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Oberwolfach Preprints

OWP 2019 - 19

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Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

DOI 10.14760/OWP-2019-19

ON CO-MINIMAL PAIRS IN ABELIAN GROUPS

ARINDAM BISWAS AND JYOTI PRAKASH SAHA

Abstract. A pair of non-empty subsets (W, W') in an abelian group G is a complement pair if $W + W' = G$. W' is said to be minimal to W if $W + (W' \setminus \{w'\}) \neq G, \forall\, w' \in W'$. In general, given an arbitrary subset in a group, the existence of minimal complement(s) depends on its structure. The dual problem asks that given such a set, if it is a minimal complement to some subset. We study tightness property of complement pairs (W, W') such that both W and W' are minimal to each other. These are termed co-minimal pairs and we show that any non-empty finite set in an arbitrary free abelian group belongs to some co-minimal pair. We also construct infinite sets forming co-minimal pairs. Finally, we remark that a result of Kwon on the existence of minimal self-complements in $\mathbb{Z}$, also holds in any abelian group.

Contents

1. Introduction

Let $(G, \cdot)$ be a group and let A, B be non-empty subsets of G with $A \cdot B = G$. Then the set A is said to be a *left complement* of B in G (respectively, B is a *right complement* of A in G) and the pair (A, B) is said to be a *complement pair* in G. A complement pair in which at least one subset is minimal will be called a *minimal pair*.

In the case of abelian groups, a left complement of a subset is also a right complement to that subset and vice versa. Also, any non-empty subset A is always a part of some complement pair (for instance, consider the pair (A, G)).

A left (resp. right) complement A of some non-empty subset B of G is said to be *minimal* if $A \cdot B = G$ (respectively $B \cdot A = G$) and $(A \setminus \{a\}) \cdot B \neq G$ (respectively $B \cdot (A \setminus \{a\}) \neq G$) for each $a \in A$. The notion of minimal complements was first introduced by Nathanson in [Nat11] in the course of his study of natural arithmetic analogues of the metric concept of nets in the setting of groups.

The situation becomes interesting when we ask whether a given subset admits a minimal complement or not, and also the dual question whether a given subset could be a minimal complement to some subset. In the case of $\mathbb{Z}$, Kwon showed that any non-empty finite subset is a minimal complement to some set in $\mathbb{Z}$ (see [Kwo19, Theorem 9]). Here, we show that

Theorem 1.1. *Any non-empty finite subset of a free abelian group (not necessarily of finite rank) is a minimal complement to some subset.*

In fact, one can go further and study tightness property of a set and its minimal complement. For this, we introduce the notion of a co-minimal pair —

2010 *Mathematics Subject Classification.* 11B13, 05E15, 05B10, 11P70.
Key words and phrases. Additive complements, Minimal complements, Sumsets, Additive number theory.

Definition 1.2 (Co-minimal pair)**.** *Let $A, B \subseteq G$ be two non-empty subsets. Then the pair (A, B) is defined to be a* co-minimal pair *if A is a left minimal complement of B and B is a right minimal complement of A.*

Thus, a co-minimal pair is a complement pair in which both the subsets are minimal. The existence of co-minimal pairs is a tricky question. We show that

Theorem 1.3. *Any non-empty finite subset of an arbitrary free abelian group is a part of some co-minimal pair. Moreover, if S is a two element subset of a group G, then $(S, R), (L, S)$ are co-minimal pairs for some subsets L, R of G.*

In section 3, we study the behaviour of co-minimal pairs under cartesian products of groups. Next, we turn our attention to infinite subsets $A, B \subseteq G$ forming co-minimal pairs. For this, the notion of spiked subsets (see Definition 4.2 and also section 4) is useful. If G_1, G_2 are subgroups of an abelian group G such that the multiplication map $G_1 \times G_2 \to G$ defined by $(g_1, g_2) \mapsto g_1 g_2$ is an isomorphism, then it turns out that the subsets of G of the form $B \times G_2$ with $B \subseteq G_1$ is a part of a co-minimal pair in G if and only if B is part of a co-minimal pair in G_1 (see Lemma 4.1). More generally, an appropriate analogue of this statement also holds for spiked subsets.

Theorem 1.4 (Theorem 4.3)**.** *Let G_1, G_2 denote two subgroups of an abelian group $\mathcal{G}$. Let X be a (u, φ)-bounded spiked subset of $\mathcal{G}$ with respect to G_1, G_2 and with base $\mathcal{B}$. If u admits a φ-moderation, then X is a part of a co-minimal pair in $G_1 G_2$ of the form (X, M_v) where M_v is the graph of the restriction of a moderation v of u to some subset M of G_1 if and only if X is equal to $\mathcal{B}G_2$ and $\mathcal{B}$ is a part of a co-minimal pair in G_1.*

Roughly speaking, the above theorem classifies all the spiked subsets which can be a part of a co-minimal pair of certain form.

On the other hand, if we take $A = \mathbb{N}^d$ in $G = \mathbb{Z}^d$, then A can never have a minimal complement and is also not a minimal complement to any set in G. Thus it is in a sense the other extreme to being a part of a co-minimal pair. We discuss generalisations of this in Proposition 5.3. This leads to a discussion on co-minimality and infinite approximate subgroups and asymptotically approximate subgroups. See section 5.

Finally, in the section on concluding remarks (see section 6), we look at self-complements[1] in arbitrary abelian groups and remark that a set A is a minimal self-complement iff A does not contain any non-trivial 3 term arithmetic progression.

2. Finite subsets and co-minimality

We begin this section by indicating a property of co-minimal pairs, and a subtle difference between the notions of minimal pair and co-minimal pair.

Lemma 2.1. *If (A, B) is a co-minimal pair in a group G, then $(g \cdot A, B \cdot h)$ forms a co-minimal pair for any two elements g, h of G.*

Proof. It follows since multiplication by an element g from the left induces a bijection from G to G and multiplication by an element h from the right also induces a bijection. □

Lemma 2.2. *There exists a non-empty subset $A \subseteq \mathbb{Z}$ such that A has a minimal complement, but A is not a minimal complement to any set. Thus, A can belong to a minimal pair, but can never belong to a co-minimal pair.*

[1]A non-empty set A is a self-complement iff $A \cdot A = G$. If, in addition, A is minimal, then it's called a minimal self-complement or a co-minimal pair (A, A).

Proof. Consider the subset $A = 2\mathbb{Z} \cup \{1\} \subseteq \mathbb{Z}$. Then $\inf A = -\infty$ and $\sup A = +\infty$. By a result of Chen–Yang [CY12, Theorem 1], the set A admits a minimal complement in $\mathbb{Z}$. However, A itself cannot be a part of a co-minimal pair. Otherwise, suppose (A, B) is a co-minimal pair for some subset B of $\mathbb{Z}$. If all the elements of B have the same parity, then replacing B by $B + 1$ if necessary, we may assume that B is a subset of $2\mathbb{Z}$. Since $A + B$ is equal to $\mathbb{Z}$, it follows that B is equal to $2\mathbb{Z}$. However, $2\mathbb{Z}\cup\{1\}$ is not a minimal complement to $2\mathbb{Z}$ since the subset $\{0, 1\}$ of $2\mathbb{Z} \cup \{1\}$ is a complement to $2\mathbb{Z}$. Let us assume that B contains two elements of different parity. Let B_e (resp. B_o) denote the subset of B consisting of the even (resp. odd) elements of B. Note that the subsets $2\mathbb{Z}+B_e$ and $2\mathbb{Z}+B_o$ of $A+B$ contains $2\mathbb{Z}$ and $1+2\mathbb{Z}$ respectively. Since B is a minimal complement to A, it follows that B_e, B_o are singleton sets. Note that the subset $2\mathbb{Z}$ of A is a complement to $B_e \cup B_o = B$. Hence A is not a minimal complement to B. Consequently, A cannot be a part of a co-minimal pair. □

Further, co-minimal pairs do not escape from subgroups in the sense that for a subset A of a group G, it is a part of a co-minimal pair or not accordingly as it is a part of a co-minimal pair in any subgroup H of G containing A.

Lemma 2.3. *Let A be a non-empty subset of a group G. Then the following statements are equivalent.*

1. *The subset A of G is a part of a co-minimal pair in G.*
2. *For any subgroup H of G containing A, the subset A of H is part of a co-minimal pair in H.*
3. *For some subgroup H of G containing A, the subset A of H is part of a co-minimal pair in H.*

Proof. Let B be a subset of G such that (A, B) is a co-minimal pair in G. Let H be a subgroup of G containing A. Let B' denote the set of elements $b \in B$ such that $A \cdot b$ is contained in H. Then (A, B') is a co-minimal pair.

Assume that for some subgroup H of G containing A, there exists a subset C of H such that (A, C) is a co-minimal pair in H. Let $\{g_\lambda\}_{\lambda\in\Lambda}$ denote a set of right coset representatives of H in G, i.e., $G = \bigsqcup_{\lambda\in\Lambda}(H \cdot g_\lambda)$. Then $(A, \bigsqcup_{\lambda\in\Lambda}(C \cdot g_\lambda))$ is a co-minimal pair in G.

Hence the implications $(1) \implies (2), (3) \implies (1)$ follow. The implication $(2) \implies (3)$ is immediate. □

Next, we will proceed to prove Theorems 1.1, 1.3.

Proposition 2.4. *Given a non-empty finite subset S of $\mathbb{Z}^n$ for $n \geq 2$, there exists an automorphism φ of the group $\mathbb{Z}^n$ such that the image of the set $\varphi(S)$ under the projection map*

$$\pi_1 : \mathbb{Z}^n \to \mathbb{Z} \text{ (onto the first coordinate)}$$

contains exactly $\#S$ elements.

Proof. We show it by induction. The base case is for $n = 2$. Suppose S is a non-empty finite subset of $\mathbb{Z}^2$. Then for some positive integer m, the image of the set $\varphi_m(S)$ under the projection map

$$\pi_1 : \mathbb{Z}^2 \to \mathbb{Z} \text{ (onto the first coordinate)}$$

contains $\#S$ elements where φ_m denotes the automorphism of $\mathbb{Z}^2$ defined by

$$\varphi_m := \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}.$$

Otherwise, there exist infinitely many positive integers $m_1 < m_2 < m_3 < \cdots$ and two distinct elements s, t in S such that

$$\pi_1(\varphi_{m_i}(s)) = \pi_1(\varphi_{m_i}(t)) \text{ for any } i \geq 1. \tag{1}$$

Let s (resp. t) be equal to (a_1, a_2) (resp. (b_1, b_2)). So any integer $i \geq 1$, we obtain

$$a_1 + m_i a_2 = b_1 + m_i b_2.$$

Note that $a_2 \neq b_2$ (otherwise $a_1 = b_1$ and hence $s = t$). Thus the equality in Equation (1) cannot hold for infinitely many positive integers m_i. So the image of the set $\varphi_m(S)$ under the projection map

$$\pi_1 : \mathbb{Z}^2 \to \mathbb{Z} \text{ (onto the first coordinate)}$$

contains $\#S$ elements for some positive integer m, i.e., the Proposition holds for $n = 2$.

Suppose the Proposition also holds for $n = r$ for some positive integer $r \geq 2$. Let S be a non-empty finite subset of $\mathbb{Z}^{r+1}$. Then for some positive integer m, the image of the set $\varphi_m(S)$ under the projection map

$$\pi : \mathbb{Z}^{r+1} \to \mathbb{Z}^r \text{ (onto the first } r\text{-coordinates)}$$

contains $\#S$ elements where φ_m denotes the automorphism of $\mathbb{Z}^{r+1}$ defined by

$$\varphi_m := \begin{pmatrix} 1 & 0 & \cdots & 0 & m \\ 0 & 1 & \cdots & 0 & m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & m \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{(r+1)\times(r+1)}.$$

Otherwise, there exist infinitely many positive integers $m_1 < m_2 < m_3 < \cdots$ and two distinct elements s, t in S such that

$$\pi(\varphi_{m_i}(s)) = \pi(\varphi_{m_i}(t)) \text{ for any } i \geq 1. \tag{2}$$

Let s (resp. t) be equal to $(a_1, a_2, \cdots, a_{r+1})$ (resp. $(b_1, \cdots, b_{r+1})$). So any integer $i \geq 1$ and $1 \leq \ell \leq r$, we obtain

$$a_\ell + m_i a_{r+1} = b_\ell + m_i b_{r+1}.$$

Note that $a_{r+1} \neq b_{r+1}$ (otherwise $a_\ell = b_\ell$ for $1 \leq \ell \leq r$ and hence $s = t$). Thus the above equality in Equation (2) cannot hold for infinitely many positive integers m_i. So the image of the set $\varphi_m(S)$ under the projection map

$$\pi : \mathbb{Z}^{r+1} \to \mathbb{Z}^r \text{ (onto the first } r\text{-coordinates)}$$

contains $\#S$ elements for some positive integer m. By the induction hypothesis, there exists an element $A \in \mathrm{GL}_r(\mathbb{Z})$ such that the image of $\pi(\varphi_m(S))$ under A contains $\#\pi(\varphi_m(S)) = \#S$ elements.

Let $\widetilde{A}$ denote the automorphism of $\mathbb{Z}^{r+1} = \mathbb{Z}^r \times \mathbb{Z}$ which acts by A on the first factor $\mathbb{Z}^r$ and acts trivially on the second factor $\mathbb{Z}$. Then the image of $(\widetilde{A} \circ \varphi_m)(S)$ under the projection map

$$\pi : \mathbb{Z}^{r+1} \to \mathbb{Z} \text{ (onto the first coordinate)}$$

contains exactly $\#S$ elements. Hence the Proposition follows. □

Lemma 2.5. *Let S be a nonempty finite subset of an abelian group G. Suppose G_1 is a group, and $\pi : G \to G_1$ is a surjective group homomorphism such that $\pi(S)$ contains $\#S$ elements. Then S is a minimal complement to some subset of G if the subset $\pi(S)$ of G_1 is a minimal complement to some subset W of G_1.*

Proof. Since G is abelian, it follows that S is a complement to $\pi^{-1}(W)$. Moreover, since S is a minimal complement to W and the image of S under $\pi : G \to G_1$ contains $\#S$ elements, the set S is a minimal complement to $\pi^{-1}(W)$. $\square$

Proof of Theorem 1.1. Let S be a nonempty finite subset of a free abelian group G. If G has finite rank, then we identify G with $\mathbb{Z}^n$ with $n = \mathrm{rk}G$. When n is equal to one, the result follows from [Kwo19, Theorem 9]. Moreover, when $n \geq 2$, by Proposition 2.4, there exists an automorphism φ of $G = \mathbb{Z}^n$ such that the image of $\varphi(S)$ under the the projection map

$$\pi : \mathbb{Z}^n \to \mathbb{Z} \text{ (onto the first coordinate)}$$

contains exactly $\#S$ elements. Note that $\pi(\varphi(S))$ contains $\#\varphi(S)$ elements and by [Kwo19, Theorem 9], the subset $\pi(\varphi(S))$ of $\mathbb{Z}$ is a minimal complement to some subset of $\mathbb{Z}$. Hence by Lemma 2.5, $\varphi(S)$ is a minimal complement to some subset W of $\mathbb{Z}^n$, and hence S is a minimal complement to $\varphi^{-1}(W)$. Consequently, any non-empty subset of any free abelian group of finite rank is a minimal complement to some subset.

Moreover, when G has infinite rank, i.e., when G is isomorphic to the direct product $\mathbb{Z}^I$ for an infinite set I, it follows that for some finite subset J of I, the image of S under the projection map $\pi : \mathbb{Z}^I \to \mathbb{Z}^J$ (obtained by restricting the elements of $\mathbb{Z}^I$ (considered as the group of all functions from I to $\mathbb{Z}$) to J) contains exactly $\#S$ elements. Indeed, if for each pair (s, t) with $s, t \in S$, choose an element $i_{s,t} \in I$ such that s, t take different values at $i_{s,t}$, then J can be taken to be

$$J = \{i_{s,t} \,|\, (s, t) \in S \times S, s \neq t\}.$$

By the conclusion of the previous paragraph, it follows that $\pi(S)$ is a minimal complement to some subset of $\mathbb{Z}^J$. Hence by Lemma 2.5, it follows that S is a minimal complement to some subset of G. This completes the proof of Theorem 1.1. $\square$

Now, we turn our attention to co-minimal pairs (cf. Definition 1.2) which measures the tightness property of a set and its complement. To show Theorem 1.3, we need a result from a prior work.

Theorem 2.6 ([BS18, Theorem 2.1])**.** *Let G be an arbitrary group with S a nonempty finite subset of G. Then every complement of S in G contains a minimal complement to S.*

This implies that a non-empty finite set $S \subseteq G$ belongs to some minimal pair. Using the above Theorem 2.6, Proposition 2.4 and Lemma 2.5, we shall show that it also belongs to a co-minimal pair when G is a free abelian group. This will establish Theorem 1.3.

Proof of Theorem 1.3. Let A be a non-empty subset of a free abelian group G. By Theorem 1.1, A is a minimal complement to some subset B of G. By Theorem 2.6, any complement C of a non-empty finite subset W of a group contains a minimal complement to W. Consequently, B contains a minimal complement B' of A. Since $A + B' = G$, A is a minimal complement to B and B contains A, it follows that A is a minimal complement to B' . Hence (A, B') is a co-minimal pair.

Let $S = \{g, h\}$ be a two-element subset of a group G. Then S is a minimal complement to $G \setminus \{hg^{-1}\}$. Indeed,

$$g \cdot (G \setminus \{g^{-1}h\}) = G \setminus \{h\}$$

and

$$h \in G \setminus \{hg^{-1}h\} = (G \cdot h) \setminus \{hg^{-1}h\} = h \cdot (G \setminus \{g^{-1}h\}),$$

which implies that S is a minimal left complement to $G \setminus \{g^{-1}h\}$. By [BS18, Theorem 2.1], $G \setminus \{g^{-1}h\}$ contains a minimal right complement to S. Hence (S, R) is a co-minimal pair for some subset R of G.

Note that the proof of [BS18, Theorem 2.1] can be suitably adapted to prove that for a non-empty finite subset S of a group G, every left complement to S contains a minimal left complement to S. Using this result and an argument similar to the above, it follows that (L, S) is a co-minimal pair for some subset L of G. □

Remark 2.7. Theorem 1.3 gives us co-minimal pairs (A, B) with $A =$ a non-empty finite set and $B =$ an infinite set. Co-minimal pairs can also occur with both A and B being infinite subsets, as we explain in the following paragraph. It is clear that the third case, i.e., A and B both finite sets forming a co-minimal pair cannot occur inside an infinite group G.

Remark 2.8. Let G be a finitely generated abelian group isomorphic to the direct product of its torsion part G_{tors} and a free abelian group F. If A is a non-empty finite subset of G such that it is contained in a single coset of F in G, then Lemma 2.1 combined with Theorems 1.1, 1.3 imply that A is a minimal complement to some subset of G and it is a part of some co-minimal pair in G.

We shall now provide constructions of infinite subsets A, B forming co-minimal pairs. Let A, B be two infinite subgroups of an abelian G such that $A \times B$ is isomorphic to G (under the product of inclusion maps), then $(A, B), (B, A)$ are co-minimal pairs. For instance, if M is an idempotent element of $\mathrm{GL}_n(\mathbb{Z})$ other than the identity, then the subsets $A = \ker M, B = \ker(I_n - M)$ of $\mathbb{Z}^n$ form a co-minimal pair. There are examples of co-minimal pairs which are not of this form. For instance, if H is a subgroup of a group G and $\{g_\lambda\}_{\lambda\in\Lambda}$ is a set of distinct left coset representatives of H in G, then $(\{g_\lambda\}_{\lambda\in\Lambda}, H)$ is a co-minimal pair.

3. Cartesian Products and co-minimality

Co-minimal pairs are preserved under cartesian products.

Proposition 3.1. *Let $n \geqslant 2$ be a natural number. If we have n co-minimal pairs (A_1, B_1), $(A_2, B_2), \cdots, (A_n, B_n)$ in groups $G_1, G_2, \cdots, G_n$ respectively, then $(A_1 \times A_2 \times \cdots \times A_n, B_1 \times B_2 \times \cdots \times B_n)$ is a co-minimal pair in $G = G_1 \times G_2 \times \cdots \times G_n$.*

Proof. The proof is similar to [BS18, Proposition 5.1]. We use induction on n. To avoid confusion, just for the proof of this proposition, we denote complement pairs and co-minimal pairs both using [,] instead of the usual (,). For $n = 2$, let $[A_1, B_1]$ and $[A_2, B_2]$ be two co-minimal pairs in the groups G_1 and G_2 respectively. Then

$$A_1 \cdot B_1 = G_1,\ (A_1 \setminus \{a\}) \cdot B_1 \subsetneq G_1\ \forall\, a \in A_1 \text{ and } A_1 \cdot (B_1 \setminus \{b\}) \subsetneq G_1\ \forall\, b \in B_1,$$
$$A_2 \cdot B_2 = G_2,\ (A_2 \setminus \{a\}) \cdot B_2 \subsetneq G_2\ \forall\, a \in A_2 \text{ and } A_2 \cdot (B_2 \setminus \{b\}) \subsetneq G_2\ \forall\, b \in B_2.$$

Now $A_1 \times A_2$, $B_1 \times B_2 \subseteq G_1 \times G_2$. It is clear that

$$(A_1 \times A_2) \cdot (B_1 \times B_2) = (A_1 \cdot B_1) \times (A_2 \cdot B_2) = G_1 \times G_2.$$

Thus $[A_1 \times A_2, B_1 \times B_2]$ forms a complement pair in $G_1 \times G_2$.

Next, we show that $[A_1 \times A_2, B_1 \times B_2]$ is co-minimal. Remove an element (b_1, b_2) from $B_1 \times B_2$ and look at the set $B := B_1 \times B_2 \setminus \{(b_1, b_2)\}$. We show that B is not a complement to $A_1 \times A_2$ in $G_1 \times G_2$, i.e., $(A_1 \times A_2) \cdot B \subsetneq G_1 \times G_2$. Since B_1 is a minimal complement to A_1, $\exists a_1 \in A_1, g_1 \in G_1$ such that the only way of representing g_1 in $A_1 \cdot B_1$ is $a_1 b_1$. Similarly, $\exists a_2 \in A_2, g_2 \in G_2$ with $g_2 = a_2 b_2$. It is clear that $(g_1, g_2) \notin (A_1 \times A_2) \cdot B$ because (g_1, g_2)

can only be represented in $(A_1 \times A_2) \cdot (B_1 \times B_2)$ as $(a_1 b_1, a_2 b_2)$. Thus $B_1 \times B_2$ is a minimal complement to $A_1 \times A_2$. An exactly similar argument shows that $A_1 \times A_2$ is also a minimal complement to $B_1 \times B_2$. Thus $[A_1 \times A_2, B_1 \times B_2]$ is a co-minimal pair.

To prove the general case we use induction. Without loss of generality, let us assume that the statement is true for k groups $G_1, G_2, \cdots, G_k$ with $k < n$. We show that the statement holds for $(k+1)$-groups. By hypothesis,

$$A_1 \cdot B_1 = G_1,\ (A_1 \setminus \{a\}) \cdot B_1 \subsetneq G_1\ \forall\, a \in A_1 \text{ and } A_1 \cdot (B_1 \setminus \{b\}) \subsetneq G_1\ \forall\, b \in B_1,$$
$$A_2 \cdot B_2 = G_2,\ (A_2 \setminus \{a\}) \cdot B_2 \subsetneq G_2\ \forall\, a \in A_2 \text{ and } A_2 \cdot (B_2 \setminus \{b\}) \subsetneq G_2\ \forall\, b \in B_2,$$
$$\vdots$$
$$A_k \cdot B_k = G_k,\ (A_k \setminus \{a\}) \cdot B_k \subsetneq G_k\ \forall\, a \in A_k \text{ and } A_k \cdot (B_k \setminus \{b\}) \subsetneq G_k\ \forall\, b \in B_k,$$
$$A_{k+1} \cdot B_{k+1} = G_{k+1},\ (A_{k+1} \setminus \{a\}) \cdot B_{k+1} \subsetneq G_{k+1}\ \forall\, a \in A_{k+1} \text{ and } A_{k+1} \cdot (B_{k+1} \setminus \{b\}) \subsetneq G_{k+1}\ \forall\, b \in B_{k+1}.$$

By the inductive assumption, $[A, B] = [A_1 \times \cdots \times A_k, B_1 \times \cdots \times B_k]$ forms a co-minimal pair in $G_1 \times \cdots \times G_k$. To show that $[A \times A_{k+1}, B \times B_{k+1}]$ is a co-minimal pair in $G_1 \times \cdots \times G_{k+1}$, we argue as in the $n = 2$ case. This proves the fact for $n = k + 1$ and by induction we are done. □

We mention that one can actually establish a stronger statement.

Proposition 3.2. *Let $\mathbb{I}$ be a non-empty indexing set and $(A_i, B_i), i \in \mathbb{I}$ be co-minimal pairs in groups $G_i, i \in \mathbb{I}$ respectively. Then $(\prod_{i \in \mathbb{I}} A_i, \prod_{i \in \mathbb{I}} B_i)$ is a co-minimal pair in $G = \prod_{i \in \mathbb{I}} G_i$.*

4. Spiked subsets and co-minimality

The above construction of co-minimal pairs was in the context of cartesian product of groups. If instead we take product groups, then the following can be established. Suppose G_1, G_2 are subgroups of an abelian group G such that the map

$$G_1 \times G_2 \to G \quad \text{defined by } (g_1, g_2) \mapsto g_1 g_2$$

is an isomorphism. We identify the group G with $G_1 \times G_2$ via this isomorphism. The subsets of G of the form $B \times B'$, more specifically, the subsets of G of the form $B \times G_2$ are one of the simplest subsets of G.

Lemma 4.1. *The subsets of G of the form $B \times G_2$ with $B \subseteq G_1$ is a part of a co-minimal pair in G if and only if B is a part of co-minimal pair in G_1.*

Proof. Suppose $B \times G_2$ is a part of a co-minimal pair $(B \times G_2, S)$ in G. Denote the projection map $G \to G_1$ by π. Note that the image $\pi(S)$ is a minimal complement to B. Since $B \times G_2$ is a minimal complement to S, it follows that B is a minimal complement to $\pi(S)$. So $(B, \pi(S))$ is a co-minimal pair.

Suppose (B, M) is a co-minimal pair for some subset M of G_1. Then $(B \times G_2, M \times \{0\})$ is a co-minimal pair. □

The subsets of G of the form $B \times G_2$ are examples of a more general class of subsets, called 'spiked subsets' as introduced in [BS19]. For the sake of completeness, we recall its definition. A subset X of $\mathbb{Z}^{k+1}$ is called a *spiked subset* if

$$\mathcal{B} \times \mathbb{Z} \subseteq X \subseteq (\mathcal{B} \times \mathbb{Z}) \bigsqcup \left(\sqcup_{x \in \mathbb{Z}^k \setminus \mathcal{B}} \left(\{x\} \times (-\infty, u(x)) \right) \right)$$

holds for some nonempty subset $\mathcal{B}$ of $\mathbb{Z}^k$ and some function $u : \mathbb{Z}^k \to \mathbb{Z}$. The set $\mathcal{B}$ is called the *base* of X. We will say that such a set X is a u-bounded spiked subset with base $\mathcal{B}$.

By [BS19, Lemma 4.5], any function $u : \mathbb{Z}^k \to \mathbb{Z}$ admits a moderation v, i.e., a function $v : \mathbb{Z}^k \to \mathbb{Z}$ such that for each $x_0 \in \mathbb{Z}^k$, the function

$$x \mapsto u(x) + v(x_0 - x)$$

defined on $\mathbb{Z}^k$ is bounded above. It turns out that any (or some) u-bounded spiked subset with base $\mathcal{B}$ admits a minimal complement in $\mathbb{Z}^{k+1}$ if and only if the base $\mathcal{B}$ admits a minimal complement in $\mathbb{Z}^k$ (see [BS19, Theorems 4.6, 5.6]).

More generally, for an abelian group $\mathcal{G}$ as above with subgroups G_1, G_2 such that G_2 is free and the multiplication map from $G_1 \times G_2 \to \mathcal{G}$ is injective, the notion of 'spiked subsets' can be extended (cf. [BS19, Definitions 5.1, 5.2]).

Definition 4.2. *A subset X of an abelian group $\mathcal{G}$ is said to be a (u, φ)-*bounded spiked subset *with respect to subgroups G_1, G_2 of $\mathcal{G}$ if*

(1) *G_2 is a finitely generated free abelian group of positive rank,*
(2) *the homomorphism $G_1 \times G_2 \to G$ defined by $(g_1, g_2) \mapsto g_1 g_2$ is injective,*

and there exists a function $u : G_1 \to G_2$ and an isomorphism $\varphi : G_2 \xrightarrow{\sim} \mathbb{Z}^{\mathrm{rk} G_2}$ such that

$$\mathcal{B}G_2 \subseteq X \subseteq \mathcal{B}G_2 \bigsqcup \left(\bigsqcup_{g_1 \in G_1 \setminus \mathcal{B}} g_1 \cdot \left(\varphi^{-1} \left(\mathbb{Z}^{\mathrm{rk} G_2}_{< \varphi(u(g_1))} \right) \right) \right)$$

holds for some non-empty subset $\mathcal{B}$ of G_1. The set $\mathcal{B}$ is called the base *of X.*

The notion of moderation extends to such a context (cf. [BS19, Definition 5.3]). Moreover, when G_1 is finitely generated, it follows that u admits a φ-moderation v (cf. [BS19, Proposition 5.4]). Furthermore, if G_1 is finitely generated, then a (u, φ)-bounded spiked subset of $\mathcal{G}$ with respect to G_1, G_2 having base $\mathcal{B}$ admits the graph of the restriction of some φ-moderation of u to some subset of G_1 as a minimal complement in $G_1 G_2$ if and only if the base $\mathcal{B}$ admits a minimal complement in G_1 (see [BS19, Theorem 5.6]).

The following result states that an appropriate formulation of Lemma 4.1 also holds for spiked subsets, and thereby classifies all the spiked subsets which can be a part of a co-minimal pair of certain form.

Theorem 4.3. *Let G_1, G_2 denote two subgroups of an abelian group $\mathcal{G}$. Let X be a (u, φ)-bounded spiked subset of $\mathcal{G}$ with respect to G_1, G_2 and with base $\mathcal{B}$. If u admits a φ-moderation, then X is a part of a co-minimal pair in $G_1 G_2$ of the form (X, M_v) where M_v is the graph of the restriction of a moderation v of u to some subset M of G_1 if and only if X is equal to $\mathcal{B}G_2$ and $\mathcal{B}$ is a part of a co-minimal pair in G_1.*

Proof. Assume that there exists a φ-moderation v of u such that (X, M_v) is co-minimal pair in $G_1 G_2$ where M_v denotes the graph of the restriction of v of u to some subset M of G_1. Since M_v is a minimal complement to X, it follows that M is a minimal complement to $\mathcal{B}$. By [BS19, Theorem 5.6], M_v is a minimal complement to $\mathcal{B}G_2$ in $G_1 G_2$. Since (X, M_v) is a co-minimal pair in $G_1 G_2$, we conclude that X cannot be larger than $\mathcal{B}G_2$. Hence X is equal to $\mathcal{B}G_2$. If $\mathcal{B}$ is not a minimal complement to M, then $\mathcal{B}G_2$ is not a minimal complement to M_v. Since (X, M_v) is a co-minimal pair in $G_1 G_2$, it follows that $\mathcal{B}$ is a minimal complement to M, i.e., $\mathcal{B}$ is a part of a co-minimal pair in G_1.

Suppose $\mathcal{B}$ is a part of a co-minimal pair $(\mathcal{B}, M)$ in G_1. Let v denote a φ-moderation of u. Then the graph M_v of the restriction of v to M is a minimal complement to $\mathcal{B}G_2$ by [BS19, Theorem 5.6]. If $\mathcal{B}G_2$ were not a minimal complement to M_v, then the set $(\mathcal{B}G_2) \setminus \{b + t\}$ would be a complement to M for some elements $b \in \mathcal{B}$ and $t \in G_2$. Since $\mathcal{B}$ is a minimal complement to M, it follows that M contains an element m such that $b+m$ does not belong to $(\mathcal{B} \setminus \{b\}) + M$. This implies that $((\mathcal{B}G_2) \setminus \{b+t\}) + M_v$ does not contain the element $b+m+t+v(m)$ of G_1G_2. Consequently, $\mathcal{B}G_2$ is a minimal complement to M_v. □

5. Semilinear sets, approximate subgroups and non co-minimality

We conclude the discussion on co-minimal pairs by mentioning subsets which are in a sense the other extreme of being part of a co-minimal pair, i.e., they *do not belong to any minimal pair*. In other words, they are not a minimal complement to any subset and also no subset can be a minimal complement to one of these sets. For this we recall the well-defined notion of an arithmetic progression in an abelian group.

Definition 5.1 (Arithmetic progressions)**.** *A subset X of an abelian group $(G,+)$ is an* unbounded arithmetic progression in G *if there exist $a \in G$ and $b \in G \setminus \{e\}$ such that*

$$X = P(a,b) := \{a + nb \,|\, n \in \mathbb{Z}_{\geq 0}\}.$$

A subset Y of G is a bounded arithmetic progression *if there exist $a, b \in G$ and $m \in \mathbb{Z}_{\geq 0}$ such that*

$$Y = P_m(a,b) := \{a + nb \,|\, n \in [0,m] \cap \mathbb{Z}\}.$$

More generally, we will use the following objects.

Definition 5.2 (Generalised arithmetic progressions)**.** *An infinite subset $\mathbf{X}$ of an abelian group $(G,+)$ is an* unbounded generalised arithmetic progression of dimension d with respect to $b_1, \cdots, b_d \in G$ *if there exists an element $a \in G$ such that $\mathbf{X}$ is equal to*

$$\{a + n_1 b_1 + \cdots + n_d b_d \,|\, n_i \text{ runs over } F_i\}$$

for some subsets $F_1, \cdots, F_d$ of $\mathbb{Z}$ where each F_i is either an unbounded arithmetic progression in $\mathbb{Z}$ or a finite set. A subset $\mathbf{Y}$ of G is a bounded generalised arithmetic progression of dimension d *if there exist $a, b_1, \cdots, b_d \in G$ and $m \in \mathbb{Z}_{\geq 0}$ such that*

$$\mathbf{Y} = P_{m_1,\cdots,m_d}(a, b_1, \cdots, b_d) := \{a + n_1 b_1 + \cdots + n_d b_d | \, n_i \in [0, m_i] \cap \mathbb{Z}, 1 \leqslant i \leqslant d\}.$$

A generalised arithmetic progression is also known as a linear set. A finite union of unbounded linear sets is called a semilinear set.

Proposition 5.3. *Let G be a free abelian group and A be a non-empty subset of G. The following are true*

1. *If $A = P_m(a,b)$ or in general if $A = P_{m_1,\cdots,m_d}(a, b_1, \cdots, b_d)$, then A is always a minimal complement to some subset of G and also A has a minimal complement.*
2. *If $A = P(a,b)$, or more generally, if A is an unbounded generalised arithmetic progression of dimension d with respect to $b_1, \cdots, b_d \in G$ and $b_1, \cdots, b_d$ generate a free subgroup of G of rank d, then A is neither a minimal complement to any subset of G, nor does it have a minimal complement in G.*
3. *If $G = \mathbb{Z}^{\mathbb{I}}$, $A = P(a,b)^{\mathbb{I}}$ where $\mathbb{I}$ is some indexing set and $P(a,b) \subseteq \mathbb{Z}$, then A is neither a minimal complement to any subset of G, nor does it have a minimal complement in G.*

Thus the subsets in (2) *and* (3) *above, can never belong to a minimal pair.*

Proof. (1) Note that A is a non-empty finite subset in G. It follows by Theorem 1.1 that A is a minimal complement to some subset and by Theorem 2.6 that A has a minimal complement.

(2) Suppose A is an unbounded generalised arithmetic progression of dimension d with respect to $b_1, \cdots, b_d \in G$ and $b_1, \cdots, b_d$ generate a subgroup of G of rank d. Then

$$A = \{a + n_1 b_1 + \cdots + n_d b_d \,|\, n_i \text{ runs over } F_i\}$$

for some element $a \in G$ and some subsets $F_1, \cdots, F_d$ of $\mathbb{Z}$ where each F_i is either an unbounded arithmetic progression in $\mathbb{Z}$ or a finite set. Reordering the elements $b_1, \cdots, b_d$ (if necessary), we assume that $F_1, \cdots, F_e$ are unbounded arithmetic progressions in $\mathbb{Z}$ with $1 \leq e \leq d$ and the remaining F_i's are finite. Replacing a by a suitable element of G, we can assume that the initial term of each of $F_1, \cdots, F_e$ is equal to zero. Let $b_1', \cdots, b_e'$ denote nonzero elements of $\mathbb{Z}$ such that F_i is equal to $\mathbb{Z}_{\geq 0} b_i'$ for $1 \leq i \leq e$. Note that $A + b_1' b_1$ is a proper subset of A and hence A is not a minimal complement to any subset of G.

It remains to show that A does not have a minimal complement in G. Replacing A by a translate of A (if necessary), we can assume $a = 0$. By [BS19, Theorem 2.3], it is enough to show that A does not admit a minimal complement in the subgroup $\langle b_1' b_1, \cdots, b_e' b_e, b_{e+1}, \cdots, b_d \rangle$. Hence, the elements $b_1' b_1, \cdots, b_e' b_e, b_{e+1}, \cdots, b_d$ could be identified with $e_1, \cdots, e_d$ and A could be thought of as a subset of $\mathbb{Z}^d$. If e is equal to d, then A is equal to the subset $\mathbb{Z}^d_{\geq 0}$ of $\mathbb{Z}^d$. By [BS18, Corollary 3.2(2)], the set A does not admit a minimal complement in $\mathbb{Z}^d$. Suppose e is less than d. Let

$$\pi_1 : \mathbb{Z}^d \to \mathbb{Z}^e \text{ and } \pi_2 : \mathbb{Z}^d \to \mathbb{Z}^{d-e}$$

denote the projections onto the first e coordinates and onto the last $d-e$ coordinates respectively. Suppose $F_{e+1}, \cdots, F_d$ are contained in the intervals $[p_{e+1}, q_{e+1}], \cdots, [p_d, q_d]$ respectively. Let us assume that $B \subseteq \mathbb{Z}^d$ is a minimal complement of A. For each

$$v \in [-q_{e+1}, -p_{e+1}] \times \cdots \times [-q_d, -p_d]$$

such that the set $B \cap \pi_2^{-1}(v)$ is non-empty, choose an element $(x_{v1}, \cdots, x_{ve})$ in $\pi_1(B \cap \pi_2^{-1}(v))$. Since B is a minimal complement to A, it follows that the set $\pi_1(B \cap \pi_2^{-1}(v))$ does not contain any point whose i-coordinate is less than or equal to $x_{vi} - 1$ for each $1 \leq i \leq e$. Since $(\underbrace{-1, \cdots, -1}_{e\text{-times}}, \underbrace{0, \cdots, 0}_{(d-e)\text{-times}})$ belongs to $\mathbb{Z}^d = B + A$, it follows that the set $B \cap \pi_2^{-1}(v)$ is non-empty for some

$$v \in [-q_{e+1}, -p_{e+1}] \times \cdots \times [-q_d, -p_d].$$

For each $1 \leq i \leq e$, define

$$x_i := \min_{v \in [-q_{e+1}, -p_{e+1}] \times \cdots \times [-q_d, -p_d], B \cap \pi_2^{-1}(v) \neq \emptyset} (x_{vi} - 1).$$

Note that $(x_1, \cdots, x_e, \underbrace{0, \cdots, 0}_{(d-e)\text{-times}})$ does not belong to $B + A = \mathbb{Z}^d$. Hence A does not admit any minimal complement.

(3) Let $f \in \mathbb{Z}^{\mathbb{I}}$ denote the constant function which takes the value b. Then $A + f$ is a proper subset of A and hence A cannot be a minimal complement to some subset of $\mathbb{Z}^{\mathbb{I}}$. Let us assume that b is positive and a is equal to 0. Suppose A admits a minimal complement B in $\mathbb{Z}^{\mathbb{I}}$. Let $\{c_i\}_{i \in \mathbb{I}}$ denote an element of B. Since B is a complement to A and $B + A$ contains $\{c_i - 1\}_{i \in \mathbb{I}}$, it follows that B contains an element $\{d_i\}_{i \in \mathbb{I}}$ with $d_i \leq c_i - 1$ for all $i \in \mathbb{I}$. Note that $B \setminus \{\{c_i\}_{i \in \mathbb{I}}\}$ is also a complement to A. Hence A does not admit any minimal complement in $\mathbb{Z}^{\mathbb{I}}$. □

As a corollary, we deduce the following fact.

Corollary 5.4. *There exist semilinear sets A such that A is neither a minimal complement to any subset of G, nor does it have a minimal complement in G.*

Proof. Take A to be one of the sets described in (2) of Proposition 5.3. □

The above Proposition 5.3 and Corollary 5.4 shed light on subtle differences in existence and inexistence of minimal complements in general abelian groups. We have seen that a proper subgroup H in any group G is always a minimal complement to any of its coset class and it also admits a minimal complement. However, the fact does not remain necessarily true when we pass to subsets which are close to being subgroups. Let us recall the notion of an approximate subgroup.

Definition 5.5 (K-approximate subgroup)**.** *Let G be a group and $K \geqslant 1$ be some parameter. A finite set $A \subseteq G$ is called a K-*approximate group *if*

1. *Identity of G, $e \in A$.*
2. *It is symmetric, i.e., if $a \in A$ then $a^{-1} \in A$.*
3. *There is a symmetric subset X lying in $A \cdot A$ with $|X| \leqslant K$ such that $A \cdot A \subseteq X \cdot A$.*

The formal definition of an approximate subgroup was introduced by Tao in [Tao08]. Informally these sort of subsets have been studied since the time of Freiman [Fre64]. Nathanson considered a more general notion of an approximate group. For him, the set A need not be finite, nor symmetric, nor contain the identity.

Definition 5.6 ((r,l)-approximate group [Nat18])**.** *Let $r, l \in \mathbb{N}$ with $r \geqslant 2$. A non-empty subset $A \subseteq G$ is an (r,l)-approximate group if there exists a set $X \subseteq G$ such that*

$$|X| \leqslant l \text{ and } A^r \subseteq X \cdot A.$$

Any finite approximate subgroup always belongs to a minimal pair (by [BS18, Theorem 2.1]), while inside free abelian groups they always belong to some co-minimal pair by Theorem 1.3. When we pass to infinite approximate subgroups (in the sense of Nathanson), this is not necessarily the case. By (2) of Proposition 5.3, there exist unbounded linear sets which can never belong to a minimal pair while Corollary 5.4 concludes the same about semilinear sets. Unbounded linear sets are however examples of approximate subgroups in the sense of Definition 5.6.

We mention briefly that in the same paper [Nat18], Nathanson introduced the notion of an asymptotic approximate group, which is a subset $A \subseteq G$ such that every sufficiently large power of A is an (r,l)-approximate group.

Definition 5.7 (Asymptotic (r,l)-approximate group)**.** *Let $r, l \in \mathbb{N}$, with $r \geqslant 2$. A subset A of a group $(G, \cdot)$ is an asymptotic (r,l)-approximate group if there exists a threshold $h_0 \in \mathbb{N}$ such that for each natural number $h \geq h_0$, there exists a subset X_h of G satisfying*

$$|X_h| \leqslant l \text{ and } A^{hr} \subseteq X_h \cdot A^h.$$

1. Nathanson in [Nat18] showed that any finite subset in an abelian group is an asymptotic (r,l)-approximate group for some $r, l \in \mathbb{N}$.
2. In [BM19], it was shown that unbounded linear sets and also semilinear sets are asymptotic (r,l)-approximate groups for some $r, l \in \mathbb{N}$.

In the first case, the sets belong to co-minimal pairs while in the second case, as a consequence of Corollary 5.4, they do not necessarily belong to even minimal pairs, let alone co-minimal pairs. Thus, in general, it is also not true that asymptotic approximate groups will be part of some co-minimal pair.

6. Concluding remarks

The above discussion motivates one to consider co-minimal pairs with $A = B$, i.e., non-empty subsets A of an abelian group G with $A + A = G$ and $A + A \setminus \{a\} \subsetneq G, \forall\, a \in A$. These type of sets have also been considered by Kwon in the context of $G = \mathbb{Z}$ (minimal self-complements). He showed that such a set exists if and only if A avoids 3 term arithmetic progressions. We remark that this holds in the context of an arbitrary abelian group.

Proposition 6.1. *Let G be an abelian group. Then for a subset A of G, (A, A) is a co-minimal pair if and only if A avoids non-trivial 3-term arithmetic progressions.*

Proof. It is clear that the proof of Kwon for $G = \mathbb{Z}$ (see [Kwo19, Theorem 10]) also extends to general abelian groups. □

7. Acknowledgements

The first author would like to acknowledge the generous support of the OWLF program and would also like to thank the Fakultät für Mathematik, Universität Wien. The second author would like to acknowledge the Initiation Grant from the Indian Institute of Science Education and Research Bhopal, and the INSPIRE Faculty Award from the Department of Science and Technology, Government of India.

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