}\right)$$

defined in Definition 8.1.8 coincides with the residue map for pairs of rational forms $\mathrm{Res}^q_{(X,D)|\mathbb{P}^n}$ *defined in Definition 8.1.14 via the isomorphism of Theorem 8.1.13.*

Proof of Corollary 8.1.15: Using Remark 8.1.4 we obtain the following diagram:

$$\begin{tikzcd}
H^k\left(\mathbb{P}^n\backslash\left(X+H\right),\mathbb{C}\right) \arrow[r] \arrow[d, "\mathrm{Res}^{LT,k}_{D|\mathbb{P}^n}"] & H^k\left(\mathcal{K}^\bullet\right) \arrow[r] \arrow[d, "\mathrm{Res}^k_{(X,D)|\mathbb{P}^n}"] & H^k\left(\mathbb{P}^n\backslash X,\mathbb{C}\right) \arrow[d, "\mathrm{Res}^k_{X|\mathbb{P}^n}"] \\
H^{k-2}\left(D,\mathbb{C}\right) \arrow[r] & H^{k-1}\left(X,D,\mathbb{C}\right) \arrow[r] & H^{k-1}\left(X,\mathbb{C}\right).
\end{tikzcd} \tag{8.1.15.1}$$

The left vertical arrow comes from the description of $\mathrm{Res}^{LT,k}_{D|\mathbb{P}^n}$ in [LT93], Chapter 2. Now the claim results from comparing Diagram 8.1.9.1 with Diagram 8.1.15.1. $\square$

Similarly to the case of hypersurfaces and complete intersections in a projective space, we will now formulate Corollary 8.1.15 in an explicit way that is accessible for computations.
We recall the maps

$$\alpha_q : H^0\left(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}\left(qd_1-n-1\right)\right) \to F^{n-q}H^{n-1}\left(X,\mathbb{C}\right),$$

defined in Theorem 7.2.4 and

$$\Psi^{LT}_p : H^0\left(\mathbb{P}^n, \bigoplus_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} \mathcal{O}_{\mathbb{P}^n}\left(p_1d_1+p_2d_2-n-1\right)\right) \to F^{n-p}H^{n-2}\left(D,\mathbb{C}\right),$$

defined in Corollary 7.3.10.

8.1.16 Notation. *We set*

$$\begin{aligned}
\mathcal{C}_{p_1,p_2} &:= \mathcal{O}_{\mathbb{P}^n}\left(p_1d_1+p_2d_2-n-1\right),\\
\mathcal{C}_q &:= \mathcal{O}_{\mathbb{P}^n}\left(qd_1-n-1\right).
\end{aligned}$$

Furthermore, we define the map

$$\begin{aligned}
\Psi^{(X,D)}_{p,q} :\ & H^0\left(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}\left(qd_1-n-1\right)\oplus \bigoplus_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=p}} \mathcal{O}_{\mathbb{P}^n}\left(p_1d_1+p_2d_2-n-1\right)\right)\\
&\to H^{n-1}\left(X,D,\mathbb{C}\right),\\
&\left(S,R_1,\ldots,R_{p-1}\right) \mapsto \mathrm{Res}^n_{(X,D)|\mathbb{P}^n}\left[\left(\frac{S}{P^q}\Delta, \bigoplus_{k=1}^{p-1}\frac{R_k}{P^kQ^{p-k}}\Delta\right)\right].
\end{aligned}$$

8.1.17 Theorem. *We get the following diagram*

$$\begin{tikzcd}
H^0\left(\mathbb{P}^n,\bigoplus_{p_1+p_2=p}\mathcal{C}_{p_1,p_2}\right) \arrow[r] \arrow[d, "\psi^{LT}_p"] & H^0\left(\mathbb{P}^n,\mathcal{C}_q\oplus\bigoplus_{p_1+p_2=p}\mathcal{C}_{p_1,p_2}\right) \arrow[r] \arrow[d, "\psi^{(X,D)}_{p,q}"] & H^0\left(\mathbb{P}^n,\mathcal{C}_q\right) \arrow[d, "\psi_q"] \\
H^{n-2}\left(D,\mathbb{C}\right) \arrow[r, "\zeta_1"] & H^{n-1}\left(X,D,\mathbb{C}\right) \arrow[r, "\zeta_2"] & H^{n-1}\left(X,\mathbb{C}\right),
\end{tikzcd}$$

where the upper row is defined in the obvious way and ζ_1 and ζ_2 are the canonical maps given by the cohomology sequence of pairs. The left square is commutative. The right square is commutative if the map $H^{n-1}(X, D, \mathbb{C}) \to H^{n-1}(X, \mathbb{C})$ is surjective.

Proof of Theorem 8.1.17: Step 1. At first we show the commutativity of the right square. Let $(S, R_1, \ldots, R_{p-1}) \in H^0\left(\mathbb{P}^n, \mathcal{C}_q \oplus \bigoplus_{(p_1+p_2=p)} \mathcal{C}_{p_1,p_2}\right)$, then $\psi_{p,q}^{(X,D)}(S, R_1, \ldots, R_{p-1})$ is the linear form

$$H_{n-1}(X, D) \to \mathbb{C}, \quad [\gamma] \mapsto \int_{\tau(\gamma)} \frac{S}{P^q}\Delta - \int_{\tau'(\partial\gamma)} \sum_{k=1}^{p-1} \frac{R_k}{P^k Q^{p-k}}\Delta.$$

The map ζ_2 restricts $\psi_{p,q}^{(X,D)}(S, R_1, \ldots, R_{p-1})$ to $H_{n-1}(X)$, which lies injective in $H_{n-1}(X, D)$, since the map $H^{n-1}(X, D, \mathbb{C}) \to H^{n-1}(X, \mathbb{C})$ is assumed to be surjective. Then obviously

$$\zeta_2 \circ \psi_{p,q}^{(X,D)}(S, R_1, \ldots, R_{p-1}) = \psi_q(S).$$

Step 2. In the second step we show that the left diagram is commutative. So let $(R_1, \ldots, R_{p-1}) \in H^0\left(\mathbb{P}^n, \bigoplus_{p_1+p_2=p} \mathcal{C}_{p_1,p_2}\right)$. Then $\psi_p^{LT}(R_1, \ldots, R_{p-1})$ is the linear form

$$H_{n-2}(D) \to \mathbb{C}, \quad [\gamma] \mapsto \int_{\tau'(\gamma)} \sum_{k=1}^{p-1} \frac{R_k}{P^k Q^{p-k}}.$$

The composition $\zeta_1 \circ \psi_p^{LT}(R_1, \ldots, R_{p-1})$ can be identified with the restriction of the linear form $\psi_p^{LT}(R_1, \ldots, R_{p-1})$ to the space

$$V := \{[\gamma] \in H_{n-2}(D) | \, H_{n-2}(\iota)[\gamma] = 0 \in H_{n-2}(X)\}.$$

For each closed cycle $\gamma \in C_{n-2}(D)$ with $[\gamma] \in V$, there exists a cycle $\tilde{\gamma} \in C_{n-1}(X)$ such that $\partial(\tilde{\gamma}) = \gamma \in C_{n-1}(D)$ and $[\tilde{\gamma}] \in H_{n-1}(X, D)$. Then

$$\zeta_1 \circ \psi_p^{LT}(R_1, \ldots, R_{p-1})([\tilde{\gamma}]) = \psi_{p,q}^{(X,D)}(0, R_1, \ldots, R_{p-1})([\tilde{\gamma}]).$$

□

8.2 Hodge and pole-order filtration

We obtain a mixed Hodge structure on $H^{n-1}(X, D, \mathbb{C})$ in the following way: The mixed Hodge structure on $H^{n-1}(X \backslash D, \mathbb{C})$ given by the logarithmic complex induces a mixed Hodge structure on the dual space $H^{n-1}(X \backslash D, \mathbb{C})^\vee$, which is canonically isomorphic to $H^{n-1}(X, D, \mathbb{C})$ (see Theorem 4.2.37). The filtration is given as follows by:

$$\begin{aligned}
F^{n-1} &: \left(H^{n-1}(X, \mathcal{O}_X)\right)^{\vee} \cong H^{n-1,0}(X) \\
F^{n-2} &: F^{n-1} \oplus \left(H^{n-2}\left(X, \Omega_X^1(\log D)\right)\right)^{\vee} \\
F^{n-3} &: F^{n-2} \oplus \left(H^{n-3}\left(X, \Omega_X^2(\log D)\right)\right)^{\vee} \\
&\vdots \\
F^{0} &: F^{1} \oplus \left(H^{0}\left(X, \Omega_X^{n-1}(\log D)\right)\right)^{\vee}.
\end{aligned}$$

On the space $H^{n-2}_{var}(D)$ we have the following filtration steps:

$$\begin{aligned}
F^{n-1}_{var}(D) &: 0 \\
F^{n-2}_{var}(D) &: H^{n-2,0}(D) \\
F^{n-3}_{var}(D) &: F^{n-2} \oplus H^{n-3,1}_{var}(D) \\
&\vdots \\
F^{0}_{var}(D) &: F^{1} \oplus H^{0,n-2}_{var}(D).
\end{aligned}$$

Using the canonical Hodge filtration on $H^{n-1}(X, \mathbb{C})$, the sequence is a sequence of mixed Hodge structures, i.e. there are exact sequences

$$0 \to F^k H^{n-2}_{var}(D) \to F^k H^{n-1}(X, D, \mathbb{C}) \to F^k \ker H^{n-1}(X, \mathbb{C}) \to 0, \tag{8.2.0.1}$$

where

$$\ker H^{n-1}(X, \mathbb{C}) := \ker\left(H^{n-1}(X, \mathbb{C}) \to H^{n-1}(D, \mathbb{C})\right).$$

We explain Sequence 8.2.0.1 for the case that $k = n-2$ and D ample in detail. The ampleness implies that the map $H^{n-1}(X, D, \mathbb{C}) \to H^{n-1}(X, \mathbb{C})$ is surjective. Modulo $H^0\left(X, \Omega_X^{n-1}\right)$, Sequence 8.2.0.1 reads

$$0 \to H^0\left(D, \Omega_D^{n-2}\right) \to H^{n-2}\left(X, \Omega_X^1(\log D)\right)^{\vee} \to H^{n-2}\left(X, \Omega_X^1\right)^{\vee} \to 0.$$

Using $H^0\left(D, \Omega_D^{n-2}\right) \cong H^{n-2}(D, \mathcal{O}_D)^{\vee}$ and $H^1\left(X, \Omega_X^{n-2}\right) \cong H^{n-2}\left(X, \Omega_X^1\right)^{\vee}$ and dualizing the sequence, we obtain

$$0 \to H^{n-2}\left(X, \Omega_X^1\right) \to H^{n-2}(D, \mathcal{O}_D) \to H^{n-2}\left(X, \Omega_X^1\right) \to 0.$$

This is just the part of the cohomology sequence associated to the residue sequence

$$0 \to \Omega_X^1 \to \Omega_X^1(\log D) \to \mathcal{O}_D \to 0.$$

In fact, $H^{n-3}(D, \mathcal{O}_D) = 0$, as D is ample and $H^{n-1}\left(X, \Omega_X^1\right) = 0$, since X is Calabi-Yau.

8.2.1 Remark. Using the isomorphism

$$H^{n-1}(X, D, \mathbb{C}) \cong H^{n-1}(X, \mathbb{C}) \oplus H^{n-2}_{var}(D) \tag{8.2.1.1}$$

we get the Hodge filtration

$$\begin{aligned}
F^{n-1} &: H^{n-1,0}(X) \\
F^{n-2} &: F^{n-1} \oplus H^{n-2,1}(X) \oplus H^{n-2,0}_{var}(D) \\
F^{n-3} &: F^{n-2} \oplus H^{n-3,2}(X) \oplus H^{n-3,1}_{var}(D) \\
&\vdots \\
F^{0} &: F^{1} \oplus H^{0,n-1}(X) \oplus H^{0,n-2}_{var}(D).
\end{aligned}$$

As an application of Theorem 8.1.17, we obtain

8.2.2 Theorem. *We assume $V = \mathbb{P}^n$ and $\deg(X) = d_1, \deg(H) = d_2$; then according to Bott's theorem, the vanishing hypothesis is satisfied.*

1. *$\operatorname{im}\left(\Psi^{(X,D)}_{p,q}\right) \in F^{n-r}H^{n-1}(X, D, \mathbb{C})$, where $n-r$ is the minimum of the numbers $n-q$ and $n-p$.*

2. *If n is odd, then $\Psi^{(X,D)}_{p,q}$ is surjective for each p, q.*

3. *If n is even, then $\operatorname{im}\left(\Psi^{(X,D)}_{p,q}\right)$ is not surjective in general; in that case it has codimension 1 in $F^{n-r}H^{n-1}(X, D, \mathbb{C})$, where r is the maximum of p and q.*

8.3 A basis for relative cohomology

We now consider a family $(\mathcal{X}, \mathcal{D})$. For carrying out the Griffiths-Dwork algorithm, we need a basis of $H^{n-1}(X, D, \mathbb{C})$, which we will set up now.

As parameter spaces for the families $\mathcal{X}$ and $\mathcal{D}$, we take local complex manifolds S_1 and S_2. We start with a hypersurface

$$\tilde{\mathcal{X}} \subset \mathbb{P}^n \times S_1.$$

More precisely, $X_z = \{P_z = 0\} \subset \mathbb{P}^n$ is given by a homogeneous polynomial

$$P_z \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(n+1)),$$

which varies holomorphically with $z \in S_1$. Let

$$\tilde{\mathcal{H}} \subset \mathbb{P}^n \times S_2$$

be a second hypersurface given analogously by $H_u = \{Q_u = 0\} \subset \mathbb{P}^n$ for each $u \in S_2$, where Q_u is a homogeneous polynomial on $\mathbb{P}^n$ for $u \in S_2$ with $Q_u \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ for some d.

In summary,

$$\begin{aligned} \tilde{\mathcal{X}} &= (X_z = \{P_z = 0\})_{z \in S_1} \xrightarrow{\pi_1} S_1, \\ \tilde{\mathcal{H}} &= (H_u = \{Q_u = 0\})_{u \in S_2} \xrightarrow{\pi_2} S_2. \end{aligned}$$

We define $\mathcal{X}$ and $\mathcal{H}$ as families over $S := S_1 \times S_2$:

$$\begin{aligned} \mathcal{X} &:= \tilde{\mathcal{X}} \times S_2 \xrightarrow{\pi_1 \times \mathrm{id}} S \\ \mathcal{H} &:= S_1 \times \tilde{\mathcal{H}}_1 \xrightarrow{\mathrm{id} \times \pi_2} S \end{aligned}$$

Finally we define a divisor $\mathcal{D}$ in $\mathcal{X}$ by

$$\mathcal{D} := \{X_z \cap H_u |\, z \in S_1, u \in S_2\} \xrightarrow{(\pi_1, \pi_2)} S_1 \times S_2$$

and set $D_{z,u} := \mathcal{D} \cap X_z \cap H_u$.

Let $[\Omega] \in H^{n-1,0}(X, D) \cong H^{n-1}(X)$ be the class of a holomorphic non-vanishing $(n-1)$-form on X, and ∇ be the Gauß-Manin connection attached to the local system $H^{n-1}(X_z, D_{z,u}, \mathbb{C})$ (cp. 9.5.1).

8.3.1 Lemma. *We suppose that $h^{p,n-1-p}(X) = 1$ for all $p = 0, \ldots, n-1$ and that $h^{q,n-2-q}_{var}(D) = 1$ for all $q = 0, \ldots, n-2$. Then a basis of $H^{n-1}(X, D, \mathbb{C})$ is given by the $2n-1$ elements*

$$[\Omega], \nabla_\psi [\Omega], \nabla_\psi^2 [\Omega], \ldots, \nabla_\psi^{n-1} [\Omega], \nabla_\phi [\Omega], \nabla_\psi \nabla_\phi [\Omega], \ldots, \nabla_\psi^{n-2} \nabla_\phi [\Omega] \\ \in H^{n-1}(X, D, \mathbb{C}).$$

Proof of Lemma 8.3.1: This is an immediate consequence of Theorem 8.2.2. □

8.4 The Griffiths-Dwork method for pairs

We continue assuming the Setup 8.0.-25 and start discussing the approach of Jockers and Soroush.

8.4.1 The approach of Jockers and Soroush

In the paper [JS09a] Jockers and Soroush introduce an integral in the case of Calabi-Yau 3-folds,

$$\int \frac{\log Q}{P} \Delta,$$

and formally compute with this integral. However, this integral needs to be defined properly. In [LLY12] Li, Lian and Yau take a slightly different approach and show that the formal computations give the correct result; see Chapter 9, where we extend the method of Li, Lian and Yau to triples. In this section we show that at least $\int \frac{\log Q}{P} \Delta$ can be defined and gives a 3-form on X.

8.4.1 Lemma. *There is a well-defined class $R(z,u) \in H^3(X,\mathbb{C})$ defined by*

$$\int_\delta R(z,u) = \int_{T(\gamma)} \frac{\log Q_u}{P_z} \Delta$$

for each $\delta \in H_3(X)$ with $\delta = [\iota_*(\gamma)]$ for a closed cycle $\gamma \in C_3(X \backslash D)$, the meaning of the integral on the right hand side being explained in the proof.

Proof of Lemma 8.4.1: We first claim that $H_3(X\backslash D) \to H_3(X)$ is surjective. To see this we consider the following diagram

$$\begin{array}{ccc} H^3(X,D,\mathbb{C}) & \longrightarrow & H^3(X,\mathbb{C}) \\ \downarrow \cong & & \downarrow \cong \\ H_3(X\backslash D) & \longrightarrow & H_3(X), \end{array}$$

where the vertical arrows are given by Poincaré duality and therefore are isomorphisms. Now it suffices that the map $H^3(X,D,\mathbb{C}) \to H^3(X,\mathbb{C})$ is surjective, which follows from the assumption that $H^3(D,\mathbb{C}) = 0$.

Let $\{a_1,\ldots,a_r\}$ be a basis of $H_3(X)$. Since the map $H_3(X\backslash D) \to H_3(X)$ is surjective, for each $k \in \{1,\ldots,r\}$ we can choose closed cycles $\gamma_k \in C_k(X\backslash D)$ such that $a_k = [\iota_*(\gamma_k)]$.
Let

$$T(\gamma_k) \in H_4\left(\mathbb{P}^4 \backslash (X \cup H)\right)$$

be the tube over γ_k in $\mathbb{P}^4 \backslash (X \cup D)$ for each k.
Let $\sigma : \mathbb{C}^5 \backslash \{0\} \to \mathbb{P}^4$ be the projection.

Claim: There exist classes

$$\tilde{T}(\gamma_j) \in H_4\left(\sigma^{-1}\left(\mathbb{P}^4 \backslash (X \cup H)\right)\right),$$

such that $\sigma_*\left(\tilde{T}(\gamma_j)\right) = T(\gamma_j)$.

Proof of Claim: Let $B := \mathbb{P}^4 \backslash (X \cup H)$ and $Z := \sigma^{-1}(B)$. We need to show that the map $\sigma_* : H_4(Z) \to H_4(B)$ is surjective. Since $\sigma : Z \to B$ is a $\mathbb{C}^*$ bundle, we have an exact sequence, see [Spa81], p. 483,

$$H_4(Z) \stackrel{\sigma_*}{\to} H_4(B) \to H_2(B).$$

So it suffices to show that $H_2(B) = 0$. By duality

$$H_2(B) \cong H^6\left(\mathbb{P}^4, X \cup H\right).$$

Now we consider the exact sequence of pairs

$$\begin{aligned} \ldots \to 0 = H^5\left(\mathbb{P}^4, \mathbb{C}\right) \to H^5(X \cup H, \mathbb{C}) \to H^6\left(\mathbb{P}^4, X \cup H, \mathbb{C}\right) \to \\ \to H^6\left(\mathbb{P}^4, \mathbb{C}\right) \cong \mathbb{C} \to H^6(X \cup H, \mathbb{C}) \to \ldots. \end{aligned}$$

The map

$$\mu : H^6\left(\mathbb{P}^4, \mathbb{C}\right) \to H^6(X \cup H, \mathbb{C})$$

is injective; in fact,

$$\mu\left(c_1\left(\mathcal{O}_{\mathbb{P}^4}(1)\right)^3\right) = c_1\left(\mathcal{O}_{\mathbb{P}^4}(1)|_{X \cup H}\right)^3 \neq 0.$$

Hence it suffices to show that $H^5(X \cup H, \mathbb{C}) = 0$. To that end we consider the Mayer-Vietoris sequence

$$\begin{aligned} \ldots \to H^4(X, \mathbb{C}) \oplus H^4(H, \mathbb{C}) \to H^4(D, \mathbb{C}) \cong \mathbb{C} \to H^5(X \cup H, \mathbb{C}) \to \\ \to H^5(X, \mathbb{C}) \oplus H^5(H, \mathbb{C}) \to \ldots. \end{aligned}$$

Obviously the map $H^4(X, \mathbb{C}) \oplus H^4(H, \mathbb{C}) \to H^4(D, \mathbb{C})$ does not vanish, and $H^5(X, \mathbb{C}) = H^5(H, \mathbb{C}) = 0$ by the Lefschetz hyperplane theorem and duality. Therefore the Mayer-Vietoris sequence implies $H^5(X \cup H, \mathbb{C}) = 0$. This proves the claim.

We view Q as a map $\mathbb{C}^5 \setminus \{0\} \to \mathbb{C}$, which we denote by $\sigma^* Q$.

Let $E := (\sigma^* Q)^{-1}(\mathbb{R}^-) \subset \mathbb{C}^5 \setminus \{0\}$ and

$$U := \left(\mathbb{C}^5 \setminus \{0\}\right) \setminus E$$

be the complement. We choose the standard branch of the logarithm, so that $\log(\sigma^* Q|_U)$ makes sense. Then we define:

$$\int_{T(\gamma_k)} \frac{\log Q}{P} \Delta := \int_{\tilde{T}(\gamma_k) \cap U} \sigma^* \left(\frac{\Delta}{P}\right)\bigg|_U \cdot \log\left((\sigma^* Q)|_U\right).$$

We have to show that this integral is finite. Since $T(\gamma_j) \cap H = \emptyset$, hence $\tilde{T}(\gamma_j) \cap \{\sigma^* Q = 0\} = \emptyset$, and therefore $\sigma^* Q\left(\tilde{T}(\gamma_j)\right)$ is compact in $\mathbb{C} \setminus \{0\}$. Therefore $\log(\sigma^* Q)$ is bounded near $\tilde{T}(\gamma_j)$ and hence the integral exists.

In summary we define a linear form on $H_3(X)$ by

$$R(a_j) := \int_{T(\gamma_j)} \frac{\log Q}{P} \Delta.$$

$\square$

8.4.2 Remark. A priori the construction of $R(z,u)$ depends on the choices made in the construction. If the approach of Jockers and Soroush works, $R(z,u)$ should be represented by a $(3,0)$-form, which is unique up to a constant. This however seems not really clear. But if $R(z,u)$ is represented by a $(3,0)$-form, then the construction of $R(z,u)$ does not depend on the choices made in the construction.

8.4.2 The work of Li, Lian and Yau

The paper [LLY12] justifies the method of Jockers and Soroush for determining a Picard-Fuchs equation mathematically. Li, Lian and Yau show that the application of the Gauß-Manin connection to the periods of the residue of a holomorphic n-form with poles along X coincides with taking partial derivatives of the class $R(z,u)$ introduced in Lemma 8.4.1. We will extend this to the case of two divisors and carry out details in the next chapter.

To start with, let $\omega_z \in H^0(\mathbb{P}^n, K_{\mathbb{P}^n}(X_z))$ be a holomorphic family of holomorphic n-forms on $\mathbb{P}^n$ with poles along X_z. Then Li, Lian and Yau show

8.4.3 Theorem. *[LLY12] Let $\Gamma_{z,u} \in H_3(X_z, D_{z,u})$ The periods*

$$\Pi : S \to \mathbb{C}, \quad \Pi(z,u) := \int_{[\Gamma_{z,u}]} \mathrm{Res}^{n-1}_{(X_z, D_{z,u})}[(\omega_{X_z}, 0)]$$

satisfy the following relations:

1. $\partial_z \Pi(z,u) = \int_{[\Gamma_{z,u}]} \mathrm{Res}^{n-1}_{(X_z,D_{z,u})}[(\partial_z \omega_{X_z}, 0)] = \int_{\tau(\Gamma_{z,u})} \partial_z \omega_z$, *where* $\tau(\partial \Gamma_{z,u}) \subset H_u$,
2. $\partial_u \Pi(z,u) = \int_{[\Gamma_{z,u}]} \mathrm{Res}^{n-1}_{(X_z,D_{z,u})}\left[\left(0, \frac{\partial_u Q_u}{Q_u}\omega_z\right)\right] =$
   $= -\int_{\partial\Gamma_{z,u}} \mathrm{Res}^{LT}_{D_{z,u}|\mathbb{P}^n}\left[\frac{\partial_u Q_u}{Q_u}\omega_z\right]$.

8.4.4 Corollary. *[LLY12] All derivatives of Π coincide with the derivatives of R, i.e.,*

$$\begin{aligned} \partial_z \Pi(z,u) &= \partial_z R(z,u), \\ \partial_u \Pi(z,u) &= \partial_u R(z,u), \\ \partial_z \partial_u \Pi(z,u) &= \partial_z \partial_u R(z,u). \end{aligned}$$

8.4.3 Griffiths-Dwork reduction for pairs

Similarly to the case of hypersurfaces or complete intersections in projective spaces, the Griffiths-Dwork algorithm for pairs uses cohomology relations that appear as residues of rational exact forms. The following is the main result in this section, compare [JS09a], Chapter 3.2:

8.4.5 Theorem. *For*

$$A_m \in H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(k \cdot \deg(P) - n\right)\right)$$

and

$$B_m \in H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(k \cdot \deg(P) + l \cdot \deg(Q) - n\right)\right)$$

the following cohomology relations hold:

1.

$$\left[\left(0, \sum_{m=1}^{n+1}\left[k\frac{B_m \partial_{x_m} P}{P^{k+1}Q^l} + l\frac{B_m \partial_{x_m} Q}{P^k Q^{l+1}}\right]\Delta\right)\right] =$$
$$= \left[\left(0, \sum_{m=1}^{n+1}\left[k\frac{B_m \partial_{x_m} P}{P^{k+1}Q^l} + l\frac{B_m \partial_{x_m} Q}{P^k Q^{l+1}}\right]\Delta\right)\right] \in H^n\left(\mathcal{K}^\bullet, \tilde{d}\right)$$

2.

$$\left[\left(\sum_{m=1}^{n+1} k\frac{A_m \partial_{x_m} P}{P^{k+1}}\Delta, \sum_{m=1}^{n+1}\frac{\partial_{x_m} Q \cdot A_m}{P^k Q}\Delta\right)\right] =$$
$$= \left[\left(\sum_{m=1}^{n+1} -\frac{\partial_{x_m} A_m}{P^k}\Delta, 0\right)\right] \in H^n\left(\mathcal{K}^\bullet, \tilde{d}\right).$$

For the following lemma, which will be used in the proof of Theorem 8.4.5, we refer to [Gri69], Theorem 7.2.2 and 7.2.3, and [LT93].

8.4.6 Lemma. *Let* $(\eta_1, \eta_2) \in H^0\left(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(kX) \oplus \Omega^{n-1}_{\mathbb{P}^n}(kX + lH)\right)$. *Then there are homogeneous polynomials*

$$A_m \in H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(k \cdot \deg(P) - n\right)\right)$$

and

$$B_m \in H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}\left(k \cdot \deg(P) + l \cdot \deg(Q) - n\right)\right)$$

such that

$$\eta_1 = \sum_{m<j}\frac{x_j A_m - x_m A_j}{P^k}(-1)^{m+j} dx_1 \wedge \ldots \wedge \widehat{dx_m} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_{n+1}$$

and

$$\eta_2 = \sum_{m<j}\frac{x_j B_m - x_m B_j}{P^k Q^l}(-1)^{m+j} dx_1 \wedge \ldots \wedge \widehat{dx_m} \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_{n+1}.$$

Consequently,

$$d\eta_1 = \sum_{m=1}^{n+1}\left[k\frac{A_m \partial_{x_m} P}{P^{k+1}} - \frac{\partial_{x_m} A_m}{P^k}\right]\Delta$$

and

$$d\eta_2 = \sum_{m=1}^{n+1}\left[k\frac{B_m \partial_{x_m} P}{P^{k+1}Q^l} + l\frac{B_m \partial_{x_m} Q}{P^k Q^{l+1}} - \frac{\partial_{x_m} B_m}{P^k Q^l}\right]\Delta.$$

Proof of Theorem 8.4.5: Let

$$(\eta_1, \eta_2) \in H^0\left(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}(kX) \oplus \Omega^{n-1}_{\mathbb{P}^n}(kX + lH)\right)$$

be as in Lemma 8.4.6.
An easy computation yields

$$\begin{aligned}
&\eta_1 \wedge \frac{dQ}{Q} = \\
&= \sum_{m=1}^{n+1} \frac{\partial_{x_m} Q \cdot A_m}{P^k Q} \Delta - \sum_{j=1}^{n+1} (-1)^j \deg Q \frac{A_j}{P^k}\ dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_{n+1}
\end{aligned}$$

and

$$\left[\sum_{j=1}^{n+1} (-1)^j \deg Q \frac{A_j}{P^k}\ dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_{n+1}\right] \in \ker\left(\mathrm{Res}^{LT,n}_{D_{z,u}|M}\right).$$

As $\tilde{d}(\eta_1, \eta_2) = \left(d\eta_1, \eta_1 \wedge \frac{dQ}{Q} - d\eta_2\right)$, we conclude:
1. For $\eta_1 = 0$, i.e., $A_m = 0$ for each m, we get the relation

$$\left[\left(0, \sum_{m=1}^{n+1}\left(k\frac{B_m \partial_{x_m} P}{P^{k+1} Q^l} + l\frac{B_m \partial_{x_m} Q}{P^k Q^{l+1}} - \frac{\partial_{x_m} B_m}{P^k Q^l}\Delta\right)\right)\right] \in H^n\left(\mathcal{K}^\bullet, \tilde{d}\right),$$

thus

$$\begin{aligned}
&\left[\left(0, \sum_{m=1}^{n+1}\left(k\frac{B_m \partial_{x_m} P}{P^{k+1} Q^l} + l\frac{B_m \partial_{x_m} Q}{P^k Q^{l+1}}\right)\Delta\right)\right] = \left[\left(0, \sum_{m=1}^{n+1} \frac{\partial_{x_m} B_m}{P^k Q^l}\Delta\right)\right] \\
&\in H^n\left(\mathcal{K}^\bullet, \tilde{d}\right). \qquad (8.4.6.1)
\end{aligned}$$

2. For $\eta_2 = 0$, we obtain

$$\begin{aligned}
\tilde{d}(\eta_1, 0) &= \left(d\eta_1, \eta_1 \wedge \frac{dQ}{Q}\right) = \\
&= \left(\sum_{m=1}^{n+1}\left(k\frac{A_m \partial_{x_m} P}{P^{k+1}} - \frac{\partial_{x_m} A_m}{P^k}\right)\Delta, \sum_{m=1}^{n+1} \frac{\partial_{x_m} Q \cdot A_m}{P^k Q}\Delta\right. \\
&\qquad \left. - \sum_{j=1}^{n+1} (-1)^j \deg Q \frac{A_j}{P^k}\ dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_{n+1}\right).
\end{aligned}$$

Thus

$$\left[\left(\sum_{m=1}^{n+1}\left(k\frac{A_m \partial_{x_m} P}{P^{k+1}} - \frac{\partial_{x_m} A_m}{P^k}\right)\Delta, \sum_{m=1}^{n+1} \frac{\partial_{x_m} Q \cdot A_m}{P^k Q}\Delta\right)\right] = 0,$$

and therefore

$$\left[\left(\sum_{m=1}^{n+1} k\frac{A_m\partial_{x_m}P}{P^{k+1}}\Delta, \sum_{m=1}^{n+1}\frac{\partial_{x_m}Q\cdot A_m}{P^kQ}\Delta\right)\right] = \left[\left(\sum_{m=1}^{n+1} -\frac{\partial_{x_m}A_m}{P^k}\Delta, 0\right)\right]$$
$$\in H^n\left(\mathcal{K}^\bullet, \tilde{d}\right). \tag{8.4.6.2}$$

□

This establishes the proof of Theorem 8.4.5.

The general procedure

We finally describe the general procedure how to compute the Picard-Fuchs equation.

We denote the ordered, $(2n-1)$-dimensional basis of the relative cohomology $H^{n-1}(X, D, \mathbb{C})$ given by Lemma 8.3.1 by $\underline{\pi}(z,u)$.

In order to calculate the Picard-Fuchs operators, we determine $\mathbb{C}$-valued $(2n-1)\times(2n-1)$-matrices $M_z(z,u)$ and $M_u(z,u)$ such that

$$\nabla_z\underline{\pi}(z,u) = M_z(z,u)\,\underline{\pi}(z,u)$$

and

$$\nabla_u\underline{\pi}(z,u) = M_u(z,u)\,\underline{\pi}(z,u).$$

In order to do this, we have to use the cohomology relations given above. Each element of $\nabla_z\underline{\pi}(z,u)$ and $\nabla_u\underline{\pi}(z,u)$ has to be written as a linear combination of the basis $\underline{\pi}$.

The matrices $M_z(z,u)$ and $M_u(z,u)$ yield differential operators, the Picard-Fuchs operators.

8.5 An example

As an application of the theory presented so far we consider now the case of quintic 3-folds X with a divisor D. However, as already mentioned, $H^{2,1}(X)$ is too large for the computation of a Picard-Fuchs eqution. Therefore it is common to consider only those quintics having a sufficiently large symmetry group G. Then we will argue on X/G instead of X. Also the divisor we consider needs to be G-invariant, so that we can consider D/G in X/G.

The case without a divisor has been carried out by Greene–Plesser in [GP90] and Batyrev. It has however to be noticed that the divisor D has to be singular, so the theory developed so far has to be adapted to take care of this difficulty.

8.5.1 A family of Calabi-Yau 3-folds

We briefly recall the mirror construction of Greene and Plesser [GP90], see also [GHJ03] and [CK99]. For each $\psi \in \mathbb{C}$ let $X_\psi := \{P_\psi = 0\} \subset \mathbb{P}^4$ be a 1-dimensional family of quintic Calabi-Yau 3-folds given by a family of homogeneous polynomials $P_\psi \in H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5)\right)$ on $\mathbb{P}^4$, namely

$$P_\psi\left([x_1 : \ldots : x_5]\right) := x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi\, x_1x_2x_3x_4x_5.$$

It is easy to compute that X_ψ is non-singular for $\psi^5 \neq 1$ and for $\psi^5 = 1$ it is singular in 125 distinct points. So from now on we will assume $\psi \neq 1$. By Batyrev's theorem, $h^{1,1}(X_\psi) = 1$ and $h^{1,2}(X_\psi) = 101$.

The group $\tilde{G} := (\mathbb{Z}/5\mathbb{Z})^5 / (\mathbb{Z}/5\mathbb{Z})$ acts on $\mathbb{P}^4$ in the following way: To begin with, there is an action of $(\mathbb{Z}/5\mathbb{Z})^5$ on $\mathbb{P}^4$ by

$$(\mathbb{Z}/5\mathbb{Z})^5 \times \mathbb{P}^4 \to \mathbb{P}^4, (a_1, \ldots, a_5) \times [x_1 : \ldots : x_5] \mapsto [\xi^{a_1} : \ldots : \xi^{a_5}],$$

where $\xi := e^{\frac{2\pi i}{5}}$. Since the subgroup $\mathbb{Z}/5\mathbb{Z} := \{(a, \ldots, a) |\, a \in \mathbb{Z}\}$ acts as the identity on $\mathbb{P}^4$, we obtain the $\tilde{G}$-action on X_ψ. Furthermore, the subgroup

$$G := \{(a_1, \ldots, a_5) |\, a_1 + \ldots + a_5 = 0\} \cong \mathbb{Z}_5^3 < \tilde{G}$$

of $\tilde{G}$ acts on $\mathbb{P}^4$; generators for the G-action on $\mathbb{P}^4$ can be given in the following way.

$$\begin{aligned} G &= \langle g_1 := (1,0,0,0,4)\,, g_2 := (0,1,0,0,4)\,, g_3 := (0,0,1,0,4)\,, \\ &\quad\; g_4 := (0,0,0,1,4)\rangle\,, \end{aligned}$$

where e.g. g_1 acts on $\mathbb{P}^4$ by

$$g_1 : [x_1 : x_2 : x_3 : x_4 : x_5] \mapsto \left[\rho x_1 : x_2 : x_3 : x_4 : \rho^4 x_5\right].$$

Here $\rho := e^{\frac{2\pi i}{5}}$.

We notice that X_ψ is G-invariant and we set

$$Y_\psi := X_\psi / G.$$

It is known that the singular locus of Y_ψ consists of 10 curves isomorphic to $\mathbb{P}^1$; three of them meeting in one point. Furthermore, Y_ψ is Gorenstein with canonical singularities, and has crepant resolutions $\pi : \check{X}_\psi \to Y_\psi$.

8.5.1 Theorem. *[Bat94] We assume $\psi^5 \neq 1$. The variety $\check{X}_\psi$ is a Calabi-Yau 3-fold with Hodge numbers $h^{1,1}\left(\check{X}_\psi\right) = 101$ and $h^{1,2}\left(\check{X}_\psi\right) = 1$.*

8.5.2 Corollary. *We obtain the following equalities:*

1. $h^{3,0}(Y_\psi) = h^{3,0}(X_\psi)^G = h^{0,3}(X_\psi)^G = h^{0,3}(Y_\psi) = 1,$
2. $h^{1,1}(Y_\psi) = h^{1,1}(X_\psi)^G = 1,$
3. $h^{2,1}(Y_\psi) = h^{2,1}(X_\psi)^G = h^{1,2}(X_\psi)^G = h^{1,2}(Y_\psi) = 1.$

Proof of Corollary 8.5.2: To begin with, we notice that by Proposition 4.2.40, we have $H^{p,q}(Y_\psi) = H^{p,q}(X_\psi)^G$. The first assertion is obvious.

Since $\dim H^{1,1}(X_\psi) = 1$, we have $\dim H^{1,1}(X_\psi)^G \leqslant 1$. On the other hand, obviously $H^{1,1}(X_\psi)^G \neq 0$.

Concerning $H^{2,1}(Y_\psi)$, we use $h^{2,1}\left(\check{X}_\psi\right) = 1$. Since $\pi_*\left(\Omega^2_{\check{X}_\psi}\right) = \tilde{\Omega}^2_{Y_\psi}$ (see Remark 4.2.42), the Leray spectral sequence gives

$$h^{2,1}(Y_\psi) = h^1\left(Y_\psi, \tilde{\Omega}^2_{Y_\psi}\right) \leqslant h^1\left(\check{X}_\psi, \Omega^2_{\check{X}_\psi}\right) = 1.$$

The non-vanishing of $H^{2,1}(Y_\psi) = H^{2,1}(X_\psi)^G$ will be clear, as it parametrizes G-invariant deformations of X_ψ and all X_ψ are G-invariant. Therefore it will be non-zero. We will furthermore construct elements in $H^{2,1}(Y_\psi)$ explicitly. Finally, by the same arguments or by duality, we obtain $H^{1,2}(Y_\psi) = 1$. □

8.5.2 A family of smooth divisors inside the family of Calabi-Yau 3-folds

Let $H_\phi := \{Q_\phi = 0\} \subset \mathbb{P}^4$ be a family of hypersurfaces in $\mathbb{P}^4$ given by

$$Q_\phi([x_1 : \ldots : x_5]) := x_5^4 - \phi\, x_1x_2x_3x_4.$$

Then for $\phi \neq 0$, the hypersurface H_ϕ is singular in the 6 lines

$$\{x_i = x_j = x_5 = 0, 1 \leqslant i < j \leqslant 4\}.$$

Let

$$D_{\psi,\phi} := X_\psi \cap H_\phi$$

for $\psi, \phi \in \mathbb{C}, \psi^5 \neq 1, \phi \neq 0$.

8.5.3 Proposition. *We suppose that* $\phi \neq \psi$ *and* $\psi(\psi-\phi)^4 + 4^4(5\phi-\psi) + 4^3 \cdot 20(\psi-\phi) = 0$. *Then*

$$\begin{aligned}\mathrm{Sing}(D_{\psi,\phi}) &= (\mathrm{Sing} H_\phi) \cap X_\psi = \\ &= \bigcup_{1\leqslant i<j\leqslant 4} \left\{[x_1 : \ldots : x_5] \in \mathbb{P}^4 \middle| x_1^5 + \ldots + x_5^5 = 0, x_i = x_j = x_5 = 0\right\}.\end{aligned}$$

Proof of Proposition 8.5.3: We write $D = D_{\psi,\phi}$ and $H = H_\phi$. It is immediately checked that

$$\begin{aligned}(\mathrm{Sing} H_\phi) \cap X_\psi &= \\ &= \bigcup_{1\leqslant i<j\leqslant 4} \left\{[x_1 : \ldots : x_5] \in \mathbb{P}^4 \middle| x_1^5 + \ldots + x_5^5 = 0, x_i = x_j = x_5 = 0\right\}.\end{aligned}$$

Therefore it suffices to show that

$$\mathrm{Sing}(D) = \mathrm{Sing}(H) \cap D,$$

i.e., that there are no singularities of D which are not singularities of H. Arguing by contradiction, we assume that there is a point $a \in \mathrm{Sing}(D)$, such that $a \notin \mathrm{Sing}(H)$.

Then the Jacobian matrix $J_{(P,Q)}$ of P and Q does not have maximal rank in a and there is a number $\lambda \in \mathbb{C}, \lambda \neq 0$, such that

$$\lambda \cdot J_P(a) = J_Q(a),$$

where J_P and J_Q denote the Jacobian matrices of P and Q.

Now, choosing the chart given by $x_1 = 1$ and assuming $x_5 = 0$, it follows easily that $a = [1 : 0 : \ldots : 0]$ in contradiction to our assumption $J_Q(a) \neq 0$. Thus $x_5 \neq 0$. The same argument holds if $x_2 = 1, x_3 = 1$ or $x_4 = 1$. Therefore we assume that $x_5 = 1$. It follows easily that $x_1^5 = \ldots = x_4^5$ and all additional singularities of D have the form

$$[x_1 : \rho^{a_1} x_1 : \rho^{a_2} x_1 : \rho^{a_3} x_1 : 1]$$

for $a_1, a_2, a_3 \in \mathbb{N}$ and $\rho = e^{\frac{2\pi i}{5}}$. Via the action of the group G on D it is possible to map each of these points to a point $[1 : 1 : 1 : 1 : x_5]$ with $x_5 \neq 0$. As the action of an element $g \in G$ on D is an automorphism of D, the points $[x_1 : \rho^{a_1} x_1 : \rho^{a_2} x_1 : \rho^{a_3} x_1 : 1]$ are singular if and only if $[1 : 1 : 1 : 1 : x_5]$ is. Now a direct computation shows that any point $[1 : 1 : 1 : 1 : x_5]$ is singular if $\psi \neq \phi$ and

$$\psi(\psi - \phi)^4 + 4^4(5\phi - \psi) + 4^3 \cdot 20(\psi - \phi) = 0. \tag{8.5.3.1}$$

□

As H_ϕ being invariant under the action of the group G, the divisor $D_{\psi,\phi}$ is invariant under G. So we can form the quotient by the group G.

8.5.4 Definition. *We set*

$$D'_{\psi,\phi} := D_{\psi,\phi}/G.$$

8.5.5 Corollary. *For general parameters ψ and ϕ, the singular locus* $\mathrm{Sing}\left(D'_{\psi,\phi}\right) = p(\mathrm{Sing}(D_{\psi,\phi}))$ *consists of* 6 *points, where $p : X_\psi \to Y_\psi$ denotes the projection.*

In the following we use the abbreviation $D = D_{\psi,\phi}$ and $D' = D'_{\psi,\phi}$.

We first determine the Hodge numbers of the G-invariant cohomology

$$H^2(D, \mathbb{C})^G = H^2(D', \mathbb{C}).$$

At the moment we have not yet shown that D has quotient singularities. Therefore we define ad hoc $H^{2,0}(D) := H^0(D, K_D)$ and $H^{0,2}(D) := H^2(D, \mathcal{O}_D)$, where $K_D := K_X \otimes \mathcal{O}(D)|_D \in \mathrm{Pic}(D)$. Of course, provided D has quotient singularities, then this coincides with the previous Definition 4.2.41 (see [Ste77]).

8.5.6 Lemma. *The Hodge groups of the G-invariant cohomology of $D = D_{\psi,\phi}$ with ψ and ϕ as in Proposition 8.5.3 satisfy the following property:*

$$H^{2,0}(D)^G \cong H^{0,2}(D)^G \cong \mathbb{C}.$$

Proof of Lemma 8.5.6: The ideal sheaf sequence

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-1) \cong \mathcal{J}_{X|\mathbb{P}^4} \otimes \mathcal{O}_{\mathbb{P}^4}(H) \to \mathcal{O}_{\mathbb{P}^4}(H) \to \mathcal{O}_X(D) \to 0$$

yields an isomorphism

$$H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(H)\right) \cong H^0(X, \mathcal{O}_X(D))$$

and therefore

$$H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(H)\right)^G \cong H^0(X, \mathcal{O}_X(D))^G.$$

We observe that $\dim H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(4)\right)^G = 2$. In fact, a basis is given by the homogeneous polynomials x_5^4 and $x_1x_2x_3x_4$.
Furthermore

$$H^0(X, \mathcal{O}_X(D))^G \cong H^0\left(Y, \mathcal{O}_Y\left(D'\right)\right) \cong H^3\left(Y, \mathcal{O}_Y\left(-D'\right)\right).$$

Since Y is a singular Calabi-Yau 3-fold, $H^2(Y, \mathcal{O}_Y) = 0$ and $H^3(Y, \mathcal{O}_Y) \cong \mathbb{C}$. Then the exact sequence

$$\begin{aligned} 0 &= H^2(Y, \mathcal{O}_Y) \to H^2\left(D', \mathcal{O}_{D'}\right) \to H^3\left(Y, \mathcal{O}_Y\left(-D'\right)\right) \to \\ &\to H^3(Y, \mathcal{O}_Y) \cong \mathbb{C} \to H^3\left(D', \mathcal{O}_{D'}\right) = 0 \end{aligned}$$

yields $\dim H^2(D', \mathcal{O}_{D'}) = 1$. Thus we know that

$$H^{0,2}(D)^G \cong H^{0,2}\left(D'\right) \cong H^2\left(D', \mathcal{O}_{D'}\right)$$

is 1-dimensional. D' being Cohen-Macaulay as a normal surface, we obtain by Serre-duality

$$H^{2,0}\left(D'\right) \cong H^{0,2}\left(D'\right),$$

hence $H^{2,0}(D)^G \cong H^{0,2}(D)^G \cong \mathbb{C}$. □

8.5.7 Lemma. *The surface $D' = D'_{\psi,\phi}$ with ψ and ϕ sufficiently general has rational singularities.*

Proof of Lemma 8.5.7: Let $\tau : \widehat{D}' \to D'$ be a minimal desingularisation. We need to show that

$$R^1\tau_*\left(\mathcal{O}_{\widehat{D}'}\right) = 0.$$

In order to prove this, we compare the cohomology of $\widehat{D}'$ and D'. We know that $H^1(D', \mathcal{O}_{D'}) = H^1(D, \mathcal{O}_D)^G = 0$ and $H^2(D', \mathcal{O}_{D'}) \cong \mathbb{C}$. Arguing by contradiction we assume that

$$R^1\tau_*\left(\mathcal{O}_{\widehat{D}'}\right) \neq 0.$$

We recall that D' has exactly 6 singularities which by symmetry are all of the same type. Hence all singularities are irrational and therefore

$$h^0\left(D', R^1\tau_*\mathcal{O}_{\widehat{D}'}\right) \geqslant 6.$$

By the Leray spectral sequence we obtain

$$h^1\left(\widehat{D}', \mathcal{O}_{\widehat{D}'}\right) \geqslant 5 \text{ and } h^2\left(\widehat{D}', \mathcal{O}_{\widehat{D}'}\right) \leqslant 1.$$

Thus

$$\chi\left(\mathcal{O}_{\widehat{D}'}\right) \leqslant -3$$

and therefore by surface classification $\widehat{D}'$ is birationally equivalent to a ruled surface. So D' is covered by rational curves. Since $D' = D'_\phi$ varies with ϕ, the variety Y is covered by rational curves and so is $\check{X}$. This is a well-known contradiction: Calabi-Yau 3-folds are not covered by rational curves. □

8.5.8 Proposition. *The surface $D = D_{\psi,\phi}$ has rational Gorenstein singularities for any ψ, ϕ as in Proposition 8.5.3.*

Proof of Proposition 8.5.8: By symmetry, it suffices to consider one singular point of D, e.g., $x = [1 : -1 : 0 : 0 : 0]$. We choose the standard chart $x_1 = 1$ and compute in $\mathbb{C}^4$. Then we apply the implicit function theorem to P_ψ and resolve P_ψ locally as $x_2 = g(x_3, x_4, x_5)$ with $g(0,0,0) = -1$. Then locally $D \subset \mathbb{C}^3$ is given by $\{f = 0\}$, where

$$f(x_3, x_4, x_5) = x_5^4 - \phi\, g(x_3, x_4, x_5)\, x_3 x_4.$$

Since the Hesse matrix of f has rank 2 in $(0,0,0)$, the point x is a rational double point of type A_n, [GLS07], Theorem I. 2.4.8. □

8.5.9 Corollary. *The surface $D' = D'_{\psi,\phi}$ has quotient singularities for any ψ, ϕ as in Proposition 8.5.3.*

Proof of Corollary 8.5.9: Let $x_0 \in D'$ be a singular point, and take $x \in p^{-1}(x_0)$ (recall that $p : D \to D'$ denotes the cover induced by $D' = D/G$). Since D has rational Gorenstein singularities, D has quotient singularities. Hence there is an open set $W = W(0) \subset \mathbb{C}^2$ such that $U \cong W/H$ for some finite subgroup $H \subset \mathrm{Gl}(2, \mathbb{C})$. By the open mapping theorem, [GR84], p.109, p is an open map, hence $p(U)$ is an open neighbourhood of x_0 in D'. The open mapping theorem can be applied since D' is normal, hence locally irreducible ([GR84], p.125). Now by a theorem of Brieskorn, [Bri68], Satz 2.8, [Ish14], Theorem 7.4.18, x_0 is a quotient singularity. □

8.5.10 Remark. Since quotient singularities are rational, we conclude again that D' has rational singularities, actually for all $D' = D'_{\psi,\phi}$ as in Proposition 8.5.3 such that D is normal.

8.5.11 Proposition. *The hypersurface H is a toric variety. In particular, the Frölicher spectral sequence with E_1-term*

$$E_1^{p,q} = H^q\left(H, \tilde{\Omega}_H^p\right)$$

degenerates at E_1 and converges to $H^*(H, \mathbb{C})$. Moreover, Poincaré-duality holds on H. Moreover H has only Gorenstein singularities.

Proof of Proposition 8.5.11: Since H is given by the equation

$$x_5^4 - \phi\, x_1 x_2 x_3 x_4 = 0,$$

is follows easily that H is a toric variety. The assertion concerning the Frölicher spectral sequence is a theorem of Danilov ([Dan78], Theorem 12.5). □

8.5.12 Lemma. $H^3(D', \mathbb{C}) = H^3(D, \mathbb{C})^G = 0$.

Proof of Lemma 8.5.12: Since D' has only quotient singularities, Poincaré-duality holds for D' (see [PS08], p.58), hence

$$H^3\left(D', \mathbb{C}\right) \cong H^1\left(D', \mathbb{C}\right).$$

So it suffices to prove that $H^1(D', \mathbb{C}) = 0$. Let $\tau : \widehat{D}' \to D'$ be a minimal desingularisation. Since $H^1(D', \mathcal{O}_{D'}) = 0$ and since D' has rational singularities, we have

$$H^1\left(\widehat{D}', \mathcal{O}_{\widehat{D}'}\right) = 0.$$

Hence $H^1\left(\widehat{D}', \mathbb{C}\right) = 0$ by Hodge decomposition. Using the Leray spectral sequence we conclude $H^1(D', \mathbb{C}) = 0$. □

8.5.13 Lemma. *For the Hodge group of type $(1,1)$ we have:*

$$H_{var}^{1,1}(D)^G = H^{1,1}(D)^G / H^{1,1}(X) \cong \mathbb{C}.$$

Proof of Lemma 8.5.13: We will make use of Section 7.3. However, the divisor H and the surface D are now singular. The results of Section 7.3 nevertheless remain true for the following reasons.

- The relevant residue maps exist. To verify this, we apply Proposition 4.2.16 to the divisors X and H in the projective manifold $\mathbb{P}^4$.

  The first two assumptions in Proposition 4.2.16 are satisfied since D has rational singularities, X and H meet transversally outside the singular locus $\operatorname{Sing}(D) = \operatorname{Sing}(H) \cap X$.

  To check the third assumption we use Remark 4.2.17 and obtain a finite set $M \subset \operatorname{Sing}(H)$ such that at every $p \in \operatorname{Sing}(D) \backslash M$, the variety H has locally the requested product form. Furthermore, in each component $\{x_i = x_j = x_5 = 0, 1 \leqslant i < j \leqslant 4\}$, e.g. $i = 3, j = 4$, of $\operatorname{Sing}(H)$ all points with $x_1, x_2 \neq 0$ can be mapped to the point $[1:1:0:0:0]$ via an automorphism of H. Therefore the structure of the singularities is the same in all these points. Hence, as this set is not finite, no point of M belongs to it. As every point contained in $\operatorname{Sing}(H) \cap X$ has two coordinates which are not zero, we obtain $(\operatorname{Sing}(H) \cap X) \cap M \neq \emptyset$.

- $\mathbb{H}^k\left(\mathbb{P}^4, \tilde{\Omega}_{\mathbb{P}^4}(\log(X \cup H))\right) = H^k\left(\mathbb{P}^4 \backslash (X \cup H), \mathbb{C}\right)$, see Theorem 4.2.43.

- Poincaré-duality holds on D (since D has quotient singularities).

- The Lefschetz hyperplane theorem holds for $H \subset \mathbb{P}^4$.

- Theorem 7.3.9 and its corollary carry over for surfaces with quotient singularities.

- The formulas of Li, Lian and Yau remain true in our setting with essentially the same proof.

We recall the map

$$\Psi_3^{LT} : H^0\left(\mathbb{P}^4, \sum_{\substack{(p_1,p_2)\in\mathbb{N}\times\mathbb{N},\\ p_1+p_2=3}} \mathcal{O}_{\mathbb{P}^4}(5p_1 + 4p_2 - 5)\right) \to F^1H^2(D, \mathbb{C})$$

defined in Corollary 7.3.10. Let

$$\kappa : F^1H^2(D,\mathbb{C}) \to F^1H^2(D,\mathbb{C})/F^2H^2(D,\mathbb{C}) \cong H^{1,1}(D)$$

be the projection and set $\overline{\Psi_3^{LT}} := \kappa \circ \Psi_3^{LT}$. Then, using Corollary 7.3.8,

$$
\begin{aligned}
H^{1,1}_{var}(D)^G &= \operatorname{im}\left(\overline{\Psi_3^{LT}}\right)^G = \\
&= \left\{ \operatorname{Res}^{LT,4}_{D|\mathbb{P}^4}\left(\left[\sum_{i=1}^{2}\frac{R_i}{P^iQ^{3-i}}\Delta\right]\right) \middle| R_i \in H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}\left(5i + 4\left(3-i\right) - 5\right)\right)\right\}^G \\
&= \left\{ \operatorname{Res}^{LT,4}_{D|\mathbb{P}^4}\left(\left[\sum_{i=1}^{2}\frac{R_i}{P^iQ^{3-i}}\Delta\right]\right) \middle| R_1 \in H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(8)\right)^{G,1}, \right. \\
&\qquad\qquad\qquad\qquad\qquad \left. R_2 \in H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(9)\right)^{G,2} \right\},
\end{aligned}
$$

where $H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d)\right)^{G,q}$ denotes the space of homogeneous polynomials R of degree d on $\mathbb{P}^4$, such that $R \circ g = \rho^q \cdot R$ for each $g \in G$. We observe that

$$R_1 \in H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(8)\right)^{G,1} = \left\langle x_5^8,\ x_1^2x_2^2x_3^2x_4^2,\ x_1x_2x_3x_4x_5^4\right\rangle$$

and

$$R_2 \in H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(9)\right)^{G,2} = \left\langle x_5^9,\ x_1^2x_2^2x_3^2x_4^2x_5,\ x_1x_2x_3x_4x_5^5\right\rangle.$$

In order to obtain the dimension of $H^{1,1}_{var}(D)^G$, we have to determine the dimension of the G-invariant part of $\ker\left(\Psi_3^{LT}\right) = \operatorname{im}(K_3)$, where K_3 is the matrix defined in Definition 7.3.19. The Singular programme in Appendix A.2 shows that the dimension of the kernel is 5. Therefore $\dim H^{1,1}_{var}(D)^G = 1$. □

8.5.3 Picard-Fuchs operators for (X, D)

For the calculation of the Picard-Fuchs operators, we use the G-invariant relative cohomology $H^3(X, D, \mathbb{C})^G$.

8.5.14 Corollary. *For general ϕ and ψ, the vector spaces*

$$H^3\left(X_\psi, D_{\psi,\phi}, \mathbb{C}\right)^G = H^3\left(Y_\psi, D'_{\psi,\phi}, \mathbb{C}\right)$$

form a local system.

Proof of Corollary 8.5.14: We note first that the groups $H^2(D_{\psi,\phi}, \mathbb{C})$ form a local sytem, since $D_{\psi,\phi}$ have only rational double points of the same type (one might argue via a simultaneous resolution). Therefore, having in mind that $H^2(X, \mathbb{C}) \cong \mathbb{C}$, the groups $H^2_{var}(D_{\psi,\phi})$ form also a local system. Since

$$H^3\left(D_{\psi,\phi}, \mathbb{C}\right) \cong H^1\left(D_{\psi,\phi}, \mathbb{C}\right) = 0,$$

the groups $H^3(X_\psi, D_{\psi,\phi}, \mathbb{C})$ form a local sytem.

Now everything remains true for the G-invariant parts. In fact, the action of G on $H^3(X_\psi, \mathbb{C})$ is independent of ψ (one could also argue on Y_ψ or one the mirror $\check{X}_\psi$). The same applies for the cohomology of $D_{\psi,\phi}$. □

This corollary shows in particular that the Gauß-Manin connection works for this singular D and is G-invariant. Therefore the theory of Li, Lian and Yau is applicable.

Putting things together using again the abbreviations $D = D_{\psi,\phi}$ and $X = X_\psi$ we have the following sequence:

8.5.15 Proposition. *We have an exact sequence*

$$0 \to F^k H^2_{var}(D)^G \to F^k H^3(X, D, \mathbb{C})^G \to F^k \ker H^3(X, \mathbb{C})^G \to 0, \tag{8.5.15.1}$$

equivalently

$$0 \to F^k H^2_{var}\left(D'\right) \to F^k H^3\left(Y, D', \mathbb{C}\right) \to F^k \ker H^3(Y, \mathbb{C}) \to 0. \tag{8.5.15.2}$$

Proof of Proposition 8.5.15: It suffices to state that $H^3(D', \mathbb{C}) = 0$, which was proved in Lemma 8.5.12. □

Proposition 8.5.15 yields the following Hodge filtration:

$$\begin{array}{ll} F^3H^3(X,\mathbb{C})^G: & H^{3,0}(X)^G \\ F^2H^3(X,\mathbb{C})^G: & F^3 \oplus H^{2,1}(X)^G \oplus H^{2,0}_{var}(D)^G \\ F^1H^3(X,\mathbb{C})^G: & F^2 \oplus H^{1,2}(X)^G \oplus H^{1,1}_{var}(D)^G \\ F^0H^3(X,\mathbb{C})^G: & F^1 \oplus H^{0,3}(X)^G \oplus H^{0,2}_{var}(D)^G \end{array}$$

We have seen that all Hodge groups $H^{3-p,p}(X)^G$ for $p = 0, \ldots, 3$ and $H^{2-p,p}_{var}(D)^G$ for $p = 0, \ldots, 2$ are 1-dimensional. Let

$$\omega_\psi := \frac{1}{P_\psi}\Delta \in H^0\left(\mathbb{P}^n, K_{\mathbb{P}^n}(X_\psi)\right).$$

According to Lemma 8.3.1 a basis of $H^3(X, D, \mathbb{C})^G$ consists apart from

$\Omega := \mathrm{Res}^4_{(X,D)}\left[(\omega,0)\right] \in H^{3,0}\left(X,D,\mathbb{C}\right)^G$ of the elements

$$\begin{aligned}
\partial_z(\Omega) &= \mathrm{Res}^4_{(X,D)}\left[\left(\frac{5\ x_1x_2x_3x_4x_5}{P^2}\ \Delta,\ 0\right)\right]\\
\partial_z^2(\Omega) &= \mathrm{Res}^4_{(X,D)}\left[\left(\frac{2\cdot 5^2\left(x_1x_2x_3x_4x_5\right)^2}{P^3}\ \Delta,\ 0\right)\right]\\
\partial_z^3(\Omega) &= \mathrm{Res}^4_{(X,D)}\left[\left(\frac{6\cdot 5^3\left(x_1x_2x_3x_4x_5\right)^3}{P^4}\ \Delta,\ 0\right)\right]\\
\partial_u(\Omega) &= \mathrm{Res}^4_{(X,D)}\left[\left(0,\ \frac{-5\ x_1x_2x_3x_4}{PQ}\ \Delta\right)\right]\\
\partial_z\partial_u(\Omega) &= \mathrm{Res}^4_{(X,D)}\left[\left(0,\ \frac{-5\ x_1x_2x_3x_4\cdot x_1x_2x_3x_4x_5}{P^2Q}\ \Delta\right)\right]\\
\partial_z^2\partial_u(\Omega) &= \mathrm{Res}^4_{(X,D)}\left[\left(0,\ \frac{-2\cdot 5^2\ x_1x_2x_3x_4\cdot\left(x_1x_2x_3x_4x_5\right)^2}{P^3Q}\ \Delta\right)\right].
\end{aligned}$$

We apply the Gauß-Manin connection with respect to ψ and ϕ to each of the seven elements of the basis of $H^3\left(X,D,\mathbb{C}\right)^G$. For each of these elements we proceed as follows:

In order to use the cohomology relations 8.4.6.1 and 8.4.6.2, we define matrices:

8.5.16 Definition.

1. *Let $\varphi_p^{(X,D)}$ for $p \geqslant 2$ be the map defined by*

$$\begin{aligned}
\varphi_p^{(X,D)}:\quad & \sum_{k=1}^{p-1} H^0\left(\mathbb{P}^n, K_{\mathbb{P}^n}\left(kY_1+(p-k)\,Y_2\right)\right) \to S^{\oplus p},\\
& \left(\frac{R}{P^p}\Delta, \sum_{k=1}^{p-1}\frac{P_k}{P^kQ^{p-k}}\Delta\right) \mapsto \left(R, P_{p-1},\dots,P_1\right).
\end{aligned}$$

2. *For each $k \geqslant 2$ we define a matrix $K_k : S^{\oplus((n+1)(k-1)+2k+2)} \to S^{\oplus k}$ by*

$$K_k := \begin{pmatrix} B_k & I_k\cdot P & I_k\cdot Q & V_{1,k} & V_{2,k}\end{pmatrix},$$

where I_k is the $k\times k$-identity matrix and B_k is the following $k\times((n+1)(k-1))$-matrix

$$B_k := \begin{pmatrix}
-J_1 & 0 & 0 & 0 & 0 & 0\\
J_2 & (k-2)\,J_1 & 0 & \dots & 0 & 0\\
0 & J_2 & (k-3)\,J_1 & \dots & 0 & 0\\
0 & 0 & 0 & \dots & \vdots & \vdots\\
0 & \vdots & \vdots & \ddots & 2J_1 & 0\\
0 & 0 & 0 & \dots & (k-3)\,J_2 & J_1\\
0 & 0 & 0 & \dots & 0 & (k-2)\,J_2
\end{pmatrix}$$

where J_1 and J_2 are defined as in Definition 7.3.19 and $V_{1,k}$ and $V_{2,k}$ are the $(k \times 1)$-matrices given by $V_{1,k} = \varphi_k^{(X,D)}\left(\partial_z^{k-1}(\Omega)\right)$ and $V_{2,k} = \varphi_k^{(X,D)}\left(\partial_z^{k-2}\partial_u(\Omega)\right)$ for $k \geqslant 2$.

8.5.17 Proposition. *([JS09a], p. 37) Using the abbreviations*

$$\begin{aligned}
D_1 &:= 1 - \psi^5, \\
D_2 &:= \phi\left(\phi - 5\psi\right)^4 - 256, \\
T_1 &:= \phi\left(8000 - \phi\left(\phi - 5\psi\right)\psi\left(61\phi^2 - 790\psi\phi + 2825\psi^2\right)\right) - 16384\psi, \\
T_2 &:= 57375\phi^2\psi^5 - 34000\phi^3\psi^4 + 7190\phi^4\psi^3 - 8\left(79\phi^5 + 14336\right)\psi^2 + \\
&\quad +\phi\left(19\phi^5 + 95936\right)\psi - 11200\phi^2, \\
T_3 &:= 22625\phi^2\psi^6 - 16325\phi^3\psi^5 + 4490\phi^4\psi^4 - 2\left(293\phi^5 + 49152\right)\psi^3 + \\
&\quad +\phi\left(37\phi^5 + 112768\right)\psi^2 - \phi^2\left(\phi^5 + 26624\right)\psi + 1920\phi^3,
\end{aligned}$$

we get the following matrices

$$M_\psi := \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\frac{\psi}{D_1} & \frac{15\psi^2}{D_1} & \frac{25\psi^3}{D_1} & \frac{10\psi^4}{D_1} & \frac{-\phi T_1}{16 D_1 D_2} & \frac{-\phi T_2}{16 D_1 D_2} & \frac{-\phi T_3}{16 D_1 D_2} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{125\phi(\phi-5\psi)}{D_2} & \frac{-175\phi(\phi-5\psi)^2}{D_2} & \frac{30\phi(\phi-5\psi)^3}{D_2}
\end{pmatrix}$$

$$M_\phi := \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{125\phi(\phi-5\psi)}{D_2} & \frac{-175\phi(\phi-5\psi)^2}{D_2} & \frac{30\phi(\phi-5\psi)^3}{D_2} \\
0 & 0 & 0 & 0 & -\frac{3}{4\phi} & -\frac{\phi-\psi}{4\phi} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2\phi} & -\frac{\phi-\psi}{4\phi} \\
0 & 0 & 0 & 0 & -\frac{125(\phi-\psi)(\phi-5\psi)}{4D_2} & \frac{175(\phi-\psi)(\phi-5\psi)^2}{4D_2} & -\frac{1}{4\phi} - \frac{15(\phi-\psi)(\phi-5\psi)^3}{2D_2}
\end{pmatrix}$$

An easy computation shows:

8.5.18 Corollary. *([JS09a], p. 37) The matrices M_ψ and M_ϕ yield the* *following Picard-Fuchs operators*

$$\begin{aligned}
\mathcal{L}_1 &= (\psi-\phi)\,\theta_\psi\partial_\phi - 4\psi\theta_\phi\partial_\phi - 3\psi\partial_\phi, \\
\mathcal{L}_2 &= \partial_\psi^2\partial_\phi^2 + \left(\frac{1}{4\phi} + \frac{15}{2D_2}(\phi-\psi)(\phi-5\psi)^3\right)\partial_\psi^2\partial_\phi \\
&\quad -\frac{175}{4D_2}(\phi-\psi)(\phi-5\psi)^2\,\partial_\psi\partial_\phi + \frac{125}{4D_2}(\phi-\psi)(\phi-5\psi)\,\partial_\phi, \\
\mathcal{L}_3 &= \mathcal{L}_3^{\text{bulk}} + \mathcal{L}_3^{\text{bdry}} = \theta_\psi(\theta_\psi-1)(\theta_\psi-2)(\theta_\psi-3) - \psi^5(\theta_\psi+1)^4 \\
&\quad +\psi^4 D_1\partial_\psi^4 + \frac{\psi^4\phi}{16D_2}\left(T_3\partial_\psi^2\partial_\phi + T_2\partial_\psi\partial_\phi + T_1\partial_\phi\right)
\end{aligned}$$

with

$$\mathcal{L}_3^{\text{bulk}} = \theta_\psi(\theta_\psi-1)(\theta_\psi-2)(\theta_\psi-3) - \psi^5(\theta_\psi+1)^4$$

and

$$\mathcal{L}_3^{\text{bdry}} = \psi^4 D_1\partial_\psi^4 + \frac{\psi^4\phi}{16D_2}\left(T_3\partial_\psi^2\partial_\phi + T_2\partial_\psi\partial_\phi + T_1\partial_\phi\right).$$

Note that the operator $\mathcal{L}_3^{\text{bulk}}$ is exactly the Picard-Fuchs operator which is satisfied by all periods of the quintic 3-fold X_ψ (see e.g. [GHJ03], Chapter 18).

Appendix A.3 contains a programme written in the Singular language ([DGPS16]) for computing the Picard-Fuchs operators of a pair consisting of a Calabi-Yau hypersurface in a projective space and a divisor that is given by intersecting the Calabi-Yau manifold with another hypersurface. Applying the programme to the example discussed above yields the matrices M_ψ and M_ϕ stated in Proposition 8.5.17.

Chapter 9

Picard-Fuchs operators for triples

We consider a Calabi-Yau 3-fold X with a smooth curve $C \subset X$. We are searching for a Picard-Fuchs equation for (X, C). In order to pursue this we need a local system, and the choice might be to consider $H^3(X, C, \mathbb{C})$.

However, we will show in the first section that $H^3(X, C, \mathbb{C}) \cong H^3(X, \mathbb{C})$ and this isomorphism respects even the natural Hodge structure on (X, C) on both sides.

One way out might be to consider the blow-up $\pi : \widehat{X} \to X$ of $C \subset X$ and then study $H^3\left(\widehat{X}, \mathbb{C}\right)$, which encodes also the genus of the curve C.

The disadvantage is that $\widehat{X}$ is no longer Calabi-Yau, although it carries a holomorphic 3-form. The deformation theories of (X, C) and $\left(\widehat{X}, E\right)$ are the same.

Owing to these difficulties we will restrict ourselves to complete intersections $C = D_1 \cap D_2$ and develop the theory for triples (X, D_1, D_2). In particular, we define a cohomology group $H^3(X, D_1, D_2, \mathbb{C})$ whose variation might lead to a Picard-Fuchs equation. We also compute an easy example. For a family of triples we set up a theory by Li, Lian and Yau. However, if the parameters of D_1 and D_2 are independent, we cannot reach the full cohomology. In a computational example we will make a first attempt to relate the cohomology of D_1 and D_2.

9.1 A topological observation

We fix a Calabi-Yau 3-fold X and a smooth curve $C \subset X$. Let $\pi : \widehat{X} \to X$ be the blow-up of $C \subset X$ and let $E := \pi^{-1}(C)$ be the exceptional divisor. Then $H^3(X \backslash C, \mathbb{C}) \cong H^3\left(\widehat{X} \backslash E, \mathbb{C}\right)$ inherits a canonical mixed Hodge structure, as we will see in the proof of Theorem 9.1.1.

9.1.1 Theorem. *Let X be a Calabi-Yau 3-fold, C a smooth curve in X and $\iota : C \hookrightarrow X$ the inclusion. Then the following cohomology groups are isomorphic*

$$H^3(X, C, \mathbb{C})^\vee \cong H^3(X \backslash C, \mathbb{C}) \cong H^3(X, \mathbb{C}),$$

and the isomorphisms respect the corresponding (mixed) Hodge structures.

Proof of Theorem 9.1.1: The first isomorphism is Lefschetz duality. Next we show that $H^3(X, C, \mathbb{C}) \cong H^3(X, \mathbb{C})$. According to Section 4.2.5 the relative cohomology of the pair (X, C) is part of a long exact sequence

$$\begin{aligned} &\ldots \to H^2(X, C, \mathbb{C}) \to H^2(X, \mathbb{C}) \overset{H^2(\iota)}{\to} H^2(C, \mathbb{C}) \to H^3(X, C, \mathbb{C}) \to \\ &\to H^3(X, \mathbb{C}) \to H^3(C, \mathbb{C}) = 0 \to \ldots. \end{aligned}$$

We observe that $H^2(C, \mathbb{C}) \cong \mathbb{C}$ and that the map $H^2(\iota) : H^2(X, \mathbb{C}) \to H^2(C, \mathbb{C})$ is not the zero map, as it maps the first Chern class of an ample line bundle on X to the first Chern class of the ample line bundle restricted to C. Thus $H^2(\iota)$ is surjective, and the map $H^3(X, C, \mathbb{C}) \to H^3(X, \mathbb{C})$ is injective. Furthermore $H^3(C, \mathbb{C}) = 0$; therefore the map $H^3(X, C, \mathbb{C}) \to H^3(X, \mathbb{C})$ is also surjective, thus bijective. This establishes the second isomorphism.

Finally we show that $H^3(X \backslash C, \mathbb{C}) \cong H^3(X, \mathbb{C})$ respects the mixed Hodge structure. To set up the mixed Hodge structures, we notice that $H^3\left(\widehat{X} \backslash E, \mathbb{C}\right) \cong H^3(X \backslash C, \mathbb{C})$ and that according to Theorem 4.2.3 there is a mixed Hodge structure on $H^3\left(\widehat{X} \backslash E, \mathbb{C}\right)$. This yields an isomorphism

$$H^3\left(\widehat{X} \backslash E, \mathbb{C}\right) \cong \bigoplus_{p+q=3} H^q\left(\widehat{X}, \Omega^p_{\widehat{X}}(\log E)\right).$$

As $H^3(X, \mathbb{C}) \cong \bigoplus_{p+q=3} H^q\left(X, \Omega^p_X\right)$, we need to show

$$H^q\left(X, \Omega^p_X\right) \cong H^q\left(\widehat{X}, \Omega^p_{\widehat{X}}(\log E)\right) \tag{9.1.1.1}$$

for each p, q with $p + q = 3$, which will prove the final statement. The isomorphy 9.1.1.1 will actually be true for any compact complex manifold and any blow-up of a submanifold C. By the Leray spectral sequence this comes down to showing

$$\pi_*\left(\Omega^p_{\widehat{X}}(\log E)\right) \cong \Omega^p_X, \tag{9.1.1.2}$$

$$R^j\pi_*\left(\Omega^p_{\widehat{X}}(\log E)\right) = 0,\ j > 0. \tag{9.1.1.3}$$

The assertion 9.1.1.2 is clear since $\pi_*\left(\Omega^p_{\widehat{X}}(\log E)\right)$ is torsionfree and contains $\pi_*\left(\Omega^p_{\widehat{X}}\right) \cong \Omega^p_X$.

For our purposes, it is sufficient to show assertion 9.1.1.3 for the case $j = 1$. Moreover, we only deal with $p = 1$ and leave the rest to the reader. We apply π_* to the exact sequence

$$0 \to \Omega^1_{\widehat{X}} \to \Omega^1_{\widehat{X}}(\log E) \to \mathcal{O}_E \to 0$$

and get

$$0 \to \pi_*\Omega^1_{\widehat{X}} \to \pi_*\Omega^1_{\widehat{X}}(\log E) \to \pi_*\mathcal{O}_E \to R^1\pi_*\Omega^1_{\widehat{X}} \to R^1\pi_*\Omega^1_{\widehat{X}}(\log E) \to$$
$$\to R^1\pi_*(\mathcal{O}_E).$$

Since $\pi_*\Omega^1_{\widehat{X}} \cong \pi_*\Omega^1_{\widehat{X}}(\log E)$ is an isomorphism, the map $\pi_*\mathcal{O}_E \cong \mathcal{O}_C \to R^1\pi_*\Omega^1_{\widehat{X}}$ is injective. The sheaf $R^1\pi_*\Omega^1_{\widehat{X}}$ can easily be seen to be locally free of rank 1 on C; we conclude the vanishing $R^1\pi_*\Omega^1_{\widehat{X}}(\log E) = 0$. □

We end the section comparing the deformations of (X, C) and the deformations of the blow-up $\left(\widehat{X}, E\right)$ and restrict ourselves to the case which is most interesting in our general setting, see also [Kod63].

9.1.2 Theorem. *Let X be a smooth projective 3-fold, $C \subset X$ a smooth curve, $\pi : \widehat{X} \to X$ the blow-up of $C \subset X$ and $E = \pi^{-1}(C)$. Let S be the germ of a complex manifold or $S = \operatorname{Spec}\left(\mathbb{C}[t]/t^2\right)$.*

Then there is a canonical bijection between the deformations of (X, C) over S and the deformations of $\left(\widehat{X}, E\right)$ over S.

Proof of Theorem 9.1.2: Step 1. Let $(\mathcal{X}, \mathcal{C})$ be a deformation of (X, C) over S. Let $\tau : \widehat{\mathcal{X}} \to \mathcal{X}$ be the blow-up of $\mathcal{C} \subset \mathcal{X}$ and $\mathcal{E} := \tau^{-1}(\mathcal{C})$ be the exceptional divisor. Then $\left(\widehat{\mathcal{X}}, \mathcal{E}\right)$ is a deformation of $\left(\widehat{X}, E\right)$ over S.
In fact, $\widehat{\mathcal{X}}_0$, the central fibre over $0 \in S$, is isomorphic to $\widehat{X}$. By [Har77], III, 7.15 and by definition of $\widehat{\mathcal{X}}_0$, the blow-up of $\mathcal{C} \cap \pi^{-1}(0) = C$ in $\mathcal{X}_0 = X$, where $\pi : \mathcal{X} \to S$ is the projection. Moreover $\mathcal{E} \cap \mathcal{X}_0 = E$.

Furthermore, since $\mathcal{X} \to S$ and $\mathcal{C} \to S$ are submersions, so is $\widehat{\mathcal{X}} \to S$; hence $\widehat{\mathcal{X}}$ is flat over S. Finally, $\mathcal{E}$ is flat over S by Lemma 3.8.3.

Step 2. Let $(\mathcal{Y}, \mathcal{E})$ be a deformation of $\left(\widehat{X}, E\right)$ over S. Since E is a $\mathbb{P}^1$-bundle over C and in fact $E = \mathbb{P}\left(\mathcal{N}^\vee_{C|X}\right)$, according to Lemma 3.7.1 the space $\mathcal{E}$ is a $\mathbb{P}^1$-bundle over a variety $\mathcal{C} \to S$, which is a deformation of C over S. Let $p : \mathcal{E} \to \mathcal{C}$ be the projection. We notice

$$\mathcal{N}^\vee_{\mathcal{E}|\mathcal{Y}}\Big|_{p^{-1}(x)} = \mathcal{N}^\vee_{E|\widehat{X}}\Big|_{p^{-1}(x)} = \mathcal{O}_{\mathbb{P}^1}(1)$$

for all $x \in \mathcal{C}$.

If S is smooth, then $\mathcal{Y}$ is smooth, and the contraction theorem of Fujiki-Nakano (see [Nak71] and [FN72]) states that there is a complex manifold $\mathcal{X}$ containing $\mathcal{C}$ with projection $\mathcal{X} \to S$ and a map $f : \mathcal{Y} \to \mathcal{X}$ such that f is the blow-up of $\mathcal{C} \subset \mathcal{X}$. The two operations are obviously inverse to each other.

Now we suppose that $S = \operatorname{Spec}\left(\mathbb{C}[t]/t^2\right)$. We recall that

$$\operatorname{Def}(X, C) = H^1(X, T_X\langle -C\rangle)$$

(which can be shown directly on X without blowing up) and

$$\operatorname{Def}\left(\widehat{X}, E\right) = H^1\left(\widehat{X}, T_{\widehat{X}}(-\log E)\right).$$

By Step 1 we obtain an injective linear map

$$H^1(X, T_X\langle -C\rangle) \to H^1\left(\widehat{X}, T_{\widehat{X}}(-\log E)\right).$$

This must be an isomorphism, because both spaces have the same dimensions. This fact comes from the definition

$$\pi_*\left(T_{\widehat{X}}(-\log E)\right) = T_X\langle -C\rangle$$

and the vanishing

$$R^1\pi_* T_{\widehat{X}}(-\log E) = 0.$$

The last equation is obtained by applying π_* to the logarithmic tangent sequence

$$0 \to T_{\widehat{X}}(-\log E) \to T_{\widehat{X}} \to \mathcal{N}_{E|\widehat{X}} \to 0$$

and using

$$\pi_*\mathcal{N}_{E|\widehat{X}} = R^1\pi_* T_{\widehat{X}} = 0.$$

9.1.3 Remark. As already said, Theorem 9.1.2 clearly generalizes to any compact complex manifold of any dimension and arbitrary compact complex submanifolds; also there will be an equivalence of deformation functors, but we will not pursue this further in this work.

9.2 Definition of a cohomology for triples

Let $\iota_1 : D_1 \hookrightarrow X$ and $\iota_2 : D_2 \hookrightarrow X$ be embeddings of compact, possibly reducible, hypersurfaces or of compact complex submanifolds into a smooth compact complex manifold. For deformations of triples (X, D_1, D_2) we refer to Definition 6.3.3.

In this section we lay down the foundations for a cohomology of triples.

9.2.1 Setup. *Let X be a projective manifold of dimension n and D_1, D_2 be smooth hypersurfaces on X which intersect transversally in the smooth submanifold C. Let $\iota_i : D_i \hookrightarrow X, i = 1, 2$, and $j : C \hookrightarrow X$ denote the inclusion maps.*

We first define a relative de Rham cohomology for the triple (X, D_1, D_2).

9.2.2 Definition. *We define the relative de Rham cohomology*

$$H^{\bullet}(X, D_1, D_2, \mathbb{C})$$

for the triple (X, D_1, D_2) to be the cohomology of the complex

$$\mathcal{A}_X^{\bullet} \oplus \mathcal{A}_{D_1}^{\bullet-1} \oplus \mathcal{A}_{D_2}^{\bullet-1}$$

with the differential

$$\tilde{d}(\alpha, \beta_1, \beta_2) := \big(d_X \alpha, \alpha|_{D_1} - d_{D_1}\beta_1, \alpha|_{D_2} - d_{D_2}\beta_2\big) \tag{9.2.2.1}$$

for $\alpha \in \mathcal{A}_X^k$ and $\beta_i \in \mathcal{A}_{D_i}^{k-1}, i = 1, 2, k \in \mathbb{N}$.

9.2.3 Remark. One easily verifies that $\tilde{d}^2 = 0$.

Furthermore, if $(\alpha, \beta_1, \beta_2) \in \mathcal{A}_X^k \oplus \mathcal{A}_{D_1}^{k-1} \oplus \mathcal{A}_{D_2}^{k-1}$ is a closed form, i.e. $\tilde{d}(\alpha, \beta_1, \beta_2) = 0$, then

$$d_{D_1 \cap D_2}(\beta_1 - \beta_2)|_{D_1 \cap D_2} = 0.$$

9.2.4 Theorem. *There is an isomorphism*

$$\begin{aligned} &H^n(X, D_1, D_2, \mathbb{C}) \cong \\ &\quad \cong \ker\left(H^n(X, \mathbb{C}) \overset{H^n(\iota_1)\oplus H^n(\iota_2)}{\longrightarrow} H^n(D_1, \mathbb{C}) \oplus H^n(D_2, \mathbb{C})\right) \\ &\quad\quad \oplus \big(H^{n-1}(D_1, \mathbb{C}) \oplus H^{n-1}(D_2, \mathbb{C})\big)_{var}, \end{aligned}$$

where

$$\begin{aligned} &\big(H^{n-1}(D_1, \mathbb{C}) \oplus H^{n-1}(D_2, \mathbb{C})\big)_{var} := \\ &\operatorname{coker}\left(H^{n-1}(X, \mathbb{C}) \overset{H^{n-1}(\iota_1)\oplus H^{n-1}(\iota_2)}{\longrightarrow} H^{n-1}(D_1, \mathbb{C}) \oplus H^{n-1}(D_2, \mathbb{C})\right). \end{aligned}$$

In particular

$$\begin{aligned} \dim H^n(X, D_1, D_2, \mathbb{C}) &= \dim H^n(X, \mathbb{C}) - \dim \operatorname{im}(H^n(\iota_1) \oplus H^n(\iota_2)) \\ &\quad + \dim H^{n-1}(D_1, \mathbb{C}) + \dim H^{n-1}(D_2, \mathbb{C}) \\ &\quad - \dim \operatorname{im}\big(H^{n-1}(\iota_1) \oplus H^{n-1}(\iota_2)\big). \end{aligned}$$

Furthermore, there is a surjective map

$$\begin{aligned} H^n(X, D_1, D_2, \mathbb{C}) &\twoheadrightarrow \ker(H^n(X, \mathbb{C}) \to H^n(D_1, \mathbb{C}) \oplus H^n(D_2, \mathbb{C})) \\ &\quad \oplus H^{n-1}_{var}(D_1) \oplus H^{n-1}_{var}(D_2), \end{aligned}$$

and there is a natural mixed Hodge structure on $H^n(X, D_1, D_2, \mathbb{C})$.

Proof of Theorem 9.2.4: The short exact sequence of complexes

$$0 \to \left(\mathcal{A}^{\bullet-1}_{D_1} \oplus \mathcal{A}^{\bullet-1}_{D_2}, (d_{D_1}, d_{D_2})\right) \xrightarrow{f} \left(\mathcal{A}^{\bullet}_{X} \oplus \mathcal{A}^{\bullet-1}_{D_1} \oplus \mathcal{A}^{\bullet-1}_{D_2}, \tilde{d}\right) \xrightarrow{g}$$
$$\to (\mathcal{A}^{\bullet}_{X}, d_X) \to 0,$$

where the maps f and g are defined in the obvious way, i.e.

$$f : \mathcal{A}^{\bullet-1}_{D_1} \oplus \mathcal{A}^{\bullet-1}_{D_2} \to \mathcal{A}^{\bullet}_{X} \oplus \mathcal{A}^{\bullet-1}_{D_1} \oplus \mathcal{A}^{\bullet-1}_{D_2}, (\beta_1, \beta_2) \mapsto (0, \beta_1, \beta_2)$$

and

$$g : \mathcal{A}^{\bullet}_{X} \oplus \mathcal{A}^{\bullet-1}_{D_1} \oplus \mathcal{A}^{\bullet-1}_{D_2} \to \mathcal{A}^{\bullet}_{X}, (\alpha, \beta_1, \beta_2) \mapsto \alpha,$$

induces a long exact sequence in cohomology

$$\begin{aligned} \ldots \ \to\ & H^{n-1}(X, D_1, D_2, \mathbb{C}) \to H^{n-1}(X, \mathbb{C}) \xrightarrow{H^{n-1}(\iota_1) \oplus H^{n-1}(\iota_2)} \\ \to\ & H^{n-1}(D_1, \mathbb{C}) \oplus H^{n-1}(D_2, \mathbb{C}) \xrightarrow{\varphi} H^{n}(X, D_1, D_2, \mathbb{C}) \to \\ \to\ & H^{n}(X, \mathbb{C}) \xrightarrow{H^{n}(\iota_1) \oplus H^{n}(\iota_2)} H^{n}(D_1, \mathbb{C}) \oplus H^{n}(D_2, \mathbb{C}) \to \ldots . \end{aligned}$$

which yields the first two statements immediately.
Obviously, there is a natural surjective map

$$\left(H^{n-1}(D_1, \mathbb{C}) \oplus H^{n-1}(D_2, \mathbb{C})\right)_{var} \twoheadrightarrow H^{n-1}_{var}(D_1) \oplus H^{n-1}_{var}(D_2).$$

As the maps $H^k(\iota_1) \oplus H^k(\iota_2)$ for $k = n-2, n-1$ respect the Hodge structures of $H^k(X, \mathbb{C})$ and $H^k(D_1, \mathbb{C}) \oplus H^k(D_2, \mathbb{C})$, we get a natural mixed Hodge structure on $H^n(X, D_1, D_2, \mathbb{C})$, which is induced by the pure Hodge structures of $H^n(X, \mathbb{C})$ and $H^{n-1}(D_i, \mathbb{C}), i = 1, 2$. □

9.2.5 Remark. By construction, $H^{n-1}(D_2, \mathbb{C}) \xrightarrow{\varphi} H^n(X, D_1, D_2, \mathbb{C})$ and $H^n(X, D_1, D_2, \mathbb{C}) \to H^n(X, \mathbb{C})$ are morphisms of mixed Hodge structures.

9.2.6 Remark. If D_1 and D_2 are ample hypersurfaces such that

$$H^n(D_1, \mathbb{C}) = H^n(D_2, \mathbb{C}) = 0,$$

e.g. X is a Calabi-Yau 3-fold, then

$$H^n(X, D_1, D_2, \mathbb{C}) \cong H^n(X, \mathbb{C}) \oplus \left(H^{n-1}(D_1, \mathbb{C}) \oplus H^{n-1}(D_2, \mathbb{C})\right)_{var}.$$

According to the Lefschetz hyperplane theorem the map $H^{n-1}(\iota_1) \oplus H^{n-1}(\iota_2)$ is injective; thus

$$\begin{aligned} &\left(H^{n-1}(D_1, \mathbb{C}) \oplus H^{n-1}(D_2, \mathbb{C})\right)_{var} \\ &\quad \cong \left(H^{n-1}(D_1, \mathbb{C}) \oplus H^{n-1}(D_2, \mathbb{C})\right) / H^{n-1}(X, \mathbb{C}). \end{aligned}$$

Furthermore

$$\begin{aligned} \dim H^n(X, D_1, D_2, \mathbb{C}) &= \dim H^n(X, \mathbb{C}) + \dim H^{n-1}(D_1, \mathbb{C}) + \\ &\quad + \dim H^{n-1}(D_2, \mathbb{C}) - \dim H^{n-1}(X, \mathbb{C}). \end{aligned}$$

9.2.7 Theorem. *There are exact sequences*

$$\begin{aligned}\dots \quad & \to H^{q-1}(D_2,\mathbb{C}) \to H^q(X,D_1,D_2,\mathbb{C}) \to H^q(X,D_1,\mathbb{C}) \to \\ & \to H^q(D_2,\mathbb{C}) \to \dots, \end{aligned} \tag{9.2.7.1}$$

respectively with D_1 and D_2 interchanged

$$\begin{aligned}\dots \quad & \to H^{q-1}(D_1,\mathbb{C}) \to H^q(X,D_1,D_2,\mathbb{C}) \to H^q(X,D_2,\mathbb{C}) \to \\ & \to H^q(D_1,\mathbb{C}) \to \dots. \end{aligned} \tag{9.2.7.2}$$

Proof of Theorem 9.2.7: The following exact sequence of complexes, defined in the obvious way,

$$0 \to \left(\mathcal{A}_{D_2}^{\bullet-1}, d_{D_2}^{\bullet-1}\right) \to \left(\mathcal{A}_X^{\bullet} \oplus \mathcal{A}_{D_1}^{\bullet-1} \oplus \mathcal{A}_{D_2}^{\bullet-1}, \tilde{d}^{\bullet}\right) \to \left(\mathcal{A}_X^{\bullet} \oplus \mathcal{A}_{D_1}^{\bullet-1}, \tilde{d}^{\bullet}\right) \to 0$$

yields the exact sequence 9.2.7.1. In the same way we get Sequence 9.2.7.2. □

9.2.8 Corollary. *Under the assumptions of Remark 9.2.6, there are exact sequences*

$$0 \to H^{n-1}(D_2,\mathbb{C}) \to H^n(X,D_1,D_2,\mathbb{C}) \to H^n(X,D_1,\mathbb{C}) \to 0$$

and

$$0 \to H^{n-1}(D_1,\mathbb{C}) \to H^n(X,D_1,D_2,\mathbb{C}) \to H^n(X,D_2,\mathbb{C}) \to 0.$$

Proof of Corollary 9.2.8: We apply Theorem 9.2.7. As D_i are ample, we know $H^n(D_i,\mathbb{C}) = H^{n-2}(D_i,\mathbb{C}) = 0$, and the maps $H^{n-1}(X,\mathbb{C}) \to H^{n-1}(D_i,\mathbb{C})$ are injective. As

$$\dim H^n(X,D_1,\mathbb{C}) = \dim H^n(X,\mathbb{C}) + \dim H^{n-1}(D_1,\mathbb{C}) - \dim H^{n-1}(X,\mathbb{C})$$

and thus

$$\begin{aligned}\dim H^n(X,D_1,D_2,\mathbb{C}) &= \dim H^n(X,\mathbb{C}) + \dim H^{n-1}(D_1,\mathbb{C}) + \\ &\quad + \dim H^{n-1}(D_2,\mathbb{C}) - \dim H^{n-1}(X,\mathbb{C}) \\ &= \dim H^n(X,D_1,\mathbb{C}) + \dim H^{n-1}(D_2,\mathbb{C}),\end{aligned}$$

we conclude that the map $H^{n-1}(D_2,\mathbb{C}) \to H^n(X,D_1,D_2,\mathbb{C})$ in Sequence 9.2.7.1 is injective. □

9.2.9 Remark. We consider a smooth divisor D in a Calabi-Yau 3-fold X. For setting up a Picard-Fuchs equation for the pair (X,D), one needs to study the variation of (X,D). This is related to the first-order deformations of the pair (X, D) as follows. We fix a Kähler class on X. By Hodge decomposition, $H^2\left(X, \Omega_X^1(\log D)\right)$ is a direct summand of $H^3(X \backslash D, \mathbb{C})$. Hence there is a canonical epimorphism

$$H^3(X \backslash D, \mathbb{C}) \rightarrow H^2\left(X, \Omega_X^1(\log D)\right).$$

By duality we obtain a canonical epimorphism

$$H^3(X, D, \mathbb{C}) \rightarrow H^1\left(X, T_X(-\log D)\right).$$

More precisely, we generalize this to the case of two divisors D_1 and D_2 meeting transversally in the smooth curve C. We will construct an epimorphism

$$H^3(X, D_1, D_2, \mathbb{C}) \rightarrow H^1\left(X, T_X\langle -C\rangle\right),$$

provided $D_1 - D_2$ is ample. This epimorphism is canonical up to a choice of a basis of $H^1(X, T_X)$. If $h^{1,1}(X) = 1$ or if the classes of the divisors D_1 and D_2 are linearly dependent, this ampleness assumption is not necessary.

9.2.10 Lemma. *We assume D_1, D_2 and $D_1 - D_2$ to be ample. Then there is an exact sequence*

$$0 \rightarrow H^0\left(C, \mathcal{N}_{C|X}\right) \rightarrow H^1\left(X, T_X\langle -C\rangle\right) \rightarrow H^1(X, T_X) \rightarrow 0.$$

Proof of Lemma 9.2.10: The exact sequence

$$0 \rightarrow T_X\langle -C\rangle \rightarrow T_X \rightarrow j_*\mathcal{N}_{C|X} \rightarrow 0$$

yields, using $H^0(X, T_X) = 0$, the exact sequence

$$0 \rightarrow H^0\left(C, \mathcal{N}_{C|X}\right) \rightarrow H^1\left(X, T_X\langle -C\rangle\right) \xrightarrow{\kappa} H^1(X, T_X).$$

We show that κ is surjective, i.e., each first-order deformation of X is the restriction of a simultaneous first-order deformation of the pair (X, C).

Let $\mathcal{X}$ be a first-order deformation of X. We look at the normal bundle sequence associated to $D_i \subset X \subset \mathcal{X}$, i.e.,

$$0 \rightarrow \mathcal{N}_{D_i|X} \rightarrow \mathcal{N}_{D_i|\mathcal{X}} \rightarrow \mathcal{N}_{X|\mathcal{X}}\big|_{D_i} \rightarrow 0.$$

We note that $\mathcal{N}_{X|\mathcal{X}}\big|_{D_i} \cong \mathcal{O}_{D_i}$. The short exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(D_i) \rightarrow \iota_{i*}\mathcal{N}_{D_i|X} \rightarrow 0$$

yields, via $H^1(X, \mathcal{O}_X(D_i)) = 0$, the equation $H^1\left(D_i, \mathcal{N}_{D_i|X}\right) = 0$. Thus we get an exact sequence

$$0 \rightarrow H^0\left(D_i, \mathcal{N}_{D_i|X}\right) \rightarrow H^0\left(D_i, \mathcal{N}_{D_i|\mathcal{X}}\right) \rightarrow H^0\left(D_i, \mathcal{O}_{D_i}\right) \rightarrow 0,$$

and therefore

$$h^0\left(D_i, \mathcal{N}_{D_i|\mathcal{X}}\right) = h^0\left(D_i, \mathcal{N}_{D_i|X}\right) + 1,$$

i.e., there is a first-order deformation $\mathcal{D}_i$ of D_i in $\mathcal{X}$.

Finally, we define $\mathcal{C} := \mathcal{D}_1 \cap \mathcal{D}_2 \subset \mathcal{X}$, so that $(\mathcal{X}, \mathcal{C})$ is a first-order deformation of (X, C) which is a preimage of the first-order deformation $\mathcal{X}$ of X under κ. □

To prepare the proof of Theorem 9.2.12, we show

9.2.11 Lemma. *We assume D_1, D_2 and $D_1 - D_2$ to be ample. There is a canonical surjective map*

$$\left(H^2\left(D_1\right) \oplus H^2\left(D_2\right)\right)_{var} \to H^0\left(C, \mathcal{N}_{C|X}\right).$$

Proof of Lemma 9.2.11: It suffices to establish a canonical epimorphism

$$H^{2,0}\left(D_1\right) \oplus H^{2,0}\left(D_2\right) \to H^0\left(C, \mathcal{N}_{C|X}\right).$$

By the adjunction formula,

$$H^{2,0}\left(D_i\right) = H^0\left(D_i, K_{D_i}\right) = H^0\left(D_i, \mathcal{N}_{D_i|X}\right).$$

Hence it suffices to construct an epimorphism

$$H^0\left(D_1, \mathcal{N}_{D_1|X}\right) \oplus H^0\left(D_2, \mathcal{N}_{D_2|X}\right) \to H^0\left(C, \mathcal{N}_{C|X}\right).$$

Due to the decomposition

$$\mathcal{N}_{C|X} = \mathcal{N}_{D_1|X}\big|_C \oplus \mathcal{N}_{D_2|X}\big|_C,$$

it remains to be shown that the restriction maps

$$H^0\left(D_i, \mathcal{N}_{D_i|X}\right) \to H^0\left(C, \mathcal{N}_{D_i|X}\big|_C\right)$$

are surjective. Using the ideal sheaf sequence for $C \subset D_i$ it is sufficient to establish the vanishing

$$H^1\left(D_i, \mathcal{J}_{C|D_i} \otimes \mathcal{N}_{D_i|X}\right) = 0.$$

We argue only for $i = 1$ and consider the ideal sheaf sequence of $D_1 \subset X$ tensorized by $\mathcal{O}_X\left(D_1 - D_2\right)$

$$\begin{aligned} \ldots &\to H^1\left(X, \mathcal{O}_X\left(D_1 - D_2\right)\right) \to H^1\left(D_1, \mathcal{O}_X\left(D_1 - D_2\right)|_{D_1}\right) \to \\ &\to H^2\left(X, \mathcal{O}_X\left(-D_2\right)\right) \to \ldots. \end{aligned}$$

As

$$H^1\left(D_1, \mathcal{J}_{C|D_1} \otimes \mathcal{N}_{D_1|X}\right) = H^1\left(D_1, \mathcal{O}_{D_1}\left(D_1 - D_2\right)\right),$$

Kodaira vanishing yields $H^1\left(D_1, \mathcal{O}_X\left(D_1 - D_2\right)|_{D_1}\right) = 0$. Analogously we get $H^1\left(D_2, \mathcal{O}_X\left(D_2 - D_1\right)|_{D_2}\right) = 0$. □

9.2.12 Theorem. *Assuming again D_1, D_2 and $D_1 - D_2$ to be ample, we fix a Kähler class $[\omega]$ on X. Then there is a surjective map*

$$H^3(X, D_1, D_2, \mathbb{C}) \to H^1(X, T_X\langle -C\rangle).$$

This map is canonical up to a choice of a basis of $H^1(X, T_X)$.

Proof of Theorem 9.2.12: According to Lemma 9.2.10 and Lemma 9.2.11 we have the following diagram of exact sequences, where the first and third vertical arrows are surjective. The map μ is still to be constructed.

$$\begin{tikzcd}
0 \arrow[r] & \left(H^2(D_1,\mathbb{C}) \oplus H^2(D_2,\mathbb{C})\right)_{var} \arrow[r] \arrow[d, "\tau"] & H^3(X, D_1, D_2, \mathbb{C}) \arrow[r] \arrow[d, dashed, "\mu"] & H^3(X,\mathbb{C}) \arrow[r] \arrow[d, "\sigma"] & 0 \\
0 \arrow[r] & H^0\left(C, \mathcal{N}_{C|X}\right) \arrow[r] & H^1(X, T_X\langle -C\rangle) \arrow[r] & H^1(X, T_X) \arrow[r] & 0,
\end{tikzcd}$$

We first establish a canonical splitting

$$\phi : H^3(X,\mathbb{C}) \to H^3(X, D_1, D_2, \mathbb{C})$$

of the upper row of the diagram.

For each class $u \in H^3(X,\mathbb{C})$ we choose the unique harmonic representative $\alpha \in \ker(\Delta_d) \subset \Gamma\left(\mathcal{A}^3_X\right)$ of the class u, where

$$\Delta_d := d^*d + dd^* : \Gamma\left(\mathcal{A}^3_X\right) \to \Gamma\left(\mathcal{A}^3_X\right)$$

is the Laplace operator. In particular $d\alpha = d^*\alpha = 0$. Let ω_i be the induced Kähler metric on D_i, furthermore Δ_i the associated Laplace operator, $\mathcal{H}_i$ the harmonic projection and G_i the Green operator; see e.g. [GH78], p. 84. Then we obtain the Hodge decomposition

$$\alpha|_{D_i} = \mathcal{H}_i\left(\alpha|_{D_i}\right) \oplus dd_i^*G_i\left(\alpha|_{D_i}\right) \oplus d_i^*dG_i\left(\alpha|_{D_i}\right).$$

As $H^3(D_i,\mathbb{C}) = 0$, we get $\mathcal{H}_i\left(\alpha|_{D_i}\right) = 0$, furthermore $d_i^*d_iG\left(\alpha|_{D_i}\right) = d_i^*G_id\left(\alpha|_{D_i}\right) = 0$. Therefore we define $\beta_i := d_i^*G_i\left(\alpha|_{D_i}\right)$ so that $\alpha|_{D_i} = d\beta_i$. Now we define

$$\phi(u) := [(\alpha, \beta_1, \beta_2)].$$

This splitting establishes an isomorphism

$$\Phi : H^3(X, D_1, D_2, \mathbb{C}) \to \left(H^2(D_1,\mathbb{C}) \oplus H^2(D_2,\mathbb{C})\right)_{var} \oplus H^3(X,\mathbb{C})$$

and therefore a map

$$\tilde{\mu} : H^3(X, D_1, D_2, \mathbb{C}) \to H^0\left(C, \mathcal{N}_{C|X}\right) \oplus H^1(X, T_X).$$

By choosing a basis of $H^1(X, T_X)$, we obtain a splitting $H^1(X, T_X) \to H^1(X, T_X\langle -C\rangle)$, which defines the map μ. $\square$

9.3 Pairing between homology and cohomology

We return to the general Setup 9.2.1 and assume $H^n(D_i, \mathbb{C}) = 0$, e.g. D_i is ample in a Calabi-Yau 3-fold X, and start defining a homology group

$$H_n(X, D_1, D_2),$$

which will be dual to $H^n(X, D_1, D_2, \mathbb{C})$. If $n \geqslant 4$, we assume additionally that the Mayer-Vietoris map $H_{n-1}(C) \to H_{n-1}(D_1) \oplus H_{n-1}(D_2)$ is injective. This is automatic if $n = 3$, since in this case, C is a curve.

We consider the following commutative diagram:

$$\begin{tikzcd}
H_n(X) \arrow[r, hook] \arrow[d] & H_n(X) \oplus H_n(X) \arrow[r, two heads] \arrow[d, hook] & H_n(X) \arrow[d, hook, "\lambda"] & \\
H_n(X,C) \arrow[r] \arrow[d] & H_n(X,D_1) \oplus H_n(X,D_2) \arrow[r] \arrow[d] & H_n(X, D_1 \cup D_2) \arrow[r, "\delta"] \arrow[d, "\tau"] & H_{n-1}(X,C) \arrow[d] \\
H_{n-1}(C) \arrow[r] \arrow[d] & H_{n-1}(D_1) \oplus H_{n-1}(D_2) \arrow[r, "\sigma"] \arrow[d] & H_{n-1}(D_1 \cup D_2) \arrow[r, two heads, "\epsilon"] \arrow[d, "\kappa"] & H_{n-2}(C) \\
H_{n-1}(X) \arrow[r, hook] \arrow[d] & H_{n-1}(X) \oplus H_{n-1}(X) \arrow[r, two heads] \arrow[d] & H_{n-1}(X) \arrow[d] & \\
H_{n-1}(X,C) \arrow[r] & H_{n-1}(X,D_1) \oplus H_{n-1}(X,D_2) \arrow[r, two heads] & H_{n-1}(X, D_1 \cup D_2). &
\end{tikzcd} \tag{9.3.0.1}$$

The first and forth rows are given by the maps

$$H_i(X) \to H_i(X) \oplus H_i(X), \quad \alpha \mapsto (\alpha, \alpha)$$

and

$$H_i(X) \oplus H_i(X) \to H_i(X), \quad (\alpha, \beta) \mapsto \alpha - \beta.$$

The second and fifth row are the relative Mayer-Vietoris sequence for pairs (see e.g. [Spa81], p. 187). The third row is the Mayer-Vietoris sequence. The columns are given by the homology sequences of pairs.

The injectivity of the map $\lambda : H_n(X) \to H_n(X, D_1 \cup D_2)$ results from our assumptions and from the vanishing $H_n(D_i) = 0$, which follows from $H^{n-2}(D_i, \mathbb{C}) = 0$.

9.3.1 Definition. *Let*

$$\begin{aligned} &H_n(X, D_1, D_2) := \\ &\qquad \{([\Gamma], [\gamma_1], [\gamma_2]) \in H_n(X, D_1 \cup D_2) \oplus H_{n-1}(D_1) \oplus H_{n-1}(D_2) | \\ &\qquad\quad \tau([\Gamma]) = \sigma([\gamma_1], [\gamma_2]) \in H_{n-1}(D_1 \cup D_2)\}, \end{aligned}$$

where σ and τ are defined as in Diagram 9.3.0.1.

9.3.2 Lemma. *If D_1, D_2 are ample divisors, then the complex vector space $H_n(X, D_1, D_2)$ has dimension*

$$\dim H_n(X, D_1, D_2) = \dim H^n(X, D_1, D_2, \mathbb{C}).$$

Proof of Lemma 9.3.2: We look at the map

$$\tau - \sigma : H_n(X, D_1 \cup D_2) \oplus H_{n-1}(D_1) \oplus H_{n-1}(D_2) \to H_{n-1}(D_1 \cup D_2),$$

then $H_n(X, D_1, D_2) = \ker(\tau - \sigma)$ and

$$\begin{aligned} \dim H_n(X, D_1, D_2) &= \dim H_n(X, D_1 \cup D_2) + \\ &+ \dim H_{n-1}(D_1) + \dim H_{n-1}(D_2) - \dim \operatorname{im}(\tau - \sigma). \end{aligned}$$

The map $\kappa : H_{n-1}(D_1 \cup D_2) \to H_{n-1}(X)$ defined in Diagram 9.3.0.1 is surjective, since in Diagram 9.3.0.1 the map

$$H_{n-1}(D_1) \oplus H_{n-1}(D_2) \to H_{n-1}(X) \oplus H_{n-1}(X)$$

is surjective by the Lefschetz hyperplane theorem. Hence

$$\dim \operatorname{im}(\tau) = \dim H_{n-1}(D_1 \cup D_2) - \dim H_{n-1}(X)$$

and

$$\begin{aligned} \dim H_n(X, D_1 \cup D_2) &= \dim H_n(X) + \dim \operatorname{im}(\tau) = \\ &= \dim H_n(X) + \dim H_{n-1}(D_1 \cup D_2) \\ &\quad - \dim H_{n-1}(X). \end{aligned}$$

We show that the map $\tau - \sigma$ is surjective. Obviously we have $\operatorname{im}(\tau) \subset \operatorname{im}(\tau - \sigma)$. Furthermore for all $w \notin \operatorname{im}(\tau)$, the diagram shows that there exists a class $\tilde{w} \in H_{n-1}(D_1) \oplus H_{n-1}(D_2)$ with $\kappa(\sigma(\tilde{w}) - w) = 0$. Thus $\sigma(\tilde{w}) - w \in \operatorname{im}(\tau)$ and $w \in \operatorname{im}(\tau - \sigma)$.
Thus

$$\begin{aligned} \dim H_n(X, D_1, D_2) &= \dim H_n(X) + \dim H_{n-1}(D_1 \cup D_2) \\ &\quad - \dim H_{n-1}(X) + \dim H_{n-1}(D_1) + \dim H_{n-1}(D_2) \\ &\quad - \dim H_{n-1}(D_1 \cup D_2) = \\ &= \dim H^n(X, D_1, D_2, \mathbb{C}). \end{aligned}$$

□

The proof actually shows the following:

9.3.3 Corollary. *The statement of Lemma 9.3.2 remains true if instead of ampleness we assume the following:*

$$H^n(D_i, \mathbb{C}) = 0 \text{ or } H^{n-2}(D_i, \mathbb{C}) = 0$$

and the maps

$$H_{n-1}(D_i) \to H_{n-1}(X)$$

for $i = 1, 2$ *are surjective.*

9.3.4 Definition. *We choose a basis* $\{e_1, \ldots, e_m\}$ *of* $H_n(X, D_1, D_2)$ *and representatives* $e_j = ([\Gamma_j], [\gamma_{j,1}], [\gamma_{j,2}])$*. Then we define a pairing*

$$\begin{aligned} \phi : H_n(X, D_1, D_2) \times H^n(X, D_1, D_2, \mathbb{C}) &\to \mathbb{C}, \\ (([\Gamma_j], [\gamma_{j,1}], [\gamma_{j,2}]), [(\alpha, \beta_1, \beta_2)]) &\mapsto \int_{([\Gamma_j],[\gamma_{1,j}],[\gamma_{2,j}])} (\alpha, \beta_1, \beta_2) \\ &:= \int_{\Gamma_j} \alpha - \int_{\gamma_{j,1}} \beta_1 + \int_{\gamma_{j,2}} \beta_2. \end{aligned}$$

9.3.5 Remark. Let $(\alpha, \beta_1, \beta_2) \in \Gamma\left(\mathcal{A}_X^{n-1} \oplus \mathcal{A}_{D_1}^{n-2} \oplus \mathcal{A}_{D_2}^{n-2}\right)$ and $([\Gamma_j], [\gamma_{j,1}], [\gamma_{j,2}]) \in H_n(X, D_1, D_2)$. Then

$$\begin{aligned} \int_{(\Gamma,\gamma_1,\gamma_2)} \tilde{d}(\alpha, \beta_1, \beta_2) &= \int_{(\Gamma,\gamma_1,\gamma_2)} \left(d\alpha, \alpha|_{D_1} - d\beta_1, \alpha|_{D_2} - d\beta_2\right) = \\ &= \int_{\Gamma} d\alpha - \int_{\gamma_1} \left(\alpha|_{D_1} - d\beta_1\right) + \int_{\gamma_2} \left(\alpha|_{D_2} - d\beta_2\right) = \\ &= \int_{\partial\Gamma} \alpha - \int_{\gamma_1} \alpha|_{D_1} + \int_{\partial\gamma_1} \beta_1 + \int_{\gamma_2} \alpha|_{D_2} - \int_{\partial\gamma_2} \beta_2 = \\ &= \int_{\gamma_1} \alpha|_{D_1} - \int_{\gamma_2} \alpha|_{D_2} - \int_{\gamma_1} \alpha|_{D_1} + \int_{\gamma_2} \alpha|_{D_2} = \\ &= 0. \end{aligned}$$

Therefore the pairing is well-defined.

9.3.6 Theorem. *We suppose that for* $j = 1, 2$ *the canonical morphisms* $\iota_{j*} : H_{n-1}(D_j) \to H_{n-1}(X)$ *are surjective; e.g., the divisors* D_i *are ample. Then the pairing* ϕ *is non-degenerate.*

Proof of Theorem 9.3.6: Let $[(\alpha, \beta_1, \beta_2)] \in H^n(X, D_1, D_2, \mathbb{C})$. We assume that

$$\int_{([\Gamma],[\gamma_1],[\gamma_2])} (\alpha, \beta_1, \beta_2) = 0$$

for all $([\Gamma], [\gamma_1], [\gamma_2]) \in H_n(X, D_1, D_2)$ and aim to show that $[(\alpha, \beta_1, \beta_2)] = 0$.

For each $[\Gamma] \in H_n(X, D_1)$ we define $\gamma_1 := \partial\Gamma \in C_{n-1}(D_1)$ and integrate over $([\Gamma], [\gamma_1], 0)$. As the pairing

$$H_n(X, D_1) \times H^n(X, D_1, \mathbb{C}) \to \mathbb{C}$$

is non-degenerate, the image of $[(\alpha, \beta_1, \beta_2)]$ under the map

$$H^n(X, D_1, D_2, \mathbb{C}) \to H^n(X, D_1, \mathbb{C})$$

is 0. The exact sequence (see Corollary 9.2.8)

$$H^{n-1}(D_2, \mathbb{C}) \to H^n(X, D_1, D_2, \mathbb{C}) \to H^n(X, D_1, \mathbb{C})$$

yields $[(\alpha, \beta_1, \beta_2)] = [(0, 0, \beta_2')]$ for a closed form $\beta_2' \in \mathcal{A}_{D_2}^{n-1}$. We need to show that $[\beta_2'] = 0 \in H^{n-1}(D_2, \mathbb{C})$.

Claim: We choose an arbitrary class $[\gamma_2] \in H_{n-1}(D_2)$. Then there exists a cycle $[\Gamma] \in H_n(X, D_1 \cup D_2)$ and $[\gamma_1] \in H_{n-1}(D_1)$ such that

$$([\Gamma], [\gamma_1], [\gamma_2]) \in H_n(X, D_1, D_2).$$

Once the claim ist proved, then

$$\int_{\gamma_2} \beta_2' = \int_{([\Gamma],[\gamma_1],[\gamma_2])} (0, 0, \beta_2') = 0$$

and we conclude $[\beta_2'] = 0 \in H^{n-1}(D_2, \mathbb{C})$.

Proof of the Claim: Using our assumption there is a class $[\gamma_1] \in H_{n-1}(D_1)$ such that $\iota_{2*}(\gamma_2) = \iota_{1*}(\gamma_1)$. Let $k_j : D_j \hookrightarrow D_1 \cup D_2$ for $j = 1, 2$ and $l : D_1 \cup D_2 \to X$ be the inclusions. Then

$$l_*(k_{1*}[\gamma_1] - k_{2*}[\gamma_2]) = 0 \in H_{n-1}(X).$$

The relative sequence in homology

$$\ldots \to H_n(X, D_1 \cup D_2) \xrightarrow{\lambda} H_{n-1}(D_1 \cup D_2) \xrightarrow{l_*} H_{n-1}(X) \to \ldots$$

yields a class $[\Gamma] \in H_n(X, D_1 \cup D_2)$ such that

$$\lambda([\Gamma]) = k_{1*}[\gamma_1] - k_{2*}[\gamma_2] \in H_{n-1}(D_1 \cup D_2),$$

i.e.,

$$\partial[\Gamma] = [k_1(\gamma_1) - k_2(\gamma_2)] \in H_{n-1}(D_1 \cup D_2).$$

This proves the claim.

Since $\dim H_n(X, D_1, D_2) = \dim H^n(X, D_1, D_2, \mathbb{C})$, the pairing is non-degenerate.

In summary, we have shown that the canonical map

$$H^n(X, D_1, D_2, \mathbb{C}) \to H_n(X, D_1, D_2)^\vee$$

given by the pairing Φ is injective. Since

$$\dim H^n(X, D_1, D_2, \mathbb{C}) = \dim H_n(X, D_1, D_2),$$

this map is an isomorphism and therefore the pairing is non-degenerate. $\square$

9.4 A residue map for triples

We next extend the definition of residues to triples.

9.4.1 Definition. *Let* $\left(\mathcal{K}^\bullet, \tilde{d}^\bullet\right)$ *be the complex defined by*

$$\mathcal{K}^k := \Gamma\left(\Omega^k_{\mathbb{P}^n}(*X) \oplus \Omega^k_{\mathbb{P}^n}(*X + *H_1) \oplus \Omega^k_{\mathbb{P}^n}(*X + *H_2)\right)$$

with differential

$$\tilde{d}^k : \mathcal{K}^k \to \mathcal{K}^{k+1},$$
$$\tilde{d}^k(\eta_1, \eta_2, \eta_3) := \left(d^k\eta_1, \eta_1 \wedge \frac{dQ_1}{Q_1} - d^k\eta_2, \eta_1 \wedge \frac{dQ_2}{Q_2} - d^k\eta_3\right).$$

The residue map for classes of rational forms is defined by

$$\mathrm{Res}^q_{(X,D_1,D_2)} : H^q\left(\mathcal{K}^\bullet, \tilde{d}\right) \to H^{q-1}(X, D_1, D_2, \mathbb{C}),$$
$$[(\eta_1, \eta_2, \eta_3)] \mapsto \Big(\mathrm{Res}^q_{(X,D_1,D_2)}[(\eta_1, \eta_2, \eta_3)] : H_{q-1}(X, D_1, D_2) \to \mathbb{C},$$
$$([\Gamma], [\gamma_1], [\gamma_2]) \mapsto \int_{\tau(\Gamma)} \eta_1 - \int_{\tau'(\partial\gamma_1)} \eta_2 + \int_{\tau'(\partial\gamma_2)} \eta_3\Big).$$

As the pairing is non-degenerate, we get a well-defined element

$$\mathrm{Res}^q_{(X,D_1,D_2)}[(\eta_1, \eta_2, \eta_3)] \in H^{q-1}(X, D_1, D_2, \mathbb{C})$$

for each $[(\eta_1, \eta_2, \eta_3)] \in H^q\left(\mathcal{K}^\bullet, \tilde{d}\right)$.

9.4.2 Remark. For each k there is a surjective map

$$\mathcal{K}^k_{(X,D_1,D_2)} \to \mathcal{K}^k_{(X,D_1)}, \quad (\alpha, \beta_1, \beta_2) \mapsto (\alpha, \beta_1),$$

which maps closed forms to closed forms and exact forms to exact forms.

9.5 Application of the Gauß-Manin connection to periods of triples

In this section we extend the theory of Li, Lian and Yau to triples and give a detailed account on the arguments of Li, Lian and Yau.

Let $\pi : \mathcal{X} \to S$ be a family of Calabi-Yau n-folds over a complex manifold S. Let $\mathcal{D}_1$ and $\mathcal{D}_2 \subset \mathcal{X}$ be families of smooth hypersurfaces over S meeting fibrewise transversally. So $\mathcal{C} := \mathcal{D}_1 \cap \mathcal{D}_2$ is a family of smooth curves over S. For $s \in S$ let $X_s, D_{i,s}, C_s$ be the corresponding fibres over $s \in S$. We assume $D_{1,s} - D_{2,s}$ to be ample for all $s \in S$.

9.5.1 Lemma. *The vector spaces* $H^n(X_s, D_{1,s}, D_{2,s}, \mathbb{C})$ *form a local system over* S*, which we denote by* $\mathcal{H}^n(\mathcal{X}, \mathcal{D}_1, \mathcal{D}_2)$.

Proof of Lemma 9.5.1: We apply the theorem of Ehresmann to the fibre product

$$\pi : \mathcal{X} \times_S \mathcal{D}_1 \times_S \mathcal{D}_2 \to S.$$

Locally over S, this fibre product is diffeomorphic to $(X_0 \times D_{1,s} \times D_{2,s}) \times S$; hence the lemma follows. □

9.5.2 Notation. We choose a smooth family

$$\left(\tilde{\alpha}, \tilde{\beta}_1, \tilde{\beta}_2\right) \in \Gamma\left(\mathcal{X}, \mathcal{A}^n_{\mathcal{X}|S} \oplus \mathcal{A}^{n-1}_{\mathcal{D}_1|S} \oplus \mathcal{A}^{n-1}_{\mathcal{D}_2|S}\right)$$

and set $\alpha_s = \iota^*_{X_s}(\tilde{\alpha})$ for all $s \in S$, where $\iota_{X_s} : X_s \hookrightarrow \mathcal{X}$ is the inclusion. In the same way we define $\beta_{i,s}$ for $i = 1, 2$.

Let

$$[\Gamma_s] \in H_n(X_s, D_{1,s} \cup D_{2,s})$$

be a smooth family of classes represented by relative n-cycles Γ_s such that

$$\partial\Gamma_s \cong \iota_{1,s}(\gamma_{1,s}) - \iota_{2,s}(\gamma_{2,s})$$

for each $s \in S$, where $(\gamma_{1,s})_{s\in S}$ and $(\gamma_{2,s})_{s\in S}$ form smooth families of cycles in $\mathcal{D}_1$ and $\mathcal{D}_2$ and $\iota_{i,s} : D_{i,s} \hookrightarrow D_{1,s} \cup D_{2,s}$ for $i = 1, 2$ denote the inclusions. Then

$$\tilde{\Gamma} := \bigcup_{s\in S} \Gamma_s \in C_n(\mathcal{X}|\, S) \text{ and } \tilde{\gamma}_i := \bigcup_{s\in S} \gamma_{i,s} \in C_{n-1}(\mathcal{D}_i|\, S).$$

Given a $\mathcal{C}^\infty$-vector field v on S, we denote by $\tilde{v}$ a lifting to $\mathcal{X}$ and by $\tilde{v}_j$ a lifting to $\mathcal{D}_j$, i.e. $\pi_*(\tilde{v}) = v$, respectively $\pi_*(\tilde{v}_j) = v$.

Let

$$\mathcal{L}_{\tilde{v}} : \mathcal{A}^k_{\mathcal{X}} \to \mathcal{A}^k_{\mathcal{X}}$$

be the Lie derivative with respect to $\tilde{v}$ and

$$\iota_{\tilde{v}} : \mathcal{A}^k_{\mathcal{X}} \to \mathcal{A}^{k-1}_{\mathcal{X}}$$

the contraction with $\tilde{v}$. Usually we write $\iota_{\tilde{v}}(\alpha) =: \tilde{v} \lrcorner\, \alpha$.

We compute the Lie derivative of the periods of the triple classes with respect to a vector field $v \in T_S$.

We aim to compute the Gauß-Manin connection on the holomorphic vector bundle associated to the local system $\mathcal{H}^n(\mathcal{X}, \mathcal{D}_1, \mathcal{D}_2)$.

9.5.3 Proposition. *The Lie derivative of the periods of the triple classes with respect to the vector field $v \in T_S$ is as follows*

$$\mathcal{L}_v \langle (\Gamma_s, \gamma_{1,s}, \gamma_{2,s}), (\alpha_s, \beta_{1,s}, \beta_{2,s}) \rangle =$$
$$= (-1)^{n+1} \left\langle (\Gamma_s, \gamma_{1,s}, \gamma_{2,s}), \left(\tilde{v} \lrcorner\, d\tilde{\alpha}, \tilde{v}_1 \lrcorner \left(\tilde{\alpha} - d\tilde{\beta}_1\right), \tilde{v}_2 \lrcorner \left(\tilde{\alpha} - d\tilde{\beta}_2\right)\right)\right\rangle.$$

For the proof of Proposition 9.5.3 we need the following

9.5.4 Remark. Let M be a smooth compact $(r+1)$-dimensional manifold with boundary and $f : M \to \mathbb{R}$ be a differentiable function. Then there is a smooth compact manifold with boundary N such that $M \cong N \times [0,1]$. For each $t > 0$ let

$$M_t := N \times [0,t].$$

Furthermore let $\omega \in \Omega^{r+1}(M) \cong \Omega^{r+1}(N \times [0,1])$ and $v \in T_M$. Then for each $t \in [0,1]$ there exists a form $\eta_t \in \Omega^r(N)$ such that we can write $\omega = \eta_t \wedge dt$. Then we obtain

$$\frac{\partial}{\partial t}\int_{M_t} \omega = \int_N v \lrcorner \omega,$$

since

$$\begin{aligned} \frac{\partial}{\partial t}\int_{M_t} \omega &= \frac{\partial}{\partial t}\int_{N\times[0,t]} \eta_t \wedge dt = \frac{\partial}{\partial t}\int_0^t \left(\int_{N\times\{\tau\}} \eta_\tau\right) d\tau = \int_{N\times\{t\}} \eta_t = \\ &= (-1)^{r+1}\int_N v \lrcorner \omega. \end{aligned}$$

The last equality follows writing $v = \left(v_0, \frac{\partial}{\partial t}\right) \in T_{N\times[0,t]}$ by contraction

$$v \lrcorner \omega = v_0 \lrcorner \eta_t \wedge dt + (-1)^{r+1}\eta_t.$$

Proof of Proposition 9.5.3: Let $s_0 \in S$ and $\sigma : [0,t] \to S$ be a smooth local curve such that $\sigma(0) = s_0$ and $\partial_t \sigma(0) = v$. For simplicity we denote the fibre of $\tilde{\Gamma}$ over $\sigma(s)$ by $\Gamma_s := \Gamma_{\sigma(s)}$ for each $s \in [0,t]$.

Let

$$\widehat{\Gamma}_t := \bigcup_{0\leqslant s\leqslant t} \Gamma_{\sigma(s)} \text{ and } \quad \widehat{\gamma}_{i,t} := \bigcup_{0\leqslant s\leqslant t} \gamma_{i,\sigma(s)} \text{ for } i = 1,2$$

be the families of cycles over $\sigma(s), 0 \leqslant s \leqslant t$. Furthermore

$$\widehat{\partial\Gamma}_t := \bigcup_{0\leqslant s\leqslant t} \partial\Gamma_{\sigma(s)} \simeq \bigcup_{0\leqslant s\leqslant t} \iota_{1,s}(\gamma_{1,s}) - \iota_{2,s}(\gamma_{2,s}) = \widehat{\gamma}_{1,t} - \widehat{\gamma}_{2,t}.$$

Then obviously we obtain

$$\partial\left(\widehat{\Gamma}_t\right) = \Gamma_t - \Gamma_0 - \widehat{\partial\Gamma}_t \simeq \Gamma_t - \Gamma_0 - (\widehat{\gamma}_{1,t} - \widehat{\gamma}_{2,t}).$$

Then we compute

$$\int_{\widehat{\Gamma}_t} d\tilde{\alpha} = \int_{\partial(\widehat{\Gamma}_t)} \tilde{\alpha} = \int_{\Gamma_t} \tilde{\alpha} - \int_{\Gamma_0} \tilde{\alpha} - \left(\int_{\widehat{\gamma}_{1,t}} \tilde{\alpha} - \int_{\widehat{\gamma}_{2,t}} \tilde{\alpha}\right),$$

thus

$$\int_{\Gamma_t} \tilde{\alpha} = \int_{\hat{\Gamma}_t} d\tilde{\alpha} + \int_{\Gamma_0} \tilde{\alpha} + \left(\int_{\widehat{\gamma}_{1,t}} \tilde{\alpha} - \int_{\widehat{\gamma}_{2,t}} \tilde{\alpha} \right)$$

and therefore

$$\frac{\partial}{\partial t} \int_{\Gamma_t} \tilde{\alpha} = \frac{\partial}{\partial t} \int_{\hat{\Gamma}_t} d\tilde{\alpha} + \frac{\partial}{\partial t} \left(\int_{\widehat{\gamma}_{1,t}} \tilde{\alpha} - \int_{\widehat{\gamma}_{2,t}} \tilde{\alpha} \right) =$$
$$= (-1)^{n+1} \int_{\Gamma_t} \tilde{v} \lrcorner\, d\tilde{\alpha} + (-1)^n \int_{\gamma_{1,t}} \tilde{v}_1 \lrcorner\, \tilde{\alpha} - (-1)^n \int_{\gamma_{2,t}} \tilde{v}_2 \lrcorner\, \tilde{\alpha}.$$

Furthermore we obtain for $i = 1, 2$

$$\int_{\widehat{\gamma}_{i,t}} d\tilde{\beta}_i = \int_{\partial(\widehat{\gamma}_{i,t})} \tilde{\beta}_i = \int_{\gamma_{i,t}} \tilde{\beta}_i - \int_{\gamma_{i,0}} \tilde{\beta}_i,$$

thus

$$\frac{\partial}{\partial t} \int_{\gamma_{i,t}} \tilde{\beta}_i = \frac{\partial}{\partial t} \int_{\widehat{\gamma}_{i,t}} d\tilde{\beta}_i = (-1)^n \int_{\gamma_{i,t}} \tilde{v}_1 \lrcorner\, d\tilde{\beta}_i.$$

Finally we get

$$\mathcal{L}_v \left\langle (\Gamma_s, \gamma_{1,s}, \gamma_{2,s}), (\alpha_s, \beta_{1,s}, \beta_{2,s}) \right\rangle = \frac{\partial}{\partial t} \left(\int_{\Gamma_t} \tilde{\alpha} - \int_{\gamma_{1,t}} \tilde{\beta}_1 + \int_{\gamma_{2,t}} \tilde{\beta}_2 \right) =$$
$$= (-1)^{n+1} \int_{\Gamma_t} \tilde{v} \lrcorner\, d\tilde{\alpha} + (-1)^n \int_{\gamma_{1,t}} \tilde{v}_1 \lrcorner\, \tilde{\alpha} - (-1)^n \int_{\gamma_{2,t}} \tilde{v}_2 \lrcorner\, \tilde{\alpha}$$
$$- (-1)^n \int_{\gamma_{1,t}} \tilde{v}_1 \lrcorner\, d\tilde{\beta}_1 + (-1)^n \int_{\gamma_{2,t}} \tilde{v}_2 \lrcorner\, d\tilde{\beta}_2$$
$$= (-1)^{n+1} \left(\int_{\Gamma_t} \tilde{v} \lrcorner\, d\tilde{\alpha} - \int_{\gamma_{1,t}} \tilde{v}_1 \lrcorner \left(\tilde{\alpha} - d\tilde{\beta}_1 \right) + \int_{\gamma_{2,t}} \tilde{v}_2 \lrcorner \left(\tilde{\alpha} - d\tilde{\beta}_2 \right) \right).$$

□

9.5.5 Corollary. *As the Gauß-Manin connection is computed by*

$$\nabla_v^{GM} \left[(\alpha_s, \beta_{1,s}, \beta_{2,s}) \right] = \left[\left(\tilde{v} \lrcorner\, d\tilde{\alpha}, \tilde{v}_1 \lrcorner \left(\tilde{\alpha} - d\tilde{\beta}_1 \right), \tilde{v}_2 \lrcorner \left(\tilde{\alpha} - d\tilde{\beta}_2 \right) \right) \right],$$

we obtain

$$\mathcal{L}_v \left\langle (\Gamma_s, \gamma_{1,s}, \gamma_{2,s}), (\alpha_s, \beta_{1,s}, \beta_{2,s}) \right\rangle =$$
$$= (-1)^{n+1} \left\langle (\Gamma_s, \gamma_{1,s}, \gamma_{2,s}), \nabla_v^{GM} \left[(\alpha_s, \beta_{1,s}, \beta_{2,s}) \right] \right\rangle.$$

In addition to the previous assumptions we assume that $M = \mathbb{P}^{n+1}$ (one might also consider a weighted projective space) and set $S = S_1 \times S_2 \times S_3$, where each S_i is a complex manifold.

Let

$$\tilde{\mathcal{X}} \subset M \times S_1$$

be a hypersurface. More precisely, $X_z = \{P_z = 0\} \subset M$ is given by a homogeneous polynomial $P_z \in H^0(M, -K_M)$ for each $z \in S_1$. Furthermore let $\tilde{\mathcal{H}}_1 \subset M \times S_2$ and $\tilde{\mathcal{H}}_2 \subset M \times S_3$ be hypersurfaces, given by $H_{1,u} = \{Q_{1,u} = 0\} \subset M$ for each $u \in S_2$ and $H_{2,v} = \{Q_{2,v} = 0\} \subset M$ for each $v \in S_3$, where $Q_{1,u}$ and $Q_{2,v}$ are homogeneous polynomials on M for all $u \in S_2, v \in S_3$.

In summary,

$$\begin{array}{rcl} M \times S_1 \supset \tilde{\mathcal{X}} & := & (X_z := \{P_z = 0\})_{z \in S_1} \xrightarrow{\pi_1} S_1 \\ M \times S_2 \supset \tilde{\mathcal{H}}_1 & := & (H_{1,u} := \{Q_{1,u} = 0\})_{u \in S_2} \xrightarrow{\pi_2} S_2 \\ M \times S_3 \supset \tilde{\mathcal{H}}_2 & := & (H_{2,v} := \{Q_{2,v} = 0\})_{v \in S_3} \xrightarrow{\pi_3} S_3. \end{array}$$

We view $\mathcal{X}, \mathcal{H}_1$ and $\mathcal{H}_2$ as families over S:

$$\begin{array}{rcl} \mathcal{X} & := & \tilde{\mathcal{X}} \times S_2 \times S_3 \xrightarrow{\pi_1 \times \mathrm{id} \times \mathrm{id}} S \\ \mathcal{H}_1 & := & S_1 \times \tilde{\mathcal{H}}_1 \times S_3 \xrightarrow{\mathrm{id} \times \pi_2 \times \mathrm{id}} S \\ \mathcal{H}_2 & := & S_1 \times S_2 \times \tilde{\mathcal{H}}_2 \xrightarrow{\mathrm{id} \times \mathrm{id} \times \pi_3} S. \end{array}$$

Moreover we define two divisors $\mathcal{D}_i$ for $i = 1, 2$, in $\mathcal{X}$:

$$\begin{array}{rcl} \tilde{\mathcal{D}}_1 & := & \{X_z \cap H_{1,u} |\, z \in S_1, u \in S_2\} \xrightarrow{(\pi_1, \pi_2)} S_1 \times S_2 \\ \tilde{\mathcal{D}}_2 & := & \{X_z \cap H_{2,v} |\, z \in S_1, v \in S_3\} \xrightarrow{(\pi_1, \pi_3)} S_1 \times S_3 \\ \mathcal{D}_1 & := & \tilde{\mathcal{D}}_1 \times S_3 \xrightarrow{(\pi_1, \pi_2) \times \mathrm{id}} S \\ \mathcal{D}_2 & := & S_2 \times \tilde{\mathcal{D}}_2 \xrightarrow{\mathrm{id} \times (\pi_1, \pi_3)} S. \end{array}$$

Let

$$(\omega_z)_{z \in S_1} \in H^0\left(M \times S_1, K_{M \times S_1 | S_1} \otimes \mathcal{O}_{M \times S_1}(\mathcal{X} \times S_1)\right)$$

be a family of rational $(n+1)$-forms on M with poles along X_z. Since

$$K_M \otimes \mathcal{O}_M(X_z) = \Omega_M^{n+1}(\log X_z),$$

we can form $\mathrm{res}_{X_z|M}(\omega_z) \in H^0(X_z, K_{X_z})$ and get a holomorphic n-form without zeros for all z.

9.5.6 Notation. Let $(U_\alpha)_\alpha$ be an open covering of M by Stein open sets U_α, and set

$$V_{\mathcal{X}}^\alpha := (U_\alpha \times S) \cap \mathcal{X}.$$

Then we may write $V_{\mathcal{X}}^\alpha = V_{\mathcal{X},0}^\alpha \times S$, where $V_{\mathcal{X},0}^\alpha = U_\alpha \cap X_{s_0}$ for $s_0 \in S$. Analogously we define $V_{\mathcal{D}_1}^\alpha := (U_\alpha \times S) \cap \mathcal{D}_1$ and $V_{\mathcal{D}_2}^\alpha := (U_\alpha \times S) \cap \mathcal{D}_2$.

Let $P_{z,\alpha}$ be the defining equation of X_z in U_α. Using a partition of unity subordinate to the covering $(U_\alpha)_\alpha$ we may write

$$\omega_z = \sum_\alpha \frac{d_M P_{z,\alpha}}{P_{z,\alpha}} \wedge \phi_{z,\alpha},$$

where $\phi_z := \operatorname{res}_{X_z|M}(\omega_z)$ and $\phi_{z,\alpha}$ is the restriction of ϕ_z to U_α.

We are going to construct liftings of holomorphic tangent vector fields $v_i \in T_{S_i}$ on S_i for $i = 1, 2, 3$ to the Calabi-Yau manifold $\mathcal{X}$ and to the smooth divisors $\mathcal{D}_1$ and $\mathcal{D}_2$. These liftings consist of a trivial lift of v to $T_{\mathcal{U}_\alpha \times S_i}$ plus a normal vector induced by a certain variation.

First we construct local holomorphic liftings of vector fields $v_i \in T_{S_i}$ to vector fields on $\mathcal{X}$.

9.5.7 Construction. 1. Let $v_1 \in T_{S_1}$. We choose a local smooth complex curve in S_1 representing v_1 and denote the variable of the curve by z such that $v_1 = \frac{\partial}{\partial z} \in T_{S_1}$. We view v_1 as an element in $T_{U_\alpha \times S_1}$.
The deformation $\tilde{\mathcal{X}}$ in M over S_1 defines for each $z \in S_1$ a section

$$h_z \in H^0\left(X_z, \mathcal{N}_{X_z|M}\right)$$

which corresponds to the first-order deformation of X_z in M given by $\tilde{\mathcal{X}}$. Putting all $h_z, z \in S_1$, together, we obtain a section $h \in H^0\left(\tilde{\mathcal{X}}, \mathcal{N}_{\tilde{\mathcal{X}}|M \times S_1}\right)$. Let $p : \mathcal{X} = \tilde{\mathcal{X}} \times S_2 \times S_3 \to \tilde{\mathcal{X}}$ be the projection. Then applying p^* to the exact sequence

$$0 \to T_{\tilde{\mathcal{X}}} \to T_{M \times S_1}|_{\tilde{\mathcal{X}}} \to \mathcal{N}_{\tilde{\mathcal{X}}|M \times S_1} \to 0, \tag{9.5.7.1}$$

we get

$$0 \to p^* T_{\tilde{\mathcal{X}}} \to p^*\left(T_{M \times S_1}|_{\tilde{\mathcal{X}}}\right) \to \mathcal{N}_{\mathcal{X}|M \times S} \to 0. \tag{9.5.7.2}$$

We restrict the exact sequence 9.5.7.1 to $V_{\mathcal{X}}^\alpha$. Since $V_{\mathcal{X}}^\alpha$ is Stein, we get a surjective map

$$\kappa_{\mathcal{X}}^\alpha : H^0\left(V_{\mathcal{X}}^\alpha, p^*\left(T_{M \times S_1}|_{V_{\mathcal{X}}^\alpha}\right)\right) \to H^0\left(V_{\mathcal{X}}^\alpha, \mathcal{N}_{\mathcal{X}|M \times S}|_{V_{\mathcal{X}}^\alpha}\right).$$

We choose

$$n_{\mathcal{X}}^\alpha \in H^0\left(V_{\mathcal{X}}^\alpha, p^*\left(T_{M \times S_1}|_{V_{\mathcal{X}}^\alpha}\right)\right) \subset H^0\left(V_{\mathcal{X}}^\alpha, T_{M \times S}|_{V_{\mathcal{X}}^\alpha}\right)$$

such that $\kappa^{\alpha}_{\mathcal{X}}\left(n^{\alpha}_{\mathcal{X}}\right) = p^{*}h|_{V^{\alpha}_{\mathcal{X}}}$.

For each $z \in S_1$ let $n^{\alpha}_{X_z} \in H^0\left(V^{\alpha}_{\mathcal{X}} \cap X_z, T_{M\times S}|_{V^{\alpha}_{\mathcal{X}}\cap X_z}\right)$ be the restriction of $n^{\alpha}_{\mathcal{X}}$ to X_z, i.e. $\kappa^{\alpha}_{X_z}\left(n^{\alpha}_{X_z}\right) = p^{*}h_z$ for each $z \in S_1$, where $\kappa^{\alpha}_{X_z}$ is the restriction of $\kappa^{\alpha}_{\mathcal{X}}$ to $V^{\alpha}_{\mathcal{X}} \cap X_z$.
Let

$$w^{\alpha}_{\mathcal{X}}\left(v_1\right) := \left.\frac{\partial}{\partial z}\right|_{V^{\alpha}_{\mathcal{X}}} + n^{\alpha}_{\mathcal{X}} \in H^0\left(V^{\alpha}_{\mathcal{X}}, T_{M\times S}|_{V^{\alpha}_{\mathcal{X}}}\right).$$

2. Let $v_2 = \frac{\partial}{\partial u} \in T_{S_2}$ and $v_3 = \frac{\partial}{\partial v} \in T_{S_3}$. Obviously we can view v_2 and v_3 as vectors in $T_{\mathcal{X}}$.

9.5.8 Lemma. *On each $U_\alpha \times S$ the following statements are valid:*

1. $\left.\left(n^{\alpha}_{X_z} \lrcorner\, d_M P_{z,\alpha}\right)\right|_{X_z} = -\,\partial_z P_{z,\alpha}|_{X_z}$ *for each $z \in S_1$,*

2. $w^{\alpha}_{\mathcal{X}}\left(v_i\right) \in H^0\left(V^{\alpha}_{\mathcal{X}}, T_{\mathcal{X}}|_{V^{\alpha}_{\mathcal{X}}}\right)$ *for $i = 1, 2, 3$.*

Notation. For each $z \in S_1$ let $w^{\alpha}_{X_z}\left(v_i\right) \in H^0\left(V^{\alpha}_{X_z}, T_{M\times S}|_{V^{\alpha}_{X_z}}\right)$ be the restriction of $w^{\alpha}_{\mathcal{X}}\left(v_i\right)$ to X_z. We will briefly write in the following $\kappa^{\alpha}_{X_z} = \kappa_{X_z}$.

Proof of Lemma 9.5.8: 1. A direct computation gives the following formula

$$n^{\alpha}_{X_z} \lrcorner\, d_M P_{z,\alpha} = \kappa_{X_z}\left(n^{\alpha}_{X_z}\right) d_M P_{z,\alpha} = -s_\alpha\left(z\right) d_M P_{z,\alpha}.$$

According to [Ser06], p. 124/125, we know $s_\alpha\left(z\right) d_M P_{z,\alpha} = \partial_z P_{z,\alpha}$. This proves the first assertion.
2. Using 1. we get

$$\begin{aligned}
\kappa_{X_z}\left(w^{\alpha}_{X_z}\left(v_1\right)\right) &= \kappa_{X_z}\left(\left.\frac{\partial}{\partial z}\right|_{V^{\alpha}_{X_z}} + n^{\alpha}_{X_z}\right) = \left(\left.\frac{\partial}{\partial z}\right|_{V^{\alpha}_{X_z}} + n^{\alpha}_{X_z}\right) \lrcorner\, dP_{z,\alpha} = \\
&= \left.\frac{\partial}{\partial z}\right|_{V^{\alpha}_{X_z}} \lrcorner\, d_S P_{z,\alpha} + n^{\alpha}_{X_z} \lrcorner\, d_M P_{z,\alpha} = \\
&= \partial_z P_{z,\alpha} - \partial_z P_{z,\alpha} = 0
\end{aligned}$$

for each $z \in S_1$, thus $w^{\alpha}_{\mathcal{X}}\left(v_i\right) \in H^0\left(V^{\alpha}_{\mathcal{X}}, T_{\mathcal{X}}|_{V^{\alpha}_{\mathcal{X}}}\right)$ for all $i = 1, 2, 3$. $\square$

Similarly to Construction 9.5.7 we construct local holomorphic liftings of the vector fields $v_i \in T_{S_i}$ for $i = 1, 2, 3$ to vector fields on $\mathcal{D}_1$:

9.5.9 Construction. **1.** Let $v_1 = \frac{\partial}{\partial z} \in T_{S_1}$ and define

$$w_{\mathcal{D}_1}^{\alpha}(v_1) := \left.\frac{\partial}{\partial z}\right|_{V_{\mathcal{D}_1}^{\alpha}} + n_{\mathcal{X}}^{\alpha}|_{\mathcal{D}_1} \in H^0\left(V_{\mathcal{D}_1}^{\alpha}, T_{M\times S}|_{V_{\mathcal{D}_1}^{\alpha}}\right).$$

2. Now consider $v_2 = \frac{\partial}{\partial u} \in T_{S_2}$. The deformation $\tilde{\mathcal{H}}_1$ in M over S_2 yields for each $u \in S_2$ a section

$$h_{1,u} \in H^0\left(H_{1,u}, \mathcal{N}_{H_{1,u}|M}\right),$$

which corresponds to the first-order deformation of $H_{1,u}$ in M given by $\tilde{\mathcal{H}}_1$. Putting all $h_{1,u}, u \in S_2$, together, we get a section $h_1 \in H^0\left(\tilde{\mathcal{H}}_i, \mathcal{N}_{\tilde{\mathcal{H}}_i|M\times S_2}\right)$.

Let $p_1 : \mathcal{H}_1 = S_1 \times \tilde{\mathcal{H}}_1 \times S_3 \to \tilde{\mathcal{H}}_1$ be the projection. Then applying $p_1{}^*$ to the exact sequence

$$0 \to T_{\tilde{\mathcal{H}}_1} \to T_{M\times S_2}|_{\tilde{\mathcal{H}}_1} \to \mathcal{N}_{\tilde{\mathcal{H}}_1|M\times S_2} \to 0, \tag{9.5.9.1}$$

we get

$$0 \to p_1{}^* T_{\tilde{\mathcal{H}}_1} \to p_1{}^*\left(T_{M\times S_2}|_{\tilde{\mathcal{H}}_1}\right) \to \mathcal{N}_{\mathcal{H}_1|M\times S} \to 0. \tag{9.5.9.2}$$

Again we restrict the exact sequence 9.5.9.2 to Stein open subsets $V_{\mathcal{H}_1}^{\alpha}$ of $\mathcal{H}_1$ and get local surjective maps

$$\kappa_{\mathcal{H}_1}^{\alpha} : H^0\left(V_{\mathcal{H}_1}^{\alpha}, p_1{}^*\left(T_{M\times S_2}\right)|_{V_{\mathcal{H}_1}^{\alpha}}\right) \to H^0\left(V_{\mathcal{H}_1}^{\alpha}, \mathcal{N}_{\mathcal{H}_1|M\times S}|_{V_{\mathcal{H}_1}^{\alpha}}\right).$$

We choose

$$n_{\mathcal{H}_1}^{\alpha} \in H^0\left(V_{\mathcal{H}_1}^{\alpha}, p_1{}^*\left(T_{M\times S_2}\right)|_{V_{\mathcal{H}_1}^{\alpha}}\right) \subset H^0\left(V_{\mathcal{H}_1}^{\alpha}, T_{M\times S}|_{V_{\mathcal{H}_1}^{\alpha}}\right),$$

such that $\kappa_{\mathcal{H}_1}^{\alpha}\left(n_{\mathcal{H}_1}^{\alpha}\right) = (p_1{}^* h_1)|_{V_{\mathcal{H}_1}^{\alpha}}$.

Furthermore for each $u \in S_2$ let

$$n_{H_{1,u}}^{\alpha} \in H^0\left(V_{\mathcal{H}_1}^{\alpha} \cap H_{1,u}, T_{M\times S}|_{V_{\mathcal{H}_1}^{\alpha} \cap H_{1,u}}\right)$$

be the restriction of $n_{\mathcal{H}_1}^{\alpha}$ to $H_{1,u}$. Then $\kappa_{H_{1,u}}^{\alpha}\left(n_{H_{1,u}}^{\alpha}\right) = (p_1^* h_{1,u})|_{V_{\mathcal{H}_1}^{\alpha} \cap H_{1,u}}$.

Let

$$n_{\mathcal{D}_1}^{\alpha} := n_{\mathcal{H}_1}^{\alpha}\big|_{\mathcal{X}} \in H^0\left(V_{\mathcal{D}_1}^{\alpha}, T_{M\times S}|_{V_{\mathcal{D}_1}^{\alpha}}\right)$$

and

$$w_{\mathcal{D}_1}^{\alpha}(v_2) := \left.\frac{\partial}{\partial u}\right|_{V_{\mathcal{D}_1}^{\alpha}} + n_{\mathcal{D}_1}^{\alpha} \in H^0\left(V_{\mathcal{D}_1}^{\alpha}, T_{M\times S}|_{V_{\mathcal{D}_1}^{\alpha}}\right).$$

3. Let $v_3 = \frac{\partial}{\partial v} \in T_{S_3}$. Obviously we can view v_3 as a vector in $T_{\mathcal{D}_1}$.

Similarly to Lemma 9.5.8 we get

9.5.10 Lemma. *For each $u \in S_2$ and each α on $U_\alpha \cap H_{1,u}$ we have*

1. $\left. \left(n^{\alpha}_{D_{1,u}} \lrcorner\, d_M Q_{1,u,\alpha} \right) \right|_{H_{1,u}} = - \partial_u Q_{1,u,\alpha}|_{H_{1,u}}$ *for each $u \in S_2$,*

2. $w^{\alpha}_{\mathcal{D}_1}(v_2) \in H^0\left(V^{\alpha}_{\mathcal{D}_1}, T_{\mathcal{D}_1}|_{V^{\alpha}_{\mathcal{D}_1}} \right)$.

9.5.11 Remark. Analogously to Construction 9.5.9 we get local liftings $w^{\alpha}_{\mathcal{D}_2}(v_i) \in H^0\left(V^{\alpha}_{\mathcal{D}_i}, T_{\mathcal{X}}|_{V^{\alpha}_{\mathcal{D}_i}} \right)$ and $w^{\alpha}_{\mathcal{D}_i}(v_3) \in H^0\left(V^{\alpha}_{\mathcal{D}_i}, T_{\mathcal{D}_i}|_{V^{\alpha}_{\mathcal{D}_i}} \right)$ for $i = 1, 2$. Of course, Lemma 9.5.10 has an analogoue for $i = 2$.

Analogously to Construction 9.5.9 we obtain local liftings $w^{\alpha}_{\mathcal{X}}(v_3) \in H^0\left(V^{\alpha}_{\mathcal{X}}, T_{\mathcal{X}}|_{V^{\alpha}_{\mathcal{X}}} \right)$ and $w^{\alpha}_{\mathcal{D}_i}(v_3) \in H^0\left(V^{\alpha}_{\mathcal{D}_i}, T_{\mathcal{D}_i}|_{V^{\alpha}_{\mathcal{D}_i}} \right)$ for $i = 1, 2$.

Now Theorem 8.4.3 of Li, Lian and Yau [LLY12] generalizes to triples:

9.5.12 Theorem. *Let $([\Gamma], [\gamma_1], [\gamma_2])_{z,u,v} \in H_n(X, D_1, D_2)$. The periods*

$$\Pi : S \to \mathbb{C}, \quad \Pi(z, u, v) := \int_{(\Gamma, \gamma_1, \gamma_2)_{z,u,v}} \mathrm{Res}_{(X, D_1, D_2)} [(\omega_z, 0, 0)]$$

satisfy the following relations:

1. $$\begin{aligned} \partial_z \Pi(z, u, v) &= \int_{(\Gamma, \gamma_1, \gamma_2)_{z,u,v}} \mathrm{Res}^{n}_{(X, D_1, D_2)} [(\omega_z, 0, 0)] = \\ &= \int_{\tau(\Gamma_{z,u,v})} \partial_z \omega_z, \end{aligned}$$

   where $\tau(\partial \Gamma_{z,u,v}) \subset H_{1,u} \cup H_{2,v}$, where $\tau(\Gamma_{z,u,v})$ denotes the tube over $\Gamma_{z,u,v}$,

2. $$\begin{aligned} \partial_u \Pi(z, u, v) &= -\int_{\gamma_{1,z,u}} \mathrm{res}_{D_{1,z,u}|H_{1,u}} \mathrm{res}_{H_{1,u}|M} \left(\frac{\partial_u Q_{1,u}}{Q_{1,u}} \omega_z \right) \\ &= -\int_{\gamma_{1,z,u}} \mathrm{res}^{LT}_{D_{1,z,u}|M} \left(\frac{\partial_u Q_{1,u}}{Q_{1,u}} \omega_z \right), \end{aligned}$$

3. $\partial_v \Pi(z, u, v) = -\int_{\gamma_{2,z,v}} \mathrm{res}_{D_{2,z,v}|H_{2,v}} \mathrm{res}_{H_{2,v}|M} \left(\frac{\partial_v Q_{2,v}}{Q_{2,v}} \omega_z \right)$.

For the proof of Theorem 9.5.12 the following will be crucial:

9.5.13 Lemma. *([LLY12], Formula 2.10) Let* $v_1 = \frac{\partial}{\partial z} \in T_{S_1}$. *Locally for all* α *and for all* $z \in S_1$, *the following formula is satisfied:*

$$w^{\alpha}_{X_z}(v_1) \lrcorner\, d\phi_z|_{X_z} = \operatorname{res}_{X_z|M}\left(\partial_z \omega_z - d_M\left(\frac{\partial_z P_{z,\alpha}}{P_{z,\alpha}}\phi_z\right)\right). \qquad (9.5.13.1)$$

9.5.14 Remark. The right hand side of Formula 9.5.13.1 is independent of the choice of the lifting of $v_1 \in T_{S_1}$ to $w^{\alpha}_{\mathcal{X}}(v_1) \in H^0\left(V^{\alpha}_{\mathcal{X}}, T_{M\times S}|_{V^{\alpha}_{\mathcal{X}}}\right)$, but the left hand side a priori depends on the choice of the lifting. So Formula 9.5.13.1 shows that the left hand side is in fact independent of the choice of the lifting. Furthermore the local expressions on the left hand side can be put together to the global expression on the right hand side.

Proof of Lemma 9.5.13: First we verify the formula

$$n^{\alpha}_{X_z} \lrcorner\, d_M\phi_z|_{X_z} = \operatorname{res}_{X_z|M}\left(-\sum_{\alpha}\frac{\partial_z P_{z,\alpha}}{P_{z,\alpha}} d_M\phi_{z,\alpha}\right).$$

Using Lemma 9.5.8 this is equivalent to the following equation

$$\begin{aligned} -\sum_{\alpha}\frac{\partial_z P_{z,\alpha}}{P_{z,\alpha}} d_M\phi_{z,\alpha} &= \frac{d_M P_z}{P_z} \wedge \left(n^{\alpha}_{X_z} \lrcorner\, d_M\phi_{z,\alpha}|_{X_z}\right) \\ \Leftrightarrow \sum_{\alpha}\left(n^{\alpha}_{X_z} \lrcorner\, d_M P_{z,\alpha}\right) \wedge d_M\phi_{z,\alpha} &= d_M P_z \wedge \left(n^{\alpha}_{X_z} \lrcorner\, d_M\phi_{z,\alpha}|_{X_z}\right). \end{aligned}$$

The last equation holds as

$$\begin{aligned} 0 &= n^{\alpha}_{X_z} \lrcorner\, (d_M P_{z,\alpha} \wedge d_M\phi_{z,\alpha}) = \\ &= n^{\alpha}_{X_z} \lrcorner\, (d_M P_z) \wedge d_M\phi_z - d_M P_z \wedge n^{\alpha}_{X_z} \lrcorner\, (d_M\phi_z). \end{aligned}$$

Therefore

$$\begin{aligned} & w^{\alpha}_{X_z}(v_1) \lrcorner\, d\phi_z|_{X_z} = n^{\alpha}_{X_z} \lrcorner\, d_M\phi_z|_{X_z} + \frac{\partial}{\partial z} \lrcorner\, d_S\phi_z|_{X_z} = \\ & = \operatorname{res}_{X_z|M}\left(-\sum_{\alpha}\frac{\partial_z P_{z,\alpha}}{P_{z,\alpha}} d_M\phi_{z,\alpha}\right) + \operatorname{res}_{X_z|M}\left(\sum_{\alpha}\frac{d_M P_{z,\alpha}}{P_{z,\alpha}} \wedge \partial_z\phi_{z,\alpha}\right) = \\ & = \operatorname{res}_{X_z|M}\left(\partial_z\omega_z - \sum_{\alpha} d_M\left(\frac{\partial_z P_{z,\alpha}}{P_{z,\alpha}}\phi_z\right)\right). \end{aligned}$$

In the last step we made use of the equation

$$\partial_z\omega_z = \sum_{\alpha}\left(-\frac{\partial_z P_{z,\alpha}}{P_{z,\alpha}} d_M\phi_{z,\alpha} + \frac{d_M P_{z,\alpha}}{P_{z,\alpha}} \wedge \partial_z\phi_{z,\alpha} + d_M\left(\frac{\partial_z P_{z,\alpha}}{P_{z,\alpha}}\phi_z\right)\right).$$

□

Proof of Theorem 9.5.14.1, Part 1: Using the variation formula 9.5.3, we obtain

$$
\begin{aligned}
&\nabla^{GM}_{\partial_z}\left[\left(\phi_z|_{X_z},0,0\right)\right] = \\
&\left[\left(w^{\alpha}_{X_z}\left(\partial_z\right) \lrcorner\, d\left(\phi_z|_{X_z}\right), -w^{\alpha}_{\mathcal{D}_{1,z}}\left(\partial_z\right) \lrcorner\, \phi_z|_{\mathcal{D}_{1,z}},\ \ -w^{\alpha}_{\mathcal{D}_{2,z}}\left(\partial_z\right) \lrcorner\, \phi_z|_{\mathcal{D}_{2,z}}\right)\right] \\
&= \left[\left(\operatorname{res}_{X_z|M}\left(\partial_z\omega_z - d_M\left(\frac{\partial_z P_z}{P_z}\phi_z\right)\right), - n^{\alpha}_{\mathcal{X}}|_{\mathcal{D}_{1,z}} \lrcorner\, \phi_z|_{\mathcal{D}_{1,z}},\right.\right. \\
&\qquad \left.\left. - n^{\alpha}_{\mathcal{X}}|_{\mathcal{D}_{2,z}} \lrcorner\, \phi_z|_{\mathcal{D}_{2,z}}\right)\right].
\end{aligned}
$$

For $\left([\Gamma],[\gamma_1],[\gamma_1]\right) \in H_3\left(X, D_1, D_2\right)$ we obtain

$$
\begin{aligned}
&\int_{\tau(\Gamma)} d_M\left(\frac{\partial_z P_z}{P_z}\phi_z\right) = \int_{\tau(\partial\Gamma)}\left.\left(\frac{\partial_z P_z}{P_z}\phi_z\right)\right|_{\mathcal{D}_{1,z}\cup\mathcal{D}_{2,z}} \\
&= \int_{\tau(\gamma_1)}\left.\left(\frac{\partial_z P_z}{P_z}\phi_z\right)\right|_{\mathcal{D}_{1,z}} - \int_{\tau(\gamma_2)}\left.\left(\frac{\partial_z P_z}{P_z}\phi_z\right)\right|_{\mathcal{D}_{2,z}} \\
&= \int_{\gamma_1}\operatorname{res}_{\mathcal{H}_1|M}\left.\left(\frac{\partial_z P_z}{P_z}\phi_z\right)\right|_{\mathcal{D}_{1,z}} - \int_{\gamma_2}\operatorname{res}_{\mathcal{H}_2|M}\left.\left(\frac{\partial_z P_z}{P_z}\phi_z\right)\right|_{\mathcal{D}_{2,z}} \\
&= -\int_{\gamma_1} n^{\alpha}_{\mathcal{X}}|_{\mathcal{D}_{1,z}} \lrcorner\, \phi_z|_{\mathcal{D}_{1,z}} + \int_{\gamma_2} n^{\alpha}_{\mathcal{X}}|_{\mathcal{D}_{2,z}} \lrcorner\, \phi_z|_{\mathcal{D}_{2,z}}.
\end{aligned}
$$

Hence we get for the pairing of the class $\left[\left(\phi_z|_{X_z},0,0\right)\right]$ with $\left([\Gamma],[\gamma_1],[\gamma_2]\right)$ the following equalities:

$$
\begin{aligned}
&\partial_z\left\langle\left[(\Gamma,\gamma_1,\gamma_2)\right],\left[\left(\operatorname{res}_{X_z|M}\left(\omega_z\right),0,0\right)\right]\right\rangle \\
&= \left\langle\left[(\Gamma,\gamma_1,\gamma_2)\right],\nabla^{GM}_{\partial_z}\left[\left(\operatorname{res}_{X_z|M}\left(\omega_z\right),0,0\right)\right]\right\rangle \\
&= \int_{\Gamma}\operatorname{res}_{X_z|M}\left(\partial_z\omega_z - d_M\left(\frac{\partial_z P_z}{P_z}\phi_z\right)\right) + \int_{\gamma_1} n^{\alpha}_{\mathcal{X}}|_{\mathcal{D}_{1,z}} \lrcorner\, \phi_z \\
&\quad - \int_{\gamma_2} n^{\alpha}_{\mathcal{X}}|_{\mathcal{D}_{2,z}} \lrcorner\, \phi_z \\
&= \int_{\tau(\Gamma)}\left(\partial_z\omega_z - d_M\left(\frac{\partial_z P_z}{P_z}\phi_z\right)\right) + \int_{\gamma_1} n^{\alpha}_{\mathcal{X}}|_{\mathcal{D}_{1,z}} \lrcorner\, \phi_z - \int_{\gamma_2} n^{\alpha}_{\mathcal{X}}|_{\mathcal{D}_{2,z}} \lrcorner\, \phi_z \\
&= \int_{\tau(\Gamma)} \partial_z\omega_z.
\end{aligned}
$$

Thus we got

$$
\begin{aligned}
\partial_z\left\langle\left[(\Gamma,\gamma_1,\gamma_2)\right],\left[\left(\operatorname{res}_{X_z|M}\left(\omega_z\right),0,0\right)\right]\right\rangle &= \\
= \left\langle\left[(\Gamma,\gamma_1,\gamma_2)\right],\left[\left(\operatorname{res}_{X_z|M}\left(\partial_z\omega_z\right),0,0\right)\right]\right\rangle&.
\end{aligned}
$$

$\square$

Proof of Theorem 9.5.14.1, Part 2:

$$\nabla^{GM}_{\partial_u}\left[\left(\phi_z|_{X_z},0,0\right)\right]= \tag{9.5.14.1}$$

$$=\left[\left(w^{\alpha}_{X_z}\left(\partial_u\right)\lrcorner\, d\left(\phi_z|_{X_z}\right),-w^{\alpha}_{D_{1,z,u}}\left(\partial_u\right)\lrcorner\,\phi_z|_{D_{1,z,u}},\right.\right. \tag{9.5.14.2}$$

$$\left.\left.-w^{\alpha}_{D_{2,z,u}}\left(\partial_v\right)\lrcorner\,\phi_z|_{D_{2,z,u}}\right)\right]$$

$$=\left[\left(\partial_u\lrcorner\, d\left(\phi_z|_{X_z}\right),-\left(\partial_u|_{V^{\alpha}_{\mathcal{D}_1}}+n^{\alpha}_{\mathcal{D}_1}\right)\lrcorner\,\phi_z|_{D_{1,z,u}},-\partial_v\lrcorner\,\phi_z|_{D_{2,z,u}}\right)\right]$$

$$=\left[\left(0,-n^{\alpha}_{\mathcal{D}_1}\lrcorner\,\phi_z|_{D_{1,z,u}},0\right)\right].$$

As

$$n^{\alpha}_{\mathcal{D}_1}\lrcorner\left(d_M Q_{1,u}\wedge\phi_z|_{X_z}\right)=0,$$

we get using Lemma 9.5.10

$$\frac{d_M Q_{1,u}}{Q_{1,u}}\wedge\left(n^{\alpha}_{\mathcal{D}_1}\lrcorner\,\phi_z|_{X_z}\right)\Big|_{D_{1,z,u}}=\frac{n^{\alpha}_{\mathcal{D}_1}\lrcorner\, d_M Q_{1,u}}{Q_{1,u}}\wedge\phi_z|_{X_z}=-\frac{\partial_u Q_{1,u}}{Q_{1,u}}\wedge\phi_z|_{X_z}.$$

Thus we have

$$\begin{aligned}\left(n^{\alpha}_{\mathcal{D}_1}\lrcorner\,\phi_z|_{X_z}\right)\Big|_{D_{1,z,u}} &= -\operatorname{res}_{D_{1,z,u}|X_z}\left(\frac{\partial_u Q_{1,u}}{Q_{1,u}}\phi_z\right)\Big|_{X_z}=\\ &= -\operatorname{res}_{D_{1,z,u}|X_z}\operatorname{res}_{X_z|M}\left(\frac{\partial_u Q_{1,u}}{Q_{1,u}}\omega_z\right)\Big|_{X_z}=\\ &= -\operatorname{res}^{LT}_{D_{1,z,u}|M}\left(\frac{\partial_u Q_{1,u}}{Q_{1,u}}\omega_z\right)\Big|_{X_z},\end{aligned}$$

therefore

$$\nabla^{GM}_{\partial_u}\left[\left(\operatorname{res}_{X_z|M}\left(\omega_z\right),0,0\right)\right]=\left[\left(0,\operatorname{res}^{LT}_{D_{1,z,u}|M}\left(\frac{\partial_u Q_{1,u}}{Q_{1,u}}\omega_z\right)\Big|_{X_z},0\right)\right].$$

Thus

$$\partial_u\Pi\left(z,u,v\right)=\int_{\gamma_1}\operatorname{res}^{LT}_{D_{1,z,u}|M}\left(\frac{\partial_u Q_{1,u}}{Q_{1,u}}\omega_z\right)\Big|_{X_z}.$$

Analogously we get

$$\nabla^{GM}_{\partial_v}\left[\left(\operatorname{res}_{X_z|M}\left(\omega_z\right),0,0\right)\right]=\left[\left(0,0,-n^{\alpha}_{D_{2,z,v}}\lrcorner\,\phi_z|_{D_{2,z,v}}\right)\right]=$$

$$=\left[\left(0,0,\operatorname{res}^{LT}_{D_{2,z,v}|M}\left(\frac{\partial_v Q_{2,v}}{Q_{2,v}}\omega_z\right)\Big|_{X_z}\right)\right]$$

and

$$\partial_v \Pi(z,u,v) = \int_{\gamma_2} \operatorname{res}^{LT}_{D_{2,z,v}|M} \left(\frac{\partial_v Q_{2,v}}{Q_{2,v}} \omega_z \right).$$

□

For the periods of the form $\left(\operatorname{res}_{X_z|M}\omega_z, 0, 0\right)$ we obtain the following relations. We obtain by direct computations:

9.5.15 Corollary.

1. $\partial_z^k \left\langle (\Gamma,\gamma_1,\gamma_2), \left(\operatorname{res}_{X_z|M}\omega_z, 0, 0\right)\right\rangle = \left\langle (\Gamma,\gamma_1,\gamma_2), \left(\operatorname{res}_{X_z|M}\left(\partial_z^k\omega_z\right), 0, 0\right)\right\rangle$
2. $\partial_z^k\partial_u \left\langle (\Gamma,\gamma_1,\gamma_2), \left(\operatorname{res}_{X_z|M}\omega_z, 0, 0\right)\right\rangle = \int_{\gamma_1} \operatorname{res}^{LT}_{D_{1,z,u}|M}\left(-\frac{\partial_u Q_1}{Q_1}\partial_z^k\left(\frac{1}{P}\right)\Delta\right)$
3. $\partial_z^k\partial_v \left\langle (\Gamma,\gamma_1,\gamma_2), \left(\operatorname{res}_{X_z|M}\omega_z, 0, 0\right)\right\rangle = \int_{\gamma_2} \operatorname{res}^{LT}_{D_{2,z,v}|M}\left(-\frac{\partial_v Q_2}{Q_2}\partial_z^k\left(\frac{1}{P}\right)\Delta\right)$
4. $\partial_v\partial_u \left\langle (\Gamma,\gamma_1,\gamma_2), \left(\operatorname{res}_{X_z|M}\omega_z, 0, 0\right)\right\rangle = 0$
5. $\partial_v\partial_z^k\partial_u \left\langle (\Gamma,\gamma_1,\gamma_2), \left(\operatorname{res}_{X_z|M}\omega_z, 0, 0\right)\right\rangle = 0.$

9.5.16 Remark. It should be noticed that with the triple method one can not automatically reach the full cohomology $H^3(X, D_1, D_2, \mathbb{C})$. The problem comes from the difference of $\left(H^2(D_1) \oplus H^2(D_2)\right)_{var}$ and $\left(H^2_{var}(D_1)\right) \oplus \left(H^2_{var}(D_2)\right)$. If $\dim H^2(X,\mathbb{C}) = 1$, the dimensions of these spaces differ by 1.

A way out might be to choose the variables not independent; a possible approach is discussed in Section 9.7.

9.6 An Example

We are now going to compute Picard-Fuchs operators in a specific example. Let X be again the quintic as in Section 8.5. Furthermore, let

$$H_{1,u} := \{Q_{1,u} = 0\} \subset \mathbb{P}^4, \text{ where } Q_{1,u} := x_1^4 - u\; x_2x_3x_4x_5$$

and

$$H_{2,v} := \{Q_{2,v} = 0\} \subset \mathbb{P}^4, \text{ where } Q_{2,v} := x_2^4 - v\; x_1x_3x_4x_5.$$

Furthermore

$$D_{1,z,u} := H_{1,u} \cap X_z \text{ and } D_{2,z,v} := H_{2,v} \cap X_z.$$

We use the Singular programme in Appendix A.2. The programme is set up individually for the pairs $(X_z, D_{1,z,u})$ and $(X_z, D_{2,z,v})$.
The result can be seen as the outcome for a Picard-Fuchs equation of (X, C), where C is a complete intersection of two linearly equivalent divisors on X.

We obtain the following result:

We get the following matrices

$$M_z \;:=\; \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{z}{D_1} & \frac{15z^2}{D_1} & \frac{25z^3}{D_1} & \frac{10z^4}{D_1} & \frac{A_1}{A} & \frac{A_2}{B} & \frac{A_3}{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{R_1}{B_1} & \frac{R_2}{B_2} & \frac{R_3}{B_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{R_4}{B_3} & \frac{R_5}{B_4} & \frac{R_6}{B_3} \end{pmatrix}$$

using the following abbreviations

$$\begin{aligned}
D_1 &:= 1 - z^5, \\
A &:= 16\,(z-1)\left(z^4+z^3+z^2+z+1\right)(-5z+u)\,(-z+u)^3 \\
&\quad \left(625z^4u - 500z^3u^2 + 150z^2u^3 - 20zu^4 + u^5 - 256\right), \\
B &:= -16\,(z-1)\left(z^4+z^3+z^2+z+1\right)(-z+u)^3 \\
&\quad \left(625z^4u - 500z^3u^2 + 150z^2u^3 - 20zu^4 + u^5 - 256\right), \\
C &:= 16\,(-z+u)\left(-3125z^{10}u + 3125z^9u^2 - 1250z^8u^3 + 250z^7u^4\right. \\
&\quad +250z^6u^5 + 276z^5u^6 + 275z^4u^7 + 275z^3u^8 + 275z^2u^9 + 275zu^{10} \\
&\quad +275u^{11} + 1280z^6 + 2869z^5u - 3125z^4u^2 + 1250z^3u^3 - 250z^2u^4 \\
&\quad \left.+25zu^5 - u^6 - 1280z + 256u\right) \\
A_1 &:= u\left(-58750000z^{14}u + 43406250z^{13}u^2 + 40462500z^{12}u^3 - 66332500z^{11}u^4\right. \\
&\quad +38288500z^{10}u^5 + 24064000z^{10} - 12356000z^9u^6 + 29689250z^9u \\
&\quad +2454700z^8u^7 - 30692825z^8u^2 - 301100z^7u^8 - 13988825z^7u^3 \\
&\quad +21100z^6u^9 + 20841875z^6u^4 - 650z^5u^{10} - 8572425z^5u^5 \\
&\quad -11665920z^5 + 1786261z^4u^6 - 4945856z^4u - 201083z^3u^7 \\
&\quad +8076288z^3u^2 + 11633z^2u^8 - 1388928z^2u^3 - 261zu^9 + 137216zu^4 \\
&\quad \left.-43200u^5\right), \\
A_2 &:= u\left(-3875000z^{14}u - 18181250z^{13}u^2 + 44957500z^{12}u^3 - 40223500z^{11}u^4\right. \\
&\quad +19534300z^{10}u^5 + 1587200z^{10} - 5812400z^9u^6 + 10573350z^9u \\
&\quad +1100980z^8u^7 - 2272455z^8u^2 - 130740z^7u^8 - 13145655z^7u^3 \\
&\quad +8940z^6u^9 + 11217965z^6u^4 - 270z^5u^{10} - 4101527z^5u^5 \\
&\quad -869888z^5 + 812235z^4u^6 - 4091200z^4u - 88373z^3u^7 \\
&\quad +3371008z^3u^2 + 4879z^2u^8 - 337024z^2u^3 - 99zu^9 \\
&\quad \left.-29696zu^4 - 9280u^5\right),
\end{aligned}$$

$$
\begin{aligned}
A_3 &:= 19687500z^{14}u^2 - 35656250z^{13}u^3 + 28812500z^{12}u^4 - 13662500z^{11}u^5 \\
&\quad +4197500z^{10}u^6 - 8104000z^{10}u - 867800z^9u^7 - 1241650z^9u^2 \\
&\quad +120860z^8u^8 + 10107315z^8u^3 - 10940z^7u^9 - 7664925z^7u^4 \\
&\quad +584z^6u^{10} + 2869807z^6u^5 + 16384z^6 - 14z^5u^{11} \\
&\quad -632773z^5u^6 + 3419712z^5u + 85241z^4u^7 - 1335936z^4u^2 - 6823z^3u^8 \\
&\quad -294272z^3u^3 + 293z^2u^9 + 183552z^2u^4 - 5zu^{10} - 24256zu^5 \\
&\quad -16384z + 896u^6 + 16384u, \\
R_1 &:= -25u\left(165z^2 - 10zu - 27u^2\right), \\
R_2 &:= -5u\left(835z^2 - 230zu - 29u^2\right), \\
R_3 &:= -2\left(8875z^4u - 6800z^3u^2 + 1830z^2u^3 - 200zu^4 + 7u^5 + 128\right), \\
B_1 &:= -\left(-5z + u\right)\left(625z^4u - 500z^3u^2 + 150z^2u^3 - 20zu^4 + u^5 - 256\right), \\
B_2 &:= 625z^4u - 500z^3u^2 + 150z^2u^3 - 20zu^4 + u^5 - 256, \\
R_4 &:= -25v\left(165z^2 - 10zv - 27v^2\right), \\
R_5 &:= -5v\left(835z^2 - 230zv - 29v^2\right), \\
R_6 &:= -17750z^4v + 13600z^3v^2 - 3660z^2v^3 + 400zv^4 - 14v^5 - 256, \\
B_3 &:= -\left(-5z + v\right)\left(625z^4v - 500z^3v^2 + 150z^2v^3 - 20zv^4 + v^5 - 256\right), \\
B_4 &:= 625z^4v - 500z^3v^2 + 150z^2v^3 - 20zv^4 + v^5 - 256,
\end{aligned}
$$

We receive the equations

$$
M_u := \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{R_1}{B_1} & \frac{R_2}{B_2} & \frac{R_3}{B_1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{3}{4u} & \frac{z-u}{4u} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2u} & \frac{z-u}{4u} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{F_1}{G_1} & \frac{F_2}{G_2} & \frac{F_3}{G_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\begin{aligned}
F_1 &:= -25\left(z - u\right)\left(165z^2 - 10zu - 27u^2\right), \\
F_2 &:= -5\left(z - u\right)\left(835z^2 - 230zu - 29u^2\right), \\
F_3 &:= -20875z^5u + 34475z^4u^2 - 18510z^3u^3 + 4310z^2u^4 - 439zu^5 \\
&\quad +1024z + 15u^6, \\
G_1 &:= -4\left(-5z + u\right)\left(625z^4u - 500z^3u^2 + 150z^2u^3 - 20zu^4 + u^5 - 256\right), \\
G_2 &:= 4\left(625z^4u - 500z^3u^2 + 150z^2u^3 - 20zu^4 + u^5 - 256\right), \\
G_3 &:= -4u\left(-5z + u\right)\left(625z^4u - 500z^3u^2 + 150z^2u^3 - 20zu^4 + u^5 - 256\right).
\end{aligned}
$$

$$M_v := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{R_4}{B_3} & \frac{R_5}{B_4} & \frac{R_6}{B_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{3}{4v} & \frac{z-v}{4v} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2v} & \frac{z-v}{4v} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{H_1}{G_4} & \frac{H_2}{G_5} & \frac{H_3}{G_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
H_1 &:= -25\,(z-v)\left(165z^2 - 10zv - 27v^2\right) \\
H_2 &:= -5\,(z-v)\left(835z^2 - 230zu - 29u^2\right) \\
H_3 &:= -20875z^5v + 34475z^4v^2 - 18510z^3v^3 + 4310z^2v^4 - 439zv^5 \\
&\quad +1024z + 15v^6, \\
G_4 &:= -4\,(-5z+v)\left(625z^4v - 500z^3v^2 + 150z^2v^3 - 20zv^4 + v^5 - 256\right) \\
G_5 &:= 4\left(625z^4v - 500z^3v^2 + 150z^2v^3 - 20zv^4 + v^5 - 256\right) \\
G_6 &:= -4v\,(-5z+v)\left(625z^4v - 500z^3v^2 + 150z^2v^3 - 20zv^4 + v^5 - 256\right)
\end{aligned}$$

We obtain the following differential equations:

1. $\partial_z^4 = \frac{z}{D_1} + \frac{15z^2}{D_1}\partial_z + \frac{25z^3}{D_1}\partial_z^2 + \frac{10z^4}{D_1}\partial_z^3 + \frac{A_1}{A}\partial_u + \frac{A_2}{B}\partial_z\partial_u + \frac{A_3}{C}\partial_z^2\partial_u;$
2. $\partial_z^3\partial_u = \frac{R_1}{B_1}\partial_u + \frac{R_2}{B_2}\partial_z\partial_u + \frac{R_3}{B_1}\partial_z^2\partial_u;$
3. $\partial_z^3\partial_v = \frac{R_4}{B_3}\partial_v + \frac{R_5}{B_4}\partial_z\partial_v + \frac{R_6}{B_3}\partial_z^2\partial_v;$
4. $\partial_u^2 = -\frac{3}{4u}\partial_u + \frac{z-u}{4u}\partial_z\partial_u;$
5. $\partial_u\partial_z\partial_u = -\frac{1}{2u}\partial_z\partial_u + \frac{z-u}{4u}\partial_z^2\partial_u;$
6. $\partial_u\partial_z^2\partial_u = \frac{F_1}{G_1}\partial_u + \frac{F_2}{G_2}\partial_z\partial_u + \frac{F_3}{G_3}\partial_z^2\partial_u;$
7. $\partial_v^2 = -\frac{3}{4v}\partial_v + \frac{z-v}{4v}\partial_z\partial_v;$
8. $\partial_v\partial_z\partial_v = -\frac{1}{2v}\partial_z\partial_v + \frac{z-v}{4v}\partial_z^2\partial_v;$
9. $\partial_v\partial_z^2\partial_v = \frac{H_1}{G_4}\partial_v + \frac{H_2}{G_5}\partial_z\partial_v + \frac{H_3}{G_6}\partial_z^2\partial_v;$

Not all all these equations are independent. Equation 5, 6, 8, 9 can be derived from the others. We obtain the following differential operators:

$$\begin{aligned}
\mathcal{L}_1 &= \mathcal{L}^{\text{bulk}} + \mathcal{L}^{\text{bdry}}; \\
\mathcal{L}_2 &= 4zu\partial_u^2 + 3z\partial_u - z(z-u)\,\partial_z\partial_u; \\
\mathcal{L}_3 &= -\frac{F_1}{G_1}\partial_u - \frac{F_2}{G_2}\partial_z\partial_u - \frac{F_3}{G_3}\partial_z^2\partial_u + \partial_u^2\partial_z^2; \\
\mathcal{L}_4 &= -\frac{H_1}{G_4}\partial_v - \frac{H_2}{G_5}\partial_z\partial_v - \frac{H_3}{G_6}\partial_z^2\partial_v + \partial_v^2\partial_z^2;
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{L}_{\text{bulk}} &= \left(1-z^5\right) z^4\partial_z^4 - 10z^8\partial_z^3 - 25z^7\partial_z^2 - 15z^6\partial_z - z^5 \\
&= \theta_z\left(\theta_z-1\right)\left(\theta_z-2\right)\left(\theta_z-3\right) - z^5\left(\theta_z+1\right)^4,
\end{aligned}$$

using the logarithmic derivative $\theta_z := z\partial_z$, and

$$\mathcal{L}^{\text{bdry}} = -\frac{A_1D_1z^4}{A}\partial_u - \frac{A_2D_1z^4}{B}\partial_z\partial_u - \frac{A_3D_1z^4}{C}\partial_z^2\partial_u.$$

9.7 A modified example

As indicated in Remark 9.5.16, the deformations of D_1 and D_2 should be linked. To be specific, let X be a quintic as in Section 8.5.

We are now going to compute Picard-Fuchs operators in a specific example. Let X be again the quintic as in Section 8.5. Furthermore, let z and u be independent parameters and

$$H_{1,z,u} := \{Q_{1,z,u} = 0\} \subset \mathbb{P}^4, \text{ where } Q_{1,z,u} := x_1^4 - (z-u)\ x_2x_3x_4x_5$$

and

$$H_{2,z,u} := \{Q_{2,z,u} = 0\} \subset \mathbb{P}^4, \text{ where } Q_{2,z,u} := x_2^4 - (z+u)\ x_1x_3x_4x_5.$$

Furthermore

$$D_{1,z,u} := H_{1,u} \cap X_z \text{ and } D_{2,z,v} := H_{2,v} \cap X_z.$$

This will carried out in further research. The method of Li, Lian and Yau has to be modified suitably.

Appendix A

Implementation of the algorithms as Singular programmes

A.1 A programme for computing the Picard-Fuchs equation of a complete intersection Calabi-Yau 3-fold

In this appendix we give a programme written in the Singular language [DGPS16] for the calculation of the Picard-Fuchs equation for the periods of a complete intersection Calabi-Yau $(n-3)$-fold in $\mathbb{P}^{n-1}$ defined by two homogeneous equations in $\mathbb{P}^{n-1}$ depending on one parameter a.

The programme is applied to the complete intersection Calabi-Yau 3-fold in $\mathbb{P}^5$ given by the two cubic polynomials considered in Example 7.3.26 in Chapter 7 respectively in [LT93].

We review the situation: Let $Q_1(\lambda), Q_2(\lambda) \in H^0\left(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)\right)$ be the following two homogeneous cubic polynomials on $\mathbb{P}^5$ depending on a parameter $\lambda = a \in \mathbb{C}$:

$$\begin{aligned} Q_1(\lambda) &:= x_1^3 + x_2^3 + x_3^3 - 3\lambda\, x_4x_5x_6, \\ Q_2(\lambda) &:= x_4^3 + x_5^3 + x_6^3 - 3\lambda\, x_1x_2x_3. \end{aligned}$$

For any details we refer to Example 7.3.26 in Chapter 7.

We briefly explain the programme:

```
LIB "general.lib";

int n = 6;
// n-1 is the dimension of the ambient projective space

int c = 2;
```

```
// c is the codimension of the Calabi-Yau manifold in the
// projective space, here c=2, i.e., the Calabi-Yau
// manifold is the intersection of two hypersurfaces

ring ra = 0,(a,x(1..n),f(1..c)),dp;
// the auxilliary ring ra is generated by a parameter a,
// the coordinates of the projective space x(1),...x(n)
// and two generators f(1), f(2) which correspond to the
// inverted homogeneous polynomials defining the
// Calabi-Yau manifold.

ideal fa = x(1)^3+x(2)^3+x(3)^3-3*a*x(4)*x(5)*x(6),
x(4)^3+x(5)^3+x(6)^3-3*a*x(1)*x(2)*x(3);
// fa is given by the two homogeneous polynomials
// defining the Calabi-Yau manifold

// For the following procedure see Remark 0.1.1

proc theta(poly g)
 {
  proc th(poly f)
   {
    return(a*diff(f,a));
   }
  poly p=0;
  for(int j=1;j<=size(g);j=j+1)
   {
    matrix cf=coeffs(g[j],f(1));
    intvec k = size(cf)-1;
    poly h=cf[k[1]+1,1];
    int i;
    for(i=2;i<=c;i=i+1)
     {
      cf=coeffs(h,f(i));
      k=intvec(k,size(cf)-1);
      h=cf[k[i]+1,1];
     }
    p = p-k[1]*h*th(fa[1])*f(1)^(k[1]+1)*f(2)^k[2]
         +th(h)*f(1)^k[1]*f(2)^k[2]
         -k[2]*h*th(fa[2])*f(1)^k[1]*f(2)^(k[2]+1);
   }
  return(p);
 }


// The following procedure associates to two matrices A
// and B with the same number of rows a matrix (A,B)
// whose number of columns is the added number of
// columns of A and B and whose number of rows coincides
```

```
// with that of A and B. It is defined by concatenating
// the rows of A and B. If the number of rows of A and B
// does not coincide, then the 1x1-matrix 0 is returned.

proc concat(matrix A, matrix B)
 {
  int rA = nrows(A);
  int cA = ncols(A);
  int rB = nrows(B);
  int cB = ncols(B);
  int i,j;
  if(rA!=rB)
   {return(0);}
  else
   {
    ideal C=A[1,1];
    for(i=1;i<=rA;i=i+1)
     {
      for(j=1;j<=cA;j=j+1)
       {
        if (i!=1 || j!=1)
        {
          C=C,A[i,j];
        }
       }
      for(j=1;j<=cB;j=j+1)
       {
        C=C,B[i,j];
       }
     }
    return(matrix(C,rA,cA+cB));
   }
 }


// The following procedure associates to an integer n the
// 1xn-matrix whose entries are all 0.

proc zero(int n)
 {
  intvec z=0;
  int i;
  for(i=2;i<=n;i=i+1)
  {
    z=z,0;
  }
  return(z);
 }
```

```
// The following procedure associates to a polynomial g
// in the ring ra the maximal total degree in f(1) and f(2)
// of all summands of g. Only summands with at least power
// one in both f(1) and f(2) are taken into account. If
// there are no summands with power at least one in both
// f(1) and f(2), then 0 is returned.

proc fdegree(poly g)
 {
  matrix cf=coeffs(g,f(1));
  intvec k = size(cf)-1;
  poly h=cf[k[1]+1,1];
  int i;
  for(i=2;i<=c;i=i+1)
  {
   cf=coeffs(h,f(i));
   k=intvec(k,size(cf)-1);
   h=cf[k[i]+1,1];
  }
  return(int(sum(k)));
 }


poly p = f(1)*f(2);

int m=4;
// m is the dimension of the (n-2)-th cohomology of the
// Calabi-Yau (n-2)-fold.

poly p(1)=theta(f(1)*f(2));

for(int k=2;k<=m;k=k+1)
 {
  poly p(k)=theta(p(k-1));
 }


// For the following procedure see Remark 0.1.2

proc maxvectf(poly p)
 {
  matrix cf=coeffs(p,f(1));
  matrix Maxvectf[size(cf) -1][1];
  for(int i=1;i<=size(cf)-1;i=i+1)
   {
    if(size(coeffs(cf[size(cf)-(i-1),1],f(2)))
      >=fdegree(p)-size(cf)+i+1)
     {Maxvectf[i,1]=coeffs(cf[size(cf)-(i-1),1],f(2))[i+1,1];}
```

```
    else
     {Maxvectf[i,1]=0;}
   }
  return(Maxvectf);
 }


// The following procedure determines the summands of a
// polynomial p in ra with maximal total degree in f(1)
// and f(2).

proc maxpolf(poly p)
 {
  matrix cf=coeffs(p,f(1));
  poly max=0;
  for(int i=1;i<=size(cf);i=i+1)
   {
    if(fdegree(p)-i+2<=size(coeffs(cf[i,1],f(2))))
     {max=max+(coeffs(cf[i,1],f(2))[fdegree(p)-i+2,1])
      *f(1)^(i-1)*f(2)^(fdegree(p)-(i-1));}
    else
     {max=max;}
   }
  return(max);
 }


// The following procedure associates to a polynomial g
// in ra a list of polynomials whose first element is g,
// the second element is the polynomial without all
// summands of the highest total degree in f(1) and f(2).
// Each following entry is the preceding one minus all
// summands of highest degree.

proc Slist(poly g)
 {
  int k=fdegree(g);
  poly S(0)=g;
  for(int l=1;l<=k-1;l=l+1)
   {
    poly S(l)=S(l-1)-maxpolf(S(l-1));
   }
  list H;
  for(l=1;l<=k-1;l=l+1)
   {
    H[l]=S(l);
   }
  return(H);
 }
```

```
// Let k be the maximal total degree in f(1) and f(2).
// The following procedure associates to a polynomial g
// in ra the list of polynomials whose i-th entry
// consists of all summands of degree k-i.

proc Mlist(poly g)
 {
  int k=fdegree(g);
  list H=Slist(g);
  list G;
  for(int l=1;l<=k-1;l=l+1)
   {
    if(size(H[l])!=0)
     {G[l]=maxvectf(H[l]);}
    else
     {G[l]=0;}
   }
  return(G);
 }


// Let A be the list of lists of polynomials in ra
// whose l-th entry is the list Mlist associated to p(m-l).

list A;
for(int l=1;l<=m-1;l=l+1)
 {
  A[l]=Mlist(p(m-l));
 }

// Let A(0) be the list of polynomials in ra
// whose l-th entry is the polynomial maxvectf(p(m-l)).

list A(0);
for(int l=1;l<=m-1;l=l+1)
 {
  A(0)[l]=maxvectf(p(m-l));
 }

list D;
D=Mlist(p(m));

for(int i=1;i<=m;i=i+1)
 {
  matrix K(i)=maxvectf(p(i));
 }

```

```
ideal ja = subst(p(1)[1],f(1),1,f(2),1);

for(int i=2;i<=size(p(1));i=i+1)
 {
  ja = ja, subst(p(1)[i],f(1),1,f(2),1);
 }


// Change of the ring:

ring r = (0,a),x(1..n),dp;

list A=imap(ra,A);
list A(0)=imap(ra,A(0));
list D=imap(ra,D);

for(int i=1;i<=m;i=i+1)
 {
  matrix P(i)=imap(ra,K(i));
 }

ideal f = imap(ra,fa);
ideal j = imap(ra,ja);

matrix z = matrix(zero(n));

int i,i1,i2,k,l;

for(int i=1;i<=m;i=i+1)
 {
  matrix J(i)[size(j^i)][1]=0;
  J(i)=J(i)+matrix(j^i,size(j^i),1);
 }


// See Remark 0.1.3

for(k=2;k<=n-1;k=k+1)
 {
  matrix E[k][k]; E = E + 1;
  if(k<n-1) {ideal jk = j^(k-1);}
  ideal B=(k-1)*jacob(f[1]);
  for(i1=1;i1<=k;i1=i1+1)
   {
    for(i2=1;i2<=k-1;i2=i2+1)
     {
      if(i1==i2)
       {
        if(i1>1) {B=B,(k-i1)*jacob(f[1]);}
```

```
        }
        else
        {
          if (i1==i2+1) {B=B,i2*jacob(f[2]);}
          else {B=B,z;}
        }
      }
    }
    if(k<n-1)
    {
      matrix M(n-k) =
      matrix(concat(concat(concat(matrix(j^(k-1),k,1),
      matrix(B,k,(k-1)*n)),f[1]*E),f[2]*E),k,1+(k-1)*n+c*k);
    }
    else
    {
      matrix M(1) =
      matrix(concat(concat(matrix(B,k,(k-1)*n),f[1]*E),
      f[2]*E),k,(k-1)*n+c*k);
    }
  }


matrix P=P(m);
list Q;


// The following division and reduction process is
// explained in Chapter 7.3.4.

matrix V[m+1][1];
for(l=1;l<=m;l=l+1)
 {
  Q = division(P,M(l));
  matrix P[m+1-l][1]=zero(m+1-l);
  if(l==1)
   {
    for(k=1;k<=m;k=k+1)
     {
      for(i=1;i<=n;i=i+1)
       {
        P[k,1]=P[k,1]+diff(Q[1][i+n*(k-1),1],x(i));
       };
      P[k,1]=P[k,1]+Q[1][n*m+k,1]+Q[1][n*(m+1)+k,1];
     };
   }
  else
   {
    for(k=1;k<=m+1-l;k=k+1)
```

```
        {
         for(i=1;i<=n;i=i+1)
          {
           P[k,1]=P[k,1]+diff(Q[1][i+n*(k-1)+1,1],x(i));
          };
         P[k,1]=P[k,1]+Q[1][n*(m+1-l)+1+k,1]
                +Q[1][n*(m+1-l)+1+(n+1-l)+k,1];
        V[m+1-l,1]=Q[1][1,1];
        V[m+1-l,1];
       }
    }
  }

V[m+1,1]=P[1,1];

// In the following part the coefficients of the
// differential operators in the Picard-Fuchs equation
// are determined (see Chapter 7.3.4):

list F;
list Q(1)=division((V[m-1,1]*J(m-1)+D[1]),A(0)[1]);
poly q(1)=poly(Q(1)[1][1,1]);
for(int b=2;b<=m-1;b=b+1)
 {
  matrix F(b)=zero(m-b+1);
  for(int v=1;v<=b-1;v=v+1)
   {
    F(b)=F(b)+q(v)*A[v][b-v];
   }
  list Q(b)=division(V[m-b,1]*J(m-b)+D[b]-F(b),A(0)[b]);
  poly q(b)=poly(Q(b)[1][1,1]);
  F[b]=F(b);
 }

for(int b=1;b<=m-1;b=b+1)
 {
  Q(b);
 }
V[m+1,1];


// The Picard-Fuchs equation is:
// \Theta^m_{lambda} - Q(1)*\Theta^{m-1}_{\lambda} -
// \ldots - Q(m-1)*\Theta^1_{\lamda} - V[m+1,1]
```

1.1.1 Remark. The procedure *theta* associates to a polynomial $f = \sum_{i,j\geqslant 1} g_{i,j} f_1^i f_2^j$ in the ring *ra* (then each $g_{i,j}$ a polynomial in the variable

$x_1, \ldots, x_n$ depending on the parameter a)

$$\Theta_a \left(\sum_{i,j \geqslant 1} g_{i,j} f_1^i f_2^j \right),$$

written in the ring ra by replacing Q_1^{-i} by f_1^i and Q_2^{-j} by f_2^j.

1.1.2 Remark. The procedure *maxvectf* associates to a polynomial p in the ring ra the summands of maximal total degree d in f_1 and f_2. These summands are written in a vector whose first entry consists of the factor of $f_1^d f_2$, the second entry consists of the factor of $f_1^{d-1} f_2^2$ and so on, until the last entry consists of the factor of $f_1 f_2^d$.

E.g., let $p = Af_1^2 f_2 + Bf_1 f_2^2 + Cf_1 f_2$ be a polynomial, where A, B, C are polynomials in the variables $x_1, \ldots, x_n$ depending on the parameter a. Then

$$\text{maxvectf}\,(p) := \quad \begin{pmatrix} A \\ B \end{pmatrix}.$$

1.1.3 Remark. In the following we define matrices $M(1)$ and $M(n-k)$ such that $M(1)$ coincides with the matrix K_n defined in Section 7.3.4 and $M(n-k)$ coincides with the matrix $\tilde{K}_{k+1}$ defined in Chapter 7.3.5. The vector $((p+1) \times 1)$-matrix $\tilde{\varrho}_{p+1} \left(\sum_{j=1}^{p+1} \frac{P^{(p)}_{p+2,j}}{Q_1^j Q_2^{p+2-j}} \Delta \right) \in S^{\oplus(p+1)}$ for $p = 1, \ldots, m-1$ is replaced by the matrices $J(i)$ for $i = 1, \ldots, m$, since the basis elements $theta(p(k)), k = 1, \ldots, m-1$ are given by the Jacobian ideal of $p(1)$.

A.2 A programme for calculating a Hodge number

This appendix contains a Singular programme ([DGPS16]) for calculating the dimension of the variational cohomology $H^{1,1}_{var}(D)$ in Lemma 8.5.13. We use the notation of Lemma 8.5.13.

This comes down to calculate the G-invariant part of the kernel of the map Ψ_3^{LT}. We aim to determine the dimension of the space of all pairs of homogeneous polynomials (R_1, R_2) with

$$R_1 \in H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(8)\right)^{G,1} = \left\langle x_5^8,\ x_1^2x_2^2x_3^2x_4^2,\ x_1x_2x_3x_4x_5^4\right\rangle$$

and

$$R_2 \in H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(9)\right)^{G,2} = \left\langle x_5^9,\ x_1^2x_2^2x_3^2x_4^2x_5,\ x_1x_2x_3x_4x_5^5\right\rangle$$

such that $(R_1, R_2) \in \operatorname{im}(K_3)$. Therefore we search homogeneous polynomials $A_1, \ldots, A_5, A_8 \in H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(5)\right), A_6, A_9 \in H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(4)\right)$ and $A_7 \in H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)\right)$ such that

$$R_1 = \sum_{i=1}^{5} \frac{\partial Q}{\partial x_k} A_k + PA_7 + QA_9$$

and

$$R_2 = \sum_{i=1}^{5} \frac{\partial P}{\partial x_k} A_k + PA_6 + QA_8.$$

We give some explanations for the following programme for Singular. For $k = 1, \ldots, 5, 8$ we write

$$A_k = \sum_{i_1+i_2+i_3+i_4+i_5=5} a^{(k)}_{i_1,i_2,i_3,i_4,i_5}\ x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4}x_5^{i_5}$$

for coefficients $a^{(k)}_{i_1+i_2+i_3+i_4+i_5} \in \mathbb{C}$, furthermore

$$A_k = \sum_{i_1+i_2+i_3+i_4+i_5=4} a^{(k)}_{i_1,i_2,i_3,i_4,i_5}\ x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4}x_5^{i_5}$$

for $k = 6, 9$ and

$$A_k = \sum_{i_1+i_2+i_3+i_4+i_5=3} a^{(k)}_{i_1,i_2,i_3,i_4,i_5}\ x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4}x_5^{i_5}$$

for $k = 7$.

We define a (2×937)-matrix M whose entries are the factors of the coefficients $a^{(k)}_{i_1,i_2,i_3,i_4,i_5}$ in the polynomials A_k. Furthermore, six entries of M consist of the generators of $H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(8)\right)^{G,1}$ and $H^0\left(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(9)\right)^{G,2}$

such that the kernel of M determines all possibilities for (R_1, R_2) with the required properties.

In the sequel we calculate all possibilities for (R_1, R_2) we have obtained. We obtain five independent solutions.

```
LIB "general.lib";
ring r = (0,b,c),x(1..5),dp;
int z=0;

poly P=x(1)^5+x(2)^5+x(3)^5+x(4)^5+x(5)^5
       -5*b*x(1)*x(2)*x(3)*x(4)*x(5);
poly Q=x(5)^4-c*x(1)*x(2)*x(3)*x(4);

matrix M[2][937];
for(int k=1;k<=5;k=k+1){
 for(int i1=0;i1<=5;i1=i1+1){
  for(int i2=0;i2<=5-i1;i2=i2+1){
   for(int i3=0;i3<=5-i1-i2;i3=i3+1){
    for(int i4=0;i4<=5-i1-i2-i3;i4=i4+1){
     z=z+1;
     M[1,z]=diff(P,x(k))*x(1)^i1*x(2)^i2*x(3)^i3*x(4)^i4
               *x(5)^(5-i1-i2-i3-i4);
    } //for-loop i4
   } //for-loop i3
  } //for-loop i2
 } //for-loop i1
} //for-loop k
//for-loop for A(6):
for(int i1=0;i1<=4;i1=i1+1){
  for(int i2=0;i2<=4-i1;i2=i2+1){
   for(int i3=0;i3<=4-i1-i2;i3=i3+1){
    for(int i4=0;i4<=4-i1-i2-i3;i4=i4+1){
     z=z+1;
     M[1,z]=P*x(1)^i1*x(2)^i2*x(3)^i3*x(4)^i4
               *x(5)^(4-i1-i2-i3-i4);
    } //for-loop i4
   } //for-loop i3
  } //for-loop i2
 } //for-loop i1
//for-loop for A(7):
for(int i1=0;i1<=3;i1=i1+1){
  for(int i2=0;i2<=3-i1;i2=i2+1){
   for(int i3=0;i3<=3-i1-i2;i3=i3+1){
    for(int i4=0;i4<=3-i1-i2-i3;i4=i4+1){
     z=z+1;
     M[1,z]=0;
    } //for-loop i4
   } //for-loop i3
  } //for-loop i2
 } //for-loop i1
```

```
//for-loop for A(8):
 for(int i1=0;i1<=5;i1=i1+1){
  for(int i2=0;i2<=5-i1;i2=i2+1){
   for(int i3=0;i3<=5-i1-i2;i3=i3+1){
    for(int i4=0;i4<=5-i1-i2-i3;i4=i4+1){
     z=z+1;
     M[1,z]=Q*x(1)^i1*x(2)^i2*x(3)^i3*x(4)^i4
            *x(5)^(5-i1-i2-i3-i4);
    } //for-loop i4
   } //for-loop i3
  } //for-loop i2
 } //for-loop i1
int v=z;
//for-loop for A(9):
 for(int i1=0;i1<=4;i1=i1+1){
  for(int i2=0;i2<=4-i1;i2=i2+1){
   for(int i3=0;i3<=4-i1-i2;i3=i3+1){
    for(int i4=0;i4<=4-i1-i2-i3;i4=i4+1){
     z=z+1;
     M[1,z]=0;
    } //for-loop i4
   } //for-loop i3
  } //for-loop i2
 } //for-loop i1
z=z+1;
M[1,z]=-x(5)^9;
z=z+1;
M[1,z]=-(x(1)*x(2)*x(3)*x(4))^2*x(5);
z=z+1;
M[1,z]=-x(1)*x(2)*x(3)*x(4)*x(5)^5;
z=z+1;
M[1,z]=0;
z=z+1;
M[1,z]=0;
z=z+1;
M[1,z]=0;
//
z=0;
for(int k=1;k<=5;k=k+1){
 for(int i1=0;i1<=5;i1=i1+1){
  for(int i2=0;i2<=5-i1;i2=i2+1){
   for(int i3=0;i3<=5-i1-i2;i3=i3+1){
    for(int i4=0;i4<=5-i1-i2-i3;i4=i4+1){
     z=z+1;
     M[2,z]=diff(Q,x(k))*x(1)^i1*x(2)^i2*x(3)^i3*x(4)^i4
            *x(5)^(5-i1-i2-i3-i4);
    } //for-loop i4
   } //for-loop i3
  } //for-loop i2
```

```
 } //for-loop i1
} //for-loop k
//for-loop for A(6):
for(int i1=0;i1<=4;i1=i1+1){
  for(int i2=0;i2<=4-i1;i2=i2+1){
   for(int i3=0;i3<=4-i1-i2;i3=i3+1){
    for(int i4=0;i4<=4-i1-i2-i3;i4=i4+1){
     z=z+1;
     M[2,z]=0;
    } //for-loop i4
   } //for-loop i3
  } //for-loop i2
 } //for-loop i1
//for-loop for A(7):
for(int i1=0;i1<=3;i1=i1+1){
  for(int i2=0;i2<=3-i1;i2=i2+1){
   for(int i3=0;i3<=3-i1-i2;i3=i3+1){
    for(int i4=0;i4<=3-i1-i2-i3;i4=i4+1){
     z=z+1;
     M[2,z]=P*x(1)^i1*x(2)^i2*x(3)^i3*x(4)^i4
            *x(5)^(3-i1-i2-i3-i4);
    } //for-loop i4
   } //for-loop i3
  } //for-loop i2
 } //for-loop i1
//for-loop for A(8):
 for(int i1=0;i1<=5;i1=i1+1){
  for(int i2=0;i2<=5-i1;i2=i2+1){
   for(int i3=0;i3<=5-i1-i2;i3=i3+1){
    for(int i4=0;i4<=5-i1-i2-i3;i4=i4+1){
     z=z+1;
     M[2,z]=0;
    } //for-loop i4
   } //for-loop i3
  } //for-loop i2
 } //for-loop i1
//for-loop for A(9):
for(int i1=0;i1<=4;i1=i1+1){
  for(int i2=0;i2<=4-i1;i2=i2+1){
   for(int i3=0;i3<=4-i1-i2;i3=i3+1){
    for(int i4=0;i4<=4-i1-i2-i3;i4=i4+1){
     z=z+1;
     M[2,z]=Q*x(1)^i1*x(2)^i2*x(3)^i3*x(4)^i4
            *x(5)^(4-i1-i2-i3-i4);
    } //for-loop i4
   } //for-loop i3
  } //for-loop i2
 } //for-loop i1
z=z+1;
```

```
M[2,z]=0;
z=z+1;
M[2,z]=0;
z=z+1;
M[2,z]=0;
z=z+1;
M[2,z]=-x(5)^8;
z=z+1;
M[2,z]=-(x(1)*x(2)*x(3)*x(4))^2;
z=z+1;
M[2,z]=-x(1)*x(2)*x(3)*x(4)*x(5)^4;


matrix V[2][1];
for(int i=1;i<=2;i=i+1){
 V[i,1]=0;
}


def C=modulo(M,V);


for(int t=1;t<=2856;t=t+1){
 matrix L(t)[937][1];
}

int w=0;
for(int t=1;t<=2856;t=t+1){
 if(ord(C[t])==0) {w=w+1; L(w)=matrix(C)[t];}
}

int d=w;

for(int k=1;k<=9;k=k+1){
 poly A(k)=0;
}

int u=0;

for(w=1;w<=105;w=w+1){

// Calculation of A(1),...,A(5):

z=0;
for(int k=1;k<=5;k=k+1){
 for(int i1=0;i1<=5;i1=i1+1){
  for(int i2=0;i2<=5-i1;i2=i2+1){
   for(int i3=0;i3<=5-i1-i2;i3=i3+1){
    for(int i4=0;i4<=5-i1-i2-i3;i4=i4+1){
```

```
      z=z+1;
      A(k)=A(k)+x(1)^i1*x(2)^i2*x(3)^i3*x(4)^i4
           *x(5)^(5-i1-i2-i3-i4)*L(w)[z,1];
     } //for-loop i4
    } //for-loop i3
   } //for-loop i2
  } //for-loop i1
} //for-loop k
//

//Calculation of the polynomial A(6):

for(int i1=0;i1<=4;i1=i1+1){
   for(int i2=0;i2<=4-i1;i2=i2+1){
    for(int i3=0;i3<=4-i1-i2;i3=i3+1){
     for(int i4=0;i4<=4-i1-i2-i3;i4=i4+1){
      z=z+1;
      A(6)=A(6)+x(1)^i1*x(2)^i2*x(3)^i3*x(4)^i4
           *x(5)^(4-i1-i2-i3-i4)*L(w)[z,1];
     } //for-loop i4
    } //for-loop i3
   } //for-loop i2
  } //for-loop i1
//int n=v;

//Calculation of the polynomial A(7):

for(int i1=0;i1<=3;i1=i1+1){
   for(int i2=0;i2<=3-i1;i2=i2+1){
    for(int i3=0;i3<=3-i1-i2;i3=i3+1){
     for(int i4=0;i4<=3-i1-i2-i3;i4=i4+1){
      z=z+1;
      A(7)=A(7)+x(1)^i1*x(2)^i2*x(3)^i3*x(4)^i4
           *x(5)^(3-i1-i2-i3-i4)*L(w)[z,1];
     } //for-loop i4
    } //for-loop i3
   } //for-loop i2
  } //for-loop i1
//

//Calculation of the polynomial A(8):

 for(int i1=0;i1<=5;i1=i1+1){
   for(int i2=0;i2<=5-i1;i2=i2+1){
    for(int i3=0;i3<=5-i1-i2;i3=i3+1){
     for(int i4=0;i4<=5-i1-i2-i3;i4=i4+1){
      z=z+1;
      A(8)=A(8)+x(1)^i1*x(2)^i2*x(3)^i3*x(4)^i4
           *x(5)^(5-i1-i2-i3-i4)*L(w)[z,1];
```

```
        }//for-loop i4
      } //for-loop i3
     } //for-loop i2
    } //for-loop i1
//

//Calculation of the polynomial A(9):

for(int i1=0;i1<=4;i1=i1+1){
  for(int i2=0;i2<=4-i1;i2=i2+1){
   for(int i3=0;i3<=4-i1-i2;i3=i3+1){
    for(int i4=0;i4<=4-i1-i2-i3;i4=i4+1){
     z=z+1;
     A(9)=A(9)+x(1)^i1*x(2)^i2*x(3)^i3*x(4)^i4
          *x(5)^(4-i1-i2-i3-i4)*L(w)[z,1];
    } //for-loop i4
   } //for-loop i3
  } //for-loop i2
 } //for-loop i1
//


poly R2(w)=0;
for(int j=1;j<=5;j=j+1){
 R2(w)=R2(w)+diff(P,x(j))*A(j);
}
R2(w)=R2(w)+P*A(6)+Q*A(8);

poly R1(w)=0;
for(int j=1;j<=5;j=j+1){
 R1(w)=R1(w)+diff(Q,x(j))*A(j);
}
R1(w)=R1(w)+P*A(7)+Q*A(9);
}

for(w=1;w<=d;w=w+1){
 matrix R(w)[2][1];
 R(w)[1,1]=R1(w);
 R(w)[2,1]=R2(w);
}

int a;

for(w=1;w<=105;w=w+1){
 if(R(w)[1,1]!=0) {a=a+1;}
 else {
  if(R(w)[2,1]!=0) {a=a+1;}
 }
}
```

```
a;
quit;
```

A.3 A programme for the computation of Picard-Fuchs operators of a pair consisting of a Calabi-Yau 3-fold and a divisor

This appendix contains a Singular programme ([DGPS16]) for the computation of Picard-Fuchs operators of a pair consisting of a Calabi-Yau 3-fold and a divisor. It is an extension of the programme given in Appendix A.1 for a complete intersection Calabi-Yau 3-fold.

```
LIB "general.lib";

int n = 5;
// n-1 is the dimension of the ambient projective space

int c = 2;
ring ra = 0,(a,b,x(1..n),f(1..c)),dp;

ideal fa = x(1)^5+x(2)^5+x(3)^5+x(4)^5+x(5)^5
-5*a*x(1)*x(2)*x(3)*x(4)*x(5),
x(5)^4-b*x(1)*x(2)*x(3)*x(4);

fa;

// fa is given by the two homogeneous polynomials
// defining the Calabi-Yau manifold and the divisor

int m=4; // m is the dimension of the (n-2)-th
// cohomology of the Calabi-Yau (n-2)-fold.

// We compute the basis of the 3-rd relative
// cohomology.

// The following procedure computes the partial derivative
// of a polynomial in ra with respect to the
// parameter a, such that f(1) and f(2) are assumed to be
// in the denominator.


proc adiff (poly g)
{
  proc th(poly f)
  {
    return(diff(f,a));
  }
  poly p=0;
  for(int j=1;j<=size(g);j=j+1)A
  {
    matrix cf=coeffs(g[j],f(1));
```

```
    intvec k = size(cf)-1;
    poly h=cf[k[1]+1,1];
    int i;
    for(i=2;i<=c;i=i+1)
    {
      cf=coeffs(h,f(i));
      k=intvec(k,size(cf)-1);
      h=cf[k[i]+1,1];
    }
    p = p-k[1]*h*th(fa[1])*f(1)^(k[1]+1)*f(2)^k[2]
        +th(h)*f(1)^k[1]*f(2)^k[2]
        -k[2]*h*th(fa[2])*f(1)^k[1]*f(2)^(k[2]+1);
  }
  return(p);
}



// The following procedure computes the partial
// derivative of a polynomial in ra with respect
// to the parameter b


proc bdiff(poly g)
{
  proc th(poly f)
  {
   return(diff(f,b));
  }
  poly p=0;
  for(int j=1;j<=size(g);j=j+1)
  {
    matrix cf=coeffs(g[j],f(1));
    intvec k = size(cf)-1;
    poly h=cf[k[1]+1,1];
    int i;
    for(i=2;i<=c;i=i+1)
    {
      cf=coeffs(h,f(i));
      k=intvec(k,size(cf)-1);
      h=cf[k[i]+1,1];
    }
    p = p-k[1]*h*th(fa[1])*f(1)^(k[1]+1)*f(2)^k[2]
        +th(h)*f(1)^k[1]*f(2)^k[2]
        -k[2]*h*th(fa[2])*f(1)^k[1]*f(2)^(k[2]+1);
  }
  return(p);
}
```

```
// Calculation of the basis:

poly PX(0)=f(1);

for(int k=1;k<=m;k=k+1)
{
 poly PX(k) = adiff(PX(k-1));
}

poly PD(1)= diff(fa[2],b)*f(1)*f(2);

for(int k=2;k<=m;k=k+1)
{
 poly PD(k) = adiff(PD(k-1));
}


// We denote the vectors we want to decompose
// by Z(1) and Z(2)
// Derivative with respect to z:

poly Z(1)=PX(4);
poly Z(2)=PD(4);

// Derivative with respect to u:

poly Z(3)=PX(3)*diff(fa[2],b)*f(2);
poly Z(4)=bdiff(PD(1));
poly Z(5)=bdiff(PD(2));
poly Z(6)=bdiff(PD(3));


proc concat(matrix A, matrix B)
{
  int rA = nrows(A);
  int cA = ncols(A);
  int rB = nrows(B);
  int cB = ncols(B);
  int i,j;
  if(rA!=rB)
    {return(0);}
  else
  {
    ideal C=A[1,1];
    for(i=1;i<=rA;i=i+1)
    {
      for(j=1;j<=cA;j=j+1)
      {
```

```
        if (i!=1 || j!=1)
        {
          C=C,A[i,j];
        }
      }
      for(j=1;j<=cB;j=j+1)
      {
        C=C,B[i,j];
      }
    }
    return(matrix(C,rA,cA+cB));
  }
}



proc zero(int n)
{
  intvec z=0;
  int i;
  for(i=2;i<=n;i=i+1)
  {
    z=z,0;
  }
  return(z);
}

// The following procedure associates to a polynomial g
// in the ring ra the maximal total degree in f(1) and f(2)
// of all summands of g. Only summands with at least power
// one in f(1) and f(2) are taken into account.
// If there are no summands with power at least
// one in both f(1) and f(2), then 0 is returned.

proc fdegree(poly g)
{
  matrix cf=coeffs(g,f(1));
  intvec k = size(cf)-1;
  poly h=cf[k[1]+1,1];
  int i;
  for(i=2;i<=c;i=i+1)
  {
    cf=coeffs(h,f(i));
    k=intvec(k,size(cf)-1);
    h=cf[k[i]+1,1];
  }
  return(int(sum(k)));
}
```

```
// The following procedure associates to an integer k the
// (k-1)x1-matrix whose i-th entry is the coefficient of
// f(1)^if(2)^(k-i) for i=1,...,k-1.

proc Fmon(int k){
 matrix F[1][k-1]=0;
 for(int i=1;i<=k-1;i=i+1)
  {
    F[1,i]=F[1,i]+f(1)^i*f(2)^(k-i);
  }
 return(F);
}


// For the following procedure see Remark

proc maxvectf(poly p){
int d=fdegree(p);
matrix cf=coeffs(p,f(1));
matrix Maxvectf[d][1];
Maxvectf[1,1]=0;
 for(int i=2;i<=d;i=i+1){
  if(size(cf)>=d-(i-2)){
   if(size(coeffs(cf[d-(i-2),1],f(2)))>=1+(i-2)){
    Maxvectf[i,1]= coeffs(cf[d-(i-2),1],f(2))[2+(i-2),1];
   }
   else {Maxvectf[i,1]=0;}
  }
  else {Maxvectf[i,1]=0;}
 }
 return(Maxvectf);
}


// The following procedure determines the summands of a
// polynomial p in ra with maximal total degree in
// f(1) and f(2).

proc maxpolf(poly p){
 matrix cf=coeffs(p,f(1));
 poly max=0;
 for(int i=1;i<=size(cf);i=i+1){
  if(fdegree(p)-i+2<=size(coeffs(cf[i,1],f(2))))
   {max=max+(coeffs(cf[i,1],f(2))[fdegree(p)-i+2,1])
        *f(1)^(i-1)*f(2)^(fdegree(p)-(i-1));}
  else
   {max=max;}
 }
```

```
return(max);
}


// The following procedure associates to a polynomial g
// in ra a list of polynomials whose first element is g,
// the second element is the polynomial without all
// summands of the highest total degree in f(1) and f(2).
// Each following entry is the preceding one minus all
// summands of highest degree.

proc Slist(poly g){
 int k=fdegree(g);
 poly S(0)=g;
 for(int l=1;l<=k-1;l=l+1){
  poly S(l)=S(l-1)-maxpolf(S(l-1));
 }
 list H;
 for(l=1;l<=k-1;l=l+1){
  H[l]=S(l);
 }
 return(H);
}


// Let k be the maximal total degree in f(1) and f(2).
// The following procedure associates to a polynomial
// g in ra the list of polynomials whose i-th entry
// consists of all summands of degree k-i.

proc Mlist(poly g){
 int k=fdegree(g);
 list H=Slist(g);
 list G;
 for(int l=1;l<=k-1;l=l+1){
  if(size(H[l])!=0)
   {G[l]=maxvectf(H[l]);}
  else
   {G[l]=0;}
 }
 return(G);
}



proc maxvectf1(poly p){
 matrix cf=coeffs(p,f(1));
 matrix K[size(cf)-1][1];
 K[1,1]=cf[size(cf),1];
```

```
  for(int i=2;i<=size(cf)-1;i=i+1){
   K[i,1]=0;
  }
  return(K);
 }


 for(int k=1;k<=m-1;k=k+1)
 {
  matrix MaxX(k)=maxvectf1(PX(k));
 }


 for(int k=1;k<=m-1;k=k+1)
 {
  matrix MaxD(k)=maxvectf(PD(k));
 }


 for(int k=1;k<=m-1;k=k+1)
 {
  list MP(k) = Mlist(PX(k));
 }


 for(int v=1;v<=6;v=v+1)
 {
  if(v==1) {matrix K(v)=maxvectf1(Z(1));}
  if(v>=2) {matrix K(v)=maxvectf(Z(v));}
 }


 ////////////////////////////////////////////////////////

 // Change of the ring

 ring r = (0,a,b),x(1..n),dp;

 int m=4;


 for(int v=1;v<=6;v=v+1)
  {
   matrix K(v)=imap(ra,K(v));
  }


 for(int v=1;v<=6;v=v+1)
  {
```

```
  matrix K=K(v);


matrix S(v)[2*m-1][1];

for(int k=1;k<=m-1;k=k+1)
{
 matrix MaxX(k)=imap(ra,MaxX(k));
}

for(int k=1;k<=m-1;k=k+1)
{
 matrix MaxD(k)=imap(ra,MaxD(k));
}


ideal f = imap(ra,fa);

matrix z = matrix(zero(n));

int i,i1,i2,i3,i4,k,l,s,o,w;
int n=5;
int m=4;


// In the following we define matrices M(n-k) with
// k rows and n(k-1)+2k lines, they coincide
// which coincide with the matrices K_{k+1}
// defined in ??

// E1(k) ist die kxk-Einheitsmatrix mit der
// letzten Spalte abgeschnitten

for(k=2;k<=n;k=k+1)
{
 matrix E1(k)[k][k-1];
 for(l=1;l<=k;l=l+1)
  {
   for(s=1;s<=k-1;s=s+1)
   {
    if(l==s) {E1(k)[l,s]=1;}
    else {E1(k)[l,s]=0;}
   }
  }
}


// E2(k) ist die kxk-Einheitsmatrix mit den ersten
// beiden Spalten abgeschnitten
```

```
for(k=2;k<=n;k=k+1)
{
 matrix E2(k)[k][k-1];
 for(l=1;l<=k;l=l+1)
  {
   for(s=1;s<=k-1;s=s+1)
   {
    if(l<=1) {E2(k)[l,s]=0;}
    else
    {
    if(l-1==s) {E2(k)[l,s]=1;}
    else {E2(k)[l,s]=0;}
    }
   }
  }
}

for(k=1;k<=n-1;k=k+1)
{
 matrix E[k+1][k+1]; E = E + 1;
 ideal B=k*jacob(f[1]);
 for(i1=1;i1<=k+1;i1=i1+1)
  {
    for(i2=1;i2<=k;i2=i2+1)
    {
      if(i1==i2)
      {
            if(i1>1) {B=B,(k+1-i1)*jacob(f[1]);}
      }
      else
      {
           if (i1==i2+1)
       {
        if(i1==2) {B=B,-jacob(f[2]);}
        else {B=B,(i2-1)*jacob(f[2]);}
       }
           else {B=B,z;}
      }
    }
  }
 if(k<=n-2) {matrix M(n-k) = matrix(concat(concat(concat
             (concat(matrix(B,k+1,k*n),f[1]*E),f[2]*E2(k+1)),
              MaxX(k)),MaxD(k)),k+1,k*n+c*(k+1)+1);}
 else        {matrix M(n-k) = matrix(concat(concat
             (matrix(B,k+1,k*n),f[1]*E),f[2]*E2(k+1)),k+1,
             k*n+c*(k+1)-1);}
}
```

```
list Q;

matrix P[m+1][1]=K;

int t=m+2-size(K);


for(l=t;l<=m;l=l+1)
 {
  if(l<m) {matrix V(l)[m-l][1]=zero(m-l);}
  Q = division(P,M(l));
  if(l>1) {poly L1(l)=Q[1][(n-l)*n+2*(n-l+1),1];
           poly L2(l)=Q[1][(n-l)*n+2*(n-l+1)+1,1];}

  matrix P[m-l+1][1]=zero(m-l+1);
  for(k=1;k<=m-l+1;k=k+1)
   {
    for(i=1;i<=n;i=i+1)
     {
      P[k,1]=P[k,1]+diff(Q[1][i+n*(k-1),1],x(i));
     };
    P[k,1]=P[k,1]+Q[1][n*(m-l+1)+k,1];
    if(k>=2) {P[k,1]=P[k,1]+Q[1][(n+1)*(m-l+1)+1+k,1];}
    if(l<m)
     {
      for(s=1;s<=m-l;s=s+1)
       {
        for(i=1;i<=n;i=i+1)
         {
          V(l)[s,1]=V(l)[s,1]-Q[1][i+n*s,1]*diff(f[2],x(i));
         }
       }
     }
   }
 }

for(int o=2;o<=t-1;o=o+1)
 {
  poly L1(o)=0; poly L2(o)=0;
 }




//Coefficient of PX(0):
S(v)[1,1]=P;

//Coefficient of PX(1):
```

```
S(v)[2,1]=L1(4);

//Coefficient of PX(2):
S(v)[3,1]=L1(3);

//Coefficient of PX(3):
S(v)[4,1]=L1(2);

//Coefficient of PD(1):
S(v)[5,1]=L2(4);

//Coefficient of PD(2):
S(v)[6,1]=L2(3);

//Coefficient of PD(3)
S(v)[7,1]=L2(2);

}


for(int i3=1;i3<=2;i3=i3+1)
{
 matrix M(i3)[2*m-1][2*m-1];
}

////////////////////////////////////////////

//matrix M(1):

for(int i4=1;i4<=7;i4=i4+1) {  if(i4==2) {M(1)[1,i4]=1;}
                               else {M(1)[1,i4]=0;}
                            }
for(int i4=1;i4<=7;i4=i4+1) {  if(i4==3) {M(1)[2,i4]=1;}
                               else {M(1)[2,i4]=0;}
                            }
for(int i4=1;i4<=7;i4=i4+1) {  if(i4==4) {M(1)[3,i4]=1;}
                               else {M(1)[3,i4]=0;}
                            }
for(int i4=1;i4<=7;i4=i4+1) { M(1)[4,i4]=S(1)[i4,1];}
for(int i4=1;i4<=7;i4=i4+1) {  if(i4==6) {M(1)[5,i4]=1;}
                               else {M(1)[5,i4]=0;}
                            }
for(int i4=1;i4<=7;i4=i4+1) {  if(i4==7) {M(1)[6,i4]=1;}
                               else {M(1)[6,i4]=0;}
                            }
for(int i4=1;i4<=7;i4=i4+1) { M(1)[7,i4]=S(2)[i4,1];}


/////////////////////////////////////////////
```

```
//matrix M(2);

for(int i4=1;i4<=7;i4=i4+1) { if(i4==5) {M(2)[1,i4]=1;}
                              else {M(2)[1,i4]=0;}
                            }
for(int i4=1;i4<=7;i4=i4+1) { if(i4==6) {M(2)[2,i4]=1;}
                              else {M(2)[2,i4]=0;}
                            }
for(int i4=1;i4<=7;i4=i4+1) { if(i4==7) {M(2)[3,i4]=1;}
                              else {M(2)[3,i4]=0;}
                            }
for(int i4=1;i4<=7;i4=i4+1) { M(2)[4,i4]=S(3)[i4,1];}
for(int i4=5;i4<=7;i4=i4+1) { M(2)[5,i4]=S(4)[i4,1];}
for(int i4=5;i4<=7;i4=i4+1) { M(2)[6,i4]=S(5)[i4,1];}
for(int i4=5;i4<=7;i4=i4+1) { M(2)[7,i4]=S(6)[i4,1];}

//////////////////////////////////////////////////

M(1);
M(2);


quit;
```

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