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Infinite Dimensional Analysis

A Hitchhiker's Guide

Third Edition

With 38 Figures
and 1 Table

 Springer

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