Forcing over ord-transitive models

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Contents

Introduction

In 1963 Paul Cohen showed the independence of the continuum hypothesis and the axiom of choice by developing the method of forcing ([3], [4]). Since then, this powerful combinatorial method for independence proofs has been widely used, generating a multitude of results in set theory of the real line, higher descriptive set theory, cardinal invariants, large cardinals, inner model theory, set-theoretic topology and infinitary combinatorics.

As in Cohen's original work, the technique of forcing is still presented and taught starting with countable transitive ground models, a classic reference being Kunen's book [19]. Transitivity, which means that every element of M is also a subset of M, is considered a convenient technical assumption as it makes many useful sentences - including all Δ_0-formulas - absolute for M.

Yet transitive models are not the only ones we can work with; indeed, as pointed out by Shelah in his *Proper forcing* [22] while explaining the technique, "assumptions [about the membership relation and transitivity of M] are not essential but it is customary to assume them, and they simplify the presentation."

Waiving the assumption about transitivity in a convenient way, one can take into consideration the so called *ord-transitive models*. Recently they have been used in Shelah's *Properness without elementaricity* [24], in Kellner's *Non-elementary proper forcing* [18] or in the *Borel conjecture and dual Borel conjecture* [14] by Goldstern et al.

Ord-transitive models are generally not transitive on ordinals, but transitive outside them, in the sense that any element of M that is not an ordinal is a subset of M. The ordinals in such models often present gaps, and they can be seen as the image of an increasing function from the natural numbers to the class of ordinals. For these models, simple concepts like "subset" and "union" are not absolute anymore. However, although fewer sentences turn out to be absolute, with some adjustment, it is still possible to define forcing extensions on ord-transitive models.

In contrast to the countable transitive models,of which there are continuum many[1], the countable ord-transitive models form a class, as ordinals in such models can be chosen almost arbitrarily. This multitude and variety of ord-transitive models in itself already constitutes a motivation to study them. Moreover, forcing over ord-transitive models has the potential to broaden the investigation of larger notions of forcing: iterations of length $> \omega_2$ could be approximated, as well as definable forcings Q by means of unions of Q^Ms for M ord-transitive.

This work is a contribution to the presentation of the forcing technique over ord-transitive models. It aims to be an accessible introduction to the method, but also a detailed description of it. We show how to generate these models, and then modify the construction of forcing extensions, adapting it to ord-transitive ground models, being cautious not to collapse ordinals. Furthermore we define forcing iterations and we discuss limits of iterations, such as ones with almost finite support. Finally we deal with systems of embeddings between M and the universe V - which can be viewed as the lines of a matrix iteration - pointing out the difficulties and some limitations.

The thesis is structured as follows:

Chapter 1: We give three ways of producing ord-transitive models. One technique is to collapse countable elementary submodels of H_λ [2] to ord-transitive models via the *ord-collapsing* map. Another method takes a pair (M, f) consisting of a countable transitive model M and an increasing function $f : \omega \to \mathrm{ON}$ and produces an ord-transitive model substituting the ordinals of M with those of $f[\omega]$. The third method constructs the smallest ord-transitive model containing a given countable model: the process is called *ord-closure*. In this chapter we also study ord-transitive models from the point of view of absoluteness: some easy formulas such as "being successor" or "subset" are not absolute for such models, whereas other ones are.

Chapter 2: Forcing extensions over countable ord-transitive models are illustrated. We modify the construction of the transitive case with special focus on the evaluation of names for ordinals, which risk being collapsed otherwise. We also show the connections between ord-transitive models, their forcing extensions and their Mostowski collapses.

[1] See A.5 for a comment about this fact.

[2] H_λ denotes here the set of sets of hereditary cardinality less than λ.

Chapter 3: We define forcing iterations with arbitrary ideals. We examine complete embeddings, dense embeddings and quotient forcings from the point of view of ord-transitive models. Natural examples generated by iterations are given.

Chapter 4: This chapter deals with limits of iterations: we analyze the smallest possible limit, the biggest one and the full countable support limit, and describe iterations whose limits are partial countable support limits.

Chapter 5: Given an ord-transitive model M, we consider coherent systems of M-complete embeddings between a partial CS-iteration $\bar{P}^M \in M$ and an iteration $\bar{P} \in V$. We show that completeness may fail when $V \models |M| = \aleph_0$. On the positive side, we prove that small limits in V, such as *almost finite support limits* together with a map called *canonical extension*, ensure the M-completeness of the system. This leads to the construction of an almost finite support iteration $\bar{P}$ over $\bar{P}^M$.

Chapter 6: In this chapter we put together the notions of essentially different functions, independent functions, autonomous sets and independent sets on ω. The existence of such families with size continuum was used in the previous chapter. Here we present the proofs in detail including the construction of a custom-tailored *triangular tree*.

Chapter 1

Ord-transitive models

We begin this chapter with the definition of an *ord-transitive* model, an example of non ord-transitive models (Remark 1.1.2) and a first way of obtaining ord-transitive models via the *ord-transitive collapse* (Definition 1.1.5 and Lemma 1.1.6). We then analyze ord-transitive models from the point of view of absoluteness (Section 1.2). We will see that some basics constructions such as union and intersection are not absolute. In Section 1.3 we introduce a second method for obtaining ord-transitive models: we uncollapse *labeled models*. As some simple formulas such as "being successor" are not absolute, in Section 1.4 we restrict our attention on ord-transitive models which are also *successor-absolute* and *cof ω-absolute*. We give properties and examples on how to obtain such models. The chapter finishes with a third technique for obtaining ord-transitive models: the *ord-closure* of an ord-absolute model is the smallest ord-transitive superset of it (Lemma 1.5.3). The chapter follows and expand the presentation of [18, Section 1.1].

1.1 Ord-transitive models and ord-transitive collapses

We introduce some notation used in the chapter. ON is the class of ordinals. $f : A \twoheadrightarrow B$ ($f : A \rightarrowtail B$) means that f is surjective (injective). For $A' \subseteq A$ then $f[A'] := \{f(a) : a \in A'\}$. When X is a definable class, say $X = \{y \in V : \phi(y)\}$ for some formula ϕ, we mean by X^M the set $\{y \in M : M \models \phi(y)\}$.

Definition 1.1.1. Let M be a countable set such that $(M, \in)$ satisfies ZFC^*, a finite fragment of ZFC, which contains at least extensionality, foundation, pairing, product, union, set difference, empty set, infinity, the existence of ω_1 and the finitely many conclusions of ZFC that we invoke in this work. Then:

- M is *ord-absolute* if $\omega^M = \omega$, $\omega \subseteq M$ and $\mathrm{ON}^M \subseteq \mathrm{ON}$ (therefore $\mathrm{ON}^M = M \cap \mathrm{ON}$).
- M is *ord-transitive* if it is ord-absolute and if $x \in M \setminus \mathrm{ON}$ then $x \subset M$.

Remark 1.1.2. That M satisfies the axiom of extensionality means that $\forall x, y \in M(\forall z \in M((z \in x)^M \leftrightarrow (z \in y)^M) \rightarrow (x = y)^M)$. The relativizations $\in^M$ and $=^M$ are defined as $\in$ and $=$. Hence, whenever we will refer to the axiom on extensionality in M it will be of the form:

$$\forall x, y \in M(\forall z \in M(z \in x \leftrightarrow z \in y) \rightarrow x = y) \tag{1.1}$$

It is very easy to run into models that are not ord-transitive. In the next remark we show that elementary submodels of H_λ are not ord-transitive.

Definition 1.1.3. Let λ be an infinite cardinal. The *family of sets of hereditary cardinality less than* λ is defined as

$$H_\lambda := \{x : |\operatorname{trcl}(x)| < \lambda\}$$

where $\operatorname{trcl}(x) := x \cup \bigcup\{\operatorname{trcl}(y) : y \in x\}$ is the *transitive closure* of x.

Remark 1.1.4. If λ is big enough[1], then any countable elementary submodel M of H_λ is not transitive, neither ord-transitive.

Proof. We can define ω_1 in M that is still uncountable in H_λ and then cannot be a subset of M. Now take the set $A := \omega_1 \setminus \omega$. $A \in M$ because it is defined with parameters in M and it is not an ordinal, because $\omega \in A$ but $\omega \nsubseteq A$. Furthermore $A \nsubseteq M$ because A is uncountable. So we have proved that there is $A \in M \setminus ORD$ such that $A \nsubseteq M$. (For more detail about countable elementary submodels of H_λ please see Section A.1 of the Appendix). □

However it is possible to construct ord-transitive models, as we can see in the next definition and lemma.

Definition 1.1.5. The *ord-transitive collapse* of a set X is defined by recursion on $(X, \in)$ as the image of the following map:

$$\tau : X \to V$$
$$x \mapsto \begin{cases} x & \text{if } x \in \mathrm{ON}; \\ \{\tau(t) : t \in x \cap X\} & \text{otherwise.} \end{cases}$$

[1] We need at least a regular $\lambda > \aleph_1$. When a notion of forcing is involved we generally work with $|\lambda| \geq (2^{|P|})^+$.

Lemma 1.1.6. *Let M be ord-absolute and let $M' := \tau[M]$ be its ord-transitive collapse. Then:*

1. $x \in \mathrm{ON} \cap M \leftrightarrow \tau(x) \in \mathrm{ON}$. *In particular,* $M \cap \mathrm{ON} = M' \cap \mathrm{ON}$;

2. τ *is an* $\in$*-isomorphism;*

3. M' *is ord-transitive;*

4. τ *is the identity if and only if* M *is ord-transitive;*

5. *Let* π *be the* transitive collapse function[2] , *i.e.* $\pi : M \to V$ *and* $\pi(x) = \{\pi(t) : t \in x \cap M\}$. *Then the following diagram commutes*[3]*:*

$$\begin{tikzcd} M \arrow[r, "\tau"] \arrow[dr, "\pi"'] & M' \arrow[d, "\pi"] \\ & M'' \end{tikzcd}$$

Proof. 1. If $x \in \mathrm{ON} \cap M$, by definition $\tau(x) = x$ is an ordinal. Conversely, if $\tau(x) \in \mathrm{ON}$ and if, by contradiction, $x \notin \mathrm{ON}$ then $\tau(x) = \{\tau(t) : t \in x \cap M\}$. Since every $\tau(t)$ is an ordinal, by induction on the rank, every $t \in x \cap M$ is an ordinal and it follows that $\tau(x) = \{t : t \in x \cap M\} = x \cap M$. That $x \cap M$ is an ordinal implies $M \models$ "x is an ordinal". Because M is an ord-absolute model, we conclude that x is an ordinal.

2. To prove the injectivity, let $x, y \in M$ such that $x \neq y$. We have three possibilities to check. Case 1: $x, y \in \mathrm{ON}$; then $\tau(x) = x \neq y = \tau(y)$. Case 2: $x, y \notin \mathrm{ON}$. By (1.1) and without loss of generality, there is a $t \in M$ such that $t \in x \setminus y$. Hence $\tau(t) \in \tau(x) \setminus \tau(y)$. Case 3: $x \in \mathrm{ON}$ and $y \notin \mathrm{ON}$. By point 1 of the Lemma, $\tau(x) \in \mathrm{ON}$ and $\tau(y) \notin \mathrm{ON}$, so $\tau(x)$ and $\tau(y)$ cannot be equal. We just proved the injectivity of τ. The surjectivity is obvious. To prove the $\in$-homomorphism, it easy to check that $x \in y$ implies $\tau(x) \in \tau(y)$. For the other direction, let $x \notin y$. As before we have different cases. Case 1: $x, y \in \mathrm{ON}$ is immediate. Case 2: $x, y \notin \mathrm{ON}$. If, by contradiction, $\tau(x) \in \tau(y)$ there would be a $t \in y \cap M$ such that $\tau(t) = \tau(x)$. By injectivity, $t = x$ and $x \in y$, a contradiction. Case 3: $x \in \mathrm{ON}, y \notin \mathrm{ON}$ same a before. Case 4: $x \notin \mathrm{ON}, y \in \mathrm{ON}$. By

[2]Also called *Mostowski collapse function.*

[3]We give the same name π for the different collapsing functions $\pi_M : M \to V$ and $\pi_{M'} : M' \to V$ defined as follows: $\forall x \in M\ \pi_M(x) = \{\pi_M(y) : y \in x \cap M\}$ and $\forall x \in M'\ \pi_{M'}(x) = \{\pi_{M'}(y) : y \in x \cap M'\}$.

point 1 $\tau(x) \notin \mathrm{ON}$ and $\tau(x) \notin \tau(y)$ because elements of an ordinal have to be ordinals.

3. Firstly, M' is ord-absolute; in fact it is easy to see - by definition, by the previous points and because M is ord-absolute - that $\omega \subseteq M', \omega^{M'} = \omega^M = \omega$ and $\mathrm{ON}^{M'} = \mathrm{ON}^M \subseteq \mathrm{ON}$. Finally, if $\tau(x) \notin \mathrm{ON}$, then $x \notin \mathrm{ON}$ and $\tau(x) = \{\tau(t) : t \in x \cap M\} \subseteq M'$, which proves the ord-transitiveness.

4. By definition, $\tau \restriction \mathrm{ON}$ is the identity map. Now let $x \in M \setminus \mathrm{ON}$. Being M ord-transitive, $x \subseteq M$ and $\tau(x) = \{\tau(t) : t \in x \cap M\} = \{\tau(t) : t \in x\} = \{t : t \in x\} = x$. The second last equality holds by induction.

5. If $x \in \mathrm{ON}$ then $\pi(\tau(x)) = \pi(x)$. If $x \notin \mathrm{ON}$, then

$$\begin{aligned}
\pi(\tau(x)) &= \{\pi(y) : y \in \tau(x) \cap M'\} \\
&= \{\pi(y) : y \in \{\tau(t) : t \in x \cap M\}\} \\
&= \{\pi(\tau(t)) : t \in x \cap M\} \\
&= \{\pi(t) : t \in x \cap M\} \text{ (by induction)} \\
&= \pi(x).
\end{aligned}$$

□

From the previous lemma it follows that although a countable elementary submodel $N \preccurlyeq H(\chi)$ is not ord-transitive, its image $\tau[N]$ is ord-transitive; and this is a first example for an ord-transitive model. For further information about countable elementary submodels of the family $H(\chi)$ of sets herediratily of cardinality less than χ we refer to the Appendix (Section A.1).

1.2 Absoluteness on ord-transitive models

We present some statements which are absolute between ord-transitive models and V. We give also some counterexamples.
Let start with the concept of absoluteness for a formula:

Definition 1.2.1. Let L be a language, let $\mathfrak{A}$ and $\mathfrak{B}$ be two L-structures, with domains A and B, such that $\mathfrak{A} \subseteq \mathfrak{B}$ (one is a substructure of the other). We say that an L-formula $\phi(\bar{x})$ is absolute if

$$\mathfrak{A} \models \phi[\sigma] \text{ if and only if } \mathfrak{B} \models \phi[\sigma]$$

for every assignment $\sigma = \{(x_1, a_1), \ldots, (x_n, a_n) : a_i \in A\}$ for ϕ in A. We say that the formula is *upwards absolute* if

$$\mathfrak{A} \models \phi[\sigma] \Rightarrow \mathfrak{B} \models \phi[\sigma]$$

for every assignment $\sigma = \{(x_1, a_1), \ldots, (x_n, a_n) : a_i \in A\}$ for ϕ in A.

In our case we will study which formulas ϕ are absolute when $L = \{\in\}$, $\mathfrak{A} = M$ (an ord-transitive model) and $\mathfrak{B} = V$.

In the next (counter)example, the reader will remark that not every simple formula is absolute.

Example 1.2.2. Let $N \preceq H(\chi)$ be countable, $H(\chi) \models \mathsf{ZFC}^*$ and $M := \tau[N]$. We claim that the formulas $x \cup y = z$, $x \subset z$ and $x \cap y = z$ are not absolute. For example, let us take $x = \omega_1$ and $y = \{\{1\}\}$. We remark that $(\omega_1)^M = \tau(\omega_1) = \omega_1$ and $\tau(\{\{1\}\}) = \{\{1\}\}$, so both x, y are also in M.

The union in M is such that $(x \cup z)^M \in M \setminus \mathrm{ON}$ and, by ord-transitivity, $(x \cup y)^M \subseteq M$. It follows that $(x \cup y)^M$ is countable, while $(x \cup y)^V$ is not. For this reason they cannot be equal, concluding that the union is not absolute. Alternatively, it can be easily proved that, for every x and y $(x \cup y)^M = \tau(x \cup y)$ while, for $x = \omega_1$ and $y = \{\{1\}\}$, $\tau(x \cup y) \neq \tau(x) \cup^V \tau(y) = x \cup^V y$.

"Being a subset" is not absolute: it holds $M \models \omega_1 \subseteq (x \cup y)^M$, in contrast to $V \models \omega_1 \nsubseteq (x \cup y)^M$, for $x = \omega_1$ and $y = \{\{1\}\}$.

Finally, call $z := (x \cup y)^M$ for $x = \omega_1$ and $y = \{\{1\}\}$. Then we have $M \models x \cap z = x$ but $V \models x \cap z \neq x$, because z is countable.

We give now a list some absolute formulas.

Property 1.2.3. Let M be an ord-transitive model. Then:

1. Finite sets are absolute: for all $n \in \omega$ $n = n^M$; $z = \{x, y\}$ is absolute; if $x \in M$ and x is finite, then $x \subset M$ and $M \models$ "x is finite".

2. $x \in \omega^\omega$ is absolute.

3. $\mathbf{\Sigma^1_1}$ formulas are absolute.

4. $H^M(\aleph_0) = H(\aleph_0)$.

5. If $M \models f : A \to B$ then $f : A \cap M \to B \cap M$. If additionally M thinks that f is injective (surjective) then f is injective (surjective with respect to the new image).

6. Let $x \in M$ and $\alpha \in M \cap \mathrm{ON}$. $x \in R_\alpha$ is upwards absolute.

7. Let $x \in M$ and $\alpha \in M \cap \mathrm{ON}$. If $x \notin \mathrm{ON}$, then $|x| \leq |\alpha|$ is upwards absolute.

8. Let $x, y \in M$. If either $x \in \mathrm{ON}$ or $x \cap \mathrm{ON} = \emptyset$, then $y \subset x$ is absolute.

Proof. 1. $\forall n \in \omega (n = n^M)$ is just proved by induction.

Let $z := \{x, y\}^M$. (Such a pair exists in M because M satisfies the pair axiom and an instance of the comprehension scheme with $\phi(t, x, y) := t = x \vee t = y$). We show that $z = \{x, y\}^V$. If $z \in M \cap \mathrm{ON}$ then $z = 2^M = 2 \subseteq M$. If $z \in M \setminus \mathrm{ON}$ then $z \subseteq M$. So in both cases z is a subset of M. So the property

$$\forall t \in z \cap M (t = x \vee t = y) \wedge x \in z \wedge y \in z$$

is equivalent to

$$\forall t \in z (t = x \vee t = y) \wedge x \in z \wedge y \in z$$

which means $z = \{x, y\}$.

If $x \in M$ and x is finite, then $x \subset M$ (it is obvious if $x \in \omega$ and it comes from ord-transitivity if $x \in M \setminus \mathrm{ON}$). Suppose $|x| = n$, by pairing and union, we can construct in a finite number of steps a function $f \in M$ such that $M \models f : n \twoheadrightarrow x$. This means that $M \models$ "x is finite".

2. Let $x \in M$ such that $M \models x \in \omega^\omega$, (or $V \models x \in \omega^\omega$). In both cases $x \notin \mathrm{ON}$ and $x \subseteq M$, thus $x \cap M = x$. We recall also that $\omega \in M$ and $\omega \subset M$. Then we can say:

$$\begin{aligned} x \in \omega^\omega &\Leftrightarrow \forall t \in x \; \exists m, n \in \omega \; (t = \{\{m\}, \{m, n\}\}) \\ &\Leftrightarrow \forall t \in M \cap x \; \exists m, n \in M \cap \omega \; (t = \{\{m\}, \{m, n\}\})^M \\ &\Leftrightarrow M \models x \in \omega^\omega \end{aligned}$$

3. A formula $\phi(x)$ is $\mathbf{\Sigma}^1_1$ iff there is an $a \in \omega^\omega$ such that $A = \{x \in \omega^\omega : \phi(x)\} \in \Sigma^1_1(a)$. We recall that $A \in \Sigma^1_1(a)$ iff there is a recursive set R such that for all $x \in \omega^\omega$ $(x \in A \Leftrightarrow \exists y \in \omega^\omega \forall n \in \omega \, R(x \upharpoonright n, y \upharpoonright n, a \upharpoonright n))$. We define now the (two-dimensional sequential) tree $T = \{(x, y) \in \omega^{<\omega} \times \omega^{<\omega} : |x| = |y| \wedge \forall n \leq |x| \; R(x \upharpoonright n, y \upharpoonright n, a \upharpoonright n)\}$. For $x \in \omega^\omega$ we define also the tree $T(x) = \{y \in \omega^{<\omega} : \forall n \leq |y| R(x \upharpoonright n, y \upharpoonright n)\}$.

As a result, $x \in A$ iff $T(x)$ is *ill-founded* with respect to the "$\supseteq$" relation (i.e. $[T(x)] \neq \emptyset$). We can thus write $A = \{x : T(x) \text{ is ill founded }\}$. Suppose now that M is an ord-transitive model such that $a, T, A \in M$. For $x \in M$ we want to show that $M \models T(x)$ is ill-founded iff $V \models T(x)$ is ill-founded. For one direction we have: $M \models T(x)$ is ill-founded $\Leftrightarrow$ $M \models \exists f \in \omega^\omega \forall n \in \omega f \upharpoonright n \in T(x)$. Such an f exists also in V and then $V \models T(x)$ is ill-founded. For the other direction, if $M \models T(x)$ is well-founded, we can construct in M the rank function $\rho : T(x) \to \mathrm{ON}^M$ such that $\forall s, t \in T(x)(s \supset t \Rightarrow \rho(s) < \rho(t))$. Since $T(x) \subseteq M$ (because $T(x) \in M \setminus \mathrm{ON}$) and $\mathrm{ON}^M \subseteq \mathrm{ON}$, it follows that $\rho : T(x) \to \mathrm{ON}$ is also a rank function in V. Therefore $T(x)$ is well-founded in V.

4. By induction and the fact that M satisfies the pairing axiom and the union axiom, and by point 1, $R_n^M = R_n$. So

$$\begin{aligned} x \in H(\aleph_0) &\Leftrightarrow \exists n \in \omega(x \in R_n) \\ &\Leftrightarrow \exists n \in \omega(x \in R_n^M) \\ &\Leftrightarrow x \in H(\aleph_0)^M \end{aligned}$$

5. Immediate.

6. We show that $R_\alpha^M \subseteq R_\alpha$ for $\alpha \in \mathrm{ON} \cap M$. If not, let $\alpha \in \mathrm{ON}^M$ be the smallest ordinal such that $R_\alpha^M \setminus R_\alpha \neq \emptyset$. For $x \in R_\alpha^M \setminus R_\alpha$, $\mathrm{rank}^M(x) < \alpha$. We call $\beta = \mathrm{rank}^M(x)$. We remark that for every $y \in x \cap M$ $\mathrm{rank}^M(y) < \beta < \alpha$. So $y \in R_\beta^M \subseteq R_\beta$ by minimality of α and $\mathrm{rank}(y) < \beta$. We get that $\mathrm{rank}(x \cap M) \leq \beta < \alpha$ and therefore $x \cap M \in R_\alpha$. Now, if $x \in M \setminus \mathrm{ON}$, then $x \cap M = x$, so $x \in R_\alpha$, a contradiction with the choice of x. If $x \in M \cap \mathrm{ON}$, then $x = \gamma$ for some $\gamma \in \mathrm{ON}^M$. Then $\mathrm{rank}^M(\gamma) < \alpha$ implies that $\mathrm{rank}(\gamma) < \alpha$, a contradiction again. For the last implication we used: $\forall \gamma \in \mathrm{ON} \cap M \;\; \mathrm{rank}^M(\gamma) = \gamma = \mathrm{rank}(\gamma)$, where $\mathrm{rank}^M(\gamma) = \sup^M\{\mathrm{rank}^M(\beta) +^M 1 : \beta \in \gamma \cap M\}$.

7. Let $x \notin \mathrm{ON}$, it implies that $x \subset M$. Now $M \models |x| \leq |\alpha|$ is equivalent to say that there is $f \in M$ such that $M \models f : x \to \alpha$ is injective. Since $x \subset M$ then f has domain x and $V \models f : x \to \alpha$ is injective.

8. If $x, y \in \mathrm{ON}$ then $M \models y \subset x$ iff $M \models y \in x$ iff $V \models x \in y$ iff $V \models x \subset y$. If $x \in \mathrm{ON}$ and $y \notin \mathrm{ON}$ then $y \subseteq M$ and $M \models y \subset x$ iff $\forall t \in M \cap y, t \in x$ iff $\forall t \in y(t \in x)$ iff $V \models y \subset x$. If $x \cap \mathrm{ON} = \emptyset$ and $y \in M \setminus \mathrm{ON}$ then it is

like the previous case. If $x \cap \mathrm{ON} = \emptyset$ and $y \in M \cap \mathrm{ON}$ then $M \models y \not\subseteq x$ and $V \models y \not\subseteq x$. $\square$

1.3 Labeled models

In this section we are going to see an equivalent construction for obtaining ord-transitive models, the *labeled models.*

Definition 1.3.1. A *labeled model* is a pair (M, f) consisting of a transitive, countable model M of ZFC^* and a strictly monotonic function $f : M \cap \mathrm{ON} \to \mathrm{ON}$ satisfying $f(\alpha) = \alpha$ for $\alpha \leq \omega$.

Definition 1.3.2. Given a labeled model (M, f), we define the *uncollapsing map* as

$$\begin{aligned} \nu_f : M &\to V \\ x &\mapsto \begin{cases} f(x) & \text{if } x \in \mathrm{ON}; \\ \{\nu_f(y) : y \in x\} & \text{otherwise.} \end{cases} \end{aligned}$$

We define the *uncollapse* of (M, f) as the set $\nu_f[M]$.

Definition 1.3.3. Given an ord-transitive model N, let $\pi_N : N \to V$ be the transitive collapse and let $\pi_N^{-1} \restriction \mathrm{ON} : \pi[N] \cap \mathrm{ON} \to \mathrm{ON}$ be the inverse of π_N restricted to the ordinals. The *labeled collapse of* N is defined as $(\pi[N], \pi_N^{-1} \restriction \mathrm{ON})$, which is also called the *associated labeled model of* N.

Lemma 1.3.4. *1. Let N be and ord-transitive model. Then π_N is an $\in$-isomorphism and $\pi[N \cap \mathrm{ON}] \subseteq \mathrm{ON}$.*

2. *If N is an ord-transitive model, then $(\pi_N[N], \pi_N^{-1} \restriction \mathrm{ON})$ is a labeled model and $\nu_{\pi_N^{-1} \restriction \mathrm{ON}}[\pi_N[N]] = N$;*

3. *If (M, f) is a labeled model, then $\nu_f[M]$ is an ord-transitive model and*

$$(\pi_{\nu_f[M]}[\nu_f[M]], \pi^{-1}_{\nu_f[M]} \restriction \mathrm{ON}) = (M, f)$$

Proof. 1. It is enough to use induction and the fact that $\in$ is extensional on N. (Compare the proof of Lemma III 5.13 in [19]).

2. Let $(M, f) := (\pi_N[N], \pi_N^{-1} \upharpoonright \mathrm{ON})$. In the following we write π instead of π_N and ν instead of $\nu_{\pi_N^{-1} \upharpoonright \mathrm{ON}}$.

   We say that (M, f) is a labeled model because $M = \pi[N]$ is transitive and $f = \pi^{-1} : M \cap \mathrm{ON} \to N \cap \mathrm{ON}$ is well defined and strictly increasing. In fact, let $\pi(x) \in \mathrm{ON}$ we want to prove that $x \in \mathrm{ON}$. By definition, $\pi(x) = \{\pi(t) : t \in x \cap N\}$ and by induction every $t \in x \cap N$ is an ordinal. Then $x \cap N$ is well ordered. We have to prove that $N \models$ "x is transitive", i.e. to show that $\forall t \in x \cap N\ N \models t \subseteq x$.
   Let $t \in x \cap N$, then $\pi(t) \in \pi(x)$ and since $\pi(x) \in \mathrm{ON}$, $\pi(t) \subseteq \pi(x)$. So from the fact that $\forall \pi(s) \in \pi(t)[\pi(s) \in \pi(t)]$, we get $\forall s \in t \cap N(s \in x)$.
   We conclude that $N \models x \in \mathrm{ON}$ and by absoluteness $x \in \mathrm{ON}$.
   $f = \pi^{-1}[\mathrm{ON} \cap M]$ is increasing because if $\pi(\alpha) < \pi(\beta)$, since π is an $\in$-isomorphism $\alpha < \beta$.
   We prove now that $\nu[\pi[N]] = N$. It is enough to remark that

$$\begin{aligned} \nu\colon \pi[N] &\longrightarrow N \\ \pi(x) &\longmapsto \begin{cases} \pi^{-1}(\pi(x)) = x & \text{if } x \in \mathrm{ON}, \\ \{\nu(\pi(y)) : \pi(y) \in \pi(x)\} & \\ \quad = \{y : \pi(y) \in \pi(x)\} = x & \text{otherwise.} \end{cases} \end{aligned}$$

3. $\nu_f[M]$ is ord-transitive: pick $\nu_f(x) \notin N \setminus \mathrm{ON}$. Of course $x \notin \mathrm{ON}$ because otherwise $\nu_f(x) = f(x) \in \mathrm{ON}$. So by definition $\nu_f(x) = \{\nu_f(y) : y \in x\} \subseteq N$.
   In the following we write π instead of $\pi_{\nu_f[M]}$. We prove now that for all $x \in M$, $\pi(\nu_f(x)) = x$. If $x \notin \mathrm{ON}$, it is clear. If $x \in \mathrm{ON}$ then $\pi(\nu_f(x)) = \pi(f(x)) = \{\pi(y) : y \in f(x) \cap \nu_f[M]\} = \{\pi(y) : y \in \nu_f[M] \cap \mathrm{ON} \wedge y < f(x)\}$, since ordinals in $\nu_f[M]$ are those of $f[\mathrm{ON} \cap M]$, then that is equal to $\{\pi(f(y)) : f(y) < f(x)\} = \{y : f(y) < f(x)\}$. Being f is strictly monotonic we get $\{y : f(y) < f(x)\} = \{y : y < x\} = x$ $\square$

The previous lemma showed that to every ord-transitive model N corresponds the associated labeled model $(\pi_N[N], \pi_N^1 \upharpoonright \mathrm{ON})$, and vice versa. As we made out formerly, some simple formulas are not absolute. By means of labeled models one can see that "being a successor ordinal" is not absolute for ord-transitive models.

Example 1.3.5. Let (M, f) a labeled model such that $f(\omega+1) = \omega+\omega$. In the corresponding ord-transitive model N we have $N \models$ "$\omega+\omega$ is the successor of ω", because $f(\omega+1) = (f(\omega)+1)^N = (\omega+1)^N$

1.4 (Nice) candidates

We are going to restrict our attention to *successor-absolute* and *cof ω-absolute* models. The terminology of this section is taken from [14, Section 3]

Definition 1.4.1. Let N be ord-transitive.

- N is *successor-absolute* if "α is a successor" and "$\alpha = \beta+1$" are both absolute between N and V.
- N is *cof ω-absolute* if it is successor-absolute and "$\mathrm{cof}(\alpha) = \omega$" and "$A$ is a countable cofinal subset of α" both are absolute between N and V.

Definition 1.4.2. We call N a *candidate* when N is successor-absolute, we call it a *nice candidate* if it is also cof ω-absolute. In the last case we also assume that ω_1 and ω_2 are in N.

In the next chapters we will tacitly often use nice candidates when referring to ord-transitive models.
We summarize the properties of a candidate:
A set N is a *candidate* if:

- N is countable;
- $(N, \in)$ is a model of ZFC^*;
- N is ord-absolute : $N \models \alpha \in \mathrm{ON}$ if and only if $\alpha \in \mathrm{ON}$, for all $\alpha \in N$;
- N is ord-transitive : if $x \in N \setminus \mathrm{ON}$, then $x \subseteq N$;
- $\omega+1 \subseteq N$;
- "α is a limit ordinal" and "$\alpha = \beta+1$" are both absolute between N and V.

A candidate N is *nice* if in addition:

- N is cof ω-absolute;
- $\omega_1, \omega_2 \in N$.

Lemma 1.4.3. *Let N be cof ω-absolute:*

1. *If $x \in N$ and $N \models$ "x is countable", then $x \subseteq N$;*

2. *If $\alpha \in N$ has countable cofinality, then $\alpha \cap N$ is cofinal in α;*

3. *If $\omega_1 \in N$, then $\omega_1^N = \omega_1$.*

Proof. 1. If $x \notin \mathrm{ON}$ then $x \subseteq N$. Otherwise $x \in \mathrm{ON}$ and, by contradiction, we can assume that x is minimal such that $x \not\subseteq N$ and $x < \omega_1^N$. Because N satisfies the set difference, $\exists y \in N(N \models y = x \setminus \{0\})$ By cof ω-absoluteness, y is countable and cofinal in x. Since $y \notin \mathrm{ON}$, then $y \subseteq N$. Furthermore, and $\forall \alpha \in y$, $\alpha < x$ and, by minimality of x, $\alpha \subseteq N$. Then $x = \bigcup_{\alpha \in y} \alpha \subseteq N$, a contradiction.

2. Let $\alpha \in N$, then:

$$\begin{aligned} &N \models \alpha \text{ has countable cofinality} && \Rightarrow \\ &N \models \exists A \subseteq \alpha : A \text{ is countable and cofinal in } \alpha && \Rightarrow \\ &V \models A \text{ is countable and cofinal in } \alpha \end{aligned}$$

where the last implication holds because N is cof ω-absolute. Since $A \in N$ and $N \models$ "A is countable" , by the previous point , $A \subseteq N$. Thus $A \subseteq \alpha \cap N$ and $\alpha \cap N$ is cofinal in α because A is.

3. If $\omega_1^N < \omega_1$ then $V \models$ "ω_1^N has countable cofinality", that implies $N \models$ "ω_1^N has countable cofinality", a contradiction. If $\omega_1 < \omega_1^N$ then $N \models$ "ω_1 has countable cofinality" that implies $V \models$ "ω_1 has countable cofinality", another contradiction. We conclude that it must be $\omega_1^N = \omega_1$. □

There is a criterion that allow us to say when an ord-transitive model is a candidate. Namely:

Lemma 1.4.4. *N is successor-absolute if and only if the associated labeled pair (M, f) is such that $f(\alpha + 1) = f(\alpha) + 1$ and $f(\delta)$ is limit for δ limit.*

Proof. Let N be successor-absolute. We remark that $f(\alpha + 1) = f(\alpha) + 1$ iff $\pi(f(\alpha + 1)) = \pi(f(\alpha) + 1)$ ($\Rightarrow$ is obvious, $\Leftarrow$ holds because π is injective). So

$$\begin{aligned}
\pi(f(\alpha+1)) = \pi(f(\alpha)+1) & \quad \Leftrightarrow \\
\alpha+1 = \pi(f(\alpha)\cup\{f(\alpha)\}) & \quad \Leftrightarrow \\
\alpha+1 = \{\pi(\gamma) : \gamma \in (f(\alpha)\cup\{f(\alpha)\})\cap N\} & \quad \Leftrightarrow \\
\alpha+1 = \pi(f(\alpha))\cup\{\pi(f(\alpha))\} & \quad \Leftrightarrow \\
\alpha+1 = \alpha\cup\{\alpha\} &
\end{aligned}$$

Let $\delta \in M$ be a limit ordinal, we prove that also $f(\delta)$ is a limit ordinal. $f(\delta)$ is the ordinal β such that $\pi(\beta) = \delta$. If β was a successor, it would be a successor also in N and there would be some $\alpha \in N$ such that $\beta = \alpha + 1$. Therefore $\delta = \pi(\beta) = \pi(\alpha) + 1$, a contradiction.

Consider now the labeled pair (M, f) with the properties: $f(\alpha+1) = f(\alpha)+1$ and $f(\delta)$ is limit for δ limit. We prove that the associated ord-transitive model N is successor-absolute. For the upwards absoluteness, let $N \models$ "β is a successor and $\beta = \gamma + 1$". Since every ordinal in N is in the image of f, there is an α such that $f(\alpha) = \gamma$. So $N \models \beta = f(\alpha) + 1$ and because $f(\alpha+1) = f(\alpha)+1$ we get $N \models \beta = f(\alpha+1)$. The last equality is true also in V: $V \models \beta = f(\alpha+1)$ and then $V \models \beta = f(\alpha+1) = f(\alpha)+1 = \gamma+1$. For the downwards direction, let $\beta \in N$ and $V \models$ "β is a successor". Then $V \models \beta = \gamma + 1$. There is some $\alpha \in M$ such that $\beta = f(\alpha)$, so $V \models f(\alpha) = \gamma + 1$. We have two cases for α : it could be a limit (but then $f(\alpha)$ would also be a limit, contradiction) or a successor. Let $\alpha = \alpha' + 1$. Then $V \models \beta = f(\alpha') + 1$ and $N \models \beta = f(\alpha'+1) = f(\alpha') + 1$ hence $N \models$ "β is a successor". □

We just saw that it is relatively easy to construct a candidate: we pick a labeled pair (M, f) such that f sends successors in successors and limits in limits. The question now arises whether we can find or construct not only candidates, but also nice ones. A positive answer is given in the next example.

Example 1.4.5.
1. Ord-collapses of elementary countable submodels of a given $H(\chi)$ are cof ω-absolute.
2. Ord-collapses of forcing extensions by proper forcing notions are cof ω-absolute.

Proof. Let $M \preceq H(\chi)$ countable and $N := \tau[M]$.

1. We have to show that for $A \in N$ and $\alpha, \beta \in N \cap \mathrm{ON}$:

(a) $N \models$ "α is successor" $\Leftrightarrow V \models$ "α is successor";

(b) $N \models \alpha = \beta + 1 \Leftrightarrow V \models \alpha = \beta + 1$;

(c) $N \models \mathrm{cof}(\alpha) = \omega \Leftrightarrow V \models \mathrm{cof}(\alpha) = \omega$;

(d) $N \models A$ is a countable cofinal subset of $\alpha \Leftrightarrow V \models A$ is a countable cofinal subset of α

(a) $N \models$ "α is successor" iff $N \models \exists\gamma(\alpha = (\gamma \cup \{\gamma\})^N)$. If, by contradiction, $V \models \alpha < (\gamma+1)^V \vee (\gamma+1)^V < \alpha$ also $H(\chi) \models \alpha < (\gamma+1)^V \vee (\gamma+1)^V < \alpha$ and by elementarity $M \models \alpha < (\gamma+1)^M \vee (\gamma+1)^M < \alpha$. Because the ordinals in N are the ordinals in M it follows that $N \models \alpha < \gamma+1 \vee \gamma+1 < \alpha$, a contradiction. Now, if $V \models$ "α is successor", then $V \models \exists\gamma(\alpha = \gamma+1)$. By elementarity we can find such a γ also in M. So $M \models \alpha = \gamma+1$ and also $N \models \alpha = \gamma+1$.

(b) Analogous to the previous point, by replacing γ with $\beta \in N$.

(c) Observe that since $\tau : M \to N$ is an $\in$-isomorphism we have for $\alpha \in M$, $M \models \mathrm{cof}(\alpha) = \omega$ iff $N \models \mathrm{cof}(\tau(\alpha)) = \tau(\omega)$. Because $\tau(\alpha) = \alpha$ and $\tau(\omega) = \omega$ we have the following:

$$\begin{aligned} N \models \mathrm{cof}(\alpha) = \omega \Leftrightarrow & \quad M \models \mathrm{cof}(\alpha) = \omega \\ \Leftrightarrow & \quad H(\chi) \models \mathrm{cof}(\alpha) = \omega \\ \Leftrightarrow & \quad V \models \mathrm{cof}(\alpha) = \omega \end{aligned}$$

For the last two implications we used the elementarity $N \preceq H(\chi)$ and χ big enough.

(d) Let us remark some useful points:

- $A \in M$ is countable iff $M \models A$ is countable. This is true by elementarity.
- If $A \in M$ is countable and $A \subseteq \mathrm{ON}$ then $\tau(A) = A$. For that, just use the fact that $A \subseteq M$ (Lemma A.1.9).
- If $A' \in N$, $A' \subseteq \mathrm{ON}$ and $N \models A'$ is countable, then $\tau^{-1}(A') = A'$.

  Proof: For $A' \in N$ there is $A \in M$ such that $\tau(A) = A'$. If $A' \in \mathrm{ON}$, also $A \in \mathrm{ON}$ and $A = A'$. If $A' \notin \mathrm{ON}$, then $A \notin \mathrm{ON}$ and, since $A \cap M \subseteq \mathrm{ON}$: $\tau(A) = \{\beta : \beta \in A \cap M\}$.

Now , by $\in$-isomorphism of τ (using only one direction) we have $N \models \tau(A)$ is countable $\Rightarrow M \models A$ is countable. By the previous points, $A \subseteq M$. This implies $A' = \tau(A) = A$.

Now, let $\phi(A, \alpha, \omega)$ be the sentence expressing that A is a countable cofinal subset of α. By $\in$-isomorphism, $N \models \phi(A, \alpha, \omega)$ iff $M \models \phi(\tau^{-1}(A), \tau^{-1}(\alpha), \tau^{-1}(\omega))$. By the previous points, $\tau^{-1}(A) = A$ and, since τ is the identity on ordinals, we can continue: $N \models \phi(A, \alpha, \omega)$ iff $M \models \phi(A, \alpha, \omega)$ iff $H(\chi) \models \phi(A, \alpha, \omega)$ iff $V \models \phi(A, \alpha, \omega)$.

2. Let M be a countable transitive model. We recall that if P is proper then every countable set of ordinals in $M[G]$ is included in a set in M, that is countable in M (see Property A.2.4). The proof that $\tau[M[G]]$[4] is a nice candidate is similar to the previous point. We only show that having a countable cofinality is absolute, i.e. $\forall \alpha \in \mathrm{ON}\, M \models \mathrm{cof}(\alpha) = \omega \Leftrightarrow \tau[M[G]] \models \mathrm{cof}(\alpha) = \omega$. We prove the following implications:

$$\begin{aligned} M \models \mathrm{cof}(\alpha) = \omega \Leftrightarrow M[G] &\models \mathrm{cof}(\alpha) = \omega && (1.2)\\ \Leftrightarrow \tau[M[G]] &\models \mathrm{cof}(\tau(\alpha)) = \tau(\omega) && (1.3)\\ \Leftrightarrow \tau[M[G]] &\models \mathrm{cof}(\alpha) = \omega && (1.4) \end{aligned}$$

One direction of 1.2 holds because $M \subset M[G]$. The direction "$\Leftarrow$" of 1.2 holds because if $f \in M[G]$ and $M[G] \models$ "$f : \omega \to \alpha$ is cofinal", then by properness of P there is a set $A \in M$ such that $M \models$ "A is countable" and $M[G] \models$ "$f[\omega] \subseteq A \wedge A \cap \alpha$ is a cofinal subset of α". So $M \models$ "$A \cap \alpha \subseteq \alpha$ is cofinal in α" and therefore $M \models \mathrm{cof}(\alpha) = \omega$.

Line 1.3 holds because τ is an $\in$-isomorphism. Finally line 1.4 is true because $\tau(\alpha) = \alpha$ and $\tau(\omega) = \omega$. □

Remark 1.4.6. As a non-example we remark that general internal forcing extensions are not always cof ω-absolute: If G is generic for a collapse forcing, for example $\mathrm{fn}(\omega_1, \omega)$, then $M[G]$ will interpret ω_1^M as countable.

[4]If not clear from the context, the double notation with square brackets is discussed in 2.5.3.

1.5 The ord-closure

In this section we give a third and last way of producing ord-transitive models, through the *ord-closure* operation. We conclude the chapter presenting some properties of the ord-closure.

Definition 1.5.1. The *ord-closure* of a set x is empty if x is an ordinal, otherwise it is defined inductively on the rank:

$$\mathrm{ordclos}(x) := \begin{cases} \emptyset & \text{if } x \in \mathrm{ON}, \\ x \cup \bigcup\{\mathrm{ordclos}(y) : y \in x\} & \text{otherwise.} \end{cases}$$

For every $\alpha \in \mathrm{ON}$ define $\mathrm{hco}(\alpha) := \{x \in R_\alpha : |\,\mathrm{ordclos}(x)| \leq \aleph_0\}$. The *hereditarily countable ord-closure* is defined as $\mathrm{hco} := \bigcup_{\alpha \in \mathrm{ON}} \mathrm{hco}(\alpha)$. We will also use another layering of hco using the $\mathrm{HCON}_\alpha := \{x : |\,\mathrm{ordclos}(x)| \leq \aleph_0 \text{ and } \forall \gamma \in \mathrm{trcl}(x) \cap \mathrm{ON}\, \gamma < \alpha\}$, where $\mathrm{trcl}(x) := x \cup \bigcup\{\mathrm{trcl}(y) : y \in x\}$. Let $\mathrm{HCON} = \bigcup_{\alpha \in \mathrm{ON}} \mathrm{HCON}_\alpha$, it is easy to see that $\mathrm{hco} = \mathrm{HCON}$.

The following example shows that not every set is in hco.

Example 1.5.2. If $\alpha > \omega_1$, then $\omega_1 \in \mathrm{hco}(\alpha)$, but neither $\omega_1 \cup \{\{1\}\}$ nor $\omega_1 \setminus \{\emptyset\}$ are elements of $\mathrm{hco}(\alpha)$.

The next lemma collect interesting properties about the ord-closure function. In particular the second point clarify how to obtain an ord-transitive model by way of the ord-closure.

Lemma 1.5.3. *1. If $y \in x$ then* $\mathrm{ordclos}(y) \subseteq \mathrm{ordclos}(x)$;

2. *If $M \notin \mathrm{ON}$ and if M is a countable ord-absolute model, then* $\mathrm{ordclos}(M)$ *is the smallest ord-transitive superset of M;*

3. *A set x is an element of some candidate iff $x \in \mathrm{hco}$;*

4. *All reals and all ordinals are in* hco, *it follows that* hco *is a proper class;*

5. $\forall \alpha \in \mathrm{ON}\ \mathrm{hco}(\alpha) \subseteq \mathrm{HCON}_\alpha$

6. *A* ZFC^**-model M is ord-transitive iff* $\mathrm{ordclos}(M) = M$;

7. *If M is ord-transitive and countable, then $M \in \mathrm{hco}$;*

8. *If M is ord-transitive and $x \in M$, then* $\mathrm{ordclos}(x) = \mathrm{ordclos}^M(x) \subseteq M$;

9. *"$x \in \mathrm{hco}(\alpha)$" is upwards absolute for ord-transitive models;*

10. $\exists \alpha \in \mathrm{ON}$ *such that* $\mathrm{HCON}_\alpha \not\subseteq \mathrm{hco}(\alpha)$.

Proof. 1. If $x \in \mathrm{ON}$ or $y \in \mathrm{ON}$, it is clear. If $y \notin \mathrm{ON}$, then $x \notin \mathrm{ON}$ and, by definition, $\mathrm{ordclos}(y) \subseteq x \cup \bigcup\{\mathrm{ordclos}(y) : y \in x\} = \mathrm{ordclos}(x)$.

2. To see the ord-transitivity, let $x \in \mathrm{ordclos}(M) \setminus \mathrm{ON}$. If $x \in M$, then $x \subseteq \mathrm{ordclos}(x) \subseteq \mathrm{ordclos}(M)$. If $x \in \mathrm{ordclos}(M) \setminus M$, there is some $y \in M$ such that $x \in \mathrm{ordclos}(y)$. By induction, $x \subseteq \mathrm{ordclos}(y) \subseteq \mathrm{ordclos}(M)$. We prove now the minimality: let A be an ord-transitive set such that $M \subseteq A$. We show by induction on the rank that $\mathrm{ordclos}(M) = M \cup \bigcup\{\mathrm{ordclos}(y) : y \in M\} \subseteq A$. $M \subseteq A$ holds by hypothesis. It remains to show that $\forall y \in M \setminus \mathrm{ON}\ \mathrm{ordclos}(y) = y \cup \bigcup\{\mathrm{ordclos}(x) : x \in y\} \subseteq A$. Since A is ord-transitive and $y \in A \setminus \mathrm{ON}$, it follows that $y \subseteq A$. By induction $\forall x \in y\ \mathrm{ordclos}(x) \subseteq A$. so $\mathrm{ordclos}(x) \subseteq A$.

3. Let $x \in M$ for some candidate M. We prove that $x \in \mathrm{hco}$. By definition we have $\mathrm{ordclos}(x) \subseteq \mathrm{ordclos}(M) = M$, so $|\mathrm{ordclos}(x)| \leq \aleph_0$. Since $x \in WF$ there is some α such that $x \in R_\alpha$. All in all $x \in \mathrm{hco}(\alpha) \subseteq \mathrm{hco}$.
To prove the converse, let $x \in \mathrm{hco}$. By the Downwards Löwenheim-Skolem theorem (Theorem A.1.5) we can build a countable elementary submodel $N \preceq H(\chi)$ such that $\{x, \mathrm{ordclos}(x)\} \subseteq N$. Because $\mathrm{ordclos}(x)$ is countable, by Lemma A.1.9, $\mathrm{ordclos}(x) \subseteq N$. The ord-transitive collapse $\tau(N)$ is the candidate we are looking for, by Example 1.4.5 1. It remains to prove that $x \in \tau(N)$. It is enough to show that $x = \tau(x)$. Clearly, $x \in \mathrm{ON}$ implies $\tau(x) = x$. If $x \notin \mathrm{ON}$, it is certainly countable because $x \subseteq \mathrm{ordclos}(x) \subseteq N$.

Claim: For every $y \in \mathrm{ordclos}(x) \subseteq N$ *we get* $\tau(y) = y$. *Moreover if* $y \notin \mathrm{ON}$, *then* $y \subseteq N$. *Proof of the claim: Let* $y \in \mathrm{ordclos}(x) \setminus \mathrm{ON}$, *then* $y \subseteq \mathrm{ordclos}(x) \subseteq N$ *because* $\mathrm{ordclos}(x)$ *is transitive for non ordinal elements. By induction on the rank, every* $z \in \mathrm{ordclos}(y)$ *has the property* $\tau(z) = z$. *So* $\tau(y) = \{\tau(z) : z \in y \cap N\} = \{z : z \in y\} = y$.

We conclude that $\tau(x) = \{\tau(y) : y \in x \cap N\} = \{y : y \in x\} = x$.

4. Let $f \in \omega^\omega$. Then $|\mathrm{ordclos}(f)| = |f| = \omega$. For $n, m \in \omega$ we get $n, m \in R_\omega$ and $\{\{n\}, \{n, m\}\} \in R_{\omega+2}$. So $f \subseteq R_{\omega+2}$ and then $f \in R_{\omega+3}$. So $f \in \mathrm{hco}(\omega+3) \subseteq \mathrm{hco}$. Therefore any real is in hco. Every ordinal is in hco because for every α, $\alpha \in R_{\alpha+1}$ and $\mathrm{ordclos}(\alpha) = \emptyset$, so we get $\alpha \in \mathrm{hco}(\alpha+1)$. Since $\mathrm{ON} \subseteq \mathrm{hco}$, we conclude that hco is a proper class.

5. For $\alpha \in \mathrm{ON}$ we show that $\mathrm{hco}(\alpha) \subseteq \mathrm{HCON}_\alpha$. Let $x \in \mathrm{hco}(\alpha)$. We have to prove that $\forall \gamma \in \mathrm{trcl}(x) \cap \mathrm{ON}$, $\gamma < \alpha$. If $\gamma \in x \cap \mathrm{ON}$ then $\gamma = \mathrm{rank}(\gamma) < \mathrm{rank}(x) \leq \alpha$. If $\gamma \in \mathrm{trcl}(y) \cap \mathrm{ON}$ for some $y \in x$, then we remark that $\mathrm{rank}(y) < \mathrm{rank}(x)$ and therefore $y \in \mathrm{hco}(\beta)$ for some $\beta < \alpha$. By induction hypothesis $\forall \gamma \in \mathrm{trcl}(y)\ \gamma < \beta < \alpha$.

6. It follows directly from point 2.

7. As M is ord-transitive and countable, $\mathrm{ordclos}(M) = M$ and $|\,\mathrm{ordclos}(M)| \leq \aleph_0$. Since $M \in WF$, there is some α such that $M \in R_\alpha$. So $M \in \mathrm{hco}_\alpha \subseteq \mathrm{hco}$.

8. Let $x \in M$. If $x \in \mathrm{ON}$, since M is ord-absolute, then $\mathrm{ordclos}(x) = \mathrm{ordclos}^M(x) = \emptyset$. If $x \notin \mathrm{ON}$, because M is ord-transitive, $x \subseteq M$. By induction on the rank, $\mathrm{ordclos}(x) = x \cup \bigcup\{\mathrm{ordclos}(y) : y \in x\} = x \cup^M \bigcup^M\{\mathrm{ordclos}^M(y) : y \in x\} = \mathrm{ordclos}(x)^M \subseteq \mathrm{ordclos}(M) = M$.

9. Let M be an ord-transitive model and $x \in M$. By Property 1.2.3, R_α is upwards absolute, and by point 8 of the current lemma, $\mathrm{ordclos}^M(x) = \mathrm{ordclos}(x)$. Therefore:

$$\begin{aligned} M \models x \in \mathrm{hco}(\alpha) &\Rightarrow M \models x \in R_\alpha \wedge |\,\mathrm{ordclos}(x)| \leq \aleph_0 \\ &\Rightarrow V \models x \in R_\alpha \wedge |\,\mathrm{ordclos}(x)| \leq \aleph_0. \\ &\Rightarrow V \models x \in \mathrm{hco}(\alpha). \end{aligned}$$

10. Consider $\alpha = \beta + 1$ and $x = \{\beta\}$, then $x \in \mathrm{HCON}_{\beta+1}$ but $x \in \mathrm{hco}(\beta + 2) \setminus \mathrm{hco}(\beta + 1)$. $\square$

Chapter 2

Forcing extensions

In the current chapter we show how to produce forcing extensions over ord-transitive models. The idea is to emulate the construction of the forcing extension for transitive models, for which we briefly review the main features (Section 2.1). However, if we repeat verbatim the procedure for ord-transitive models, a first issue arises: the evaluation of a check name of a given ordinal can collapse the ordinal itself (Remark 2.5.4). We would then have an extension of the ground model with new ordinals. This wouldn't be a good construction, as a an important characteristic of the forcing extension is that the ordinals in the extension remain the same as the ones in the ground model. We give a solution to the problem and show that the new construction for ord-transitive models has good properties (Theorem 2.5.6). The reader will also learn the connections between ord-transitive models, their forcing extensions and their transitive collapses (Lemma 2.4.3 and Theorem 2.5.5).

2.1 Review of the transitive case

We refer to [19, Chapter VII] for the construction of forcing extensions over countable transitive models. We recall nevertheless some definitions and properties useful for the ord-transitive case.

Definition 2.1.1. Let M be a countable transitive model (also shortened c.t.m.) of a sufficiently large fragment of ZFC. A *partial order* $\langle P, \leq_P, 1_P\rangle \in M$ is a triple where $\leq_P$ is a binary relation on P which is transitive and reflexive. 1_P is the weakest element: $\forall p \in P\ (p \leq_P 1_P)$. Moreover we say that q is *stronger* than p if $q \leq_P p$.
We write $p \parallel q$ to say that the conditions p and q are *compatible*, i.e. $\exists r \in P$

such that $r \leq p$ and $r \leq q$. We write $p \perp q$ to say that the two conditions are not compatible. A filter $G \subseteq P$ is *M-generic* if it intersects every dense set of P contained in the ground model M.
A set $\dot{\sigma}$ is a *P-name* iff it is a relation and

$$\forall \langle \dot{\tau}, p \rangle \in \dot{\sigma} \ \dot{\tau} \text{ is a } P\text{-name and } p \in P.$$

Given $x \in M$ its *check name* is defined as

$$\check{x} := \{\langle \check{y}, 1_P \rangle : y \in x\}$$

The *evaluation* of a P-name $\dot{\sigma}$ is the set

$$\mathrm{val}_G(\dot{\sigma}) := \{\mathrm{val}_G(\dot{\tau}) : \exists p \in G(\langle \tau, p \rangle \in \sigma)\}$$

The notions of P-names and check names are absolute for transitive models. The *forcing extension* of M is defined as $M[G] := \{\mathrm{val}_G(\dot{\sigma}) : \dot{\sigma} \in M^P\}$, where M^P is the sets of all P-names in M.
Let $\phi(x_1, \ldots, x_n)$ be a formula with all free variables shown, let $\dot{\tau}_1, \ldots, \dot{\tau}_n \in M^P$ and $p \in P$, then $p \Vdash \phi(\dot{\tau}_1, \ldots, \dot{\tau}_n)$ iff

$$\forall G((G \text{ is } P\text{-generic over } M \wedge p \in G) \rightarrow \phi^{M[G]}(\mathrm{val}_G(\dot{\tau}_1), \ldots, \mathrm{val}_G(\dot{\tau}_n))).$$

In [19] it is shown that the relation $\Vdash$ is equivalent to the statement[1] $\Vdash^*$ relativized to M and the following theorem is also proved:

Theorem 2.1.2. *Let M be a c.t.m. and P a partial order in M. Let $\phi(x_1, \ldots, x_n)$ be a formula with all free variables shown; let $\dot{\tau}_1, \ldots, \dot{\tau}_n \in M^P$. Then:*

- *For all $p \in P$,*

$$p \Vdash \phi(\dot{\tau}_1, \ldots, \dot{\tau}_n) \text{ iff } (p \Vdash^* \phi(\dot{\tau}_1, \ldots, \dot{\tau}_n))^M.$$

- *For all G that are P-generic over M,*

$$\phi(\mathrm{val}_G(\dot{\tau}_1), \ldots, \mathrm{val}_G(\dot{\tau}_n))^{M[G]} \text{ iff } \exists p \in G(p \Vdash \phi(\tau_1, \ldots, \tau_n)).$$

2.2 The ord-transitive case

We deal now with ord-transitive models. We start with general definitions, then control what is absolute for ord-transitive models and simplify our definitions.

[1]Please look at [19, Ch.VII Def.3.3] for the definition of $\Vdash^*$.

Definition 2.2.1. Let M be a countable ord-transitive model. Let $(P, \leq_P, 1_P) \in M$ such that $M \models$ "$(P, \leq_P, 1_P)$ is a partial order". A set G is called *P-generic* over M if:

- $G \cap P \cap M$ is a filter on $P \cap M$;
- For every $D \in M$ such that $M \models$ "D is a dense subset of P", the intersection $G \cap D \cap M$ is not empty.

Since most of the time domains of partial orders are not ordinals (for example domains may contain partial functions, or subtrees, hence they aren't ordinals), it is acceptable to consider forcing notions $(P, \leq_P, 1_P) \in M$ where $P \in M \setminus \mathrm{ON}$. In this case some definitions became absolute, as described in the following lemma.

Lemma 2.2.2. *Let M be ord-transitive and let $\langle P, \leq_P, 1_P\rangle \in M$ such that $P \in M \setminus \mathrm{ON}$. Then:*

1. *"P is a poset"* [2] *is absolute between M and V;*
2. *If P is a poset, let $D \in M \setminus \mathrm{ON}$. Then "$D$ is a dense subset of P" is absolute between M and V.*

Proof. For $P \in M \setminus \mathrm{ON}$ and $D \in M \setminus \mathrm{ON}$ it follows that $P \subseteq M$ and $D \subseteq M$. We also remark that $\leq_P \subseteq M$, as $\leq_P \in M \setminus \mathrm{ON}$. So we can say that:

1.

$$\begin{aligned}
M \models P \text{ is a poset } &\Leftrightarrow M \models \leq_P \text{ is transitive and reflexive}\\
&\Leftrightarrow \forall p, q, r \in P \cap M\\
&\quad ((p,q) \in \leq_P \cap M \wedge (q,r) \in \leq_P \cap M) \to (p,r) \in \leq_P \cap M\\
&\quad \text{and } \forall p \in P (p,p) \in \leq_P \cap M\\
&\Leftrightarrow \forall p, q, r \in P((p,q) \in \leq_P \wedge (q,r) \in \leq_P \to (p,r) \in \leq_P)\\
&\quad \text{and } \forall p \in P (p,p) \in \leq_P\\
&\Leftrightarrow V \models P \text{ is a poset .}
\end{aligned}$$

2.

$$\begin{aligned}
M \models D \text{ is dense in } P &\Leftrightarrow \forall p \in P \cap M \exists d \in D \cap M \ (d,p) \in \leq_P \cap M\\
&\Leftrightarrow \forall p \in P \exists d \in D \ (d,p) \in \leq_P\\
&\Leftrightarrow V \models D \text{ is dense in } P
\end{aligned}$$

□

[2] From now on we just write P when it is clear that we refer to $\langle P, \leq_P, 1_P\rangle$.

Then we have the following simplified definition.

Definition 2.2.3. Let M be an ord-transitive model and $P \in M \setminus \mathrm{ON}$ be a poset. We say that a set $G \subseteq P$ is *P-generic over M* iff G is a filter and for every dense subset D of P, such that $D \in M$, it holds: $G \cap D \neq \emptyset$.

In the next lemma we also make sure that the notion of antichain is absolute for ord-transitive models. We say that $A \subseteq P$ is an *antichain* if $\forall p, q \in A(p \perp_P q)$. We remark that a maximal antichain of P is dense in P.

Lemma 2.2.4. *Let M be a countable ord-transitive model, let $P \in M \setminus \mathrm{ON}$, $\leq_P \in M \setminus \mathrm{ON}$ and $A \in M$. If $A \subseteq P$, then $A \subseteq M$ and "A is an antichain of P" is absolute for M.*

Proof. If $A \subseteq P$ and $P \subseteq M$, then $A \subseteq M$. Moreover, $M \models A$ is an antichain in P if and only if

$$\forall a \in A \cap M(a \in P) \text{ and } \forall a, b, \in A \cap M \forall p \in P((p, a) \in \leq_P \Rightarrow (p, b) \notin \leq_P)$$

Since $A \cap M = A$, the last sentence is equivalent to $V \models$ "A is an antichain of P". $\square$

Lemma 2.2.5. *With the settings of the previous lemma, if $A \subseteq P$ or if $A \subseteq M$ then "A is a maximal antichain of M" is absolute for M.*

Proof. By Lemma 2.2.4, we just need to show the absoluteness of maximality. So $M \models$ "A is maximal" iff $\forall p \in P \cap M \exists a \in A \cap M \exists r \in P \cap M((r, a) \in \leq_P \cap M \wedge (r, p \in \leq_P \cap M))$ Since $P, A, \leq_P \subseteq M$ we get that the previous sentence is equivalent to $\forall p \in P \exists a \in A(p \parallel a)$, i.e. $V \models A$ is a maximal. $\square$

2.3 P-names

We are going to define P-names in M and to show that the definition is absolute.

Definition 2.3.1. By recursion, a P-name $\dot{\sigma}$ is a relation such that $\forall y \in \dot{\sigma}\ \exists \dot{\tau}, p$ such that $\dot{\tau}$ is a P name, $p \in P$ and $\langle \dot{\tau}, p \rangle = y$.

Lemma 2.3.2. *If M is ord-transitive, definitions by recursion are well defined in M. In other words, we can construct in M sets by mean of recursive definitions.*

Proof. M satisfies the axiom of regularity and extensionality, hence we can apply Theorem III.5.6 in [19] for $R = \in$. □

Lemma 2.3.3. *Let $\dot{\sigma} \in M$, then $M \models$ "$\dot{\sigma}$ is a P-name" iff $V \models$ "$\dot{\sigma}$ is a P-name".*

Proof. The fact that $\dot{\sigma}$ is a P-name (in V or in M) implies that $\dot{\sigma}$ is not an ordinal. Hence $\dot{\sigma} \in M \setminus \mathrm{ON}$ and therefore $\dot{\sigma} \subseteq M$. We also recall that $P \subseteq M$, as $P \in M \setminus \mathrm{ON}$. Moreover for $a, p \in M$ the ordered pair is absolute for M, i.e. $\langle a, p \rangle^M = \langle a, p \rangle$. So by induction on the rank we can say:

$$\begin{aligned} M \models \text{“}\dot{\sigma} \text{ is a } P\text{-name''} &\Rightarrow \forall x \in \dot{\sigma} \cap M \exists a \in M\, \exists p \in P \cap M\; x = \langle a, p \rangle^M \wedge \\ &\quad (a \text{ is a } P\text{- name})^M \\ &\Rightarrow \forall x \in \dot{\sigma} \exists a \in M\, \exists p \in P\; x = \langle a, p \rangle \wedge a \text{ is a } P\text{-name} \\ &\Rightarrow V \models \dot{\sigma} \text{ is a } P\text{-name.} \end{aligned}$$

For the other direction if $\dot{\sigma} \in V$ and if $V \models \dot{\sigma}$ is a P-name, then $\dot{\sigma} \in M \setminus \mathrm{ON}$, so $\dot{\sigma} \subseteq M$. Moreover $\forall x \in \dot{\sigma} x \in M \setminus \mathrm{ON}$ and hence $x \subseteq M$. Therefore for $x = \langle a, p \rangle$ also $a, p \in M$. By induction on the rank:

$$\begin{aligned} V \models \dot{\sigma} \text{ is a } P\text{-name} &\Rightarrow \forall x \in \dot{\sigma} \exists a \exists p \in P\; x = \langle a, p \rangle \wedge a \text{ is a } P\text{-name} \\ &\Rightarrow \forall x \in \dot{\sigma} \exists a \in M \exists p \in P\; x = \langle a, p \rangle \wedge (a \text{ is a } P\text{-name})^M \\ &\Rightarrow M \models \dot{\sigma} \text{ is a } P\text{-name.} \end{aligned}$$ □

2.4 Standard names

We present the notion of a check name constructed in an ord-transitive model. We also show that check names are not absolute, but they commute with the Mostowski collapse.

Definition 2.4.1. Let M be ord-transitive and $x \in M$. An *ord-transitive check name* in M is defined as

$$\check{x}^M := \{\langle \check{y}^M, 1_P \rangle : y \in x \cap M\}$$

Remark 2.4.2.

- For $x \in M$, we have $\check{x}^M \in M$. It is indeed a recursive definition in M.
- If $x \in M$ has the property $\mathrm{trcl}(x) \subseteq M$, then $\check{x}^M = \check{x}$. In particular, this is true when $\mathrm{trcl}(x) \cap \mathrm{ON} \subseteq \omega \cup \{\omega\}$.

- The check name is not absolute. If $x \in M \cap \mathrm{ON}$, it can happen that $\check{x}^M \neq \check{x}$. Take for instance the ord-transitive model corresponding to (M', f) and such that $f(\omega+1) = 2\omega+1$. Then $(2\omega+1)\check{}^M = (\omega+1)\check{}$.

- However, if π is the transitive collapse, then $\pi(\check{x}^M) = (\pi(x))\check{}$, as we show in the following lemma.

Lemma 2.4.3. *Let $x \in M$, then the check name commutes with the transitive collapse π of M:*

$$\pi(\check{x}^M) = (\pi(x))\check{}.$$

In other words the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\pi"] \arrow[d, "\check{}^M"'] & M' \arrow[d, "\check{}"] \\ \check{M}^M \arrow[r, "\pi"] & \check{M}' \end{tikzcd}$$

where $M' := \pi[M]$, $\check{M}^M := \{\check{x}^M : x \in M\}$ and $\check{M}' := \{\check{x} : x \in M'\}$.

Proof. We define $\langle P', \leq_{P'}, 1_{P'}\rangle := \langle \pi(P), \pi(\leq_P), \pi(1_P)\rangle$. By induction we see that

$$\begin{aligned} \pi(\check{x}^M) &= \{\pi(y) : y \in \check{x}^M \cap M\} \\ &= \{\pi(y) : y \in \check{x}^M\} \\ &= \{\langle \pi(\check{a}^M), 1_{P'}\rangle : \langle \check{a}^M, 1_P\rangle \in \check{x}^M\} \\ &= \{\langle (\pi(a))\check{}, 1_{P'}\rangle : a \in x \cap M\} \\ &= \{\langle (\pi(a))\check{}, 1_{P'}\rangle : \pi(a) \in \pi(x)\} \\ &= (\pi(x))\check{}. \end{aligned}$$

□

Lemma 2.4.4. *The transitive collapse of M sends P-names into P'-names, it is actually a bijection $\pi : M^P \to M'^{P'}$.*

Proof. The proof follows from the fact that π is an $\in$-isomorphism (Lemma 1.3.4), and two isomorphic structures are elementary equivalent. Hence if $\phi(\sigma, P)$ is the sentence "σ is a P name" then $M \models \phi(\sigma, P)$ iff $M' \models \phi(\pi(\sigma), \pi(P))$.

2.5 Ord-evaluation of names

We are ready now to define the evaluation of names for an ord-transitive model M, that we will call *ord-evaluation*. We have to be careful when a P-name $\dot{\tau}$ represents a check name for an ordinal in M, because in that case the "traditional" evaluation (Definition 2.1.1) would collapse the cardinal and it could happen that for $x \in M \cap \mathrm{ON}$ $\mathrm{val}_G(\check{x}^M) < x$. For example, from the Remark 2.4.2 we would have $\mathrm{val}_{G'}((2\omega+1)^{\check{}M}) = \omega + 1$. Clearly we do not want that to happen, since we would like that the forcing extension has the same ordinals as the ground model. The following definition avoids the problem.

Definition 2.5.1. Let $\dot{\tau} \in M$ be a P-name and G be P-generic over M. Let $G' := \pi[G]$, where π is the Mostowski-collapsing function for M. Let $f := \pi^{-1} \restriction \mathrm{ON}$. The *ord-evaluation* of $\dot{\tau}$ is defined as

$$\mathrm{ordval}_G(\dot{\tau}) := \begin{cases} f(\mathrm{val}_{G'}(\pi(\dot{\tau}))) & \text{if } \mathrm{val}_{G'}(\pi(\dot{\tau})) \in \mathrm{ON} \\ \{\mathrm{ordval}_G(\dot{\sigma}) : \exists p \in G(\langle \dot{\sigma}, p\rangle \in \dot{\tau})\} & \text{otherwise.} \end{cases}$$

We are ready now to define the forcing extension of an ord-transitive model.

Definition 2.5.2. Let M be an ord-transitive model, $P \in M \setminus \mathrm{ON}$ a poset and let G be P-generic over M. The *forcing extension* of M is defined as

$$M[G] := \{\mathrm{ordval}_G(\dot{\tau}) : \dot{\tau} \in M^P\}$$

Notation 2.5.3. Before we continue, we want to warn the reader about some notation. Although it should be clear from the context, we want to be sure there is no misunderstanding in the use of the square brackets. When $f : A \to B$ is a function, the square brackets are used to refer to the image of f (or to a subset of it): $\forall A' \subseteq A f[A'] = \{f(a) : a \in A'\}$.
When G is M-generic, the square brackets indicate the forcing extension $M[G]$.

Remark 2.5.4. Let $\pi : M \to \pi[M]$ be the Mostowski collapse function for M and $M' := \pi[M]$. Let $\nu : (M', f) \to M$ the uncollapsing function, where $f = \pi^{-1} \restriction \mathrm{ON}$. Given G generic and $G' := \pi[G]$, we will often implicitly extend π and ν to $M[G]$ and $M'[G']$ in the obvious way:

$$\begin{aligned} \pi : M[G] &\to M'[G'] \\ x &\mapsto \{\pi(y) : y \in x \cap M[G]\} \end{aligned}$$

and

$$\nu : (M'[G'], f) \to M[G]$$
$$x \mapsto \begin{cases} f(x) & \text{if } x \in ON; \\ \{\nu(y) : y \in x\} & \text{otherwise.} \end{cases}$$

The uncollapsing map is well defined as $M' \cap \mathrm{ON} = M'[G'] \cap \mathrm{ON}$.

The next theorem shows that the ord-evaluation,the evaluation, the transitive collapse and the uncollapsing map form a commutative diagram.

Theorem 2.5.5. *Given M, P and G as in the previous definition, and $G' := \pi[G]$, the extension $M[G]$ corresponds uncollapsing the transitive forcing extension of the transitive collapse of M. Namely:*

$$M[G] = \nu[M'[G']]$$

where $M' := \pi[M]$ and $G' := \pi[G]$. In fact, for every $\dot{\tau} \in M^P$

$$\mathrm{ordval}_G(\dot{\tau}) = \nu(\mathrm{val}_{G'}(\pi(\dot{\tau}))).$$

The following diagram is therefore commutative:

$$\begin{tikzcd} M \arrow[r, hook, "\check{}"] & M^P \arrow[r, "\pi"] \arrow[d, "\mathrm{ordval}_G"] & M'^{P'} \arrow[d, "\mathrm{val}_{G'}"] \\ & M[G] & M'[G'] \arrow[l, "\nu"'] \end{tikzcd}$$

$$\begin{tikzcd} \dot{\tau} \arrow[r, "\pi"] \arrow[d, "\mathrm{ordval}_G"'] & \pi(\dot{\tau}) \arrow[d, "\mathrm{val}_{G'}"] \\ \mathrm{ordval}_G(\dot{\tau}) & \mathrm{val}_{G'}(\pi(\dot{\tau})) \arrow[l, "\nu"'] \end{tikzcd}$$

Proof. We show that for every $\dot{\tau} \in M^P$ $\mathrm{ordval}_G(\dot{\tau}) = \nu(\mathrm{val}_{G'}(\pi(\dot{\tau})))$.
If $\mathrm{val}_{G'}(\pi(\dot{\tau})) \in \mathrm{ON}$, then the theorem follows from the definition of ordval.
If $\mathrm{val}_{G'}(\pi(\dot{\tau})) \notin \mathrm{ON}$, then by definition of ordval, by induction on the rank of $\dot{\tau}$ and by definition of the maps π and ν we get:

$$\begin{aligned} \mathrm{ordval}_G(\dot{\tau}) &= \{\mathrm{ordval}_G(\dot{\sigma}) : \exists p \in G(\langle \dot{\sigma}, p\rangle \in \dot{\tau})\} \\ &= \{\nu(\mathrm{val}_{G'}(\pi(\dot{\sigma}))) : \exists p \in G(\langle \pi(\dot{\sigma}), \pi(p)\rangle \in \pi(\tau))\} \\ &= \{\nu(\mathrm{val}_{G'}(\dot{\xi})) : \exists q \in G'(\langle \dot{\xi}, q\rangle \in \pi(\dot{\tau}))\} \\ &= \{\nu(\mathrm{val}_{G'}(\dot{\xi})) : \mathrm{val}_{G'}(\dot{\xi}) \in \mathrm{val}_{G'}(\pi(\dot{\tau}))\} \\ &= \nu(\mathrm{val}_{G'}(\pi(\dot{\tau}))). \end{aligned}$$

□

The construction of $M[G]$ that we have seen so far is what we are looking for, as the next theorem points out.

Theorem 2.5.6. *Let M be an ord-transitive model, $P \in M$ a partial order and let $G \subseteq P$ be M-generic. The "ord-forcing" extension $M[G]$ satisfies the following properties:*

1. *$M \subseteq M[G]$;*
2. *$M \cap \mathrm{ON} = M[G] \cap \mathrm{ON}$;*
3. *$M[G]$ is ord-transitive;*
4. *$M[G]$ end-extends M, i.e. $\forall x \in M \forall y \in M[G]$ $(y \in x \to y \in M)$;*
5. *For all P-names $\dot{\tau} \in M$*
$$M[G] \models \phi(\mathrm{ordval}_G(\dot{\tau})) \text{ iff for some } p \in G\ M \models p \Vdash^* \phi(\dot{\tau}).$$
6. *Let N be an ord-transitive model such that $M \subseteq N$, $G \in N$ and $M \cap \mathrm{ON} = N \cap \mathrm{ON}$. Then $M[G] \subseteq N$.*

Proof. We strongly employ the equivalent definition of $M[G]$ given by Lemma 2.5.5. Commutativity of check names with transitive collapses is also used (Lemma 2.4.3)

1. Let $x \in M$. Then
$$\begin{aligned}\mathrm{ordval}_G(\check{x}^M) &= \nu(\mathrm{val}_{G'}(\pi(\check{x}^M)))\\ &= \nu(\mathrm{val}_{G'}(\pi(x)\check{\ }))\\ &= \nu(\pi(x))\\ &= x.\end{aligned}$$
Therefore $x = \mathrm{ordval}_G(\check{x}^M) \in M[G]$.

2.
$$\begin{aligned}M[G] \cap \mathrm{ON} &= f[M'[G'] \cap \mathrm{ON}]\\ &= f[M' \cap \mathrm{ON}]\\ &= \pi^{-1}[M' \cap \mathrm{ON}]\\ &= M \cap \mathrm{ON}\end{aligned}$$
The last equality holds, because π is an $\in$-isomorphism between M and M' (Lemma 1.3.4(1)), therefore $\forall \alpha \in M(M \models \alpha \text{ is an ordinal} \Leftrightarrow M' \models \pi(\alpha) \text{ is an ordinal})$.

3. $M[G]$ is ord-transitive because it is the image of an uncollapsing map [3] (Lemma 1.3.4 (3)).

4. Let $x \in M$ and $y \in x \cap M[G]$. We prove that $y \in M$. If $x \in M \setminus \mathrm{ON}$, it is clear since $x \subseteq M$.
If $x \in M \cap \mathrm{ON}$, then in particular y is an ordinal because it is an element of the ordinal x. So $y \in M[G] \cap \mathrm{ON} = M \cap \mathrm{ON}$, by point 2. We conclude that $M[G]$ is an end-extension of M.

5. We observe that $\pi : M \to M'$ can be extended to $\pi : M[G] \to M'[G']$ in the following way: for $x \in M[G]$ $\pi(x) = \{\pi(y) : y \in x \cap M[G]\}$. By the same way of Lemma 1.3.4 it can be proved that π is an $\in$-isomorphism between $M[G]$ and $M'[G']$. Moreover, by induction of the rank, by the definitions of π and ν and by Theorem 2.5.5

$$\pi(\mathrm{ordval}_G(\dot{\tau})) = \pi(\nu(\mathrm{val}_{G'}(\pi(\dot{\tau})))) = \mathrm{val}_{G'}(\pi(\dot{\tau}))$$

We are now ready to prove equivalences below.

$$\begin{aligned} M[G] \models \phi(\mathrm{ordval}_G(\dot{\tau})) \Leftrightarrow & M'[G'] \models \phi(\pi(\mathrm{ordval}_G(\dot{\tau}))) \\ & \quad \text{because } \pi \text{ is an } \in \text{-isomorphism} \\ \Leftrightarrow & M'[G'] \models \phi(\mathrm{val}_{G'}(\pi(\dot{\tau}))) \\ & \quad \text{because } \pi(\mathrm{ordval}_G(\dot{\tau})) = \mathrm{val}_{G'}(\pi(\dot{\tau})) \\ \Leftrightarrow & M' \models q \Vdash^* \phi(\pi(\dot{\tau})) \text{ for some } q \in G' \\ & \quad \text{by the Forcing theorem for c.t.m.} \\ \Leftrightarrow & M' \models \pi(p) \Vdash^* \phi(\pi(\dot{\tau})) \text{ for some } p \in G \\ \Leftrightarrow & M \models p \Vdash^* \phi(\dot{\tau}) \text{ for some } p \in G \\ & \quad \text{because } \pi \text{ is an } \in \text{-isomorphism} \end{aligned}$$

6. For any $\dot{\tau} \in M^P$, $\dot{\tau} \in N$. Since $G \in N$ and $\pi_M^{-1} \restriction \mathrm{ON} = \pi_N^{-1} \restriction \mathrm{ON}$ (this comes from $M \cap \mathrm{ON} = N \cap \mathrm{ON}$) we see, by induction on the rank, that:

$$\mathrm{ordval}_G(\dot{\tau}) = (\mathrm{ordval}_G(\dot{\tau}))^N \in N. \qquad \square$$

[3] It is actually the extended uncollapsing map, as we pointed out in the Remark 2.5.4.

Chapter 3

Forcing iterations

This chapter consists of three sections. In the first section we present the notion of forcing iteration on a family of ideals. The second section investigates complete and dense embeddings from the point of view of ord-transitive models. Examples of natural embeddings generated by iterations are also given. Moreover we will observe that substituting partial functions with complete functions produces a forcing-equivalent definition of iteration. The last section deals with quotient forcings: we give definitions and properties useful for the next chapters.

3.1 Definition of a forcing iteration

Our definition of forcing iteration is inspired by [14, Section 3]. We distinguish between iterations with final limit and iteration without final limit. We recall some notation: when p is a partial function, $p \restriction \alpha$ designates the sequence $\langle p(\beta) : \beta \in \alpha \rangle$ and the *support* relatively to α is the set $\{\beta \in \alpha : p(\beta) \text{ is defined and } p(\beta) \neq 1_{Q_\beta}\}$. We say that $I \subseteq \mathcal{P}(\alpha)$ is an *ideal* on α if the following conditions are satisfied: $\emptyset \in I$, $\alpha \notin I$, $(\forall x, y \in I \rightarrow x \cup y \in I)$ and $\forall x \in I \wedge \forall y \subseteq \alpha (y \subseteq x \rightarrow y \in I)$.

Definition 3.1.1. Let $\mathcal{I} = \langle I_\alpha : \alpha \leq \epsilon \rangle$ be a family of ideals such that:

- $\forall \alpha < \epsilon \; I_\alpha \subseteq \mathcal{P}(\alpha)$ is an ideal on $\alpha + 1$ containing all the finite subsets of α;
- $\forall \alpha < \beta \leq \epsilon \; I_\beta \cap \mathcal{P}(\alpha) = I_\alpha$.

A *forcing iteration* on $\mathcal{I}$ of length ϵ without (or with) final limit is an object of the form:

$$\langle\langle\langle P_\alpha, \leq_{P_\alpha}, 1_{P_\alpha}\rangle : \alpha < \epsilon\rangle, \langle\langle Q_\alpha, \leq_{Q_\alpha}, 1_{Q_\alpha}\rangle : \alpha < \epsilon\rangle\rangle,$$

(respectively, of the form:

$$\langle\langle\langle P_\alpha, \leq_{P_\alpha}, 1_{P_\alpha}\rangle : \alpha \leq \epsilon\rangle, \langle\langle Q_\alpha, \leq_{Q_\alpha}, 1_{Q_\alpha}\rangle : \alpha < \epsilon\rangle\rangle),$$

which satisfy the following conditions:[1]

- Each $\langle P_\alpha, \leq_{P_\alpha}, 1_{P_\alpha}\rangle$ is a partial order;
- Each $\langle Q_\alpha, \leq_{Q_\alpha}, 1_{Q_\alpha}\rangle$ is a P_α-name for a partial order;
- For α limit $P_\alpha = \{p \subseteq \alpha \times \bigcup_{\gamma<\alpha} \mathrm{dom}(Q_\gamma) : p$ is a partial function with $\mathrm{dom}(p) \subseteq \alpha$ and $\forall \beta \in \mathrm{dom}(p)\ p(\beta) \in \mathrm{dom}(Q_\beta)$ and $\mathrm{supp}(p) \in I_\alpha\}$;
- $p \in P_{\alpha+1}$ iff $p \restriction \alpha \in P_\alpha \wedge p(\alpha) \in \mathrm{dom}(Q_\alpha) \wedge p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \in Q_\alpha$;
- For any $p \in P_\beta$, for all $\alpha < \beta$ we have $p \restriction \alpha \in P_\alpha$;
- For all $\alpha < \beta$, $P_\alpha \subseteq P_\beta$;
- For all $\beta \in \mathrm{dom}(p)$ $p \restriction \beta \Vdash_{P_\beta} p(\beta) \in Q_\beta$;
- Given a condition $r \in P_\beta$ and $\alpha < \beta$, we define

  $$r^{\mathrm{tot}}_\beta(\alpha) := \begin{cases} r(\alpha) & \alpha \in \mathrm{dom}(r), \\ 1_{Q_\alpha} & \alpha \in \beta \setminus \mathrm{dom}(r) \end{cases}$$

  When it is clear from the context we omit β and write r^{tot}.
- For all $p, q \in P_\beta$, we say that q is *stronger* than p (and write $q \leq_{P_\beta} p$) if $\forall \alpha < \beta\ q \restriction \alpha \Vdash_{P_\alpha} q^{\mathrm{tot}}(\alpha) \leq_{Q_\alpha} p^{\mathrm{tot}}(\alpha)$;
- $P_0 := \{\emptyset\}$;
- For all β, $1_{P_\beta} := \langle 1_{Q_\alpha} : \alpha < \beta\rangle$;
- For all $\alpha < \beta$, for all $p \in P_\beta$, if $q \leq_{P_\alpha} p \restriction \alpha$, then

  $$q \wedge p := q \cup p \restriction [\alpha, \beta) \in P_\beta.$$

  We call $\bar{P}$ a *finite support iteration* if $\bar{P}$ is a forcing iteration on the family $\langle \alpha^{<\omega} : \alpha \leq \epsilon\rangle$. The iteration $\bar{P}$ is a *countable support iteration* if it is an iteration on the ideals $\langle \alpha^{\leq\omega} : \alpha \leq \epsilon\rangle$.

[1] The indices of P in the next points have to be considered $< \epsilon$ for the iterations without final limit and $\leq \epsilon$ otherwise. The indices of Q are always considered $< \epsilon$.

From the definition it is immediate to see that:

Property 3.1.2. 1. For all β, the empty set is an element of P_β;

2. For all β, for all $p, q \in P_\beta$ if $q \leq_{P_\beta} p$, then $\forall \alpha < \beta \; (q \restriction \alpha \leq_{P_\alpha} p \restriction \alpha)$;

3. For all $\alpha < \beta$, given $p \in P_\beta$ and $q \in P_\alpha$ such that $q \leq_{P_\alpha} p \restriction \alpha$, then $q \wedge p \leq_{P_\beta} q$ and $q \wedge p \leq_{P_\beta} p$;

4. For all $\alpha < \beta$, for all $p, q \in P_\beta$ $(p \restriction \alpha \perp_{P_\alpha} q \restriction \alpha \to p \perp_{P_\beta} q)$;

5. $\forall \alpha < \beta \forall p, q \in P_\beta$
$$\operatorname{supp}(p) \cap \operatorname{supp}(q) \subseteq \alpha \to (p \restriction \alpha \perp_{P_\alpha} q \restriction \alpha \leftrightarrow p \perp_{P_\beta} q).$$

Proof. 1. $\emptyset \in P_\beta$ follows from $\emptyset \in P_0$ and $P_0 \subseteq P_\beta$.

2. Let $q \leq_{P_\beta} p$. By definition,
$$\forall \gamma < \beta \; q \restriction \gamma \Vdash_{P_\gamma} q^{\text{tot}}(\gamma) \leq_{Q_\gamma} p^{\text{tot}}(\gamma). \tag{3.1}$$
Now let $\alpha < \beta$. For $\gamma < \alpha$ $(q \restriction \alpha) \restriction \gamma = q \restriction \gamma$, $(p^{\text{tot}} \restriction \alpha)(\gamma) = p^{\text{tot}}(\gamma)$ and $(p^{\text{tot}} \restriction \alpha)(\gamma) = p^{\text{tot}}(\gamma)$. Therefore from (3.1) we can say that
$$\forall \gamma < \alpha (q \restriction \alpha) \restriction \gamma \Vdash_{P_\gamma} (q^{\text{tot}} \restriction \alpha)(\gamma) \leq_{Q_\gamma} (p^{\text{tot}} \restriction \alpha)(\gamma),$$
hence $q \restriction \alpha \leq_{P_\alpha} p \restriction \alpha$.

3. We show that $q \wedge p$ extends both q and p. Given $\gamma < \beta$ if $\gamma < \alpha$, then by hypothesis $(q \wedge p) \restriction \gamma = q \restriction \gamma \Vdash_{P_\gamma} q^{\text{tot}}(\gamma) \leq_{Q_\gamma} p^{\text{tot}}(\gamma) \wedge q^{\text{tot}}(\gamma) \leq_{Q_\gamma} q^{\text{tot}}(\gamma)$. If $\gamma \geq \alpha$, then $(q \wedge p)(\gamma) = p(\gamma)$. Therefore $(q \wedge p) \restriction \gamma \Vdash_{P_\gamma} (q \wedge p)^{\text{tot}}(\gamma) \leq_{Q_\gamma} p^{\text{tot}}(\gamma)$ and also $(q \wedge p) \restriction \gamma \Vdash_{P_\gamma} (q \wedge p)^{\text{tot}}(\gamma) \leq_{Q_\gamma} q^{\text{tot}}(\gamma) = 1_{Q_\gamma}$.

4. If $p \parallel_{P_\beta} q$ then a common extension r would generate the extension $r \restriction \alpha$ common to $p \restriction \alpha$ and $q \restriction \alpha$.

5. One direction has already been proved in the previous point. For the other direction, let $\operatorname{supp}(p) \cap \operatorname{supp}(q) \subseteq \alpha$ and assume $p \restriction \alpha \parallel_{P_\alpha} q \restriction \alpha$ Let $s \in P_\alpha$ be a common extension. Define $r \in P_\beta$ coordinate-wise:
$$r(\gamma) := \begin{cases} s(\gamma) & \gamma \in \operatorname{supp}(p) \cap \operatorname{supp}(q) \\ p(\gamma) & \gamma \in \operatorname{supp}(p) \setminus \operatorname{supp}(q) \\ q(\gamma) & \gamma \in \operatorname{supp}(q) \setminus \operatorname{supp}(p) \\ 1_{Q_\gamma} & \gamma \in \beta \setminus (\operatorname{supp}(p) \cup \operatorname{supp}(q)). \end{cases}$$
By induction on the length, r is a common extension of p and q. $\square$

The requirement that the supports of the conditions stay in an ideal well defines the limit steps of the iteration and is essential to preserve the κ-(anti)chain condition in a finite support iterated forcing, for any regular $\kappa > \omega$ (see for example [19, Lemma VIII 5.12]).

3.2 Complete embeddings

We show that our definition of forcing iteration produces, as expected, an increasing chain of completely embedded posets (Lemma 3.2.6). We also observe that in the definition of forcing iteration it does not matter if we consider only total functions on the α's instead of partial functions (Lemma 3.2.7 and Lemma 3.2.8). We begin therefore with the definition of a *complete embedding* and discuss two equivalent forms (Definition 3.2.1 and Lemma 3.2.2). The said definitions can be seen inside an ord-transitive model under certain assumptions (Remark 3.2.3). We also present the version for ord-transitive models of two consequences of a complete embedding in forcing extensions (Lemma 3.2.4 and Lemma 3.2.5).

Definition 3.2.1. Given two forcing notions $\langle P, \leq_P, 1_P\rangle$ and $\langle Q, \leq_Q, 1_Q\rangle$, we say that $i : P \to Q$ is a *complete embedding* if the following conditions are satisfied:

a) $\forall p, q \in P(q \leq_P p \to i(q) \leq_Q i(p))$;

b) $\forall p, q \in P(q \perp_P p \leftrightarrow i(q) \perp_Q i(p))$;

c) If A is a maximal antichain in P then $i[A]$ is a maximal antichain in Q.

If i is the identity map, we say that P is a *complete subforcing* of Q and write $P \subset_C Q$. If moreover $i[P]$ is dense in Q we call i a *dense embedding*.
We say that P and Q are *forcing equivalent* if there is a dense embedding $i : P \to Q$.

When we check the completeness of an embedding we can also examine the equivalent properties:

Lemma 3.2.2. *Let $\langle P, \leq_P, 1_P\rangle$ and $\langle Q, \leq_Q, 1_Q\rangle$ be two partial orders. Let $i : P \to Q$ be a relation which satisfies the first two points of Definition 3.2.1. Then the following points are equivalent:*

d) If A is a maximal antichain in P then $i[A]$ is a maximal antichain in Q;

d') $\forall q \in Q \exists p' \in P \forall p \in P(i(p) \perp_Q q \to p \perp_P p')$;

d") $\forall q \in Q \exists p' \in P \forall p \in P(p \leq_P p' \to i(p) \parallel_Q q)$. We call p' a reduction *of q to P.*

Proof. d) $\Rightarrow$ d'): Fix $q \in Q$. Define $B_q := \{p \in P : i(p) \perp_Q q\}$. Consider an antichain A maximal for B_q (i.e. $A \subseteq B_q$ and $\forall p \in B_q \exists a \in A(a \parallel_P p)$). Then $i[A]$ is not maximal in P, as $\forall a \in A\ i(a) \perp_Q q$. Therefore, by hypothesis, A is not maximal in P and there is $p' \in P$ such that $\forall a \in A(p' \perp_P a)$. By the maximality of A in B_q it follows $\forall p \in P(i(p) \perp_Q q \to p \perp_P p')$.

d') $\Rightarrow$ d"): The proof just follows from the following implications:

$$\begin{aligned}
&\forall q \in Q \exists p' \in P \forall p \in P(i(p) \perp_Q q \to p \perp_P p') && \Rightarrow \\
&\forall q \in Q \exists p' \in P \forall p \in P(i(p) \perp_Q q \to p \nleq_P p') && \Rightarrow \\
&\forall q \in Q \exists p' \in P \forall p \in P(p \leq_P p' \to i(p) \parallel_Q q)
\end{aligned}$$

d") $\Rightarrow$ d): Let A be a maximal antichain in P. If by contradiction $i[A]$ is not maximal, then there is a $q \in Q$ such that $\forall a \in A\ i(a) \perp_Q q$. By hypothesis, there is a reduction p' of q, therefore $\forall p \in P(i(p) \perp_Q q \to p \nleq_P p')$. It follows in particular that $\forall p \in A{\downarrow} := \{p \in P : \exists a \in A p \leq_P a\}$ $p \nleq_P p'$. This means that A is not maximal, a contradiction. □

Remark 3.2.3. Given an ord-transitive model M and $i, P, Q \in M \setminus \mathrm{ON}$, the notion of complete embedding is absolute for M. Furthermore, Lemma 3.2.2 is provable in M if all the B_q are definable in M (an instance of the comprehension scheme is needed) and if a maximal antichain $A \subseteq B_q$ can be found (AC needed). The notion of antichain is absolute for M if $A \in M \setminus \mathrm{ON}$.

Complete embeddings admit backwards-genericity and they generate increasing extensions of ord-transitive models. The next lemma states it more precisely:

Lemma 3.2.4. *Let M be a countable ord-transitive model, let $i, P, Q \in M \setminus \mathrm{ON}$ such that $i : P \to Q$ is a (complete embedding)^M. If H is Q-generic over M, then $i^{-1}(H)$ is P-generic over M and $M[i^{-1}(H)] \subseteq M[H]$. Moreover, if i is a dense embedding then $M[i^{-1}(H)] = M[H]$.*

Proof. We refer to [19, Theorem VII 7.5] for the proof, for which we just make some comments. We need to ensure that the version for transitive models works also for ord-transitive models. For the genericity of $i^{-1}(H)$ [19, Lemma VII 7.4] is applied. In our case we have to require that for any $p, q \in P$ the set

$$D := \{r \in P : r \perp_P p \vee r \perp_P q \vee (r \leq_P p \wedge r \leq_P q)\}$$

belongs to M. Therefore M has to satisfy an instance of the comprehension scheme for the formula $\phi(r, P, \leq_P, p, q) := \forall s \in P((s,r) \in \leq_P \rightarrow (s,p) \notin \leq_P) \vee \forall s \in P((s,r) \in \leq_P \rightarrow (s,q) \notin \leq_P) \vee ((r,p) \in \leq_P \wedge (r,q) \in \leq_P)$. But this is easily solved because the ord-transitive models we work with satisfy enough axioms of ZFC (see Definition 1.1.1).

To show that $M[i^{-1}(H)] \subseteq M[H]$, we remark that $i \in M \subseteq M[H]$ and $H \in M[H]$. It follows that $i^{-1}(H) \in M[H]$. Thus $M[i^{-1}(H)] \subseteq M[H]$ by minimality of $M[i^{-1}(H)]$ (see Theorem 2.5.6, point 6). If i is a dense embedding we proceed analogously (see also [19, Theorem VII 7.11]). □

A further property of complete embeddings for (ord-)transitive models is that we can force with the "bigger" poset the same sentences forced by the "smaller" poset:

Lemma 3.2.5. *Let M be an ord-transitive model. Let $P, Q, i \in M \setminus \mathrm{ON}$ and let $i : P \to Q$ be a complete embedding. Define $i_*(\dot{\tau}) := \{\langle i_*(\dot{\sigma}), i(p)\rangle : \langle \dot{\sigma}, p\rangle \in \dot{\tau}\}$. If $\phi(x_1, \ldots, x_n)$ is absolute for ord-transitive models then*

$$p \Vdash_P \phi(\dot{\tau}_1, \ldots, \dot{\tau}_n) \text{ iff } i(p) \Vdash_Q \phi(i_*(\dot{\tau}_1), \ldots, i_*(\dot{\tau}_n))$$

for $\dot{\tau}_1, \ldots, \dot{\tau}_n \in M^P$. If i is a dense embedding the previous equivalence holds for any formula ϕ.

Proof. The proof is analogue to the one of [19, Theorem VII 7.13], which is done for countable transitive models. According to Theorem 2.5.6 the forcing relation for countable ord-transitive models enjoys the same properties as the forcing relation for countable transitive models. □

Our definition of forcing iteration represent a concrete example of complete subforcings:

Lemma 3.2.6. *Let $\bar{P}$ be a forcing iteration of length ϵ. For $\alpha < \beta < \epsilon$ (or for $\alpha < \beta \leq \epsilon$ if the iteration has a final limit) we define:*

$$\begin{aligned} i_{\alpha\beta} : P_\alpha &\to P_\beta \\ p &\mapsto p \end{aligned}$$

Then $i_{\alpha\beta}$ is a complete embedding and therefore $P_\alpha \subset_C P_\beta$.

Proof. It is clear that $p \leq_{P_\alpha} q \to i_\alpha(p) \leq_{P_\beta} i_\alpha(q)$. We consider now $p \perp_{P_\alpha} q$. If, by contradiction, $i_{\alpha\beta}(p) \parallel_{P_\beta} i_{\alpha\beta}(q)$, then $\exists r \in P_\beta\ r \leq_{P_\beta} p \wedge r \leq_{P_\beta} q$. This implies $r \restriction \alpha \leq_{P_\alpha} p \wedge r \restriction \alpha \leq_{P_\alpha} q$, a contradiction. Let $p \in P_\beta$, then $p \restriction \alpha$ is a reduction to P_α: for any $r \leq_{P_\alpha} p \restriction \alpha$ we have $r \parallel_{P_\beta} p$ as $r \wedge p \in P_\beta$ is a common extension. □

In the definition of iterated forcing we could have chosen total functions instead of partial functions to define the conditions. In fact, the two versions are *forcing-equivalent*, i.e. one can be densely embedded into the other, as we show in the next lemma. Therefore, according to Lemma 3.2.7, they produce the same forcing extension.

Lemma 3.2.7. *Let $\bar{P}$ be a forcing iteration of length ϵ. For $\alpha < \epsilon$ (or for $\alpha \leq \epsilon$ if the iteration has a final limit), let $P_\alpha^{\mathrm{tot}} := \{p_\alpha^{\mathrm{tot}} : p \in P_\alpha\}$. We define*

$$\begin{aligned} \mathrm{tot}_\alpha : P_\alpha &\to P_\alpha^{\mathrm{tot}} \\ p &\mapsto p_\alpha^{\mathrm{tot}}. \end{aligned}$$

Then the set $P_\alpha^{\mathrm{tot}} := \mathrm{tot}_\alpha[P_\alpha]$ is a partial order with $1_{P_\alpha^{\mathrm{tot}}} = 1_{P_\alpha}$ and with the relation $q \leq_{P_\alpha^{\mathrm{tot}}} p$ defined as $\forall \gamma < \alpha\ p \restriction \gamma \Vdash_{P_\gamma^{\mathrm{tot}}} p(\gamma) \leq_{Q_\gamma} q(\gamma)$. Moreover $\mathrm{tot}_\alpha : P_\alpha \to P_\alpha^{\mathrm{tot}}$ is a dense embedding.

Proof. It is clear that $\langle P_\alpha^{\mathrm{tot}}, 1_{P_\alpha^{\mathrm{tot}}}, \leq_{P_\alpha^{\mathrm{tot}}}\rangle$ is a partial order. We prove that $\mathrm{tot}_\alpha : P_\alpha \to P_\alpha^{\mathrm{tot}}$ is a dense embedding. The weakest element is preserved: $\mathrm{tot}_\alpha(1_{P_\alpha})$ $= 1_{P_\alpha} = 1_{P_\alpha^{\mathrm{tot}}}$. It is also clear that $p \leq_{P_\alpha} q \to \mathrm{tot}_\alpha(p) \leq_{P_\alpha^{\mathrm{tot}}} \mathrm{tot}_\alpha(q)$. We consider now $p \perp_{P_\alpha} q$. If, by contradiction, $\exists r \in P_\alpha^{\mathrm{tot}}(r \leq_{P_\alpha^{\mathrm{tot}}} \mathrm{tot}_\alpha(p) \wedge r \leq_{P_\alpha^{\mathrm{tot}}} \mathrm{tot}_\alpha(q))$, then $\exists A \subseteq \alpha$ such that $r \restriction A \in P_\alpha$ and $\forall \gamma \in \alpha \setminus A\ r(\gamma) = 1_{Q_\gamma}$. Then $r \restriction A \leq_{P_\alpha} p \wedge r \restriction A \leq_{P_\alpha} q$, a contradiction. Finally $\mathrm{tot}_\alpha[P_\alpha]$ is dense in P_α^{tot}: let $p \in P_\alpha^{\mathrm{tot}}$, then, by definition, there is an $A \subseteq \alpha$ such that $p \restriction A \in P_\alpha$ and $\forall \gamma \in \alpha \setminus A\ p(\gamma) = 1_{Q_\alpha}$. Then $\mathrm{tot}_\alpha(p \restriction A) \leq_{P_\alpha^{\mathrm{tot}}} p$. □

Analogously to Lemma 3.2.7,we show that for all $\alpha < \beta$ the poset P_α^{tot} is completely embedded in P_β^{tot}.

Lemma 3.2.8. *Let $\bar{P}$ be a forcing iteration of length ϵ. For $\alpha < \beta < \epsilon$ (or for $\alpha < \beta \leq \epsilon$) define:*

$$\begin{aligned} i_{\alpha\beta}^{\mathrm{tot}} : P_\alpha^{\mathrm{tot}} &\to P_\beta^{\mathrm{tot}} \\ p &\mapsto p^\frown\langle 1_{Q_\gamma} : \alpha \leq \gamma < \beta\rangle \end{aligned}$$

Then $i^{\text{tot}}_{\alpha\beta}$ is a complete embedding and satisfies the properties of [19, Lemma VII 5.11].

Proof. First of all we remark that $i^{\text{tot}}_{\alpha\beta}(1_{P^{\text{tot}}_\alpha}) = 1_{P^{\text{tot}}_\alpha}{}^\frown\langle 1_{Q_\gamma} : \alpha \leq \gamma < \beta\rangle = 1_{P^{\text{tot}}_\beta}$. Moreover for $q \leq_{P^{\text{tot}}_\alpha} p$, by induction on the length $q^\frown\langle 1_{Q_\gamma} : \alpha \leq \gamma < \beta\rangle \leq_{P^{\text{tot}}_\beta} p^\frown\langle 1_{Q_\gamma} : \alpha \leq \gamma < \beta\rangle$. Consider now $q \perp_{P^{\text{tot}}_\alpha} p$. We show that $i^{\text{tot}}_{\alpha\beta}(q) \perp_{P^{\text{tot}}_\beta} i^{\text{tot}}_{\alpha\beta}(p)$. If not, an element $r \in P^{\text{tot}}_\beta$ such that $r \leq_{P^{\text{tot}}_\beta} i^{\text{tot}}_{\alpha\beta}(p) \wedge r \leq_{P^{\text{tot}}_\beta} i^{\text{tot}}_{\alpha\beta}(q)$ generates a common extension of p and q: $r \restriction \alpha \leq_{P^{\text{tot}}_\alpha} i^{\text{tot}}_{\alpha\beta}(q)q \wedge r \restriction \alpha \leq_{P^{\text{tot}}_\alpha} p$. Let $q \in P^{\text{tot}}_\beta$. Then $q \restriction \alpha$ is a reduction of q to P^{tot}_α. In fact, for any $p \leq_{P^{\text{tot}}_\alpha} q \restriction \alpha$ we have $i^{\text{tot}}_{\alpha\beta}(p) \parallel_{P^{\text{tot}}_\alpha} q$ because $p \wedge q$ extends in P^{tot}_β both $i^{\text{tot}}_{\alpha\beta}(p)$ and q. $\square$

The last two lemmata just ensured that it does not make any difference if in Definition 3.1.1 we take partial functions or total functions. Both choices are forcing-equivalent and when we take total functions, the definition coincides with the one presented in [19]. We can therefore use all the properties listed in [19, Definitions VII 5.8-5.10 and Lemma VII 5.11)].

3.3 Quotient forcing

We present in this section three properties related to quotient forcings generated by complete embeddings. Properties 3.3.3, 3.3.6 and 3.3.7 correspond to Exercises D3, D4, D5 of [19, Chapter VII], which are proposed for countable transitive models. We decided to keep the notation for countable transitive models in these proofs, knowing that the properties work also for countable ord-transitive models M. Namely, for ord-transitive models we just need to translate all the occurrences of check names $\check{x}$ to ord-transitive check names $\check{x}^M$, consider ord-evaluations of names and interpret the forcing relation inside the ord-transitive model. From these properties it will be clear that extending a model by P_β is equivalent to extend via P_α and then via the quotient P_β/H_α.

Definition 3.3.1. Let $\bar{P}$ be a forcing iteration of length ϵ (with or without final limit) . For $\alpha < \beta < \epsilon$ (or for $\alpha < \beta \leq \epsilon$ if the iteration has a final limit) let $H_\alpha \subseteq P_\alpha$ be generic and let define the *quotient of P_β over H_α* as follows:

$$P_\beta/H_\alpha := \{p \in P_\beta : p \restriction \alpha \in H_\alpha\}$$

with order $\leq_{P_\beta/H_\alpha} = \leq_{P_\beta} \cap (P_\beta/H_\alpha \times P_\beta/H_\alpha)$ and weakest element $1_{P_\beta/H_\alpha} := 1_{P_\beta}$.

Lemma 3.3.2. *In the settings of the previous definition, for $\alpha < \beta$ let $i_{\alpha\beta} : P_\alpha \to P_\beta$ be the identity embedding defined in Lemma 3.2.6. Let $X := \{q \in P_\beta : \forall p \in H_\alpha\ q \parallel_{P_\beta} i_{\alpha\beta}(p)\}$. Then:*

$$X = P_\beta / H_\alpha.$$

Proof. Let $q \in X$. Then $\forall p \in H_\alpha\ q \parallel_{P_\beta} p$ (by definition $i_{\alpha\beta}(p) = p$). In particular $\forall p \in H_\alpha\ q \restriction \alpha \parallel_{P_\alpha} p$. Consider a maximal antichain $A_{q\restriction\alpha} \supseteq \{q \restriction \alpha\}$. Then $H_\alpha \cap A_{q\restriction\alpha} \neq \emptyset$. Let $r \in H_\alpha \cap A_{q\restriction\alpha}$, then $r \parallel_{P_\alpha} q \restriction \alpha$ and hence $r = q \restriction \alpha$ because both r and $q \restriction \alpha$ are elements of the antichain $A_{q\restriction\alpha}$. Hence $q \restriction \alpha \in H_\alpha$. This concludes the proof that $q \in P_\beta / H_\alpha$.
Let now $q \in P_\beta / H_\alpha$. Then $q \restriction \alpha \in H_\alpha$. Hence $\forall p \in H_\alpha\ q \restriction \alpha \parallel_{P_\alpha} p$. For a fixed $p \in H_\alpha$ let $r \in P_\alpha$ extend both $q \restriction \alpha$ and p. Then $r \wedge q \in P_\beta$ extends in P_β both q and p. We have proved that $\forall p \in H_\alpha\ q \parallel_{P_\beta} p$. □

Property 3.3.3. Let M be a countable (ord-)transitive model. Let $P, Q, i \in M \setminus \mathrm{ON}$ and let $i : P \to Q$ be a complete embedding. For any generic $G \subseteq P$ define $Q/G := \{q \in Q : \forall p \in G(q \parallel i(p))\}$. Then p is a reduction of q to P iff $p \Vdash \check{q} \in \check{Q}/\Gamma$, where Γ is the name for the generic filter.

Proof. From left to right: we suppose that $p \nVdash \check{q} \in \check{Q}/\Gamma$. Then there is a $r \leq p$ such that $r \Vdash \check{q} \notin \check{Q}/\Gamma$. Let G be generic such that $r \in G$. Then $M[G] \models \exists g \in G\ i(g) \perp_Q q$. In particular for $s \leq_P p$ such that $s \leq_P g \wedge s \leq_P r$ we have $i(s) \perp_Q q$. This contradicts the definition of p being a reduction of q to P.

From right to left: $p \Vdash \check{q} \in \check{Q}/\Gamma$ implies $\forall p' \leq_P p(p' \Vdash \check{q} \in \check{Q}/\Gamma)$. We fix $p' \leq_P p$ and a G generic such that $p' \in G$. Then $M[G] \models q \in Q/G$, i.e. $M[G] \models \forall g \in G\ i(g) \parallel_Q q$. In particular this must be true for p': $i(p') \parallel_Q q$. Therefore p is a reduction of q to P. □

Lemma 3.3.4. *With the settings of the previous property, let $D \subseteq Q$ be dense in Q. Then the set $R_D := \{r \in P : \exists s \in D\ \forall r' \leq r(i(r') \parallel_Q s)\}$ of all reductions of elements of D is open and dense in P. Moreover if D_q is a set* dense under q *(i.e. $\forall q' \leq q \exists d \in D_q\ d \leq q'$), then R_{D_q} is open dense under p, where p is a reduction of q to P.*

Proof. We just prove the second part of the lemma, because the first corresponds to $p = 1_P$ and $q = 1_Q$. Let $q \in Q$, let $p \in P$ be a reduction of q and let $D_q \subseteq Q$ be dense under q. We prove that

$$R_{D_q} := \{r \in P : \exists s \in D_q \forall r' \leq_P r(i(r') \parallel_Q s)\}$$

is open dense under p.
From the definition, it is clear that R_{D_q} is open. We prove that R_{D_q} is predense under p. Let $p' \leq_P p$. Then $i(p') \parallel_Q q$ because p is a reduction of q. Thus there is $s \in Q$ such that $s \leq_Q i(p') \wedge s \leq_Q q$. Since D_q is dense under q, this implies that $\exists d \in D_q$ with $d \leq_Q s$. Consider a reduction $p_d \in R_{D_q}$ of d. The reduction has the property $i(p_d) \parallel_Q d$ and therefore also $i(p_d) \parallel_Q i(p')$. This implies $p_d \parallel_P p'$. The set R_{D_q} is therefore predense under p. To conclude the proof, we just remark that an open predense set is (open) dense. $\square$

Lemma 3.3.5. *Let $G, H \subseteq P$ be generic. If $\forall h \in H \forall g \in G (h \parallel_P g)$, then $H = G$.*

Proof. If by contradiction $H \neq G$, w.l.o.g. there is $g \in G \setminus H$. Fix a maximal antichain $A_g \supseteq \{g\}$, then $A_g \cap H \neq \emptyset$. For any $h \in A_g \cap H$, by hypothesis, we have $h \parallel_P g$. This is a contradiction with A_g being an antichain. $\square$

Property 3.3.6. Let M be a countable (ord-)transitive model. Let $P, Q, i \in M \setminus \mathrm{ON}$ and let $i : P \to Q$ be a complete embedding. Let G be P-generic over M. Let K be Q/G generic over $M[G]$. Then K is Q-generic over M and $M[K]_Q = M[G][K]_{Q/G}$, where $M[K]_Q = \{\tau_K : \tau \in M^Q\}$ and $M[G][K]_{Q/G} = \{\tau_K : \tau \in M[G]^{Q/G}\}$;

Proof. It is clear that K is a filter also in Q. We show that K is Q-generic over M. Let $D \in M$, such that $D \subseteq Q$ is dense in Q. We show that $D \cap Q/G$ is dense in Q/G which will imply that $D \cap K \neq \emptyset$. Let $q_0 \in Q/G$. Then these is a $p_0 \in G$ such that $p_0 \Vdash q_0 \in \check{Q}/\Gamma$. p_0 is therefore also a reduction of q_0. Moreover $D \restriction q_0 := \{d \in D : d \leq q_0\}$ is dense under q_0. By the previous point, $R_{D \restriction q_0} := \{p \in P : \exists d \in D \restriction q_0 \; p \Vdash d \in \check{Q}/\Gamma\}$ is open dense under p_0. Hence $p_0 \Vdash \exists d \in \check{D}, d \leq \check{q_0} \wedge d \in \check{Q}/\Gamma$, this means that $D \cap Q/G$ is dense in Q/G. To show that $M[K]_Q = M[G][K]_{Q/G}$, consider that $M[K]_Q$ is the smallest transitive model containing K as an element and M as a subset. Therefore $M[K]_Q \subseteq M[G][K]_{Q/G}$.

For the other inclusion, consider the generic $i^{-1}[K] \subseteq P$. By hypothesis $\forall k \in K \forall g \in G \; i(g) \parallel_Q k$. Hence $\forall g' \in i^{-1}[K] \forall g \in G (g' \parallel_Q g)$. By Lemma 3.3.5 $i^{-1}[K] = G$. $\square$

Property 3.3.7. Let M be a countable (ord-)transitive model. Let $P, Q, i \in M$ and let $i : P \to Q$ be a complete embedding. Let H be Q-generic over M and let $G := i^{-1}(H)$. Then $H \subseteq Q/G$, H is Q/G-generic over $M[G]$, and $M[H]_Q = M[G][H]_{Q/G}$

Proof. We show that $H \subseteq Q/G$. In fact, $\forall p \in G = i^{-1}(H)\ i(p) \in H$. Hence $\forall h \in H \forall p \in G i(p) \parallel_Q h$ because H is a filter. We prove now that H is Q/G-generic over $M[G]$. Let $D \in M[G]$ such that $D \subseteq Q/G$ is dense in Q/G. There is a P-name τ such that $\tau_G = D$ and there is $p_0 \in G$ such that $p_0 \Vdash \tau \subseteq Q/\Gamma$ is dense in $\check{Q}/\Gamma$.

Claim 3.3.7.1.

$$D_0 := \{q \in Q : \exists p \in P, \exists q_1 \in Q\ (p \Vdash q_1 \in \tau) \wedge q \leq_Q i(p) \wedge q \leq_Q q_1\}$$

is dense below $i(p_0)$.

Proof of the claim. Proof of the claim: Let $r \leq_Q i(p_0)$. Then we can chose a reduction $p_r \in P$ of r such that $p_r \leq_P p_0$. (p is a reduction of r if $\forall p' \leq_P p\ i(p') \parallel_Q r$. Because $r \leq_Q i(p_0)$, then $\forall p' \leq_P p\ i(p') \parallel_Q i(p_0)$. Hence $\forall p' \leq p\ p' \parallel p_0$. In particular, $p \parallel_P p_0$. Take p_r such that $p_r \leq_P p \wedge p_r \leq p_0$. Then p_r is still a reduction of r). Thus $p_r \Vdash r \in Q/\Gamma \wedge \tau$ is dense in Q/Γ. This implies $p_r \Vdash \exists d \in \tau\ d \leq_Q r$, which implies $\exists d \in Q(d \leq_Q r \wedge p_r \Vdash d \in \tau)$. For that d it also holds $p_r \Vdash d \parallel_Q i(p_r)$, by definition of Q/G since $p_r \Vdash i(p_r) \in H$ and $d \in Q/\Gamma$. Thus a $q \in Q$ extending both $i(p_r)$ and d also extends r and belongs to D_0. □

We show that $D \cap H \neq \emptyset$. We know that $i(p_0) \in H$ and D_0 is dense under $i(p_0)$. This implies that $\exists q \in D_0 \cap H$. By definition of D_0, $\exists p \in P \exists q_1 \in Q\ q \leq_Q i(p) \wedge q \leq_Q q_1 \wedge p \Vdash q_1 \in \tau$. Being H a filter, $i(p)$ and q_1 are both elements of H. This implies that $p \in G$ and $q_1 \in D \cap H$. We have therefore proved that H is Q/G-generic over $M[G]$.

Now $M[G][H] \subseteq M[H]$ comes from the fact that $i, H \in M[H]$ (and therefore $G = i^{-1}[H] \in M[H]$). $M[H] \subseteq M[G][H]$ is due to the minimality of $M[H]$. □

Chapter 4

Limits of iterations

Given a forcing iteration without final limit, we examine suitable limits: the direct, the inverse and full countable support limits. We analyze also partial countable support limits and look at partial countable support iterations. We will see that preservation properties impose constraints on the choice of the limits.

4.1 Direct, inverse and full countable support limits

For a forcing iteration of length ϵ without final limit, we want to define a limit P_ϵ for $\bar{P}$. We are going to describe the smallest possible limit as well as the biggest one and the full countable support limit.

Definition 4.1.1. Let $\bar{P}$ be a forcing iteration of length ϵ without final limit, as in Definition 3.1.1. We define three limits of $\bar{P}$:

a) The *direct limit:*

$$P^d_\epsilon := \bigcup_{\alpha<\epsilon} P_\alpha.$$

b) The *inverse limit:*

$$P^i_\epsilon := \{p \subseteq \epsilon \times \bigcup_{\alpha<\epsilon} \operatorname{dom} Q_\alpha : p \text{ is a partial function} \wedge \\ \forall \alpha < \epsilon\ (p \restriction \alpha \in P_\alpha \wedge p(\alpha) \in \operatorname{dom} Q_\alpha)\}$$

c) The *full countable support limit (full CS-limit):*

$$P^{CS}_\epsilon := \begin{cases} P^i_\epsilon & \text{if } \operatorname{cof}(\epsilon) = \omega; \\ P^d_\epsilon & \text{if } \operatorname{cof}(\epsilon) > \omega. \end{cases}$$

Property 4.1.2.
1. The direct limit P^d_ϵ is the smallest possible limit of the iteration.
2. The inverse limit P^i_ϵ is the largest one.

Proof.
1. If P_ϵ is another limit, by definition of forcing iteration, $\forall \alpha < \epsilon$ $P_\alpha \subseteq P_\epsilon$. Hence $P^d_\epsilon = \bigcup_{\alpha<\epsilon} P_\alpha \subseteq P_\epsilon$.
2. If P_ϵ is another limit of the iteration, then any $p \in P_\epsilon$, by definition of forcing iteration, satisfies $\mathrm{dom}(p) \subseteq \epsilon$ and $p \restriction \alpha \in P_\alpha$ $\forall \alpha < \epsilon$. Hence $p \in P^i_\epsilon$. □

Example 4.1.3. Limits in a finite support iteration $\bar{P}$ of length ϵ are direct limits. In fact, let $\beta \leq \epsilon$ be a limit ordinal and let $p \in P_\beta$. Then $|\operatorname{supp}(p)| < \omega$ and there is an ordinal $\alpha < \beta$ such that $\operatorname{supp}(p) \subseteq \alpha$. So we can consider $p \in P_\alpha$. This shows that $P_\beta = \bigcup_{\alpha<\beta} P_\alpha$.

Remark 4.1.4. Let $\bar{P}$ be a forcing iteration of length ϵ. If a condition belongs to a direct limit P^d_β for some $\beta \leq \epsilon$, then its support may be infinite. For example, if P^i_ω is the inverse limit (and admits conditions with infinite support) and $P^d_{\omega+\omega}$ is the direct limit, a condition $p \in P^d_{\omega+\omega}$ can come from P^i_ω and can have infinite support.

Example 4.1.5. Let $\bar{P}$ be a countable support iteration of length ϵ. Then for every limit ordinal $\beta \leq \epsilon$ we have $P_\beta = P^{\mathrm{CS}}_\beta$. This is showed by induction: let for all limits $\alpha < \beta$ be $P_\alpha = P^{\mathrm{CS}}_\alpha$.
If $\operatorname{cof}(\beta) = \omega$, we just need to show that $P^{\mathrm{CS}}_\beta = P^i_\beta \subseteq P_\beta$, as $P_\beta \subseteq P^i_\beta$ is clear. Let $p \in P^i_\beta$, we show that $|\operatorname{supp}(p)| \leq \omega$. Let A be a countable cofinal subset of β, then:

$$|\operatorname{supp}(p)| \leq \sum_{\alpha\in A} |\operatorname{supp}(p \restriction \alpha)| \leq |A| \cdot \omega = \omega$$

as $p = \bigcup_{\alpha\in A} p \restriction \alpha$ and $p \restriction \alpha \in P_\alpha$.
If $\operatorname{cof}(\beta) > \omega$ then any condition $p \in P_\beta$ has its support entirely contained in some $\alpha < \beta$. Therefore $p \in P_\alpha$. Hence $P_\beta \subseteq \bigcup_{\alpha<\beta} P_\alpha = P^d_\beta = P^{\mathrm{CS}}_\beta$.

Generalizing, we can also define the κ-support limit:

Definition 4.1.6. Let $\bar{P}$ be a forcing iteration of length ϵ. Let $\kappa \leq \epsilon$ be a regular infinite cardinal. The *full κ-support limit* is:

$$P^{\kappa S}_\epsilon := \begin{cases} P^i_\epsilon & \text{if } \operatorname{cof}(\epsilon) \leq \kappa; \\ P^d_\epsilon & \text{if } \operatorname{cof}(\epsilon) > \kappa. \end{cases}$$

Analogously to Example 4.1.5 a forcing iteration of length $\epsilon \geq \kappa$ on the family $\mathcal{I} = \langle \alpha^{\leq\kappa} : \alpha \leq \epsilon \rangle$ is a forcing where all the limits are full κ-support limits.

4.2 Cardinalities of limits

The next lemma collects some facts about the cardinality of the direct limit, the cardinality of the indirect limit and their relation.

Lemma 4.2.1. *Let $\bar{P}$ be a forcing iteration of length ϵ without final limit. Then*

1. *If $\forall\alpha < \epsilon\ P_\alpha \Vdash |\mathrm{RO}(Q_\alpha)| > 1$, then $P^d_\epsilon \subsetneq P^i_\epsilon$;*
2. *$|P^d_\epsilon| = \sup_{\alpha<\epsilon} |P_\alpha|$;*
3. *If $\forall\alpha < \epsilon\ P_\alpha \Vdash |\mathrm{RO}(Q_\alpha)| > 1$, then $|\epsilon| \leq |P^d_\epsilon|$;*
4. *$|P^i_\epsilon| \leq |P^d_\epsilon|^{|\epsilon|} \leq |P^d_\epsilon|^{|P^d_\epsilon|} = 2^{|P^d_\epsilon|}$.*

Here with $\mathrm{RO}(Q)$ we denote the Boolean algebra generated by Q.

Proof.

1. A condition $p : \epsilon \to \bigcup_{\alpha<\epsilon} \mathrm{dom}\, Q_\alpha$, such that $\forall\alpha < \epsilon\ P_\alpha \Vdash p(\alpha) \neq_{\mathrm{RO}(Q_\alpha)} 1_{Q_\alpha}$, belongs to $P^i_\epsilon \setminus P^d_\epsilon$.
2. By definition, $P^d_\epsilon = \bigcup_{\alpha<\epsilon} P_\alpha$.
3. For every $\alpha < \epsilon$ we can find a condition $p^\alpha \in P_{\alpha+1} \setminus P_\alpha$ by choosing $p^\alpha(\alpha) \in \mathrm{dom}(Q_{\alpha+1})$ such that $P_\alpha \Vdash p^\alpha(\alpha) \neq_{\mathrm{RO}(P_\alpha)} 1_{q_\alpha}$. The set $\{p^\alpha : \alpha < \epsilon\} \subseteq P^d_\epsilon$ has cardinality ϵ and contains conditions pairwise not forcing-equivalent.
4. This follows from the third point and the fact that to every condition $p \in P^i_\epsilon$ we can associate the relative function:

$$\begin{aligned} p' : \epsilon &\to \bigcup_{\alpha<\epsilon} P_\alpha \\ \alpha &\mapsto p \upharpoonright \alpha. \end{aligned}$$

□

4.3 Partial CS-limit

In the settings of Definition 4.1.1 let us consider as a limit for the forcing iteration a subset of the full CS-limit. This will define the so called *partial* CS-*limit*. We show that despite the terminology a partial or full CS-limit does not contain necessarily all conditions with countable support (Example 4.3.3). Moreover a partial CS-limits can contain conditions with uncountable supports (Example 4.3.4).

Definition 4.3.1. Given an iteration $\bar{P}$ of length ϵ without final limit we say that P_ϵ is a *partial CS-limit* of the iteration if:

$$P_\epsilon \subseteq P_\epsilon^{CS} \text{ and } \langle P_\alpha, Q_\alpha : \alpha \leq \epsilon \rangle \text{ is a forcing iteration with final limit.}$$

Notation: for $\beta < \epsilon$ $\bar{P} \restriction \beta := \langle P_\alpha, Q_\alpha : \alpha < \beta \rangle$.

Example 4.3.2. P_ϵ^d and P_ϵ^i are both partial CS-limits.

We show in the next example that the size of supports in a CS-limit is not part of the definition: a CS-limit doesn't contain every condition with countable support.

Example 4.3.3. Let $\bar{P}$ be an iteration of length $\omega + \omega$ with final limit such that $P_\omega = \bigcup_{n\in\omega} P_n$ and $P_{\omega+\omega} = P_{\omega+\omega}^{\mathrm{CS}}$, i.e. $P_{\omega+\omega} = P_{\omega+\omega}^i$. In this case $\bar{P}$ behaves like a finite support iteration until P_ω. For $n \in \omega$ the conditions in $P_{\omega+n}$ have finite supports as well, while in $P_{\omega+\omega}$ there are conditions with infinite supports. Therefore $\bar{P}$ is not a countable support iteration neither a finite support iteration. We remark moreover that any $p \in (\bar{P} \restriction \omega)^i$, with $\mathrm{supp}(p)$ unbounded in ω is not an element of $P_{\omega+\omega}$.

From the last example we saw that not every countable subset of ϵ is the support of a condition in P_ϵ^{CS}. Conversely, given an iteration $\bar{P}$ of length ϵ, it does not follow that all the conditions in P_ϵ^{CS} have countable supports: their support can be even bigger, as in the next example.

Example 4.3.4. Let $\epsilon > \omega_1$ and let $\bar{P}$ be an iteration of length ϵ with final limit such that $P_\epsilon = P_\epsilon^{\mathrm{CS}}$ and such that for some $\omega_1 \leq \alpha < \epsilon$:

$$\forall \gamma \leq \alpha P_\gamma \Vdash |\mathrm{RO}(Q_\gamma)| > 1 \text{ and}$$
$$\forall \gamma \leq \alpha (\gamma \text{ limit ordinal } \rightarrow P_\gamma = P_\gamma^i).$$

Then there is a $p \in P_\alpha$ with $\mathrm{supp}(p) = \alpha$. It follows that P_ϵ^{CS} doesn't contain only conditions with countable supports.

4.4 Partial CS-iterations

In the previous paragraph we saw that a partial CS-limit can contain conditions with uncountable support. Following [14, Definition 3.6] we restrict our attention to iterations where at each limit stage only partial CS-limits are allowed. Such construction is called a *partial* CS-*iteration*. In this case, the supports of conditions are countable. In this section we also show that separativity is preserved (Lemma 4.4.2) whereas properties like properness may not be maintained (Example 4.4.5).

Definition 4.4.1. A forcing iteration $\bar{P}$ of length ϵ with final limit is called *partial CS-iteration* if:

a) For every limit ordinal $\alpha \leq \epsilon$, $P_\alpha \subseteq P_\alpha^{\mathrm{CS}}$;

b) For every $\alpha < \epsilon$, $P_\alpha \Vdash$ "Q_α is separative".

A forcing notion P is *separative* if for $p, q \in P$ $p \not\leq q \to \exists r \leq p(r \perp q)$.

From the definition it follows immediately that in a partial CS-iteration all the iterands are separative:

Lemma 4.4.2. *Let $\bar{P}$ be a partial* CS*-iteration of length ϵ, then $\forall \alpha \leq \epsilon$ P_α is separative.*

Proof. For P_0 it is immediate as it contains only the condition $\emptyset$. For successor ordinals we have $P_{\alpha+1} = P_\alpha * Q_\alpha$. If $p \not\leq_{P_{\alpha+1}} q$ we have two cases. The first case is when $p \restriction \alpha \not\leq_{P_\alpha} q \restriction \alpha$. By induction, P_α is separative, so $\exists r \in P_\alpha(r \leq_{P_\alpha} p \restriction \alpha \wedge r \perp_{P_\alpha} q \restriction \alpha)$. Then $r^\frown p(\alpha) \leq_{P_{\alpha+1}} p$ and $r^\frown p(\alpha) \perp_{P_{\alpha+1}} q$. The second case holds when $p \restriction \alpha \leq_{P_\alpha} q \restriction \alpha \wedge p \restriction \alpha \nVdash_{P_\alpha} p(\alpha) \leq q(\alpha)$. So there is some $s \leq_{P_\alpha} p \restriction \alpha$ such that $s \Vdash_{P_\alpha} p(\alpha) \not\leq q(\alpha)$. It follows that $s \Vdash_{P_\alpha} \exists r(r \leq p(\alpha) \wedge r \perp q(\alpha))$. Then $s^\frown r \leq_{P_{\alpha+1}} p$ and $s^\frown r \perp_{P_{\alpha+1}} q$.
If α is a limit, $p \not\leq_{P_\alpha} q$ implies $\exists \beta < \alpha$ such that $p \restriction \beta \nVdash_{P_\beta} p(\beta) \leq_{Q_\beta} q(\beta)$ $\exists \beta < \alpha(p \restriction \beta \not\leq_{P_\beta} q \restriction \beta)$. By induction, $\exists r \in P_\beta(r \leq_{P_\beta} p \restriction \beta \wedge r \perp_{P_\beta} q \restriction \beta)$. We conclude that $r^\frown p[\beta, \alpha) \leq_{P_\alpha} p$ and $r^\frown p[\beta, \alpha) \perp_{P_\alpha} q$. □

In the following we present three equivalent statements that are satisfied by CS-iterations. This will be useful later to distinguish if a condition is contained in the generic filter.

Lemma 4.4.3. *Let $\bar{P}$ be a partial CS-iteration of length ϵ with final limit. Let H be P_ϵ-generic and for $\alpha < \epsilon$ let H_α the P_α-generic filter induced by H. We write $q \leq_\alpha^* p$*

to say that $q \Vdash_{P_\alpha} p \in H_\alpha$ *(equivalently* $q \leq^*_{P_\alpha} p$ *iff* $\forall r \leq_{P_\alpha} q\ r \parallel_{P_\alpha} p$*). Then the following statements are equivalent:*

a) For all $p \in P_\epsilon$*:* $p \in H$ *iff* $p \restriction \alpha \in H_\alpha$*, for all* $\alpha < \epsilon$*;*

b) For all $q, p \in P_\epsilon$*: if* $q \restriction \alpha \leq^*_{P_\alpha} p \restriction \alpha$ *for each* $\alpha < \epsilon$*, then* $q \leq^*_{P_\epsilon} p$*;*

c) For all $q, p \in P_\epsilon$*: if* $q \restriction \alpha \leq^*_{P_\alpha} p \restriction \alpha$ *for each* $\alpha < \epsilon$*, then* $q \parallel_{P_\epsilon} p$*;*

Proof.

a) →b) : Let $q \restriction \alpha \leq^*_{P_\alpha} p \restriction \alpha$ for each $\alpha < \epsilon$. By contradiction we assume that $q \nleq^*_{P_\epsilon} p$. So we get

$$\begin{aligned} q \nleq^*_{P_\epsilon} p &\Rightarrow q \nVdash_{P_\epsilon} p \in H \\ &\Rightarrow \exists r \leq_{P_\epsilon} q\ r \Vdash_{P_\epsilon} p \notin H \\ &\Rightarrow \exists r \leq_{P_\epsilon} q\ r \Vdash_{P_\epsilon} \exists \alpha < \epsilon\ p \restriction \alpha \notin H_\alpha \qquad \text{[by a)]} \\ &\Rightarrow \exists r \leq_{P_\epsilon} q \exists \alpha < \epsilon\ r \Vdash_{P_\epsilon} p \restriction \alpha \notin H_\alpha. \end{aligned}$$

We consider the following dense subset of P_α:

$$D := \{t \in P_\alpha : (t \perp_{P_\alpha} p \restriction \alpha) \vee (t \perp_{P_\alpha} q \restriction \alpha) \vee (t \leq_{P_\alpha} p \restriction \alpha \wedge t \leq_{P_\alpha} q \restriction \alpha)\}$$

In particular, $D' := D \cap ((r \restriction \alpha) \downarrow)$ is dense under $r \restriction \alpha$, where $(r \restriction \alpha) \downarrow := \{p \in P_\alpha : p \leq_{P_\alpha} r \restriction \alpha\}$. Let H be P_ϵ-generic such that $r \in H$. Then, since $r \restriction \alpha \in H_\alpha$, there is some $t \in D' \cap H_\alpha$. Such a t has the following properties: $t \leq_{P_\alpha} r \restriction \alpha$ and $t \nleq_{P_\alpha} p \restriction \alpha$ (because $p \restriction \alpha \notin H_\alpha$). By definition of D it follows that $t \perp_{P_\alpha} p \restriction \alpha$. Since $r \restriction \alpha \leq_{P_\alpha} q \restriction \alpha$, we get $t \leq_{P_\alpha} q \restriction \alpha$. All in all, $q \restriction \alpha \nleq^*_{P_\alpha} p \restriction \alpha$, a contradiction.

b) →c) :

$$\begin{aligned} q \leq^*_{P_\epsilon} p &\Rightarrow \forall r \leq_{P_\epsilon} q (r \parallel_{P_\epsilon} p) \\ &\Rightarrow \forall r \leq_{P_\epsilon} q \exists s (s \leq_{P_\epsilon} r \wedge s \leq_{P_\epsilon} p) \\ &\Rightarrow \exists s (s \leq_{P_\epsilon} q \wedge s \leq_{P_\epsilon} p) \\ &\Rightarrow q \parallel_{P_\epsilon} p \end{aligned}$$

c) →a) : We prove that if $\forall \alpha < \epsilon\ p \restriction \alpha \in H_\alpha$ then $p \in H$. The other direction holds by definition. If, by contradiction, $p \notin H$, there is some $q \in H$

such that $q \perp_{P_\epsilon} p$. We have the implications:

$$\begin{aligned} q \perp_{P_\epsilon} p &\Rightarrow \forall r \leq_\epsilon q (r \perp_{P_\epsilon} p) \\ &\Rightarrow \forall r \leq_{P_\epsilon} q \exists \alpha < \epsilon (r \upharpoonright \alpha \not\leq^*_\alpha p \upharpoonright \alpha) && \text{[by c)]} \\ &\Rightarrow \forall r \leq_{P_\epsilon} q \exists \alpha < \epsilon (r \upharpoonright \alpha \not\Vdash_{P_\alpha} p \upharpoonright \alpha \in H_\alpha) \\ &\Rightarrow \forall r \leq_{P_\epsilon} q \exists \alpha < \epsilon \exists s \leq_{P_\alpha} r \upharpoonright \alpha (s \Vdash_{P_\alpha} p \upharpoonright \alpha \notin H_\alpha) \end{aligned}$$

From the last line, we remark that $(s \wedge r) \leq_{P_\epsilon} r$. Hence the following subset of P_ϵ is dense under q:

$$D := \{s \in P_\epsilon : \exists \alpha < \epsilon (s \upharpoonright \alpha \leq_{P_\alpha} r \upharpoonright \alpha \text{ and } s \upharpoonright \alpha \Vdash_{P_\alpha} p \upharpoonright \alpha \notin H_\alpha)\}$$

We can find a condition $s \in D \cap H \cap (q \downarrow)$. Now since $s \in H$, we get that $s \upharpoonright \alpha \in H_\alpha$ and therefore $p \upharpoonright \alpha \notin H_\alpha$, a contradiction. □

Property 4.4.4. Let ϵ be a limit ordinal and let $\bar{P}$ be a partial CS-iteration with final limit. Then P_ϵ satisfies any (all) conditions of Lemma 4.4.3.

Proof. By Lemma 4.4.2 for all $\alpha \leq \epsilon$ P_α is separative. Thus $\leq^*_{P_\alpha} = \leq_{P_\alpha}$ for all $\alpha \leq \epsilon$. (That $\leq_{P_\alpha} \subseteq \leq^*_{P_\alpha}$ is clear. For the other direction, if $q \not\leq_{P_\alpha} p$ then $\exists r \leq_{P_\alpha} q$ such that $r \perp_{P_\alpha} p$, which means that $q \not\leq^*_{P_\alpha} p$). Condition b) of Lemma 4.4.3 holds because of the definition of $\leq_\epsilon$ and the fact that ϵ is a limit ordinal. □

We have seen that the property of being separative is preserved in a partial CS-iteration. However there are other features which may not be maintained, like properness or the preservation of ω_1. We conclude the section with an example that shows that a finite support iteration of Sacks[1] forcings (Sacks forcing is proper) is not proper: namely it collapses ω_1.

Example 4.4.5. A finite support iteration of length ω, whose iterands are Sacks forcings, is not proper.

Let $(P_n, Q_n)_{n \in \omega}$ be an iterated forcing with direct limit and such that $P_n \models$ "Q_n is a Sacks forcing". We remark that every iterand is (forced to be) proper: Sacks forcing satisfies Axiom A which in turn implies properness (see in the Appendix Property A.4.7 and Property A.3.2). By Lemma A.4.8 Sacks forcing is not ccc.

In order to show that P_ω is not proper, we just need to show that it collapses ω_1. The next lemma gives us the result.

[1]For the definition of Sacks forcing please see Section A.4 in the Appendix.

Lemma 4.4.6. *Let $\bar{P}$ be a finite support forcing iteration of length ω with final limit. If for each $n < \omega$ $1 \Vdash_{P_n}$ "Q_n is not ccc", then P_ω collapses ω_1.*

Proof. We are going to use the evaluation function "val" of Definition 2.1.1. The proof works however also for the function "ordval" of Definition 2.5.1.
If for some $n < \omega$ the partial order P_n collapses ω_1, we are done. Otherwise $\omega_1^{M[G_n]} = \omega_1^M$ for every $n < \omega$, where G_n is P_n-generic. For every $n \in \omega$ let A_n, τ_n be P_n-names such that

$$1 \Vdash_{P_n} A_n \text{ is a maximal antichain in } Q_n \wedge \tau_n : A_n \twoheadrightarrow \omega_1. \tag{4.1}$$

Let G be a P_ω-generic filter and let consider the induced filters $G_n := \{p \in P_n : p \in G\}$ and $H_n := \{\mathrm{val}(\rho, G_n) : \rho \in \mathrm{dom}(Q_n) \wedge \exists p \in P_n\ p^\frown\langle\rho\rangle \in G_{n+1}\}$.

In the extension $M[G]$ we have $H_n \cap \mathrm{val}(A_n, G_n) = \{q_n\}$. We claim that the following function is onto ω_1^M:

$$\begin{aligned} f : \omega &\to \omega_1^M \\ n &\mapsto \mathrm{val}(\tau_n, G_n)(q_n) \end{aligned}$$

We first show that for every $\alpha < \omega_1$ we have appropriate dense sets:

$$D_\alpha := \{p \in P_\omega : \exists n \in \omega \exists q \in \mathrm{dom}(Q_n) p \restriction n \Vdash_{P_n} \text{"}q \in A_n \wedge \tau_n(q) = \alpha\text{"} \wedge p(n) = q\}.$$

We prove that D_α is dense. Let $p \in P_\omega$, then there is some $n \in \omega$ such that $p \in P_n$. From (4.1) we have

$$1 \Vdash_{P_n} A_n \text{ is a maximal antichain in } Q_n \wedge \exists a \in A_n(\tau_n(a) = \alpha).$$

Hence in particular

$$\exists a \in \mathrm{dom}(A_n)\ \exists r \leq_{P_n} p\ r \Vdash_{P_n} \tau_n(a) = \alpha$$

Let $q := r^\frown a$. We remark that $q \leq_{P_\omega} p$ and $q \in D_\alpha$, hence D_α is dense.

We show now that in $M[G]$ the function f is surjective. For every $\alpha \in \omega_1$, we pick $p \in D_\alpha \cap G$. We have in particular:

- $\forall n \in \omega\ p \restriction n \in G$;
- $\forall n \in \omega\ \ \mathrm{val}(p(n), G_n) \in H_n$;
- $\exists n \in \omega\ p \restriction n \Vdash \tau(p(n)) = \alpha$.

All in all there is some $n \in \omega$ such that $v_n := \mathrm{val}(p(n), G_n) \in H_n \cap \mathrm{val}(A_n, G_n)$ and in $M[G]$ we conclude that $f(n) = \mathrm{val}(\tau_n, G_n)(v_n) = \alpha$. ☐

Chapter 5

The canonical extension and the AFS-iteration

In the current chapter we will examine coherent systems of complete embeddings. More specifically, for an ord-transitive model M and a partial CS-iteration $\bar{P}^M \in M$ of length ϵ we analyze iterations $\bar{P} \in V$ such that for every $\alpha \in \epsilon \cap M$ there is an M-complete embedding $i_\alpha : P^M_\alpha \to P_\alpha$ with the property of being coherent with the previous maps (i.e. for $\beta < \alpha$ $i_\alpha \restriction \beta = i_\beta$).
In the first section we introduce the definition of a *canonical extension of a system* and investigate its properties. We will then observe that the M-completeness of a canonical extension may not work when $V \models |M| = \aleph_0$.
In the second section we define the *almost finite support limit* (AFS-limit) of $\bar{P}$ over $\bar{P}^M$, describe its characteristics and we verify that such a limit preserves the M-completeness.
Finally, in the last section, we construct by induction an iteration $\bar{P}$ and a M-complete coherent system from $\bar{P}^M$ to $\bar{P}$. The resulting $\bar{P}$ will be an *almost finite support iteration* (AFS-iteration). We achieve the chapter establishing the ccc-preservation in a AFS-iteration and, under specific conditions, the preservation of the σ-centered property. This chapter follows and expands part of [14, Sections 3.B and 3.C]

5.1 Canonical extensions and embeddings

Let M be a nice candidate, $\bar{P}^M \in M$ and $M \models$ "$\bar{P}^M$ is a partial CS-iteration of length ϵ with final limit". We also ask that ϵ is a limit ordinal (since M is nice, the notion of being a limit ordinal is absolute). Let $\bar{P} \in V$ be a partial

CS-iteration of length $\epsilon' := \sup(\epsilon \cap M)$ without final limit.

Given a *coherent system of M-complete embeddings* $\langle i_\beta : P^M_\beta \to P_\beta | \beta < \epsilon \cap M\rangle$ between $\bar{P}^M$ and $\bar{P}$ (Definition 5.1.3) it is possible to define a *canonical extension* $j : P^M_\epsilon \to P^{CS}_{\epsilon'}$ (Definition 5.1.4). We will analyze the map j and see that it is coherent, it preserves the order relation (Property 5.1.5) but it may fail to be M-complete (Example 5.1.6).
We then try to give an inductive procedure to construct a *canonical embedding* from $\bar{P}^M$ to $\bar{P}$ (Definition 5.1.7). The successor step works: this is proved by a folklore lemma adapted to ord-transitive models (Proposition 5.1.8). The limit step does not preserve maximal antichains if the upper model sees that M is countable (Example 5.1.9). Before moving into the next section - where an explicit construction for a canonical embedding is given - we analyze two cases for which incompatibility is preserved at limit steps (Lemma 5.1.10).

Remark 5.1.1. We recall that, being M a nice candidate, $\mathrm{cof}(\epsilon) = \omega$ is absolute between M and V and that

$$\epsilon' = \epsilon \Leftrightarrow \mathrm{cof}(\epsilon) = \omega.$$

In fact, if $\sup(\epsilon \cap M) = \epsilon$, we get $\mathrm{cof}(\epsilon) = \omega$ because the countable set $\epsilon \cap M$ is cofinal in ϵ. The other direction is proved in Lemma 1.4.3: If $\mathrm{cof}(\epsilon) = \omega$ then $\epsilon \cap M$ is cofinal in ϵ. Hence $\epsilon' = \sup(\epsilon \cap M) = \epsilon$.

Definition 5.1.2. Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two notions of forcing such that $P, \leq_P \in M \setminus \mathrm{ON}$. We say that a map $i : P \to Q$ is an *M-complete embedding* if:

1. $\forall p, q \in P(q \leq_P p \Rightarrow i(q) \leq_Q i(p))$;
2. For any $A \subseteq P$ such that $A \in M$, if $M \models$ "A is a maximal antichain[1] in P", then $V \models$ "$i[A]$ is a maximal antichain in Q".

or equivalently

2'. For $D \subseteq P$ and $D \in M$ if $M \models$ "D is predense[2] in P", then $V \models$ "$i[D]$ is predense in Q.

[1]By Lemma 2.2.4 the notion of antichain is absolute for M. Remark that if we just assume $(A \subseteq P)^M$, it could happen that $A \not\subseteq P$, as "being a subset" is not absolute for ord-transitive models, and Lemma 2.2.4 cannot be used.

[2]Under the conditions of the definition, $M \models$ "D is predense in P" is equivalent to $V \models$ "D is predense in P"

Definition 5.1.3. We call $\langle i_\beta : P_\beta^M \to P_\beta | \beta \in \epsilon \cap M \rangle$ a *system of M-complete coherent embeddings* if every i_β is an M-complete embedding and if for $\alpha, \beta \in \epsilon \cap M$, for $p \in P_\beta^M$, if $\alpha < \beta$, then:

$$i_\alpha(p \upharpoonright \alpha) = i_\beta(p) \upharpoonright \alpha.$$

Given a system of M-complete coherent embeddings, the aim is to describe a possible limit $P_{\epsilon'}$ of the iteration in V and a coherent map $i_\epsilon : P_\epsilon^M \to P_{\epsilon'}$

Definition 5.1.4. We define the *canonical extension* of the system as the following map:

$$\begin{aligned} j : P_\epsilon^M &\to P_{\epsilon'}^{\mathrm{CS}} \\ p &\mapsto \bigcup_{\beta \in \epsilon \cap M} i_\beta(p \upharpoonright \beta) \end{aligned}$$

Property 5.1.5. Let $i_\epsilon := j$. Then:

1. i_ϵ is coherent, i.e. for all $\alpha \in \epsilon \cap M$ $i_\epsilon(p) \upharpoonright \alpha = i_\alpha(p \upharpoonright \alpha)$;

2. For every $\alpha \in \epsilon' = \sup(\epsilon \cap M)$ there is a $\beta \in \epsilon \cap M$ such that $\alpha \leq \beta$ and $i_\epsilon(p) \upharpoonright \alpha = i_\beta(p \upharpoonright \beta) \upharpoonright \alpha$;

3. $p \leq_{P_\epsilon^M} q \Rightarrow i_\epsilon(p) \leq_{P_{\epsilon'}^{\mathrm{CS}}} i_\epsilon(q)$

Proof. 1.

$$\begin{aligned} i_\epsilon(p) \upharpoonright \alpha &= \Big(\bigcup_{\beta \in \epsilon \cap M} i_\beta(p \upharpoonright \beta)\Big) \upharpoonright \alpha \\ &= \Big(\bigcup_{\beta \in \epsilon \cap M \wedge \beta < \alpha} i_\beta(p \upharpoonright \beta) \upharpoonright \alpha\Big) \cup i_\alpha(p \upharpoonright \alpha) \cup \Big(\bigcup_{\gamma \in \epsilon \cap M \wedge \gamma > \alpha} i_\gamma(p \upharpoonright \gamma) \upharpoonright \alpha\Big) \\ &= \Big(\bigcup_{\beta \in \epsilon \cap M \wedge \beta < \alpha} i_\alpha(p \upharpoonright \alpha) \upharpoonright \beta\Big) \cup i_\alpha(p \upharpoonright \alpha) \\ &= i_\alpha(p \upharpoonright \alpha) \end{aligned}$$

We made use of the coherence of the i_β for $\beta \in \epsilon \cap M$ and the fact that $i_\beta(p \upharpoonright \beta) \upharpoonright \alpha = i_\alpha(p \upharpoonright \alpha) \upharpoonright \beta \subseteq i_\alpha(p \upharpoonright \alpha)$ for $\beta \leq \alpha$.

2. If $\alpha \in \epsilon \cap M$ just pick $\beta = \alpha$. If $\alpha \in \epsilon' \setminus M$ we consider a $\beta \in \epsilon \cap M$ such that $\beta \geq \alpha$. Now by the previous point we get

$$i_\epsilon(p) \upharpoonright \alpha = (i_\epsilon(p) \upharpoonright \beta) \upharpoonright \alpha = i_\beta(p \upharpoonright \beta) \upharpoonright \alpha.$$

3.

$$\begin{aligned}
p \leq_{P^M_\epsilon} q &\Rightarrow \forall \alpha \in \epsilon \cap M \; p \restriction \alpha \leq_{P^M_\alpha} q \restriction \alpha \\
&\Rightarrow \forall \alpha \in \epsilon \cap M \; i_\alpha(p \restriction \alpha) \leq_{P_\alpha} i_\alpha(q \restriction \alpha) \\
&\Rightarrow \forall \alpha \in \epsilon \cap M \; i_\epsilon(p) \restriction \alpha \leq_{P_\alpha} i_\epsilon(q) \restriction \alpha \\
&\Rightarrow \forall \alpha \in \epsilon' \; i_\epsilon(p) \restriction \alpha \leq_{P_\alpha} i_\epsilon(q) \restriction \alpha \\
&\Rightarrow i_\epsilon(p) \leq_{P^{CS}_{\epsilon'}} i_\epsilon(q)
\end{aligned}$$

The second last implication holds by point 2. □

The canonical extension $j : P^M_\epsilon \to P^{CS}_\epsilon$ is not always M-complete, as the next example shows:

Example 5.1.6. Consider a forcing iteration $\bar{P}^M \in M$ of length ω with final limit. Let $\bar{P}$ be a forcing iteration of length ω without final limit such that $\forall n \in \omega \; i_n : P^M_n \to P_n$ is an M-complete embedding. For all $n \in \omega$ let $P^M_n = 2^{<n}$ and let P^M_ω be the direct limit, i.e. $P^M_\omega = \bigcup_{n\in\omega} P^M_n = 2^{<\omega}$. The order is the inverse inclusion. Let $j : (P^d_\omega)^M \to P^i_\omega$ be as in Definition 5.1.4. For every $n \in \omega$ we define a finite sequence $q_{n+1} \in P_{n+1}$ such that:

$$\forall m \leq n \qquad q_{n+1}(m) := \begin{cases} 0 & \text{if } m < n; \\ 1 & \text{if } m = n. \end{cases}$$

Then $A := \{q_{n+1} : n \in \omega\}$ is an element of M (we recall that M satisfies enough axioms of ZFC to define A) and is a predense subset of P^M_ω. Namely, let $p \in P^M_\omega$ then $\exists n \in \omega (p \in P^M_n)$. If $\exists m \leq n$ such that $p \restriction m = q_m$ then $p \leq_{P^M_\omega} q_m$ and therefore $p \parallel_{P^M_\omega} q_m$. If $\forall m \leq n (p \restriction m \neq q_m)$ then $q_{n+1} \leq_{P^M_\omega} p$ and therefore $q_{n+1} \parallel_{P^M_\omega} p$. (See Figure 5.1 for $n = 2$). For every $n \in \omega$ let $z_n \in P^M_n$ be the sequence of length n consisting of 0's. Then the infinite sequence $\bigcup_{n\in\omega} i_n(z_n) \in P^{CS}_\omega$ is incompatible with every element of $j[D]$. Hence $j[D]$ is not predense and j is not M-complete.

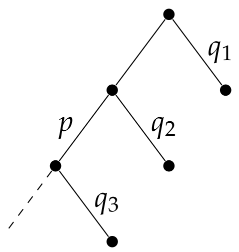

Figure 5.1: A representation of the construction where $p \in P^M_2$ and $q_3 \leq_{P^M_\omega} p$.

We just saw that the following diagram may not be complete

$$\begin{tikzcd}
P_0^M \arrow[r, hook] \arrow[d, "i_0"] & P_1^M \arrow[r, hook] \arrow[d, "i_1"] & \dots \arrow[r, hook] & P_\omega^M \arrow[d, dotted, "j"] \\
P_0 \arrow[r, hook] & P_1 \arrow[r, hook] & \dots \arrow[r, hook] \arrow[u, leftarrow, "i_n"] & P_\omega^{\mathrm{CS}}
\end{tikzcd}$$

To construct partial CS-limits $P_{\epsilon'}$ such that $j : P_\epsilon^M \to P_{\epsilon'}$ is M-complete, we restrict our attention to the *canonical system of embeddings* presented in [14, Definition 3.10].

Definition 5.1.7. Let $\bar{P}^M$ be a partial CS-iteration in M and let $\bar{P}$ be a partial CS-iteration in V, both with final limit and of length ϵ. Let $Q_\alpha^M \subseteq Q_\alpha$ for all $\alpha \in \epsilon + 1 \cap M$. We say that $\bar{P}^M$ *canonically embeds in* $\bar{P}$ if for all $\alpha \in (\epsilon+1) \cap M$ the following points hold:

- If $\alpha = \beta + 1$, by induction we assume that i_β is M-complete, which implies that a V-generic filter $H_\beta \subseteq P_\beta$ induces a M-generic filter $H_\beta^M := i_\beta^{-1}[H_\beta] \subseteq P_\beta^M$. We require that (in the H_β extension) the set $Q_\beta^M[H_\beta^M]$ is an $M[H_\beta^M]$-complete subforcing of $Q_\beta[H_\beta]$. Therefore $i_{\alpha+1}(p) = i_\alpha(p \restriction \alpha)^\frown p(\alpha)$.
- If α is a limit, then i_α is the canonical extension of the family $(i_\beta)_{\beta \in \alpha \cap M}$. Moreover we require $i_\alpha[P_\alpha^M] \subseteq P_\alpha$ and i_α to be M-complete. (If $\alpha' := \sup(\alpha \cap M) < \alpha$, then the image of i_α will be in $P_{\alpha'}$).

The successor point in the previous definition is justified by the following proposition, whose version for transitive models can be found in [10, Lemma 13].

Proposition 5.1.8. *Let M, N be ord-transitive models such that $M \in N$. Let $S \in M \setminus \mathrm{ON}$ and $T \in N \setminus \mathrm{ON}$ be partial orders such that $S \lessdot_M T$ (i.e. S is an M-complete subforcing of T). Let $B \in M \setminus \mathrm{ON}$ be a S-name for a partial order and $C \in N \setminus \mathrm{ON}$ be a T-name for a partial order such that $\mathrm{dom}(B) \subseteq \mathrm{dom}(C)$ and $T \Vdash_T B \lessdot_{M[G^M]} C$.*
If for all $q \in S$, for all $a, b \in \mathrm{dom}(B)$, such that $S \Vdash_S^M a, b \in B$ the following points are satisfied:

- $M \cap \mathrm{ON} \subseteq N \cap \mathrm{ON}$,
- $q \Vdash_S^M b \leq_B a \to q \Vdash_T^N b \leq_B a$,

- $q \Vdash_S^M b \perp_B a \to q \Vdash_T^N b \perp_B a$
- *if* $q \Vdash_S^M A$ *is a maximal antichain in* B *then* $q \Vdash_T^N A$ *is a maximal antichain in* C*,*
- *For any* S*-name* $\dot{a}$, $\mathrm{ordval}_H^M(\dot{a}) = \mathrm{ordval}_G^N(\dot{a})$ *(or* $\mathrm{ordval}_H^M(\dot{a}) = \mathrm{val}_G^N(\dot{a})$ *if* N *is transitive). We will shorten the notation of the evaluation with* $\dot{a}[H]$ *and* $\dot{a}[G]$*,*

then $S * B \lessdot_M T * C$.

Proof. Let $\langle q, b\rangle, \langle p, a\rangle \in S * B$. The order is preserved:

$$\begin{aligned}\langle q, b\rangle \leq_{S*B} \langle p, a\rangle &\Rightarrow q \leq_S p \wedge q \Vdash_S^M b \leq_B a\\ &\Rightarrow q \leq_T p \wedge q \Vdash_T^N b \leq_B a\\ &\Rightarrow q \leq_T p \wedge q \Vdash_T^N b \leq_C a\\ &\Rightarrow \langle q, b\rangle \leq_{T*C} \langle p, a\rangle.\end{aligned}$$

Let now $\langle q, b\rangle \perp_{S*B} \langle p, a\rangle$, we show that the incompatibility is preserved. If $q \perp_S p$, then $q \perp_T p$ and therefore $\langle q, b\rangle \perp_{T*C} \langle p, a\rangle$.
If $q \parallel_S p$, then

$$\forall r \in S \exists r' \in S (r \leq_S q \wedge r \leq_S p \to r' \leq_S r \wedge r' \Vdash_S^M b \perp_B a). \tag{5.1}$$

The previous line together with separativity implies that the following set is dense in S:

$$D := \{r \in S : r \perp_S q \vee r \perp_S p \vee r \Vdash_S^M b \perp_B a\}.$$

Namely let $s \in S$, then if $s \not\leq_S q$ (or $s \not\leq_S p$), then by separativity $\exists r \in S(r \leq_S s \wedge r \perp_S q)$ (or $r \leq_S s \wedge r \perp_S p$). This implies that $r \in D$.
If otherwise $s \leq_S q$ and $s \leq_S p$, then by (5.1) $\exists r \leq_S s$ such that $r \Vdash_S^M b \perp_B a$, which implies that $r \in D$.
The set D is in particular predense in S and belongs to M (as M satisfies enough axioms of ZFC). It follows that D is a predense subset of T. Now, since $q \parallel_T p$, then there is a $s \in T$ such that $s \leq_T q \wedge s \leq_T p$ and therefore there is a $r \in D$ such that $r \parallel_T s$. Therefore $r \parallel_T q \wedge r \parallel_T p$, which implies $r \parallel_S q \wedge r \parallel_S p$. Since $r \in D$, this implies $r \Vdash_S^M b \perp_B a$. By hypothesis we get $r \Vdash_T b \perp_B a$ and then $r \Vdash_T b \perp_C a$. This means that $s \nVdash_T b \parallel_C a$.
All in all, we just saw that $\forall s \in T(s \leq_T q \wedge s \leq_T p \to s \nVdash_T b \parallel_C a)$. This means that $\langle q, b\rangle \perp_{T*C} \langle p, a\rangle$, i.e. the incompatibility is preserved.
We show now that the antichains of M are preserved. Let $A = \{\langle s_\alpha, \dot{b}_\alpha\rangle :$

$\alpha < \kappa\} \in M$ be a maximal antichain of $S * B$. From the previous paragraph we already know that A is an antichain in $T * C$. We just have to prove the maximality. If A is not maximal in $T * C$, then let $\langle t, \dot{c}\rangle \in T * C$ be a condition such that $\forall \alpha < \kappa\ \langle t, \dot{c}\rangle \perp_{T*C} \langle s_\alpha, \dot{b}_\alpha\rangle$. Let $\dot{H}$ be the canonical S-name for the S-generic filter and let $\dot{\Omega}$ be a S-name such that $S \Vdash_S^M \dot{\Omega} = \{\alpha : \dot{s}_\alpha \in \dot{H}\}$.

Claim 5.1.8.1. *$S \Vdash_S^M \{\dot{b}_\alpha : \alpha \in \dot{\Omega}\}$ is a maximal antichain of B.*

Proof of the claim. We assume not, then there are $s \in S$ and a S-name $\dot{b}$ for an element of B such that $s \Vdash_S^M \forall \alpha(\alpha \in \dot{\Omega} \to \dot{b} \perp_B \dot{b}_\alpha)$. Then $\langle s, \dot{b}\rangle \in S * B$ and therefore there is a condition $\langle s_\alpha, \dot{b}_\alpha\rangle \in A$ such that $\langle s_\alpha, \dot{b}_\alpha\rangle \parallel_{S*B} \langle s, \dot{b}\rangle$. A common extension $\langle s', \dot{b}'\rangle$ satisfies $s' \Vdash_S^M \dot{b}' \leq_B \dot{b}_\alpha \wedge \dot{b}' \leq_B \dot{b}$ and also $s' \Vdash_S^M \alpha \in \dot{\Omega}$, as $s' \leq b_\alpha$. All in all $s' \Vdash_S^M \alpha \in \dot{\Omega} \wedge \dot{b} \parallel_B \dot{b}_\alpha$, contradicting our assumption. $\square$

Let now G be a T-generic filter such that $t \in G$. Since $S \lessdot_M T$, there is a S-generic filter H such that $M[H] \subseteq N[G]$ as a subset. Let $c := \dot{c}[G]$ and $b_\alpha := \dot{b}_\alpha[G] = \dot{b}_\alpha[H]$ for α such that $s_\alpha \in H$. Let $\Omega := \dot{\Omega}[G] = \{\alpha < \kappa : s_\alpha \in H\}$. By the above claim $\{b_\alpha : \alpha \in \Omega\}$ is a maximal antichain of B in $M[H]$ and by hypothesis of the lemma a maximal antichain of B in $N[G]$. Hence a maximal antichain of C, as $T \Vdash B \lessdot_{M[H]} C$. So there is $\alpha \in \Omega$ such that $c \parallel_C b_\alpha$. Therefore there is $t' \in G$ such that $t' \leq_T s_\alpha$ and $t' \leq_T t$ and $t' \Vdash_T \alpha \in \dot{\Omega} \wedge \dot{c} \parallel_C \dot{b}_\alpha$. Thus there is a T-name $\dot{c}'$ such that $t' \Vdash_T \dot{c}' \leq_C \dot{c} \wedge \dot{c}' \leq_C \dot{b}_\alpha$ and so $\langle t', \dot{c}'\rangle$ is a common extension of $\langle t, \dot{c}\rangle$ and $\langle s_\alpha, \dot{b}_\alpha\rangle$, which is in contradiction with the initial assumption. $\square$

Lifting any forcing iteration to any forcing iteration whose limit in the upper model is bigger than the almost finite support limit, can cause problems. In particular, if the upper model sees that the lower model is countable, we have the following example:

Example 5.1.9. Let M be an ord-transitive model and let $\bar{P} \in M$ be an iteration of length ω with support in some ideal $I \subseteq [\omega]^{\leq\omega}$ such that $I \in M$. Let N be an ord-transitive model such that $M \in N$ and $N \models |M| = \aleph_0$. Let $\bar{P}^N \in N$ be an iteration of length ω such that $N \models \forall n \in \omega P_n^M \lessdot_M P_n^N$ and $P_n^N \models$ "$Q_n^M \lessdot_{M[G_n^M]} Q_n^N$ and Q_n^M has two incompatible elements". Let P_ω^N be the indirect limit. If $\forall n \in \omega \forall a, b \in \mathrm{dom}(Q_n^M) \forall r \in P_n^M (r \Vdash_{P_n^M} a \perp_{Q_n^M} b \Leftrightarrow r \Vdash_{P_n^N} a \perp_{Q_n^N} b)$, then $P_\omega^M \not\lessdot_M P_\omega^N$.

Proof. Let $\langle p_n \in P_\omega^M : n \in \omega \rangle$ be an enumeration in N of all the elements of P_ω^M. We construct a chain of conditions $r_n \in P_n^M \subseteq P_n^N$ inductively on ω. Let $r_0 := 1_{P_0^M}$. Let $\dot{a}_n \in \operatorname{dom} Q_n^M$ be a P_n^M-name such that

$$r_n \Vdash_{P_n^M} \dot{a}_n \perp_{Q_n^M} \dot{p}_n(n) \wedge \dot{a}_n \in Q_n^M.$$

We define $r_{n+1} := r_n{}^\frown \dot{a}_n \in P_{n+1}^M$. Now the sequence $\langle r_n : n \in \omega \rangle \in N$ generates the condition $r := \bigcup_{n \in \omega} r_n \in P_\omega^N$. We remark that $\forall n \in \omega\ p_n \perp_{P_\omega^N} r$. This implies that whenever $A \in M$ is a maximal antichain in P_ω^M then A is not more maximal in P_ω^N, i.e. $P_\omega^M \not\lessdot_M P_\omega^N$. □

We have seen that the canonical extension is not always complete. In the following lemma we present two cases where incompatibility is preserved. We follow the proof of [14, Lemma 3.12].

Lemma 5.1.10. *Let $\bar{P}_\epsilon^M \in M$ be an (iteration of length ϵ with final limit)^M and let $\bar{P} \in V$ be an iteration of length $\epsilon' = \sup(\epsilon \cap M)$ with final limit. For $\alpha \in \epsilon \cap M$ let $i_\alpha : P_\alpha^M \to P_\alpha$ be a system of M-complete coherent embeddings. Let*

$$\begin{aligned} i_\epsilon : P_\epsilon^M &\to P_{\epsilon'}^{\mathrm{CS}} \\ p &\mapsto \bigcup_{\alpha \in \epsilon \cap M} i_\alpha(p \restriction \alpha) \end{aligned}$$

be the canonical extension. Then:

1. *If P_ϵ^M is a (direct limit)^M then $i_\epsilon[P_\epsilon^M] \subseteq P_{\epsilon'}$ and*

$$\begin{aligned} p \leq_{P_\epsilon^M} q &\Rightarrow i_\epsilon(p) \leq_{P_{\epsilon'}} i_\epsilon(q) \\ p \perp_{P_\epsilon^M} q &\Rightarrow i_\epsilon(p) \perp_{P_{\epsilon'}} i_\epsilon(q) \end{aligned}$$

2. *If $i_\epsilon[P_\epsilon^M] \subseteq P_{\epsilon'}^{\mathrm{CS}}$ and if i_ϵ maps (predense sets $D \subseteq P_\epsilon^M$)^M in predense sets of $P_{\epsilon'}$, then i_ϵ also preserves incompatibility.*

Proof. 1. We show that $i_\epsilon[P_\epsilon^M]$ is included in the direct limit, more precisely $i_\epsilon[P_\epsilon^M] \subseteq \bigcup_{\alpha \in \epsilon \cap M} P_\alpha \subseteq \bigcup_{\alpha < \epsilon'} P_\alpha$. We can easily see it because $P_\epsilon^M = (\bigcup_{\alpha \in \epsilon} P_\alpha^M)^M$ and for $p \in P_\epsilon^M$ there is a $\alpha \in \epsilon \cap M$ such that $p \in P_\alpha^M$. Hence, by coherence, $i_\epsilon(p) = i_\alpha(p) \in P_\alpha$.

Suppose now that $p \leq_{P_\epsilon^M} q$. P_ϵ^M is a direct limit, then there is $\alpha \in \epsilon \cap M$ such that $p, q \in P_\alpha^M$. So from $p \leq_{P_\alpha^M} q$ we get $i_\epsilon(p) = i_\alpha(p) \leq_{P_\alpha} i_\alpha(q) = i_\epsilon(q)$. If $p \perp_{P_\epsilon^M} q$, then $p \perp_{P_\alpha^M} q$ and $i_\alpha(p) \perp_{P_\alpha} i_\alpha(q)$ by assumption on i_α. Hence $i_\epsilon(p) \perp_{P_\epsilon} i_\epsilon(q)$.

2. Let $p, q \in P^M_\epsilon$ such that $p \perp_{P^M_\epsilon} q$. If by contradiction, $i_\epsilon(p) \parallel_{P_{\epsilon'}} i_\epsilon(q)$, let $G \subseteq P_{\epsilon'}$ be the generic filter containing $i_\epsilon(p)$ and $i_\epsilon(q)$. In M we define

$$D := \{r \in P^M_\epsilon : (r \leq_{P^M_\epsilon} p \wedge r \leq_{P^M_\epsilon} q) \vee (r \perp_{P^M_\epsilon} p) \vee (r \perp_{P^M_\epsilon} q)\}$$

$(D$ is dense $)^M$ and by absoluteness D is dense in V. By lemma 4.4.4 the following set is also dense (and belongs to M, as M satisfies enough instances of the comprehension scheme):

$$\begin{aligned} D' :=& \{r \in P^M_\epsilon : (r \leq_{P^M_\epsilon} p \wedge r \leq_{P^M_\epsilon} q) \vee (\exists \alpha \in \epsilon \cap M \; r \restriction \alpha \perp_{P^M_\alpha} p \restriction \alpha) \\ & \vee (\exists \alpha \in \epsilon \cap M \; r \restriction \alpha \perp_{P^M_\alpha} q \restriction \alpha)\} \end{aligned}$$

Then there is a $r \in D'$ such that $i_\epsilon(r) \in i_\epsilon[D'] \cap G$, as i_ϵ preserves predense sets. We want to conclude that r is a common extension of p and q. If not, for some $\alpha < \epsilon$ we would have, wlog, $r \restriction \alpha \perp_{P^M_\alpha} p \restriction \alpha$. We remark however that the filter $G \restriction \alpha$ contains $i_\alpha(r \restriction \alpha) = i_\epsilon(r) \restriction \alpha$ and $i_\alpha(p \restriction \alpha) = i_\epsilon(p) \restriction \alpha$. This contradicts the fact that i_α preserves incompatibility. □

5.2 Almost finite support limit

We have seen in the previous paragraph that, when M is countable in the upper model, the system of embeddings may fail to be complete at limits. This is also due to the fact that the iterations in the upper model can be much bigger. This suggest to restrict the attention to *almost finite support (*AFS*) limits* (Definition 5.2.1). After showing that the image of the canonical embedding is namely a subset of the AFS-limit and that it corresponds to the direct limit when the index of the limit has uncountable cofinality (Property 5.2.3), we prove that an AFS-limit makes the system of iterations M-complete (Lemma 5.2.5).

In this section we adopt the notation and analyze the ideas of [14, Definition 3.13 and Lemma 3.14].

Definition 5.2.1. Let M be an ord-transitive model. Let ϵ be a limit ordinal in M, and let $\epsilon' = \sup(\epsilon \cap M)$. Let $\bar{P}^M \in M$ be a partial countable support iteration of length ϵ in M, with final limit. Let $\bar{P}$ be an iteration in V of length ϵ' without final limit. Assume that the embeddings $i_\alpha : P^M_\alpha \to P_\alpha$ work for all $\alpha \in \epsilon \cap M = \epsilon' \cap M$. Let $i_\epsilon := j$ be the canonical extension as defined in

Definition 5.1.4. The *almost finite support limit of $\bar{P}$ over $\bar{P}^M$* (AFS-limit) is the following subforcing of $P^{\mathrm{CS}}_{\epsilon'}$:

$$P^{\mathrm{AFS}}_{\epsilon'} := \{q \wedge i_\epsilon(p) \in P^{\mathrm{CS}}_{\epsilon'} : p \in P^M_\epsilon \wedge \exists \alpha \in \epsilon \cap M(q \in P_\alpha \wedge q \leq_{P_\alpha} i_\alpha(p \restriction \alpha))\}$$

Remark 5.2.2. For $\alpha \in \epsilon$ let I_α be the ideal relative to P_α and for $\alpha \in \epsilon \cap M$ let I^M_α be the ideal relative to P^M_α. We remark that if $\forall \alpha \in \epsilon \cap M\ I_\alpha = I^M_\alpha$, then the ideal I relative to $P^{\mathrm{AFS}}_{\epsilon'}$ is $I = \bigcup_{\alpha \in \epsilon \cap M} I_\alpha$.

Property 5.2.3. With the settings of the previous definition, it follows that:

1. $i_\epsilon[P^M_\epsilon] \subseteq P^{\mathrm{AFS}}_{\epsilon'}$.
2. If $\mathrm{cof}(\epsilon) > \omega$ then
$$P^{\mathrm{AFS}}_{\epsilon'} = \bigcup_{\alpha \in \epsilon'} P_\alpha$$

Proof. 1. Let $p \in P^M_\epsilon$, for any $\alpha \in \epsilon \cap M$ let $q = i_\alpha(p \restriction \alpha)$. Then $i_\epsilon(p) = \bigcup_{\beta \in \epsilon \cap M} i_\beta(p \restriction \beta) = i_\alpha(p \restriction \alpha) \cup \bigcup_{\beta \in \epsilon \cap M \wedge \alpha < \beta} i_\beta(p \restriction \beta) = q \wedge i_\epsilon(p) \in P^{\mathrm{AFS}}_{\epsilon'}$.

2. Let $q \wedge i_\epsilon(p) \in P^{\mathrm{AFS}}_{\epsilon'}$, we show that $q \wedge i_\epsilon(p) \in \bigcup_{\alpha \in \epsilon'} P_\alpha$. Since $\mathrm{cof}(\epsilon) > \omega$ we know that $P^M_\epsilon = \bigcup_{\alpha \in \epsilon \cap M} P^M_\alpha$. So there is $\alpha \in \epsilon \cap M$ such that $p \in P^M_\alpha$. Hence $\forall \gamma \in \epsilon \cap M\ (\gamma \geq \alpha \to i_\epsilon(p) = i_\gamma(p))$. We also know that $q \leq_{P_\beta} i_\beta(p \restriction \beta)$ for some $\beta \in \epsilon \cap M$. Now taking $\gamma \geq \max\{\alpha, \beta\}$ we get $q \wedge i_\epsilon(p) = q \wedge i_\gamma(p) \in P_\gamma$.
For the other direction, if $q \in P_\alpha$ then there is some $\beta \in \epsilon \cap M$ such that $\beta \geq \alpha$. Hence $q \in P_\beta$ and $q \leq_{P_\beta} i_\beta(1_{P^M_\beta})$. So $q \wedge i_\epsilon(1_{P^M_\epsilon}) = q \in P_{\epsilon'}$. □

The following lemma shows that maximal antichains in P_ϵ are still maximal in the quotient forcing P_ϵ / G_α. The idea of the proof comes from [12]. This fact is applied in Lemma 5.2.5 to show that $i_\epsilon[P^M_\epsilon]$ is M-completely embedded in $P^{\mathrm{AFS}}_{\epsilon'}$.

Lemma 5.2.4. *Let G_ϵ be generic for P_ϵ and let $G_\alpha := G_\epsilon \cap P_\alpha$ for $\alpha < \epsilon$. If $A \subseteq P_\epsilon$ is a maximal antichain, then*

$$P_\alpha \Vdash_{P_\alpha} A \cap P_\epsilon / G_\alpha \textit{ is a maximal antichain in } P_\epsilon / G_\alpha.$$

Proof. Let assume the converse: let assume that there is some $r \in P_\alpha$ and $p \in P_\epsilon$ such that

$$r \Vdash_{P_\alpha} p \in P_\epsilon / G_\alpha \wedge \forall a \in A \cap P_\epsilon / G_\alpha (p \perp_{P_\epsilon / G_\alpha} a). \tag{5.2}$$

Without loss of generality we can assume that $r \leq_\alpha p \restriction \alpha$. Let call $s := r \wedge p \in P_\epsilon$ the common extension of r and p. Since A is a maximal antichain, we can find some $a \in A$ such that $a \parallel_{P_\epsilon} s$. So there is a $t \in P_\epsilon$ such that $t \leq_{P_\epsilon} a$ and $t \leq_{P_\epsilon} s$. In particular $t \restriction \alpha \leq_{P_\alpha} a \restriction \alpha$, so together with $t \leq_{P_\epsilon} a$ and $t \leq_{P_\epsilon} s \leq_{P_\epsilon} p$ we get

$$t \restriction \alpha \Vdash_{P_\alpha} a \in A \cap P_\epsilon / G_\alpha \wedge a \parallel_{P_\epsilon / G_\alpha} p$$

But from $t \restriction \alpha \leq_{P_\alpha} s \restriction \alpha = r$ and (5.2) we get a contradiction:

$$t \restriction \alpha \Vdash_{P_\alpha} \forall a \in A \cap P_\epsilon / G_\alpha (a \perp_{P_\epsilon / G_\alpha} p).$$ □

We can now prove that the almost finite support limit works, which means that $i_\epsilon[P^M_\epsilon]$ is an M-complete embedding in $P^{\mathrm{AFS}}_{\epsilon'}$ (cf. [14, Lemma 3.14]).

Lemma 5.2.5. *Assume that $\bar{P}$ and $\bar{P}^M$ are as in Definition 5.2.1 and let $P_{\epsilon'} := P^{\mathrm{AFS}}_{\epsilon'}$ be the almost finite support limit of $\bar{P}$. Then:*

1. *$\bar{P}^\frown P_{\epsilon'}$ is a partial* CS*-iteration;*

2. *i_ϵ works, i.e. i_ϵ is an M-complete embedding from P^M_ϵ to $P_{\epsilon'}$.*

Proof. 1. From the definition it follows easily that $P_{\epsilon'} \subseteq P^{\mathrm{CS}}_{\epsilon'}$. Moreover for any $\alpha < \epsilon'$, P_α is a complete subforcing of $P_{\epsilon'}$: It is easy to see that the order relation and the incompatibility relation are preserved. If $A \subseteq P_\alpha$ is a maximal antichain, then A is maximal in $P_{\epsilon'}$: if not, then there is $q \wedge i_\epsilon(p) \in P_{\epsilon'}$ such that $\forall a \in A \; q \wedge i_\epsilon(p) \perp_{P_{\epsilon'}} a$. We remark that for $q \wedge i_\alpha(p \restriction \alpha) \in P_\alpha$ there is $a \in A$ such that $a \parallel_{P_\alpha} q \wedge i_\alpha(p \restriction \alpha)$. Therefore for $r \in P_\alpha$ such that $r \leq_{P_\alpha} a$ and $r \leq_{P_\alpha} q \wedge i_\alpha(p \restriction \alpha)$ we get $r \wedge (q \wedge i_\epsilon(q)) \leq_{P_{\epsilon'}} a$ and $r \wedge (q \wedge i_\epsilon(q)) \leq_{P_{\epsilon'}} q \wedge i_\epsilon(p)$.

2. The function $i_\epsilon : P^M_\epsilon \to P_{\epsilon'}$ is well defined because $i_\epsilon[P^M_\epsilon] \subseteq P_{\epsilon'}$ by Property 5.2.3. Moreover it is order-preserving by the third point of Property 5.1.5. For the incompatibility preservation we use the second point of Lemma 5.1.10 and show that i_ϵ preserves predense sets of M. This will give us also the preservation of maximal antichains of M. Let $D \in M$ be predense in P^M_ϵ. Let $D{\downarrow} := \{p \in P^M_\epsilon : \exists d \in D \; p \leq_{P^M_\epsilon} d\}$ be the dense set generated by D. Let $A \subseteq D{\downarrow}$ be a maximal antichain. We suppose that M satisfies enough ZFC, so that $D{\downarrow} \in M$ and hence $A \in M$. We fix some $q \wedge i_\epsilon(p) \in P_{\epsilon'}$ where $q \in P_\alpha$ and $q \leq_{P_\alpha} i_\alpha(p \restriction \alpha)$. We show that there is some $a \in A$ such that $q \wedge i_\epsilon(p) \parallel_{P_{\epsilon'}} i_\epsilon(a)$. This proves that $i_\epsilon[D]$ is predense in $P_{\epsilon'}$ (because $\exists d \in D \; a \leq_{P^M_\epsilon} d$ and therefore

$q \wedge i_\epsilon(p) \parallel_{P_{\epsilon'}} i_\epsilon(d))$. When G_α is P_α-generic, then $G_\alpha^M := i_\alpha^{-1}[G_\alpha]$ is P_α^M-generic, since i_α is M-complete. By Lemma 5.2.4, since A is maximal in P_α^M we get:

$$P_\alpha^M \Vdash_{P_\alpha^M} \text{"}A \cap P_\epsilon^M / G_\alpha^M \text{ is a maximal antichain in } P_\epsilon^M / G_\alpha^M\text{"}$$

This implies:

$$P_\alpha \Vdash_{P_\alpha} \text{"}A \cap P_\epsilon^M / G_\alpha^M \text{ is a maximal antichain in } P_\epsilon^M / G_\alpha^M\text{"}$$

by Lemma 3.2.5, because "being a maximal antichain" is absolute for ord-transitive models.

In particular, $q \Vdash_{P_\alpha}$ "$A \cap P_\epsilon^M / G_\alpha^M$ is predense in $P_\epsilon^M / G_\alpha^M$". We recall that $q \leq_{P_\alpha} i_\alpha(p \restriction \alpha)$, hence:

$$q \Vdash_{P_\alpha} \text{"}p \in P_\epsilon^M / G_\alpha^M \wedge (p \parallel_{P_\epsilon^M / G_\alpha^M} a) \text{ for some } a \in A \cap P_\epsilon^M / G_\alpha^M.\text{"}$$

So we can say that $\exists a \in A \exists t \in P_\epsilon^M \exists q' \leq_{P_\alpha} q$ such that

$$q' \Vdash_{P_\alpha} t \leq_{P_\epsilon^M / G_\alpha^M} p \wedge t \leq_{P_\epsilon^M / G_\alpha^M} a$$

Without loss of generality we can choose $q' \leq_{P_\alpha} i_\alpha(t \restriction \alpha)$. We can finally conclude that $q' \wedge i_\epsilon(t) \leq_{P_{\epsilon'}} q \wedge i_\epsilon(p)$ and $q' \wedge i_\epsilon(t) \leq_{P_{\epsilon'}} i_\epsilon(a)$. $\square$

5.3 Almost finite support iteration

Given a partial CS-iteration $\bar{P}^M \in M$ we present the definition of an *almost finite support iteration* $\bar{P}$ *over* $\bar{P}^M$ (Definition 5.3.1). One of the peculiarities of the sequence $\bar{P}$ is that, at limit steps $\beta \in M$ of countable cofinality, AFS-limits are considered. The resulting iteration is indeed a partial CS-iteration and $\bar{P}^M$ is M-completely embedded into $\bar{P}$ (Lemma 5.3.2). We also show that an AFS-iteration preserves the countable chain condition (Property 5.3.5) and, under specific hypothesis, it preserves the σ-centered property (Property 5.3.7 and Lemma 5.3.8).

The following definition follows [14, Definition and Claim 3.15].

Definition 5.3.1. Let $\bar{P}^M$ be a partial CS-iteration in M of length ϵ with final limit. We can construct by induction on $\beta \in \epsilon + 1$ an *almost finite support iteration (*AFS*-iteration)* $\bar{P}$ *over* $\bar{P}^M$ as follows (We refer to Figure 5.2 for a sketch of the construction):

1. As induction hypothesis we assume that for all $\alpha \in \beta \cap M$ the canonical embedding i_α works.

2. If $\beta = \alpha + 1$ and $\alpha \in M$, then we pick some Q_α, such that Q_α^M is (forced to be) an $M[H_\alpha^M]$-complete subforcing of Q_α. If $\alpha \notin M$ there is no restriction about Q_α. We also ask that $P_\alpha \Vdash_{P_\alpha} Q_\alpha$ is separative.

3. If $\beta \in M$ and $\mathrm{cof}(\beta) = \omega$, then P_β is the AFS-limit of $(P_\alpha, Q_\alpha)_{\alpha<\beta}$ over P_β^M.

4. If $\beta \in M$ and $\mathrm{cof}(\beta) > \omega$, then P_β is the AFS-limit of $(P_\alpha, Q_\alpha)_{\alpha<\beta}$ over P_β^M, which corresponds to the direct limit.

5. For limit ordinals β not in M, let P_β be the direct limit.

$$\begin{tikzcd}[column sep=small]
& \epsilon \in M \arrow[dr, dashed] \arrow[d, no head] & \\
& {} \arrow[d, no head] & \epsilon' = \sup(\epsilon \cap M) \arrow[d, no head] \\
\beta \notin M & {} \arrow[r] \arrow[d, no head] & \bigcup_{\alpha<\beta} P_\alpha \arrow[d, no head] \\
\beta \in M \wedge \mathrm{cof}(\beta) > \omega & P^M_\beta \arrow[r] \arrow[d, no head] & \bigcup_{\alpha<\beta} P_\alpha \arrow[d, no head] \\
\beta \in M \wedge \mathrm{cof}(\beta) = \omega & P^M_\beta \arrow[r] \arrow[d, no head] & P^{\mathrm{AFS}}_\beta \arrow[d, no head] \\
\alpha + 1 \in M & P^M_{\alpha+1} = P^M_\alpha * Q^M_\alpha \arrow[r] \arrow[d, no head] & P_{\alpha+1} = P_\alpha * Q_\alpha \arrow[d, no head] \\
& \vdots \arrow[r] & \vdots
\end{tikzcd}$$

Figure 5.2: A sketch of the construction of an almost finite support iteration over $\bar{P}^M$.

Lemma 5.3.2. *The resulting sequence $\bar{P}$ in the previous definition is a partial* CS-*iteration and $\bar{P}^M$ embeds into $\bar{P}$.*

Proof. $\bar{P}$ is a partial CS-iteration if for every limit β, $P_\beta \subseteq P^{\mathrm{CS}}_\beta$ and if for every $\alpha < \epsilon$ $P_\alpha \Vdash Q_\alpha$ is separative. The last point is fulfilled, because we decided to work with separative forcings (otherwise we can always take the separative quotient). So we show that for every $\beta \leq \epsilon$ $P_\beta \subseteq P^{\mathrm{CS}}_\beta$. If $\beta \in M$ and $\mathrm{cof}(\beta) = \omega$, then by definition $P_\beta = P^{\mathrm{AFS}}_\beta$ which is a subset of P^{CS}_β by Lemma 5.2.5. If $\beta \in M$ and $\mathrm{cof}(\beta) > \omega$ or if $\beta \notin M$ then $P_\beta = \bigcup_{\alpha<\beta} P_\alpha \subseteq P^{\mathrm{CS}}_\beta$.

We recall that $\bar{P}^M$ embeds in $\bar{P}$ if $\forall \alpha \in \epsilon \cap M$ $i_\alpha : P^M_\alpha \to P_\alpha$ is M-complete. If $\alpha = \beta + 1$, i_α is M-complete by Proposition 5.1.8 with $S := P^M_\beta$, $B := Q^M_\beta$, $T := P_\beta$ and $C := Q_\beta$. If α is a limit ordinal, we just use Lemma 5.2.5. $\square$

In the next pages we show that AFS-iterations over $\bar{P}^M$ preserve the countable chain condition and also the property of being σ-centered. To prove that the c.c.c. is preserved we need the following two lemmata which establish the c.c.c. preservation in iterations where direct limits often occur.

Lemma 5.3.3. *Let κ be an uncountable regular cardinal. Let $\bar{P}$ be a forcing iteration of length ϵ satisfying:*

- $\forall \alpha < \epsilon$, *$P_\alpha$ satisfies the k-chain condition;*
- $\mathrm{cof}(\epsilon) < \kappa$*;*
- $P_\epsilon = \bigcup_{\alpha<\epsilon} P_\alpha$.

Then P_ϵ satisfies the κ-chain condition.

Proof. Let $A \subseteq P_\epsilon$ such that $|A| = \kappa$. Let $\langle \epsilon_\eta : \eta \in \mathrm{cof}(\epsilon) \rangle$ be a cofinal increasing sequence on ϵ. Then $P_\epsilon = \bigcup_{\eta<\mathrm{cof}(\epsilon)} P_{\epsilon_\eta}$. For every $\eta \in \mathrm{cof}(\epsilon)$ we define $A_\eta := P_{\epsilon_\eta} \cap A$. Then $A = \bigcup_{\eta\in\mathrm{cof}(\epsilon)} A_\eta$. Since κ is regular and $\mathrm{cof}(\epsilon) < \kappa$ it follows that $\exists \eta \in \mathrm{cof}(\epsilon)$ such that $|A_\eta| = \kappa$. By hypothesis P_{ϵ_η} satisfies the k-chain condition, hence A_η is not an antichain in P_{ϵ_η} and therefore A is not an antichain in P_ϵ. □

The proof of the next lemma comes from [1, Theorem 2.2] and it is reported with more details.

Lemma 5.3.4. *Let κ be a regular uncountable cardinal and let $\bar{P}$ be an iteration of length ϵ without final limit such that*

- $\forall \alpha \in \epsilon$, *$P_\alpha$ satisfies the κ-chain condition;*
- *if* $\mathrm{cof}(\epsilon) = \kappa$, *then $S := \{\beta < \epsilon : P_\beta = \bigcup_{\gamma<\beta} P_\gamma\}$ is stationary in ϵ.*

Then $P_\epsilon = \bigcup_{\alpha<\epsilon} P_\alpha$ satisfies the κ-chain condition.

Proof. Let A be a subset of P_ϵ of cardinality κ. If $\mathrm{cof}(\epsilon) > \kappa$, by regularity of $\mathrm{cof}(\epsilon)$, there exists $\alpha < \mathrm{cof}(\epsilon)$ such that $A \subseteq P_\alpha$. We conclude that A is not an antichain, because P_α satisfies the k-chain condition.
If $\mathrm{cof}(\epsilon) < \kappa$, this is Lemma 5.3.3.
If $\mathrm{cof}(\epsilon) = \kappa$ let $\langle \epsilon_\eta : \eta < \kappa \rangle$ be a normal sequence with limit ϵ, i.e. the cofinal map

$$\begin{aligned} g : \kappa &\to \epsilon \\ \eta &\mapsto \epsilon_\eta \end{aligned}$$

is increasing and continuous. We remark then that $g[\kappa]$ is a club in ϵ, hence $S' := S \cap g[\kappa]$ is in particular a stationary subset of ϵ. Let enumerate $A = \{p_\xi : \xi < \kappa\}$. We show now that the the following subset of κ is a club:

$$C := \{\eta \in \kappa : \forall \xi < \eta \; p_\xi \in P_{g(\eta)}\}.$$

C is closed: Let $\alpha < \kappa$ such that $\sup(C \cap \alpha) = \alpha$. Therefore for any $\xi < \alpha$ there is $\eta \in C \cap \alpha$ such that $\xi < \eta$, hence $p_\xi \in P_{g(\eta)} \subseteq P_{g(\alpha)}$. We just showed that $\forall \xi < \alpha \; p_\xi \in P_{g(\alpha)}$. So $\alpha \in C$.
C is unbounded in κ: Let $\beta < \kappa$, we show that there is $\gamma \in C$ such that $\beta \leq \gamma$. We construct inductively on $n \in \omega$ an increasing chain of γ_n's such that

$$\beta \leq \gamma_n \text{ and } \forall \xi < \gamma_n \; p_\xi \in P_{\gamma_{n+1}}$$

and define $\gamma := \bigcup_{n \in \omega} \gamma_n$. As $P_\gamma \subseteq P_{g(\gamma)}$, it is easy to see that $\gamma \in C$.
Let $\gamma_0 := \beta$. We suppose that the γ_i's are defined for every $i \leq n$. At step $n+1$, since P_ϵ is a direct limit, for every $\xi < \gamma_n$ there is $\beta_\xi < \kappa$ such that $p_\xi \in P_{\beta_\xi}$ and $\beta_\xi \geq \gamma_n$. Then $\gamma_{n+1} := \bigcup_{\xi < \gamma_n} \beta_\xi$.

The set $S'' := S' \cap C$ is again stationary. Let now f be the function

$$\begin{aligned} f : \kappa &\to \kappa \\ \xi &\mapsto \min\{\eta : \operatorname{supp}(p_\xi) \cap \epsilon_\xi \subseteq \epsilon_\eta\} \end{aligned}$$

Because $\operatorname{supp}(p_\xi) \cap \epsilon_\xi \subseteq \epsilon_\xi$ we have immediately that $f(\xi) \leq \xi$ for all $\xi \in \kappa$. Moreover f is regressive on S' (and therefore on S''): let $\xi \in S'$, then in particular P_{ϵ_ξ} is a direct limit and can be written as $P_{\epsilon_\xi} = \bigcup_{\eta < \xi} P_{\epsilon_\eta}$. Therefore, there is a $\eta < \xi$ such that $p_\xi \upharpoonright \epsilon_\xi \in P_{\epsilon_\eta}$. This implies $\operatorname{supp}(p_\xi) \cap \epsilon_\xi \subseteq \epsilon_\eta$ and thus $f(\xi) \leq \eta < \xi$.

By Fodor's lemma f is constant on some stationary subset $S''' \subseteq S''$. That implies that for some $\eta \in \kappa$

$$\forall \xi \in S''' \operatorname{supp}(p_\xi) \cap \epsilon_\xi \subseteq \epsilon_\eta.$$

Without loss of generality we can assume that $\forall \xi \in S''' \; \eta < \xi$. The set $B := \{p_\xi \upharpoonright \epsilon_\eta : \xi \in S'''\}$ is a subset of P_{ϵ_η} and has cardinality $\leq \kappa$. If $|B| < \kappa$, then there are $\gamma, \xi \in S'''$ such that $p_\xi \upharpoonright \epsilon_\eta = p_\gamma \upharpoonright \epsilon_\eta$.

If $|B| = \kappa$, since P_{ϵ_η} satisfies the κ-c.c., there are two compatible conditions $p_\xi \upharpoonright \epsilon_\eta$ and $p_\gamma \upharpoonright \epsilon_\eta$, let say for $\gamma < \xi$. Let $r \in P_{\epsilon_\eta}$ be a common extension. Then we can construct an extension $q \in P_\epsilon$ common to p_ξ and p_γ:

$$q(\beta) := \begin{cases} r(\beta) & \text{if } \beta < \epsilon_\eta \\ p_\gamma(\beta) & \text{if } \epsilon_\eta < \beta < \epsilon_\xi \\ p_\xi(\beta) & \text{if } \epsilon_\xi < \beta < \epsilon \end{cases}$$

We conclude that A is not an antichain. $\square$

Property 5.3.5. Let $\bar{P}^M$ be a partial CS-iteration in M of length ϵ with final limit. Let $\bar{P} = \langle P_\alpha, Q_\alpha \rangle_{\alpha \leq \epsilon}$ be the AFS-iteration over $\bar{P}^M$ where $P_\alpha \Vdash_{P_\alpha}$ "Q_α is ccc". Then P_ϵ is ccc.

Proof. By induction on $\gamma \leq \epsilon$, we show that P_γ satisfies the chain condition. If γ is a successor, then the two step iteration of κ-cc forcings is again κ-cc for any uncountable regular cardinal κ (for a proof see [15, Theorem 16.4]).
If $\mathrm{cof}(\gamma) > \omega$ then $P_\gamma = \bigcup_{\beta<\gamma} P_\beta$. Moreover $\gamma' := \sup(\gamma \cap M) < \gamma$, therefore $S := \{\beta \in \gamma : P_\beta \text{ is a direct limit } \wedge \gamma' < \beta\} \supseteq \{\beta \in \gamma : \beta \text{ limit } \wedge \gamma' < \beta\}$ is in particular stationary. We can apply Lemma 5.3.4, which implies that P_γ is ccc.
If $\mathrm{cof}(\gamma) = \omega$ and $\gamma \notin M$, then P_γ is ccc by Lemma 5.3.3 with $\kappa = \aleph_1$ and $\epsilon = \gamma$.
If $\mathrm{cof}(\gamma) = \omega$ and $\gamma \in M$, then P_γ is the AFS-limit. Let $A \subseteq P_\gamma$ be uncountable, let enumerate $A = \{q_\nu \wedge i_\gamma(p_\nu) : \nu \in \omega_1\}$. Then for every $\nu \in \omega_1$ $q_\nu \in \bigcup_{\alpha\in\gamma\cap M} P_\alpha$ and $p_\nu \in P^M_\gamma$. Since $P^M_\gamma \subset M$ and $V \models |M| = \aleph_0$, then V models that P^M_γ is countable, so there is $p \in P^M_\gamma$ such that $|\{\nu \in \omega_1 : p_\nu = p\}| = \aleph_1$. Let $B := \{\nu \in \omega_1 : p_\nu = p\}$, consider $A' := \{q_\nu \wedge i_\gamma(p) : \nu \in B\}$. Since $\mathrm{cof}(\gamma) = \omega$ and ω_1 is regular, there is $\alpha \in \gamma \cap M$ and $B' \subseteq B$ such that $|B'| = \aleph_1$ and $\forall \nu \in B'$ $q_\nu \in P_\alpha$. Now P_α is ccc, therefore there must be two compatible conditions p_ν and p_η with $\nu, \eta \in B'$. Let $r \in P_\alpha$ be a common extension, then $r \wedge i_\gamma(p)$ is a common extension of $p_\nu \wedge i_\gamma(p)$ and $p_\eta \wedge i_\gamma(p)$. We conclude that A cannot be an antichain. $\square$

Definition 5.3.6. Let P be a notion of forcing. A subset A of P is *centered* if for every $a_1, \ldots, a_n \in A$ there is some $p \in P$ such that for all $1 \leq i \leq n$ $p \leq a_i$. We say that P is *σ-centered* if it can be written as a countable union of centered sets, i.e.

$$P = \bigcup_{n\in\omega} A_n$$

where, for any $n \in \omega$, $A_n \subseteq P$ and A_n is centered.

For an AFS-iteration, if we require that only countable many iterands are non trivial -with no restrictions on the length of the iteration- then the property of being σ-centered is preserved. The proof of the next Property is based on Lemma 3.17 in [14].

Property 5.3.7. Assume that $\bar{P}$ is an AFS-iteration over $\bar{P}^M$ where only countably many Q_α are non-trivial (for example those with $\alpha \in M$) and where each Q_α is forced to be σ-centered. Then P_ϵ is σ-centered as well.

Proof. By induction on the length we prove that P_β is σ-centered for all $\beta \leq \epsilon$. If $\beta = \alpha + 1$ we know that $P_\alpha = \bigcup_{n\in\omega} A_n$ and $P_\alpha \Vdash \dot{Q}_\alpha = \bigcup_{n\in\omega} \dot{B}_n$ where for all $n \in \omega$ A_n is centered and $P_\alpha \Vdash \dot{B}_n$ is σ-centered. Then $P_\alpha * \dot{Q}_\alpha = \bigcup_{n,m\in\omega}(A_n * \dot{B}_m)$ is σ-centered: namely for every $n, m \in \omega$ the set $A_n * \dot{B}_m = \{\langle a, \dot{b}\rangle : a \in A_n \wedge a \Vdash_{P_\alpha} \dot{b} \in \dot{B}_m\}$ is centered. In fact, for any $k \in \omega$, for any $\langle a_1, \dot{b}_1\rangle, \langle a_2, \dot{b}_2\rangle, \ldots, \langle a_k, \dot{b}_k\rangle \in A_n * \dot{B}_m$ there is an $a \in P_\alpha$ which is a common extension of all the a_i's. Since $a \Vdash_{P_\alpha} \dot{b}_i \in \dot{B}_m$ then $\exists \dot{b} \in \mathrm{dom}(\dot{Q}_\alpha)$ such that $a \Vdash_{P_\alpha} \forall i \leq k, \dot{b} \leq_{Q_\alpha} \dot{b}_i$. Then $\langle a, \dot{b}\rangle$ extends every $\langle a_i, \dot{b}_i\rangle$ for $i \leq k$.

If $\mathrm{cof}(\beta) = \omega$ and P_β is a direct limit, let $\langle \beta_m : m \in \omega\rangle$ be a cofinal increasing sequence in β. We assumed that for all $m \in \omega$ $P_{\beta_m} = \bigcup_{n\in\omega} A^m_n$ where A^m_n is centered. Then $P_\beta = \bigcup_{n,m\in\omega} A^m_n$ is σ-centered.

If $\mathrm{cof}(\beta) > \omega$, since there are countably many non trivial $\dot{Q}_\alpha$, there is a $\beta' < \beta$ such that $\forall \alpha > \beta' P_\alpha \Vdash \dot{Q}_\alpha$ is trivial. Then P_β is forcing equivalent to $P_{\beta'}$ and hence σ-centered.

If $\mathrm{cof}(\beta) = \omega$, $\beta \in M$ and $P_\beta = P^{\mathrm{AFS}}_\beta$, let $B := \beta \cap M$. Then $V \models |B| = \aleph_0$. We remark that B is cofinal in β. We remark also that $V \models |P^M_\beta| = \aleph_0$. For $\alpha \in B$ we write $P_\alpha = \bigcup_{n\in\omega} A^\alpha_n$, with A^α_n centered. For $p \in P^M_\beta, \alpha \in B$ and $n \in \omega$ we consider

$$A^{\alpha,n}_p := \{q \wedge i_\beta(p) : q \in A^n_\alpha \wedge q \leq_{P_\alpha} i_\alpha(p \restriction \alpha)\}$$

Then every $A^{\alpha,n}_p$ is centered. Namely consider any $q_1 \wedge i_\beta(p), \ldots, q_k \wedge i_\beta(p) \in A^{\alpha,n}_p$ and take a common extension r of q_i for $i \leq k$ (this is possible because A^α_n is centered). Then $r \wedge i_\beta(p)$ extends all $q_j \wedge i_\beta(p)$ for $j \leq k$. Because $V \models |\omega \cup B \cup P^M_\beta| = \aleph_0$ then $P_\beta = \bigcup_{n\in\omega,\alpha\in B,p\in P^M_\beta} A^{\alpha,n}_p$ is σ-centered. □

We conclude the section with the following lemma: a finite support iteration of σ-centered forcings is again σ-centered if the length of the iteration is smaller than the continuum. The proof is guided by [13] and makes use of the existence of a family of independent functions (see Theorem 6.1.8 in Chapter 6).

Lemma 5.3.8. *Let $\langle P_\alpha, Q_\alpha : \alpha \leq \epsilon\rangle$ be a finite support iteration of length ϵ such that $|\epsilon| \leq 2^{\aleph_0}$ and such that for every $\alpha < \epsilon$ $P_\alpha \Vdash Q_\alpha$ is σ-centered. Then P_ϵ is σ-centered.*

Proof. We start by taking ϵ-many independent functions[3] $\mathcal{W} = \{F_\alpha : \alpha \in \epsilon \wedge F_\alpha \in \omega^\omega\}$. By hypothesis we know that there is a sequence $\langle C_\alpha : \alpha \in \epsilon\rangle$ such that for each $\alpha \in \epsilon$:

[3]We say that a collection $\mathcal{W} \subseteq \omega^\omega$ of functions on ω is a *family of independent functions* if

- C_α is a P_α-name;
- $P_\alpha \Vdash_{P_\alpha}$ "$C_\alpha : \omega \to \{C_\alpha(n) : n \in \omega\}$ where $C_\alpha(n)$ are centered subsets of Q_α and $Q_\alpha = \bigcup_{n\in\omega} C_\alpha(n)$";
- $P_\alpha \Vdash_{P_\alpha}$ "$\forall n \in \omega\ C_\alpha(n)$ is upwards closed, hence $1_{Q_\alpha} \in C_\alpha(n)$.

For any condition $p \in P_\epsilon$, since $\mathrm{supp}(p)$ is finite and since $\forall n \in \omega\ 1_{Q_\alpha} \in C_\alpha(n)$, there is a condition $q \leq_{P_\epsilon} p$ such that $\forall \alpha < \epsilon \exists n \in \omega q \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \in C_\alpha(n)$. This shows that the following set is dense:

$$D := \{q \in P_\epsilon : \exists p \in P_\epsilon \forall \alpha < \epsilon \exists n \in \omega\ q \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \in C_\alpha(n)\}.$$

We now consider for $k \in \omega$ the following set:

$$D_k := \{q \in D : \forall \alpha < \epsilon\ q \restriction \alpha \Vdash_{P_\alpha} q(\alpha) \in C_\alpha(F_\alpha(k))\}.$$

Claim. *D_k is centered.*

Proof of the claim. Given $q_0, \dots, q_n \in D_k$ let $S := \bigcup_{i\leq n} \mathrm{supp}(q_i)$. S is clearly finite. We want to find a condition $r \in P_\epsilon$ such that $r \leq_{P_\epsilon} q_i$ for all $i \leq n$. Inductively on $\alpha \leq \epsilon$ we define

$$r(\alpha) := \begin{cases} 1_{Q_\alpha} & \text{if } \alpha \notin S \\ t \in \mathrm{dom}(Q_\alpha) \text{ such that } r \restriction \alpha \Vdash_{P_\alpha} t \leq q_i(\alpha) (\forall i \leq n) & \text{if } \alpha \in S. \end{cases}$$

The condition r is a common extension of the q_i's and it is well defined. We observe that, by induction, $r \restriction \alpha \leq_{P_\alpha} q_i \restriction \alpha$ and, since $q_i \restriction \alpha \Vdash_{P_\alpha} q_i(\alpha) \in C_\alpha(F_\alpha(k))$, also $r \restriction \alpha \Vdash_{P_\alpha} q_i(\alpha) \in C_\alpha(F_\alpha(k))$. Because $C_\alpha(F_\alpha(k))$ is centered and because of the maximality principle, we can find a $t \in \mathrm{dom}(Q_\alpha)$ such that $r \restriction \alpha \Vdash_{P_\alpha}$ "$\forall i \leq n\ t \leq_{P_\alpha} q_i(\alpha)$". □

We now show that $D = \bigcup_{k\in\omega} D_k$. Given a condition $q \in D$, the set

$$X_q := \{(\alpha, n) : \alpha \in \mathrm{supp}(q) \wedge n = \min\{m : q \restriction \alpha \Vdash_{P_\alpha} q(\alpha) \in C_\alpha(m)\}\}$$

---

for any $l \in \omega$, for any $f_0, \dots, f_l \in \mathcal{W}$ and any $n_0, \dots, n_l \in \omega$ there is a point $x \in \omega$ such that:

$$\forall i \leq l\ f_i(x) = n_i.$$

. The existence of continuum -and hence of ϵ- many independent functions is discussed in the next chapter.

is finite. Because the $F_\alpha' s$ are independent functions, we can find a $k \in \omega$ such that for all $(\alpha, n) \in X_q$ we have $F_\alpha(k) = n$. Therefore $q \in D_k$. Finally it remains to show that P_ϵ is σ-centered. This follows directly from the fact that P_ϵ is the upwards closure of D. $\square$

Chapter 6

Independent functions and independent sets

We have shown in the previous chapter how the property of a poset being σ-centered is preserved in iterations. In particular we reported the proof[1] for the finite support iteration of length $\leq 2^{\omega}$. There, one point was taken for granted: the existence of a family of size continuum of independent functions on ω. This fact - and the proof of it - inspired the creation of the present chapter, whose aim is to put together the notions of *essentially different functions*, *independent functions*, *autonomous sets* and *independent sets* on ω. It is worth mentioning that such notions are related to the area of combinatorial cardinal characteristics of the continuum (see [2] and Figure 6.1). Namely independent sets behave similarly to the *almost disjoint sets* (see [11]) and they are involved in the definition of the *independence number* $\mathfrak{i}$, which is the smallest size of a maximal independent family.

In the next pages, among others, we have rewritten in more details and modern notation some proofs from [8] , [7] and [13]. Furthermore in Lemma 6.1.6 a new construction is presented - the so-called *triangular tree* - to show, without help of independent sets, the existence of continuum-many essentially different functions.

[1] This topic was discussed by Goldstern and Blass in [13]

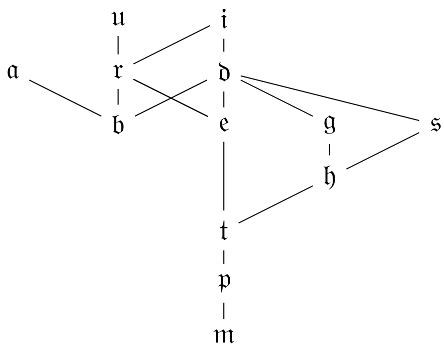

Figure 6.1: Hasse's diagram of combinatorial characteristics of the continuum.

We give a list of the principal symbols and notations used in this chapter.

$x\Delta y$: The symmetric difference $x\Delta y := (x \setminus y) \cup (y \setminus x)$.

B^A: For A and B non-empty, B^A is the set of all functions with domain A and image in B.

$f[X]$: The set of the images of elements in X when f is a function $f[X] := \{f(x) : x \in X\}$.

$A^{\complement}_X$: Let $A \subseteq X$, the complement of A in X is $A^{\complement}_X := \{x \in X : x \notin A\}$. When X is clear from the context we will write $A^{\complement}$.

$A^{<\omega}$: The set of finite sequences of A, i.e. $A^{<\omega} := \{f : \exists n \in \omega\ f : n \to A\}$.

A^{ω}: The set of infinite sequences on A, i.e. $A^{\omega} := \{f : f : \omega \to A\}$.

$[A]^{<\omega}$: The set of all finite subsets of A, i.e. $[A]^{<\omega} := \{x \in \mathcal{P}(A) : |x| < \omega\}$.

$[A]^{\omega}$: The set of all infinite countable subsets of A, i.e. $[A]^{\omega} := \{x \in \mathcal{P}(A) : |x| = \omega\}$.

$[T]$: The branches of the tree $T \subseteq \omega^{<\omega}$, i.e. $[T] := \{x \in \omega^{\omega} : \forall n \in \omega\ x \restriction n \in T\}$.

$\mathrm{lh}(t)$: The length of the sequence $t \in \omega^{<\omega}$.

6.1 Essentially different and independent functions

In this section we introduce the *essentially different functions* (Definition 6.1.1) and we show that there are continuum many of them on $[\omega]^{<\omega}$ and on ω (Theorem 6.1.3 and Corollary 6.1.4). Furthermore we restrict the assumptions and define the notion of *independent functions* (Definition 6.1.5). Analogously, we show that there are continuum many such functions on ω (Theorem 6.1.8); the proof makes use of the existence of the *triangular tree* (Lemma 6.1.6).

Definition 6.1.1. Let A be an infinite set. We say that a collection[2] $\mathcal{W} \subseteq A^A$ of functions on A is a *family of essentially different functions* if, whenever we pick finitely many of them, they differ in a common point:

$$\forall m \in \omega\ \forall f_0, \ldots, f_m \in \mathcal{W} \exists x \in A \forall i < j \leq m(f_i(x) \neq f_j(x)).$$

In the next pages A will be either $[\omega]^{<\omega}$ or ω. The following short lemma is used in Corollary 6.1.4 with $B = \omega$ and $C = [\omega]^{<\omega}$.

Lemma 6.1.2. *Given non empty sets B and C and a bijection $g : B \to C$, the* conjugating function *defined as:*

$$\begin{aligned} G : C^C &\to B^B \\ f &\mapsto g^{-1} \circ f \circ g \end{aligned}$$

is injective. Therefore for any $X \subseteq C^C$ there is a bijection between X and $G[X]$.

Proof. Let $f_1 \neq f_2$ be two different functions in C^C, then there is $x \in C$ such that $f_1(x) \neq f_2(x)$. Hence $G(f_1) \neq G(f_2)$ because $G(f_1)(g^{-1}(x)) = g^{-1} \circ f_1 \circ g(g^{-1}(x)) = g^{-1} \circ f_1(x) \neq g^{-1} \circ f_2(x) = g^{-1} \circ f_2 \circ g(g^{-1}(x)) = G(f_2)(g^{-1}(x))$, as g is a bijection. □

The original proof of the following theorem comes from [8].

Theorem 6.1.3. *There are continuum many essentially different functions on $[\omega]^{<\omega}$, i.e.*

$$\exists \mathcal{W} \subseteq ([\omega]^{<\omega})^{([\omega]^{<\omega})} \text{ such that } |\mathcal{W}| = 2^{\aleph_0} \text{ and } \mathcal{W} \text{ is a family of essentially different functions.}$$

[2] We chose the letter $\mathcal{W}$ because it remembers the original German definition *wesentlich verschiedene Abbildungen* given by Hausdorff in [8].

Proof. Recall that $|[\omega]^\omega| = 2^{\aleph_0}$. So we will enumerate the essentially different functions by an index t running on $[\omega]^\omega$, giving us continuum many essentially different functions in the resulting $\mathcal{W}$. For every infinite set $t \in [\omega]^\omega$ we define the functions e_t on $[\omega]^{<\omega}$:

$$\begin{aligned} e_t : [\omega]^{<\omega} &\to [\omega]^{<\omega} \\ x &\mapsto x \cap t. \end{aligned}$$

We claim that the family $\mathcal{W} = \{e_t : t \in [\omega]^\omega\}$ has cardinality continuum and consists of essentially different functions. Consider the following function:

$$\begin{aligned} E : [\omega]^\omega &\to \mathcal{W} \\ t &\mapsto e_t \end{aligned}$$

The map E is surjective by definition of $\mathcal{W}$. It is also injective: let $t_1, t_2 \in [\omega]^\omega$ be different. Without loss of generality we can pick some $m \in t_1 \setminus t_2$. By the choice of m we have $e_{t_1}(\{m\}) = \{m\} \cap t_1 = \{m\}$ but $e_{t_2}(\{m\}) = \emptyset$. Hence $E(t_1) \neq E(t_2)$. We conclude that $|\mathcal{W}| = |[\omega]^\omega| = 2^\omega$.
We take now finitely many functions $e_{t_0}, e_{t_1}, \ldots, e_{t_m} \in \mathcal{W}$, we show that they are essentially different, i.e. that there is a finite set x such that $e_{t_i}(x) \neq e_{t_j}(x)$ for all $i < j \leq m$. For every $i < j \leq m$ we pick an $n_{ij} \in t_i \Delta t_j$ (as in Figure 6.1), and define $x := \{n_{ij} : i < j \leq m\}$.

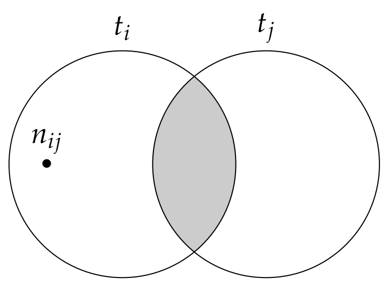

Figure 6.2: The element n_{ij} is taken outside the intersection of t_i and t_j.

Then $e_{t_i}(x) = t_i \cap x \neq t_j \cap x = e_{t_j}(x)$ because n_{ij} is either in $t_i \cap x$ or in $t_j \cap x$ (n_{ij} was chosen outside of $t_i \cap t_j$). $\square$

Corollary 6.1.4. *There are continuum many essentially different functions on ω, i.e. $\exists \mathcal{W}' \subseteq \omega^\omega$ such that $|\mathcal{W}'| = 2^{\aleph_0}$ and $\mathcal{W}'$ contains only essentially different functions.*

Proof. We apply Lemma 6.1.2 with $B = \omega$, $C = [\omega]^{<\omega}$ and a bijection $g : \omega \to \omega^{<\omega}$, getting $G : ([\omega]^{<\omega})^{([\omega]^{<\omega})} \to \omega^\omega$.

$$\begin{tikzcd}
\omega \arrow[r, dashed, "f:=G(e)"] \arrow[d, "g"] & \omega \\
{[\omega]^{<\omega}} \arrow[r, "e"] & {[\omega]^{<\omega}} \arrow[u, "g^{-1}"']
\end{tikzcd}$$

Figure 6.3: Each $e \in \mathcal{W}$ corresponds to a unique $f \in \mathcal{W}'$ such that $f = G(e) := g^{-1} \circ e \circ g$.

Consider the family $\mathcal{W}$ of Theorem 6.1.3, then

$$\mathcal{W}' := G[\mathcal{W}]$$

is a family of essentially different functions on ω such that $|\mathcal{W}'| = 2^{\aleph_0}$. $\square$

We consider now a restriction of the initial definition and present the notion of *independent functions*:

Definition 6.1.5. Let A be an infinite set. We say that a collection $F \subseteq A^A$ of functions on A is a *family of independent functions on A* if for any $l \in \omega$, for any $f_0, \dots, f_l \in F$ and any $n_0, \dots, n_l \in A$ there is a point $x \in A$ such that:

$$\forall i \leq l \ f_i(x) = n_i.$$

Our proof of the existence of continuum many independent functions is based on the existence of the *triangular tree*:

Lemma 6.1.6. *There is a perfect*[3] *tree $T \subseteq \omega^{<\omega}$ whose branches contain infinitely many often any natural number, i.e. $\forall b \in [T]$, $\forall n \in \omega$ $|\{i \in \omega : b(i) = n\}| = \omega$.*

Proof. We will take the nodes of $\omega^{<\omega}$ such that the branches have the form:

$$\langle \square, \square, 0, \square, 0, 1, \square, 0, 1, 2, \square, 0, 1, 2, 3, \square, \dots, 0, 1, \dots, n, \square, \dots, \rangle$$

where every time that some $\square$ appears we can choose any value of ω. More precisely, we say that a finite sequence x of natural numbers is in T if and

[3] A tree is *perfect* if every node has an extension node that is splitting. A node is splitting if it has at least two immediate successor nodes.

only if for all $n < \mathrm{lh}(x)$

$$x(n) = \begin{cases} \text{some element of } \omega, & \text{if } \exists k \in \omega \ (n = \frac{k(k+1)}{2}); \\ (n-1) - \frac{k(k+1)}{2}, & \text{if } \exists k \in \omega \ (\frac{k(k+1)}{2} < n < \frac{(k+1)(k+2)}{2}). \end{cases} \qquad \square$$

We call such a tree the *triangular tree.* [4]

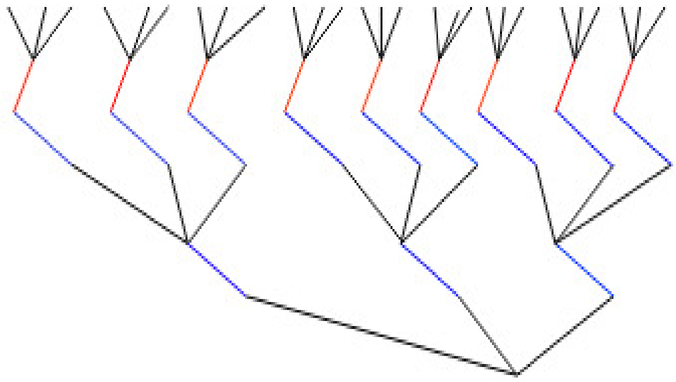

Figure 6.4: A finite sub-tree of the triangular tree.

Lemma 6.1.7. *Let $T \subseteq \omega^{<\omega}$ be the triangular tree. Let denote with $\mathrm{fin}({}^{<\omega}\omega, \omega)$ the set of partial finite functions $p : \omega^{<\omega} \to \omega$. Then there is a $F : [T] \times \mathrm{fin}(\omega^{<\omega}, \omega) \to \omega$ such that for every $f_0, \ldots, f_m \in [T]$ for $m \in \omega$ and every $n_0, \ldots, n_m \in \omega$ there is a $p \in \mathrm{fin}(\omega^{<\omega}, \omega)$ such that $F(f_i, p) = n_i$ for $i \leq m$.*

Proof. We define F in the following way:

$$F(f,p) := \begin{cases} f(\min\{l : p(f \restriction l) = f(l)\}), & \text{if } \{l \in \omega : p(f \restriction l) = f(l)\} \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

Given $f_0, \ldots, f_m \in [T]$ and $n_0, \ldots, n_m \in \omega$, let $n \in \omega$ be the least level where all the $f_i \restriction n$ differ:

$$n := \min\{k \in \omega : \forall i < j \leq m(f_i \restriction n \neq f_j \restriction n)\}$$

Now for every $i \leq m$ there is some level $a_i \geq n$ such that $f_i(a_i) = n_i$ (because every natural number appears infinitely often in every branch of the tree). Let $p := \{\langle f_i \restriction a_i, n_i \rangle : i \leq m\}$. Then $\{l : p(f_i \restriction l) = f_i(l)\} = \{a_i\}$ and $F(f_i, p) = f_i(a_i) = n_i$ for every $i \leq k$. $\square$

[4] We call it triangular because the the positions where $\square$ appear correspond to the triangular numbers $k(k+1)/2 = \binom{k+1}{2}$.

Theorem 6.1.8. *There is a family $\mathcal{G} \subseteq \omega^\omega$ of independent functions of size continuum.*

Proof. Let T be the triangular tree and $F : [T] \times \mathrm{fin}(\omega^{<\omega}, \omega) \to \omega$ defined as in the previous lemma. Let $g : \omega \to \mathrm{fin}(\omega^{<\omega}, \omega)$ be a bijection. For every branch $b \in [T]$ we define a function on ω:

$$\begin{aligned} g_b : \omega &\to \omega \\ n &\mapsto F(b, g(n)). \end{aligned}$$

We claim that the family $\mathcal{G} := \{g_b : b \in [T]\}$ contains essentially different functions and $|\mathcal{G}| = 2^\omega$. If $b_1 \neq b_2$, then, by the previous lemma, there is $n \in \omega$ such that $g_{b_1}(n) = F(b_1, g(n)) \neq F(b_2, g(n)) = g_{b_2}(n)$. Fix $b_0, \dots, b_m \in [T]$ and $n_0, \dots, n_m \in \omega$. There is $n \in \omega$ such that $g_{b_i}(n) = F(b_i, g(n)) = n_i$ for all $i \leq m$. $\square$

6.2 Autonomous sets and independent sets

We define now the *autonomous sets* and show that there are continuum many of them on $\omega \times [\omega]^{<\omega}$ and on ω (Theorem 6.2.2 and Corollary 6.2.3). The proofs make use of the existence of continuum many essentially different functions. We will notice that the autonomous sets in those proofs are also *independent sets* (Theorem 6.2.5), which implies that there are continuum many independent sets on ω (Corollary 6.2.6).

Definition 6.2.1. Suppose that we have a set U, that we call "universe", and some subsets A and B. We remark that the intersections of A, or its complement, with B, or its complement, constitute a partition of the universe, as in Figure 6.5.

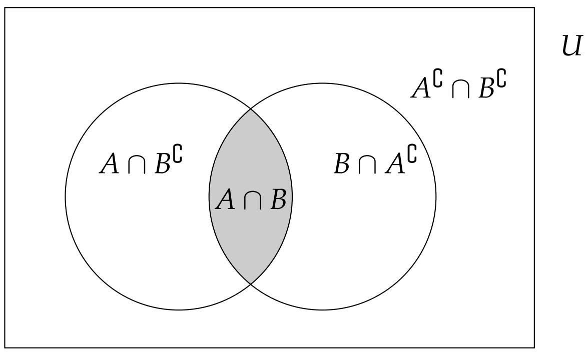

Figure 6.5: A maximal partition of the universe with two sets.

We say that finitely many sets $A_1, A_2, \ldots, A_n \subseteq U$ are *autonomous* if the related partition $P_{A_1,A_2,\ldots,A_n} := \{A_1^{i_1} \cap A_2^{i_2} \cap \cdots \cap A_n^{i_n} : i_k \in \{0,1\} \forall k \leq n\}$ (where $A^0 := A$ and $A^1 = A^{\complement}$) has maximal size 2^n. The following definition generalizes the concept:

Let U be an infinite set. We say that a family $\mathcal{S} \subset \mathcal{P}(U)$ is a *family of autonomous sets* if for every pair $\mathcal{A}, \mathcal{B} \in [\mathcal{S}]^{<\omega}$ of finite subsets of $\mathcal{S}$, such that $\mathcal{A} \cap \mathcal{B} = \emptyset$, the following holds:

$$\bigcap \mathcal{A} \cap (\omega \setminus \bigcup \mathcal{B}) \neq \emptyset.$$

The proof of the following theorem takes advantage of the existence of continuum many essentially different functions on ω and comes from [8].

Theorem 6.2.2. *There is a family of autonomous sets $\mathcal{S} \subseteq \mathcal{P}_\omega(\omega \times [\omega]^{<\omega})$ of size continuum.*

Proof. Let $\{f_t : t \in 2^\omega\}$ be the family of continuum many essentially different functions of Corollary 6.1.4. We remark that the set $\omega \times [\omega]^{<\omega}$ is countable. We consider the following function:

$$\begin{aligned} Z : 2^\omega &\to \mathcal{P}(\omega \times [\omega]^{<\omega}) \\ t &\mapsto \{(n, y) : f_t(n) \in y\} \end{aligned}$$

We show that $\mathcal{S} := \mathrm{im}(Z)$ has size continuum and is a family of autonomous sets on $\omega \times [\omega]^{<\omega}$. Z is injective: Let $t_1 \neq t_2 \in 2^\omega$, then $f_{t_1} \neq f_{t_2}$ and so there is some $n \in \omega$ such that $f_{t_1}(n) \neq f_{t_2}(n)$. Therefore $Z(t_1) \neq Z(t_2)$ because $(n, \{f_{t_1}(n)\}) \in Z(t_1) \setminus Z(t_2)$. The family $\mathcal{S}$ has size continuum: $2^\omega \leq |\mathcal{S}| \leq 2^{|\omega \times [\omega]^{<\omega}|} = 2^\omega$.
We pick now two disjoint $A, B \in [\mathcal{S}]^{<\omega}$ and we enumerate them:

$$\begin{aligned} A &= \{Z(t_i) : i \leq n\} \\ B &= \{Z(t'_j) : j \leq m\}. \end{aligned}$$

Because the f's are essentially different, there is $x \in \omega$ such that $f_{t_i}(x)$ and $f_{t'_j}(x)$ are all distinct for $i \leq n$ and $j \leq m$. We define $y := \{f_{t_i}(x) : i \leq n\}$. It follows that $\forall i \leq n$ and $\forall j \leq m$:

$$f_{t_i}(x) \in y \wedge f_{t'_j}(x) \notin y,$$

therefore

$$(x, y) \in Z(t_i) \wedge (x, y) \notin Z(t'_j).$$

Hence
$$\bigcap_{i\leq n} Z(t_i) \cap \bigcap_{j\leq m} (Z(t'_j))^{\complement} \neq \emptyset.$$
□

Corollary 6.2.3. *There is an autonomous family $\mathcal{S}' \subseteq [\omega]^\omega$ of size continuum.*

Proof. Let $b : \omega \times [\omega]^\omega \to \omega$ be a bijection. Then $\mathcal{S}' := \{b[X] : X \in \mathcal{S}\}$ is the family we are looking for. □

If we look closer to the proof of Theorem 6.2.2, the set $\mathcal{S}$ is actually a family of *independent sets* as we will show in Theorem 6.2.5.

Definition 6.2.4. Let U be an infinite and countable set. We say that a family $\mathcal{S} \subseteq \mathcal{P}(U)$ is an *independent family* (or a *family of independent sets*) if for every disjoint pair $\mathcal{A}, \mathcal{B} \in [\mathcal{S}]^{<\omega}$ of finite subsets of $\mathcal{S}$ the following holds:
$$|\bigcap \mathcal{A} \cap (\omega \setminus \bigcup \mathcal{B})| = \omega.$$

We can immediately remark that independent families are also autonomous.

Theorem 6.2.5. *There is an independent family $\mathcal{S} \subseteq \omega \times [\omega]^{<\omega}$ of size continuum.*

Proof. The family $\mathcal{S}$ constructed in Theorem 6.2.2 satisfies the conditions. For $k \in \omega$ define the sets $y_k := y \cup \{k\}$. With the notation of the proof of Theorem 6.2.2, we get
$$\{(x, y_k) : k > \max\{y \cup \{f_{t'_j}(x) : j \leq m\}\}\} \subseteq \bigcap_{i\leq n} Z(t_i) \cap \bigcap_{j\leq m} (Z(t'_j))^{\complement}.$$
□

Analogously as before:

Corollary 6.2.6. *There is an independent family $\mathcal{S}' \subseteq [\omega]^\omega$ of size continuum.*

Proof. Let $b : \omega \times [\omega]^{<\omega} \to \omega$ be a bijection, then $S' := \{b[X] : X \in S\}$ is an independent family of subsets of ω of size continuum. □

There are many other ways of proving Corollary 6.2.6: see for example [11] for a collection of eight different proofs.

6.3 From independent sets to independent functions

We present two proofs for showing that for any independent family on ω there is a family of independent functions of the same size. The first proof comes from [7, Theorem 3]. The sketch for second proof, which contains a nice construction, is to be found in [21].

Theorem 6.3.1. *Given an independent family $G \subseteq [\omega]^\omega$ of infinite subsets of ω, there is a family $F \subseteq \omega^\omega$ of independent functions such that $|F| = |G|$. In particular, by Corollary 6.2.6 there is an independent family of functions of size continuum.*

Proof. We define $A := \{(n_0, \ldots, n_k, F_0, \ldots, F_k) : k \in \omega, \forall i \leq k\, n_i \in \omega \wedge F_i \in [\omega]^{<\omega}\}$. We remark that $|A| = \omega$, hence we can fix a bijection $h : \omega \to A$. We assign to every $S \in G$ a function $f_S \in {}^\omega\omega$ in the following way: for $n \in \omega$ let $h(n) = (n_0, \ldots, n_k, F_0, \ldots, F_k)$, then

$$f_S(n) := \begin{cases} n_i, & \text{if } \exists i \leq k (F_i \subseteq S \wedge F_j \nsubseteq S\ \forall j \neq i); \\ 0, & \text{otherwise.} \end{cases}$$

Let $F := \{f_S : S \in G\}$. We show that $|F| = |G|$: let $S_1 \neq S_2 \in G$, then $f_{S_1} \neq f_{S_2}$. In fact, for $n \in S_1 \setminus S_2$, let $s := (1, \{n\})$. For $m := h^{-1}(s)$ we have

$$f_{S_i}(m) = \begin{cases} 1, & \text{if } \{n\} \subseteq S_i \\ 0, & \text{otherwise} \end{cases}$$

and therefore $f_{S_1}(m) = 1$ while $f_{S_2}(m) = 0$.

We prove now that F is a family of independent functions. Let $f_{S_0}, \ldots, f_{S_k} \in F$ be different functions and $n_0, \ldots, n_k \in \omega$.

For $i, j = 0, \ldots, k$, if $i \neq j$ we pick $n_{ij} \in S_i \setminus S_j$ and define

$$F_i := \begin{cases} \{n_{i0}, n_{i1}, \ldots, n_{ii-1}, n_{ii+1}, \ldots, n_{ik}\} & \text{for } 1 \leq i \leq k-1, \\ \{n_{i1}, n_{i2}, \ldots, n_{ik}\} & \text{for } i = 0, \\ \{n_{i0}, n_{i2}, \ldots, n_{ik-1}\} & \text{for } i = k. \end{cases}$$

It is immediate to see that $F_i \subseteq S_i$ and $F_i \nsubseteq S_j\ \forall j \neq i$.

We conclude that for

$$a = h^{-1}((n_0, n_1, \ldots, n_k, F_0, F_1, \ldots, F_k))$$

we have $f_{S_i}(a) = n_i$ for $i = 0, 1, \ldots, k$. $\square$

Theorem 6.3.2. *Given a family A of independent subsets of ω such that $|A| > \omega$ there is a family F of independent functions on ω of the same size as A.*

Proof. We enumerate $A = \{a_\alpha : \alpha \in \gamma\}$ and we consider a bijection $h : \gamma \times \omega \to \gamma$. For $\alpha \in \gamma$ we define the following set

$$A_\alpha := \{a_{h(\alpha,n)} \in A : n \in \omega\}.$$

The A_α's have size ω and form a partition of A in γ parts. We remark that given $\alpha \in \gamma$, if $m < n$ the following intersection is empty:

$$\bigcap_{i<m} a^{\complement}_{h(\alpha,i)} \cap a_{h(\alpha,m)}) \cap (\bigcap_{i<n} a^{\complement}_{h(\alpha,i)} \cap a_{h(\alpha,n)} = \emptyset.$$

So for every $k \in \omega$, if there is an $n \in \omega$ such that $k \in \bigcap_{i<n} a^{\complement}_{h(\alpha,i)} \cap a_{h(\alpha,n)}$ then such n is unique. For every $\alpha \in \gamma$ the following function is therefore well defined:

$$f_\alpha : \omega \to \omega$$
$$f_\alpha(k) := \begin{cases} n & \text{if } k \in \bigcap_{i<n} a^{\complement}_{h(\alpha,i)} \cap a_{h(\alpha,n)}; \\ 0 & \text{otherwise.} \end{cases}$$

We remark that f_α is surjective and that $\forall n \in \omega$ the set $f_\alpha^{-1}(n)$ is infinite. We show that $F := \{f_\alpha : \alpha \in \gamma\}$ is a family of independent functions and therefore has size $|\gamma|$ (the f_α's are pairwise different, as they are independent). Let $(\alpha_1, n_1), (\alpha_2, n_2), \dots, (\alpha_l, n_l) \in \gamma \times \omega$ have distinct first coordinates. We want to find a $k \in \omega$ such that $f_{\alpha_i}(k) = n_i$ for all $i = 1, \dots, l$.

Consider the intersection

$$I := \bigcap_{1 \le j \le l} (\bigcap_{i<n_j} a^{\complement}_{h(\alpha_i,i)} \cap a_{h(\alpha_i,n_i)})$$

by independence of the sets I is not empty, so we can take a $k \in I$ and, by definition, we get $f_{\alpha_i}(k) = n_i$ for $1 \le i \le l$. □

6.4 From independent functions to independent sets

We conclude the chapter with a last implication: for any family of independent functions there is a family of independent sets of the same size. To prove the result we need the following lemma that characterizes the independent functions.

Lemma 6.4.1. *Let $F \subseteq \omega^\omega$ be a family of independent functions on ω. Then for any $l \in \omega$, for any $f_0, \ldots, f_l \in F$ and for any $n_0, \ldots, n_l \in \omega$ there are infinitely many $k \in \omega$ such that*

$$\forall i \leq l \ f_i(k) = n_i.$$

Proof. We fix $l \in \omega$, $f_0, \ldots, f_l \in F$ and $n_0, \ldots, n_l \in \omega$. For any $k \in \omega$ we show that there is a $k' \in \omega$ such that $k' > k$ and $f_i(k') = n_i$ for all $i \leq l$. We pick any $f_{l+1} \in F$ and define

$$n_{l+1} := \sum_{i=0}^{k} f_{l+1}(i) + 1.$$

In particular for every $i \leq k$

$$n_{l+1} \neq f_{l+1}(i). \tag{6.1}$$

By independence of the functions, $\exists k' \in \omega$ such that $\forall i \leq l+1 \ f_i(k') = n_i$. By (6.1) it has to be $k' > k$. □

Corollary 6.4.2. *For every $l \in \omega$, for every $f_0, \ldots, f_l \in F$, for every $n_0 \ldots, n_l \in \omega$ the previous lemma implies that*

$$|f_0^{-1}[\{n_0\}] \cap f_1^{-1}[\{n_1\}] \cdots \cap f_l^{-1}[\{n_l\}]| = \omega.$$

We are now ready to prove the theorem:

Theorem 6.4.3. *Let $F \subseteq \omega^\omega$ be a family of independent functions, then there is a family $G \subseteq [\omega]^\omega$ of independent sets such that $|G| = |F|$.*

Proof. For every $f \in F$ we define $A_f := f^{-1}[\{0\}]$. By Corollary 6.4.2, $|A_f| = \omega$. We show that the family $G := \{A_f : f \in F\}$ is independent. Given $\mathcal{A} := \{A_{f_0}, \ldots, A_{f_n}\}$ and $\mathcal{B} := \{A_{g_0}, \ldots, A_{g_m}\}$, such that $\mathcal{A} \cap \mathcal{B} = \emptyset$, we prove that the intersection $\bigcap_{i \leq n} A_{f_i} \cap \bigcap_{j \leq m} A_{g_j}^{\complement}$ is infinite. This is immediate by Corollary 6.4.2:

$$\begin{aligned} \omega &= |\bigcap_{i \leq n} f_i^{-1}[\{0\}] \cap \bigcap_{j \leq m} g_j^{-1}[\{1\}]| \\ &\leq |\bigcap_{i \leq n} f_i^{-1}[\{0\}] \cap \bigcap_{j \leq m} (\bigcup_{i>0} g_j^{-1}[\{i\}])| \\ &= |\bigcap_{i \leq n} A_{f_i} \cap \bigcap_{j \leq m} A_{g_j}^{\complement}|. \end{aligned}$$

□

Appendix

A.1 H_λ and its countable elementary submodels

We collect some facts about countable elementary submodels of H_λ, that can help to better understand the examples and constructions of some ord-transitive models in Chapter 1.

Definition A.1.1. A *model* for a given language L is a pair $\mathfrak{A} = (A, I)$, where A is the universe of $\mathfrak{A}$ and I is the interpretation function which maps the symbols of L to appropriate relations, functions and constants in A. It is usually also displayed as $\mathfrak{A} = (A, P^{\mathfrak{A}}, ..., F^{\mathfrak{A}}, ..., c^{\mathfrak{A}})$. By recursion on the length of terms and formulas one defines the value of a term $t^{\mathfrak{A}}[a_1, ...a_n]$ and satisfaction $\mathfrak{A} \models \phi[a_1, ..., a_n]$

Definition A.1.2. A *submodel* of $\mathfrak{A}$ is a subset $B \subset A$ endowed with the relations $P^{\mathfrak{A}} \cap B^n$, functions $F^{\mathfrak{A}} \restriction B^m$, and constants $c^{\mathfrak{A}}$. All $c^{\mathfrak{A}}$ belong to B and B is closed under all $F^{\mathfrak{A}}$.

Definition A.1.3. A submodel $\mathfrak{B} \subset \mathfrak{A}$ is an *elementary submodel* $\mathfrak{B} \prec \mathfrak{A}$ if for every formula ϕ, and every $b_1, ..., b_n \in B$,

$$\mathfrak{B} \models \phi[b_1, ..., b_n] \text{ iff } \mathfrak{A} \models \phi[b_1, ..., b_n]$$

Lemma A.1.4 (Tarsky-Vaught criterion). *Let $\mathfrak{A}$ a submodel of $\mathfrak{B}$. Then $\mathfrak{A} \preceq \mathfrak{B}$ iff for all existential formulas $\phi(\vec{x})$ (of the form $\exists y \psi(\vec{x}, y)$), all $\vec{a} \in A$: if $\mathfrak{B} \models \phi[\vec{a}]$, then there is some $b \in A$ such that $\mathfrak{B} \models \psi[\vec{a}, b]$*

Theorem A.1.5 (Downward Löwenheim-Skolem-Tarski Theorem). *ZFC^-. Let $\mathfrak{B}$ be any structure for a given language $\mathcal{L}$. Fix κ such that $max(|\mathcal{L}|, \aleph_0) \leq \kappa \leq |B|$, and fix $S \subseteq B$ with $|S| \leq \kappa$. Then there is an $\mathfrak{A} \preceq \mathfrak{B}$ such that $S \subseteq A$ and $|A| = \kappa$.*

Definition A.1.6. For any cardinal λ

$$H_\lambda = \{x : |\operatorname{trcl}(x)| < \lambda\}$$

is the family of all *sets hereditarily of cardinality less than* λ, where the *transitive closure* $T = \operatorname{trcl}(x)$ is defined by induction as $T_0 = x$, $T_{n+1} = \bigcup T_n$ and $T = \bigcup_{n=0}^{\infty} T_n$.

Lemma A.1.7. *For any infinite cardinal* λ, $|H_\lambda| = 2^{<\lambda}$.

Proof. For all $\kappa < \lambda$ we have $P(\kappa) \subseteq H_\lambda$, thus $2^\kappa \leq |H_\lambda|$. To show that $2^\kappa \leq |H_\lambda|$, we construct an injective map $F : H(\lambda) \to \bigcup\{P(\kappa \times \kappa) : \kappa < \lambda\}$ as follows: if $x \in H(\lambda)$ let $\kappa = |\operatorname{trcl}(x) \cup \{x\}| < \lambda$ and by AC choose a relation $F(x) \subseteq \kappa \times \kappa$ such that $(\kappa; F(x)) \cong (\operatorname{trcl}(x) \cup \{x\}; \in)$. F is injective because x is determined by the isomorphism type of $\in$ on $\operatorname{trcl}(x) \cup \{x\}$ $\square$

The proof of the next lemma can be found in [20, Theorem II.2.1].

Lemma A.1.8. (ZFC^-)[1] *The set* $(H_\lambda, \in)$ *is a model of all* ZFC *except for the instances of the power-set axiom whenever* λ *is a regular uncountable cardinal and* $H_\lambda \models ZFC$ *if* λ *is strongly inaccessible.*[2]

In the following lemma we collect properties of countable elementary submodels of H_λ.

Lemma A.1.9. *Let* $M \preceq H_\lambda$ *be countable. Then:*

1. *Every finite ordinal, as well as* ω, *is a member of* M;
2. *Every finite subset of* M *is a member of* M;
3. *Every set definable with parameters in* M *is a member of* M. *Therefore the definable ordinals* $\omega, \omega_1, \omega_2, \ldots, \omega_\omega, \omega_{\omega_1}$ *can all be assumed to be members of* M;
4. *Suppose that* $\alpha < \lambda$ *and* $\alpha \subseteq M$, *then for every* $x \in M$ *if* $H_\lambda \models |x| = \alpha$ *then* $x \subseteq M$;
5. *Every countable element of* M *is a subset of* M, *but not every countable ordinal is an element of* M;

[1] ZFC^- denote ZFC with the Foundation Axiom deleted.

[2] $H_\lambda \models$ ZFC is actually a scheme in the metatheory, i.e. we prove that ϕ^{H_λ} is a theorem of ZFC^- whenever ϕ is any axiom of ZFC.

6. *The relation $\in$ is transitive on the set of elementary countable submodels of H_λ, i.e. for any countable $A, B, C \preceq H_\lambda$ if $A \in B \in C$ then $A \in C$;*

7. *Let $y := M \cap \omega_1$, then:*

   (a) *$y = \delta$ for some countable ordinal δ;*

   (b) *If $x \in M$, $x \subseteq \omega_1$ and $\delta \in x$, then x is stationary in ω_1. In particular, x is not countable;*

8. *If $\lambda > \omega_1$ then $\omega_1 \in M$ and $\omega_1 \subsetneq M$; if $\delta = M \cap \omega_1$ and π is the Mostowski isomorphism from M onto a transitive T, then $\pi(\omega_1) = \delta$ and $\pi(\xi) = \xi$ for all $\xi < \delta$. Also, $T \models ZFC - P$ (if λ regular) and $\delta = (\omega_1)^T$.*

9. *If $M \preceq H_{\aleph_1}$, then M is transitive.*

The Mostowski's function *on (A, R), with R well-founded and set-like on A, is defined recursively: for $y \in A$ we define $\pi(y) = \{\pi(x) : x \in y \downarrow\}$, where $y \downarrow = pred_{A,R}(y)$.*

Proof. Points 1-6 are easy consequences of elementarity and of the definition of H_λ (for more details see [17, Section 24.2]).
Point 7: $M \cap \omega_1$ is an ordinal because it is linearly ordered and transitive (every $x \in M \cap \omega_1$ is countable and thus $x \subseteq M \cap \omega_1$). Because in particular $\delta \subseteq M$ it follows that δ is countable. If δ were a successor ordinal, there would be some α such that $\delta = \alpha \cup \{\alpha\} = M \cap \omega_1$ but then $\alpha \in M \cup \omega_1$ and since the set $S(\alpha) \in \omega_1$ is definable with parameters in M, it follows that $\delta = S(\alpha) \in M \cap \omega_1$, a contradiction.
To prove the stationarity of x if, by contradiction,

$$H_\lambda \models \exists C[(C \text{ club in } \omega_1) \wedge (C \cap x = \emptyset)]$$

then, by elementarity,

$$M \models \exists C[(C \text{ club in } \omega_1) \wedge (C \cap x = \emptyset)].$$

Let $C \in M$ such that $M \models (C \text{ club in } \omega_1) \wedge (C \cap x = \emptyset)$. We show that $\delta \in C$, in order to get the contradiction $\delta \in C \cap x$. Now, $M \models (C \text{ is a club})$, implies in particular that $M \models (C \text{ is unbounded in } \omega_1)$, that is $C \cap M$ is unbounded $\omega_1 \cap M$. It implies that $\sup(C \cap \delta) = \delta$. Since $H_\lambda \models C$ is a club, we get that $H_\lambda \models \delta \in C$.
Point 8: $\omega_1 \nsubseteq M$ is clear. $\omega_1 \in M$ as ω_1 is definable in H_λ as the first

uncountable ordinal. Remark that $M \cap (\omega_1 \cup \{\omega_1\}) = \delta \cup \{\omega_1\}$, therefore $\pi(\xi) = \xi$ for all $\xi < \delta$ and $\pi(\omega_1) = \delta$. The rest follows from the fact that $T \cong M \preceq H_\lambda \models \mathsf{ZFC\text{-}P}$.
Point 9 is a direct consequence of point 5. $\square$

A.2 Proper forcing

Definition A.2.1. Let λ be an uncountable cardinal. Let $[\lambda]^\omega = \{x \subseteq \lambda : |x| = \omega\}$. We say that $C \subseteq [\lambda]^\omega$ is a *club* if

i) C is *unbounded*: $\forall x \in [\lambda]^\omega \exists y \in C\; x \subseteq y$;

ii) C is *closed*: for any chain $x_0 \subseteq x_1 \subseteq \cdots \subseteq x_n \subseteq \ldots, n < \omega$ of sets in C, $\bigcup_{n\in\omega} x_n \in C$.

A set $S \subseteq [\lambda]^\omega$ is *stationary* if it intersects every club, i.e.,

$$\forall C \subseteq [\lambda]^\omega (C \text{ club } \Rightarrow S \cap C \neq \emptyset).$$

Definition A.2.2. A notion of forcing $(P, \leq)$ is *proper* if for every uncountable cardinal λ, for every set $S \in V$

$$(S \subseteq [\lambda]^\omega \wedge S \text{ stationary })^V \Rightarrow (S \text{ stationary })^{V[G]}.$$

The following lemma gives an equivalent definition of properness. Its proof can be found in [15, Lemma 31.16].

Lemma A.2.3. *P is proper if and only if for every $p \in P$, every sufficiently large λ and every countable $M \preceq (H_\lambda, \in, <)$ containing P and p, there exists a $q \leq p$ that is (M, P)-generic. (q is (M, P)-generic if for every maximal antichain $A \in M$, the set $A \cap M$ is predense below q.)*

Property A.2.4.

- If P is c.c.c. then P is proper;
- If P is ω-closed then P is proper;
- If P is proper then $\forall X \in V[G] (|X| = \omega)^{V[G]} \exists Y \in V (|Y| = \omega)^V \wedge (X \subseteq Y)^{V[G]}$;
- If P is proper then P preserves $\aleph_1$.

Proof. For the first three points please refer to Lemma 31.2, 31.3 and 31.4 of [15]. The third point implies the last one. $\square$

A.3 Axiom A

Axiom A was first introduced by Baumgartner in [1].

Definition A.3.1. A notion of forcing $P, \leq$ satisfies *Axiom A* if there are partial orderings $\{\leq_n, n \in \omega\}$ on P such that:

i) $\forall p, q \in P\ q \leq_0 p \Rightarrow q \leq p$ and for every $n \in \omega\ q \leq_{n+1} p \Rightarrow q \leq_n p$;

ii) For every descending sequence $p_0 \geq_0 p_1 \geq_1 \cdots \geq_{n-1} p_n \geq_n \ldots$ there is a q such that $q \leq_n p_n$ for all n;

iii) For every $p \in P$, for every $n \in \omega$ and for every ordinal name[3] $\dot{\alpha}$ there exists a $q \leq_n p$ and a (countable set $B)^V$ such that $q \Vdash \dot{\alpha} \in B$.

Property A.3.2. If P satisfies Axiom A, then P is proper.

Proof. This proof make use of an equivalent definition of properness via the *proper game*. We refer to [15, Lemma 31.11] for more details. □

Property A.3.3. 1. Every ω-closed forcing satisfies Axiom A;

2. Every c.c.c. forcing satisfies Axiom A.

Proof. 1. Let $p \leq_n q$ iff $p \leq q$ for all n.

2. Let $p \leq_n q$ iff $p = q$ for all $n > 0$. □

A.4 Sacks forcing

Definition A.4.1. $p \subseteq 2^{<\omega}$ is a *perfect tree* if:

- (initial segments closure) $\forall t \in p, \forall n < \mathrm{lh}(t)\ t \restriction n \in p$,
- (perfect) $\forall t \in p \exists t' \in p (t \subseteq t' \wedge t' \in \mathrm{split}(p))$,

where $\mathrm{split}(p)$ is the set of all *splitting nodes* of p, i.e. $\mathrm{split}(p) := \{t \in p : t^\frown 0 \in p$ and $t^\frown 1 \in p\}$.
We define the *Sacks forcing* $(\mathbb{S}, \leq_{\mathbb{S}})$ as $\mathbb{S} := \{p \subseteq 2^{<\omega} : p$ is a perfect tree$\}$ and $p \leq_{\mathbb{S}} r$ iff $p \subseteq r$. The maximal element is $1_{\mathbb{S}} := 2^{<\omega}$.

[3] $\dot{\alpha}$ is an ordinal name if $\forall p \exists q \leq p \exists \beta \in V \cap ON(q \Vdash \dot{\alpha} = \beta)$

Definition A.4.2. We say that $t \in p$ is a *n-splitting node* if there are exactly n splitting nodes $s \subseteq t$.

Definition A.4.3. Let $p, q \in \mathbb{S}$. $p \leq_n q$ if and only if $p \leq q$ and every n-th splitting node of q is an n-th splitting node of p.

Definition A.4.4. Let $p \in \mathbb{S}$. For $s \in p$, let $p \restriction s$ denote the tree $\{t \in p : t \subseteq s \text{ or } t \supseteq s\}$. If A is a set of incompatible nodes of p and for each $s \in A$, q_s is a perfect tree such that $q_s \subseteq p \restriction s$, then the *amalgamation* of $\{q_s : s \in A\}$ into p is the perfect tree

$$\{t \in p : \text{ if } t \supseteq s \text{ for some } s \in A, \text{ then } t \in q_s\}.$$

This construction basically replaces in p each $p \restriction s$ by q_s.

Definition A.4.5. A *fusion sequence* is a sequence of conditions $\{p_n\}_{n=0}^{\infty}$ such that $p_n \leq_n p_{n-1}$ for all $n \geq 0$.

Lemma A.4.6 (Fusion lemma). *If $\{p_n\}_{n=0}^{\infty}$ is a fusion sequence, then $p := \bigcap_{n=0}^{\infty} p_n$ is a perfect tree.*

Proof. Let $s \in p$, we show that there is a splitting node in p extending s. Let $m = \mathrm{lh}(s)$, we choose an m-splitting node $t \in p_m$ extending s. Then t is a splitting node of p: namely for every $n \geq m$, since $p_n \leq_n p_m$ and t is m-splitting in p_m, we have $t, t^\frown 0, t^\frown 1 \in p_n$. $\square$

Property A.4.7. Sacks forcing satisfies Axiom A.

Proof. For $n \in \omega$ we define $p \leq_n q$ if and only if $p \leq q$ and every n-th splitting node of q is an n-th splitting node of p. If $\{p_n\}_{n=0}^{\infty}$ is a fusion sequence then $p := \cap_{n=0}^{\infty} p_n$ is a perfect tree (see Lemma A.4.6).
Let now $\dot{\alpha}$ be an ordinal name and p be a condition of the Sacks forcing. Fix $n \in \omega$, we find a $q \leq_n p$ and a countable B such that $q \Vdash \dot{\alpha} \in B$. Let S_n be the set of the nth splitting nodes of p. For every $s \in S_n$ pick $q_{s^\frown i} \leq p \restriction s^\frown i$ such that $q_{s^\frown i} \Vdash \dot{\alpha} = \beta_{s^\frown i}$, for some ordinal $\beta_{s^\frown i}$. Let q be the amalgamation of $\{q_{s^\frown i} : s \in S_n \wedge i = 0, 1\}$ and $B = \{\beta_{s^\frown i} : s \in S_n \wedge i = 0, 1\}$. We conclude that $q \leq_n p$ and $q \Vdash \dot{\alpha} \in B$. $\square$

Lemma A.4.8. *Sacks forcing does not satisfy the countable chain condition.*

Proof. We show that there are antichains of size $2^{\aleph_0}$. Fix an almost disjoint family $\{A_\alpha : \alpha < 2^{\aleph_0}\}$ of subsets of ω (see Lemma A.4.9 for the existence of such a family). For every $\alpha < 2^{\aleph_0}$ choose a perfect tree T_α whose splitting levels are exactly the elements of A_α. For example

$$T_\alpha = \{s \in 2^{<\omega} : \forall n < |s| (n \notin A_\alpha \to s(n) = 0)\}$$

Then the intersection $T_\alpha \cap T_\beta$ does not contain any perfect tree. □

Lemma A.4.9. *There is an almost disjoint family of subsets of ω of size $2^{\aleph_0}$.*

Proof. Take a bijection $f : 2^{<\omega} \to \omega$. For every $x \in 2^\omega$ let $A_x := \{x \restriction n : n \in \omega\}$. Then the set $\{f[A_x] : x \in 2^\omega\}$ is an almost disjoint set of size $2^{\aleph_0}$. □

A.5 Counting countable transitive models

In this section we consider countable transitive structures. Here the word *transitive* means that the domain A of the structure is transitive in V, i.e. $\forall x \in^V A, \forall y \in^V x, x \in^V A$.

About countable transitive domains we can say that:

Lemma A.5.1. 1. *If A is countable and transitive, then $A \in H(\aleph_1) := \{x : |\operatorname{trcl}(x)| < \aleph_1\}$, where $\operatorname{trcl}(x) := \bigcup_{n\in\omega}(\bigcup^n x), \bigcup^0 x := x$ and $\bigcup^{n+1} x = \bigcup \bigcup^n x$.*

2. *$|H(\aleph_1)| = 2^{<\aleph_1}$ and hence $|H(\aleph_1)| = 2^{\aleph_0}$.*

3. *The $\mathcal{L}$-structures $(A, R^{\mathfrak{A}})$ such that A is countable and transitive are at most $2^{\aleph_0}$-many.*

Proof. 1. Being transitive, $A = \operatorname{trcl}(A)$. Moreover $|\operatorname{trcl}(A)| = |A| = \omega$ implies $A \in H(\aleph_1)$.

2. $|H(\aleph_1)| \geq 2^{<\aleph_1}$ because $\mathcal{P}(\alpha) \subseteq H(\aleph_1)$ for all $\alpha < \aleph_1$ and therefore $|H(\aleph_1)| \geq 2^\alpha$. $|H(\aleph_1)| \leq 2^{<\aleph_1}$ because we can define an injective function $F : H(\aleph_1) \to \bigcup\{\mathcal{P}(\alpha \times \alpha) : \alpha < \aleph_1\}$ as follows: For $x \in H(\aleph_1)$ let $\alpha := |\operatorname{trcl}(x) \cup \{x\}| < \aleph_1$ and by AC choose a relation $F(x) \subseteq \alpha \times \alpha$ such that $(\alpha; F(x)) \cong (\operatorname{trcl}(x) \cup \{x\}; \in)$. The function F is injective as x is determined by the isomorphism type of $\in$ on $\operatorname{trcl}(x) \cup \{x\}$. We remark also that $2^{<\aleph_1} = \sup\{2^\alpha : \alpha < \aleph_1\}$ hence $|H(\aleph_1)| = 2^{\aleph_0}$.

3. There are at most $|H(\aleph_1)|$ many countable transitive domains and for each domain at most $2^{\aleph_0}$ binary relations. Therefore there are at most $|H(\aleph_1)| \cdot 2^{\aleph_0} = 2^{\aleph_0}$ many countable transitive structures. $\square$

About models of ZFC we can say:

Lemma A.5.2. *If there is a countable transitive model of* ZFC *then there are at least* $2^{\aleph_0}$*-many.*

Proof. Let M be a countable transitive model of ZFC and consider $\mathbb{C} = 2^{<\omega}$ ordered by reverse end-extension. Let $\langle D_n : n \in \omega \rangle$ be an enumeration of all open dense subsets of $\mathbb{C}$ that are elements of M.
We construct the sequences $\langle \langle p_h \in \mathbb{C} : h \in 2^n \rangle : n \in \omega \rangle$ and $\langle k_n \in \omega : n \in \omega \rangle$ by induction on n. For $n = 0$, let $p_\emptyset := \emptyset$ and $k_0 := 0$.
At step $n+1$, suppose that k_n and $A_n := \{p_h : h \in 2^n\}$ are defined. Since A_n is finite, there is a natural number $k_{n+1} \in \omega$ such that for every $p_h \in A_n$ there is an extension $p'_h \in D_n$ of length exactly k_{n+1}. For $i = 0, 1$ we define

$$p_{h^\frown i} := p_h'^\frown i.$$

For $n \in \omega$ and $h \in 2^n$ we get the following properties:

- $\forall m \leq n \ p_{h^\frown i} \in D_m$ for $i = 0, 1$, since all D_m's are open;
- $\mathrm{lh}(p_{h^\frown i}) = k_{n+1} + 1$;
- $p_g \leq p_h$ if and only if g is an end-extension of h;
- $p_{h^\frown i}(k_{n+1}) = h^\frown i(n) = i$ for $i = 0, 1$.

For $f \in 2^\omega$ let G_f be the filter generated by $\{p_{f \restriction n} : n \in \omega\}$. From the previous points it follows that for $f \neq f'$ there is $n \in \omega$ such that

$$\bigcup G_f(k_{n+1}) = f(n) \neq f'(n) = \bigcup G_{f'}(k_{n+1}).$$

Therefore for $f \neq f'$ we have $G_f \neq G_{f'}$. Now, since $M[G_f]$ is countable, it can contain only countably many of the $G_{f'}$ for $f' \neq f$. So among the $M[G_f], f \in (2^\omega)^V$ there are $2^{\aleph_0}$ different models. $\square$

Putting together Lemma A.5.1 and Lemma A.5.2 and we get:

Corollary A.5.3. *If there is a countable transitive model of* ZFC*, there are exactly* $2^{\aleph_0}$*-many.*

Bibliography

[1] James E. Baumgartner. Iterated forcing. In *Surveys in set theory*, volume 87 of *London Math. Soc. Lecture Note Ser.*, pages 1–59. Cambridge Univ. Press, Cambridge, 1983.

[2] Andreas Blass. Combinatorial cardinal characteristics of the continuum. In *Handbook of set theory. Vols. 1, 2, 3*, pages 395–489. Springer, Dordrecht, 2010.

[3] Paul Cohen. The independence of the continuum hypothesis I. *Proc. Nat. Acad. Sci. U.S.A.*, 50:1143–1148, 1963.

[4] Paul Cohen. The independence of the continuum hypothesis II. *Proc. Nat. Acad. Sci. U.S.A.*, 51:105–110, 1964.

[5] Paul Cohen. The discovery of forcing. In *Proceedings of the Second Honolulu Conference on Abelian Groups and Modules (Honolulu, HI, 2001)*, volume 32, pages 1071–1100, 2002.

[6] A. R. D. Mathias (editor). *Surveys in Set Theory*. London Mathematical Society Lecture Note Series 87. Cambridge University Press, 1983.

[7] Ryszard Engelking and M. Karłowicz. Some theorems of set theory and their topological consequences. *Fundamenta Mathematicae*, 57(3):275–285, 1965.

[8] Hausdorff Felix. Über zwei Sätze von G. Fichtenholz und L. Kantorovich. *Studia Mathematica*, 6:18–19, 1936.

[9] Grigori Fichtenholz and Kantorovič Leonid. Sur les opérations linéaires dans l'espace des fonctions bornées. *Studia Mathematica*, 5:69–98, 1934.

[10] Vera Fischer and Jörg Brendle. Mad families, splitting families and large continuum. *Journal of symbolic logic*, 76, 1:198–208, 2011.

[11] Stefan Geschke. Almost disjoint and indipendent families. 2011.

[12] Martin Goldstern. Tools for your forcing construction. In *Set theory of the reals (Ramat Gan, 1991)*, volume 6 of *Israel Math. Conf. Proc.*, pages 305–360. Bar-Ilan Univ., Ramat Gan, 1993.

[13] Martin Goldstern and Andrea Blass. Finite support iterations of σ-centered forcing notions. `https://mathoverflow.net/questions/84124/finite-support-iterations-of-sigma-centered-forcing-notions`, version of 21.11.16.

[14] Martin Goldstern, Jakob Kellner, Saharon Shelah, and Wolfgang Wohofsky. Borel conjecture and dual Borel conjecture. *Transactions of the American Mathematical Society*, 366(1):245–307, 2014.

[15] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. Springer, Berlin, 2003. The third millennium edition, revised and expanded.

[16] Martin Weese Winfried Just. *Discovering Modern Set Theory. II: Set-Theoretic Tools for Every Mathematician (Graduate Studies in Mathematics, Vol. 18)*. American Mathematical Society, 1997.

[17] Winfried Just and Martin Weese. *Discovering modern set theory. II*, volume 18 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. Set-theoretic tools for every mathematician.

[18] Jakob Kellner. Non-elementary proper forcing. *Rend. Sem. Mat. Univ. Politec. Torino Vol. 70, 3 (2012), 207 246*, 2012.

[19] Kenneth Kunen. *Set theory. An introduction to independence proofs*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam-New York, 1980.

[20] Kenneth Kunen. *Set theory*, volume 34 of *Studies in Logic (London)*. College Publications, London, 2011.

[21] Mathoverflow. Independent families of functions on ω of size continuum. `https://mathoverflow.net/questions/271116/independent-families-of-functions-on-omega-of-size-continuum`, version of 01.06.17.

[22] Saharon Shelah. *Proper forcing*, volume 940 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1982.

[23] Saharon Shelah. *Proper and improper forcing*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1998.

[24] Saharon Shelah. Properness without elementaricity. *J. Appl. Anal.*, 10(2):169–289, 2004.

[25] Stevo Todorcevic. *Notes on Forcing Axioms*. World Scientific Publishing Co. Pte. Ltd., 2014.

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