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Centralisers of fundamental subgroups

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Incidence geometry and group theory

Incidence geometry arises from the points, lines and planes of elementary geometry, in accordance to some properties, which can be stated in terms of inclusion and intersection. Of course this subject can be generalised in various directions like projective spaces, linear spaces or buildings, etcetera.

The role of symmetry in science and of its mathematical counterparts, i.e. automorphisms, isomorphisms, morphisms, groups and categories is well understood today. In particular the use of symmetry may simplify certain problems and be the key to their solutions.

The idea of describing a geometry $\mathcal{G}$ by a transformation group (symmetry group) leaving the geometry $\mathcal{G}$ invariant, was pointed out by Felix Klein, see [62]. He demanded an interaction between geometry and group theory. Klein realized that a certain class of geometries $\mathcal{G}$ could be described via invariants of the automorphism group $\mathrm{Aut}(\mathcal{G})$ of $\mathcal{G}$.

The program of Klein, the Erlangen Program, provides a wonderful device to create geometries. Consider a geometry $\mathcal{G}$ consisting of points, lines and planes and an incidence relation (a binary symmetric relation on the objects of $\mathcal{G}$, such that two different objects in relation are not of the same type) and some set of axioms of $\mathcal{G}$, which will not be specified for the moment. Suppose there exists a subgroup G of the automorphism group of $\mathcal{G}$, which acts transitively on the triples (p, l, P), called flags, where p is a point, l is a line, P is a plane and the objects p, l and P are pairwise incident. We fix such a triple say (p_0, l_0, P_0) and consider the stabilisers G_{p_0}, G_{l_0} and G_{P_0} in G. Our next step is to describe all objects of $\mathcal{G}$ using only the group G and the subgroups G_{p_0}, G_{l_0} and G_{P_0}. We can identify the set of points, denoted by $\mathcal{P}$, of the geometry $\mathcal{G}$ via the bijection $G/G_{p_0} \to \mathcal{P}$ with $gG_p \mapsto g(p)$. More generally we have a bijection between all objects of the geometry $\mathcal{G}$, so between the points, lines and plane of $\mathcal{G}$, and the left coset of the subgroups G_{p_0}, G_{l_0} and G_{P_0} in G. We also want to recover the incidence relations between the points, lines and planes of $\mathcal{G}$. Suppose p is a point incident to the line l, then we can determine a plane P incident to p and l. By the assumption that G acts transitively on the flags of $\mathcal{G}$, we find a group element g in G, such that $g(p_0), g(l_0), g(P_0)$ coincides with p, l, P implying $g \in gG_{p_0} \cap gG_{l_0} \neq \varnothing$. Therefore the objects represented by cosets, whose intersection is nonempty, are incident and vice versa. Using this program we have turned a geometry in the spirit of Klein into group theoretical data.

We can also reverse this idea, in the sense of describing groups by geometries. We start with a group G and subgroups G_0, G_1, G_2. We define an incidence system Γ

with respect to G_0, G_1, G_2. The objects of Γ are the cosets gG_0 of G_0, the cosets gG_1 of G_1 and the cosets gG_2 of G_2 for $g \in G$. Two different coset gG_i and hG_j are incident for $i \neq j$ and some $g, h \in G$ if and only if $gG_i \cap hG_j \neq \varnothing$. Note that Γ is not necessarily a geometry, by theorem 8.3.10 of [16]. It turns out that G acts on the incidence system Γ as a group of automorphisms by left translation, but this action might not be flag transitive. Therefore the technique for deriving from a geometry some group theoretical data describing this geometry is only partially reversed by turning a group with certain subgroups into a geometry, as studied by M. Stroppel in [83].

Another idea is to search for defining relations of groups by means of simple connectedness of certain geometries. Tits' lemma, see [94], states that a flag transitive geometry is simply connected if and only if the selected subgroup of automorphisms of the geometry is the universal completion of the amalgam of its maximal parabolic subgroups with respect to some maximal flag, i.e. stabilisers of elements of this maximal flag. Tits' lemma is a fantastic tool for proving group theoretical statements in terms of geometric results. Sometimes it is possible to use local characterisations of a geometry instead of proving simple-connectivity for this geometry. Such a local approach to geometries is also inspired by group theory. In particular finite simple groups are sometimes characterised by local information. The point is the following: If there is a unique geometry up to isomorphism with certain local properties, then a characterisation of this geometry obtained from these local properties implies simple connectedness of the geometry. Indeed the universal cover of a geometry also has the local properties and is by definition simply connected. If one also proves that a subgroup of automorphisms of the geometry acts flag transitively then the characterisation obtain from the local properties allows an identification of this subgroup from local conditions like centralisers of involutions. The local characterisation of the Kneser graph $K(n,2)$ for $n \geq 7$ in [48] can be used to characterise alternating and symmetric groups (of sufficiently large degree) by centralisers of various of their elements via theorem 27.1, part I chapter 1 of [29]. If one starts with a connected graph Γ having certain local properties, like that all induced subgraphs on the neighbours of a vertex are isomorphic, then a local characterisation of this graph has also an application in group theory. Again via theorem 27.1, part I chapter 1 of [29], if a locally recognisable graph Γ admits a subgroup G of automorphisms that acts transitively on the set of ordered triangles of Γ, then the local recognition of the graph Γ implies a local recognition of the subgroup G. As example we refer to the results in [38] and [39].

The present thesis

In this thesis we present several local characterisation results for different graphs and geometries. The group theoretical consequences of these results can be obtained in the same way as its respective analogues in [38] and [39]. In chapter 2, we study graphs on lines of a complex vector space endowed with the inner product, where two different lines are adjacent if one line is contained in the orthogonal space of the other line and vice versa. These graphs can also be described in terms of fundamental $SU_2(\mathbb{C})$ subgroups of the group $SU_n(\mathbb{C})$. The graph on the fundamental $SU_2(\mathbb{C})$ subgroups of $SU_n(\mathbb{C})$ with the commutation relation as adjacency relation is also a description of a line graph over a complex vector space with the standard scalar product. In the first couple sections of this chapter we concentrate on the line graph of a complex vector space with an anisotropic form of dimension greater or equal to five. We show that from the line graph we can determine a geometry, which is isomorphic to the point-line geometry of the complex polar space in certain dimension. Hence the reconstruction of the complex projective space and the polarity is possible. In the last section of chapter 2 we look at connected graphs which are locally the line graph of an at least seven dimensional complex vector space with an anisotropic form. A local recognition theorem will be obtained for these graphs via the computation of the diameter.

Chapter 3 is very similar to chapter 2. Here we study the graph on the hyperbolic lines (non-degenerate lines) of a unitary vector space over a finite field $\mathbb{F}_{q^2}$ of dimension at least seven, where two different hyperbolic lines are adjacent if and only if one hyperbolic line is in the polar space of the other hyperbolic line. As before in the first couple sections of this chapter we focus on the hyperbolic line graph obtained from a unitary vector space over $\mathbb{F}_{q^2}$ of dimension at least five. In particular we study all hyperbolic lines of an at least five-dimensional unitary vector space over $\mathbb{F}_{q^2}$ and their relative positions to each other. As result of this process we obtain a point-line-geometry which we can identify via theorem 3.4.1 proved by H. Cuypers, see also theorem 1.2 of [25]. The last section of chapter 3 deals with connected locally hyperbolic lines graphs. We prove that the diameter of the considered graphs is two, which enables us to classify all these graphs by their internal properties. Also this local recognition theorem is achieved under the use of theorem 3.4.1. The hyperbolic line graphs are of interest as the hyperbolic lines of a unitary n-dimensional vector space over $\mathbb{F}_{q^2}$ are in one-to-one correspondence with fundamental $SU_2(\mathbb{F}_{q^2})$ subgroups of the special unitary group $SU_n(\mathbb{F}_{q^2})$. Moreover the two different fundamental $SU_2(\mathbb{F}_{q^2})$ subgroups of $SU_n(\mathbb{F}_{q^2})$ commute if and only if the corresponding hyperbolic lines are adjacent. The local recognition of connected graphs that are locally the hyperbolic line graph of a unitary vector space over $\mathbb{F}_{q^2}$ of dimension at least seven implies a local recognition of the group $PSU_n(\mathbb{F}_{q^2})$ for $n \geq 7$.

In the last chapter of this thesis, we return to the setting of chapter 2. We start with a connected graph which is locally the line graph of a six-dimensional complex vector space endowed with an anisotropic form. Section 4.3 of this chapter describes one way to find automorphisms of such a graph. Thus we obtain some subgroup of the automorphism group of the considered graphs. This subgroup has furthermore the special property that it contains for each vertex of the graph a subgroup isomorphic to $\mathrm{SU}_2(\mathbb{C})$ fixing the neighbourhood graph of this vertex elementwise and acting naturally on the two-dimensional vector subspace, to which the fixed vertex belongs to. It is possible to identify this subgroup of automorphisms via a locally $\mathbf{W}(A_5)$ reflection graph, which will be constructed in section 4.4, and the theory of weak Phan systems. The result is that the subgroup will be isomorphic to some central quotient of either $\mathrm{SU}_8(\mathbb{C})$ or of the simple connected version of $E_{6,-78}$ is achieved in section 4.6. Furthermore we study in section 4.7 the fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups graph of $\mathrm{SU}_8(\mathbb{C})$ and $E_{6,-78}$. We prove for both graphs, that the induced graph on the neighborhood for every vertex is isomorphic to the line graph of a six-dimensional complex vector space endowed with an anisotropic form. We also achieve the result that both graphs are simply connected. Taking this result we obtain the following: Let Γ be any connected graph, which is locally the line graph of a six dimensional complex vector space endowed with an anisotropic form satisfying some further technical property. Then the universal cover of the graph Γ is isomorphic either to the fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups graph of $\mathrm{SU}_8(\mathbb{C})$ or to the fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups graph of a compact real form of $E_6(\mathbb{C})$. As before the local recognition of these graphs implies a local recognition of the groups $\mathrm{PSU}_8(\mathbb{C})$ and the adjoint version of $E_{6,-78}$.

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For my son Bjarne Grothe

Given him a page, where he can start to read this thesis. That is a poem of James Krüss.

Was denken in der Neujahrsnacht die Tiere und die Menschen?

Was denken in der Neujahrsnacht
Die Kater und die Katzen?
Sie denken, daß im alten Jahr
Der Mausefang bescheiden war,
Und strecken in das neue Jahr
Begehrlich ihre Tatzen.

Was denken in der Neujahrsnacht
Die Pudel und die Möpse?
Sie denken, daß nicht jeden Tag
Ein Knochen auf dem Teller lag,
Und wünschen für den Neujahrstag
Sich Leberwurst und Klöpse.

Was denken in der Neujahrsnacht
Die Vögel hierzulande?
Sie denken an die Storchenschar,
Die hier im Sommer fröhlich war
Und die nun wandelt, Paar um Paar,
Im warmen Wüstensande.

Was denken in der Neujahrsnacht
Die Knäblein und die Knaben?
Sie denken, ob der Frost bald weicht
Und ob ein Mensch den Mond erreicht
Und ob sie nächstes Jahr vielleicht
Schuhgröße vierzig haben.

Was denken in der Neujahrsnacht
In aller Welt die Mädchen?
Die Mädchen denken unentwegt
Und angeregt und aufgeregt
An das, was man im Sommer trägt,
Ob Gretchen oder Kätchen.

Was denken in der Neujahrsnacht
Die alten, alten Leute?
Sie denken unterm weißen Haar,
Wie sonderbar das Leben war
Und daß das Glück sie wunderbar
Geleitet hat bis heute.

Inzidenzgeometrie und Gruppentheorie

Die Theorie der Inzidenzgeometrie wird ausgehend von den Grundobjekten der elementaren Geometrie, also den Punkten, Geraden und Ebenen, durch einige weitere Eigenschaften definiert, welche sich mit Hilfe von Durchschnitten und Vereinigungen von Mengen beschreiben lassen. Durch die allgemeine Definition von Inzidenzgeometrien ist die abstrakte Beschreibung verschiedener weiterer Strukturen wie die des Projektiven Raumes, die des Linearen Raumes oder die der Gebäuden, et cetera möglich.

Die Rolle von Symmetrien in naturwissenschaftlichen Fragestellungen und die mathematische Darstellung von Symmetrien in Form von Automorphismen, Isomorphismen, Morphismen, Gruppen oder Kategorien ist heutzutage gut verstanden. Insbesondere können Symmetrien in hohem Maße Problemstellungen vereinfachen oder zur Lösung von Fragestellungen beizutragen.

Auf die Möglichkeit eine Geometrie $\mathcal{G}$ durch eine Transformationsgruppe, welche die Geometrie $\mathcal{G}$ invariant läßt, zu beschreiben, wurde erstmals von Felix Klein in [62] hingewiesen. Er suchte nach einer Verbindung zwischen der Gruppentheorie und der Theorie der Geometrie. Klein stellte für einige Klassen von Geometrien fest, daß Invariante der Automorphismengruppe $\mathrm{Aut}(\mathcal{G})$ die Geometrie $\mathcal{G}$ eindeutig beschreiben können.

Felix Klein formuliert im Erlanger Programm einen wunderbaren Algorithmus um Geometrien mittels Gruppentheorie zu definieren. Dazu betrachteten wir eine Geometrien $\mathcal{G}$ bestehend aus Punkten, Geraden und Ebenen und einer Inzidenzrelation (eine binäre symmetrische Relation auf den Objekten von $\mathcal{G}$ mit der Eigenschaft, daß zwei verschiedene Objekte von $\mathcal{G}$ nur dann in Relation zu einander sein können, wenn sie von verschiedenem Typ sind) sowie einigen weiteren Bedingungen an $\mathcal{G}$.

Angenommen es gibt eine Untergruppe G der Automorphismengruppe $\mathrm{Aut}(\mathcal{G})$ von $\mathcal{G}$, die transitive auf der Menge aller Tripels (p, l, P), auch Fahnen genannt, wirkt, wobei p ein Punkt, l eine Gerade und P eine Ebene ist und die Objekte p, l und P paarweise inzident sind. Wir wählen eine Fahne (p_o, l_o, P_o) und betrachten die Stabilisatoren G_{p_o}, G_{l_o} and G_{P_o} von p_o, l_o und P_o in der Gruppe G. Man versucht nun alle Objekte und Relationen der Geometrie $\mathcal{G}$ durch die Untergruppen G_{p_o}, G_{l_o} and G_{P_o} zu beschreiben. Die Menge aller Punkte $\mathcal{P}$ von $\mathcal{G}$ kann eindeutig durch die Bijektion $G/G_{p_o} \to \mathcal{P}$ mit $gG_p \mapsto g(p)$ identifiziert werden. Allgemein gilt, es gibt ein Bijektion zwischen den Objekten von $\mathcal{G}$, also allen Punkten, Geraden und Ebenen, und den Linksnebenklassen der Untergruppen G_{p_o}, G_{l_o} und G_{P_o}

in G. Auch die Inzidenzrelation zwischen den Punkten, Geraden und Ebenen der Geometrie $\mathcal{G}$ kann durch Eigenschaften der Untergruppen G_{p_0}, G_{l_0} und G_{P_0} zueinander repräsentiert werden. Sei p ein Punkt inzident zur Geraden l und P eine Ebene inzident zu p und l. Da die Gruppe G transitiv auf den Fahnen von $\mathcal{G}$ wirkt, gibt es ein Gruppenelement g in G, so daß die Fahne $(g(p_0), g(l_0), g(P_0))$ mit der Fahne (p, l, P) übereinstimmt, woraus folgt daß $g \in gG_{p_0} \cap gG_{l_0} \neq \emptyset$. Daher gilt, daß zueinander inzidente Objekte der Geometrie durch Linksnebenklassen, deren Schnittmenge nicht leer ist, repräsentiert werden und umgekehrt. Dieser Algorithmus verwandelt also eine Geometrie mit fahnentransitiver Automorphismusgruppe im Sinne von Klein in gruppentheoretische Daten.

Natürlich kann man versuchen diese Algorithmus umzukehren um Gruppen mit Hilfe von Geometrien zu beschreiben . Wir starten mit einer Gruppe G und Untergruppen G_0, G_1, G_2 von G und definieren ein Inzidenzsystem Γ bezüglich der Untergruppen G_0, G_1, G_2. Die Objekte von Γ sind die Linksnebenklassen gG_0,gG_1 und gG_2 von G_0, G_1 und G_2 für $g \in G$ und zwei verschiedene Linksnebenklassen gG_i and hG_j sind genau dann inzident für $i \neq j$ und $g, h \in G$ wenn $gG_i \cap hG_j \neq \emptyset$. Satz 8.3.10 in [16] zeigt, daß das Inzidenzsystem Γ nicht notwendigerweise eine Geometrie ist. Die Grupppe G wirkt auf dem Inzidenzsystem Γ durch Linksmultiplikation als Automorphismengruppe. Jedoch ist diese Wirkung nicht immer fahnentransitiv ist. Daher ist die Technik von einer Geometrie gruppentheoretische Daten zu extrahieren, die diese Geometrie eindeutig beschreibt, nur unvollständig umkehrbar. Die Idee ausgehend von einer Gruppe und einigen Untergruppen eine Geometrie zu definieren wurde von M. Stroppel in [83] studiert.

Ein anderer Ansatz um Geometrien mittels Gruppen und Untergruppen zu beschreiben ist die Suche nach definierenden Relationen mit Hilfe des Einfachenzusammenhanges. Insbesondere besagt Tits' Lemma, sieh [94], daß eine fahnentransitive Geometrie genau dann einfachzusammenhängend ist, wenn eine fahnentransitive Untergruppe der Automorphismengruppe der Geometrie der universelle Abschluß des Amalgans der maximal parabolischen Untergruppen bezüglich einer maximalen Fahne, daher der universelle Abschluß des Amalgans der Stabilisatoren der Elemente einer maximalen Fahne, ist. Tits' Lemma ist ein ideales Werkzeug um gruppentheoretische Aussagen mit Hilfe von geometrischen Resultaten zu beweisen.

Manchmal ist es auch möglich lokale Charakerisierungen von Geometrien zu benutzen anstatt den Einfachenzusammenhang für diese Geometrien zu beweisen. Solch eine lokale Betrachtungsweise wurde durch die Gruppentheorie angeregt. Insbesondere können einige endlichen einfachen Gruppen durch lokale Informationen charakterisiert werden. Der entscheidene Punkt ist der folgende: Gibt es eine eindeutige Geometrie bis auf Isometrie mit lokale Bedingungen, dann impliziert die Charakterisierung der Geometrie durch diese lokalen Bedingungen den Einfachenzusammenhang der Geometrie, da die universelle Überlagerung der Geometrie auch die lokalen Bedingungen erfüllt und per Definition einfach zusammenhängend ist. Kann man weiterhin zeigen, daß es eine fahnentransitive Untergruppe der Automorphismengruppe der Geometrie gibt, dann erlaubt die lokale Erkennung der Geometrie eine Klassifizierung der Untergruppe aus lokalen Bedingungen, wie zum Beispiel den Zentralisator von Involutionen.

Ein Beispiel für das beschriebene Vorgehen ist die lokale Charakterisierung des Knesergraphens $K(n, 2)$ für $n \geq 7$ in [48]. Die Beschreibung des Knesergraphens $K(n, 2)$ für $n \geq 7$ durch lokale Daten wird dazu benutzt alternierende und symmetrische Gruppen (von ausreichend Grad) durch Zentralisatoren von verschiedenartig Elementen dieser Gruppen mittels Theorem 27.1, Abschnitt I Kapitel 1 von [29], zu charakterisieren.

Wenn man mit einem zusammenhängende Graphen Γ beginnt, der bestimmte lokale Eigenschaften, etwa daß alle induzierten Nachbarschaftsgraphen einer Ecke isomorph sind, besitzt, dann hat eine Charakterisierung von Γ durch die lokalen Eigenschaften einige gruppentheoretische Anwendungen. Wiederum mittels Theorem 27.1, Abschnitt I Kapitel 1 von [29], erhalten wir folgende Aussage: Sei Γ ein lokal erkennbarer Graph mit Automorphismenuntergruppe G, die transitiv auf der Menge der geordneten Dreiecken von Gamma wirkt. Dann impliziert die lokale Charakterisierung des Graphens Γ eine lokale Erkennung der Gruppe G. Beispiele für dieses Vorgehen sind die Resultate in [38] und [39].

Zusammenfassung

In dieser Dissertation werden einige lokale Erkennungsresultate für verschiedene Graphen und Geometrien vorgestellt. Die gruppentheoretischen Konsequenzen von diesen Resultaten erhält man in der gleichen Art und Weise wie ihr entsprechenden Analoga in [38] und [39].

In Kapitel 2 studieren wir den Geradengraphen des komplexen Vektorraumes mit einer anisotropen Form, wobei genau dann zwei verschiedene Geraden im Graphen benachbart sind, wenn eine Gerade im Senkrechtraum der anderen Gerade liegt und umgekehrt. Diese Graphen können auch mittels fundamentaler $SU_2(\mathbb{C})$ Untergruppen der Gruppe $SU_n(\mathbb{C})$ definiert werden. Der Graph auf den fundamentalen $SU_2(\mathbb{C})$ Untergruppen der Gruppe $SU_n(\mathbb{C})$ mit der Kommutatorrelation als Nachbarschaftsrelation ist daher eine weitere Beschreibung des Geradengraphen des komplexen Vektorraumes mit einer anisotropen Form. In den ersten Abschnitten dieses Kapitels fokussieren wie auf den Geradengraphen des komplexes Vektorraumes der Dimension größer gleich fünf mit einer anisotropen Form. Ausgehend vom Geradengraphen definieren wir eine Geometrie, welche isomorph zur Punkt-Geraden-Geometrie des komplexen Polarraumes einer bestimmten Dimension ist.

Folglich ist eine Rekonstruktion des komplexen Projektiven Raumes und der Polarität möglich. Im letzten Abschnitt dieses Kapitels werden zusammenhängende Graphen, welche lokal der Geradengraph des mindestens sieben-dimensionalen komplexen Vektorraumes mit einer anisotropen Form ist, studiert. Durch die Bestimmung der Durchmessers kann ein lokaler Erkennungssatz für diese Graphen bewiesen werden.

Kapitel 3 ist sehr ähnlich zu dem vorangegangenen Kapitel 2. Wir studieren den Graphen auf den hyperbolischen Geraden (nicht degenerierten Geraden) des unitären Vektorraums über den endlichen Körpers $\mathbb{F}_{q^2}$ von Dimension mindestens sieben, wobei genau dann zwei verschiedene hyperbolische Geraden benachbart sind, wenn eine hyperbolische Gerade im Senkrechtraum der anderen hyperbolischen Gerade ist und umgekehrt. In den ersten Abschnitten dieses Kapitels werden grundlegende Eigenschaften des hyperbolischen Geradengraphens eines mindestens fünfdimensionalen unitären Vektorraumes über $\mathbb{F}_{q^2}$ bewiesen. Insbesondere wird der Abstand und die Lage zweier verschiedener hyperbolischer Geraden zueinander in einem unitären Vektorraumes der Dimension größer gleich fünf über $\mathbb{F}_{q^2}$ bestimmt. Ausgehend von diesen grundlegenden Eigenschaften des hyperbolischen Geradengraphens ist es mögliche eine Punkt-Geraden Geometrie zu definieren, welche mittels dem Satz 3.4.1 von H. Cuypers, Theorem 1.2 in [25], klassifiziert werden kann. Im letzten Abschnitt dieses Kapitels werden zusammenhängende lokal hyperbolische Geradengraphen studiert. Wir beweisen, daß der Durchmesser der betrachteten Graphen zwei ist. Dieses Resultat ermöglicht die Klassifizierung aller zusammenhängenden lokal hyperbolische Geradengraphen ab der Dimension sieben. Auch dieser lokale Erkennungssatz wird mittels dem Satz 3.4.1 von H. Cuypers erreicht.

Die hyperbolischen Geradengraphen sind von Bedeutung, da die hyperbolischen Geraden eines n-dimensionalen unitären Vektorraumes über $\mathbb{F}_{q^2}$ in Bijektion zu den fundamentalen $\mathrm{SU}_2(\mathbb{F}_{q^2})$ Untergruppen von $\mathrm{SU}_n(\mathbb{F}_{q^2})$ stehen. Darüber hinaus gilt, daß zwei verschiedene fundamentale $\mathrm{SU}_2(\mathbb{F}_{q^2})$ Untergruppen genau dann kommutieren, wenn die korrespondierenden hyperbolischen Geraden im hyperbolischen Geradegraphen benachbart sind. Daher impliziert die lokale Erkennung eines zusammenhängenden Graphens, der lokal der hyperbolische Geradengraph eines mindestens sieben-dimensionalen unitären Vektorraumes über $\mathbb{F}_{q^2}$ ist, die lokale Erkennung der Gruppe $\mathrm{PSU}_n(\mathbb{F}_{q^2})$ für $n \geq 7$.

Im letzten Kapitel dieser Dissertation gehen wir noch einmal zurück zur Situation in Kapitel 2. Wir starten mit dem zusammenhängenden Graphen, der lokal der Geradengraph eines sechs-dimensionalen komplexen Vektorraumes mit einer anisotropen Form ist. Abschnitt 4.3 dieses Kapitels beschreibt eine Methode Automorphismen dieses Graphen zu konstruieren. Mit dieser Methode erhalten wir eine Untergruppe der Automorphismengruppe des Graphens mit folgender speziellen Eigenschaft: Die Untergruppe enthält für jede Ecke des Graphens eine Untergrupe isomorph zu $\mathrm{SU}_2(\mathbb{C})$, welche den induzierten Nachbarschaftsgraphen der ausgewählten Ecke fixiert und auf natürliche Art und Weise auf dem zwei-dimensionalen Vektorraum, welcher zu der ausgewählten Ecke gehört, wirkt. Diese Untergruppe von Automorphismen ist isomorph zu einem zentralen Quotienten der Gruppe $\mathrm{SU}_8(\mathbb{C})$ oder der Gruppe $E_{6,-78}$. Dieses Resultat wird in Abschnitt 4.6 mit Hilfe eines lokal $\mathbf{W}(A_5)$-Spiegelungsgraphens, welche in Abschnitt 4.4 definiert wird, und der Theorie über schwache Phan-Systeme bewiesen.

In Abschnitt 4.7 wird der fundamentale $\mathrm{SU}_2(\mathbb{C})$ Untergruppengraph von $\mathrm{SU}_8(\mathbb{C})$ und von $E_{6,-78}$ studiert. Wir beweisen für diese beiden Graphen, daß der induzierte Nachbarschaftsgraph einer jeden Ecke isomorph ist zum Geradengraph eines sechs-dimensionalen komplexen Vektorraumes mit einer anisotropen Form. Desweiteren wird gezeigt, daß diese Graphen einfachzusammenhängend sind.

Damit erhalten wir folgendes Resultat: Für jeden zusammenhängenden Graphen, welcher lokal der Geradengraph eines sechs-dimensionalen komplexen Vektorraumes mit einer anisotropen Form ist und der eine weitere technische Eigenschaft besitzt, gilt, daß die universelle Überlagerung des Graphens isomorph ist zum fundamentalen $\mathrm{SU}_2(\mathbb{C})$ Untergruppengraph von $\mathrm{SU}_8(\mathbb{C})$ oder zum fundamentalen $\mathrm{SU}_2(\mathbb{C})$ Untergruppengraph von $E_{6,-78}$. Dieser lokalen Erkennungsatz impliziert die lokale Erkennung der Gruppen $\mathrm{PSU}_8(\mathbb{C})$ und der adjungierten Version von $E_{6,-78}$.

Contents

CHAPTER ONE

Introduction

A central problem in synthetic geometry is the characterisation of graphs and geometries. The local recognition of locally homogeneous graphs forms one category of such characterisations, which works as follows:

Let Δ be a graph. A graph Γ is called **locally** Δ if for each vertex x of Γ the graph Γ_x is isomorphic to Δ, where Γ_x is the induced subgraph of Γ on the set of vertices adjacent to x. It is a natural question to ask for all (connected) graphs, which are locally some graph Δ. This classification question is called the local recognition problem for graphs that are locally Δ, which can be found in great quantities in the literature. One of the earliest and most influential is [17]. The present thesis solves a number of open local recognition problems.

Projective geometry

Let V be a finite dimensional vector space over a field $\mathbb{F}$. The set of subspaces of V of dimension k is known as the Grassmannian $G_k(V)$. The **projective geometry** $\mathbb{P}(V) = \mathbb{P}(V, \mathbb{F})$ resp. the **projective space** $\mathbb{P}(V)$, see also section A.1, is the partially ordered set of all subspaces of V. We call the elements of $G_1(V)$ the **points** of $\mathbb{P}(V)$, the elements of $G_2(V)$ the **lines** of $\mathbb{P}(V)$ and the elements of $G_3(V)$ the **planes** of $\mathbb{P}(V)$. If V is an n-dimensional vector space then the members of $G_{n-1}(V)$ are the **hyperplanes** of $\mathbb{P}(V)$ and the elements of $G_{n-2}(V)$ are the **hyperlines** of $\mathbb{P}(V)$.

Let U and W be subspaces of V then $U + W = \langle U, W \rangle$ is the subspace spanned by

U and W, so the smallest subspace of V containing U and W. Similar $\langle v_1, \ldots, v_n \rangle$ is the subspace of V spanned by the vectors $v_1, \ldots, v_n$.

We can also use the following axiomatic approach to finite dimensional projective geometries, which by the fundamental theorem of projective geometry, see theorem 1.1.1 or [60], describes a finite dimensional projective geometry $\mathcal{P}$ isomorphic to $\mathbb{P}(V, \mathbb{F})$ for sufficiently large dimension. We start with some set P, whose elements we call points, together with a collection L of subsets of P of size at least two, whose members we call lines. Two different point a and b are **collinear** if there is a line l containing a and b. The pair $\mathcal{P} = (P, L)$ is a **projective space** resp. **projective geometry** if the following axioms are satisfied:

- If a and b are distinct points, then there is exactly one line l, which contains both a and b.
- Each line contains at least three points.
- There is at least one line and not all points are on that line.
- **Veblen-Young Axiom**
  If a, b, c, d are pairwise distinct point such that the line through a and b and the line through c and d are incident to some point e, then also the line through a and c and the line through b and d are incident to some point f.

A set of points is said to be a **subspace** , if, whenever it contains two different points a and b on a line l, then it contains the entire line l passing through a and b. We call a chain of distinct non-empty subsets

$$B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_n$$

a **flag of length** n. A projective geometry is of **finite dimension** if the length of all its flags has an upper bound. The **dimension** of a finite dimensional projective geometry is the length of a maximal flag. We have the following classification by the fundamental theorem of projective geometry, theorem E.11 of [60].

Theorem 1.1.1 (fundamental theorem of projective geometry)
Existence Part: Given an projective geometry $\mathcal{P}$ of dimension $3 \leq n \leq \infty$, there is a division ring $\mathbb{D}$ and a right vector space V over $\mathbb{D}$ such that $\mathcal{P}$ is isomorphic to $\mathbb{P}(V)$.

Uniqueness Part: Suppose V is a right vector space over a division ring $\mathbb{D}$, where $3 \leq \dim(V) \leq \infty$, and W is a right vector space over a division ring $\mathbb{S}$ such that there exists an isomorphism $\varphi : \mathbb{P}(V, \mathbb{D}) \to \mathbb{P}(W, \mathbb{S})$. Then there exists a ring isomorphism $\sigma : \mathbb{D} \to \mathbb{S}$ and an invertible semi-linear transformation $T : V \to W$, i.e. $T(x\lambda) = T(x)\sigma(\lambda)$ and $T(x + y) = T(x) + T(y)$ for all $\lambda \in \mathbb{D}$ and $x, y \in V$, such that $\varphi(x\mathbb{D}) = T(x)\mathbb{S}$.

Polarities and forms

For the next part let V be a finite dimensional vector space over $\mathbb{F}$. A **correlation**, also called **duality**, of a projective geometry $\mathbb{P}(V)$ is an incidence preserving bijection from $\mathbb{P}(V)$ to $\mathbb{P}(V)$, which reverses inclusion. Thus a correlation sends points to hyperplanes and hyperplanes to points. A **polarity** of $\mathbb{P}(V)$ is a correlation π of order two, so $\pi \cdot \pi = \mathrm{id}$ and the pair $(\mathbb{P}(V), \pi)$ is called a **polar geometry**.

The next criterion gives us easy conditions to check whether a bijection π of $\mathbb{P}(V)$ is a duality resp. polarity or not.

Lemma 1.1.2 (lemma 9.1.5 of [16]) *Any duality of a projective space $\mathbb{P}$ of finite dimension n, is determined by a mapping δ of the set of points of $\mathbb{P}$ onto the set of hyperplanes of $\mathbb{P}$ with the property that the points a, b, c are collinear in $\mathbb{P}$ if and only if $\delta(a), \delta(b)$ and $\delta(c)$ are hyperplanes having pairwise the same intersection.*

Lemma 1.1.3 (lemma 9.1.6 of [16]) *Any polarity of a projective space $\mathbb{P}$ of finite dimension n, is determined by a mapping π of the set of points of $\mathbb{P}$ onto the set of hyperplanes of $\mathbb{P}$ with the property that $a \in \pi(b)$ implies $b \in \pi(a)$ for all points $a, b \in \mathbb{P}$.*

The polarities of $\mathbb{P}(V)$ can be classified for $\dim(V) \geq 3$ in terms of sesquilinear forms. This leads to our next topic.

An **anti-automorphism** of a skew field $\mathbb{F}$, is a bijection $\sigma : \mathbb{F} \to \mathbb{F}$ with the properties

$$\sigma(c_1 + c_2) = \sigma(c_1) + \sigma(c_2),$$

$$\sigma(c_1 \cdot c_2) = \sigma(c_2) \cdot \sigma(c_1).$$

Every automorphism of a (commutative) field is an anti-automorphism.

If σ is an anti-automorphism of a (commutative) field $\mathbb{F}$, then a σ-**sesquilinear form** on a vector space V over $\mathbb{F}$ is a map $\beta : V \times V \to \mathbb{F}$ such that

$$\beta(v_1 + v_2, u) = \beta(v_1, u) + \beta(v_2, u),$$

$$\beta(u, v_1 + v_2) = \beta(u, v_1) + \beta(u, v_2) \text{ and}$$

$$\beta(\lambda u, \mu v) = \lambda\sigma(\mu)\beta(u, v)$$

for all $\lambda, \mu \in \mathbb{F}, u, v, u_1, u_2, v_1, v_2 \in V$.

A non-zero vector $v \in V$ is **isotropic** or **singular** if $\beta(v, v) = 0$ and otherwise if $\beta(v, v) \neq 0$ then v is a **regular** vector. Two different vectors u, v are said to be **orthogonal** resp. **perpendicular** if $\beta(u, v) = 0$. For a subspace $X \subseteq V$ the subspace

$X^\pi := \{v \in V \mid \beta(v,u) = 0 \text{ for all } u \in X\}$ is called the **orthogonal subspace**, also **polar subspace** or **perpendicular subspace, of** X. Certainly the following identities are satisfied for all subspaces U, W of V.

$$U \subseteq W \text{ implies } W^\pi \subseteq U^\pi$$

$$(W+U)^\pi = \langle W, U\rangle^\pi = W^\pi \cap U^\pi \text{ and } (W \cap U)^\pi = W^\pi + U^\pi$$

A σ-sesquilinear form β is called **non-degenerate** if $\beta(u,v) = 0$ for all $u \in V$ implies $v = 0$ or, equivalently, if $\beta(u,v) = 0$ for all $v \in V$ implies $u = 0$. If β is a non-degenerate σ-sesquilinear form of a vector space V, then

$$\dim(U) + \dim(U^\pi) = \dim(V)$$

for every subspace U of V and we refer to the subspace X^π as **orthogonal complement of** X for any subspace X of V. Furthermore a σ-sesquilinear form β such that $\beta(u,v) = 0$ implies $\beta(v,u) = 0$ for all $u, v \in V$ is said to be **reflexive**. A non-degenerate sesquilinear β is reflexive if and only if $(U^\pi)^\pi = U^{\pi\pi} = U$ for very subspace $U \subseteq V$. We call a subspace U of V **non-degenerate** resp. **regular** if $\beta_{|U\times U} : U \times U \to \mathbb{F}$ is non-degenerate. On the other hand a subspace U of V is **degenerate** resp. **singular** if $\beta_{|U\times U}$ is not a non-degenerate form, so $\beta_{|U\times U}$ is a **degenerate** or **singular** form. A subspace U of V on which β vanishes identically, is called **totally singular**. The vector space V is **anisotropic** if $\beta(v,v) \neq 0$ for all $v \in V\backslash\{0\}$. Moreover a **hyperbolic line** of V is a two-dimensional subspace l spanned by some singular non-zero vectors x and y such that $\beta(x,y) = 1$.

Let β be a σ-sesquilinear form on V, then the **radical** of V, denoted by $\mathrm{rad}(V)$, is the subspace $\{v \in V \mid \beta(v,u) = 0 \text{ for all } u \in V\}$. We call two σ-sesquilinear forms α and β of V **isometric** if there is a linear map $\varphi : V \to V$ such that $\beta(\varphi(v), \varphi(u)) = \alpha(v,u)$ for all $u, v \in V$.

The relation between polarities of $\mathbb{P}(V)$ and non-degenerate reflexive σ-sesquilinear forms β of finite dimensional vector spaces V over $\mathbb{F}$ can be described as follows.

Theorem 1.1.4 (theorem 9.2.10 and theorem 9.3.7 of [16]) *Let V be a vector space of finite dimension and β be a non-degenerate reflexive σ-sesquilinear form on V. Then the mapping $\delta_\beta : \langle a\rangle \mapsto \langle a\rangle^\pi$ for all non-zero vectors $a \in V$ determines a polarity of $\mathbb{P}(V)$.*

Theorem 1.1.5 (Birkhoff-von Neumann, theorem 7.1 of [86]) *If $\dim(V) \geq 3$ and if π is a polarity of $\mathbb{P}(V)$, then π arises from a non-degenerate reflexive σ-sesquilinear from β of one of the following types:*

- **alternating or symplectic**
  in this case $\mathbb{F}$ is a field, $\sigma = \mathrm{id}_{\mathbb{F}}$ and $\beta(v,v) = 0$ for all $v \in V$.

- **symmetric**
  in this case $\mathbb{F}$ *is a field,* $\sigma = \mathrm{id}_{\mathbb{F}}$ *and* $\beta(u, v) = \beta(v, u)$ *for all* $u, v \in V$.

- **hermitian**
  in this case $\sigma^2 = \mathrm{id}_{\mathbb{F}}$, $\sigma \neq \mathrm{id}_{\mathbb{F}}$ *and* $\beta(u, v) = \sigma(\beta(v, u))$ *for all* $u, v \in V$.

The polar geometry $(\mathbb{P}(V), \pi)$ is known as a **symplectic**, **orthogonal** or **unitary** geometry according to which case of the theorem from above holds.

Next we will concentrate on hermitian forms of an n-dimensional vector space V over $\mathbb{F}$. Thus let β be a σ-hermitian form of V and $\kappa : v_1, \ldots, v_n$ be a basis of V. The **Gram matrix** of β with respect to the basis κ is the matrix

$$G_\kappa^\beta = (u_{ij})_{1\le i,j\le n} = (\beta(v_i, v_j))_{1\le i,j\le n}.$$

Certainly G_κ^β is a hermitian matrix, thus $G_\kappa^\beta = \sigma(G_\kappa^\beta)^T$. Let W be a vector subspace complement of $\mathrm{rad}(V)$, i.e. $V = W \oplus \mathrm{rad}(V)$. Then the restriction $\beta_{|W\times W}$ of β to the subspace W is a non-degenerate σ-hermitian form of W.

Let $\mathbb{F}_0$ denote the fix field of σ, i.e.

$$\mathbb{F}_0 = \{x \in \mathbb{F} \mid x = \sigma(x) = \overline{x}\}.$$

Thus $\mathbb{F}_0$ is a subfield of $\mathbb{F}$ with $\dim_{\mathbb{F}_0}(\mathbb{F}) = 2$, where we consider $\mathbb{F}$ as a vector space over $\mathbb{F}_0$. Furthermore the group homomorphisms

$$\mathrm{tr} : \mathbb{F} \to \mathbb{F}_0 : x \mapsto x + \sigma(x) \text{ and } N : \mathbb{F}^\times \to \mathbb{F}_0^\times : x \mapsto x \cdot \sigma(x)$$

are called trace and norm, respectively. Since tr is an $\mathbb{F}_0$-linear map and not every element s of $\mathbb{F}$ maps under tr to zero, the trace maps $\mathbb{F}$ onto $\mathbb{F}_0$. In this context it follows immediately that any non-degenerate subspace U of V with respect to the σ-hermitian form β, contains some non-degenerate one-dimensional subspace $p \subseteq U$. Indeed, suppose for all vectors $u \in U$ we have $\beta(u, u) = 0$, then we fix two non-zero vectors u and w such that $\beta(u, w) \neq 0$, which is possible as U is a regular subspace of V. We have $0 = \beta(u + \lambda w, u + \lambda w) = \sigma(\lambda)\beta(u, w) + \lambda\sigma(\beta(w, u))$ for all $\lambda \in \mathbb{F}$ implying $\frac{\beta(w,u)}{\beta(u,w)} = -1$. Therefore $\sigma(\lambda) = \lambda$ for all $\lambda \in \mathbb{F}$, a contradiction to the fact that $\dim_{\mathbb{F}_0}(\mathbb{F}) = 2$.

Next we choose a regular vector w_1 in the finite dimensional subspace W, then $W = \langle w_1\rangle \oplus \langle w_1\rangle^\pi$ and by induction we have $W = \langle w_1\rangle \oplus \ldots \oplus \langle w_m\rangle$, where the vectors $w_1, \ldots, w_m$ are regular and mutually orthogonal. Thus with respect to an orthogonal basis $\delta : w_1, \ldots, w_m, r_1, \ldots, r_{n-m}$ of V, where $\langle r_1, \ldots, r_{n-m}\rangle = \mathrm{rad}(V)$, the Gram matrix of β with respect to δ is of the form

$$G_\delta^\beta = \mathrm{diag}(\beta(w_1, w_1), \ldots, \beta(w_m, w_m), 0, \ldots, 0).$$

Recall, that for each vector $v \in V$ the scalar $\beta(v, v)$ is in $\mathbb{F}_0$, as $\beta(v, v) = \sigma(\beta(v, v))$, cf. theorem 1.1.5 hermitian case. Let $\alpha_1, \dots, \alpha_p$ be a transversal of the coset partition $\mathbb{F}_0^\times / \operatorname{im}(N)$, where the subgroup $\operatorname{im}(N)$ is the image of the norm map. Then for each index $i \in \{1, \dots, m\}$ we can scale the basis vector w_i of the basis δ in the following sense. As $\beta(w_i, w_i) = \lambda\sigma(\lambda)\alpha_j$ for a unique $j \in \{1, \dots, p\}$ and $\lambda\sigma(\lambda) \in \operatorname{im}(N)$ we set $u_i = \frac{w_1}{\lambda}$ and determine that $\beta(u_i, u_i) = \lambda\sigma(\lambda)\beta(w_i, w_i) = \alpha_j =: \alpha_{j_i}$. Therefore with respect to the scaled orthogonal basis $\delta_{\text{scaled}} : u_1, \dots, u_m, r_1, \dots, r_{n-m}$ of V, the Gram matrix of β with respect to δ_{scaled} is

$$G^{\beta}_{\delta_{\text{scaled}}} = \operatorname{diag}(\alpha_{j_1}, \dots, \alpha_{j_m}, 0, \dots, 0).$$

Thus we have the following general result.

Proposition 1.1.6 (Sylvester) *Let V be an n-dimensional vector space over a field $\mathbb{F}$ and β a σ-hermitian form on V for an anti-automorphism σ on $\mathbb{F}$. Then there exists an orthogonal basis $\delta : v_1, \dots, v_n$ of V such that $\beta(v_i, v_i) = \alpha_{j_i} \in \{\alpha_1, \dots, \alpha_p\}$ for $1 \le i \le m \le n$, where $\alpha_1, \dots, \alpha_m$ is a transversal of $\mathbb{F}_0^\times / \operatorname{im}(N)$, $\beta(v_i, v_i) = 0$ for $m + 1 \le i \le n$ and the Gram matrix is*

$$G^{\beta}_{\delta} = \operatorname{diag}(\alpha_{j_1}, \dots, \alpha_{j_m}, 0, \dots, 0).$$

Furthermore, the number n_0 of zero-diagonal entries in G^{β}_{δ} does not depend on the choice of δ.

We obtain a similar result for an orthogonal form b on an n-dimensional vector space V over a field $\mathbb{F}$ with $\operatorname{char}(\mathbb{F}) \neq 2$ via investigation of $\mathbb{F}^\times / (\mathbb{F}^\times)^2$, cf. [71].

A better result is possible if $\mathbb{F}$ is an ordered field. Two $n \times n$ matrices A and B with entries in $\mathbb{F}$ are cogredient if there exists an invertible $n \times n$ matrix P with entries in $\mathbb{F}$ such that $A = PBP^T$.

Theorem 1.1.7 (Sylvester, theorem 6.7 of [58]) *Let $\mathbb{F}$ be an ordered field and suppose the diagonal matrices*

$$D = \operatorname{diag}(b_1, \dots, b_r, 0 \dots, 0) \text{ with } b_i \neq 0$$

$$M = \operatorname{diag}(\hat{b}_1, \dots, \hat{b}_r, 0 \dots, 0) \text{ with } \hat{b}_i \neq 0$$

are called **cogredient**. *Then the number of positive b_i is the same as the number of positive $\hat{b}_i$.*

We have a classification of anisotropic resp. non-degenerate σ-hermitian forms of V over $\mathbb{F}$, if $\mathbb{F}$ is a finite field or $\mathbb{F} = \mathbb{C}$.

First let $\mathbb{F}$ be a finite field. Then there is a unique anisotropic σ-hermitian form of the n-dimensional vector space V over $\mathbb{F} = \mathbb{F}_{q^2}$ if and only if $n \leq 1$, see Theorem 6.3.4 of [20]. Let σ be an automorphism of order two, so $\sigma^2 = \mathrm{id}_{\mathbb{F}}$ and $\mathbb{F} = \mathbb{F}_{q^2}$ for some prime power q, implying σ is the field automorphism $a \mapsto a^q = \overline{a}$ by § 12, chapter V of [13]. Moreover the norm map N is surjective, since the multiplicative group of a finite field is cyclic and $\ker(N) = \{a \in \mathbb{F}^\times \mid a^{q+1} = 1\}$. Therefore $\mathrm{im}(N) = \mathbb{F}_0^\times$ and $\mathbb{F}_0^\times / \mathrm{im}(N) = \{1 \cdot \mathrm{im}(N)\}$. Under these conditions, proposition 1.1.6 implies the next proposition.

Proposition 1.1.8 *Let β be a non-degenerate σ-hermitian form of an n-dimensional vector space V over a finite field $\mathbb{F}_{q^2}$. Then there exists an orthonormal basis δ of V such that $G_\delta^\beta = I$, and furthermore all non-degenerate σ-hermitian forms of V over $\mathbb{F}_{q^2}$ are isometric.*

For $\mathbb{F} = \mathbb{C}$ we have the following results. Let σ be the complex conjugation of the complex numbers, then $N(\mathbb{C}^\times) = \mathrm{im}(N) = \mathbb{R}^+ = \{r \in \mathbb{R} \mid r > 0\}$, thus $\mathbb{R}^\times / \mathrm{im}(N) = \{1\mathbb{R}^+, -1\mathbb{R}^+\}$. By proposition 1.1.6 and theorem 1.1.7 we obtain the following classification.

Proposition 1.1.9 *The only anisotropic σ-hermitian forms of a finite dimensional vector space V over $\mathbb{C}$ are positive or negative definite. Thus either $G_\delta^\beta = I$ or $G_\delta^\beta = -I$ for some suitable orthonormal basis δ of V.*

A non-degenerate σ-hermitian form of an n-dimensional vector space V over $\mathbb{C}$ has the Gram matrix

$$G_\delta^\beta = \begin{pmatrix} I & \\ & -I \end{pmatrix}$$

with respect to an orthonormal basis $\delta : v_1, \ldots, v_p, w_1, \ldots, w_{n-p}$ of V, where the number of positive diagonal entries is p and the number of negative diagonal entries is $n - p$.

Graph theory

In the next little part we will just collect some notions of graph theory which we will use later on.

A **graph** Γ is a set $\mathcal{V} = \mathcal{V}(\Gamma)$ together with a distinguished collection $E = E(\Gamma) \subseteq \mathcal{V} \times \mathcal{V}$ of (unordered) pairs of elements of $\mathcal{V}$. The elements of $\mathcal{V}$ are the **vertices** of the graph Γ and the elements of E are its **edges**. A graph with vertex set $\mathcal{V}$ is said to be a graph on $\mathcal{V}$. Thus a graph is completely described by specifying the pair $(\mathcal{V}, E)$. Two vertices x and y of Γ are **adjacent** or **neighbours** if $\{x, y\}$ is an edge of Γ. Let $\Gamma = (\mathcal{V}, E)$ and $\Gamma' = (\mathcal{V}', E')$ be two graphs. We call the both graphs

Γ and Γ' **isomorphic** and write $\Gamma \cong \Gamma'$ if there is a bijection $\varphi : \mathcal{V} \to \mathcal{V}'$ with $\{x, y\} \in E$ if and only if $\{\varphi(x), \varphi(y)\} \in E'$. Furthermore let $\Gamma = (\mathcal{V}, E)$ be a graph and $\mathcal{X} \subseteq \mathcal{V}$, then we define $E_{\mathcal{X}}$ to be the collection of all edges $\{x, y\} \in E$ with $x, y \in \mathcal{X}$. A graph $\Gamma' = (\mathcal{X}, E')$ is a **subgraph** of Γ if and only if $\mathcal{X} \subseteq \mathcal{V}$ and $E' \subseteq E_{\mathcal{X}}$. A subgraph $\Gamma' = (\mathcal{X}, E')$ is called an **induced subgraph** if and only if $E' = E_{\mathcal{X}}$, that is, any adjacency among vertices of $\mathcal{X}$ is already represented by an edge in E'. Thus induced subgraphs are completely determined by their set of vertices. For that reason, we will often denote induced subgraphs by their vertex set. Let $\Gamma_i = (\mathcal{X}_i, E_i)$ for $i = 1, 2$ be two subgraphs of the graph $\Gamma = (\mathcal{V}, E)$. The **intersection** of these two subgraphs is the subgraph $\Gamma_1 \cap \Gamma_2 = (\mathcal{X}_1 \cap \mathcal{X}_2, E_1 \cap E_2)$. Certainly if $\Gamma_i = (\mathcal{X}_i, E_i = E_{\mathcal{X}_i}) = \mathcal{X}_i$ is an induced subgraph for $i = 1, 2$ then $\Gamma_1 \cap \Gamma_2 = \mathcal{X}_1 \cap \mathcal{X}_2$ is the induced subgraph on $\mathcal{X}_1 \cap \mathcal{X}_2$.

A **path** in a graph $\Gamma = (\mathcal{V}, E)$ is a finite sequence of vertices

$$\gamma = (v_0, v_1, \cdots, v_n)$$

such that $(v_i, v_{i+1}) \in E$ for $0 \le i \le n-1$. The length of the path γ is the natural number n. Two vertices x and y of Γ are **connected** if there is a path $\gamma = (v_0, \cdots, v_n)$ in Γ with $v_0 = x$ and $y = v_n$. The **distance** between two connected vertices x and y of Γ is the length of the shortest path between x and y and denoted by $d_\Gamma(x, y)$. A graph $\Gamma = (\mathcal{V}, E)$ is **connected** if any two different vertices of Γ are connected by a path in Γ. For a connected graph Γ we define the **diameter** of Γ to be the supremum of the numbers $\{d_\Gamma(x, y) \mid x, y \in \mathcal{V}\}$.

Moreover for each vertex $x \in \mathcal{V}$ of a graph $\Gamma(\mathcal{V}, E)$ we have the induced subgraph Γ_x, also called the **neighbourhood graph** of x, on the vertices which are adjacent to x, hence $\{y \in \mathcal{V} \mid x \text{ is adjacent to } y\} = \Gamma_x = \mathcal{V}_x$. For a subset $\mathcal{X} \subseteq \mathcal{V}$ the induced subgraph on the common neighbours of $\mathcal{X}$ is the subgraph $\Gamma_{\mathcal{X}} = \bigcap_{x \in \mathcal{X}} \Gamma_x$. Let Σ be some graph. We define a graph Γ **locally** Σ if for each vertex x of Γ the graph Γ_x is isomorphic to Σ.

The present thesis

In this thesis we will study on one hand the graphs which are locally the line graph of an n-dimensional unitary vector space of $\mathbb{C}$ with respect to the scalar product $(\cdot, \cdot)$ for $n \ge 6$. On the other hand we consider all graphs which are locally the hyperbolic line graph of an n-dimensional unitary vector space of $\mathbb{F}_{q^2}$ with respect to an hermitian form $(\cdot, \cdot)$ for $n \ge 7$. In each case we will describe and classify all possibilities for a connected graph Γ, which is either locally $\mathbf{S}(V_n)$ for $n \ge 6$ or locally $\mathbf{G}(U_n)$ for $n \ge 7$.

Definition 1.1.10 Let $n \in \mathbb{N}$, let $V = V_n$ be an n-dimensional vector space over the complex numbers and let $(\cdot, \cdot)$ be an anisotropic form (the scalar product or the negative of the scalar product) on $V \times V$. For a subspace $U \subseteq V$ the polar of U is $U^\pi = \{x \in V \mid (x, u) = 0 \text{ for all } u \in U\}$. The **line graph** $\mathbf{S}(V_n)$ of the complex vector space V_n is the graph on the two-dimensional subspaces of V_n, where two distinct lines l and k of V_n are adjacent, in symbols $k \perp l$, if and only if $l \subseteq k^\pi$ or, equivalently, if $k \subseteq l^\pi$.

Let $U_n = U$ denote an n-dimensional vector space over $\mathbb{F}_{q^2}$ endowed with a non-degenerate hermitian form $(\cdot,\cdot)$. Certainly for a subspace $W \subseteq U$ the orthogonal space of W is $W^\pi = \{x \in U \mid (x, w) = 0 \text{ for all } w \in W\}$. The **hyperbolic line graph** $\mathbf{G}(U_n)$ is the graph on the hyperbolic lines, i.e., the non-degenerate two-dimensional subspaces, of U_n, where hyperbolic lines l and m are adjacent, in symbols $l \perp m$, if and only if l is perpendicular to m with respect to the unitary form.

Now let x be a vertex of $\mathbf{S}(V_n)$, the local graph $\mathbf{S}(V_n)_x =: x^\perp$ is the subgraph induced by $\mathbf{S}(V_n)$ on the set of vertices $\{y \in \mathbf{S}(V_n) \mid x \perp y\}$, the neighbours of x in the graph $\mathbf{S}(V_n)$. For a set of vertices X of $\mathbf{S}(V_n)$ the graph $X^\perp$ is defined as $\bigcap_{x \in X} x^\perp$.

We will use the same notation for the hyperbolic line graph $\mathbf{G}(U_n)$. Therefore for a vertex x in $\mathbf{G}(U_n)$ we denote the neighbourhood graph of x with $\mathbf{G}(U_n)_x = x^\perp$ and for a set of vertices X of $\mathbf{G}(U_n)$ we define $\mathbf{G}(U_n)_X = \bigcap_{x \in X} x^\perp = X^\perp$.

Fundamental $\mathrm{SL}_2(\mathbb{F})$ subgroups and fundamental $\mathrm{SU}_2(\mathbb{F})$ subgroups

In the last part of this introduction we will define the notion of a **fundamental** $\mathrm{SL}_2(\mathbb{F})$ **subgroup** of the group $\mathrm{SL}_n(\mathbb{F})$ respectively a **fundamental** $\mathrm{SU}_2(\mathbb{F})$ **subgroup** of the group $\mathrm{SU}_n(\mathbb{F})$.

We consider the n-dimensional vector space V over $\mathbb{F}$ and let $\mathrm{GL}(V)$ be all $\mathbb{F}$-linear invertible endomorphisms of V, the **general linear group** of V. Certainly $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{F})$, the group of all invertible $n \times n$ matrices with entries in $\mathbb{F}$. The determinant map from $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{F})$ to the multiplicative group $\mathbb{F}^\times$ is a group homomorphism onto $\mathbb{F}^\times$. The kernel of this map is the group $\mathrm{SL}(V)$, consisting of all invertible automorphism of V of determinant one, which is isomorphic to $\mathrm{SL}_n(\mathbb{F})$, where $\mathrm{SL}_n(\mathbb{F})$ is the group of all invertible $n \times n$ matrices of determinant one. We call $\mathrm{SL}(V)$ the **special linear group** of V. Certainly the groups $\mathrm{GL}(V)$ and $\mathrm{SL}(V)$ act naturally on the vector space V. For $g \in \mathrm{GL}(V)$, we set $[g, V] = \{gv - v \mid v \in V\}$ and $C_V(g) = \{v \in V \mid gv = v\}$. The subspace $[g, V]$ is the **centre** of g and $C_V(g)$ is its **axis** if $\dim([g, V]) = 1$. For a subgroup $U \subseteq \mathrm{GL}(V)$ we define the **commutator** to be

$$[U, V] = \{gv - v \mid g \in U, v \in V\}$$

and the **centraliser** to be the subspace

$$C_V(U) = \{v \in V \mid gv = v \text{ for all } g \in U\}.$$

A **fundamental** $\mathrm{SL}_2(\mathbb{F})$ **subgroup** of $\mathrm{SL}(V)$ is a subgroup F of $\mathrm{SL}(V)$ isomorphic to $\mathrm{SL}_2(\mathbb{F})$ such that $\dim([F,V]) = 2$ and $\dim(C_V(F)) = n-2$. There is a one-to-one correspondence between the non-intersecting line-hyperline pairs (l,L) of the vector space V and the fundamental $\mathrm{SL}_2(\mathbb{F})$ subgroups of $\mathrm{SL}(V)$. Indeed the mapping $F \mapsto ([F,V], C_V(F))$, where F is a fundamental $SL_2(\mathbb{F})$ subgroup of $\mathrm{SL}(V)$ describes the claimed correspondence.

Let β be a σ-hermitian form on the vector space V, so $\beta(u,v) = \sigma(\beta(v,u))$ for all $u,v \in V$. The **general unitary group** $\mathrm{GU}(V)$ consists of all linear transformation $\varphi \in \mathrm{GL}(V)$ such that $\beta(\varphi(v),\varphi(u)) = \lambda\beta(v,u)$ for some $\lambda \in \mathbb{F}_0$. The elements $\varphi \in \mathrm{GL}(V)$ with $\beta(\varphi(v),\varphi(u)) = \beta(v,u)$ form the **unitary group** $\mathrm{U}(V)$ of V. This group is isomorphic to $\mathrm{U}_n(\mathbb{F}) = \{M \in \mathrm{GL}_n(\mathbb{F}) \mid \beta(M(v),M(u)) = \beta(v,u) \text{ for all } u,v \in V\}$. The **special unitary group** $\mathrm{SU}(V)$ of V is defined to be the kernel of the surjective group homomorphism $\varphi \mapsto \det(\varphi)$ between $U(V)$ and the group $\{a \in \mathbb{F}^\times \mid a\sigma(a) = 1\}$, so $\mathrm{SU}(V) = U(V) \cap \mathrm{SL}(V) = \{\varphi \in \mathrm{U}(V) \mid \det(\varphi) = 1\} \cong \mathrm{SU}_n(\mathbb{F}) = \{M \in \mathrm{U}_n(\mathbb{F}) \mid \det(M) = 1\}$. A **fundamental** $\mathrm{SU}_2(\mathbb{F})$ **subgroup** of $\mathrm{SU}(V)$ is a subgroup F of $\mathrm{SU}(V)$ isomorphic to $\mathrm{SU}_2(\mathbb{F})$ such that $\dim([F,V]) = 2$ and $\dim(C_V(F)) = n-2$. Here we have a one-to-one correspondence between the non-degenerate lines of the vector space V with respect to β and the fundamental $\mathrm{SU}_2(\mathbb{F})$ subgroups of $\mathrm{SU}(V)$. The map $F \mapsto [F,V]$, where F is a fundamental $\mathrm{SU}_2(\mathbb{F})$ subgroup of $\mathrm{SU}(V)$ determines this bijective correspondences.

The **fundamental** $\mathrm{SU}_2(\mathbb{F})$ **subgroups graph** $\mathbf{F}(\mathrm{SU}(V)) = \mathbf{F}(V)$ is the graph on the fundamental $\mathrm{SU}_2(\mathbb{F})$ subgroups of $\mathrm{SU}(V)$, where two different fundamental $\mathrm{SU}_2(\mathbb{F})$ subgroups F and H are adjacent if and only if F and H commute in $\mathrm{SU}(V)$, so $[F,H] = 1$.

Proposition 1.1.11 *Let V_n be an n-dimensional vector space over $\mathbb{F}$ with a non-degenerate reflexive σ-sesquilinear from β. Then the line graph $\mathbf{S}(V_n)$ and the fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups graph $\mathbf{F}(V_n)$ are isomorphic if $\mathbb{F} = \mathbb{C}$ and β is an anisotropic hermitian form on V_n.*

Also the hyperbolic line graph $\mathbf{G}(V_n)$ and the fundamental $\mathrm{SU}_2(\mathbb{F}_{q^2})$ subgroups graph $\mathbf{F}(V_n)$ are isomorphic if $\mathbb{F} = \mathbb{F}_{q^2}$ and β is a non-degenerate hermitian form on V_n.

Proof: See proposition 4.7.30 and the pages 176, 178. ■

CHAPTER TWO

On the complex unitary geometry for $n \geq 7$

2.1 Local recognition of the line graph of complex unitary space for $n \geq 7$

In this chapter we focus on the line graph of an n-dimensional unitary vector space over $\mathbb{C}$. Thus we recall the definition of the graph $\mathbf{S}(V_n)$ for $n \in \mathbb{N}$.

Definition 2.1.1 Let $n \in \mathbb{N}$, let $V = V_n$ be an n-dimensional vector space over the complex numbers and let $(\cdot,\cdot)$ be an anisotropic form (the scalar product or the negative of the scalar product) on $V \times V$. For a subspace $U \subseteq V$ the polar of U is $U^\pi = \{x \in V \mid (x,u) = 0 \text{ for all } u \in U\}$. The **line graph** $\mathbf{S}(V_n)$ of the complex vector space V_n is the graph on the two-dimensional subspaces of V_n, the lines of V_n, where two distinct lines l and k of V_n are adjacent, in symbols $k \perp l$, if and only if $l \subseteq k^\pi$ or, equivalently, if $k \subseteq l^\pi$.

For a vertex x of $\mathbf{S}(V_n)$, the neighbourhood graph $\mathbf{S}(V_n)_x = x^\perp$ is the subgraph induced by $\mathbf{S}(V_n)$ on the set of vertices $\{y \in \mathbf{S}(V_n) \mid x \perp y\}$. For a set of vertices X of $\mathbf{S}(V_n)$ the graph $X^\perp$ is defined as $\bigcap_{x \in X} x^\perp$.

For $n \geq 5$ it is possible to reconstruct the space $\mathbb{P}(V)$ and the unitary vector space V_n over $\mathbb{C}$ from a graph Σ isomorphic to the line graph $\mathbf{S}(V_n)$. From this reconstruction we will obtain the automorphism group of $\mathbf{S}(V_n)$. Later on we study connected graphs Γ, which are locally $\mathbf{S}(V_n)$ for $n \geq 7$. The main result of this part is the following local recognition theorem.

Theorem 2.1.2 *Let $n \geq 7$ and let Γ be a connected locally $\mathbf{S}(V_n)$ graph. Then Γ is isomorphic to $\mathbf{S}(V_{n+2})$.*

This result depends on the computation of the diameter of a connected locally $\mathbf{S}(V_n)$ graph Γ for $n \geq 7$. This statement is optimal in the following sense. The graph $\mathbf{S}(V_8)$ is a connected locally $\mathbf{S}(V_6)$ graph but also the graph on the fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups of the compact real form ${}^2E_6(\mathbb{C}) := E_{6,-78}$ of the group $E_6(\mathbb{C})$, see §4, chapter IX of [11] or proposition 7.18 in chapter 3 of [91]. In chapter 4 we will classify all connected locally $\mathbf{S}(V_6)$ graphs.

From a group theoretical point of view, theorem 2.1.2 implies a local recognition of $\mathrm{PGU}_{n+2}(\mathbb{C})$:

Theorem 2.1.3 *Let $n \geq 7$ and G be a group with subgroups A and B isomorphic to $\mathrm{SU}_2(\mathbb{C})$, and denote the central involution of A by x and the central involution of B by y. Moreover, we assume that the following is satisfied:*

- *$C_G(x) = X \times K$ with $K \cong \mathrm{GU}_n(\mathbb{C})$ and $A \leq X$;*
- *$C_G(y) = Y \times J$ with $J \cong \mathrm{GU}_n(\mathbb{C})$ and $B \leq Y$;*
- *A is a fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroup of J;*
- *B is a fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroup of K;*
- *the subgroup $J \cap K$ contains a central involution z of a fundamental $\mathrm{SU}_2(\mathbb{C})$ group of both J and K.*

If $G = \langle J, K \rangle$ then (up to isomorphism) $\mathrm{PSU}_{n+2}(\mathbb{C}) \leq G/Z(G) \leq \mathrm{PGU}_{n+2}(\mathbb{C})$.

Theorem 2.1.3 is deduced from theorem 2.1.2 in exactly the same way as its respective counterparts in [22] and [38].

2.2 The line graph of the unitary vector space V_5

In section 2.5 we will focus on connected graphs Γ, which are locally $\mathbf{S}(V_7)$. To obtain control over the intersection of two induced subgraphs $x^\perp$ and $y^\perp$ inside Γ for two adjacent vertices x and y, we need to study the line graph $\mathbf{S}(V_5)$ of the five-dimensional unitary vector space V_5 over $\mathbb{C}$. Our intention is to reconstruct the vector space V_5 from the graph $\mathbf{S}(V_5)$. More precisely, we will construct a point-line geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \supset)$ from $\mathbf{S}(V_5)$ which is isomorphic to the geometry on points and lines of V_5 along with its natural anisotropic polarity.

First we determine the diameter of $\mathbf{S}(V_5)$. Any two different vertices l and m of $\mathbf{S}(V_5)$ can have distance one, i.e., they are adjacent, distance two, three or four. For each case we describe precisely in which configuration the lines l and m are in the vector space V_5.

Observation 2.2.1 Let l and m be two lines of V_5. Then l and m have distance one in the graph $\mathbf{S}(V_5)$ if and only if $l \subseteq m^\pi$.

Certainly this is only another formulation of the definition of the adjacency relation for the line graph $\mathbf{S}(V_5)$.

Lemma 2.2.2 *Let l and m be two lines of V_5. Then l and m have distance two in $\mathbf{S}(V_5)$ if and only if $\langle l, m\rangle$ is a three-dimensional subspace, a plane, in V_5.*

Proof: Let l and m be two lines of V_5 which have distance two in $\mathbf{S}(V_5)$. Therefore we find a vertex z of $\mathbf{S}(V_5)$, which is a line in V_5, adjacent to l and m. The orthogonal space z^π is a three-dimensional space, a plane of V_5, and contains the lines l and m. Since l and m are at distance two, they are different lines in V_5 implying that $\langle l, m\rangle = z^\pi$.

Now we assume that the subspace $\langle l, m\rangle$ is a plane in V_5. Since V_5 is a vector space of dimension five, the orthogonal space $\langle l, m\rangle^\pi = h$ is a line in V_5. So h is a vertex in $\mathbf{S}(V_5)$ adjacent to the vertices l and m, which shows that the distance between l and m is at most two. The fact that $\langle l, m\rangle^\pi$ is a two-dimensional subspace forces that $m \nsubseteq l^\pi$. Thus l and m have distance at least two. ∎

Lemma 2.2.3 *Let l and m be two lines of V_5. Then l and m have distance three in $\mathbf{S}(V_5)$ if and only if l and m are two non-intersecting non-perpendicular lines in V_5 such that $l^\pi \cap m$ is a point of V_5.*

Proof: Suppose the vertices l and m have distance three in $\mathbf{S}(V_5)$, then by observation 2.2.1 and lemma 2.2.2 we obtain that $m \nsubseteq l^\pi$ and the lines l and m do not intersect each other in V_5. Moreover we find a chain $l \perp z \perp y \perp m$ in the graph $\mathbf{S}(V_5)$. By the statement of lemma 2.2.2 the lines z and m contain a common point $d = z \cap m$ and by observation 2.2.1 the line z is contained in the orthogonal space of l, hence $z \subseteq l^\pi$, which implies that d is a point of the subspace $m \cap l^\pi$. In particular, the intersection of the subspaces l^π and m is the point d.

Conversely let l and m be two non-intersecting non-perpendicular lines in V_5 satisfying the condition that $l^\pi \cap m$ is a point d. By observation 2.2.1 and lemma 2.2.2 the vertices l and m do not have distance one or two in the graph $\mathbf{S}(V_5)$. To prove the statement, we will identify two different lines z and y in the vector space V_5 such that $l \perp z \perp y \perp m$ in $\mathbf{S}(V_5)$. Therefore we consider the orthogonal space l^π of l, the point $d = l^\pi \cap m$ and choose any line z in l^π incident with the point d. The lines z and m are different and intersect in the point d, thus by lemma 2.2.2

the vertices z and m have distance two in the graph $\mathbf{S}(V_5)$. Therefore we can pick a vertex y in $\mathbf{S}(V_5)$ with the property that $z \perp y \perp m$. Clearly, the line y is different from the line l and we are done. ∎

Lemma 2.2.4 *Let l and m be two lines of V_5. Then l and m have distance four in $\mathbf{S}(V_5)$ if and only if l and m are two different non-intersecting non-perpendicular lines with the property that $l^\pi \cap m$ is empty.*

Proof: By the three previous lemmata the vertices l and m have at least distance four in the graph $\mathbf{S}(V_5)$. To prove the statement it is enough to show that the graph $\mathbf{S}(V_5)$ contains a path of length four between the vertex l and the vertex m.

Fix a point d in the subspace $l^\pi \cap m^\pi$, which is at least of dimension one in V_5 and choose a line z in l^π containing the point d. Certainly the vertices z and l are adjacent in the graph $\mathbf{S}(V_5)$, moreover the vertices m and z have distance three in $\mathbf{S}(V_5)$ by lemma 2.2.3 and the fact that $z \cap m^\pi$ is the point a. Thus there is path of length four between l and m in $\mathbf{S}(V_5)$. ∎

Proposition 2.2.5 *The line graph $\mathbf{S}(V_5)$ is connected and its diameter is four. Moreover $\mathbf{S}(V_5)$ is locally the line graph $\mathbf{S}(V_3)$.*

Proof: The first statement is immediate from observation 2.2.1 and lemma 2.2.2 to lemma 2.2.4 since the line graph $\mathbf{S}(V_5)$ contains vertices l and m at distance one to four. To show the second statement we fix any vertex l of $\mathbf{S}(V_5)$ which is the line l in V_5. The set of points which are contained in l^π span a three-dimensional subspace U of V_5 and each line m which is perpendicular to l is contained in U, in particular the graph $\mathbf{S}(V_5)$ is locally the line graph of V_3. ∎

Our goal is to reconstruct the vector space V_5 from the graph $\mathbf{S}(V_5)$. Therefore we will describe the line graph $\mathbf{S}(V_5)$ in more detail to establish properties of $\mathbf{S}(V_5)$ that can be used to achieve our goal. In particular we will investigate how subspaces of V_5 correlate to the induced subgraphs $X^{\perp\perp} = \{X^\perp\}^\perp$ of certain sets X of vertices of the graph $\mathbf{S}(V_5)$.

Definition 2.2.6 Let U be a subspace of the unitary vector space V_5. The set of all lines in U is denoted by $\mathbf{L}(U)$.

Lemma 2.2.7 *Let l and m be two distinct vertices of $\mathbf{S}(V_5)$ with $\{l, m\}^\perp \neq \emptyset$. Then any vertex contained in the graph $\{l, m\}^{\perp\perp}$ is contained as a line in $\mathbf{L}(\langle l, m\rangle)$ and vice versa.*

Proof: Since $\{l, m\}^{\perp\perp} = (\{l, m\}^\perp)^\perp = \bigcap_{z\in\{l,m\}^\perp} z^\perp = \bigcap_{z\in\{l,m\}^\perp} \mathbf{L}(z^\pi)$ and since for every $z \in \{l, m\}^\perp$ it follows from the definition that $\mathbf{L}(\langle l, m\rangle) \subseteq \mathbf{L}(z^\pi)$, we get that each line of $\mathbf{L}(\langle l, m\rangle)$ is contained as a vertex in the set $\{l, m\}^{\perp\perp}$.

Conversely, let p be a point not contained in the subspace spanned by the lines l and m whence $\langle l, m\rangle^\pi \nsubseteq p^\pi$. Since $\dim(\langle l, m\rangle^\pi \cap p^\pi) \geq 1$ we fix a point $s \in \langle l, m\rangle^\pi \cap p^\pi$. Now let t be an one-dimensional subspace of $\langle l, m\rangle^\pi$ which is not contained in the subspace p^π and regard the line $h = \langle s, t\rangle$. The line h is not a subspace of $\langle l, m\rangle^\pi \cap p^\pi$ but a line in $\langle l, m\rangle^\pi$, therefore $h \in \{l, m\}^\perp$. The statement is proved since for each point p not contained in $\langle l, m\rangle$ we can fix an element h of $\mathbf{L}(\langle l, m\rangle^\pi)$ such that $p \notin h^\pi$ implying that no line through p is adjacent to h. ■

A similar statement can be proved about three different vertices under some additional premises.

Lemma 2.2.8 *Let k, l and m be three distinct vertices of $\mathbf{S}(V_5)$. If the lines k, l and m intersect in a common point in V_5 and $\{k, l, m\}^\perp$ is not empty then $\mathbf{L}(\langle k, l, m\rangle) = \{k, l, m\}^{\perp\perp}$.*

Proof: Since the lines k, l and m intersect in a common point in V_5 the subspace $\langle k, l, m\rangle$ has dimension three or dimension four.

Suppose $\dim(\langle k, l, m\rangle) = 3$ then the line m is contained in $\langle k, l\rangle$. Using that $\{k, l, m\}^\perp$ is not empty it follows also that $\{k, l\}^\perp \neq \varnothing$ and by lemma 2.2.7 we observe $\mathbf{L}(\langle k, l, m\rangle) = \mathbf{L}(\langle k, l\rangle) = \{k, l\}^{\perp\perp}$. The equality between $\{k, l\}^{\perp\perp}$ and $\{k, l, m\}^{\perp\perp}$ is obtained from the following identities: $\{k, l, m\}^{\perp\perp} = (\{k, l, m\}^\perp)^\perp = \bigcap_{z\in\{k,l,m\}^\perp} z^\perp = \bigcap_{z\in\{k\}^\perp\cap\{l\}^\perp\cap\{m\}^\perp} z^\perp = \bigcap_{z\in\mathbf{L}(k^\pi)\cap\mathbf{L}(l^\pi)\cap\mathbf{L}(m^\pi)} z^\perp = \bigcap_{z\in\mathbf{L}(\langle k,l,m\rangle^\pi)} z^\perp = \bigcap_{z\in\mathbf{L}(\langle k,l\rangle^\pi)} z^\perp = \bigcap_{z\in\{k,l\}^\perp} z^\perp = (\{k, l\}^\perp)^\perp = \{k, l\}^{\perp\perp}$. If $\dim(\langle k, l, m\rangle) = 4$, then $\langle k, l, m\rangle^\pi$ is a one-dimensional subspace of the vector space V_5 and therefore the induced subgraph $\{k, l, m\}^\perp$ is empty, so we are done. ■

Lemma 2.2.9 *Let k, l and m be three distinct vertices in $\mathbf{S}(V_5)$ then $\{k, l, m\}^\perp = \{k, l, m\}^{\perp\perp\perp}$.*

Proof: In the case that $\{k, l, m\}^\perp$ is empty, then by definition $\{k, l, m\}^{\perp\perp} = \varnothing^\perp = \mathbf{S}(V_5)$. Thus $\{k, l, m\}^{\perp\perp\perp} = \mathbf{S}(V_5)^\perp = \varnothing = \{k, l, m\}^\perp$.

Alternatively if $\{k, l, m\}^\perp$ is not empty, then by lemma 2.2.8 we have equality between the induced subgraph $\{k, l, m\}^{\perp\perp}$ in $\mathbf{S}(V_5)$ and all lines contained in the subspace $\langle k, l, m\rangle$ of V_5. That leads to the following containments: $\{k, l, m\}^{\perp\perp\perp} = (\{k, l, m\}^{\perp\perp})^\perp = \bigcap_{z\in\{k,l,m\}^{\perp\perp}} z^\perp = \bigcap_{z\in\mathbf{L}(\langle k,l,m\rangle)} z^\perp \subseteq \bigcap_{z\in\{k,l,m\}} z^\perp = k^\perp \cap l^\perp \cap m^\perp = \{k, l, m\}^\perp$. The other direction is given directly from the definition, since $\{k, l, m\}^\perp = \{z \in \Gamma \mid z \perp k, z \perp l, z \perp m\} \subseteq \{z \in \Gamma \mid z \perp k, z \perp l, z \perp m\}^{\perp\perp} = \{k, l, m\}^{\perp\perp\perp}$. ■

If we can identify all points and lines of the unitary vector space V_5 in terms of $\mathbf{S}(V_5)$ as well as their incidence relation to each other, then a reconstruction of V_5 from the graph $\mathbf{S}(V_5)$ is possible. Certainly we have an obvious one-to-one correspondence between the lines of V_5 and the vertices of $\mathbf{S}(V_5)$, thus the idea is to recover points of the unitary space V_5 as pencils of the lines. Hence we have to determine under which conditions two different lines intersect in a point.

Lemma 2.2.10 *Two distinct lines l and m intersect in a common point in the unitary space V_5 if and only if the graph $\{l, m\}^{\perp}$ is not empty.*

Proof: The claim follows directly from lemma 2.2.2. ■

Remark 2.2.11 If two distinct lines l and m intersect in a common point in the unitary space V_5 then the induced subgraph $\{l, m\}^{\perp\perp}$ of the graph $\mathbf{S}(V_5)$ is minimal with respect to inclusion (i.e. for any distinct vertices s_1, s_2 in $\{l, m\}^{\perp\perp}$ we have the equality $\{s_1, s_2\}^{\perp\perp} = \{l, m\}^{\perp\perp}$). This claim follows directly from lemma 2.2.7 and from lemma 2.2.10.

Obtaining every point of the unitary space V_5 as a pencil of lines, we need some condition to check in the graph $\mathbf{S}(V_5)$ whether three distinct lines of V_5 intersect in one point or not. We propose the following: Three different pairwise intersecting lines k_1, k_2 and k_3 of V_5 intersect in one point if we can find a line s in V_5 such that

- the line s intersects the line k_i, if $s \neq k_i$, for $1 \leq i \leq 3$,
- $\langle s, k_1, k_2 \rangle$ is a four-dimensional subspace in V_5.

Rephrasing these in terms of induced subgraphs yields the following conditions: Three different vertices k_1, k_2 and k_3 of $\mathbf{S}(V_5)$ with $\{k_i, k_j\}^{\perp} \neq \emptyset$, $1 \leq i \neq j \leq 3$, intersect in one point if we can find a vertex s of $\mathbf{S}(V_5)$ with the two properties:

- the induced subgraph $\{s, k_i\}^{\perp}$ is not empty, if $s \neq k_i$, for $i \in \{1, 2, 3\}$ (cf. lemma 2.2.10),
- the graph $\{s, k_1, k_2\}^{\perp}$ is empty (cf. lemma 2.2.8).

To verify that every point of V_5 can be realized by three pairwise intersecting lines as above, we only have to show that for any two distinct vertices k and l of $\mathbf{S}(V_5)$, we can find a vertex s of $\mathbf{S}(V_5)$ such that $\{s, k, l\}^{\perp} = \emptyset$ and $\{s, k\}^{\perp} \neq \emptyset \neq \{s, l\}^{\perp}$. That statement will be proved in the next lemma.

Lemma 2.2.12 *For any distinct intersecting lines k and l of V_5 there is a line s in V_5, intersecting the lines k and l and the subspace spanned by k, l and s is of dimension four.*

Proof: Let k and l be two distinct intersecting lines in V_5 and denote the intersection point with d. Next we choose a point p inside the two-dimensional subspace $\langle k, l \rangle^{\pi}$. Certainly the subspace generated from the two different points p and d is a line $s = \langle p, d \rangle$ of V_5. By construction the line s intersects the lines k and l in the point d. We are done if the subspace $\langle k, l, s \rangle$ of V_5 has dimension four. Since $p \notin \langle k, l \rangle$, $d \in \langle k, l \rangle$

and $\langle k, l \rangle$ is a three-dimensional subspace of V_5 it follows that $\langle k, l, s \rangle = \langle k, l, p \rangle$ is of dimension four. ■

Now we want define a geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \supset)$ consisting of the recovered points and the vertices of $\mathbf{S}(V_5)$.

Definition 2.2.13 Let Γ be a graph isomorphic to $\mathbf{S}(V_5)$. Two different vertices k and l of Γ are defined to **intersect** if the graph $\{k, l\}^\perp$ is not empty. Three different pairwise intersecting vertices k_1, k_2 and k_3 of Γ are defined to **intersect in one point** if the graph Γ contains a vertex s with the following properties:

- the vertex s intersects each vertex k_i, if $s \neq k_i$, for $1 \leq i \leq 3$,
- the induced subgraph $\{k_1, k_2, s\}^\perp$ of Γ is empty.

An **interior point** of the graph Γ is a maximal set $\mathcal{S}$ of different pairwise intersecting vertices of Γ such that any three elements of $\mathcal{S}$ intersect in one point. These maximal sets exist by Zorn's lemma. We denote the set of all interior points of Γ by $\mathcal{P}$. Moreover, a vertex of the graph Γ is also called an **interior line** of the graph Γ. The set of all interior lines of Γ is denoted by $\mathcal{L}$.

Proposition 2.2.14 *Let Γ be a connected graph isomorphic to $\mathbf{S}(V_5)$. The geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \supset)$ on the interior points and interior lines of Γ with symmetrised containment as incidence relation is isomorphic to the geometry of points and lines of the unitary vector space V_5.*

Proof: Let $\mathcal{G}_{V_5} = (\mathcal{P}_{V_5}, \mathcal{L}_{V_5})$ be the geometry of points and lines of V_5, thus $\mathcal{P}_{V_5}$ is the set of all one-dimensional subspaces of V_5 and $\mathcal{L}_{V_5}$ is the set of all two-dimensional subspaces of V_5.

Now we consider the map $\mu : \mathcal{G}_{V_5} \to \mathcal{G}$, which maps each line l of V_5 to the interior line l, in symbols $\mu(l) = l$, for every $l \in \mathcal{L}_{V_5}$ and the image of a point p of V_5 is the set of all interior lines of $\mathcal{G}$ such that p is a point of each line, in symbols $\mu(p) = \{l \in \mathcal{L} \mid p \text{ is a point of } l \text{ in } V_5\}$.

We claim that μ is a homomorphism of geometries, in fact a correlation, and the map $\varphi : \mathcal{G} \to \mathcal{G}_{V_5}$ defined as below is the inverse map of μ. The image of each interior line l under the map φ is the line l of V_5, so $\varphi(l) = l$ for each $l \in \mathcal{L}$, and for each $p \in \mathcal{P}$ we define the image of p under φ to be the intersection point of all interior lines of p, in symbols $\varphi(p) = \bigcap_{l \in p} l =: q_p$.

First we show that the maps φ and μ are homomorphisms between the geometries $\mathcal{G}$ and $\mathcal{G}_{V_5}$. Therefore let p be a point of $\mathcal{G}_{V_5}$ and l be a line of the set $\mathcal{L}_{V_5}$ incident to the point p, then $\mu(p) = \{k \in \mathcal{L} \mid p \text{ is a point of } k \text{ in } V_5\}$ and $\mu(l) = l$. Since the line l contains the point p it follows that $l \in \mu(p)$, thus $\mu(l)$ is incident to $\mu(p)$ in $\mathcal{G}$. Now we consider an interior point p and an interior line l such that l is incident to p, which means that $l \in p$. Since $\varphi(l) = l$ and $\varphi(p) = q_p$ with the property that each line k of p contains the point q_p, we have that q_p is a point of l and thus $\varphi(l)$ and $\varphi(p)$ are incident in the geometry $\mathcal{G}_{V_5}$.

Next we prove the two identities $\varphi \circ \mu = id_{\mathcal{G}_{V_5}}$ and $\mu \circ \varphi = id_{\mathcal{G}}$. Thus let l be a line and p be a point of $\mathcal{G}_{V_5}$, then $(\varphi \circ \mu)(l) = l$ and $(\varphi \circ \mu)(p) = \varphi(\{l \in \mathcal{L} \mid p \text{ is a point of } l \text{ in } V_5\}) = q_p$ such that every line k of the set $\{l \in \mathcal{L} \mid p \text{ is a point of } l \text{ in } V_5\}$ contains the point q_p. If $q_p = p$ then of course $(\varphi \circ \mu)(p) = p$, otherwise if $p \neq q_p$, then every line k of the line set $\{l \in \mathcal{L} \mid p \text{ is a point of } l \text{ in } V_5\}$ contains the point q_p and the point p, hence $k = \langle q_p, p \rangle$, contradiction. It follows that $\varphi \circ \mu = id_{\mathcal{G}_{V_5}}$.

Now let p be an interior point and l be an interior line l of $\mathcal{G}$, then $(\mu \circ \varphi)(l) = l$ and $(\mu \circ \varphi)(p) = \mu(q_p) = \{l \in \mathcal{L} \mid q_p \text{ is a point of } l \text{ in } V_5\}$. Of course $p = \{l \in \mathcal{L} \mid q_p \text{ is a point of } l \text{ in } V_5\}$ by the definition of φ and μ and the definition of an interior point. This finishes the proof of $\mu \circ \varphi = id_{\mathcal{G}}$.

It follows that μ is an isomorphism and that $\mathcal{G}_{V_5} \cong \mathcal{G}$. As consequence we proved that the geometry $\mathcal{G}$ is a projective space. ∎

Remark 2.2.15 Directly from the proposition above we get that two interior points p and q of the projective space $\mathcal{G}$ have a unique interior line l in common, $l \in p \cap q$. Moreover two intersecting lines k and l of the space $\mathcal{G}$ determine a unique point p, by $\{l, k\} \subseteq p$.

If the graph Γ is isomorphic to $\mathbf{S}(V_5)$, then we call the projective space $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ the **interior space on** Γ. The final step is to define an endomorphism π on the interior geometry $\mathcal{G}$, which is an anisotropic polarity, and prove the following corollary.

Corollary 2.2.16 *The automorphism group of* $\mathbf{S}(V_5)$ *is isomorphic to the automorphism group of the projective unitary space* $\mathbb{P}(V_5)$.

Consider on the projective geometry $\mathcal{G}_{V_5}$ the polarity $\beta : \mathcal{G}_{V_5} \to \mathcal{G}_{V_5}$, which arises from the scalar product $(\cdot,\cdot)$ as follows $\beta(l) = \{p \in \mathcal{P}_{V_5} \mid (p, q) = 0 \text{ for every point } q \text{ incident to the line } l\}$ for any line $l \in \mathcal{L}_{V_5}$ and for every point $p \in \mathcal{P}_{V_5}$ we define $\beta(p) = \{q \in \mathcal{P}_{V_5} \mid (q, p) = 0\}$. Let π be the endomorphism of $\mathcal{G}$ defined as the following composition map $\pi = \mu \circ \beta \circ \varphi$.

Lemma 2.2.17 *The endomorphism* π *is an anisotropic polarity of the projective geometry* $\mathcal{G}$.

Proof: We need to show that π is a correlation of order two, which means that π is a order-reversing bijection with the property that $\pi \circ \pi = id$.

Of course the map π is a bijection since μ, φ and β are bijections. The second claim is clear by the fact that β is an anisotropic polarity of $\mathcal{G}_{V_5}$ and the identities $(\mu \circ \beta \circ \varphi) \circ (\mu \circ \beta \circ \varphi) = \mu \circ \beta \circ \varphi \circ \mu \circ \beta \circ \varphi = \mu \circ \beta \circ id_{\mathcal{G}_{V_5}} \circ \beta \circ \varphi = \mu \circ id_{\mathcal{G}_{V_5}} \circ \varphi = id_{\mathcal{G}}$.

Therefore it is left to show that the image of an interior point is a hyperplane and the image of an interior line is a hyperline in $\mathcal{G}$. Thus let p be an interior point of $\mathcal{G}$. The image of p under the endomorphism π is $\pi(p) = (\mu \circ \beta \circ \varphi)(p) = (\mu \circ \beta)(q_p) = \mu(\{a \in \mathcal{P}_{V_5} \mid (a, q_p) = 0\})$, which is a hyperplane in $\mathcal{G}$ using the fact that β is a polarity, hence $\{a \in \mathcal{P}_{V_5} \mid (a, q_p) = 0\}$ is a hyperplane, and μ is a isomorphism of projective geometries.

With the same arguments we get that $\pi(l) = (\mu \circ \beta \circ \varphi)(l) = (\mu \circ \beta)(l) = \mu(\{a \in \mathcal{P}_{V_5} \mid (a, q) = 0$ for every points q incident to the line $l\})$ is a hyperline in $\mathcal{G}$. ■

Proof of corollary 2.2.16: The statement follows now from the fact that the map $\delta : \mathrm{Aut}(\mathbf{S}(V_5)) \to \mathrm{Aut}(\mathbb{P}(V_5))$ with $\delta(\alpha) = \varphi \circ \alpha \circ \mu$ is an isomorphism. ■

2.3 The line graph of the unitary spaces V_n for $n \geq 6$

The purpose of the following section is to reconstruct the n-dimensional unitary vector space V_n over $\mathbb{C}$ for $n \geq 6$ from the line graph $\mathbf{S}(V_n)$. As in the preceding section we will build a geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \supset)$ from the graph $\mathbf{S}(V_n)$ and prove that $\mathcal{G}$ is isomorphic to the geometry of points and lines of the unitary vector space V_n. The results obtained in this section are very similar to the results of the previous section, however the methods of proof are a bit different.

As before we start by determining the diameter of the line graph $\mathbf{S}(V_n)$ for $n \geq 6$.

Proposition 2.3.1 *Let $n \geq 6$. The line graph $\mathbf{S}(V_n)$ of V_n is connected, its diameter is two and the graph $\mathbf{S}(V_n)$ is locally $\mathbf{S}(V_{n-2})$. Moreover for any two distinct lines l and m of $\mathbf{S}(V_n)$ the induced subgraph $\{l, m\}^{\perp}$ is not empty.*

Proof: Let l and m be two different lines of V_n. Since $\dim(l) = \dim(m) = 2$, certainly $\dim(l^{\pi}) = \dim(m^{\pi}) = \dim(V_n) - 2 = n - 2$ and the dimension formula implies that $\dim(l^{\pi} \cap m^{\pi}) \geq n - 4 \geq 6 - 4 = 2$. Hence we can choose a line h in the subspace $l^{\pi} \cap m^{\pi}$. In particular any two different vertices of the graph $\mathbf{S}(V_n)$ are connected by a path of length at most two in the graph $\mathbf{S}(V_n)$.

To establish the local property, we fix any vertex l of $\mathbf{S}(V_n)$ which is the two-dimensional subspace l in the vector space V_n. The points and lines which are contained in the orthogonal space l^{π} of l span an $(n-2)$-dimensional subspace U of V_n. Furthermore each line m perpendicular to l is contained in U. Since the restriction of the scalar product $(\cdot, \cdot)$ to the subspace l^{π} is again a scalar product, the graph $\mathbf{S}(V_n)$ is locally the line graph of the unitary vector space V_{n-2}.

To prove the last claim let l and m be any two different lines in V_n. Since the diameter of $\mathbf{S}(V_n)$ is two, they are connected either by a third line z in $\mathbf{S}(V_n)$, so $l \perp z \perp m$ or the two vertices l and m are adjacent. Certainly in the first case the line z lies in the intersection subspace $l^{\pi} \cap m^{\pi}$ implying $z \in \{l, m\}^{\perp}$. Otherwise if $l \perp m$ then the space spanned by the lines l and m has dimension four, consequently

$z = \langle l, m\rangle^\pi$ is a two-dimensional subspace of V_n. It follows again that $z \in \{l, m\}^\perp$. ■

Since we want to recover the unitary space V_n from the line graph $\mathbf{S}(V_n)$ we will describe a translation from graph language into vector space notation. Therefore recall definition 2.2.6 for the unitary vector space V_n, $n \geq 6$.

Lemma 2.3.2 *Let $n \geq 6$ and let l and m be two distinct vertices of $\mathbf{S}(V_n)$. Any vertex of $\{l, m\}^{\perp\perp}$ is contained as a line in $\mathbf{L}(\langle l, m\rangle)$ and vice versa.*

Proof: Use the arguments from the proof of lemma 2.2.7. ■

We will show an analogous lemma dealing with three different vertices under similar conditions.

Lemma 2.3.3 *Let $n \geq 6$ and let k, l and m be three different vertices in $\mathbf{S}(V_n)$. Suppose k, l and m intersect in a common point in the unitary space V_n and suppose that the subgraph $\{k, l, m\}^\perp$ is not empty, then $\mathbf{L}(\langle k, l, m\rangle) = \{k, l, m\}^{\perp\perp}$.*

Proof: Since the lines k, l and m intersect in a common point in V_n, they span a subspace of dimension three or four in V_n.

If $\langle k, l, m\rangle$ is of dimension three, then the line m is properly contained in the subspace $\langle k, l\rangle$, thus $\langle k, l, m\rangle = \langle k, l\rangle$. From the previous lemma 2.3.2 it follows that $\mathbf{L}(\langle k, l, m\rangle) = \mathbf{L}(\langle k, l\rangle) = \{k, l\}^{\perp\perp}$.

In the other case if $\dim(\langle k, l, m\rangle) = 4$ then we choose a lines s in $\langle k, l, m\rangle$ and consider its two-dimensional polar subspace $t = s^\pi \cap \langle k, l, m\rangle$ inside $\langle k, l, m\rangle$. Now, again by lemma 2.3.2 and the fact that $\langle k, l, m\rangle = \langle s, t\rangle$, it follows that $\mathbf{L}(\langle k, l, m\rangle) = \mathbf{L}(\langle s, t\rangle) = \{s, t\}^{\perp\perp}$.

It is left to show in the first case that $\{k, l, m\}^{\perp\perp}$ is equal to $\{k, l\}^{\perp\perp}$ and in the second case that the induced subgraph $\{s, t\}^{\perp\perp}$ is equal to $\{k, l, m\}^{\perp\perp}$. But this can be obtained under the assumption that $g = k$ and $d = l$ in the case that $\dim(\langle k, l, m\rangle) = 3$ and $g = s$ and $d = t$ if $\dim(\langle k, l, m\rangle) = 4$ from the next identities: $\{k, l, m\}^{\perp\perp} = \{\{k, l, m\}^\perp\}^\perp = \bigcap_{z\in\{k,l,m\}^\perp} z^\perp = \bigcap_{z\in\{k\}^\perp\cap\{l\}^\perp\cap\{m\}^\perp} z^\perp = \bigcap_{z\in\mathbf{L}(k^\pi)\cap\mathbf{L}(l^\pi)\cap\mathbf{L}(m^\pi)} z^\perp = \bigcap_{z\in\mathbf{L}(\langle k,l,m\rangle^\pi)} z^\perp = \bigcap_{z\in\mathbf{L}(\langle g,d\rangle^\pi)} z^\perp = \bigcap_{z\in\{g,d\}^\perp} z^\perp = \{\{g, d\}^\perp\}^\perp = \{g, d\}^{\perp\perp}$. ■

The next part of this section will describe the reconstruction of any point of the vector space V_n using only the vertices of the graph $\mathbf{S}(V_n)$, which are the lines of the unitary vector space V_n. First we give a criterion to decide whether two different lines intersect or not. Recall from remark 2.2.11 that for two distinct vertices l and m the induced subgraph $\{l, m\}^{\perp\perp}$ in $\mathbf{S}(V_n)$ is minimal with respect to inclusion if for any pair of distinct vertices s_1, s_2 in the induced subgraph $\{l, m\}^{\perp\perp}$ it is satisfied that $\{s_1, s_2\}^{\perp\perp} = \{l, m\}^{\perp\perp}$.

Lemma 2.3.4 *Let $n \geq 6$. Two distinct lines l and m of V_n intersect in a common point if and only if the induced subgraph $\{l, m\}^{\perp\perp}$ in the graph $\mathbf{S}(V_n)$ is minimal with respect to inclusion.*

Proof: First, let l and m be two intersecting lines in V_n, then $\dim(\langle l, m\rangle) = 3$. Take any two distinct vertices s_1 and s_2 of $\{l, m\}^{\perp\perp}$. Of course, s_1 and s_2 are different lines of the set $\mathbf{L}(\langle l, m\rangle)$ by lemma 2.3.2, thus $\langle s_1, s_2\rangle$ is a three-dimensional subspace of $\langle l, m\rangle$ in V_n implying that $\langle s_1, s_2\rangle = \langle l, m\rangle$. Using lemma 2.3.2 again, we obtain $\mathbf{L}(\langle l, m\rangle) = \{l, m\}^{\perp\perp}$ and $\mathbf{L}(\langle s_1, s_2\rangle) = \{s_1, s_2\}^{\perp\perp}$, which leads to the identities $\{s_1, s_2\}^{\perp\perp} = \mathbf{L}(\langle s_1, s_2\rangle) = \mathbf{L}(\langle l, m\rangle) = \{l, m\}^{\perp\perp}$.

Suppose the subspace $\langle l, m\rangle$ has dimension four so the two distinct lines l and m are skew to each other in V_n. Pick a point p on the line l and a point q on the line m. The two-dimensional subspace $\langle p, q\rangle$ is contained in $\langle l, m\rangle$ and intersects the distinct line l. Furthermore the subspace $\langle\langle p, q\rangle, l\rangle = \langle q, l\rangle$ has dimension three and is properly contained in $\langle l, m\rangle$, thus $\{\langle p, q\rangle, l\}^{\perp\perp} = \mathbf{L}(\langle\langle p, q\rangle, l\rangle) \subsetneq \mathbf{L}(\langle l, m\rangle) = \{l, m\}^{\perp\perp}$ by lemma 2.3.2, which proves that $\langle l, m\rangle^{\perp\perp}$ is not minimal in $\mathbf{S}(V_n)$ with respect to inclusion. ∎

Our next goal is to recover all points of the space V_n as pencils of lines. Three different pairwise intersecting lines k_1, k_2 and k_3 intersect in one point in the unitary vector space V_n if we can find a line s in V_n such that

- the line s intersects k_i, if $s \neq k_i$, for $1 \leq i \leq 3$,
- $\langle s, k_1, k_2\rangle$ is a four-dimensional space in V_n.

The same statement in terms of induced subgraphs is the following: Three different vertices k_1, k_2 and k_3 of $\mathbf{S}(V_n)$, where the vertex set $\{k_i, k_j\}^{\perp\perp}$ is minimal with respect to inclusion in $\mathbf{S}(V_n)$ for $1 \leq i \neq j \leq 3$, intersect in one point, if in the line graph $\mathbf{S}(V_n)$ is a vertex s with the properties:

- the induced subgraph $\{s, k_i\}^{\perp\perp}$ is minimal with respect to inclusion in $\mathbf{S}(V_n)$, if $s \neq k_i$, for $i \in 1, 2, 3$ (cf. lemma 2.3.4),
- $\{k_1, k_2\}^{\perp\perp} = \mathbf{L}(\langle k_1, k_2\rangle) \subsetneq \mathbf{L}(\langle k_1, k_2, s\rangle) = \{k_1, k_2, s\}^{\perp\perp}$ (cf. lemma 2.3.2, lemma 2.3.3 and lemma 2.3.5).

The identities $\{k_1, k_2\}^{\perp\perp} = \mathbf{L}(\langle k_1, k_2\rangle)$ and $\mathbf{L}(\langle k_1, k_2, s\rangle) = \{k_1, k_2, s\}^{\perp\perp}$ have been proved in lemma 2.3.2 and in lemma 2.3.3. Therefore let k and l be two different intersecting lines in V_n. By this assumption the subspace $\langle k, l\rangle$ has dimension three, thus the orthogonal space $\langle k, l\rangle^\pi$ has dimension greater or equal to three in the unitary space V_n. Now we consider the line $s = \langle p, q\rangle$ where p is a point in the space $\langle k, l\rangle^\pi$ and q is the intersecting point of the lines l and k. The space $\langle k, l, s\rangle = \langle k, l, p\rangle$ has dimension four, since $p \notin \langle k, l\rangle$. Thus the dimension of the orthogonal space $\langle l, k, p\rangle^\pi$ is greater or equal to two in V_n. Therefore we can choose a line in subspace $\langle k, l, s\rangle^\pi$ which shows that the graph $\{k, l, s\}^\perp$ is not empty. So we have proved the next lemma.

Lemma 2.3.5 *Let $n \geq 6$. For any two different intersecting lines k and l of V_n there is a line s in V_n intersecting the lines l and k such that $\langle k, l, s\rangle$ is a four-dimensional space, in particular the induced subgraph $\{k, l, s\}^{\perp}$ of $\mathbf{S}(V_n)$ is not empty.*

Definition 2.3.6 Let $n \geq 6$ and Γ be a graph isomorphic to the line graph $\mathbf{S}(V_n)$. Two different vertices k and l of Γ are defined to **intersect** if the induced subgraph $\{k, l\}^{\perp\perp}$ is minimal in Γ with respect to inclusion.

Three distinct pairwise intersecting vertices k_1, k_2 and k_3 of Γ are defined to **intersect in one point** if there is a vertex s in Γ satisfying the following conditions:

- the vertex s intersects the vertex k_i, if $s \neq k_i$, for $1 \leq i \leq 3$,
- the induced subgraph $\{k_1, k_2, s\}^{\perp}$ is non-empty and $\{k_1, k_2\}^{\perp\perp} = \mathbf{L}(\langle k_1, k_2\rangle) \subsetneq \mathbf{L}(\langle k_1, k_1, s\rangle) = \{k_1, k_2, s\}^{\perp\perp}$.

An **interior point** of the graph Γ is a maximal set $\mathcal{S}$ of distinct pairwise intersecting vertices of Γ such that any three elements of $\mathcal{S}$ intersect in one point. We denote the set of all interior points of Γ by $\mathcal{P}$. Moreover, an **interior line** of the graph Γ is a vertex of the graph Γ. The set of all interior lines of Γ is denoted by $\mathcal{L}$.

Proposition 2.3.7 *Let $n \geq 6$ and let Γ be a graph isomorphic to $\mathbf{S}(V_n)$. The geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \supset)$ on the interior points and interior lines of Γ with symmetrised containment as incidence relation is isomorphic to the geometry on points and lines of the unitary vector space V_n.*

Proof: The proof is a analogue to the proof of proposition 2.2.14. ■

If the graph Γ is isomorphic to $\mathbf{S}(V_n)$, then we call the geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ the **interior space on** Γ.

Corollary 2.3.8 *Let $n \geq 6$. The automorphism group of $\mathbf{S}(V_n)$ is isomorphic to the automorphism group of the projective unitary space $\mathbb{P}(V_n)$.*

Proof: See the proof of corollary 2.2.16. ■

2.4 The graph $\mathrm{S}(V_{n-2})$ inside the graph $\mathrm{S}(V_n)$ for $n \geq 7$

We will concentrate on the line graph $\mathbf{S}(V_n)$ for $n \geq 7$. By proposition 2.3.7 we can construct the interior space $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ on $\mathbf{S}(V_n)$, which is isomorphic to the geometry of points and lines of the complex unitary vector space V_n. If we fix a vertex x of the graph $\mathbf{S}(V_n)$, the induced subgraph $x^{\perp}$ is isomorphic to the graph $\mathbf{S}(V_{n-2})$ by proposition 2.3.1, so by proposition 2.2.14 and proposition 2.3.7 we can construct the interior space $\mathcal{G}_x = (\mathcal{P}_x, \mathcal{L}_x)$ on $x^\perp \cong \mathbf{S}(V_{n-2})$, which is isomorphic to the geometry of points and lines of the complex unitary vector space V_{n-2}. In this section we will prove that the unitary projective space $\mathcal{G}_x$ is a subspace of $\mathcal{G}$.

Notation: We will index every local object of the interior space $\mathcal{G}_x$ by the vertex x. In particular, for vertices k, l, m of the subgraph $x^\perp$ we use the notations $\{k, l, m\}^\perp_x = \{k, l, m\}^\perp \cap x^\perp$ and $\{k, l, m\}^{\perp\perp}_x = \{\{k, l, m\}^\perp_x\}^\perp_x = \{\{k, l, m\}^\perp_x\}^\perp \cap x^\perp$.

Obviously the line set $\mathcal{L}_x$ is properly contained in the line set $\mathcal{L}$ since every line of $\mathcal{G}_x$ is a vertex in the graph $x^\perp$ and therefore also a vertex in the graph $\mathbf{S}(V_n)$. We will prove a similar statement for points, i.e., for each point p of the interior space $\mathcal{G}$ the set of lines $p \cap \mathcal{L}_x$ is either a point p_x of the projective space $\mathcal{G}_x$ or empty.

Let us start with the case $n = 7$.

Lemma 2.4.1 *Let p be a point of $\mathcal{G}$. Any two different elements l and m of $p \cap \mathcal{L}_x$ admit an intersection in $\mathcal{G}_x$.*

Proof: Take two different lines l and m of $p \cap \mathcal{L}_x$. The claim follows from lemma 2.2.10 if we can show that $\{l, m\}^\perp_x \neq \varnothing$ in the graph $x^\perp$. Since the vertices l and m are adjacent to vertex x in the graph $\mathbf{S}(V_7)$, the plane $\langle l, m\rangle$ of the interior space $\mathcal{G}$ is orthogonal to the line x. Thus $\langle l, m\rangle$ is a three-dimensional subspace of the five-dimensional subspace x^π in $\mathcal{G}$. The orthogonal space of $\langle l, m\rangle$ inside the subspace x^π is the two-dimensional space $h = \langle l, m\rangle^\pi \cap x^\pi$, thus the line h is perpendicular to the lines l, m and x in $\mathcal{G}$. It follows that $h \in \{x, l, m\}^\perp = \{l, m\}^\perp \cap x^\perp = \{l, m\}^\perp_x$, which implies that $\{l, m\}^\perp_x \neq \varnothing$ and proves that l and m intersect in the projective space $\mathcal{G}_x$. ■

Lemma 2.4.2 *Let p be a point in $\mathcal{G}$. Any three different elements k_1, k_2 and k_3 of $p \cap \mathcal{L}_x$ intersect in one point in $\mathcal{G}_x$.*

Proof: Let k_1, k_2 and k_3 be three different elements in the set $p \cap \mathcal{L}_x$. By definition 2.2.13 we have to find a vertex s in the graph $x^\perp$ such that

- the graph $\{s, k_i\}^\perp_x$ is not empty, if $s \neq k_i$, for $i \in \{1, 2, 3\}$,
- the graph $\{s, k_1, k_2\}^\perp_x$ is empty.

Using lemma 2.4.1 we conclude that k_1, k_2 and k_3 are mutually intersecting lines in $\mathcal{G}_x$. Since k_1, k_2 and k_3 are vertices of the graph $\mathbf{S}(V_7)$, adjacent to x, the span of k_1, k_2 and k_3 in $\mathcal{G}$ is a subspace of x^π. As the vertices k_1, k_2 and k_3 are elements of the point $p \in \mathcal{P}$, the point $d := k_1 \cap k_2 = k_2 \cap k_3 = k_1 \cap k_3$ is a one-dimensional subspace of x^π, as well.

Suppose $\langle k_1, k_2, k_3\rangle$ is a subspace of dimension four in $\mathcal{G}$. Then we choose s equal to the line k_3. The orthogonal space of $\langle k_1, k_2, s\rangle = \langle k_1, k_2, k_3\rangle$ inside the subspace x^π is a point, say z. Any line l of $\langle k_1, k_2, s\rangle^\pi$ in $\mathcal{G}$ intersects the subspace x^π either in the point z or not at all. Thus the induced subgraph $\{s, k_1, k_2\}_x^\perp = \{s, k_1, k_2\}^\perp \cap x^\perp$ is empty. Moreover the graphs $\{s, k_1\}_x^\perp$ and $\{s, k_2\}_x^\perp$ are not empty, since the vertices k_1, k_2 and k_3 are mutually intersecting lines in $\mathcal{G}_x$. Hence, in this case, the claim follows.

If, on the other hand, the subspace $\langle k_1, k_2, k_3\rangle$ is a plane in $\mathcal{G}$, then fix a point y in $\langle k_1, k_2, k_3\rangle^\pi \cap x^\pi$ and consider the line $s = \langle d, y\rangle$ of the space $\mathcal{G}$. The line s is contained in the subspace x^π and intersects each line k_i in the point d, for $i \in \{1, 2, 3\}$. By lemma 2.2.10, the induced subgraph $\{s, k_i\}_x^\perp$ of $x^\perp$ is not empty for $i \in \{1, 2, 3\}$. Because $y \notin \langle k_1, k_2\rangle$, the subspace $\langle s, k_1, k_2\rangle$ has dimension four inside the subspace x^π and with a similar argument as above it follows that $\{s, k_1, k_2\}_x^\perp$ is empty. ∎

Proposition 2.4.3 *Let p be a point in $\mathcal{G}$. Then either the set of lines $p \cap \mathcal{L}_x$ is a point p_x of $\mathcal{G}_x$ or the empty set.*

Proof: If $p \cap \mathcal{L}_x \neq \emptyset$ then we fix an element $l \in p \cap \mathcal{L}_x$. The vertex l is adjacent to the vertex x in $\mathbf{S}(V_7)$, thus $l \subseteq x^\pi$ in the projective space $\mathcal{G}$. Let m be an element of the point p different from l. Due to remark 2.2.15 the two distinct lines l and m define the unique point $d := l \cap m$ in the projective space $\mathcal{G}$. Certainly, we can also find line $n \subseteq x^\pi$, different from l and containing the point d, which implies that n is a line of p. Moreover since $n \perp x$ we observe that n is an element of $p \cap \mathcal{L}_x$. By lemma 2.4.1, the interior lines l and n intersect in $\mathcal{G}_x$, say in the point p_x.

Let k be an arbitrary line of the interior point p_x. We will show that k is a line of the interior point p of $\mathcal{G}$, thus completing the proof. By definition the vertex k is a line in the subspace x^π of $\mathcal{G}$ incident to the point d. Also, any vertex g of p is a line in the space $\mathcal{G}$ incident to the point d. Therefore the space spanned by k and some line g is a plane in $\mathcal{G}$. By lemma 2.3.2 we obtain $\{k, g\}^{\perp\perp} = \mathbf{L}(\langle k, g\rangle)$. Due to lemma 2.3.2, again, and the fact that the span of two lines s_1, s_2 of the plane $\langle k, g\rangle$ equals that plane, we verify the identities $\{k, g\}^{\perp\perp} = \mathbf{L}(\langle k, g\rangle) = \mathbf{L}(\langle s_1, s_2\rangle) = \{s_1, s_2\}^{\perp\perp}$, which is the first condition in definition 2.3.6. To establish the second condition, choose $h, f \in p$. By the above the lines h, f and k mutually intersect in the projective space $\mathcal{G}$. The subspace $\langle h, f, k\rangle$ is either of dimension three or four in $\mathcal{G}$. If $\dim(\langle h, f, k\rangle) = 4$, then certainly $\{k, h\}^{\perp\perp} = \mathbf{L}(\langle k, h\rangle) \subsetneq \mathbf{L}(\langle k, f, h\rangle) = \{k, f, h\}^{\perp\perp}$ by lemma 2.3.2 and lemma 2.3.3. If $\langle h, f, k\rangle$ is a plane in $\mathcal{G}$, then $\langle h, f, k\rangle = \langle h, f\rangle$. By lemma 2.3.5 we find a line s intersecting the lines h, f and k in the point d such that $\langle s, h, f\rangle$ is a four-dimensional space in $\mathcal{G}$. Hence $k \in p$, which proves the claim. ∎

Next we want to show the converse, i.e. for each point p_x of the geometry $\mathcal{G}_x$ there is a unique point p in $\mathcal{G}$ such that $p_x \subseteq p$.

Lemma 2.4.4 *Let p_x be a point in $\mathcal{G}_x$. Any two distinct elements k and l of p_x are intersecting vertices of $\mathbf{S}(V_7)$.*

Proof: Let k and l be two distinct elements of the point p_x. By definition 2.3.6 we have to check that $\{k,l\}^\perp$ is not empty and the double perp $\{k,l\}^{\perp\perp}$ in $\mathbf{S}(V_7)$ is minimal with respect to inclusion. Since $k \perp x \perp l$ in $\mathbf{S}(V_7)$ the first condition is obvious. By lemma 2.3.2, the subgraph $\{k,l\}^{\perp\perp}$ in $\mathbf{S}(V_7)$ is minimal with respect to inclusion, if and only if the lines k and l span a three-dimensional subspace of V_7. By way of contradiction, suppose k and l span a four-dimensional subspace of V_7. Using now that k and l are two different lines of the point p_x, we get the chain $k \perp x \perp l$ in the graph $\mathbf{S}(V_7)$. Since the subspace $\langle k,l\rangle$ is supposed to be four-dimensional, the space x^π does not contain a line orthogonal to $\langle k,l\rangle$, whence $\{k,l\}^\perp \cap x^\perp = \varnothing$, a contradiction to $k,l \in p_x$, cf. definition 2.2.13. ■

Lemma 2.4.5 *Let p_x be a point in $\mathcal{G}_x$. Any three distinct elements k_1, k_2 and k_3 of p_x intersect in one point in $\mathcal{G}$.*

Proof: Lemma 2.4.4 implies that the lines k_1, k_2 and k_3 mutually intersect in $\mathcal{G}$. It remains to show, cf. definition 2.3.6, that in the graph $\mathbf{S}(V_7)$ is a vertex s such that

- the induced subgraph $\{s,k_i\}^{\perp\perp}$ is minimal in $\mathbf{S}(V_7)$ with respect to inclusion, if $s \neq k_i$, for $i \in \{1,2,3\}$,
- $\{k_1,k_2\}^{\perp\perp} = \mathbf{L}(\langle k_1,k_2\rangle) \subsetneq \mathbf{L}(\langle k_1,k_2,s\rangle) = \{k_1,k_2,s\}^{\perp\perp}$.

By definition 2.2.13 there is a line s in $\mathcal{G}_x$ such that $\{k_i,s\}^\perp \neq \varnothing$ if $s \neq k_i$, $1 \leq i \leq 3$ and $\{k_1,k_2,s\}^\perp = \varnothing$. The induced subgraph $\{s,k_i\}^{\perp\perp}$ is minimal in $\mathbf{S}(V_7)$ with respect to inclusion, if $s \neq k_i$ for $1 \leq i \leq 3$, by lemma 2.2.11 and 2.4.4. Thus it is left to show that $\{k_1,k_2\}^{\perp\perp} \subsetneq \{k_1,k_2,s\}^{\perp\perp}$. But this is obvious, because $s \in \{k_1,k_2,s\}^{\perp\perp}$ and $s \notin \{k_1,k_2\}^{\perp\perp}$. ■

From the preceding lemmata we obtain the following result:

Proposition 2.4.6 *Let p_x be a point of $\mathcal{G}_x$. The interior space on $\mathbf{S}(V_7)$ contains a unique point p such that $p_x \subseteq p$. In particular, the interior space $\mathcal{G}_x$ on $x^\perp$ is isomorphic a codimension two subspace of the space $\mathcal{G}$.*

Proof: It remains to prove the claim about the codimension. This, however, follows from the fact that $\mathcal{G}_x \cong \mathbb{P}(V_5)$ and $\mathcal{G} \cong \mathbb{P}(V_7)$. ■

The analogue holds for arbitrary $n \geq 8$.

Proposition 2.4.7 *Let $n \geq 8$ and let x be a vertex of the graph $\mathbf{S}(V_{n-2})$. The interior space $\mathcal{G}_x$ on the subgraph $x^\perp$ is isomorphic to a hyperline of the interior space $\mathcal{G}$ on the graph $\mathbf{S}(V_n)$.*

The statement follows if for each vertex x of the line graph $S(V_n)$ the interior space $\mathcal{G}_x$ is isomorphic to a subspace of $\mathcal{G} \cong \mathbb{P}(V_n)$. The difference to the preceding part of this section is the identical definition of an interior point in $\mathcal{G}_x$ and in $\mathcal{G}$. In order to observe that for each interior point p and every line x of $\mathcal{G}$ the line set $p \cap \mathcal{L}_{\mathbf{x}}$ is either an interior point p_x or empty and also that each interior point p_x of $\mathcal{G}_x$ is contained in a unique interior point p, the next lemma is important.

Lemma 2.4.8 *Let l and m be two different intersecting lines of $\mathcal{L}_x$. The induced subgraph $\{l, m\}^{\perp\perp}$ of $\mathbf{S}(V_n)$ is equal to $\{l, m\}_x^{\perp\perp}$ of $x^\perp$.*

Proof: The vertices l and m are adjacent to the vertex x in the line graph $\mathbf{S}(V_n)$, thus the plane $\langle l, m\rangle$ is a subspace of x^π and $\{l, m\}^{\perp\perp} = \mathbf{L}(\langle l, m\rangle)$ in $\mathcal{G}$ by lemma 2.3.2. Therefore all vertices of the induced subgraph $\{l, m\}^{\perp\perp}$ are adjacent to x in $\mathbf{S}(V_n)$. This fact combined with $\{\{l, m\}^\perp \cap x^\perp\}^\perp \subseteq \{l, m\}^{\perp\perp}$ implies the containment of $\{l, m\}^{\perp\perp}$ in $\{l, m\}_x^{\perp\perp}$.

On the other hand let k be a vertex of the graph $\{l, m\}_x^{\perp\perp}$. The following construction shows the incidence between the line k and the plane $\langle l, m\rangle$ in $\mathcal{G}$. Let $\langle l, m\rangle_x^\pi = \langle l, m\rangle^\pi \cap x^\pi$ be the orthogonal space of the plane $\langle l, m\rangle$ inside the subspace x^π. Since $\dim(\langle l, m\rangle_x^\pi) = n - 5$ we fix a point p in $\langle l, m\rangle_x^\pi$ and choose $n - 6$ different lines h_i, $i = 1, \ldots, n - 6$, in the subspace $\langle l, m\rangle_x^\pi$, mutually intersecting in the point p. Certainly $\langle h_1, \ldots h_{n-6}\rangle = \langle l, m\rangle_x^\pi$ and moreover every line h_i with $i = 1, \ldots, n - 6$, corresponds to a vertex adjacent to the vertices x, l and m in the graph $\mathbf{S}(V_n)$. Using the definition of the perp relation we obtain that the vertex k is an element of $z_x^\perp$ for all $z \in \{l, m\}_x^\perp$. In particular the vertex k is connected with each vertex h_i, $i = 1, \ldots, n - 6$, in the graph $\mathbf{S}(V_n)$. Thus the line k is contained in the subspace $x^\pi \cap \langle h_1, \ldots, h_{n-6}\rangle^\pi = x^\pi \cap (\langle l, m\rangle_x^\pi)^\pi = \langle l, m\rangle$ implying that $\{l, m\}_x^{\perp\perp} \subseteq \{l, m\}^{\perp\perp}$ and we are done. ∎

Lemma 2.4.9 *Let $n \geq 8$ and p be a point in $\mathcal{G}$. Any two distinct elements l and m of $p \cap \mathcal{L}_x$ intersect in a common point in $\mathcal{G}_x$.*

Proof: For two different elements l and m of the line set $p \cap \mathcal{L}_x$ in view of definition 2.3.6 we have to verify that $\{l, m\}_x^\perp \neq \emptyset$ and that the induced subgraph $\{l, m\}_x^{\perp\perp}$ is minimal in $x^\perp$ with respect to inclusion.

In the line graph $\mathbf{S}(V_n)$ both vertices l and m are adjacent to x and $\langle l, m\rangle$ is a plane in the interior space $\mathcal{G}$. Thus $\langle l, m\rangle$ is a subspace of the $(n-2)$-dimensional space x^π. Since $n \geq 8$ the subspace $\langle l, m\rangle^\pi \cap x^\pi$ has dimension $n - 5 \geq 3$ and we can pick a line h inside $\langle l, m\rangle^\pi \cap x^\pi$. Thus h is a vertex of the subgraph $\{l, m\}_x^\perp$ implying the first condition $\{l, m\}_x^\perp \neq \emptyset$.

Since l and m intersect in the interior space $\mathcal{G}$ we obtain that $\{l, m\}^{\perp\perp}$ is minimal in $\mathbf{S}(V_n)$ w.r.t. inclusion. By the statement of lemma 2.4.8 we conclude that the induced subgraph $\{l, m\}_x^{\perp\perp}$ is minimal in $x^\perp$ with respect to inclusion, as well. ∎

Lemma 2.4.10 *Let $n \geq 8$ and p be a point in $\mathcal{G}$. Any three pairwise distinct elements k_1, k_2 and k_3 of $p \cap \mathcal{L}_x$ intersect in one point in $\mathcal{G}_x$.*

Proof: Let k_1, k_2 and k_3 be three pairwise different elements of the line set $p \cap \mathcal{L}_x$. Using lemma 2.4.12 we conclude that k_1, k_2 and k_3 are pairwise intersecting lines in $\mathcal{G}_\mathbf{x}$. As the vertices k_1, k_2 and k_3 are elements of the point $p \in \mathcal{P}$, the three points $k_1 \cap k_2$, $k_1 \cap k_3$ and $k_2 \cap k_3$ coincide with a unique point d in $\mathcal{G}$.

By definition 2.3.6 the claim follows if we find a vertex s in the graph $x^\perp$ such that

- the induced subgraph $\{s, k_i\}_x^{\perp\perp}$ is minimal in $x^\perp$ with respect to inclusion if $s \neq k_i$, $i = 1, 2, 3$
- $\{k_1, k_2\}_x^{\perp\perp} \subsetneq \{s, k_1, k_2\}_x^{\perp\perp}$.

Since k_1, k_2 and k_3 are pairwise different vertices of the line set $p \cap \mathcal{L}_x$, the vertices k_1, k_2 and k_3 are adjacent to x in the line graph $\mathbf{S}(V_n)$ thus $\langle k_1, k_2, k_3 \rangle$ is a subspace of the space x^π inside the interior space $\mathcal{G}$.

Suppose that the subspace $\langle k_1, k_2, k_3 \rangle$ has dimension four in $\mathcal{G}$. Then we choose $s = k_3$ and we are done by lemma 2.3.3 and the fact that $\{s, k_1, k_2\}^{\perp\perp} = \{k_1, k_2, k_3\}^{\perp\perp} = \{k_1, k_2\}^\perp \cap \{k_2, k_3\}^{\perp\perp} = \{k_1, k_2\}_x^\perp \cap \{k_2, k_3\}_x^{\perp\perp} = \{k_1, k_2, k_3\}_x^{\perp\perp} = \{k_1, k_2, s\}_x^{\perp\perp}$ by way of lemma 2.4.8.

Alternatively, if $\langle k_1, k_2, k_3 \rangle$ is a plane in $\mathcal{G}$, then we can choose a line h of x^π through the point d not contained in the plane $\langle k_1, k_2, k_3 \rangle$. The line h is a vertex of the graph $\mathbf{S}(V_n)$, adjacent to x. Moreover the induced subgraphs $\{h, k_i\}^{\perp\perp}$ are minimal in $\mathbf{S}(V_n)$ w.r.t. inclusion for $i = 1, 2, 3$ and $\{k_1, k_2\}^{\perp\perp} \subsetneq \{h, k_1, k_2\}^{\perp\perp}$ since h is an element of the point p. Using again lemma 2.4.8 we obtain the identities $\{k_1, k_2\}^{\perp\perp} = \{k_1, k_2\}_x^{\perp\perp}$, $\{h, k_1, k_2\}^{\perp\perp} = \{h, k_1, k_2\}_x^{\perp\perp}$ as well as $\{h, k_i\}^{\perp\perp} = \{h, k_i\}_x^{\perp\perp}$ for $i = 1, 2, 3$, in particular for each vertex $z \in \{k_1, k_2, k_3\}$ the induced subgraph $\{h, k_i\}_x^{\perp\perp}$ is minimal in $x^\perp$ with respect to inclusion and $\{k_1, k_2\}_x^{\perp\perp} \subsetneq \{h, k_1, k_2\}_x^{\perp\perp}$. ∎

Proposition 2.4.11 *For $n \geq 8$ let p be a point in $\mathcal{G}$. The set of lines $p \cap \mathcal{L}_x$ is either a point p_x of the interior space $\mathcal{G}_x$ or the empty set.*

Proof: The claim follows by lemma 2.4.9, lemma 2.4.10 and a similar argumentation as used in the proof of proposition 2.4.3. ∎

We also show the opposite statement that each point p_x of the interior space $\mathcal{G}_x$ is contained in a unique point p of the interior space $\mathcal{G}$ if $n \geq 8$

Lemma 2.4.12 *Let $n \geq 8$ and p_x be a point of $\mathcal{G}_x$. Any two different lines k and l of p_x intersect in a common point in $\mathcal{G}$.*

Proof: By lemma 2.3.4 if the induced subgraph $\{l, m\}^{\perp\perp}$ is minimal in the line graph $\mathbf{S}(V_n)$ with respect to inclusion then the lines k and l intersect in a common point in $\mathcal{G}$. Using the statement of lemma 2.4.8 $\{k, l\}^{\perp\perp} = \{k, l\}_x^{\perp\perp}$ and since k and l are elements of the point p_x the induced subgraph $\{k, l\}_x^{\perp\perp}$ is minimal in $x^\perp$ with respect to inclusion. By lemma 2.4.8 again we obtain the identities $\{s_1, s_2\}^{\perp\perp} =$

$\{s_1, s_2\}_x^{\perp\perp} = \{k, l\}_x^{\perp\perp} = \{k, l\}^{\perp\perp}$ for any $s_1, s_2 \in \{k, l\}^{\perp\perp} = \{k, l\}_x^{\perp\perp}$ which confirms that the graph $\{k, l\}^{\perp\perp}$ is minimal in $\mathbf{S}(V_n)$ w.r.t. inclusion. ∎

Lemma 2.4.13 *Let $n \geq 8$ and p_x be a point of $\mathcal{G}_x$. Any three distinct elements k_1, k_2 and k_3 of p_x intersect in one point in the space $\mathcal{G}$.*

Proof: The previous lemma 2.4.12 implies that any three different elements k_1, k_2 and k_3 of a point $p_x \in \mathcal{P}_x$ are mutually intersecting lines of the interior space $\mathcal{G}$.

The subgraph $x^\perp$ is isomorphic to the line graph $\mathbf{S}(V_{n-2})$ and $n - 2 \geq 6$ thus $x^\perp$ contains a vertex s intersecting each line k_i in $\mathcal{G}_x$ if $s \neq k_i$ for $i \in \{1, 2, 3\}$ and $\{k_1, k_2\}^{\perp\perp} \subsetneq \{k_1, k_2, s\}^{\perp\perp}$ by definition 2.3.6, in particular the line s is an element of the point p_x. Since $x^\perp$ is an induced subgraph of $\mathbf{S}(V_n)$ it follows that s is also a line of the interior space $\mathcal{G}$ intersecting each line k_i in $\mathcal{G}$ if $s \neq k_i$ for $i \in \{1, 2, 3\}$ by lemma 2.4.12. The statement of this lemma is proved if we can verify the condition $\{k_1, k_2, s\}^{\perp\perp} = \{k_1, k_2, s\}_x^{\perp\perp}$. However $\{k_1, k_2, s\}^{\perp\perp} = \{k_1, s\}^{\perp\perp} \cap \{k_2, s\}^{\perp\perp}$ and $\{k_i, s\}^{\perp\perp} = \{k_i, s\}_x^{\perp\perp}$ for $i = 1, 2$ by lemma 2.4.8 consequently $\{k_1, k_2, s\}^{\perp\perp} = \{k_1, s\}^{\perp\perp} \cap \{k_2, s\}^{\perp\perp} = \{k_1, s\}_x^{\perp\perp} \cap \{k_2, s\}_x^{\perp\perp} = \{k_1, k_2, s\}_x^{\perp\perp}$ ∎

A conclusion from the previous lemmata is the following proposition.

Proposition 2.4.14 *Let $n \geq 8$ and let p_x be a point of the interior space $\mathcal{G}_x$. The interior space of the line graph $\mathbf{S}(V_n)$ contains a unique point p with $p_x \subseteq p$.*

Notice that the interior space $\mathcal{G}_x$ is isomorphic to a subspace of $\mathcal{G}$ since every line l of $\mathcal{G}_x$ is a line of $\mathcal{G}$ and any point $p_x \in \mathcal{P}_x$ is contained in a unique point p of the interior space of $\mathbf{S}(V_n)$ by proposition 2.4.14. The claim about the dimension of $\mathcal{G}_x$ inside $\mathcal{G}$ follows from the fact that $\mathcal{G}_x \cong \mathbb{P}(V_{n-2})$ and $\mathcal{G}_x \cong \mathbb{P}(V_n)$, which proves proposition 2.4.7.

2.5 The global space

In this section we will study the following situation: Let $n \geq 7$ and let Γ be a connected graph which is locally isomorphic to the line graph $\mathbf{S}(V_n)$. Our goal is to show that the graph Γ is isomorphic to $\mathbf{S}(V_{n+2})$.

Since the graph Γ is locally $\mathbf{S}(V_n)$, for every vertex $\mathbf{x}$ of Γ, we can construct the interior space $\mathcal{G}_\mathbf{x}$ isomorphic to $\mathbb{P}(V_n)$ from the induced subgraph $\mathbf{x}^\perp$, see proposition 2.3.7. The idea to prove the main theorem 2.1.2 is to construct a global geometry on Γ using the family $(\mathcal{G}_\mathbf{x})_{\mathbf{x}\in\Gamma}$ of interior spaces and to identify this global geometry as a projective space. Observe that any local object (point, line, plane, etc.) only exists in an interior space $\mathcal{G}_\mathbf{x}$ for some vertex $\mathbf{x}$ in the graph Γ, so one task will be to show that there are well-defined global objects, in order to define our global geometry.

In the last part of this section we prove that the global geometry on Γ is a projective space over the complex numbers with an anisotropic polarity and that the line graph of that global geometry is isomorphic to Γ.

Notation: To avoid confusion, we will index every local object by the vertex $\mathbf{x}$ whose interior space it belongs to. For example, if $\mathbf{x} \perp \mathbf{y}$ in the graph Γ, then $\mathbf{y}$ is a vertex of the subgraph $\mathbf{x}^\perp$ corresponding to the local object $y_\mathbf{x}$, an interior line, in the space $\mathcal{G}_\mathbf{x}$. By $\mathbf{y}_\mathbf{x}$ we denote the vertex $\mathbf{y}$ considered as a vertex of the subgraph $\mathbf{x}^\perp$. With the symbol $\mathbf{y}_\mathbf{x}^\perp$ we denote the subgraph $\{\mathbf{x},\mathbf{y}\}^\perp$ which is of course an induced subgraph of $\mathbf{x}^\perp$. The interior space obtained from the graph $\mathbf{y}_\mathbf{x}^\perp$ will be denoted with $\mathcal{G}_{\mathbf{y}_\mathbf{x}}$.

Definition 2.5.1 A **global line** of Γ is a vertex of the graph Γ. The set of all global lines of Γ is denoted by $\mathcal{L}_\Gamma$.

Let $\mathbf{x},\mathbf{y},\mathbf{z}$ and $\mathbf{w}$ be four vertices of Γ such that $\mathbf{z}\perp\mathbf{x}\perp\mathbf{w}\perp\mathbf{y}$. Notice that $U^\mathbf{x}_{z_\mathbf{x},w_\mathbf{x}} := \langle z_\mathbf{x}, w_\mathbf{x}\rangle^\pi = z_\mathbf{x}^\pi \cap w_\mathbf{x}^\pi$ is a subspace of the interior spaces $\mathcal{G}_\mathbf{x}$ and $\mathcal{G}_{\mathbf{x}_\mathbf{z}}$. Since $\mathcal{G}_{\mathbf{x}_\mathbf{z}} = \mathcal{G}_{\mathbf{z}_\mathbf{x}}$, by propositions 2.4.6 and 2.4.7 this space $U^\mathbf{x}_{z_\mathbf{x},w_\mathbf{x}}$ can also be considered as a subspace of the interior space $\mathcal{G}_\mathbf{z}$, in fact of $x_\mathbf{z}^\pi$. For emphasis we denote this space $U^\mathbf{x}_{z_\mathbf{x},w_\mathbf{x}}$ by $U^\mathbf{z}_{z_\mathbf{x},w_\mathbf{x}}$ when considering it in $\mathcal{G}_\mathbf{z}$. Notice also that $\dim(U^\mathbf{x}_{z_\mathbf{x},w_\mathbf{x}}) = \dim(U^\mathbf{z}_{z_\mathbf{x},w_\mathbf{x}})$. For the same reasons as above any local object contained in $U^\mathbf{x}_{z_\mathbf{x},w_\mathbf{x}} \cap U^\mathbf{x}_{x_\mathbf{w},y_\mathbf{w}}$ is a local object of the interior spaces $\mathcal{G}_\mathbf{z}, \mathcal{G}_\mathbf{x}, \mathcal{G}_\mathbf{y}$ and $\mathcal{G}_\mathbf{w}$.

Proposition 2.5.2 *Let $n \geq 7$ and let Γ be a connected graph which is locally isomorphic to $\mathbf{S}(V_n)$. Then the graph Γ has diameter two.*

Proof: Let $\mathbf{z}\perp\mathbf{x}\perp\mathbf{y}\perp\mathbf{w}$ be a chain of different vertices in Γ. We will distinguish between the cases $n \geq 8$ and $n = 7$.

In case $n \geq 8$, the dimension of the intersection of $U^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}$ and $U^\mathbf{y}_{x_\mathbf{y},w_\mathbf{y}}$ is greater than or equal to two, because the codimension of $U^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}$ and of $U^\mathbf{y}_{x_\mathbf{y},w_\mathbf{y}}$ is at most two in the space $x_\mathbf{y}^\pi$ of dimension at least six. Therefore we can choose an interior line $l_\mathbf{y}$ in $U^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}} \cap U^\mathbf{y}_{x_\mathbf{y},w_\mathbf{y}}$. The interior line $l_\mathbf{y}$ corresponds to a vertex $\mathbf{l}$ in the graph Γ and is adjacent to $\mathbf{x},\mathbf{y},\mathbf{z}$ and $\mathbf{w}$. It follows by induction that the connected graph Γ has diameter two.

Now we turn to the case $n = 7$. Either $\dim(U^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}} \cap U^\mathbf{y}_{x_\mathbf{y},w_\mathbf{y}}) \geq 2$ — in which case we fix an interior line $l_\mathbf{y}$ in the intersection of the spaces $U^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}}$ and $U^\mathbf{y}_{x_\mathbf{y},w_\mathbf{y}}$ as before — or $U^\mathbf{y}_{z_\mathbf{x},y_\mathbf{x}} \cap U^\mathbf{y}_{x_\mathbf{y},w_\mathbf{y}} = p_\mathbf{y}$ is an interior point of the interior spaces $\mathcal{G}_\mathbf{y}$, $\mathcal{G}_\mathbf{z}$ (denoted by $p_\mathbf{z}$), $\mathcal{G}_\mathbf{x}$ (denoted by $p_\mathbf{x}$) and $\mathcal{G}_\mathbf{w}$ (denoted by $p_\mathbf{w}$).

Assuming the latter case, fix an interior line $n_\mathbf{x}$ in the subspace $U^\mathbf{x}_{z_\mathbf{x},y_\mathbf{x}}$ which has at least dimension three, incident to the interior point $p_\mathbf{x}$. Since $n_\mathbf{x}$ corresponds to a vertex $\mathbf{n}$ in Γ adjacent to $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, we can consider the chain $\mathbf{z}\perp\mathbf{n}\perp\mathbf{y}\perp\mathbf{w}$ in the graph Γ. With the construction as above we obtain an interior point $q_\mathbf{n} = U^\mathbf{n}_{z_\mathbf{n},y_\mathbf{n}} \cap U^\mathbf{n}_{n_\mathbf{y},w_\mathbf{y}}$ in $\mathcal{G}_\mathbf{n}$ which is also an interior point of $\mathcal{G}_\mathbf{y}$ (denoted by $q_\mathbf{y}$). The interior points $q_\mathbf{y}$ and $p_\mathbf{y}$ are different by the fact that $p_\mathbf{y}$ is a point on the line $n_\mathbf{y}$ and $q_\mathbf{y}$ is a point in the orthogonal subspace $n_{\mathbf{y}}^{\pi}$ of $n_{\mathbf{y}}$. Thus we regard the interior line $l_{\mathbf{y}} = \langle p_{\mathbf{y}}, q_{\mathbf{y}} \rangle$ generated from the two points $q_{\mathbf{y}}$ and $p_{\mathbf{y}}$, which is contained in the subspace $w_{\mathbf{y}}^{\pi}$ of $\mathcal{G}_{\mathbf{y}}$. Furthermore the interior line $l_{\mathbf{y}}$ matches with a vertex $\mathbf{l}$ in the graph Γ adjacent to the vertices $\mathbf{y}$ and $\mathbf{w}$ by construction.

At this point we have constructed the chain $\mathbf{z} \perp \mathbf{n} \perp \mathbf{y} \perp \mathbf{l}$ in the graph Γ with the property that the subspace $\langle n_{\mathbf{y}}, l_{\mathbf{y}} \rangle$ has dimension three in the interior space $\mathcal{G}_{\mathbf{y}}$. Therefore $\dim(U^{\mathbf{y}}_{z_{\mathbf{n}}, y_{\mathbf{n}}} \cap U^{\mathbf{y}}_{n_{\mathbf{y}}, l_{\mathbf{y}}}) \geq 2$, so using arguments from above we find a vertex $\mathbf{t}$ in Γ adjacent to the vertices $\mathbf{z}, \mathbf{n}, \mathbf{y}, \mathbf{l}$.

In the seven-dimensional interior space $\mathcal{G}_{\mathbf{t}}$ the vertices $\mathbf{y}$ and $\mathbf{z}$ correspond to the interior lines $y_{\mathbf{t}}$ and $z_{\mathbf{t}}$. The intersection of $y_{\mathbf{t}}^{\pi}$ and $z_{\mathbf{t}}^{\pi}$ is a three-dimensional subspace of $\mathcal{G}_{\mathbf{t}}$ which contains the unique interior point $p_{\mathbf{t}}$ induced from the point $p_{\mathbf{y}} \cap \mathcal{L}_{\mathbf{t}}$ and the unique interior point $q_{\mathbf{t}}$ such that $q_{\mathbf{y}} \cap \mathcal{L}_{\mathbf{t}} \subseteq q_{\mathbf{t}}$. Thus the interior line $l_{\mathbf{y}}$ is incident to the subspace $U^{\mathbf{t}}_{z_{\mathbf{t}}, y_{\mathbf{t}}}$. It follows that the vertices $\mathbf{l}$ and $\mathbf{z}$ are adjacent in the graph Γ. Again by induction the connected graph Γ has diameter two. ∎

Our next goal is to construct a notion of global points for Γ such that each interior point $p_{\mathbf{x}}$ for some $\mathbf{x} \in \Gamma$ is contained in a unique global point.

Lemma 2.5.3 *Let* $\mathbf{x}$, $\mathbf{y}$ *and* $\mathbf{z}$ *be three vertices of* Γ *and* $p_{\mathbf{x}}$ *be an interior point of* $\mathcal{G}_{\mathbf{x}}$ *such that* $\mathbf{x} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x}$ *and* $y_{\mathbf{x}}, z_{\mathbf{x}} \in p_{\mathbf{x}}^{\pi}$. *Denote the unique interior point of* $\mathbf{y}^{\perp}$ *induced from the point* $p_{\mathbf{y}_{\mathbf{x}}}$ *by* $p_{\mathbf{y}}$ *and the unique interior point of* $\mathbf{z}^{\perp}$ *induced from the point* $p_{\mathbf{z}_{\mathbf{x}}}$ *by* $p_{\mathbf{z}}$. *Then* $z_{\mathbf{y}} \in p_{\mathbf{y}}^{\pi}$ *and the unique interior point* $q_{\mathbf{z}}$ *of* $\mathbf{z}^{\perp}$ *induced by* $p_{\mathbf{z}_{\mathbf{y}}}$ *is equal to the interior point* $p_{\mathbf{z}}$.

Proof: Consider the unique interior point $p_{\mathbf{y}} \supseteq p_{\mathbf{y}_{\mathbf{x}}}$ in $\mathcal{G}_{\mathbf{y}}$ and the unique interior point $p_{\mathbf{z}} \supseteq p_{\mathbf{z}_{\mathbf{x}}}$ in $\mathcal{G}_{\mathbf{z}}$, cf. section 2.4. Since $y_{\mathbf{x}}^{\pi} \cap z_{\mathbf{x}}^{\pi}$ is of dimension at least three and incident to the point $p_{\mathbf{x}}$, we can find distinct interior lines $g_{\mathbf{x}}^{1}$ and $g_{\mathbf{x}}^{2}$ of the point $p_{\mathbf{x}}$ in the subspace $y_{\mathbf{x}}^{\pi} \cap z_{\mathbf{x}}^{\pi}$, so that $g_{\mathbf{x}}^{1}, g_{\mathbf{x}}^{2} \in p_{\mathbf{y}_{\mathbf{x}}}$ and $g_{\mathbf{x}}^{1}, g_{\mathbf{x}}^{2} \in p_{\mathbf{z}_{\mathbf{x}}}$. Thus the global lines $\mathbf{g}^{1}$ and $\mathbf{g}^{2}$ are also lines of the point $p_{\mathbf{y}}$ and $p_{\mathbf{z}}$. Moreover $g_{\mathbf{y}}^{1}, g_{\mathbf{y}}^{2} \in p_{\mathbf{z}_{\mathbf{y}}}$ and, thus, the unique interior point $q_{\mathbf{z}}$ induced from $p_{\mathbf{y}_{\mathbf{z}}}$ also contains both interior lines $g_{\mathbf{z}}^{1}$ and $g_{\mathbf{z}}^{2}$. Due to remark 2.2.15 we have $q_{\mathbf{z}} = p_{\mathbf{z}}$. ∎

Definition 2.5.4 A **global point** p of Γ is a set of vertices of the graph Γ such that

$$p = p_{\mathbf{x}} \cup \bigcup_{\mathbf{h} \in \mathbf{x}^{\perp}} \{ p_{\mathbf{h}} \in \mathcal{P}_{\mathbf{h}} \mid p_{\mathbf{x}_{\mathbf{h}}} \subseteq p_{\mathbf{h}} \}$$

for some vertex $\mathbf{x}$ of Γ and some interior point $p_{\mathbf{x}}$ of the interior space $\mathcal{G}_{\mathbf{x}}$. The set of all global points of Γ is denoted by $\mathcal{P}_{\Gamma}$.

In order to understand the above definition better, we will show that a global point is stable under iteration of the above process. To this end, let $\mathbf{x}$ be a vertex of Γ and $p_{\mathbf{x}}$ be an interior point of the point set $\mathcal{P}_{\mathbf{x}}$. We set $p_0 := p_{\mathbf{x}} \cup \bigcup_{\mathbf{h} \in \mathbf{x}^{\perp}} \{ p_{\mathbf{h}} \in \mathcal{P}_{\mathbf{h}} \mid p_{\mathbf{x}_{\mathbf{h}}} \subseteq p_{\mathbf{h}} \}$ and by the above definition, p_0 is a global point.

Define

$$p_1 = p_{\mathbf{x}} \cup \bigcup_{\mathbf{h} \in \mathbf{x}^\perp} \{p_{\mathbf{h}} \in \mathcal{P}_{\mathbf{h}} \mid p_{\mathbf{x}_\mathbf{h}} \subseteq p_{\mathbf{h}}\} \cup \bigcup_{\mathbf{k} \perp \mathbf{h} \perp \mathbf{x}} \{p_{\mathbf{k}} \in \mathcal{P}_{\mathbf{k}} \mid p_{\mathbf{x}_\mathbf{h}} \subseteq p_{\mathbf{h}}, p_{\mathbf{h}_\mathbf{k}} \subseteq p_{\mathbf{k}}\}.$$

Certainly $p_0 \subseteq p_1$. Assume $p_1 \neq p_0$ and let $\mathbf{k}$ be a global line in p_1 such that $\mathbf{k}$ is not contained in the vertex set p_0. Then by construction we can find a chain of vertices $\mathbf{x} \perp \mathbf{y} \perp \mathbf{w} \perp \mathbf{k}$ in Γ and interior points $p_{\mathbf{y}}$ and $p_{\mathbf{w}}$ with $p_{\mathbf{x}_\mathbf{y}} \subseteq p_{\mathbf{y}}$, $p_{\mathbf{y}_\mathbf{w}} \subseteq p_{\mathbf{w}}$ and $\mathbf{k} \in p_{\mathbf{w}}$. Due to the proof of proposition 2.5.2 there is a vertex $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}, \mathbf{k}\}^\perp$ and a path $\mathbf{w} \perp \mathbf{c}^1 \perp \cdots \perp \mathbf{c}^n \perp \mathbf{z}$ in the induced subgraph $\{\mathbf{y}, \mathbf{k}\}^\perp$. Since the interior line $c^1_{\mathbf{w}}$ is in the orthogonal space of $k_{\mathbf{w}}$, we can find an interior point $p_{\mathbf{c}^1} \supseteq p_{\mathbf{w}_{\mathbf{c}^1}}$ containing the interior line $k_{\mathbf{c}^1}$ in the interior space $\mathcal{G}_{\mathbf{c}^1}$. Using lemma 2.5.3 we obtain $p_{\mathbf{y}_{\mathbf{c}^1}} \subseteq p_{\mathbf{c}^1}$. Arguing along the path $\mathbf{w} \perp \mathbf{c}^1 \perp \cdots \perp \mathbf{c}^n \perp \mathbf{z}$, we end up with $p_{\mathbf{y}_\mathbf{z}} \subseteq p_{\mathbf{z}}$, $k_{\mathbf{z}} \in p_{\mathbf{z}}$. This implies $p_{\mathbf{x}_\mathbf{z}} \subseteq p_{\mathbf{z}}$, so $\mathbf{k} \in p_0$. Hence $p_0 = p_1$. This consideration has two immediate consequences.

Proposition 2.5.5 *Let p be a global point and $\mathbf{x}$ be vertex of Γ. The intersection of the line set $\mathcal{L}_{\mathbf{x}}$ of the interior space $\mathcal{G}_{\mathbf{x}}$ and the global point p is either an interior point $p_{\mathbf{x}}$ of $\mathcal{G}_{\mathbf{x}}$ or the empty set.*

Proposition 2.5.6 *The notion of a global point p is well-defined and does not depend on the starting interior point $p_{\mathbf{x}} \subseteq p$.*

Proof: If $\mathbf{x} \perp \mathbf{y}$ and $p_{\mathbf{x}}, p_{\mathbf{y}} \subseteq p$, then

$$\begin{aligned} & p_{\mathbf{y}} \cup \bigcup_{\mathbf{l} \in \mathbf{y}^\perp} \{p_{\mathbf{l}} \in \mathcal{P}_{\mathbf{l}} \mid p_{\mathbf{y}_\mathbf{l}} \subseteq p_{\mathbf{l}}\} \\ = {} & p_{\mathbf{y}} \cup \bigcup_{\mathbf{l} \in \mathbf{y}^\perp} \{p_{\mathbf{l}} \in \mathcal{P}_{\mathbf{l}} \mid p_{\mathbf{y}_\mathbf{l}} \subseteq p_{\mathbf{l}}\} \cup \bigcup_{\mathbf{k} \perp \mathbf{l} \perp \mathbf{y}} \{p_{\mathbf{k}} \in \mathcal{P}_{\mathbf{k}} \mid p_{\mathbf{y}_\mathbf{l}} \subseteq p_{\mathbf{l}}, p_{\mathbf{l}_\mathbf{k}} \subseteq p_{\mathbf{k}}\} \\ \supseteq {} & p_{\mathbf{x}} \cup \bigcup_{\mathbf{l} \in \mathbf{x}^\perp} \{p_{\mathbf{l}} \in \mathcal{P}_{\mathbf{l}} \mid p_{\mathbf{x}_\mathbf{l}} \subseteq p_{\mathbf{l}}\}, \end{aligned}$$

so by symmetry $p_{\mathbf{x}}$ and $p_{\mathbf{y}}$ define the same global point. The general case is proved by an iteration of this argument along each path between the vertices $\mathbf{x}$ and $\mathbf{z}$ with $p_{\mathbf{x}}, p_{\mathbf{z}} \subseteq p$ in the connected graph Γ. ∎

The triple $\mathcal{G}_\Gamma = (\mathcal{P}_\Gamma, \mathcal{L}_\Gamma, \supset)$ is a point-line geometry, called the **global geometry** on Γ.

Proposition 2.5.7 *The point-line geometry $\mathcal{G}_\Gamma = (\mathcal{P}_\Gamma, \mathcal{L}_\Gamma, \supset)$ is a projective space.*

Proof: We have to show that the geometry $\mathcal{G}_\Gamma$ satisfies the axioms of a projective space, see definition A.1.2. Let p and q be two distinct global points of $\mathcal{G}_\Gamma$ and let $\mathbf{l} \in p$ and $\mathbf{m} \in q$, where $\mathbf{l}$ and $\mathbf{m}$ are two distinct global lines of $\mathcal{G}_\Gamma$. Due to proposition 2.5.2 we find a vertex $\mathbf{z} \in \Gamma$ adjacent to $\mathbf{l}$ and $\mathbf{m}$. Using proposition 2.5.5 and the condition that $\mathbf{l}, \mathbf{m} \in \mathbf{z}^\perp$ we get two distinct interior points $p \cap \mathbf{z}^\perp = p \cap \mathcal{L}_\mathbf{z} = p_\mathbf{z}$ and $q \cap \mathbf{z}^\perp = q \cap \mathcal{L}_\mathbf{z} = q_\mathbf{z}$ of $\mathcal{P}_\mathbf{z}$. Since $\mathcal{G}_\mathbf{z}$ is a linear space by proposition 2.3.7 it contains a unique interior line $k_\mathbf{z}$ connecting the interior points $p_\mathbf{z}$ and $q_\mathbf{z}$. This interior line $k_\mathbf{z}$ corresponds to a unique vertex $\mathbf{k}$ of the graph Γ and we have found a global line $\mathbf{k}$ joining the global points p and q. If $\mathbf{h}$ were another global line joining the global points p and q, again by proposition 2.5.2 we would find a vertex $\mathbf{m} \in \{\mathbf{k}, \mathbf{h}\}$, yielding a contradiction to the linearity of $\mathcal{G}_\mathbf{m}$. So the global geometry $\mathcal{G}_\Gamma$ is a linear space.

Next we verify the axiom of Veblen-Young in $\mathcal{G}_\Gamma$, thus let a, b, c, d and e be global points of $\mathcal{G}_\Gamma$ such that the global points a, b, c and a, d, e are collinear triples on distinct global lines, say $\mathbf{l}^{abc}$, the joining line of a, b and c and $\mathbf{k}^{ade}$, the joining line of a, d and e. We claim that the joining line $\mathbf{m}^{bd}$ of the points b and d intersects the joining line $\mathbf{n}^{ce}$ of the points c and e. Due to proposition 2.5.2 we fix a vertex $\mathbf{z}$ in Γ such that $\mathbf{m}^{bd} \perp \mathbf{z} \perp \mathbf{n}^{ce}$. The claim follows now from local analysis of the interior space $\mathcal{G}_\mathbf{z}$ and from the propositions 2.3.7 and 2.5.5. Indeed the interior space $\mathcal{G}_\mathbf{z}$ contains the interior lines $m_\mathbf{z}^{bd}$ and $n_\mathbf{z}^{ce}$ and the interior points $b_\mathbf{z}, c_\mathbf{z}, d_\mathbf{z}$ and $e_\mathbf{z}$. Moreover, the space $\mathcal{G}_\mathbf{z}$ contains also the connecting line of $b_\mathbf{z}$ and $c_\mathbf{z}$ and the line joining $d_\mathbf{z}$ and $e_\mathbf{z}$, thus $l_\mathbf{z}^{abc}$ and $k_\mathbf{z}^{ade}$ are interior lines in $\mathcal{G}_\mathbf{z}$. Therefore $c \cap \mathcal{L}_\mathbf{z}$ is an interior point $c_\mathbf{z}$ in the space $\mathcal{G}_\mathbf{z}$. Since $\mathcal{G}_\mathbf{z}$ is a projective space by proposition 2.3.7 and the five interior points $a_\mathbf{z}, b_\mathbf{z}, c_\mathbf{z}, d_\mathbf{z}$ and $e_\mathbf{z}$ satisfy the axiom of Veblen-Young cf. page 2 or definition A.1.2, we conclude that the interior lines $m_\mathbf{z}^{bd}$ and $n_\mathbf{z}^{ce}$ have a point in common, say $f_\mathbf{z}$. Now we can extend the interior point $f_\mathbf{z}$ to a global point f of the geometry $\mathcal{G}_\Gamma$ and, of course, $\mathbf{m}^{bd}, \mathbf{n}^{ce} \in f$ by the fact that $\mathbf{m}^{bd}, \mathbf{n}^{ce} \in f_\mathbf{z}$. ■

Definition 2.5.8 With the symbol $\langle \mathbf{x}^\perp \rangle$ for a vertex $\mathbf{x}$ in the graph Γ we will denoted the pair of sets $(\mathcal{P}^\mathbf{x}, \mathcal{L}_\mathbf{x}) \subseteq (\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ such that $\mathcal{P}^\mathbf{x}$ contains all global points of the graph Γ that have a non-empty intersection with the line set $\mathcal{L}_\mathbf{x}$ of the interior space $\mathcal{G}_\mathbf{x}$, i.e. $p \in \mathcal{P}^\mathbf{x}$ if and only if $p \cap \mathcal{L}_\mathbf{x} \neq \emptyset$.

Let p be a global point of the geometry $\mathcal{G}_\Gamma$. We denote with $\langle p^\perp \rangle$ the pair of sets

$$(\mathcal{P}^p, \mathcal{L}^p) := \Big(\bigcup_{\mathbf{x} \in p} \mathcal{P}^\mathbf{x}, \bigcup_{\mathbf{x} \in p} \mathcal{L}_\mathbf{x}\Big) = \bigcup_{\mathbf{x} \in p} \langle \mathbf{x}^\perp \rangle \subseteq (\mathcal{P}_\Gamma, \mathcal{L}_\Gamma).$$

We claim that $\langle \mathbf{x}^\perp \rangle$ is a hyperline of $\mathcal{G}_\Gamma$ for each global line $\mathbf{x}$ and $\langle p^\perp \rangle$ is a hyperplane of the geometry $\mathcal{G}_\Gamma$ for each global point p of $\mathcal{G}_\Gamma$.

Lemma 2.5.9 *Let $n \geq 7$. The geometries $\langle \mathbf{x}^\perp \rangle$ and $\langle p^\perp \rangle$ are subspaces of the global space $\mathcal{G}_\Gamma$.*

Proof: We start to prove that for any vertex $\mathbf{x}$ of the graph Γ the point-line geometry $\langle \mathbf{x}^\perp \rangle = (\mathcal{P}^\mathbf{x}, \mathcal{L}_\mathbf{x})$ is a subspace, so for any two global points $p, q \in \mathcal{P}^\mathbf{x}$ the global line $\mathbf{l}$ joining both points has to be an interior line of $\mathcal{L}_\mathbf{x}$.

Let p, q be two different global points of $\mathcal{P}^{\mathbf{x}}$. Then by proposition 2.5.5 we derive two interior points $p_{\mathbf{x}} = p \cap \mathcal{L}_{\mathbf{x}}$ and $q_{\mathbf{x}} = q \cap \mathcal{L}_{\mathbf{x}}$ of the interior space $\mathcal{G}_{\mathbf{x}}$. By the linearity of the space $\mathcal{G}_{\mathbf{x}}$ there exists a unique interior line $l_{\mathbf{x}}$ which is incident to both interior points $p_{\mathbf{x}}$ and $q_{\mathbf{x}}$. In particular the global line $\mathbf{l}$ is contained in $\mathcal{L}_{\mathbf{x}}$ and connects the two global points p and q, which proves the claim.

Next we show that for any two different global points a and b of the point set $\mathcal{P}^p$ of some global point p of $\mathcal{G}_\Gamma$, the global line $\mathbf{l}$ joining a and b is an element of the line set $\mathcal{L}^p$. Let $\mathbf{x}$ be a vertex of p such that $a \in \mathcal{P}^{\mathbf{x}}$ and $\mathbf{y}$ be a vertex of p with $b \in \mathcal{P}^{\mathbf{y}}$ and fix two interior lines $h^a_{\mathbf{x}} \in a_{\mathbf{x}}$ and $h^b_{\mathbf{y}} \in b_{\mathbf{y}}$. By lemma 2.5.2 we find a vertex $\mathbf{z}$ in Γ adjacent to $\mathbf{x}$ and $\mathbf{y}$, thus in the graph Γ we have the chain $\mathbf{h}^a \perp \mathbf{x} \perp \mathbf{z} \perp \mathbf{y} \perp \mathbf{h}^b$. In the interior space $\mathcal{G}_{\mathbf{x}}$ we regard the intersection of the $(n-3)$-dimensional subspace $U^{\mathbf{x}}_{x_{\mathbf{z}},y_{\mathbf{z}}}$ and the hyperline $(h^a_{\mathbf{x}})^\pi$. Since this intersection space has dimension at least three, we can fix an interior line $g_{\mathbf{x}}$ inside this space and get the chain $\mathbf{g} \perp \mathbf{h}^a \perp \mathbf{x} \perp \mathbf{g} \perp \mathbf{y} \perp \mathbf{h}^b$ in Γ. Therefore, without loss of generality, we can assume that $a \cap \mathcal{L}_{\mathbf{z}}$ is an interior point $a_{\mathbf{z}}$ and $h^a_{\mathbf{x}} = h^a_{\mathbf{z}}$.

In the interior space $\mathcal{G}_{\mathbf{z}}$ we obtain the following spaces and its intersections. The space $U^{\mathbf{z}}_{x_{\mathbf{z}},y_{\mathbf{z}}}$ is of dimension $n-3$ and $a^\pi_{\mathbf{z}}$ is a hyperplane of $\mathcal{G}_{\mathbf{z}}$, thus $a^\pi_{\mathbf{z}} \cap U^{\mathbf{z}}_{x_{\mathbf{z}},y_{\mathbf{z}}} =: H^{\mathbf{z}}_{x_{\mathbf{z}},y_{\mathbf{z}},a_{\mathbf{z}}}$ is at least an $(n-4)$-dimensional subspace of the hyperplane $p^\pi_{\mathbf{z}}$ in $\mathcal{G}_{\mathbf{z}}$. Since $H^{\mathbf{z}}_{x_{\mathbf{z}},y_{\mathbf{z}},a_{\mathbf{z}}}$ is a subspace of $y^\pi_{\mathbf{z}}$ whence $H^{\mathbf{y}}_{x_{\mathbf{z}},y_{\mathbf{z}},a_{\mathbf{z}}}$ is a subspace of dimension at least $n-4$ in the interior space $\mathcal{G}_{\mathbf{y}}$. Considering also the hyperplane $b^\pi_{\mathbf{y}}$ in $\mathcal{G}_{\mathbf{y}}$ we can choose a line $t_{\mathbf{y}}$ in the intersection of $H^{\mathbf{y}}_{x_{\mathbf{z}},y_{\mathbf{z}},a_{\mathbf{z}}}$ and $b^\pi_{\mathbf{y}}$ and regard the chain of vertices $\mathbf{t} \perp \mathbf{h}^a \perp \mathbf{x} \perp \mathbf{t} \perp \mathbf{y} \perp \mathbf{h}^b \perp \mathbf{t}$. Therefore w.l.o.g. we can also assume that $b \cap \mathcal{L}_{\mathbf{z}}$ is an interior point $b_{\mathbf{z}}$ and $h^b_{\mathbf{y}} = h^b_{\mathbf{z}}$.

Consider the path $\mathbf{z} \perp \mathbf{h}^a \perp \mathbf{x} \perp \mathbf{z} \perp \mathbf{y} \perp \mathbf{h}^b \perp \mathbf{z}$ in Γ to see that $p \cap \mathcal{L}_{\mathbf{z}}$ is the interior point $p_{\mathbf{z}}$ and the joining line $l^{a,b}_{\mathbf{z}}$ of the interior points $a_{\mathbf{z}}$ and $b_{\mathbf{z}}$ is contained in the hyperplane $p^\pi_{\mathbf{z}}$. Thus there is an interior line $w_{\mathbf{z}}$ in the interior point $p_{\mathbf{z}}$ such that $l^{a,b}_{\mathbf{z}} \subseteq w^\pi_{\mathbf{z}}$. It follows directly that $\mathbf{l}^{a,b} \in \mathcal{L}_{\mathbf{w}} \subseteq \mathcal{L}^p$, hence the joining line $\mathbf{l}^{a,b}$ is contained in $\langle p^\perp \rangle$. ∎

Proposition 2.5.10 *Let $n \geq 7$. The subspace $\langle \mathbf{x}^\perp \rangle$ is a hyperline of the projective space $\mathcal{G}_\Gamma$ for each vertex $\mathbf{x}$ in Γ.*

Proof: The set $\langle \mathbf{x}^\perp \rangle$ is not a hyperplane or the whole space, due to the fact that none of the global lines $\mathbf{h} \in \mathcal{L}_{\mathbf{x}}$ intersects the global line $\mathbf{x}$.

Let E be any global plane of $\mathcal{G}_\Gamma$. We claim that the space $\mathcal{G}_\Gamma$ contains a global point p in the subspace $\langle x^\perp \rangle$ such that p is a also a point of the global plane E.

Since $\mathcal{G}_\Gamma$ is a projective space we can choose in the plane E two distinct global lines $\mathbf{k}$ and $\mathbf{l}$, which span the global plane E. Certainly, the vertices $\mathbf{k}$ and $\mathbf{l}$ are not adjacent in Γ but by the fact that the diameter of Γ is two, due to proposition 2.5.2, we find a vertex $\mathbf{z}$ in the graph Γ adjacent to the vertices $\mathbf{k}$ and $\mathbf{l}$. Therefore in $\mathcal{G}_{\mathbf{z}}$ we consider the orthogonal space $k^\pi_{\mathbf{z}}$ of the interior line $k_{\mathbf{z}}$. The intersection of the spaces $k^\pi_{\mathbf{z}}$ and the interior plane $\langle k_{\mathbf{z}}, l_{\mathbf{z}} \rangle$ contains an interior point $p_{\mathbf{z}}$. We choose an interior line $h_{\mathbf{z}}$ in $k_{\mathbf{z}}^{\pi}$ through the point $p_{\mathbf{z}}$. Obviously, the interior line $h_{\mathbf{z}}$ corresponds to a unique vertex $\mathbf{h}$ adjacent to $\mathbf{k}$ and we can assume that the vertices $\mathbf{k}$ and $\mathbf{h}$ are not adjacent to $\mathbf{x}$, as otherwise $\mathbf{h}, \mathbf{k} \in \mathcal{L}_{\mathbf{x}}$ and we are done.

However using proposition 2.5.2 again we find a vertex $\mathbf{y}$ such that $\mathbf{h} \perp \mathbf{y} \perp \mathbf{x}$ and with the construction from proposition 2.5.2 we choose also a vertex $\mathbf{s}$ in the subgraph $\{\mathbf{k}, \mathbf{h}, \mathbf{x}\}^{\perp}$.

In the interior space $\mathcal{G}_{\mathbf{s}}$ the orthogonal subspace $x_{\mathbf{s}}^{\pi}$ of the interior line $x_{\mathbf{s}}$ has dimension $n-2 \geq 5$ and the space spanned by the interior lines $k_{\mathbf{s}}$ and $h_{\mathbf{s}}$ is a four-dimensional subspace which contains the global plane $\langle k_{\mathbf{s}}, p_{\mathbf{s}} \rangle = \langle \mathbf{k}, \mathbf{l} \rangle = E$. Thus the intersection space of E and $x_{\mathbf{s}}^{\pi}$ contains an interior point $q_{\mathbf{s}}$. We conclude that the global plane E and the space $\langle x^{\perp} \rangle$ contain the global point q, which shows that $\langle x^{\perp} \rangle$ intersects each global plane E in at least one global point. Consequently $\langle x^{\perp} \rangle$ is a hyperline of the projective space $\mathcal{G}_{\Gamma}$. ∎

Proposition 2.5.11 *Let $n \geq 7$. The subspace $\langle p^{\perp} \rangle$ is a hyperplane of the projective space $\mathcal{G}_{\Gamma}$ for each global point p of Γ.*

Proof: Certainly the subspace $\langle p^{\perp} \rangle$ is properly contained in the projective space $\mathcal{G}_{\Gamma}$, as for each vertex $\mathbf{x} \in p$ the intersection $p \cap \mathcal{L}_{\mathbf{x}} = \varnothing$ implying that $p \notin \langle p^{\perp} \rangle$.

Furthermore let $\mathbf{l}$ be any global line of Γ, we will give a proof that the projective space $\mathcal{G}_{\Gamma}$ contains a global point q with the property that $\mathbf{l} \in q$ and $q \in \langle p^{\perp} \rangle$.

We fix a global line $\mathbf{k}$ of the point p and from lemma 2.5.2 we can find a line $\mathbf{z}$ in the graph Γ such that $\mathbf{k} \perp \mathbf{z} \perp \mathbf{l}$. Since the orthogonal space $p_{\mathbf{z}}^{\pi}$ of the interior point $p_{\mathbf{z}}$ has dimension $n-1$ in $\mathcal{G}_{\mathbf{z}}$, the interior line $l_{\mathbf{z}}$ intersects $p_{\mathbf{z}}^{\pi}$ in at least one interior point $q_{\mathbf{z}}$. Therefore the global point q contains the global line $\mathbf{l}$ and the global point p contains a global line $\mathbf{m}$ such that $q_{\mathbf{z}} \subseteq m_{\mathbf{z}}^{\pi}$ in $\mathcal{G}_{\mathbf{z}}$. Consequently $\mathbf{l} \in q \cap \mathcal{L}_{\mathbf{m}} \neq \varnothing$ and q is a point of the subspace $\langle p^{\perp} \rangle$. ∎

Remark 2.5.12 Notice that the hyperline $\langle \mathbf{x}^{\perp} \rangle$ is isomorphic to the interior space $\mathcal{G}_{\mathbf{x}}$. Indeed the map $\alpha : \langle \mathbf{x}^{\perp} \rangle \to \mathcal{G}_{\mathbf{x}}$ with $\alpha(\mathbf{l}) = \mathbf{l}$ for every $\mathbf{l} \in \mathcal{L}_{\mathbf{x}}$ and $\alpha(p) = p \cap \mathcal{L}_{\mathbf{x}}$ for $p \in \mathcal{P}^{\mathbf{x}}$, is a bijective incidence preserving map. So α is an isomorphism between the point-line geometries $\langle \mathbf{x}^{\perp} \rangle$ and $\mathcal{G}_{\mathbf{x}}$ for every vertex $\mathbf{x}$ of Γ. Because $\mathcal{G}_{\mathbf{x}} \cong \mathbb{P}(V_n)$ by proposition 2.3.7, we observe that $\langle \mathbf{x}^{\perp} \rangle$ is isomorphic to the point-line geometry of the complex unitary vector space V_n.

The fundamental theorem of projective geometry, see [16], implies that $\mathcal{G}_{\Gamma}$ is isomorphic to the projective space of the $(n+2)$-dimensional complex vector space.

Proposition 2.5.13 *Let $n \geq 7$ and let Γ be a connected graph locally $\mathbf{S}(V_n)$. The point-line geometry $\mathcal{G}_{\Gamma} = (\mathcal{P}_{\Gamma}, \mathcal{L}_{\Gamma})$ is isomorphic to the point-line geometry of the complex vector space V_{n+2}.*

Proof: Using again the fundamental theorem of projective geometry and the fact that $\mathcal{G}_\Gamma$ contains projective subspaces defined over the field of the complex numbers, the space $\mathcal{G}_\Gamma$ itself is a projective space over the complex numbers $\mathbb{C}$. Its dimension is determined as follows: For every line $\mathbf{x} \in \mathcal{L}_\Gamma$ the space $\mathcal{G}_\Gamma$ contains the codimension two subspace $\langle \mathbf{x}^\perp \rangle$ of dimension n, so the space $\mathcal{G}_\Gamma$ has dimension $n+2$. ∎

The last step to prove our main theorem will be to define an endomorphism π_Γ of the global space $\mathcal{G}_\Gamma$, which turns out to be an anisotropic polarity. This polarity allows us to define a line graph on the projective space $\mathcal{G}_\Gamma$, which will be isomorphic to the graph Γ. To define the polarity on $\mathcal{G}_\Gamma$ we need two statements about the classification of hyperplanes and hyperlines in $\mathcal{G}_\Gamma$.

Lemma 2.5.14 *Let H be a hyperline of $\mathcal{G}_\Gamma$, then $H = \langle \mathbf{h}^\perp \rangle$ for some unique global line* $\mathbf{h}$ *in $\mathcal{G}_\Gamma$.*

Proof: Let $\mathbf{k}$ be a global line in the hyperline H. Then due to lemma 2.5.10 the projective space $\mathcal{G}_\Gamma$ contains the hyperline $\langle \mathbf{k}^\perp \rangle$. Moreover the hyperline $\langle \mathbf{k}^\perp \rangle$ does not intersect the global line $\mathbf{k}$. Indeed $\langle \mathbf{k}^\perp \rangle \cap H = L$ is either a hyperline or a hyperplane of the interior space $\mathcal{G}_\mathbf{k}$. Suppose L is a subspace of codimension one in $\mathcal{G}_\mathbf{k}$. Then L is a hyperplane of the hyperline H, which leads to the fact that L intersects each line of H in at least one point. Thus L intersects also the global line $\mathbf{k}$, contradiction.

Therefore L is a hyperline of the interior space $\mathcal{G}_\mathbf{k}$ and it follows with the polarity π from lemma 2.2.17 that $\pi(L) = l_\mathbf{k}$ is an interior line in $\mathcal{G}_\mathbf{k}$ and of course $\mathbf{k} \in \langle \mathbf{l}^\perp \rangle$ as well as $L \subseteq \langle \mathbf{l}^\perp \rangle$. Therefore $H = \langle \mathbf{k}, L \rangle = \langle \mathbf{l}^\perp \rangle$ for the unique global line $\mathbf{l}$. ∎

Lemma 2.5.15 *Let P be a hyperplane of $\mathcal{G}_\Gamma$, then $P = \langle p^\perp \rangle$ for a unique global point p in $\mathcal{G}_\Gamma$.*

Proof: Now let $\mathbf{k}$ be a global line of the hyperplane P, then $\langle \mathbf{k}^\perp \rangle$ is a hyperline of the projective space $\mathcal{G}_\Gamma$, which does not intersect the global line $\mathbf{k}$. In fact the hyperline $\langle \mathbf{k}^\perp \rangle$ is not properly contained in the hyperplane P. Hence by the dimension formula $\langle \mathbf{k}^\perp \rangle \cap P = L$ is a hyperplane of the interior space $\mathcal{G}_\mathbf{k}$.

With the polarity π of the projective space $\mathcal{G}_\mathbf{k}$ we obtain that $\pi(L) = p_\mathbf{k}$ is an interior point of the interior space $\mathcal{G}_\mathbf{k}$. As before we have that $\mathbf{k} \in \langle p^\perp \rangle$ and $L \subseteq \langle p^\perp \rangle$. This leads to the statement that the space spanned by the line $\mathbf{k}$ and the space L is incident to the subspace $\langle p^\perp \rangle$, in particular $\langle \mathbf{k}, L \rangle = \langle p^\perp \rangle$ using a dimension argument. Therefore $P = \langle \mathbf{k}, L \rangle = \langle p^\perp \rangle$ for the unique global point p, which contains the interior point $p_\mathbf{k}$. ∎

Denote the set of all hyperplanes of the projective geometry $\mathcal{G}_\Gamma$ with $\mathcal{H}_\Gamma$ and define the transformation $\pi_\Gamma : \mathcal{P}_\Gamma \cup \mathcal{H}_\Gamma \to \mathcal{P}_\Gamma \cup \mathcal{H}_\Gamma$ to map each global point p to the hyperplane $\langle p^\perp \rangle$ and each hyperplane $\langle p^\perp \rangle$ to the global point p. This is a well-defined map by the two preceding lemmas.

To conclude that π_Γ is a polarity on the finite dimensional projective space $\mathcal{G}_\Gamma$, due to lemma 9.1.6 of [16], it suffices to show that for all global points p, q in $\mathcal{P}_\Gamma$ the map π_Γ satisfies the property that $p \in \langle q^\perp \rangle$ implies $q \in \langle p^\perp \rangle$. Let p be a point incident to the hyperplane $\langle q^\perp \rangle$, then we can find a global line $\mathbf{k}$ of the global point q with the property that $p \cap \mathcal{L}_\mathbf{k} = p_\mathbf{k}$ is an interior point of $\mathcal{G}_\mathbf{k}$ by definition 2.5.8. So fix an interior line $h_\mathbf{k}$ incident to the interior point $p_\mathbf{k}$ in $\mathcal{G}_\mathbf{k}$. The global line $\mathbf{h}$ is adjacent to $\mathbf{k}$ in the graph Γ and thus $\mathbf{k} \in \mathcal{L}_\mathbf{h}$. Due to proposition 2.5.5 we know that $q \cap \mathcal{L}_\mathbf{h}$ is an interior point in $\mathcal{G}_\mathbf{h}$ and $\pi_\Gamma(\langle q^\perp \rangle) = q$ is incident to $\pi_\Gamma(p) = \langle p^\perp \rangle$ by definition 2.5.8 again. The polarity π_Γ is anisotropic, because $p \notin \langle p^\perp \rangle$ for all $p \in \mathcal{P}_\Gamma$. Thus we have proved the following.

Proposition 2.5.16 *The endomorphism π_Γ is an anisotropic polarity on the complex projective space $\mathcal{G}_\Gamma$. In particular, π_Γ is induced by a scalar product on some complex vector space.*

Proof: It remains to prove the final claim. By proposition 2.5.13, the projective space $\mathcal{G}_\Gamma$ is isomorphic to the complex projective space $\mathbb{P}(V_{n+2})$. By the classification of polarities, see theorem 1.1.5, π_Γ is induced by a symmetric or an alternating bilinear form or a hermitian sesquilinear form on V_{n+2}. Since π_Γ is anisotropic, it has to be induced by an anisotropic hermitian sesquilinear form, i.e., a complex scalar product. ∎

The final step is to define the line graph $\mathbf{S}(\mathcal{G}_\Gamma)$ of the unitary projective space $\mathcal{G}_\Gamma = (\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$. The vertex set of $\mathbf{S}(\mathcal{G}_\Gamma)$ is the set of global lines of $\mathcal{G}_\Gamma$ where two distinct global lines $\mathbf{k}$ and $\mathbf{l}$ are adjacent, in symbols $\mathbf{k} \perp_\Gamma \mathbf{l}$ if and only if $\mathbf{k} \in \langle \mathbf{l}^\perp \rangle = \pi_\Gamma(\mathbf{l})$ or equivalently if and only if $\mathbf{l} \in \langle \mathbf{k}^\perp \rangle = \pi_\Gamma(\mathbf{k})$.

Proposition 2.5.17 *The graph Γ is isomorphic to the line graph $\mathbf{S}(\mathcal{G}_\Gamma)$ of $\mathcal{G}_\Gamma$.*

Proof: Let φ be the map $\Gamma \to \mathbf{S}(\mathcal{G}_\Gamma)$ defined by $\mathbf{x} \mapsto \mathbf{x}$. Since the vertex set of Γ is equal to $\mathcal{L}_\Gamma$, the map φ is a bijection. Take two adjacent vertices $\mathbf{x}$ and $\mathbf{y}$ in Γ, obviously $\mathbf{x}$ is contained in $\langle \mathbf{y}^\perp \rangle = \pi_\Gamma(\mathbf{y})$ and so $\mathbf{x}$ is adjacent to $\mathbf{y}$ in the graph $\mathbf{S}(\mathcal{G}_\Gamma)$. It follows that φ is an isomorphism between the graph Γ and the line graph of $\mathbf{S}(\mathcal{G}_\Gamma)$. ∎

Theorem 2.1.2 *Let $n \geq 7$ and let Γ be a connected locally $\mathbf{S}(V_n)$ graph. Then the graph Γ is isomorphic to $\mathbf{S}(V_{n+2})$.*

Proof: The statement follows immediately from proposition 2.5.17, because the global point-line geometry $\mathcal{G}_\Gamma$ on Γ together with the anisotropic polarity π_Γ is isomorphic to the projective space $\mathbb{P}(V_{n+2})$ with the polarity induced by the complex scalar product, cf. propositions 2.5.13 and 2.5.16. ∎

CHAPTER THREE

On the finite hyperbolic unitary geometry for $n \geq 7$

3.1 Local recognition of the hyperbolic line graph of finite unitary space for $n \geq 7$

The geometry on the points and hyperbolic lines of a non-degenerate finite unitary polar space (or, short, **hyperbolic unitary geometry**) is interesting for a number of reasons. One reason is the fact that every pair of intersecting hyperbolic lines either spans a **dual affine plane** (sometimes also called a **symplectic plane**), for a definition see page 64, or some well-understood **linear plane** related to a classical unital, cf. [72], [85]. With some additional technical hypotheses this observation can actually be used to characterise the hyperbolic unitary geometries over finite fields. This characterisation of the geometry on singular points and hyperbolic lines of a finite unitary space — the hyperbolic unitary geometry — is described in the unpublished manuscript [25] of H. Cuypers. In section 3.4 we will study Cuypers' approach in detail, his result is restated as theorem 3.4.1. Another observation is the 1-1 correspondence between the set of long root subgroups, resp. fundamental $\mathrm{SU}_2(\mathbb{F}_{q^2})$ subgroups of a $\mathrm{SU}_n(\mathbb{F}_{q^2})$ and the points, resp. hyperbolic lines of the corresponding unitary geometry via the map that assigns the respective groups to their commutator in the module. This correspondence is well-known, see e.g. [91, chapter 2]. Our interest in the unitary hyperbolic geometry stems from this second observation. This chapter can be viewed as a cousin to chapter 4 of [35], see also [39] where the same setting is studied for the groups $Sp_{2n}(\mathbb{F}_{\shortparallel})$ for arbitrary fields and to chapter 2.

However these first-grade relatives are much better behaved and a lot easier to handle than the current chapter. The increased difficulty compared to [39] originates from the fact that we prove theorem 3.1.2 for $n \geq 7$ instead of $n \geq 8$ (odd-dimensional non-degenerate symplectic forms do not exist), while the increased difficulty compared to chapter 2 comes from the fact that subspaces of non-degenerate subspaces can be very far from being non-degenerate, whereas subspaces of anisotropic subspaces are anisotropic.

We recall the definition of the hyperbolic line graph.

Definition 3.1.1 Let $U_n = U$ denote an n-dimensional vector space over $\mathbb{F}_{q^2}$ endowed with a non-degenerate hermitian form. Certainly for a subspace $U \subseteq V$ the orthogonal space of U is $U^\pi = \{x \in V \mid (x,u) = 0 \text{ for all } u \in U\}$. The **hyperbolic line graph** $\mathbf{G}(U_n)$ is the graph on the hyperbolic lines, i.e., the non-degenerate two-dimensional subspaces, of U_n, where hyperbolic lines l and m are adjacent (in symbols $l \perp m$) if and only if l is perpendicular to m with respect to the unitary form.

For a vertex x in $\mathbf{G}(U_n)$ we denote the neighbourhood graph of x with $\mathbf{G}(U_n)_x = x^\perp$ and for a set of vertices X of $\mathbf{G}(U_n)$ we define $\mathbf{G}(U_n)_X = \bigcap_{x \in X} x^\perp = X^\perp$.

The first part of this chapter focuses exclusively on the hyperbolic lines and their relative positions.

It is easily seen (cf. proposition 3.3.3) that the graph $\mathbf{G}(U_n)$ is locally $\mathbf{G}(U_{n-2})$. Conversely, this property is characteristic for this graph for sufficiently large n:

Theorem 3.1.2 *Let $n \geq 7$, let $q \geq 3$ be a prime power, and let Γ be a connected graph that is locally $\mathbf{G}(U_n)$. Then Γ is isomorphic to $\mathbf{G}(U_{n+2})$.*

The requirement in the preceding theorem that Γ be connected results from the fact that a graph is locally Δ if and only if each of its connected components is locally Δ. So in fact, its primary role is to provide irreducibility.

For $n \geq 8$ this result has been stated without proof in the PhD thesis, Theorem 4.5.3 of [35], of R. Gramlich. Comparing the proofs of lemmata 3.6.5, 3.6.6 and 3.6.7 with the proof of lemma 3.6.8, the reader will understand why the case $n = 7$ is so much more difficult than the case $n \geq 8$. Counter-examples to the local recognition are only known for $n = 6$. They come from the exceptional groups of type ${}^2E_6(\mathbb{F}_{q^2})$, see [91].

As mentioned before, the motivation of our research was of group-theoretic nature. If the field $\mathbb{F}$ has characteristic distinct from 2, translating theorem 3.1.2 into the language of group theory yields the following.

Theorem 3.1.3 *Let $n \geq 7$ and let q be an odd prime power. Let G be a group with subgroups A and B isomorphic to $\mathrm{SU}_2(\mathbb{F}_{q^2})$, and denote the central involution of A by x and the central involution of B by y. Furthermore, assume the following holds:*

- $C_G(x) = X \times K$ *with* $K \cong \mathrm{GU}_n(\mathbb{F}_{q^2})$ *and* $A \leq X$;
- $C_G(y) = Y \times J$ *with* $J \cong \mathrm{GU}_n(\mathbb{F}_{q^2})$ *and* $Y \leq B$;
- *A is a fundamental $\mathrm{SU}_2(\mathbb{F}_{q^2})$ subgroup of J;*
- *B is a fundamental $\mathrm{SU}_2(\mathbb{F}_{q^2})$ subgroup of K;*
- *there exists an involution in $J \cap K$ that is the central involution of a fundamental $\mathrm{SU}_2(\mathbb{F}_{q^2})$ subgroup of both J and K.*

If $G = \langle J, K \rangle$, then $\mathrm{PSU}_{n+2}(\mathbb{F}_{q^2}) \leq G/Z(G) \leq \mathrm{PGU}_{n+2}(\mathbb{F}_{q^2})$.

In sections 3.2 and 3.3 we study properties of the hyperbolic line graph $\mathbf{G}(U_n)$ for $n \geq 5$. Section 3.4 deals with the interaction of the graph $\mathbf{G}(U_n)$ with the hyperbolic unitary geometry. In particular, we investigate how to reconstruct the hyperbolic unitary geometry using intrinsic properties of $\mathbf{G}(U_n)$ only. In section 3.5 we study embeddings of $\mathbf{G}(U_{n-2})$ in $\mathbf{G}(U_n)$, which provides us with valuable information for the proof of theorem 3.1.2 that we give in section 3.6. Most of our arguments are based on counting in subspaces of U_n of various dimensions and ranks, so that for the convenience of the reader we include a collection of known counting results in appendix B. For quick reference we also give some tables containing the necessary information at the end of appendix B on pages 262 and 264. A proof of theorem 3.1.3 is not included in this thesis, because the problem of how to deduce a result like theorem 3.1.3 from a result like theorem 3.1.2 has been thoroughly studied in [35] and, thus, is well-understood.

3.2 The hyperbolic line graph of U_5

Let $q \geq 3$ be a prime power and let U_5 be a five-dimensional non-degenerate unitary vector space over $\mathbb{F}_{q^2}$ with polarity π. So U_5 is equipped with a non-degenerate hermitian form $(\cdot,\cdot)$. We have defined the graph $\mathbf{G}(U_5)$ with the set of non-degenerate two-dimensional subspaces of U_5 as the set of vertices where two vertices l and m are adjacent if and only if $l \subset m^\pi$. The aim of this section is to reconstruct the unitary vector space U_5 from the graph $\mathbf{G}(U_5)$. To this end we will define a point-line geometry $G = (\mathcal{P}, \mathcal{L}, \subset)$ using intrinsic properties of the graph $\mathbf{G}(U_5)$ and establish an isomorphism between G and the geometry on singular points and hyperbolic lines of U_5. From there U_5 is easily recovered.

Using this strategy as our road map we first determine the diameter of $\mathbf{G}(U_5)$. Any two different vertices l and m of $\mathbf{G}(U_5)$ can have distance one, l and m are adjacent, two, three, or four. In the following few lemmata, for each case we will thoroughly investigate the configuration of the lines l and m in the unitary vector space U_5.

A reformulation of the adjacency relation for the hyperbolic line $\mathbf{G}(U_5)$ is the following observation.

Observation 3.2.1 Let l and m be two different hyperbolic lines of U_5. Then l and m have distance one in the graph $\mathbf{G}(U_5)$ if and only if $l \subset m^\pi$.

Lemma 3.2.2 *Let l and m be two different hyperbolic lines of U_5. Then l and m have distance two in $\mathbf{G}(U_5)$ if and only if the subspace $\langle l, m\rangle$ is a non-degenerate plane in U_5.*

Proof: Let l and m be two hyperbolic lines of U_5 which have distance two in $\mathbf{G}(U_5)$. Thus the graph $\mathbf{G}(U_5)$ contains a vertex z, which is a hyperbolic line in U_5, adjacent to the vertices l and m. Its orthogonal space z^π, a regular plane of U_5, contains the two different hyperbolic lines l and m, whence the hyperbolic lines l and m span the non-degenerate plane z^π.

Conversely, suppose that $\langle l, m\rangle$ is a regular three-dimensional subspace of U_5. Since U_5 is a five-dimensional non-degenerate unitary vector space, the polar space of $\langle l, m\rangle$ is a hyperbolic line $h = \langle l, m\rangle^\pi$ of U_5. By definition the vertex h is adjacent to the vertices l and m in $\mathbf{G}(U_5)$. Since the hyperbolic lines l and m intersect in U_5 and, thus, span the plane $\langle l, m\rangle$, it follows that $l \not\perp m$ and therefore the vertices l and m have distance two in $\mathbf{G}(U_5)$. ∎

Lemma 3.2.3 *Let l and m be two different hyperbolic lines of U_5. Then l and m have distance three in $\mathbf{G}(U_5)$ if and only if l and m are two non-intersecting hyperbolic lines such that $l^\pi \cap m$ is a one-dimensional subspace p of U_5.*

Proof: Suppose the vertices l and m have distance three in the graph $\mathbf{G}(U_5)$. Then we find a vertex z in the graph $\mathbf{G}(U_5)$ adjacent to l such that $\langle z, m\rangle$ is a regular plane of U_5 using lemma 3.2.2. Since the hyperbolic lines z and m span a plane in U_5 the intersection of the subspaces z and m is a one-dimensional space $p = m \cap z$. As $z \subseteq l^\pi$, the hyperbolic line m intersects the subspace l^π in at least the point p. Since the vertices l and m are not adjacent in $\mathbf{G}(U_5)$, we have $m \nsubseteq l^\pi$, so $m \cap l^\pi = p$.

In order to prove the first implication of the statement it is left to show that the hyperbolic lines l and m do not intersect in U_5. By way of contradiction we assume that $\langle l, m\rangle$ is a three-dimensional subspace. The plane $\langle l, m\rangle$ is degenerate by lemma 3.2.2, thus $l^\pi \cap m^\pi$ is a singular two-dimensional subspace of U_5. Since p, the intersection point of m and l^π, is incident to the hyperbolic line m, we have $p \notin \mathrm{rad}(\langle m, l\rangle)$ and $m^\pi \subseteq p^\pi$, whence $m^\pi \cap l^\pi \subseteq p^\pi \cap l^\pi$. Of course p is either singular or regular. Furthermore $\dim(m^\pi \cap l^\pi) = 2 = \dim(p^\pi \cap l^\pi)$, consequently $m^\pi \cap l^\pi = p^\pi \cap l^\pi$.

If p is a regular point, then $p^\pi \cap l^\pi$ is a regular line, contradicting the fact that $m^\pi \cap l^\pi$ is degenerate. If p is a singular point, then of course $p^\pi \cap l^\pi$ is a singular two-dimensional subspace s of rank one containing the point p itself and the radical of $p^\pi \cap l^\pi$. Therefore $p = \mathrm{rad}(p^\pi \cap l^\pi) = \mathrm{rad}(m^\pi \cap l^\pi) = \mathrm{rad}(\langle m, l\rangle) \neq p$, a contradiction. Thus $\langle m, l\rangle$ has to be a four-dimensional space and the two hyperbolic lines l and m have a trivial intersection in U_5.

Now for the other implication. If l and m are two non-intersecting hyperbolic lines in U_5 such that $l^\pi \cap m$ is a one-dimensional subspace p, then, due to lemma 3.2.1 and lemma 3.2.2, the vertices l and m do not have distance one or two in the graph $\mathbf{G}(U_5)$. To prove the statement, we construct a hyperbolic line z in the subspace l^π with the property that the subspace $\langle m, z\rangle$ is a non-degenerate plane in U_5, implying that $l \perp z$ and that the distance between the vertices z and m in $\mathbf{G}(U_5)$ is two, by lemma 3.2.2.

Consider the orthogonal subspace l^π of the hyperbolic line l and two points $p \in l^\pi \cap m$ and $x \in l^\pi \cap m^\pi$. Note that p and x are uniquely determined by the assumptions that $\dim(l^\pi \cap m) = 1$ and $\dim(\langle l, m\rangle) = 4$ in U_5. Moreover $p \in x^\pi$ since $p \in m$ and $x \in m^\pi$.

If both, the point p and the point x are regular then $z = \langle p, x\rangle$ is a hyperbolic line contained in l^π, since p and x are perpendicular to each other as noted before. Furthermore $\langle m, x\rangle$ is a non-degenerate plane of U_5 due to the fact that $x \in m^\pi$, proving the statement in this special case.

If p is singular and x is regular, then $\langle p, x\rangle$ is a singular line of rank one, because p is an element of x^π as mentioned above. We consider the $q^2 - q$ hyperbolic lines h_i, $1 \leq i \leq q^2 - q$, in l^π incident to x, cf. lemma B.1.5. Any two different hyperbolic lines h_i and h_j span the plane l^π and each subspace h_i^π is a regular plane in x^π for $1 \leq i < j \leq q^2 - q$. Moreover the intersection of h_i^π with the hyperbolic line m is a point $r_i = h_i^\pi \cap m$ in x^π with $r_i \neq r_j$ for $1 \leq i < j \leq q^2 - q$. Indeed, $m \notin h_i^\pi$, because $h_i \cap m^\pi$ is the one-dimensional subspace x for each hyperbolic line h_i. If $r_i = r_j$ for $i \neq j$, then we obtain that $r_i = h_i^\pi \cap m = r_j = h_j^\pi \cap m = h_i^\pi \cap h_j^\pi \cap m = \langle h_i, h_j\rangle^\pi \cap m = l \cap m = \{0\}$, a contradiction. Thus we have $q^2 - q$ different one-dimensional subspaces r_i on the hyperbolic line m, whence on the line m is a regular point $r = r_k$ for some $k \in \{1, \ldots, q^2 - q\}$, because $q^2 - q > q + 1$ for $q \geq 3$, where $q + 1$ is the number of singular points on a hyperbolic line (cf. the formula B.3 on page 255 and table B.3 on page 264). Note that the points r and p span the hyperbolic line m. Note also that $r^\pi \cap l^\pi = h_k$. For, $r_i^\pi = (h_i^\pi \cap m)^\pi = \langle h_i, m^\pi\rangle$, so r^π contains h_k; since l^π contains h_k as well and since $r^\pi \cap l^\pi$ is two-dimensional, we have $r^\pi \cap l^\pi = h_k$. Due to lemma B.1.5 each point on the hyperbolic line h_k different from the point x generates with the point p a regular two-dimensional subspace of l^π. Therefore the hyperbolic line h_k contains $q^2 - q - 1$ different regular points y_i such that $\langle y_i, p\rangle$ is a hyperbolic line. Furthermore the span of the two hyperbolic lines m and h_k is a four-dimensional space of rank at least three, since $\langle r, h_k \rangle \subseteq \langle m, h_k \rangle$ and $\mathrm{rk}(\langle r, h_k \rangle) = \mathrm{rk}(\langle r, r^\pi \cap l^\pi \rangle) = 3$.

If the four-dimensional space $\langle m, h_k \rangle$ is non-degenerate, then, by the statement of lemma B.1.6, we obtain that the hyperbolic line h_k contains at least $q^2 - 2q - 2 > 0$ (recall that $q \geq 3$) different regular points z_i such that $\langle z_i, p \rangle = z$ is a hyperbolic line and $\langle m, z \rangle = \langle r, p, z_i \rangle$ is a non-degenerate plane. Alternatively, if the rank of the four-dimensional space $\langle m, h_k \rangle$ is three, then due to lemma B.1.7 on the hyperbolic line h_k are at least $q^2 - q - 2 > 0$ different regular points z_i, which satisfy the conditions that $\langle z_i, p \rangle = z$ is a hyperbolic line and $\langle m, z \rangle = \langle r, p, z_i \rangle$ is a non-degenerate plane and we are done in this case.

Next we assume the point p to be regular and the point x to be singular. Then the hyperbolic line $h = l^\pi \cap p^\pi$ is incident to the singular point $x = l^\pi \cap m^\pi$, because p is incident to m. Moreover the regular point $r = p^\pi \cap m$ and the hyperbolic line h span a plane P of rank two or three. Due to lemma B.1.4 and lemma B.1.5 the plane P contains at least $q^2 - q$ different hyperbolic lines incident to the point r. Certainly, the intersections of these $q^2 - q$ hyperbolic lines with h are pairwise distinct using a similar argument as above. At least $q^2 - 2q - 1$ of those intersection points are regular. Choosing one of those, say a, the line $z = \langle a, p \rangle \subset l^\pi$ is a hyperbolic line, as $a \in p^\pi$. The plane $\langle m, z \rangle = \langle r, p, a \rangle$ has the Gram matrix (with respect to some suitably chosen basis in r, p, and a) of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \overline{\gamma} \\ 0 & \gamma & 1 \end{pmatrix}$. This matrix has a non-zero determinant as z is a hyperbolic line, so $\langle m, z \rangle$ is regular. Again, by lemma 3.2.2 we are finished in this case.

The case that both points x and p are singular does not occur, as otherwise the regular plane l^π would contain the totally singular line $\langle x, p \rangle$, a contradiction. ■

Lemma 3.2.4 *Let l and m be two different hyperbolic lines of U_5. Then l and m have distance four in $\mathbf{G}(U_5)$ if and only if either*

- *l and m are two non-intersecting lines such that $l^\pi \cap m$ is trivial in U_5, or*
- *l and m are two intersecting lines spanning a degenerate plane in U_5.*

Proof: Let l and m be two vertices in the graph $\mathbf{G}(U_5)$ of distance four. If the subspace $\langle l, m \rangle$ is a non-degenerate plane, then l and m have distance two by lemma 3.2.2. Therefore, if $\langle l, m \rangle$ is a plane, then $\langle l, m \rangle$ is a degenerate subspace of U_5. Alternatively, if $\langle l, m \rangle$ is a four-dimensional subspace in U_5 and $l^\pi \cap m \neq \{0\}$, then l and m have distance one in $\mathbf{G}(U_5)$ by definition or distance three in $\mathbf{G}(U_5)$ by lemma 3.2.3, a contradiction again. It follows that, if the subspace $\langle l, m \rangle$ is of dimension four, then $l^\pi \cap m$ is trivial.

In order to show the converse implication of the statement let $\langle l, m \rangle$ be either a degenerate plane or a four-dimensional subspace such that $l^\pi \cap m = \{0\}$. By the previous results, (observation 3.2.1, lemma 3.2.2 and lemma 3.2.3), the vertices l and m do not have distance one, two or three in $\mathbf{G}(U_5)$. Therefore it is enough to find a path of length four in $\mathbf{G}(U_5)$ between the vertices l and m to finish the proof of this lemma.

We choose a hyperbolic line z in the polar space l^π intersecting the orthogonal space m^π in a point. Such a choice is possible, because l^π is a regular subspace of U_5 and the non-trivial subspace $l^\pi \cap m^\pi \subsetneq l^\pi$. By construction the vertices l and z are adjacent in $\mathbf{G}(U_5)$. Moreover, $m \cap l^\pi$ is trivial either by assumption (case 1) or since $\langle l, m \rangle$ is a degenerate plane (case 2) and the facts that only the radical of the singular plane $\langle l, m \rangle$ is contained in l^π as well as that $\mathrm{rad}(\langle l, m \rangle) \cap m = \{0\}$. Hence m and z do not intersect, but satisfy the condition $\dim(z \cap m^\pi) = 1$. So m and z have distance three in the hyperbolic line graph $\mathbf{G}(U_5)$ by lemma 3.2.3 and, thus, the distance between the vertices l and m is four in $\mathbf{G}(U_5)$. ∎

Proposition 3.2.5 *The graph* $\mathbf{G}(U_5)$ *is a connected locally* $\mathbf{G}(U_3)$ *graph of diameter four.*

Proof: For any singular point p in the orthogonal space l^π of a hyperbolic line l in U_5, the subspace $\langle l, p \rangle$ is of dimension three and rank two. By the formula B.4 on page 255 it is possible to choose a hyperbolic line m different from l in the plane $\langle l, p \rangle$, thus l and m span the degenerate plane $\langle l, p \rangle$ and hence the vertices l and m have distance four in $\mathbf{G}(U_5)$ by lemma 3.2.4. The statement about the diameter now follows from the fact that two hyperbolic lines cannot form a configuration other than the ones described in 3.2.1 to 3.2.4. The local property is obvious. ∎

Remark 3.2.6 Let l and m be two arbitrary vertices of the hyperbolic line graph $\mathbf{G}(U_5)$. An important induced subgraph of $\mathbf{G}(U_5)$ is the common perp of the vertices l and m. If the induced subgraph $\{l, m\}^\perp$ is not empty then the subspace $\langle l, m \rangle^\pi$ of U_5 contains some hyperbolic line. We observe that $\{l, m\}^\perp \neq \{0\}$ in $\mathbf{G}(U_5)$ if and only if l and m have distance two in $\mathbf{G}(U_5)$. Indeed, if l and m are at distance two in $\mathbf{G}(U_5)$, then the hyperbolic lines l and m span a regular plane in U_5, thus $\langle l, m \rangle^\pi$ is a hyperbolic line, by lemma 3.2.2. In all other cases, i.e., if the vertices have distance one, three or four, then either $\langle l, m \rangle$ is a four-dimensional subspace and therefore $\langle l, m \rangle^\pi$ is a single point of U_5 or the hyperbolic lines l and m span a degenerate planes, which implies that $\langle l, m \rangle^\pi$ is a rank one line. Thus in these cases the subgraph $\{l, m\}^\perp$ is the empty graph. Of course if the vertices l and m have distance one in $\mathbf{G}(U_5)$, then $\langle l, m \rangle$ is a four-dimensional non-degenerate space in the unitary vector space U_5 and $l^\pi \cap m^\pi = \langle l, m \rangle^\pi$ is a regular point of U_5.

Definition 3.2.7 Let W be a subspace of U_5. The set of all hyperbolic lines of W is denoted by $\mathbf{L}(W)$.

Lemma 3.2.8 *Let l and m be two distinct vertices of $\mathbf{G}(U_5)$ with $\{l,m\}^\perp \neq \emptyset$. Then any vertex in $\{l,m\}^{\perp\perp}$ is a hyperbolic line of the subspace $\langle l,m\rangle$ in U_5 and vice versa.*

Proof: Let l and m be two distinct vertices in $\mathbf{G}(U_5)$ such that $\{l,m\}^\perp$ is not empty. Due to remark 3.2.6 the vertices l and m have distance two in $\mathbf{G}(U_5)$ and it follows that the graph $\{l,m\}^\perp$ is the single vertex $\langle l,m\rangle^\pi$. Thus we obtain the equalities $\{l,m\}^{\perp\perp} = \{\{l,m\}^\perp\}^\perp = \bigcap_{z\in\{m,l\}^\perp} z^\perp = \mathbf{L}(\langle l,m\rangle^\pi)^\perp = \mathbf{L}(\langle l,m\rangle^{\pi\pi}) = \mathbf{L}(\langle l,m\rangle)$. ■

It will prove useful to know whether two hyperbolic lines intersect in the projective space (i.e., the two hyperbolic lines span a plane in the projective space) or not (i.e., they span a four-dimensional space in the projective space). Lemmas 3.2.2 to 3.2.4 show that in order to distinguish the above two cases, we have to study vertices of $\mathbf{G}(U_5)$ at distance three and four more thoroughly.

Lemma 3.2.9 *If l and m are two non-intersecting hyperbolic lines of U_5 such that $l^\pi \cap m$ is a point p, then in the graph $\mathbf{G}(U_5)$ the number of different paths of length three between l to m is at most q^2. On the other hand, this number is at least q^2-q-1, if p is a singular point, and at least q^2-2q-1, if p is a regular point.*

Proof: Let h be an arbitrary neighbour of l in $\mathbf{G}(U_5)$, i.e., $h \subset l^\pi$. By lemma 3.2.2 there exists a common neighbour k of h and m (and, thus, a path of length three from l to m through h) if and only if $\langle h,m\rangle$ is a regular plane. In fact, if $\langle h,m\rangle$ is a regular plane, then k is uniquely determined as $\langle h,m\rangle^\pi$. Therefore it suffices to study all regular planes E with $m \subseteq E \subseteq \langle m,l^\pi\rangle$ such that $E \cap l^\pi$ is a regular line.

Let us first deduce the upper bound in the statement of the lemma from the observations made in the above paragraph. If $p = l^\pi \cap m$ is a singular point, then q^2 different hyperbolic lines and exactly one singular line of the orthogonal space l^π run through the point p by lemma B.1.5. If $p = l^\pi \cap m$ is a regular point, then q^2-q different hyperbolic lines and $q+1$ distinct singular lines are incident to the point p in the subspace l^π. Hence there are at most q^2 paths from l to m.

Next we want to establish the respective lower bounds. Regard the four-dimensional subspace $W = \langle m,l^\pi\rangle$, which is of rank three or four. In the subspace W the hyperbolic line m is contained in q^2+1 different planes E_i by the formula B.3 on page 255. Each plane E_i of W intersects the regular plane l^π in a line, by the dimension formula and because $m \nsubseteq l^\pi$. Since $p \in l^\pi$ is incident to each plane E_i, every line $h_i = E_i \cap l^\pi$ runs through p. Moreover the lines h_i are mutually distinct, because the identity $h_i = h_j$ implies $E_i = \langle h_i,m\rangle = \langle h_j,m\rangle = E_j$.

If the subspace W is of rank four, then the hyperbolic line m lies on q^2-q different non-degenerate planes E_i^m by lemma B.1.6. Therefore we obtain q^2-q different lines $E_i^m \cap l^\pi = h_i^m$ incident to the point p in the subspace l^π. At least q^2-q-1 lines of the q^2-q lines h_i^m are hyperbolic lines, if p is a singular point, due to lemma B.1.5. On the other hand, if p is a regular point, then at least $q^2-q-(q+1) = q^2-2q-1$ lines of the $q^2 - q$ lines h_i^m are hyperbolic lines by lemma B.1.5 again. Alternatively if W is of rank three, then exactly q^2 different non-degenerate planes E_i^m are incident to the hyperbolic line m by lemma B.1.7. Hence we obtain q^2 different lines $E_i^m \cap l^\pi = h_i^m$ in the regular plane l^π containing the point p. By lemma B.1.5, at least $q^2 - 1$ of these q^2 lines h_i^m are regular, if p is a singular point and at least $q^2 - (q+1) = q^2 - q - 1$ lines are regular, if p is a regular point. ∎

Lemma 3.2.10 *If l and m are two non-intersecting hyperbolic lines of U_5 which are at distance four in the graph $\mathbf{G}(U_5)$, then there are at most q^4 different paths of length four from l to m.*

Proof: By lemma 3.2.4 we have $\dim(\langle l, m\rangle) = 4$ with $l^\pi \cap m = \{0\}$. A neighbour h of l in $\mathbf{G}(U_5)$ is at distance three from the vertex m if and only if $\dim(\langle h, m\rangle) = 4$ and $\dim(h \cap m^\pi) = 1$ by lemma 3.2.3. Thus h is a hyperbolic line in l^π running through the point $x := \langle l, m\rangle^\pi$. If the one-dimensional subspace x is singular, then l^π contains q^2 different hyperbolic lines h_i^l incident with x by lemma B.1.5. If x is regular point, then, by lemma B.1.5 again, there are $q^2 - q$ hyperbolic lines through x in l^π. By lemma 3.2.3 the vertices m and h_i^l are at distance three in $\mathbf{G}(U_5)$. Combining the above numbers with lemma 3.2.9 we obtain at most $q^2 \cdot q^2 = q^4$ paths from l to m. ∎

Lemma 3.2.11 *If l and m are two intersecting hyperbolic lines spanning a degenerate plane, then the hyperbolic line graph $\mathbf{G}(U_5)$ contains at least $q^6 - 3q^5 + 2q^4 - q^2$ different paths of length four from l to m.*

Proof: If h is a neighbour of l, then the vertex h is at distance three from m in $\mathbf{G}(U_5)$ if and only if $\dim(\langle h, m\rangle) = 4$ and $\dim(h \cap m^\pi) = 1$ in U_5 by lemma 3.2.3. Consequently h is a hyperbolic line in the polar space l^π of l such that $\langle l, m\rangle^\pi \cap h$ is a one-dimensional subspace. Since the rank one line $\langle l, m\rangle^\pi$ contains exactly one singular point x and q^2 regular points p_i by the formula B.3 on page 255 and table B.3 on page 264, thus, due to lemma B.1.5, the regular plane l^π contains q^2 hyperbolic lines h_{x_i} incident to the point x. Each regular point p_i admits $q^2 - q$ incident hyperbolic lines $h_{p_i,j}$ of l^π. Certainly, all those hyperbolic lines h_{x_i} and $h_{p_i,j}$ are pairwise distinct as otherwise they would coincide with the line $\langle l, m\rangle^\pi$. By lemma 3.2.9 we have at least $q^2 - q - 1$ different paths of length three in $\mathbf{G}(U_5)$ between each vertex h_{x_i} and the vertex m and not less than $q^2 - 2q - 1$ different paths of length three in $\mathbf{G}(U_5)$ from each vertex $h_{p_i,j}$ to the vertex m. Accordingly we obtain at least $q^2(q^2 - q - 1) + q^2(q^2 - q)(q^2 - 2q - 1) = q^6 - 3q^5 + 2q^4 - q^2$ different paths of length four from l and m in the graph $\mathbf{G}(U_5)$. ∎

Lemma 3.2.12 *Two different vertices l and m of distance four in $\mathbf{G}(U_5)$ intersect in a point in the vector space U_5 if and only if the number of different paths of length four between l and m in $\mathbf{G}(U_5)$ is greater than q^4.*

Proof: Since $q \geq 3$, we have $q^6 - 3q^5 + 2q^4 - q^2 > q^4$, so the claim follows from lemma 3.2.10 and lemma 3.2.11. ■

Lemma 3.2.13 *Two distinct vertices l and m of the hyperbolic line graph $\mathbf{G}(U_5)$ intersect in a common point in U_5 if and only if either*

- *the subgraph $\{l, m\}^\perp$ is not empty, or*
- *the vertices l and m have distance four in $\mathbf{G}(U_5)$ and there are more than q^4 different paths of length four from l to m.*

Proof: This is an immediate consequence of lemma 3.2.2 to lemma 3.2.4 together with the statements of lemma 3.2.12, lemma 3.2.8 and remark 3.2.6. ■

In the next step we want to recover all points of the space U_5 as pencils of hyperbolic lines. Therefore we need a construction to check in the graph $\mathbf{G}(U_5)$ whether three distinct lines of U_5 intersect in one point or not. Therefore take the following characterisation: Three different hyperbolic lines k_1, k_2 and k_3 of U_5 intersect in one point if we can find a hyperbolic line s in U_5 such that

- the plane $\langle s, k_i \rangle$ is regular for $1 \leq i \leq 3$ and $s \neq k_i$,
- $\langle s, k_1, k_2 \rangle$ is a four-dimensional space in U_5.

The same statement in graph language is that three different vertices k_1, k_2 and k_3 of $\mathbf{G}(U_5)$ intersect in one point if we can find a vertex s of $\mathbf{G}(U_5)$ with the following properties:

- the induced subgraph $\{s, k_i\}^\perp$ is not empty for $i \in \{1, 2, 3\}$ and $s \neq k_i$,
- $\{s, k_1, k_2\}^\perp$ is the empty graph.

To verify the claim that every one-dimensional subspace of the U_5 can be detected by three pairwise intersecting distinct vertices k_1, k_2 and k_3 of $\mathbf{G}(U_5)$ as stated above, we have to show that we can find a vertex s of the graph $\mathbf{G}(U_5)$ such that $\{s, k_1, k_2\}^\perp = \varnothing$ and $\{s, k_i\}^\perp \neq \varnothing$ for $i = 1, 2, 3$ and $s \neq k_i$. This will be proved in the next lemma.

Lemma 3.2.14 *Let k_1, k_2 and k_3 be three distinct hyperbolic lines of U_5, which intersect in a unique one-dimensional subspace p. Then the unitary polar space U_5 contains a hyperbolic line l with the properties that $\langle k_1, k_2, l \rangle$ is a four-dimensional space and that $\langle l, k_i \rangle$ is a non-degenerate plane for $i = 1, 2, 3$ and $l \neq k_i$.*

Proof: First we consider the case that p is a regular point. Therefore the orthogonal space p^π is a non-degenerate four-dimensional subspace of U_5. The three lines k_1, k_2 and k_3 intersect the subspace p^π in the three different regular points $p_{k_1} = k_1 \cap p^\pi$, $p_{k_2} = k_2 \cap p^\pi$ and $p_{k_3} = k_3 \cap p^\pi$. Furthermore, $\langle p_{k_i} \mid 1 \leq i \leq 3\rangle$ is either a plane or a line of p^π containing some regular points. If $\dim(\langle p_{k_i} \mid 1 \leq i \leq 3\rangle)$ is a subspace of dimension two, then $\langle p_{k_i} \mid 1 \leq i \leq 3\rangle^\pi \cap p^\pi$ is a line of rank at least one. Thus we choose a regular point m_p of $\langle p_{k_i} \mid 1 \leq i \leq 3\rangle^\pi \cap p^\pi$ and see that $s = \langle m_p, p\rangle$ is a hyperbolic line through the point p, the plane $\langle s, k_i\rangle$ is of rank three for each $i \in \{1,2,3\}$ and that $\dim(\langle s, k_1, k_2\rangle) = \dim(\langle p, m_p, p_{k_1}, p_{k_2}\rangle) = 4$. On the other hand, if $\dim(\langle p_{k_i} \mid 1 \leq i \leq 3\rangle) = 3$, then we consider the regular plane $P_{k_1} = p_{k_1}^\pi \cap p^\pi$, which contains $q^4 - q^3 + q^2$ regular points r_i. Either at least $q^4 - q^3 + q^2 - (q^3 + 1) = q^4 - 2q^3 + q^2 - 1$ different regular points of the plane P_{k_1} span together with p_{k_2} a hyperbolic line if $p_{k_2} \notin P_{k_1}$ or at least $q^4 - q^3 + q^2 - 1 - ((q+1)(q^2-1)) = q^4 - q^3 + q^2 - 1 - (q^3 + q^2 - q - 1) = q^4 - 2q^3 + q + 1$ different regular points of the plane P_{k_1} span together with p_{k_2} a hyperbolic line if $p_{k_2} \in P_{k_1}$. This implies that at least

$$\begin{array}{lr} q^4 - 2q^3 + q^2 - 1 - (q^3 + 1) = q^4 - 3q^3 + q^2 - 2 & \text{if } p_{k_2}, p_{k_3} \notin P_{k_1}, \\ q^4 - 2q^3 + q^2 - 1 - (q^3 + q^2 - q) = q^4 - 3q^3 + q - 1 & \text{if } p_{k_2} \notin P_{k_1}, p_{k_3} \in P_{k_1}, \\ q^4 - 2q^3 + q + 1 - (q^3 + 1) = q^4 - 3q^3 + q & \text{if } p_{k_2} \in P_{k_1}, p_{k_3} \notin P_{k_1}, \\ q^4 - 2q^3 + q + 1 - (q^3 + q^2 - q) = q^4 - 3q^3 - q^2 + 2q + 1 & \text{if } p_{k_2}, p_{k_3} \in P_{k_1} \end{array}$$

different regular points of the plane P_{k_1} span together with p_{k_2} and also together with p_{k_3} a hyperbolic line in p^π. Since the first three numbers are greater than zero for $q \geq 3$ in these cases we find a regular point $m_p \in P_{k_1}$ such that $s = \langle m_p, p\rangle$ is a hyperbolic line through the point p, $\langle s, k_i\rangle$ is a rank three plane for each $i \in \{1,2,3\}$ and that $\dim(\langle s, k_1, k_2\rangle) = \dim(\langle p, m_p, p_{k_1}, p_{k_2}\rangle) = 4$.

Therefore we consider the case that the two different regular points p_{k_2} and p_{k_3} are contained in the regular plane P_{k_1}. The line $h = \langle p_{k_2}, p_{k_3}\rangle$ contains regular points, thus the rank of h is one or two. If the rank of h is two then we choose $s = k_3$ and conclude that the planes $\langle k_1, k_3\rangle$ and $\langle k_2, k_3\rangle$ are of rank three. Certainly the three lines k_1, k_2 and k_3 span a subspace of dimension four. Otherwise if h is a singular line of rank one, then both the point p_{k_2} is incident to q different singular line different from h in P_{k_1} and the point p_{k_3} is incident to q different singular line different from h in P_{k_1}. It follows that at most $q(q^2-1) + q(q^2-1) + q^2 - 2 = 2q^3 + q^2 - 2q - 2$ different regular points distinct from p_{k_2} and p_{k_3} of the plane P_{k_1} span either with p_{k_2} or with p_{k_3} a singular in line in P_{k_1}. So at least $q^4 - q^3 + q^2 - 2 - (2q^3 + q^2 - 2q - 2) = q^4 - 3q^3 + 2q$ different regular points of the plane P_{k_1} span with p_{k_2} a hyperbolic line and with p_{k_3} a rank two line in p^π. As $q^4 - 3q^3 + 2q \geq 0$ for $q \geq 3$ we can find a regular point $m_p \in P_{k_1}$ such that $s = \langle m_p, p\rangle$ is a hyperbolic line through the point p, $\langle s, k_i\rangle$ is a rank three plane for each $i \in \{1,2,3\}$ and that $\dim(\langle s, k_1, k_2\rangle) = \dim(\langle p, m_p, p_{k_1}, p_{k_2}\rangle) = 4$.

For the next part of this proof we assume that p is a singular point. In the unitary polar space U_5 every hyperbolic line k is incident to $q^4 - q^3 + q^2$ different non-degenerate planes E_j^k and to $q^3 + 1$ different singular planes S_i^k in U_5 by the formula B.3 on page 255. For the hyperbolic line k_1 we obtain the $q^4 - q^3 + q^2$ different non-degenerate planes $E_{k_1}^j$, $1 \leq j \leq q^4 - q^3 + q^2$ and consider in each of these planes the hyperbolic lines running through the point p. As p is a singular point each plane $E_{k_1}^j$ has $q^2 - 1$ different hyperbolic lines $h^r_{E_{k_1}^j}$, $1 \leq r \leq q^2 - 1$, incident to p and different from the hyperbolic line k_1 by lemma B.1.5. Recall that $E_{k_1}^i \cap E_{k_1}^j = k_1$ if and only if the planes are different, which leads to the fact that a hyperbolic line $h^r_{E_{k_1}^j}$ with $r \in \{1, \ldots, q^2 - 1\}$ is not incident to the regular plane $E_{k_1}^i$ if $i \neq j$. Therefore the polar space U_5 contains $(q^4 - q^3 + q^2 - 1)(q^2 - 1) = q^6 - q^5 + q^3 - 2q^2 - 1$ different hyperbolic lines $h^r_{E_{k_1}^j}$, $1 \leq j \leq^4 - q^3 + q^2$ and $1 \leq r \leq q^2 - 1$ with the same properties as above.

Next we consider the singular planes $S_{k_2}^j$ and $S_{k_3}^i$ in U_5. The point p is not contained in the radicals of the planes $S_{k_n}^j$ for $n = 2, 3$, because the hyperbolic lines k_2 and k_3 are running through the point p, and thus in each rank two plane $S_{k_n}^j$ are $q^2 - 1$ different hyperbolic lines $l^r_{S_{k_n}^j}$, $1 \leq r \leq q^2 - 1$ incident to p and different from the hyperbolic line k_n for $n = 2, 3$. Therefore in the planes $S_{k_2}^j$ and $S_{k_3}^j$ are together at most $2(q^2 - 1)(q^3 + 1) = 2q^5 - 2q^3 + 2q^2 - 2$ different hyperbolic lines $l^r_{S_{k_n}^j}$, $n = 2, 3$ and $1 \leq r \leq q^2 - 1$, with the assumed properties. Thus if p is singular in U_5 and $q \geq 3$, then $q^6 - q^5 + q^3 - 2q^2 - 1 - (2q^5 - 2q^3 + 2q^2 - 2) = q^6 - 3q^5 + 3q^3 - 4q^2 + 1 > 0$, which implies that we find a hyperbolic line s in U_5 with the claimed properties. ∎

Definition 3.2.15 Let Γ be a graph isomorphic to $\mathbf{G}(U_5)$. Two different vertices k and l of Γ are defined to **intersect** if

- either the induced subgraph $\{k, l\}^{\perp}$ is not empty, or
- the vertices k and l have distance four in Γ and the number of different paths of length four between l and m in $\mathbf{G}(U_5)$ is greater than q^4.

Three distinct pairwise intersecting vertices k_1, k_2 and k_3 of Γ are defined to **intersect in one point** if there is a vertex s of Γ with the following properties:

- the induced subgraph $\{s, k_i\}^{\perp}$ is not empty for $i \in \{1, 2, 3\}$ and $s \neq k_i$,
- $\{s, k_1, k_2\}^{\perp}$ is the empty graph.

An **interior point** of the graph Γ is a maximal set p of distinct pairwise intersecting vertices of Γ such that any three elements of p intersect in one point. We denote the set of all interior points of Γ by $\mathcal{I}$. Moreover, an **interior line** of the graph Γ is a vertex of the graph Γ. The set of all interior lines of Γ is denoted by $\mathcal{L}$.

The discussions in this section imply the following result.

Proposition 3.2.16 *Let Γ be a connected graph isomorphic to $\mathbf{G}(U_5)$. Then the geometry $(\mathcal{I}, \mathcal{L}, \supset)$ is isomorphic to the geometry on arbitrary one-dimensional subspaces and regular two-dimensional subspaces of the unitary polar space U_5.*

3.3 The hyperbolic line graph of U_n for $n \geq 6$

Let $q \geq 3$ be a prime power, let $n \geq 6$, and let U_n be an n-dimensional non-degenerate unitary vector space over $\mathbb{F}_{q^2}$ with polarity π, more precisely U_n is endowed with a non-degenerate hermitian form $(\cdot,\cdot)$. Let $\mathbf{G}(U_n)$ be the graph with the set of all non-degenerate two-dimensional subspaces of U_n as set of vertices. Two vertices l and m are adjacent if and only if $l \subset m^\pi$. In analogy to the preceding section, the aim of this section is to reconstruct the unitary vector space U_n from the hyperbolic line graph $\mathbf{G}(U_n)$.

Proposition 3.3.1 *Let $n \geq 8$. Then $\mathbf{G}(U_n)$ is a connected graph of diameter two.*

Proof: Let l and k be two distinct vertices of the graph $\mathbf{G}(U_n)$. The space $H = \langle l, k \rangle$ has dimension three or four. Since it contains the hyperbolic lines l and m, the rank of H is at least two. In particular the radical of H has dimension at most two. The space H^π has dimension at least four and rank at least two, since $\mathrm{rad}(H^\pi) = \mathrm{rad}(H)$. Therefore $H^\pi = \langle k, l \rangle^\pi = k^\pi \cap l^\pi$ contains a hyperbolic line h, so that the distance between the vertices l and k is at most two in $\mathbf{G}(U_n)$, for $n \geq 8$. Because the unitary vector space U_n contains two intersecting hyperbolic lines l and m, it follows that $l \not\perp m$ by definition and the diameter of $\mathbf{G}(U_n)$ is two. ■

Proposition 3.3.2 *The graphs $\mathbf{G}(U_6)$ and $\mathbf{G}(U_7)$ are connected of diameter three.*

Proof: Let l and m be two distinct vertices of the graph $\mathbf{G}(U_6)$. Then $P = \langle l, m \rangle$ is a three- or four-dimensional subspace of U_6. Suppose $P = \langle l, m \rangle$ is a plane. Then the planes P and P^π have rank two or three, because the hyperbolic line l and m are proper subspaces of P. Therefore the plane P^π contains a hyperbolic line h and thus the vertices l and m have distance two in $\mathbf{G}(U_6)$.

If $P = \langle l, m \rangle$ is a four-dimensional subspace of U_6, then P is of rank two, three, or four. In the case that P is a regular subspace, then of course P^π is a hyperbolic line and the vertices l and m have distance two. Finally, we assume that the four-dimensional space P is a singular subspace of U_6. We fix a point x in the radical of P.

Then x is incident to the orthogonal space l^π of the hyperbolic line l, which is a regular four-dimensional subspace of U_6. We choose a hyperbolic line h in l^π running through the point x in l^π. Certainly the vertex h is adjacent to l in the hyperbolic line graph $\mathbf{G}(U_6)$. If $\langle h, m\rangle$ is a plane, then there exists a common neighbour of h and m by the above, yielding a path of length three from l to m in $\mathbf{G}(U_6)$. Hence we can assume that subspace of $\langle h, m\rangle$ is of dimension four. The rank of this space is four as well. Indeed, the Gram matrix of $\langle m, h\rangle$ is $G = \begin{pmatrix} 0 & 1 & 0 & \alpha \\ 1 & 0 & 0 & \beta \\ 0 & 0 & 0 & \delta \\ \overline{\alpha} & \overline{\beta} & \overline{\delta} & \gamma \end{pmatrix}$ with respect to a basis v_1^m, v_2^m, x^h, v_2^h of $\langle m, h\rangle$ such that the pair of vectors v_1^m, v_2^m is a hyperbolic pair of is line m, the vector x^h is some non-trivial vector of the point x and v_2^h is a non-trivial vector of a regular point of the line h, in particular $(x^h, v_2^h) = \delta \neq 0$. This Gram matrix has determinant $\delta\overline{\delta} \neq 0$ and, thus, the space $\langle m, h\rangle$ is of rank four. By the above h and m have distance two, so the vertices l and m have distance three in $\mathbf{G}(U_6)$.

Now let l and m be two vertices of the graph $\mathbf{G}(U_7)$. Since the subspace $\langle l, m\rangle$ has dimension at most four and rank at least two, there exists a non-degenerate six-dimensional subspace W of U_7 containing l and m. By the above, the vertices l and m have distance at most three in the hyperbolic line graph $\mathbf{G}(W)$, which is a subgraph of $\mathbf{G}(U_7)$. Hence the diameter of $\mathbf{G}(U_7)$ is at most three.

To establish diameter three for the graphs $\mathbf{G}(U_6)$ and $\mathbf{G}(U_7)$, we have to find two vertices that are not at distance one or two. Choose a four-dimensional rank two subspace H of U_6 respectively of U_7. By the formula B.4 on page 255 and the formula B.2 on page 254 the subspace H contains q^8 hyperbolic lines and any point of this space is incident to q^4+q^2+1 different lines. Since $q^8 \geq (q^2+1)\cdot(q^4+q^2+1) = q^6+q^4+q^2+1$ we find two non-intersecting hyperbolic lines l and m of U_6 resp. U_7 spanning the subspace H. The polar space $\langle l, m\rangle^\pi = H^\pi$ has dimension two resp. three and rank zero resp. one in the vector space U_6 resp. U_7, thus there is no common neighbour of l and m in the hyperbolic line graph $\mathbf{G}(U_6)$ resp. $\mathbf{G}(U_7)$. Hence the diameter of $\mathbf{G}(U_n)$ with $6 \leq n \leq 7$ is three. ∎

The next proposition describes two key properties of the hyperbolic line graph $\mathbf{G}(U_n)$ which will turn out to characterise $\mathbf{G}(U_n)$ for $n \geq 9$, cf. theorem 3.1.2.

Proposition 3.3.3 *Let $n \geq 5$. Then the hyperbolic line graph $\mathbf{G}(U_n)$ is connected and locally $\mathbf{G}(U_{n-2})$.*

Proof: See propositions 3.2.5, 3.3.1, 3.3.2. The local property is obvious. ∎

Lemma 3.3.4 *Let $n \geq 6$ and let l and m be two distinct vertices of the graph $\mathbf{G}(U_n)$ such that $\{l, m\}^\perp \neq \emptyset$. Then any element of $\{l, m\}^{\perp\perp}$ is a hyperbolic line in the subspace $\langle l, m\rangle$ of U_n and vice versa.*

Proof: Since $\{l,m\}^{\perp\perp} = \{\{l,m\}^{\perp}\}^{\perp} = \bigcap_{z\in\{l,m\}^{\perp}} z^{\perp} = \bigcap_{z\in\{l,m\}^{\perp}} \mathbf{L}(z^{\pi})$, obviously $\mathbf{L}(\langle l,m\rangle) \subseteq \{l,m\}^{\perp\perp}$.

Conversely, let k be a hyperbolic line of U_n not incident to the subspace $\langle l,m\rangle$, then of course $\langle l,m\rangle^{\pi} \nsubseteq k^{\pi}$ implying $\langle l,m\rangle^{\pi}\backslash k^{\pi} \neq \varnothing$. The statement is proved if we can find a hyperbolic line $h \subseteq \langle l,m\rangle^{\pi}$, which is not incident to the orthogonal space k^{π} of the hyperbolic line k. From the assumption that the induced subgraph $\{l,m\}^{\perp}$ is not empty it follows that $\mathrm{rad}(\langle l,m\rangle^{\pi})$ is properly contained in the subspace $\langle l,m\rangle^{\pi}$. We claim that the unitary space U_n contains some point y in the set $\langle l,m\rangle^{\pi}\backslash(k^{\pi}\cup \mathrm{rad}(\langle l,m\rangle^{\pi}))$. If $\mathrm{rad}(\langle l,m\rangle^{\pi}) \subseteq k^{\pi}$ then $k^{\pi}\cup\mathrm{rad}(\langle l,m\rangle^{\pi}) = k^{\pi}$ and we can fix some point y in the set $\langle l,m\rangle^{\pi}\backslash \mathrm{rad}(\langle l,m\rangle^{\pi}) = \langle l,m\rangle^{\pi}\backslash k^{\pi}$. On the other hand if $\mathrm{rad}(\langle l,m\rangle^{\pi}) \nsubseteq k^{\pi}$ then $\mathrm{rad}(\langle l,m\rangle^{\pi}) \cup k^{\pi}$ is not a subspace of the vector space U_n and $\langle l,m\rangle^{\pi}$ is neither a subspace of $\mathrm{rad}(\langle l,m\rangle^{\pi})$ nor a subspace of k^{π}, thus the set $\langle l,m\rangle^{\pi}\backslash(k^{\pi}\cup\mathrm{rad}(\langle l,m\rangle^{\pi}))$ contains a point y.

An arbitrary line g of U_n through the point y intersects the set $k^{\pi}\cup\mathrm{rad}(\langle l,m\rangle^{\pi})$ in at most two points by the fact that $\dim(k^{\pi}\cap g)$ as well as $\dim(\mathrm{rad}(\langle l,m\rangle^{\pi})\cap g)$ is at most one. Therefore, we choose a hyperbolic line running through y in $\langle l,m\rangle^{\pi}$ and find a singular point $x \in \langle l,m\rangle^{\pi}\backslash(k^{\pi}\cup\mathrm{rad}(\langle l,m\rangle^{\pi}))$. Using $x \notin \mathrm{rad}(\langle l,m\rangle^{\pi})$ we obtain a hyperbolic line h in $\langle l,m\rangle^{\pi}$ incident to the point x which is not contained in the subspace k^{π}. The lemma is proved. ■

A similar conclusion can be proved for three different vertices in the graph $\mathbf{G}(U_n)$.

Lemma 3.3.5 *Let $n \geq 6$ and l, k and m be three distinct vertices in $\mathbf{G}(U_n)$. Suppose the hyperbolic lines l, k and m intersect in a common point U_n and satisfy $\{l,k,m\}^{\perp} \neq \varnothing$. Then $\mathbf{L}(\langle l,k,m\rangle) = \{l,k,m\}^{\perp\perp}$.*

Proof: By assumption the subspace spanned by the hyperbolic lines l, k and m is of dimension three or four. Denote the common intersection of the three hyperbolic lines by p.

Suppose $\langle l,k,m\rangle$ is a plane. Then m is a hyperbolic line of the subspace $\langle l,k\rangle$ and, thus, $\langle l,k,m\rangle = \langle l,k\rangle$. Using lemma 3.3.4 we obtain the following equality, $\mathbf{L}(\langle l,k,m\rangle) = \mathbf{L}(\langle l,k\rangle) = \{l,k\}^{\perp\perp}$.

If $\langle l,k,m\rangle$ is a four-dimensional subspace, then we need to find a hyperbolic line h such that $\langle l,k,m\rangle = \langle l,h\rangle$. In the case that the subspace $\langle l,k,m\rangle$ has rank four, we choose $h = l^{\pi} \cap \langle l,k,m\rangle$. If the subspace $\langle l,k,m\rangle$ has rank two, take as h an arbitrary line in the complement of both l and $\mathrm{rad}(\langle l,k,m\rangle)$. Indeed, we can find such a line h in $\langle l,k,m\rangle$ by the fact that at most $2q^6 + 4q^4 + 4q^2 + 2$ of the $q^8 + q^6 + 2q^4 + q^2 + 1$ different lines of $\langle l,k,m\rangle$ intersect l or $\mathrm{rad}(\langle l,k,m\rangle)$. Certainly h is a hyperbolic line since every line not intersecting the radical of $\langle l,k,m\rangle$ is regular by lemma B.1.2. Finally, if $\langle l,k,m\rangle$ has rank three, then we choose the rank two plane $P = \langle k,\mathrm{rad}(\langle l,k,m\rangle)\rangle$. Since the hyperbolic lines k and l are distinct and intersect in a common point we have $\dim(l\cap P) = 1$. Moreover the radical of P coincides with the point $\mathrm{rad}(\langle l,k,m\rangle)$. In the plane P we choose the line h in the complement of both $\mathrm{rad}(\langle l,k,m\rangle)$ and $l \cap P$. Certainly the subspace h is regular by lemma B.1.2. It follows from the construction that $\langle l,h\rangle = \langle l,P\rangle = \langle l,k,\mathrm{rad}(\langle l,k,m\rangle)\rangle = \langle l,k,m\rangle$ and, by lemma 3.3.4, that $\mathbf{L}(\langle l,k,m\rangle) = \mathbf{L}(\langle l,h\rangle) = \{l,h\}^{\perp\perp}$.

For $g \in \{h,k\}$ as appropriate, the equality between the sets $\{l,k,m\}^{\perp\perp}$ and $\{l,g\}^{\perp\perp}$ follows as $\{l,k,m\}^{\perp\perp} = \{\{l,k,m\}^{\perp}\}^{\perp} = \bigcap_{z\in\{l,k,m\}^{\perp}} z^{\perp} = \bigcap_{z\in l^{\perp}\cap k^{\perp}\cap m^{\perp}} z^{\perp} = \bigcap_{z\in\mathbf{L}(l^{\pi})\cap\mathbf{L}(k^{\pi})\cap\mathbf{L}(m^{\pi})} z^{\perp} = \bigcap_{z\in\mathbf{L}(\langle l,k,m\rangle^{\pi})} z^{\perp} = \bigcap_{z\in\mathbf{L}(\langle l,g\rangle^{\pi})} z^{\perp} = \bigcap_{z\in\{l,g\}^{\perp}} z^{\perp} = \{l,g\}^{\perp\perp}$. ∎

Our main goal in this section is to construct a point-line geometry from the graph $\mathbf{G}(U_n)$ which is isomorphic to the geometry on arbitrary one-dimensional subspaces and regular two-dimensional subspaces of U_n. We want to use the vertices of $\mathbf{G}(U_n)$ as lines and to define the points as pencils of lines. Therefore we will study properties of vertices of $\mathbf{G}(U_n)$ that correspond to intersecting hyperbolic lines of $\mathbf{G}(U_n)$.

Lemma 3.3.6 *Let $n \geq 6$. Two hyperbolic lines l and m intersect in a common point in the unitary polar space U_n if and only if $\{l,m\}^{\perp}$ is non-empty and $\{l,m\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_n)$ with respect to inclusion (i.e. for any pair of distinct hyperbolic lines $s_1, s_2 \in \{l,m\}^{\perp\perp}$ we have $\{s_1,s_2\}^{\perp\perp} = \{l,m\}^{\perp\perp}$).*

Proof: Assume that two distinct hyperbolic lines l and m intersect in the point p in U_n, so that $\langle l,m\rangle$ is a plane of rank two or three. Since $n \geq 6$, the polar space $\langle l,m\rangle^{\pi}$ of $\langle l,m\rangle$ is a subspace of dimension at least three, which contains a hyperbolic line, since $\dim(\mathrm{rad}(\langle l,m\rangle^{\pi})) = \dim(\mathrm{rad}(\langle l,m\rangle)) \leq 1$. Hence $\{l,m\}^{\perp} \neq \varnothing$. Using lemma 3.3.4, we have $\{l,m\}^{\perp\perp} = \mathbf{L}(\langle l,m\rangle)$. It follows immediately that $\{l,m\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_n)$ with respect to inclusion.

Conversely, assume l and m do not intersect. Therefore $\langle l,m\rangle$ is four-dimensional of rank two, three, or four. By the formulas B.3 and B.4 on page 255 we can choose a plane P of rank two in $\langle l,m\rangle$ containing distinct hyperbolic lines s_1 and s_2 spanning P itself. Lemma 3.3.4 yields $\{s_1,s_2\}^{\perp\perp} = \langle s_1,s_2\rangle \subsetneq \langle l,m\rangle$, thus $\{l,m\}^{\perp\perp}$ is not minimal in $\mathbf{G}(U_n)$ with respect to inclusion. ∎

For the same problem with three different vertices of $\mathbf{G}(U_n)$, we propose the following. Three different mutually intersecting hyperbolic lines intersect in one point in u_n if we can find a hyperbolic line s such that:

- the hyperbolic line s intersects each hyperbolic line k_i with $s \neq k_i$ for $1 \leq i \leq 3$, and
- the space $\langle s, k_1, k_2\rangle$ is of dimension four.

Translated into graph language the above conditions say that three different mutually intersecting vertices k_1, k_2, k_3 intersect in one point if there exists a vertex s of $\mathbf{G}(U_n)$ such that:

- $\{s, k_i\}^\perp \neq \emptyset$ and $\{s, k_i\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_n)$ with respect to inclusion, if $k_i \neq s$, for $1 \leq i \leq 3$ (cf. lemma 3.3.6), and
- $\{k_1, k_2\}^{\perp\perp} = \mathbf{L}(\langle k_1, k_2\rangle) \subsetneq \mathbf{L}(\langle k_1, k_1, s\rangle) = \{k_1, k_2, s\}^{\perp\perp}$ (cf. lemma 3.3.7).

For the above definition to be useful we only need to verify that for any two intersecting hyperbolic lines l and m, there exists a hyperbolic line s in the vector space U_n such that $\langle l, m, s\rangle$ is a four-dimensional space and $\{l, m, s\}^\perp \neq \emptyset$.

Lemma 3.3.7 *Let $n \geq 6$. For any two intersecting hyperbolic lines k and l there is a hyperbolic line g, intersecting k and l, such that $\langle k, l, g\rangle$ is four-dimensional subspace with $\{k, l, g\}^\perp \neq \emptyset$.*

Proof: Suppose we can find a hyperbolic line g such that $W = \langle k, l, g\rangle$ is a four-dimensional subspace. If W is non-degenerate, then W^π is of dimension at least two and non-degenerate, so the subspace W^π contains some hyperbolic line. Therefore we will construct a regular four-dimensional subspace of U_n containing the plane $\langle l, k\rangle$ in order to prove the lemma.

If the plane $\langle k, l\rangle$ is non-degenerate, then a non-degenerate four-dimensional subspace containing this plane obviously exists. We simply choose an arbitrary regular one-dimensional subspace $y \in \langle k, l\rangle^\pi$ and consider $\langle k, l, y\rangle$. If the subspace $\langle k, l\rangle$ is degenerate, then it has rank two. Denote the one-dimensional radical of $\langle k, l\rangle$ by x. In this case we choose an arbitrary one-dimensional space $y \in l^\pi \backslash x^\pi$ and obtain a Gram matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & \alpha \end{pmatrix}$ for $\langle l, x, y\rangle$ with respect to an orthonormal basis of l and arbitrary non-trivial vectors in x and y. This matrix has determinant distinct from 0 for any value of α. Hence $\langle l, x, y\rangle = \langle k, l, y\rangle$ is a four-dimensional non-degenerate space as required.

As g we choose in both cases some hyperbolic line in $\langle k, l, y\rangle$ running through the point p, which is different from k and l. This is possible since the regular four-dimensional space $\langle l, k, y\rangle$ contains at least $q^4 - q^3 + q^2$ distinct hyperbolic lines incident to p by lemma B.1.6. ∎

Definition 3.3.8 Let $n \geq 6$ and let Γ be a graph isomorphic to $\mathbf{G}(U_n)$. Two vertices k and l of Γ are defined to **intersect** if $\{k, l\}^\perp \neq \emptyset$ and if $\{k, l\}^{\perp\perp}$ is minimal in Γ with respect to inclusion. Three mutually intersecting vertices k_1, k_2, k_3 of Γ are defined to **intersect in one point** if there exists a vertex s in Γ with the following properties:

- the vertex s intersects with each vertex k_i, if $s \neq k_i$, for $1 \leq i \leq 3$,

- $\{k_1, k_2, s\}^\perp \neq \varnothing$ and $\{k_1, k_2\}^{\perp\perp} = \mathbf{L}(\langle k_1, k_2 \rangle) \subsetneq \mathbf{L}(\langle k_1, k_1, s \rangle) = \{k_1, k_2, s\}^{\perp\perp}$.

An **interior point** of the graph Γ is a maximal set p of distinct pairwise intersecting vertices of Γ such that any three elements of p intersect in one point. We denote the set of all interior points of Γ by $\mathcal{I}$. Moreover, an **interior line** of the graph Γ is a vertex of the graph Γ. The set of all interior lines of Γ is denoted by $\mathcal{L}$.

The discussions in this section yield the following statement.

Proposition 3.3.9 *Let $n \geq 6$ and let Γ be a graph isomorphic to $\mathbf{G}(U_n)$. Then the point-line geometry $(\mathcal{I}, \mathcal{L}, \supset)$ is isomorphic to the geometry on arbitrary one-dimensional subspaces and regular two-dimensional subspaces of U_n.*

3.4 The hyperbolic geometry

In this section we study the point-line geometry $(\mathcal{I}, \mathcal{L})$ of a graph Γ isomorphic to $\mathbf{G}(U_n)$ for $n \geq 5$, which has been defined in definition 3.2.15 for $n = 5$ and in definition 3.3.8 for $n \geq 6$. By construction it is clear that $(\mathcal{I}, \mathcal{L})$ is isomorphic to the geometry on arbitrary one-dimensional subspaces and regular two-dimensional subspaces of U_n. In section 3.6 we will define a similar geometry on an arbitrary connected locally $\mathbf{G}(U_n)$ graph for $n \geq 7$ and prove that it is isomorphic to the point-line geometry of singular one-dimensional subspaces and regular two-dimensional subspaces of some unitary polar geometry over some finite field. This result will be achieved via theorem 1.2 of [25], restated below as theorem 3.4.1. In order to be able to deal with the setting of an arbitrary connected locally $\mathbf{G}(U_n)$ graph in section 3.6, we will in this section investigate the interaction of the graph $\Gamma \cong \mathbf{G}(U_n)$, the geometry $(\mathcal{I}, \mathcal{L})$, and the geometry on (singular) one-dimensional subspaces and non-degenerate two-dimensional subspaces of U_n in the context of Cuypers' theorem. In this thesis non-collinearity is denoted by the symbol $\sim$ and by convention, a point is non-collinear to itself.

Theorem 3.4.1 (Cuypers [25], theorem 1.2) *Let $G = (P, L)$ be a non-linear, planar connected partially linear space of finite order $q \geq 3$. Suppose the following holds in G:*

1. *all planes are finite and either isomorphic to a dual affine plane or linear plane;*
2. *in a linear plane no four lines intersect in six points;*
3. *for all points x and y the inclusion $x^\sim \subseteq y^\sim$ implies $x = y$;*
4. *if E is a linear plane and x a point, then $x^\sim \cap E \neq \varnothing$.*

Then q is a prime power and G is isomorphic to the geometry of singular points and hyperbolic lines of a non-degenerate symplectic or unitary polar space over the field $\mathbb{F}_q$, respectively $\mathbb{F}_{q^2}$.

The first step we have to take in order to apply theorem 3.4.1 to the geometry $(\mathcal{I}, \mathcal{L})$ is to distinguish singular one-dimensional subspaces from regular one-dimensional subspaces. In particular, we need to reconstruct the concept of perpendicularity.

Definition 3.4.2 Two interior points a and b of the graph Γ are **orthogonal**, denoted by $a \perp b$, if there exist a vertex $m \in a$ and a vertex $l \in b$ satisfying $m \perp l$, i.e., m and l are adjacent in Γ. Also, we say that an interior point p is orthogonal to an interior line l, if p contains a vertex m adjacent to l.

Let $p \in \mathcal{I}$. Denote the set of all interior points orthogonal to p by $\mathcal{I}^p$ and the set of all interior lines orthogonal to p by $\mathcal{L}^p$.

For an interior line l of Γ we denote by $\mathcal{I}_l$ the set of all interior points of Γ orthogonal to l.

Furthermore, if X is a set of interior points, then by $\mathcal{I}^X$ denote the set of all interior points orthogonal to each interior point in X. Similarly, $\mathcal{L}^X$ stands for the set of all vertices orthogonal to each point in X. We have $\mathcal{I}^X = \bigcap_{p \in X} \mathcal{I}^p$ and $\mathcal{L}^X = \bigcap_{p \in X} \mathcal{L}^p$.

In lemma 3.2.8 and lemma 3.3.4 we showed an equivalence between the hyperbolic lines of the plane $\langle k, l \rangle$ and the vertices of $\{k, l\}^{\perp\perp}$ for any two vertices $k, l \in \Gamma$ with $\{k, l\}^\perp \neq \varnothing$. For $n \geq 6$ any pair of intersecting hyperbolic lines k and l of U_n satisfies the condition $\{k, l\}^\perp \neq \varnothing$, since $\mathrm{rk}(\langle k, l \rangle^\pi) \geq 2$. In that case we have full control over the hyperbolic lines contained in a plane P generated by two intersecting hyperbolic lines k and l and, thus, over the elements of $\{k, l\}^{\perp\perp}$. However in case $n = 5$ the induced subgraph $\{k, l\}^\perp$ is not empty if and only if the span of k and l is a regular plane due to remark 3.2.6. In the case that two hyperbolic lines k and l generate a singular plane, so far we have no method to describe the hyperbolic lines contained in $\langle k, l \rangle$ as vertices of the graph Γ. This defect is fixed in the next lemma, thus enabling us to define planes of the graph Γ.

Lemma 3.4.3 *Let k and l be two intersecting vertices of $\mathbf{G}(U_5)$ such that $\{k, l\}^\perp = \varnothing$. Then any vertex of $\mathcal{L}^{\mathcal{I}_k \cap \mathcal{I}_l}$ is a hyperbolic line of the space $\langle l, k \rangle$ in U_5 and vice versa.*

Proof: Since $\mathcal{L}^{\mathcal{I}_k \cap \mathcal{I}_l} = \bigcap_{p \in \mathcal{I}_k \cap \mathcal{I}_l} \mathcal{L}^p = \bigcap_{p \in \mathcal{I}_k \cap \mathcal{I}_l} (\bigcup_{m \in p} m^\perp) = \bigcap_{p \in \mathcal{I}_k \cap \mathcal{I}_l} (\bigcup_{m \in p} \mathbf{L}(m^\pi))$, it is enough to show that for any hyperbolic line $h \subseteq \langle k, l \rangle$ each interior point $p \in \mathcal{I}_k \cap \mathcal{I}_l$ contains a vertex m such that $h \subseteq m^\pi$, to prove the inequality $\mathbf{L}(\langle k, l \rangle) \subseteq \mathcal{L}^{\mathcal{I}_k \cap \mathcal{I}_l}$

Let h be an arbitrary hyperbolic line in the degenerate plane $\langle l, k \rangle$, so $l^\pi \cap k^\pi \subseteq h^\pi$ and h^π is a non-degenerate plane of U_5. We choose some point g of the singular line

$l^\pi \cap k^\pi$ and some hyperbolic line m_g in h^π incident to g, in particular $h \in \mathbf{L}(m_g^\pi)$. Thus $h \in \bigcup_{m \in p} \mathbf{L}(m^\pi)$ for each point $p \in \mathcal{I}_k \cap \mathcal{I}_l$. Hence $\mathbf{L}(\langle k, l \rangle) \subseteq \bigcup_{m \in p} \mathbf{L}(m^\pi)$ for each point $p \in \mathcal{I}_k \cap \mathcal{I}_l$, so $\mathbf{L}(\langle k, l \rangle) \subseteq \mathcal{L}^{\mathcal{I}_k \cap \mathcal{I}_l}$.

Conversely, let h be a hyperbolic line of U_5 not contained in the plane $\langle k, l \rangle$. It suffices to realize that the singular line $k^\pi \cap l^\pi$ contains a point s not incident to h^π. But since $h^\pi \cap k^\pi \cap l^\pi$ is one-dimensional, this is obvious. Hence the identity $\mathcal{L}^{\mathcal{I}_k \cap \mathcal{I}_l} = \mathbf{L}(\langle l, k \rangle)$ follows. ∎

We come now to the definition of a plane of the graph Γ.

Definition 3.4.4 Let $n \geq 5$ and let Γ be a graph isomorphic to $\mathbf{G}(U_n)$. Two vertices k and l of Γ are said to **span a graphical plane** $E_\Gamma = \langle k, l \rangle_\Gamma$, if k and l are two intersecting vertices of Γ; cf. definitions 3.2.15 and 3.3.8 for the notion of intersecting vertices. The graphically plane $\langle k, l \rangle_\Gamma$ is the point-line geometry whose line set equals either

- the set $\{k, l\}^{\perp\perp} = \mathbf{L}(\langle k, l \rangle)$ in case $\{k, l\}^\perp \neq \varnothing$ (thus k and l have distance two in Γ) or
- the set $\mathcal{L}^{\mathcal{I}_k \cap \mathcal{I}_l} = \mathbf{L}(\langle k, l \rangle)$ in case $\{k, l\}^\perp = \varnothing$ (so $n = 5$ and k and l have distance four in Γ).

The point set of $\langle k, l \rangle_\Gamma$ contains all interior points d which are incident to some line of $\langle k, l \rangle_\Gamma$, i.e., $d \in \langle k, l \rangle_\Gamma$ if and only if there exists a line $m \in d$ contained in $\langle k, l \rangle_\Gamma$.

Let p be an interior point of Γ then we consider all graphical planes $E_\Gamma^{ij} = \langle k_i, k_j \rangle_\Gamma$ for $k_i, k_j \in p$ and $k_i \neq k_j$ and count the number $\mathcal{N}_{E_\Gamma^{ij}}$ of interior lines in E_Γ^{ij}. By the formulas B.3 and B.4 on page 255 the number $\mathcal{N}_{E_\Gamma^{ij}}$ is either equal to q^4 or to $q^4 - q^3 + q^2$. In fact by lemma B.1.11 we find a plane $\langle k_i, k_j \rangle = E_\Gamma^{ij}$ such that $\mathcal{N}_{E_\Gamma^{ij}} = q^4 - q^3 + q^2$. In this graphical plane we count the number $\mathcal{N}_p$ of hyperbolic lines incident to p, in other words $\mathcal{N}_p = |p \cap E_\Gamma^{ij}|$ and due to lemma B.1.5 the number $\mathcal{N}_p$ is either $q^2 - q$ or q^2.

We call p an **interior singular point** of Γ if $\mathcal{N}_p = q^2$. An interior point that is not singular, is called **regular**.

Two intersecting vertices k and l of Γ are defined to **intersect in a singular point** if the intersecting point of k and l is an interior singular point.

Definition 3.4.5 Let $n \geq 5$ and let Γ be a graph isomorphic to $\mathbf{G}(U_n)$. By $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ we denote the point-line geometry which consists of all interior singular points and all interior lines of the geometry $(\mathcal{I}, \mathcal{L})$ from definitions 3.2.15 and 3.3.8.

It is called the **interior hyperbolic geometry**. Two interior lines k and l of $\mathcal{G}$ are said to span an **algebraic plane** $\langle k, l\rangle_{\mathcal{G}}$ of the geometry $\mathcal{G}$, if k and l intersect in an interior singular point of Γ. The algebraic plane $\langle k, l\rangle_{\mathcal{G}}$ consists of all interior lines and all interior singular points of the graphical plane $\langle k, l\rangle_{\Gamma}$.

Lemma 3.4.6 *The geometry $\mathcal{G}$ is a connected partially linear space of order q.*

Proof: Since the interior hyperbolic lines of the geometry $\mathcal{G}$ are in one-to-one correspondence to the hyperbolic lines of a unitary polar space U_n the order of $\mathcal{G}$ is q. Indeed by the formula B.3 on page 255 each hyperbolic line of U_n contains exactly $q+1$ singular points and so each interior line of $\mathcal{G}$ is incident to exactly $q+1$ interior singular points.

Let p and d be two interior singular points, which are collinear to more than one common line, so let $k, l \in p \cap d$. Then the interior lines k and l intersect in the interior singular point p implying that k and l span the algebraic plane $\langle k, l\rangle_{\mathcal{G}}$. By definition 3.4.5 the algebraic plane $\langle k, l\rangle_{\mathcal{G}}$ contains the two different interior points p and d and the line set of $\langle k, l\rangle_{\mathcal{G}}$ coincides with $\mathbf{L}(\langle l, k\rangle)$, so we conclude that $l = \langle p, d\rangle = k$, contradiction. By definition 3.2.15 and definition 3.3.8 the interior singular points are pencils of hyperbolic lines and the diameter of Γ is two, three or four by lemma 3.2.5, proposition 3.3.1 and proposition 3.3.2. In particular the geometry $\mathcal{G}$ is a connected space. ∎

Recall that we denote by $p^{\sim}$ all interior singular point of $\mathcal{G}$ which are not collinear to a given interior singular point p of $\mathcal{G}$. Therefore for any interior singular point $p \in \mathcal{P}$ we obtain that $p^{\sim} = (\mathcal{I}^p \cap \mathcal{P}) \cup p$.

Lemma 3.4.7 *The geometry $\mathcal{G}$ is a non-linear space, so the space $\mathcal{G}$ contains two non-collinear interior points.*

Proof: To verify the statement, it is sufficient to show that $\mathcal{I}^p \cap \mathcal{P}$ is not empty for some interior singular point p of $\mathcal{G}$. This is certainly satisfied since the order of the geometry $\mathcal{G}$ is q and any interior singular point incident to k with $k \in h^{\perp}$ and $h \in p$ is an element of $\mathcal{I}^p \cap \mathcal{P}$. ∎

Definition 3.4.8 Let $n \geq 5$ and let Γ be a graph isomorphic to $\mathbf{G}(U_n)$. Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be the interior hyperbolic geometry and let k and l be two interior lines of $\mathcal{G}$ spanning an algebraic plane $\langle k, l\rangle_{\mathcal{G}}$. Denote the intersection point of k and l, which is singular, by p.

If $\langle k, l\rangle_{\mathcal{G}}$ contains $q^4 - q^3 + q^2$ interior lines, then the **geometric plane** $\langle k, l\rangle_{\mathcal{G}}^{g}$ spanned by k and l is defined to be the algebraic plane $\langle k, l\rangle_{\mathcal{G}}$.

If otherwise $\langle k, l\rangle_{\mathcal{G}}$ contains q^4 interior lines, then the **geometric plane** $\langle k, l\rangle_{\mathcal{G}}^{g}$ spanned by k and l is defined as the following object: Its line set consists of all interior lines of $\langle k, l \rangle_{\mathcal{G}}$ intersecting both l and k in interior singular points distinct from p (those lines are called connecting lines), and all interior lines of $\langle k, l \rangle_{\mathcal{G}}$ that contain p and intersect one of the connecting lines in an interior singular point. The points of $\langle k, l \rangle_{\mathcal{G}}^{g}$ are all interior singular points d incident with some of the above lines.

The next results are fundamental and will be needed in the proof of proposition 3.4.20.

Lemma 3.4.9 *Let $E_{\mathcal{G}}^{g} = \langle k, l \rangle_{\mathcal{G}}^{g}$ be a geometric plane of $\mathcal{G}$ coinciding with the algebraic plane $\langle k, l \rangle_{\mathcal{G}}$. Then $\langle k, l \rangle_{\mathcal{G}}^{g}$ is a finite linear subspace of $\mathcal{G}$ generated by any two of its intersecting lines.*

Proof: Let p be the intersection of k and l. Since $\langle k, l \rangle_{\mathcal{G}}^{g}$ coincides with the algebraic plane $\langle k, l \rangle_{\mathcal{G}}$ the geometric plane $\langle k, l \rangle_{\mathcal{G}}^{g}$ contains $q^4 - q^3 + q^2$ interior lines and the subspace $\langle k, l \rangle \subseteq U_n$ is a non-degenerate plane. Hence $\{k, l\}^{\perp} \neq \emptyset$, and by definitions 3.4.4, 3.4.5, 3.4.8 the line set of $\langle k, l \rangle_{\mathcal{G}}^{g}$ equals $\{k, l\}^{\perp\perp}$. If an interior line m intersects the geometric plane $E_{\mathcal{G}}^{g}$ in two interior singular points, then the hyperbolic line m intersects also the plane $\langle k, l \rangle$ in at least two singular points and thus $m \subseteq \langle k, l \rangle$. It follows that $m \in \{k, l\}^{\perp\perp}$, whence m is an interior line of $\langle k, l \rangle_{\mathcal{G}}^{g}$. Therefore $\langle k, l \rangle_{\mathcal{G}}^{g}$ is a subspace of the geometry $\mathcal{G}$. The subspace $\langle k, l \rangle_{\mathcal{G}}^{g}$ is a linear subspace, since the interior singular points of $\langle k, l \rangle_{\mathcal{G}}^{g}$ are in one-to-one correspondence with the singular points of the regular plane $\langle k, l \rangle$ of U_n and every pair of singular points contained in a regular plane of U_n spans a hyperbolic line of this plane. Certainly the geometric plane $\langle k, l \rangle_{\mathcal{G}}^{g}$ is a finite plane because $\langle k, l \rangle_{\mathcal{G}}^{g}$ contains a finite number of interior lines and $\mathcal{G}$ has order q.

It remains to prove that $\langle k, l \rangle_{\mathcal{G}}^{g}$ is spanned by any two of its intersecting lines, say s and t. Any subspace A of $\langle k, l \rangle_{\mathcal{G}}^{g}$ containing s and t has at least $q^2 + q + 1$ points. Indeed, let x be a point on s and not on t. Then each of the q lines connecting x with a point y, which is on t and not on s, contains q points distinct from x, totalling q^2. Together with the $q + 1$ points on s this gives $q^2 + q + 1$ points contained in A. If there exists a point z in $\langle k, l \rangle_{\mathcal{G}}^{g}$ outside A, then, by linearity of $\langle k, l \rangle_{\mathcal{G}}^{g}$, there are at least $q^2 + q + 1$ interior lines in $\langle k, l \rangle_{\mathcal{G}}^{g}$ through z, contradicting the fact that there exist exactly q^2 interior lines of $\langle k, l \rangle_{\mathcal{G}}^{g}$ through z by lemma B.1.5. Hence a point z of $\langle k, l \rangle_{\mathcal{G}}^{g}$ not contained in A does not exist, so $A = \langle k, l \rangle_{\mathcal{G}}^{g}$ and any two intersecting lines of $\langle k, l \rangle_{\mathcal{G}}^{g}$ in fact span the geometric plane $\langle k, l \rangle_{\mathcal{G}}^{g}$. ∎

Lemma 3.4.10 *Let $E_{\mathcal{G}}^{g} = \langle k, l \rangle_{\mathcal{G}}^{g}$ be a geometric plane of $\mathcal{G}$ such that the algebraic plane $\langle k, l \rangle_{\mathcal{G}}$ contains exactly q^4 different interior lines and let c be a connecting line of $\langle k, l \rangle_{\mathcal{G}}^{g}$ intersecting the interior line k in the interior singular point k_c. Moreover let t be an interior line through the interior point p of $\langle k, l \rangle_{\mathcal{G}}^{g}$, where p is the intersection* *of k and l, which intersects c in an interior singular point. Then the interior line t intersects each connecting line of $\langle k, l\rangle_{\mathcal{G}}^{g}$ going through k_c in an interior singular point.*

Proof: Since $\langle k, l\rangle_{\mathcal{G}}^{g} \neq \langle k, l\rangle_{\mathcal{G}}$ the hyperbolic lines k and l span a rank two plane $P = \langle k, l\rangle$ in U_n. Thus for any singular point s on the hyperbolic line k the subspace $s^{\pi} \cap P$ is a totally singular line and $s^{\pi} \cap l$ is a unique singular point on l. Moreover $\langle s, d\rangle$ is a hyperbolic line in P for any singular point d on l distinct from $s^{\pi} \cap l$.

As c is a connecting line of $\langle k, l\rangle_{\mathcal{G}}^{g}$ through the point k_c, the interior line c is spanned by the interior singular points k_c and l_c, where l_c is incident to l and collinear to k_c. We denote the intersection point of c and t by h. Let g be connecting line of $\langle k, l\rangle_{\mathcal{G}}^{g}$ incident to k_c and different from c, so $g = \langle k_c, l_g\rangle$ for some interior singular point $l_g \in l$, which is collinear to k_c and different from l_c.

In order to prove the statement we will show that in the rank two plane $\langle k, l\rangle$ of U_n the hyperbolic line t intersects the hyperbolic line g in a singular point. Therefore we fix a vector $k_c^{v} \in k_c$ and a vector p^{v} of the singular point p such that $(k_c^{v}, p^{v}) = 1$, where $(\cdot,\cdot)$ denotes the hermitian form on U_n. Furthermore let $s_{k_c} = (k_c)^{\pi} \cap l$ and $s_{k_c}^{v}$ be some vector of the point s_{k_c}.

By lemma B.1.12 the vector $s_{k_c}^{v} - \mu(s_{k_c}^{v}, p^{v})p^{v} = l_g^{v}$ spans the singular point l_g for some $\mu \in \mathbb{F}^{\sigma,1\times}$, for the definition of $\mathbb{F}^{\sigma,1\times}$ see appendix B and $s_{k_c}^{v} - \nu(s_{k_c}^{v}, p^{v})p^{v} = l_c^{v}$ spans the singular point l_c for some $\nu \in \mathbb{F}^{\sigma,1\times}$. Also the vector $h^{v} = l_c^{v} - \delta(l_c^{v}, k_c^{v})k_c^{v}$ spans the singular point h for some scalar $\delta \in \mathbb{F}^{\sigma,1\times}$. Hence every singular point of the hyperbolic line t is spanned by a vector of the form

$$
\begin{aligned}
& h^{v} - \lambda(h^{v}, p^{v})p^{v} \\
={}& l_c^{v} - \delta(l_c^{v}, k_c^{v})k_c^{v} - \lambda(l_c^{v} - \delta(l_c^{v}, k_c^{v})k_c^{v}, p^{v})p^{v} \\
={}& l_c^{v} - \delta(l_c^{v}, k_c^{v})k_c^{v} - \lambda(l_c^{v}, p^{v})p^{v} + \lambda\delta(l_c^{v}, k_c^{v})(k_c^{v}, p^{v})p^{v} \\
={}& s_{k_c}^{v} - \nu(s_{k_c}^{v}, p^{v})p^{v} - \delta(s_{k_c}^{v} - \nu(s_{k_c}^{v}, p^{v})p^{v}, k_c^{v})k_c^{v} - \lambda(s_{k_c}^{v} - \nu(s_{k_c}^{v}, p^{v})p^{v}, p^{v})p^{v} + \\
& \lambda\delta(s_{k_c}^{v} - \nu(s_{k_c}^{v}, p^{v})p^{v}, k_c^{v})(k_c^{v}, p^{v})p^{v} \\
={}& s_{k_c}^{v} - \nu(s_{k_c}^{v}, p^{v})p^{v} - \delta(s_{k_c}^{v}, k_c^{v})k_c^{v} + \delta\nu(s_{k_c}^{v}, p^{v})(p^{v}, k_c^{v})k_c^{v} - \lambda(s_{k_c}^{v}, p^{v})p^{v} + \\
& \lambda\nu(s_{k_c}^{v}, p^{v})(p^{v}, p^{v})p^{v} + \lambda\delta(s_{k_c}^{v}, k_c^{v})(k_c^{v}, p^{v})p^{v} - \nu\lambda\delta(s_{k_c}^{v}, p^{v})(p^{v}, k_c^{v})(k_c^{v}, p^{v})p^{v} \\
={}& s_{k_c}^{v} - \nu(s_{k_c}^{v}, p^{v})p^{v} - \delta(s_{k_c}^{v}, k_c^{v})k_c^{v} + \delta\nu(s_{k_c}^{v}, p^{v})k_c^{v} - \lambda(s_{k_c}^{v}, p^{v})p^{v} + \delta\lambda(s_{k_c}^{v}, k_c^{v})p^{v} - \\
& \lambda\nu\delta(s_{k_c}^{v}, p^{v})p^{v}
\end{aligned}
$$

for some $\lambda \in \mathbb{F}^{\sigma,1}$.

Suppose $\delta \cdot \nu = -1$ then for any non-zero scalar $\lambda \in \mathbb{F}^{\sigma,1}$ we determine that

$$h^{v} - \lambda(h^{v}, p^{v})p_{v} = s_{k_c}^{v} - \nu(s_{k_c}^{v}, p^{v})p^{v} - (s_{k_c}^{v}, p^{v})k_c^{v} = l_c^{v} - k_c^{v},$$

thus $(h^\nu, p^\nu) = 0$, contradiction. Therefore $\delta \cdot \nu \neq -1$ and due to lemma B.1.13 for the three non-zero scalars μ, ν and δ of $\mathbb{F}^{\sigma,1}$ we can determine three scalars λ, ε and $\alpha \in \mathbb{F}^{\sigma,1\times}$ such that

$$\begin{aligned}
& s^\nu_{k_c} - \nu(s^\nu_{k_c}, p^\nu)p^\nu + \delta\nu(s^\nu_{k_c}, p^\nu)k^\nu_c - \lambda(s^\nu_{k_c}, p^\nu)p^\nu - \lambda\nu\delta(s^\nu_{k_c}, p^\nu)p^\nu \\
= \; & s^\nu_{k_c} - \nu(s^\nu_{k_c}, p^\nu)p^\nu - \delta(s^\nu_{k_c}, k^\nu_c)k^\nu_c + \delta\nu(s^\nu_{k_c}, p^\nu)k^\nu_c - \lambda(s^\nu_{k_c}, p^\nu)p^\nu + \\
& \delta\lambda(s^\nu_{k_c}, k^\nu_c)p^\nu - \lambda\nu\delta(s^\nu_{k_c}, p^\nu)p^\nu \\
= \; & s^\nu_{k_c} - \varepsilon(s^\nu_{k_c}, k^\nu_c)k^\nu_c - \mu(s^\nu_{k_c}, p^\nu)p^\nu + \mu\varepsilon(s^\nu_{k_c}, k^\nu_c)p^\nu - \alpha(s^\nu_{k_c}, k^\nu_c)k^\nu_c + \\
& \alpha\mu(s^\nu_{k_c}, p^\nu)k^\nu_c - \alpha\mu\varepsilon(s^\nu_{k_c}, k^\nu_c)k^\nu_c \\
= \; & s^\nu_{k_c} - \mu(s^\nu_{k_c}, p^\nu)p^\nu + \alpha\mu(s^\nu_{k_c}, p^\nu)k^\nu_c \\
= \; & l^\nu_g - \alpha(l^\nu_g, k^\nu_c)k^\nu_c \, ,
\end{aligned}$$

which implies that the hyperbolic line $t = \langle h, p \rangle$ intersects $\langle k_c, l_d \rangle = g$ in a singular point. ∎

A consequence of lemma 3.4.10 is the following result.

Lemma 3.4.11 *Let $E^g_{\mathcal{G}} = \langle k, l \rangle^g_{\mathcal{G}}$ be a geometric plane of $\mathcal{G}$ such that the algebraic plane $\langle k, l \rangle_{\mathcal{G}}$ contains exactly q^4 different interior lines and let p be the intersection point of k and l. Any line through the interior singular point p intersects each connecting line of $\langle k, l \rangle^g_{\mathcal{G}}$ in an interior singular point.*

Let $E^g_{\mathcal{G}} = \langle k, l \rangle^g_{\mathcal{G}}$ be a geometric plane of $\mathcal{G}$ such that the algebraic plane $\langle k, l \rangle_{\mathcal{G}}$ contains exactly q^4 different interior lines and let p be the intersection point of k and l. We claim that any interior point d of $\langle k, l \rangle^g_{\mathcal{G}}$ is on a line through p in $\langle k, l \rangle^g_{\mathcal{G}}$ or an element of $p^\sim$.

To prove this we assume that d is some interior point of $\langle k, l \rangle^g_{\mathcal{G}}$ not on a line through p, otherwise there is nothing to prove. So by definition 3.4.8, d is incident to some connecting line c. Since the plane $\langle k, l \rangle$ in U_n is of rank two, only the singular point $p^\pi \cap c$ of the hyperbolic line c is not incident to some hyperbolic line running through p in $\langle l, k \rangle$ by lemma B.1.4. Therefore if $d = p^\pi \cap c$ then $d \in p^\sim$. If on the other hand d is a singular point distinct from $p^\pi \cap c$ then $\langle c, d \rangle$ is a hyperbolic line in $\langle l, k \rangle$ and thus by definition 3.4.8, the interior singular point d is a line through p in $\langle k, l \rangle^g_{\mathcal{G}}$.

Lemma 3.4.12 *Let $E^g_{\mathcal{G}} = \langle k, l \rangle^g_{\mathcal{G}}$ be a geometric plane of $\mathcal{G}$ such that the algebraic plane $\langle k, l \rangle_{\mathcal{G}}$ contains exactly q^4 different interior lines and let k_p be some interior singular point incident to k. Then any interior point d of $\langle k, l \rangle^g_{\mathcal{G}}$ not on k is either an element of $k^\sim_p$ or incident with a connecting line through k_p, i.e., an interior line through k_p intersecting l in an interior singular point.*

Proof: Let d be an interior singular point of $\langle k, l\rangle_G^g$ not incident to a connecting line through the interior point k_p, since otherwise d is collinear to k_p and there is nothing to prove.

Let d be not contained in $k_p^\sim$ and suppose d is on an interior line t through p, where p is the intersecting point of k and l. Then by lemma 3.4.11, the interior line t intersects a connecting line c going through k_p say in the interior point h. Moreover we consider the interior singular point $l_c = l \cap c$ of l. We will show the claim by the method used in lemma 3.4.10. So let k_p^v be a vector of k_p and p^v be a vector of the singular point p such that $(p^v, k_p^v) = 1$. Furthermore let $s_p = p^\pi \cap c$ and s_p^v be some vector of the point s_p.

By lemma B.1.12, for some $\mu, \nu \in \mathbb{F}^{\sigma,1\times}$ the vector $s_p^v - \mu(s_p^v, k_p^v)k_p^v = l_c^v$ spans the singular point l_c and the singular point h is generated by $s_p^v - \nu(s_p^v, k_p^v)k_p^v = h^v$. Moreover for some scalar $\delta \in \mathbb{F}^{\sigma,1\times}$ the span of the vector $d^v = h^v - \delta(h^v, p^v)p^v$ is the singular point d. Therefore the vector

$$\begin{aligned} & d^v - \lambda(d^v, k_p^v)k_p^v \\ = \;& s_p^v - \nu(s_p^v, k_p^v)k_p^v - \delta(s_p^v, p^v)p^v + \delta\nu(s_p^v, k_p^v)p^v - \lambda(s_p^v, k_p^v)k_p^v + \\ & \delta\lambda(s_p^v, p^v)k_p^v - \lambda\nu\delta(s_p^v, k_p^v)k_p^v \end{aligned}$$

generates a singular point of the hyperbolic line $\langle k_p, d\rangle$ for every $\lambda \in \mathbb{F}^{\sigma,1}$. If $\delta \cdot \nu = -1$ then $d^v - \lambda(d^v, k_p^v)k_p^v = h^v - (s_p^v, k_p^v)p^v$, thus $(d^v, k_p^v) = 0$, contradiction. Hence $\delta \cdot \nu \neq -1$ and due to lemma B.1.13 for the three non-zero scalars μ, ν and δ of $\mathbb{F}^{\sigma,1}$ we can determine three scalars λ, ε and $\alpha \in \mathbb{F}^{\sigma,1\times}$ such that

$$\begin{aligned} & s_p^v - \nu(s_p^v, k_p^v)k_p^v - \delta(s_p^v, p^v)p^v + \delta\nu(s_p^v, k_p^v)p^v - \lambda(s_p^v, k_p^v)k_p^v + \\ & \delta\lambda(s_p^v, p^v)k_p^v - \lambda\nu\delta(s_p^v, k_p^v)k_p^v \\ = \;& s_{p_v}^v - \varepsilon(s_{p_v}^v, p^v)p^v - \mu(s_p^v, k_p^v)k_p^v + \mu\varepsilon(s_p^v, p^v)k_p^v - \alpha(s_p^v, p^v)p^v + \\ & \alpha\mu(s_p^v, k_p^v)p^v - \alpha\mu\varepsilon(s_p^v, p^v)p^v \\ = \;& l_c^v - \alpha(l_c^v, p^v)p^v , \end{aligned}$$

which implies that the hyperbolic line $t = \langle k_p, d\rangle$ intersects l in a singular point.

Next we turn to the case that d is a singular point on some connecting line c, which intersects the interior line k in an interior singular point k_c different from k_p, and an element of $p^\sim$. Let l_c be the intersecting point of l and c. If the interior points k_p and l_c are on a common connecting line m, then we have the same configuration as above, we just exchange the roles of the singular point p and l_c, and replace the point h by the point k_c. Therefore the hyperbolic line $\langle k, d\rangle$ intersects the hyperbolic line l in an singular point implying that d is on a connecting line through k_p in the geometric plane $\langle k, l\rangle_G^g$.

On the other hand if $l_c \in k_p^\sim$ then we consider the singular point d in the rank two plane $\langle k, l\rangle$ of U_n. As $d \in p^\sim$ the totally singular line $p^\pi \cap \langle k, l\rangle$ is going through d.

By lemma B.1.4 any line through d in $\langle k, l\rangle$ different from $p^\pi \cap \langle k, l\rangle$ is regular. Thus let k_g be a singular point on k, which is not contained in l_c^π. Then $g = \langle k_g, d\rangle$ is a hyperbolic line implying that $d \notin k_g^\sim$. Furthermore $l_c^\pi \cap k$ is the unique singular point k_p, so $l_c \notin k_g^\sim$ and by the argumentation from above the hyperbolic line $g = \langle k_g, d\rangle$ intersects the hyperbolic line l in a singular point l_g different from l_c. It follows also that l_g is not a subspace of $k_p^\pi \cap \langle k, l\rangle$, as $k_p^\pi \cap l = l_c$.

Next we use the connecting line g in place of c. So d is on the connecting line g, which intersects k in the interior point k_g and the interior line l in l_g, which is not an element of $k_p^\sim$. Thus the interior line $\langle k_p, d\rangle$ intersects the interior line l in an interior singular point and we are done. ∎

Lemma 3.4.13 *Let $E_\mathcal{G}^g = \langle k, l\rangle_\mathcal{G}^g$ be a geometric plane of $\mathcal{G}$ such that the algebraic plane $\langle k, l\rangle_\mathcal{G}$ contains exactly q^4 different interior lines and let p be the intersecting point of k and l. Then any interior point d of $\langle k, l\rangle_\mathcal{G}^g$ distinct from p is incident to some connecting line of $E_\mathcal{G}^g$.*

Proof: Let d be some interior point of the geometric plane $\langle k, l\rangle_\mathcal{G}^g$ then d is on some interior line of $\langle k, l\rangle_\mathcal{G}^g$, thus d is either on a connecting line or an interior singular point of a line through p.

Suppose d is incident to some line t through p. We choose some interior singular point k_p on k, then by lemma 3.4.12 the point d is either collinear to k_p, implying that d is on a connecting line through the point k_p and we are done, or $d \in k_p^\sim$.

If $d \in k_p^\sim$ then d is a point on the totally singular line $k_p^\pi \cap \langle k, l\rangle$ in U_n. Moreover d is not the radical of the rank two plane, as $d \in t$. It follows by lemma B.1.4 that $\langle d, k_b\rangle$ is a hyperbolic line for every singular point $k_b \in k$ different from k_p. In fact d is collinear to the interior singular point k_b by lemma 3.4.12, thus d is on a connecting line through k_b in $\langle k, l\rangle_\mathcal{G}^g$. ∎

Let x and y be two different interior singular points of a geometric plane $\langle k, l\rangle_\mathcal{G}^g$ of $\mathcal{G}$, which does not coincide with the algebraic plane $\langle k, l\rangle_\mathcal{G}$. Then $x^\pi \cap k = k_x$ and $y^\pi \cap k = k_y$ are unique singular points on the hyperbolic line k in the rank two plane $\langle k, l\rangle$ of U_n. The hyperbolic line k contains $q+1 \geq 4$ different singular points, thus k contains a singular point k_p such that $\langle x, k_p\rangle$ and $\langle y, k_p\rangle$ are hyperbolic lines. This argumentation together with lemma 3.4.12 and lemma 3.4.13 proves the statement of the next lemma.

Lemma 3.4.14 *Let $E_\mathcal{G}^g = \langle k, l\rangle_\mathcal{G}^g$ be a geometric plane of $\mathcal{G}$ such that the algebraic plane $\langle k, l\rangle_\mathcal{G}$ contains exactly q^4 different interior lines. Then any two interior singular points of $\langle k, l\rangle_\mathcal{G}^g$ not both on k are incident to a connecting line of some interior singular point k_p of the interior line k.*

Lemma 3.4.15 *Let $E_\mathcal{G}^g = \langle k, l\rangle_\mathcal{G}^g$ be a geometric plane of $\mathcal{G}$ such that the algebraic plane $\langle k, l\rangle_\mathcal{G}$ contains exactly q^4 different interior lines. Then $\langle k, l\rangle_\mathcal{G}^g$ is a finite plane* *of $\mathcal{G}$.*

Proof: Let h be an interior line of $\mathcal{G}$ intersecting the geometric plane $\langle k,l\rangle^g_{\mathcal{G}}$ in two different interior singular points x and y. We may assume that neither x nor y are on the line k nor coincide with the point p. Indeed if $x \in k$ or $y \in k$ then by lemma 3.4.12 the interior singular points are on a connecting line of $\langle k,l\rangle^g_{\mathcal{G}}$ implying $h \in \langle k,l\rangle^g_{\mathcal{G}}$ and moreover if $x = p$ or $y = p$ then x and y are on a line through p, again $h \in \langle k,l\rangle^g_{\mathcal{G}}$.

By lemma 3.4.14 there is an interior singular point k_p on k such that x as well as y are on some connecting line through k_p, we denote the lines by g_x ($x \in g_x$) and g_y ($y \in g_y$). From the fact that $x, y \in h$ we get that either $x \notin p^{\sim}$, or $y \notin p^{\sim}$. So without loss of generality we can assume that $x \notin p^{\sim}$, thus the interior singular point x is on an interior line t through p in $\langle k,l\rangle^g_{\mathcal{G}}$. Due to lemma 3.4.10 the intersection of t and g_x is an interior singular point d.

To prove the statement it is sufficient to verify that the hyperbolic line h intersects the line k resp. the line l in a singular point in U_n. Therefore we fix a vector $x^v \in x$ and a vector k^v_p of the singular point k_p such that $(k^v_p, x^v) = 1$. Furthermore let $s_{k_p} = (k_p)^\pi \cap t$ and $s^v_{k_p}$ be some vector of the point s_{k_p}. For some $\mu, \nu \in \mathbb{F}^{\sigma,1\times}$ the vector $s^v_{k_p} - \mu(s^v_{k_p}, x^v)x^v = p^v$ spans the singular point p and the point d is generated by the vector $s^v_{k_p} - \nu(s^v_{k_p}, x^v)x^v = d^v$ due to lemma B.1.12. Moreover $y = \langle y^v\rangle$ with $y^v = d^v - \delta(d^v, k^v_p)k^v_p$ for some $\delta \in \mathbb{F}^{\sigma,1\times}$. Certainly every singular point of h is spanned by a vector of the form

$$\begin{aligned} & y^v - \lambda(y^v, x^v)x^v \\ = \; & s^v_{k_p} - \nu(s^v_{k_p}, x^v)x^v - \delta(s^v_{k_p}, k^v_p)k^v_p + \delta\nu(s^v_{k_p}, x^v)k^v_p - \lambda(s^v_{k_p}, x^v)x^v + \\ & \delta\lambda(s^v_{k_p}, k^v_p)x^v - \lambda\nu\delta(s^v_{k_p}, x^v)x^v \end{aligned}$$

for some $\lambda \in \mathbb{F}^{\sigma,1}$. If $\delta \cdot \nu = -1$ then for any non-zero scalar $\lambda \in \mathbb{F}_{+0}$ we have the equation

$$\begin{aligned} & y^v - \lambda(y^v, x^v)x_v \\ = \; & s^v_{k_p} - \nu(s^v_{k_p}, x^v)x^v - (s^v_{k_p}, x^v)k^v_p \\ = \; & l^v_c - (s^v_{k_p}, x^v)k^v_p\,, \end{aligned}$$

thus $(y^v, x^v) = 0$, contradiction. Therefore $\delta \cdot \nu \neq -1$ and due to lemma B.1.13 for the three non-zero scalars μ, ν and δ of $\mathbb{F}^{\sigma,1}$ we determine three scalars λ, ε and

$\alpha \in \mathbb{F}^{\sigma,1\times}$ such that

$$\begin{aligned}
& s_{k_p}^{\nu} - \nu(s_{k_p}^{\nu}, x^{\nu})x^{\nu} + \delta\nu(s_{k_p}^{\nu}, x^{\nu})k_p^{\nu} - \lambda(s_{k_p}^{\nu}, x^{\nu})x^{\nu} - \lambda\nu\delta(s_{k_p}^{\nu}, x^{\nu})x^{\nu} \\
= \; & s_{k_p}^{\nu} - \nu(s_{k_p}^{\nu}, x^{\nu})x^{\nu} - \delta(s_{k_p}^{\nu}, k_p^{\nu})k_p^{\nu} + \delta\nu(s_{k_p}^{\nu}, x^{\nu})k_p^{\nu} - \lambda(s_{k_p}^{\nu}, x^{\nu})x^{\nu} + \\
& \delta\lambda(s_{k_p}^{\nu}, k_p^{\nu})x^{\nu} - \lambda\nu\delta(s_{k_p}^{\nu}, x^{\nu})x^{\nu} \\
= \; & s_{k_p}^{\nu} - \varepsilon(s_{k_p}^{\nu}, k_p^{\nu})k_p^{\nu} - \mu(s_{k_p}^{\nu}, x^{\nu})x^{\nu} + \mu\varepsilon(s_{k_p}^{\nu}, k_p^{\nu})x^{\nu} - \alpha(s_{k_p}^{\nu}, k_p^{\nu})k_p^{\nu} + \\
& \alpha\mu(s_{k_p}^{\nu}, x^{\nu})k_p^{\nu} - \alpha\mu\varepsilon(s_{k_p}^{\nu}, k_p^{\nu})k_p^{\nu} \\
= \; & s_{k_p}^{\nu} - \mu(s_{k_p}^{\nu}, x^{\nu})x^{\nu} + \alpha\mu(s_{k_p}^{\nu}, x^{\nu})k_p^{\nu} \\
= \; & p^{\nu} - \alpha(p^{\nu}, k_p^{\nu})k_p^{\nu},
\end{aligned}$$

which implies that the hyperbolic line h intersects k in a singular point. By symmetry also the intersection of l and h is a singular point and thus $\langle k, l\rangle_{\mathcal{G}}^{g}$ is a subspace of $\mathcal{G}$.

The geometry $\mathcal{G}$ is of order q. By the formula B.4 on page 255 we have $|\mathcal{L}^{\mathcal{I}_k \cap \mathcal{I}_l}| = |\mathbf{L}(\langle k, l\rangle)| = q^4$, so that the subspace $\langle k, l\rangle_{\mathcal{G}}^{g}$ is finite. To complete the proof let P the plane of the geometry $\mathcal{G}$ spanned by k and l. As P is a subspace of $\mathcal{G}$, the plane P contains all interior lines intersecting k and l in an interior singular point. Thus P contains all connecting lines of the $\langle k, l\rangle_{\mathcal{G}}^{g}$. Because every point of $\langle k, l\rangle_{\mathcal{G}}^{g}$ is on a connecting line of $\langle k, l\rangle_{\mathcal{G}}^{g}$ by lemma 3.4.13, we have $\langle k, l\rangle_{\mathcal{G}}^{g} = P$, which shows that $\langle k, l\rangle_{\mathcal{G}}^{g}$ is a plane of $\mathcal{G}$. ∎

Before we determine the isomorphism type of the geometric plane $\langle k, l\rangle_{\mathcal{G}}^{g} \neq \langle k, l\rangle_{\mathcal{G}}$, we recall the definition of a dual affine plane. A subspace P of a geometry is isomorphic to a **dual affine plane** (also called a **symplectic plane**) if

- any two lines of P intersect in a point;
- to any line m of P and any point d of P not incident to m, there exists a unique point p on m not collinear to d;
- the subspace P contains at triangle, i.e. three different lines that do not intersect in a common point.

Thus P is isomorphic to a projective plane from which a point and all lines through that point have been removed.

Lemma 3.4.16 *Let $E_{\mathcal{G}}^{g} = \langle k, l\rangle_{\mathcal{G}}^{g}$ be a geometric plane of $\mathcal{G}$ such that the algebraic plane $\langle k, l\rangle_{\mathcal{G}}$ contains exactly q^4 different interior lines. Then $\langle k, l\rangle_{\mathcal{G}}^{g}$ is isomorphic to a dual affine plane. In particular, any two lines of $\langle k, l\rangle_{\mathcal{G}}^{g}$ generate $\langle k, l\rangle_{\mathcal{G}}^{g}$.*

Proof: The plane $\langle k,l\rangle_{\mathcal{G}}^{g}$ clearly contains a triangle, so that the last condition in the definition of a dual affine plane is satisfied. Let m be an interior line and d be an interior singular point not incident to m of the plane $\langle l,k\rangle_{\mathcal{G}}^{g}$. The hyperbolic lines l and k span the degenerate plane $\langle l,k\rangle$ in U_n, thus $\langle k,l\rangle = \langle m,d\rangle$, where d is singular point different from the radical of $\langle k,l\rangle$. It follows that the totally singular line $d^\pi \cap \langle k,l\rangle$ intersects the hyperbolic line m in the unique singular point $m \cap d^\pi = m_d$. Hence the interior singular point m_d is an element of $d^{\sim}$ and all other interior points incident to m are collinear to d by lemma 3.4.12 and lemma 3.4.13, which proves the second condition for the subspace $\langle k,l\rangle_{\mathcal{G}}^{g}$.

Finally let m and n be two interior lines of $\langle k,l\rangle$. If m and n are two lines through p or if one of them is a line through p and the other a connecting line, then by lemma 3.4.11 we have nothing to prove. So we can assume that m and n are connecting lines of $\langle k,l\rangle$ not intersecting in an interior singular point of k or l, as otherwise there is nothing to prove. We fix the following notation: let $k \cap m = k_m$, $k \cap n = k_n$, $l \cap m = l_m$ and $l \cap n = l_n$.

If the interior singular points k_m and l_n are collinear in $\mathcal{G}$, then let k_m^v be a vector of the singular point k_m and choose $l_n^v \in l_n$ in such a way that $(l_n^v, k_m^v) = 1$. Moreover let $s_{l_n} = (l_n)^\pi \cap k$ and $s_{l_n}^v$ be some vector of the singular point s_{l_n}.

As before we get that the vector $s_{l_n}^v - \mu(s_{l_n}^v, k_m^v)k_m^v = k_n^v$ spans the singular point k_n for some $\mu \in \mathbb{F}^{\sigma,1\times}$ and $s_{l_n}^v - \nu(s_{l_n}^v, k_m^v)k_m^v = p^v$ spans the singular point p for some $\nu \in \mathbb{F}^{\sigma,1\times}$. Furthermore the vector $l_m^v = p^v - \delta(p^v, l_n^v)l_n^v$ generates the singular point l_m for some scalar $\delta \in \mathbb{F}^{\sigma,1\times}$ and every singular point of the hyperbolic line m is spanned by a vector of the form

$$\begin{aligned} & l_m^v - \lambda(l_m^v, k_m^v)k_m^v \\ = \; & s_{l_n}^v - \nu(s_{l_n}^v, k_m^v)k_m^v - \delta(s_{l_n}^v, l_n^v)l_n^v + \delta\nu(s_{l_n}^v, k_m^v)l_n^v - \lambda(s_{l_n}^v, k_m^v)k_m^v + \\ & \delta\lambda(s_{l_n}^v, l_n^v)k_m^v - \lambda\nu\delta(s_{l_n}^v, k_m^v)k_m^v \end{aligned}$$

for some $\lambda \in \mathbb{F}^{\sigma,1}$. Certainly $\delta \cdot \nu \neq -1$. Indeed if $\delta \cdot \nu = -1$ then for any non-zero scalar $\lambda \in \mathbb{F}^{\sigma,1}$ we have that $l_m^v - \lambda(l_m^v, k_m^v)k_m^v = p^v - (s_{l_n}^v, k_m^v)l_n^v$, thus $(l_m^v, k_m^v) = 0$, contradiction. By lemma B.1.13 for the three non-zero scalars μ, ν and δ of $\mathbb{F}^{\sigma,1}$ we can determine three scalars λ, ε and $\alpha \in \mathbb{F}^{\sigma,1\times}$ such that

$$\begin{aligned} & s_{l_n}^v - \nu(s_{l_n}^v, k_m^v)k_m^v + \delta\nu(s_{l_n}^v, k_m^v)l_n^v - \lambda(s_{l_n}^v, k_m^v)k_m^v - \lambda\nu\delta(s_{l_n}^v, k_m^v)k_m^v \\ = \; & s_{l_n}^v - \mu(s_{l_n}^v, k_m^v)k_m^v + \alpha\mu(s_{l_n}^v, k_m^v)l_n^v \\ = \; & k_n^v - \alpha(k_n^v, l_n^v)l_n^v , \end{aligned}$$

which implies that the hyperbolic line n intersects m in a singular point.

On the other hand if $k_m \in l_n^{\sim}$ then we choose an interior singular point w of n collinear to k_m and p. Thus w is on a connecting line z through k_m and on the line t through p in the geometric plane $\langle k,l\rangle_{\mathcal{G}}^{g}$. By lemma 3.4.11 the line t intersects m in an interior singular point d. Replacing in the above calculation l_n by w and l_m by d, we see that the interior lines m and n also intersect in an interior singular point.

The fact that any two lines of $\langle k, l\rangle^g_{\mathcal{G}}$ generate $\langle k, l\rangle^g_{\mathcal{G}}$ follows immediately from the fact that any two lines intersect. Indeed let m and n be two different interior lines of $\langle k, l\rangle^g_{\mathcal{G}}$. Then these two interior lines intersect in an interior singular point by the argumentation above. Let P be the subspace generated by m and n in $\mathcal{G}$. It is sufficient to show that $k, l \in P$ to conclude that P coincides with the geometric plane $\langle k, l\rangle^g_{\mathcal{G}}$. Certainly if m, n are two connecting lines or if either of them is a connecting line and the other is a line through p then by the property that P is a subspace of $\mathcal{G}$ the interior lines k and l are elements of P. If otherwise m and n are two different lines through p then by lemma 3.4.11, the subspace P contains some connecting line of $\langle k, l\rangle^g_{\mathcal{G}}$ and the interior point p, thus $k, l \in P$. ∎

Proposition 3.4.17 *The point-line geometry $\mathcal{G}$ is a connected planar non-linear partially linear space of order q whose planes are linear or symplectic.*

Proof: The statement follows from lemma 3.4.6, lemma 3.4.7 and the sequence of lemmata from 3.4.9 to 3.4.16. ∎

In the next lemmata we check that $\mathcal{G}$ satisfies condition three and condition four of theorem 3.4.1.

Lemma 3.4.18 *Let x and y be distinct points of $\mathcal{G}$. Then $x^\sim \nsubseteq y^\sim$.*

Proof: If $x \perp y$, then we find interior lines $l \in x$ and $m \in y$ such that $l \perp m$ in the graph Γ. Since l and m by definition correspond to orthogonal hyperbolic lines of the unitary space U_n, any point on l is on a singular line going through y in U_n, so for every point $p \in l$ distinct from x we have $p \notin x^\sim$ but $p \in y^\sim$. If $x \not\perp y$, then there exists an interior line l incident to x and y. By definition $x \in x^\sim$ and $x \notin y^\sim$. ∎

Lemma 3.4.19 *Let E be a linear plane and x be an interior singular point of the geometry $\mathcal{G}$. Then the linear plane E intersects the point set $x^\sim$, in symbols $E \cap x^\sim \neq \emptyset$.*

Proof: If x is a point on the plane E, then from $x \in x^\sim$ we obtain $x \in E \cap x^\sim \neq \emptyset$.

If x is not in E, then let k, l be two interior lines of $\mathcal{G}$ such that $E = \langle l, k\rangle^g_{\mathcal{G}}$. We can consider the subspace $\langle k, l\rangle \cap x^\pi$ of the regular plane $\langle k, l\rangle$ orthogonal to x, which is at least of dimension two, so it contains a singular point y, whence $y \in E \cap x^\sim$. ∎

We have now reached our goal.

Notation: We denote with $\mathbb{H}(U_n)$ the geometry of all singular points and all hyperbolic lines of an n-dimensional non-degenerate unitary polar space U_n over the field $\mathbb{F}_{q^2}$.

Proposition 3.4.20 *The point-line geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ is isomorphic to the geometry of singular points and hyperbolic lines of an n-dimensional non-degenerate unitary polar space over $\mathbb{F}_{q^2}$.*

Proof: By proposition 3.4.17 the point-line geometry $\mathcal{G}$ is a connected planar non-linear partially linear space of order q whose planes are finite and linear or symplectic. The geometry $\mathcal{G}$ satisfies hypothesis three of theorem 3.4.1 by lemma 3.4.18 and the validity of hypothesis four of theorem 3.4.1 is proved by lemma 3.4.19. Any geometrical plane of the geometry $\mathcal{G}$ isomorphic to a linear space in fact is isomorphic to the geometry on the singular points and the hyperbolic lines of a classical hermitian unital, so it satisfies hypothesis two, which was observed by O'Nan [72]. Hence we can apply theorem 3.4.1. As the geometry $\mathcal{G}$ contains linear planes, it is isomorphic to $\mathbb{H}(U_m)$. The number of lines of $\mathcal{G}$ equals the number of vertices of Γ which equals the number of hyperbolic lines of U_n. Hence $m = n$. ∎

Definition 3.4.21 Let $n \geq 5$ and let Γ be a graph isomorphic to $\mathbf{G}(U_n)$. Then the point-line geometry $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ is called the **interior space** on Γ.

Corollary 3.4.22 *The automorphism group of $\mathbf{G}(U_n)$ is isomorphic to the automorphism group of the projective unitary space $\mathbb{P}(U_n)$.*

3.5 The graph $\mathrm{G}(U_{n-2})$ inside the graph $\mathrm{G}(U_n)$ for $n \geq 7$

In this section we will study the hyperbolic line graph $\mathbf{G}(U_n)$ for $n \geq 7$ and the interior space $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ on $\mathbf{G}(U_7)$, which is isomorphic to the geometry of singular points and hyperbolic lines of the non-degenerate unitary vector space U_n by proposition 3.4.20. We denote the non-degenerate unitary form $U_n \times U_n \to \mathbb{F}_{q^2}$ by $(\cdot,\cdot)$. Let x be a vertex of the graph $\mathbf{G}(U_n)$, then the induced subgraph $x^\perp$ of $\mathbf{G}(U_n)$ is isomorphic to the hyperbolic line graph $\mathbf{G}(U_{n-2})$ by proposition 3.3.3. Using the results of section 3.4 we can construct the interior space $\mathcal{G}_x = (\mathcal{P}_x, \mathcal{L}_x)$ of the graph $x^\perp \cong \mathbf{G}(U_{n-2})$. Moreover the geometry $\mathcal{G}_x$ is isomorphic to the geometry of singular points and hyperbolic lines of the non-degenerate unitary space U_{n-2} by proposition 3.4.20. The corresponding non-degenerate unitary form $(\cdot,\cdot)_x$ of $\mathcal{G}_x$ can be identified with the restriction $(\cdot,\cdot)_{|x^\pi}$ of $(\cdot,\cdot)$. In this context the elements of the geometry $\mathcal{G}_x$ are called **local**.

Notation: We index every local object of the interior space $\mathcal{G}_x$ with the vertex x. In particular, for vertices k, l, m of $x^\perp$ we use the notations $\{k,l,m\}_x^\perp = \{k,l,m\}^\perp \cap x^\perp$ and $\{k,l,m\}_x^{\perp\perp} = \{\{k,l,m\}_x^\perp\}_x^\perp = \{\{k,l,m\}_x^\perp\}^\perp \cap x^\perp = \{\{k,l,m\}^\perp \cap x^\perp\}^\perp \cap x^\perp$. With $\langle k, l\rangle_x$ we denote the vector subspace of $x^\pi \cong U_{n-2}$ generated by the two interior lines k and l of $\mathcal{G}_x$.

In this section we show that the interior space $\mathcal{G}_x$ is isomorphic to a subspace of codimension two of the interior space $\mathcal{G}$. We obtain the result that each singular interior point $p_x \in \mathcal{G}_x$ is contained in a unique singular interior point of $\mathcal{G}$ and, conversely, that for any singular interior point p of the geometry $\mathcal{G}$ either $p \cap \mathcal{L}_x$ is empty or a singular interior point of $\mathcal{G}_x$.

In this first part we concentrate on the case $n = 7$.

Lemma 3.5.1 *Let p be a singular interior point in $\mathcal{G}$. If $l, m \in p \cap \mathcal{L}_x$ are two distinct elements, then the interior lines l and m intersect in a singular interior point of $\mathcal{G}_{\mathbf{x}}$.*

Proof: We need to establish the defining properties from definition 3.2.15 for l and m. Therefore we have to verify that either $\{l, m\}_x^\perp \neq \varnothing$ or the vertices l and m have distance four in $x^\perp$ with more than q^4 different paths of length four between these two vertices in $x^\perp$. In the graph $\mathbf{G}(U_7)$ both vertices l and m are adjacent to x as $l, m \in \mathcal{L}_x$. Furthermore $\langle l, m \rangle$ is a three-dimensional subspace and contained in x^π. If the plane $\langle l, m \rangle$ is non-degenerate, then $\{l, m\}_x^\perp \neq \varnothing$ by remark 3.2.6. If on the other hand the subspace $\langle l, m \rangle$ is degenerate, then by lemma 3.2.4 the vertices l and m have distance four in the induced subgraph $x^\perp$. By lemma 3.2.12, the graph $x^\perp$ contains more than q^4 different paths of length four between l and m. Hence the interior lines l and m intersect in the interior space $\mathcal{G}_x$. ■

Lemma 3.5.2 *Let p be a singular interior point of $\mathcal{G}$ and k_1, k_2, and k_3 be three pairwise distinct elements of $p \cap \mathcal{L}_x$. Then the interior lines k_1, k_2, and k_3 intersect in one interior singular point of $\mathcal{G}_{\mathbf{x}}$.*

Proof: In order to prove the claim we show that k_1, k_2, k_3 satisfy the properties of definition 3.2.15. By lemma 3.5.1 the interior lines k_1, k_2, k_3 intersect pairwise in a singular interior point of $\mathcal{G}_x$. Furthermore the vector subspace of U_7 spanned by the hyperbolic lines k_1, k_2 and k_3 is a subspace of x^π and since the vertices k_1, k_2, k_3 are elements of p, the one-dimensional subspace $d = k_1 \cap k_2 \cap k_3$ is contained in x^π as well. This setup satisfies the hypothesis of lemma 3.2.14 implying that the subspace x^π contains a hyperbolic line s such that

- $\{s, k_i\}_x^\perp \neq \varnothing$, if $s \neq k_i$, for $i = 1, 2, 3$,
- $\{s, k_1, k_2\}_x^\perp = \varnothing$.

Hence by definition 3.2.15 the three vertices k_1, k_2 and k_3 of $p \cap \mathcal{L}_x$ intersect in one interior point of $\mathcal{G}_x$, which is singular, cf. definition 3.4.4. ■

Proposition 3.5.3 *Let p be an interior singular point in $\mathcal{G}$. The interior line set $p \cap \mathcal{L}_x$ is either an interior singular point p_x in $\mathcal{G}_x$ or the empty set.*

Proof: Suppose $p \cap \mathcal{L}_x \neq \emptyset$, then let l be some element of $p \cap \mathcal{L}_x$ and m be an interior line of the point p different from l. Since $l \perp x$ in $\mathbf{G}(U_7)$ it follows that the hyperbolic line l is a subspace of x^π in $\mathcal{G}$ which intersects the hyperbolic line m in a one-dimensional singular subspace d. Hence the singular point d is also a subspace of x^π. Let p_x be the interior point of $\mathcal{G}_x$ containing all hyperbolic lines of x^π incident to the point d.

Let k be an arbitrary hyperbolic line of the interior point p_x. The proposition is proved, if the vertex k is an element of $p \cap \mathcal{L}_x$. Since $k \subseteq x^\pi$ it suffices to prove $k \in p$.

Any element n of the interior point p is a hyperbolic line of U_7 incident to the point d. Thus we choose a vertex $n \in p$ distinct from k and intend to prove that $\{k, n\}^\perp \neq \emptyset$ and that $\{k, n\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_7)$ with respect to inclusion, cf. definition 3.3.8. Since both hyperbolic lines k and n contain the point d in U_7, the vector space spanned by both is a plane of rank at least two. Hence $\langle k, n\rangle^\pi$ is four-dimensional subspace of rank at least three, thus $\langle k, n\rangle^\pi$ contains a hyperbolic line. In particular, $\{k, n\}^\perp \neq \emptyset$ and due to lemma 3.3.4 we have $\{k, n\}^{\perp\perp} = \mathbf{L}(\langle k, n\rangle)$. By lemma 3.3.4 again and the fact that the span of two different hyperbolic lines s_1, s_2 of the three-dimensional subspace $\langle k, n\rangle$ equals this plane, we obtain the equality $\{k, n\}^{\perp\perp} = \mathbf{L}(\langle k, n\rangle) = \mathbf{L}(\langle s_1, s_2\rangle) = \{s_1, s_2\}^{\perp\perp}$. Therefore $\{k, n\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_7)$ with respect to inclusion.

Next, we choose two different elements n and m of p. By the argumentation above n, m and k are three mutually intersecting interior lines in the unitary space $\mathcal{G}$ and the subspace $\langle n, m, k\rangle$ of U_7 is of dimension three or four. If $\langle n, m, k\rangle$ is a non-degenerate four-dimensional subspace, then $\langle n, m, k\rangle^\pi$ is a regular plane in $\mathcal{G}$ containing some hyperbolic line. Hence the subgraph $\{n, m, k\}^\perp$ is not empty and by lemma 3.3.4 and lemma 3.3.5 it follows directly that $\{n, k\}^{\perp\perp} = \mathbf{L}(\langle n, k\rangle) \subsetneq \mathbf{L}(\langle n, m, k\rangle) = \{m, n, k\}^{\perp\perp}$. If otherwise the subspace $\langle n, m, k\rangle$ is of dimension three or degenerate and of dimension four, then by lemma 3.3.7 applied to the plane $\langle n, m\rangle$ spanned by n and m, there exists a hyperbolic line s in the unitary vector space $\mathcal{G}$ intersecting the lines n and m (and consequently k) in d such that $\langle s, n, m\rangle$ is four-dimensional non-degenerate subspace. This implies that $\{s, n, m\}^\perp \neq \emptyset$ and again we get the inequality $\{n, m\}^{\perp\perp} = \mathbf{L}(\langle n, m\rangle) \subsetneq \mathbf{L}(\langle s, n, m\rangle) = \{s, n, m\}^{\perp\perp}$, thus $k \in p$ by definition 3.3.8. ∎

Lemma 3.5.4 *Let p_x be a singular interior point of $\mathcal{G}_x$ for some vertex x in $\mathbf{G}(U_7)$ and let l and m be two distinct elements of p_x. Then l and m intersect in a singular interior point of $\mathcal{G}$.*

Proof: By definition 3.3.8 the vertices $l, m \in p_x$ intersect in $\mathbf{G}(U_7)$, if $\{l, m\}^\perp \neq \emptyset$ and $\{l, m\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_7)$ with respect to inclusion. Since l and m are vertices of the induced subgraph $x^\perp$ of $\mathbf{G}(U_7)$, we conclude that $x \in \{l, m\}^\perp$. By lemma 3.3.4 we have $\{l, m\}^{\perp\perp} = \mathbf{L}(\langle l, m\rangle)$. The plane $\langle l, m\rangle$ is a subspace of x^π, since l and m are incident to x^π. This implies $\langle l, m\rangle = \langle l, m\rangle_x$ and $\mathbf{L}(\langle l, m\rangle) = \mathbf{L}(\langle l, m\rangle_x)$.

Next, let s and t be two different vertices of $\{l, m\}^{\perp\perp}$. By the identities above $s, t \in \{l, m\}^{\perp\perp} = \mathbf{L}(\langle l, m \rangle) = \mathbf{L}(\langle l, m \rangle_x) = \langle l, m \rangle_x$. In fact the interior lines s and t span the plane $\langle l, m \rangle$ in $\mathcal{G}$. Moreover $\{l, m\}^{\perp\perp} = \bigcap_{z \in \{k,l\}^\perp} z^\perp$, so $\{l, m\}^{\perp\perp} \subseteq x^\perp$, which implies that s and t are vertices of the subgraph $x^\perp$. Again, $\langle s, t \rangle = \langle s, t \rangle_x$ and $\{s, t\}^{\perp\perp} = \mathbf{L}(\langle s, t \rangle) = \mathbf{L}(\langle s, t \rangle_x)$. Therefore $\{s, t\}^{\perp\perp} = \mathbf{L}(\langle s, t \rangle) = \mathbf{L}(\langle l, m \rangle) = \{l, m\}^{\perp\perp}$, which shows that the double perp $\{l, m\}^{\perp\perp}$ is minimal in the graph $\mathbf{G}(U_7)$ with respect to inclusion. ∎

Lemma 3.5.5 *Let p_x be an interior point of $\mathcal{G}_x$. Any three distinct vertices k_1, k_2 and k_3 of p_x intersect in one point in $\mathcal{G}$.*

Proof: By the previous lemma 3.5.4 any three distinct lines k_1, k_2 and k_3 of an interior point $p_x \in \mathcal{P}_x$ are mutually intersecting interior lines in the interior space $\mathcal{G}$. Moreover the induced subgraph $x^\perp$ contains a vertex s with the properties that $\{s, k_i\}_x^\perp \neq \varnothing$ in $x^\perp$ if $k_i \neq s$ for $i \in \{1, 2, 3\}$ and $\{s, k_1, k_2\}_x^\perp = \varnothing$. Thus the plane $\langle k_1, k_2 \rangle_x$ is properly contained in the four-dimensional subspace $\langle k_1, k_2, s \rangle_x$ of x^π in $\mathcal{G}$, so $\langle k_1, k_2 \rangle = \langle k_1, k_2 \rangle_x \subsetneq \langle k_1, k_2, s \rangle_x = \langle k_1, k_2, s \rangle$.

Furthermore the vertex s is also an interior line of the space $\mathcal{G}$ and by lemma 3.5.4 the interior line s intersects each interior line k_i different from s in $\mathcal{G}$ for $i \in \{1, 2, 3\}$. The proof of the statement is finished if we can show that $\{k_1, k_2\}^{\perp\perp} \subsetneq \{k_1, k_2, s\}^{\perp\perp}$ in $\mathbf{G}(U_7)$. The interior lines k_1, k_2 and s are vertices of $x^\perp$ thus $\{k_1, k_2, s\}^{\perp\perp} = \mathbf{L}(\langle k_1, k_2, s \rangle) = \mathbf{L}(\langle k_1, k_2, s \rangle_x)$ by lemma 3.3.5 and the fact that $x \in \{k_1, k_2, s\}^\perp$. Using lemma 3.3.4 we get equality between the vertex set of the induced subgraph $\{k_1, k_2\}^{\perp\perp}$ and the hyperbolic lines set $\mathbf{L}(\langle k_1, k_2 \rangle) = \mathbf{L}(\langle k_1, k_2 \rangle_x)$. Finally we obtain the equalities $\{k_1, k_2\}^{\perp\perp} = \mathbf{L}(\langle k_1, k_2, \rangle_x) \subsetneq \mathbf{L}(\langle k_1, k_2, s \rangle_x) = \{k_1, k_2, s\}^{\perp\perp}$, and we are done. ∎

Proposition 3.5.6 *Let p_x be an interior point of $\mathcal{G}_x$. There is a unique interior point p in the interior space of $\mathbf{G}(U_7)$ such that $p_x \subseteq p$.*

Proof: The uniqueness of the interior point p follows directly from the fact that the interior space of $\mathbf{G}(U_7)$ is isomorphic to U_7. For, suppose the interior space $\mathcal{G}$ contains two different interior points p and g such that $p_x \subseteq p$ and $p_x \subseteq g$. Then let k be an interior line of p which is not incident to g and let l_1 and l_2 be two different interior lines of p_x. In the unitary polar space $\mathcal{G}$ the two different hyperbolic lines l_1 and l_2 intersect in the point p, but on the other hand $p = k \cap l_1 = l_2 \cap k = l_1 \cap l_2 = g$, contradiction. ∎

The line set $\mathcal{L}_x$ of the interior space $\mathcal{G}$ is a subset of the interior line set $\mathcal{L}$. Also every interior point p_x of $\mathcal{P}_x$ is contained in a unique point p of the interior space $\mathcal{G}$, thus the interior space $\mathcal{G}_x$ is a subspace of the interior space $\mathcal{G}$. In the next proposition we also determine the dimension of the subspace $\mathcal{G}_x$ in the interior space $\mathcal{G}$.

Proposition 3.5.7 *Let x be a vertex of the graph $\mathbf{G}(U_7)$. The interior space $\mathcal{G}_x$ on $x^\perp$ is isomorphic to a codimension two subspace of the interior space $\mathcal{G}$ on $\mathbf{G}(U_7)$.*

Proof: Since $\mathcal{G}_x \cong \mathbb{H}(U_5)$ and $\mathcal{G} \cong \mathbb{H}(U_7)$ the claim follows. ■

The same result holds for $n \geq 8$.

Proposition 3.5.8 *Let $n \geq 8$ and let x be a vertex of the graph $\mathbf{G}(U_n)$. The interior space $\mathcal{G}_x$ on $x^\perp$ is isomorphic to a codimension two subspace of the interior space $\mathcal{G}$ on $\mathbf{G}(U_n)$.*

The proof of this statement is similar to the proof of proposition 3.5.7. The remainder of this section is devoted to this proof.

Lemma 3.5.9 *Let l and m be two intersecting interior lines of $\mathcal{G}_x$. Then the induced subgraph $\{l,m\}^{\perp\perp}$ of $\mathbf{G}(U_n)$ is equal to $\{l,m\}_x^{\perp\perp}$.*

Proof: Let l and m be two intersecting lines in $\mathcal{G}_x$. Then by lemma 3.3.6 we have $\{l,m\}_x^\perp \neq \varnothing$ and $\{l,m\}_x^{\perp\perp} = \mathbf{L}(\langle l,m\rangle_x)$ in $\mathcal{G}_x$. Moreover $l \perp x \perp m$ in $\mathbf{G}(U_n)$, so $x \in \{l,m\}^\perp$ and $\{l,m\}^{\perp\perp} \subseteq x^\perp$ implying the identity $\{l,m\}^{\perp\perp} \cap x^\perp = \{l,m\}^{\perp\perp}$. Since $\{l,m\}_x^\perp = \{l,m\}^\perp \cap x^\perp \subseteq \{l,m\}^\perp$ we obtain that $(\{l,m\}^\perp \cap x^\perp)^\perp \supseteq \{l,m\}^{\perp\perp}$ and conclude that $(\{l,m\}^\perp \cap x^\perp)^\perp \cap x^\perp \supseteq \{l,m\}^{\perp\perp} \cap x^\perp$, hence $\{l,m\}^{\perp\perp} \subseteq \{l,m\}_x^{\perp\perp}$.

Suppose $\{l,m\}_x^{\perp\perp} \nsubseteq \{l,m\}^{\perp\perp}$. Let k be a vertex in $\{l,m\}_x^{\perp\perp} \setminus \{l,m\}^{\perp\perp}$ and ${}^x\langle l,m\rangle := \langle l,m\rangle^\pi \cap x^\pi$ be the orthogonal space of $\langle l,m\rangle$ inside x^π in $\mathcal{G}$. Since $\dim(x^\pi) = n-2$ and $\langle l,m\rangle$ is a plane of rank at least two, the subspace ${}^x\langle l,m\rangle$ contains at most a one-dimensional radical and is of dimension $n-5$. Thus we fix a regular point y in ${}^x\langle l,m\rangle$ and consider the $(n-6)$-dimensional subspace $y^\pi \cap {}^x\langle l,m\rangle$, which is again either non-degenerate or contains the one-dimensional radical $\mathrm{rad}(y^\pi \cap {}^x\langle l,m\rangle) = \mathrm{rad}({}^x\langle l,m\rangle) = r$.

If $y^\pi \cap {}^x\langle l,m\rangle$ is regular then by [20, theorem 6.3.1] or [86, chapter 10] the subspace $y^\pi \cap {}^x\langle l,m\rangle$ is either the direct sum of $\frac{n-6}{2}$ hyperbolic lines h_i, $y^\pi \cap {}^x\langle l,m\rangle = \bigoplus_{i=1}^{\frac{n-6}{2}} h_i$ (in case $n-6$ is even) or the direct sum of $\lfloor \frac{n-6}{2} \rfloor$ hyperbolic lines h_i and a regular point c, $y^\pi \cap {}^x\langle l,m\rangle = \bigoplus_{i=1}^{\lfloor \frac{n-6}{2} \rfloor} h_i \oplus c$ (in case $n-6$ odd).

On the other hand if the subspace $y^\pi \cap {}^x\langle l,m\rangle$ is degenerate then we choose a rank one line s through the point r in $y^\pi \cap {}^x\langle l,m\rangle$. The orthogonal space $s^\pi \cap (y^\pi \cap {}^x\langle l,m\rangle)$ is an $(n-7)$-dimensional degenerate subspace with radical r, so $s^\pi \cap (y^\pi \cap {}^x\langle l,m\rangle) = \langle s_1,\dots,s_{n-8},r\rangle$ for some points s_i. Hence $S = \langle s_1,\dots,s_{n-8}\rangle$ is an $(n-8)$-dimensional non-degenerate subspace implying together with [20, theorem 6.3.1] or [86, chapter 10] that S is either the direct sum of $\frac{n-8}{2}$ hyperbolic lines h_i, $S = \bigoplus_{i=1}^{\frac{n-8}{2}} h_i$ if $n-8$ is even or otherwise the direct sum of $\lfloor \frac{n-8}{2} \rfloor$ hyperbolic lines h_i and a regular point c, $S = \bigoplus_{i=1}^{\lfloor \frac{n-8}{2} \rfloor} h_i \oplus c$, if $n-8$ is odd. We also denote the line s by $h_{\lfloor \frac{n-8}{2} \rfloor +1} = h_{\lfloor \frac{n-6}{2} \rfloor}$. Then either $y^\pi \cap {}^x\langle l, m\rangle = \bigoplus_{i=1}^{\frac{n-6}{2}} h_i$ if $n-6$ is even or $y^\pi \cap {}^x\langle l, m\rangle = \bigoplus_{i=1}^{\lfloor \frac{n-6}{2} \rfloor} h_i \oplus c$ if $n-6$ is odd.

Next, in both cases, we fix on each line h_i for $1 \leq i \leq \lfloor \frac{n-6}{2} \rfloor$ two different regular points h_{1i} and h_{2i}. Thus the two-dimensional subspaces $h_{ij,y} = \langle y, h_{ji}\rangle$ are hyperbolic lines for $1 \leq i \leq \lfloor \frac{n-6}{2} \rfloor$,$1 \leq j \leq 2$ and ${}^x\langle l, m\rangle = \langle y, y^\pi \cap {}^x\langle l, m\rangle\rangle$ is either equal to the subspace

$$\langle h_{ji,y} \mid \text{ for } j = 1, 2; i = 1, \ldots, \frac{n-6}{2}\rangle$$

if $\frac{n-6}{2}$ is even or to the subspace

$$\langle h_{ji,y}, \langle y, c\rangle \mid \text{ for } j = 1, 2; i = 1, \ldots, \lfloor \frac{n-6}{2} \rfloor\rangle$$

if $\frac{n-6}{2}$ is odd. Every hyperbolic line $h_{ji,y}$, $1 \leq i \leq \frac{n-6}{2}$, $j = 1, 2$ as well as the hyperbolic line $c_y = \langle y, c\rangle$ belongs to a vertex in the graph $\mathbf{G}(U_n)$ adjacent to the vertices x, l and m.

Since k is an element of $\{l, m\}_x$, the vertex k is contained in each induced subgraph $z_x^\perp$ for every $z \in \{l, m\}_x^\perp$. Therefore k is adjacent to each vertex $h_{ji,y}$, $i = 1, \ldots, \lfloor \frac{n-6}{2} \rfloor$, $j = 1, 2$, and to the vertex c_y in the graph $\mathbf{G}(U_n)$. Hence the vertex k is either a hyperbolic line of the subspace $x^\pi \cap \langle h_{ji,y} \mid \text{ for } j = 1, 2; i = 1, \ldots, \frac{n-6}{2}\rangle^\pi = x^\pi \cap ({}^x\langle l, m\rangle)^\pi = \langle l, m\rangle$ (in case that $n-6$ is even) or a hyperbolic line of the subspace $x^\pi \cap \langle h_{ji,y}, c_y \mid \text{ for } j = 1, 2; i = 1, \ldots, \lfloor \frac{n-6}{2} \rfloor\rangle^\pi = x^\pi \cap ({}^x\langle l, m\rangle)^\pi = \langle l, m\rangle$ (in case that $n-6$ is odd), contradiction. ■

On occasion we will use this result in the following form.

Lemma 3.5.10 *Let k, l and m be three different mutually intersecting interior lines of the space $\mathcal{G}_x$. Then $\{k, l, m\}^{\perp\perp} = \{k, l, m\}_x^{\perp\perp}$.*

Proof: By lemma 3.5.9 from above, we know $\{k, l\}^{\perp\perp} = \{k, l\}_x^{\perp\perp}$ and $\{l, m\}^{\perp\perp} = \{l, m\}_x^{\perp\perp}$. Since $\{k, l, m\}^{\perp\perp} = \{k, l\}^{\perp\perp} \cap \{l, m\}^{\perp\perp}$ we regard the following identities $\{k, l, m\}^{\perp\perp} = \{k, l\}^{\perp\perp} \cap \{l, m\}^{\perp\perp} = \{k, l\}_x^{\perp\perp} \cap \{l, m\}_x^{\perp\perp} = (\{k, l\}^\perp \cap x^\perp)^\perp \cap x^\perp \cap (\{l, m\}^\perp \cap x^\perp)^\perp \cap x^\perp = (\{k, l\}^\perp \cap x^\perp)^\perp \cap (\{l, m\}^\perp \cap x^\perp)^\perp \cap x^\perp = (\{k, l\}^\perp \cap \{l, m\}^\perp \cap x^\perp)^\perp \cap x^\perp = (\{k, l, m\}^\perp \cap x^\perp)^\perp \cap x^\perp = \{k, l, m\}_x^{\perp\perp}$. ■

We now start to prove the claim that for each interior singular point p of $\mathcal{G}$ and each vertex x of $\mathbf{G}(U_n)$ the line set $p \cap \mathcal{L}_x$ is either an interior singular point of $\mathcal{G}_x$ or empty.

Lemma 3.5.11 *Let $n \geq 8$ and p be an interior singular point of $\mathcal{G}$. Any two distinct elements l and m of $p \cap \mathcal{L}_x$ intersect in the space $\mathcal{G}_x$.*

Proof: For two distinct elements l and m of the set $p \cap \mathcal{L}_x$ we need to verify by definition 3.3.8 the two conditions that $\{l, m\}_x^\perp \neq \varnothing$ and that the induced subgraph $\{l, m\}_x^{\perp\perp}$ is minimal in $x^\perp$ with respect to inclusion.

Since the vertices l and m are adjacent to x in $\mathbf{G}(U_n)$, the plane $\langle l, m\rangle$, which is of rank at least two, is contained in the $(n-2)$-dimensional non-degenerate subspace x^π of U_n. As $n \geq 8$ the $(n-5)$-dimensional subspace $\langle l, m\rangle^\pi \cap x^\pi$ has at least two. So $\langle l, m\rangle^\pi \cap x^\pi$ contains some hyperbolic line h implying that $h \in \{l, m\}_x^\perp \neq \emptyset$.

The induced subgraph $\{l, m\}^{\perp\perp}$ is minimal in $\mathbf{G}(U_n)$ with respect to inclusion by definition 3.3.8. By lemma 3.5.9, $\{l, m\}_x^{\perp\perp} = \{l, m\}^{\perp\perp}$, and for two distinct vertices s and t of $\{l, m\}_x^{\perp\perp}$ we obtain the identities $\{s, t\}^{\perp\perp} = \{l, m\}^{\perp\perp} = \{l, m\}_x^{\perp\perp}$. Since s and t are two vertices of $\{l, m\}^{\perp\perp}$ they are two different intersecting hyperbolic lines of the plane $\langle k, l\rangle \subseteq x^\pi$. By lemma 3.3.6 and definition 3.3.8 the interior lines s and t intersect in the space $\mathcal{G}$, which implies that $\{s, t\}^{\perp\perp} = \{s, t\}_x^{\perp\perp}$ by lemma 3.5.9. In fact $\{s, t\}_x^{\perp\perp} = \{s, t\}^{\perp\perp} = \{l, m\}^{\perp\perp} = \{l, m\}_x^{\perp\perp}$ proving the minimality with respect to inclusion in $x^\perp$ of $\{l, m\}_x^{\perp\perp}$. ∎

Lemma 3.5.12 *Let $n \geq 8$ and p be an interior singular point in $\mathcal{G}$. Any three different elements k_1, k_2 and k_3 of $p \cap \mathcal{L}_x$ intersect in one point in $\mathcal{G}_x$.*

Proof: Let k_1, k_2 and k_3 be three different vertices of $p \cap \mathcal{L}_x$. By lemma 3.5.11 we get that k_1, k_2 and k_3 are pairwise intersecting lines in $\mathcal{G}_x$. By definition 3.3.8 the interior lines k_1, k_2 and k_3 intersect in one point in $\mathcal{G}_x$ if we find an interior line $s \in \mathcal{L}_x$ such that $\{s, k_i\}_x^\perp$ is not empty, $\{s, k_i\}_x^{\perp\perp}$ is minimal in $x^\perp$ with respect to inclusion if $s \neq k_i$ for $i = 1, 2, 3$ and that $\{k_1, k_2\}_x^{\perp\perp} \subsetneq \{s, k_1, k_2\}_x^{\perp\perp}$ in $x^\perp$.

The span of the hyperbolic lines k_1, k_2 and k_3 in the unitary space $\mathcal{G} \cong U_n$ is a subspace either of dimension three or four of the subspace x^π since k_1, k_2 and k_3 are vertices of $x^\perp$. Moreover as k_1, k_2 and k_3 are interior lines of the point $p \in \mathcal{P}$ the points $k_1 \cap k_2, k_2 \cap k_3$ and $k_1 \cap k_3$ coincide with a unique singular point d in $\mathcal{G}$. Certainly this point is incident to x^π.

Suppose $\langle k_1, k_2, k_3\rangle$ is a regular four-dimensional subspace in $\mathcal{G}$ then let $s = k_3$ and we are done by the fact that $\{k_1, k_2\}_x^{\perp\perp} = \{k_1, k_2\}^{\perp\perp} \subsetneq \{k_3, k_1, k_2\}^{\perp\perp} = \{k_3, k_1, k_2\}_x^{\perp\perp}$, by lemma 3.5.9 and lemma 3.5.10.

If $\langle k_1, k_2, k_3\rangle$ is either a plane or a singular four-dimensional subspace in U_n then we consider the plane $P = \langle k_1, k_2\rangle \subseteq \langle k_1, k_2, k_3\rangle \subseteq x^\pi$, which is of rank at least two and contains the intersecting point d. Since $\dim(x^\pi) = n-2 \geq 6$ we can fix a hyperbolic line s in the subspace x^π intersecting the regular two-dimensional subspaces k_1 and k_2 such that $\langle k_1, k_2, s\rangle$ is a rank four four-dimensional space in x^π by lemma 3.3.7. We can also conclude that $s \cap k_1 = k_1 \cap k_2 = k_2 \cap s = d$ implying that the $\{s, k_i\}_x^\perp$ are not empty, the induced subgraphs $\{s, k_i\}_x^{\perp\perp}$ are minimal with respect to inclusion in $x^\perp$ if $s \neq k_i$ and $i = 1, 2, 3$ and also that $\{k_1, k_2\}_x^{\perp\perp} \subsetneq \{s, k_1, k_2\}_x^{\perp\perp}$ in $x^\perp$ due to lemma 3.5.9, lemma 3.5.10 and lemma 3.3.6. ∎

Proposition 3.5.13 *Let $n \geq 8$ and let p be an interior singular point in $\mathcal{G}$. The line set $p \cap \mathcal{L}_x$ is either an interior singular point p_x in $\mathcal{G}_x$ or the empty set.*

Proof: This statement follows under the use of lemma 3.5.11 and lemma 3.5.12 by an argumentation similar to the one used in the proof of proposition 3.5.3. ■

Finally we show the converse direction, so we verify that each interior point p_x of $\mathcal{G}_x$ is contained in a unique point p of $\mathcal{G}$.

Lemma 3.5.14 *Let $n \geq 8$ and let p_x be an interior singular point of $\mathcal{G}_x$. Any two distinct lines k and l of p_x intersect in a common point in $\mathcal{G}$.*

Proof: By definition 3.3.8 the vertices k and l of the interior point p_x intersect in $\mathcal{G}$ if $\{k,l\}^\perp$ is not empty and $\{k,l\}^{\perp\perp}$ is minimal in the graph $\mathbf{G}(U_n)$ with respect to inclusion in the graph $\mathbf{G}(U_n)$.

Note that k and l are vertices of the graph $x^\perp$, so $k \perp x \perp l$ in $\mathbf{G}(U_n)$ and we conclude that $x \in \{l,m\}^\perp \neq \emptyset$. By lemma 3.5.9 we have equality between the graphs $\{k,l\}^{\perp\perp}$ and $\{k,l\}_x^{\perp\perp}$. Moreover, $\{k,l\}_x^{\perp\perp}$ is minimal with respect to inclusion in $x^\perp$. Thus, for any two different vertices s_1, s_2 of $\{k,l\}_x^{\perp\perp} = \{k,l\}^{\perp\perp}$ we obtain with lemma 3.5.9 the identities $\{k,l\}^{\perp\perp} = \{k,l\}_x^{\perp\perp} = \{s_1,s_2\}_x^{\perp\perp} = \{s_1,s_2\}^{\perp\perp}$, which shows that $\{k,l\}^{\perp\perp}$ is minimal with respect to inclusion in $\mathbf{G}(U_n)$. ■

Lemma 3.5.15 *Let $n \geq 8$ and p_x be an interior singular point of $\mathcal{G}_x$. Then any three distinct lines k_1, k_2 and k_3 of p_x intersect in one interior point in $\mathcal{G}$.*

Proof: By lemma 3.5.14 any three distinct elements k_1, k_2 and k_3 of $p_x \in \mathcal{P}_x$ are pairwise intersecting interior lines of $\mathcal{G}$. Since $n \geq 8$ we can fix a vertex s in $x^\perp$ with the properties that s intersects each line k_i in $\mathcal{G}_x$ if $s \neq k_i$ for $i \in \{1,2,3\}$ and $\{k_1,k_2\}_x^{\perp\perp} \subsetneq \{k_1,k_2,s\}_x^{\perp\perp}$, in particular s is also an element of the interior point p_x. The subspace s is also an interior line in $\mathcal{G}$ and this regular line intersects each line k_i in $\mathcal{G}$ if $s \neq k_i$ for $i \in \{1,2,3\}$ by lemma 3.5.14. The proof of the statement is complete since $\{k_1,k_2,s\}^{\perp\perp} = \{k_1,k_2,s\}_x^{\perp\perp}$ by lemma 3.5.10. Indeed the hyperbolic lines k_1 and k_2 are vertices of the subgraph $x^\perp$. By lemma 3.5.9 we obtain that $\{k_1,k_2\}_x^{\perp\perp} = \{k_1,k_2\}^{\perp\perp}$, which leads to the fact that $\{k_1,k_2\}^{\perp\perp} = \{k_1,k_2\}_x^{\perp\perp} \subsetneq \{k_1,k_2,s\}_x^{\perp\perp} = \{k_1,k_2,s\}_x^{\perp\perp}$. ■

Proposition 3.5.16 *Let $n \geq 8$ and let p_x be an interior point of G_x. There is a unique point p in the interior space of $\mathbf{G}(U_n)$ such that $p_x \subseteq p$.*

Proof: A similar argument as used in proposition 3.5.6 implies the statement. ■

Proof of proposition 3.5.8: Recall that every interior line of $\mathcal{G}_x$ is an interior line of $\mathcal{G}$ and any interior point p_x of $\mathcal{G}_x$ is contained in a unique interior point p of $\mathcal{G}$, thus $\mathcal{G}_x$ is isomorphic to a subspace of the interior space on $\mathbf{G}(U_n)$. As $\mathcal{G}_x \cong \mathbb{H}(U_{n-2})$ and $\mathcal{G}$ is isomorphic to the geometry $\mathbb{H}(U_n)$ the codimension of the isomorphic image of $\mathcal{G}_x$ inside $\mathcal{G}$ is two, proving the claim. ■

3.6 The global space

In this section we analyse the following situation. Let $n \geq 7$ and let Γ be a connected graph which is locally isomorphic to the hyperbolic line graph $\mathbf{G}(U_n)$. At the end of this section we prove theorem 3.1.2, i.e., that Γ is isomorphic to the hyperbolic line graph $\mathbf{G}(U_{n+2})$.

Due to the property that for every vertex $\mathbf{x}$ of Γ the induced subgraph $\mathbf{x}^\perp$ is isomorphic to $\mathbf{G}(U_n)$, we can construct the interior spaces $\mathcal{G}_\mathbf{x}$ on $\mathbf{x}^\perp$, see proposition 3.3.9 and proposition 3.4.20. We use this family $(\mathcal{G}_\mathbf{x})_{\mathbf{x}\in\Gamma}$ of local interior spaces to construct a global geometry $\mathcal{G}_\Gamma$ on Γ, which via theorem 3.4.1 will turn out to be isomorphic to the geometry on the singular points and the hyperbolic lines of some unitary polar space.

Interior objects are a priori only defined in some interior space $\mathcal{G}_x$, $x \in \Gamma$. They are called **local objects**. Therefore one problem we have to tackle in this section is to introduce well-defined global points and lines for our point-line geometry $\mathcal{G}_\Gamma$. After that we will establish the validity of the hypothesis of theorem 3.4.1 for $\mathcal{G}_\Gamma$.

Notation: To avoid confusion, we will index every local object by the vertex $\mathbf{x}$ whose interior space it belongs to. For example, if $\mathbf{x} \perp \mathbf{y}$ in the graph Γ, then $\mathbf{y}$ is a vertex of the subgraph $\mathbf{x}^\perp$ corresponding to the local object $y_\mathbf{x}$, an interior line, in the space $\mathcal{G}_\mathbf{x}$. By $\mathbf{y}_\mathbf{x}$ we denote the vertex $\mathbf{y}$ considered as a vertex of the subgraph $\mathbf{x}^\perp$. With the symbol $\mathbf{y}_\mathbf{x}^\perp$ we denote the subgraph $\{\mathbf{x},\mathbf{y}\}^\perp$ which is of course an induced subgraph of $\mathbf{x}^\perp$. The interior space obtained from the graph $\mathbf{y}_\mathbf{x}^\perp$ will be denoted with $\mathcal{G}_{\mathbf{y}_\mathbf{x}}$. Furthermore by $\langle y_\mathbf{x}, z_\mathbf{x}\rangle$ we denote the subspace of $\mathcal{G}_\mathbf{x}$ spanned by the two interior lines $y_\mathbf{x}$ and $z_\mathbf{x}$ as a subspace of the underlying projective space.

Definition 3.6.1 A **global line** of Γ is a vertex of the graph Γ. The set of all global lines of Γ is denoted by $\mathcal{L}_\Gamma$.

Lemma 3.6.2 *Let $n \geq 7$ and let* $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ *be vertices of* Γ *with the property that* $\mathbf{z}\perp\mathbf{x}\perp\mathbf{w}\perp\mathbf{y}\perp\mathbf{z}$. *Assume that the vertices* $\mathbf{w}$ *and* $\mathbf{z}$ *are connected by a path in the induced subgraph* $\{\mathbf{x},\mathbf{y}\}^\perp$ *of* Γ. *Then* $\{\mathbf{x}_\mathbf{w},\mathbf{y}_\mathbf{w}\}_\mathbf{w}^{\perp\perp} = \{\mathbf{x}_\mathbf{z},\mathbf{y}_\mathbf{z}\}_\mathbf{z}^{\perp\perp}$. *In particular, the spaces* $\langle x_\mathbf{w}, y_\mathbf{w}\rangle$ *and* $\langle x_\mathbf{z}, y_\mathbf{z}\rangle$ *have equal line sets and can be identified.*

Proof: By assumption there exist vertices $\mathbf{c}^1,\dots,\mathbf{c}^n$ of the graph Γ such that $\mathbf{z} \perp \mathbf{c}^1 \perp \mathbf{c}^2 \perp \dots \perp \mathbf{c}^n \perp \mathbf{w}$ is a path from $\mathbf{z}$ to $\mathbf{w}$ in $\{\mathbf{x},\mathbf{y}\}^\perp$. Since $\mathbf{c}^1 \in \{\mathbf{x},\mathbf{y},\mathbf{z}\}^\perp$ the hyperbolic lines $x_\mathbf{z}$ and $y_\mathbf{z}$ are perpendicular to the line $c_\mathbf{z}^1$ in the interior space $\mathcal{G}_\mathbf{z}$. Hence the space spanned by $x_\mathbf{z}$ and $y_\mathbf{z}$ is orthogonal to the hyperbolic line $c_\mathbf{z}^1$ in $\mathcal{G}_\mathbf{z}$. In particular, every vertex $\mathbf{u}_{bz}$, which belongs to a hyperbolic line $u_\mathbf{z}$ contained in $\langle x_\mathbf{z}, y_\mathbf{z}\rangle$, is adjacent to $\mathbf{c}_\mathbf{z}^1$. Therefore the space $\langle x_\mathbf{z}, y_\mathbf{z}\rangle$ can be identified with a subspace of $\mathcal{G}_{\mathbf{c}_\mathbf{z}^1}$, whence with a subspace of $\mathcal{G}_{\mathbf{c}^1}$, cf. Propositions 3.5.7 and 3.5.8. Hence, by lemma 3.3.4, we have $\{\mathbf{x_z}, \mathbf{y_z}\}_{\mathbf{z}}^{\perp\perp} = \{\mathbf{x_{c^1}}, \mathbf{y_{c^1}}\}_{\mathbf{c}^1}^{\perp\perp}$. Repeating the above argument along the path $\mathbf{z} \perp \mathbf{c}^1 \perp \ldots \perp \mathbf{c}^n \perp \mathbf{w}$, we obtain $\{\mathbf{x_w}, \mathbf{y_w}\}_{\mathbf{w}}^{\perp\perp} = \{\mathbf{x_z}, \mathbf{y_z}\}_{\mathbf{z}}^{\perp\perp}$. ■

Lemma 3.6.3 *Let $n \geq 7$ and $\mathbf{z} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$ be a chain in Γ such that the rank of the subspace $\langle z_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi}, x_{\mathbf{y}} \rangle$ is at least $\max\{n-4, 6\}$. Then there is a vertex $\mathbf{h} \in \{\mathbf{z}, \mathbf{y}, \mathbf{w}\}^{\perp}$ in the same connected component as $\mathbf{x}$ in $\{\mathbf{y}, \mathbf{z}\}^{\perp}$.*

Proof: We consider the subspace $\langle z_{\mathbf{x}}, y_{\mathbf{x}} \rangle$ and the polar space $H^{\mathbf{x}}_{z_\mathbf{x}, y_\mathbf{x}} = z_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi} = \langle z_{\mathbf{x}}, y_{\mathbf{x}} \rangle^{\pi}$ in the interior space $\mathcal{G}_{\mathbf{x}}$. The subspace $H^{\mathbf{x}}_{z_\mathbf{x}, y_\mathbf{x}}$ has dimension $m \geq n-4$ and is of rank $r \geq n-6$. Since this subspace is perpendicular to the hyperbolic line $y_{\mathbf{x}}$ in $\mathcal{G}_{\mathbf{x}}$, it can be identified with a unique subspace of dimension m and rank r of $\mathcal{G}_{\mathbf{y}}$, denoted by $H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$. The subspace $W_{\mathbf{y}} = \langle x_{\mathbf{y}}, H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}} \rangle \subseteq \mathcal{G}_{\mathbf{y}}$ has dimension $d = m + 2 \geq n - 2$ and rank $r + 2 \geq n - 4$. By hypothesis the rank of $W_{\mathbf{y}}$ is at least six, thus $W_{\mathbf{y}}$ contains a six-dimensional non-degenerate space $V_{\mathbf{y}}$. It follows that the space $V_{\mathbf{y}} \cap H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$ contains a four-dimensional subspace of rank at least two, as $H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$ has codimension two in $W_{\mathbf{y}}$. Hence there exists a hyperbolic line $k_{\mathbf{y}}$ in $V_{\mathbf{y}} \cap H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$. Also, the intersection $V_{\mathbf{y}} \cap w_{\mathbf{y}}^{\pi}$ contains a four-dimensional subspace of rank at least two, so there also exists a hyperbolic line $h_{\mathbf{y}}$ in $V_{\mathbf{y}} \cap w_{\mathbf{y}}^{\pi}$. The local line $h_{\mathbf{y}}$ leads to a vertex $\mathbf{h} \in \{\mathbf{y}, \mathbf{w}\}^{\perp}$ and the local line $k_{\mathbf{y}}$ corresponds to a vertex $\mathbf{k} \in \{\mathbf{y}, \mathbf{x}\}^{\perp}$. Local analysis of $\mathbf{x}^{\perp}$ and $\mathcal{G}_{\mathbf{x}}$ shows $\mathbf{k} \perp \mathbf{z}$. Indeed $k_{\mathbf{y}}$ is a hyperbolic line of $V_{\mathbf{y}} \cap H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}} \subseteq H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$ and $\mathbf{k} \perp \mathbf{x}$ in Γ, it follows that the hyperbolic line $k_{\mathbf{x}}$ is contained in $H^{\mathbf{x}}_{z_\mathbf{x}, y_\mathbf{x}} \subseteq z_{\mathbf{x}}^{\pi}$, thus $\mathbf{k}$ and $\mathbf{z}$ are two adjacent vertices of Γ. By proposition 3.3.3 we can find a path from $\mathbf{k}$ to $\mathbf{h}$ in the graph $\mathbf{G}(V_{\mathbf{y}}) \subseteq \mathbf{y}^{\perp}$. In particular, the vertex $\mathbf{h}$ lies in the same connected component of $\mathbf{y}^{\perp}$ as the vertex $\mathbf{x}$.

Let $\mathbf{s}^0 \perp \mathbf{s}^1 \perp \cdots \perp \mathbf{s}^m$ be a path from $\mathbf{k} = \mathbf{s}^0$ to $\mathbf{h} = \mathbf{s}^m$ in $\mathbf{G}(V_{\mathbf{y}})$. To finish the proof it suffices to prove that γ is a path in the induced subgraph $\mathbf{z}^{\perp}$. We proceed by induction. The vertex $\mathbf{k}$ is adjacent to $\mathbf{z}$ by construction. We have $M_{\mathbf{y}} := k_{\mathbf{y}}^{\pi} \cap W_{\mathbf{y}} = k_{\mathbf{y}}^{\pi} \cap \langle x_{\mathbf{y}}, H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}} \rangle = \langle x_{\mathbf{y}}, k_{\mathbf{y}}^{\pi} \cap H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}} \rangle$, because $\mathbf{x} \perp \mathbf{k}$. Notice that $M_{\mathbf{y}} = \langle x_{\mathbf{y}}, k_{\mathbf{y}}^{\pi} \cap H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}} \rangle$ is a $\dim(H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}})$-dimensional subspace of $k_{\mathbf{y}}^{\pi} \subseteq \mathcal{G}_{\mathbf{y}}$. Considering this space inside the interior space $\mathcal{G}_{\mathbf{k}}$, denoted by $M_{\mathbf{k}}$, we obtain $\dim(M_{\mathbf{k}}) = \dim(M_{\mathbf{y}}) = \dim(H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}) = \dim(H^{\mathbf{x}}_{z_\mathbf{x}, y_\mathbf{x}}) = \dim(H^{\mathbf{k}}_{z_\mathbf{k}, y_\mathbf{k}})$ by lemma 3.6.2, where $H^{\mathbf{k}}_{z_\mathbf{k}, y_\mathbf{k}} = z_{\mathbf{k}}^{\pi} \cap y_{\mathbf{k}}^{\pi} = \langle z_{\mathbf{k}}, y_{\mathbf{k}} \rangle^{\pi}$. Furthermore, $M_{\mathbf{k}} = \langle x_{\mathbf{k}}, (k_{\mathbf{y}}^{\pi} \cap H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}})^{\mathbf{k}} \rangle \subseteq H^{\mathbf{k}}_{z_\mathbf{k}, y_\mathbf{k}}$, whence $M_{\mathbf{k}} = H^{\mathbf{k}}_{z_\mathbf{k}, y_\mathbf{k}}$. Here $(k_{\mathbf{y}}^{\pi} \cap H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}})^{\mathbf{k}}$ denotes the subspace of $\mathcal{G}_{\mathbf{k}}$ corresponding to the subspace $k_{\mathbf{y}}^{\pi} \cap H^{\mathbf{y}}_{z_\mathbf{x}, y_\mathbf{x}}$ of $\mathcal{G}_{\mathbf{y}}$. Consequently, $k_{\mathbf{y}}^{\pi} \cap W_{\mathbf{y}} = M_{\mathbf{k}} = H^{\mathbf{y}}_{z_\mathbf{k}, y_\mathbf{k}}$ and in particular, $W_{\mathbf{y}} = \langle k_{\mathbf{y}}, H^{\mathbf{y}}_{z_\mathbf{k}, y_\mathbf{k}} \rangle$.

By induction we assume that the vertices $\mathbf{s}^i$ with $i \leq n$, $n \in \mathbb{N}$, are adjacent to $\mathbf{z}$. Then an argument as in the paragraph above yields $W_{\mathbf{y}} = \langle s_{\mathbf{y}}^i, H^{\mathbf{y}}_{z_{\mathbf{s}^i}, y_{\mathbf{s}^i}} \rangle$ and $(s_{\mathbf{y}}^i)^{\pi} \cap W_{\mathbf{y}} = H^{\mathbf{y}}_{z_{\mathbf{s}^i}, y_{\mathbf{s}^i}}$, whence $W_{\mathbf{y}} = \langle s_{\mathbf{y}}^i, H^{\mathbf{y}}_{z_{\mathbf{s}^i}, y_{\mathbf{s}^i}} \rangle$ for $i = 1, \ldots, n$. The vertex $\mathbf{s}^{n+1}$ is adjacent to $\mathbf{y}$ and $\mathbf{s}_n$ in the graph Γ. Moreover, $s_{\mathbf{y}}^{n+1}$ is a hyperbolic line of the subspace $V_{\mathbf{y}}$ in the interior space $\mathcal{G}_{\mathbf{y}}$. Thus $s_{\mathbf{y}}^{n+1}$ is a hyperbolic line of the $(\dim(V_{\mathbf{y}}) - 2)$-dimensional subspace $(s_{\mathbf{y}}^n)^{\pi} \cap V_{\mathbf{y}}$ in $\mathcal{G}_{\mathbf{y}}$. Since $(s_{\mathbf{y}}^n)^{\pi} \cap V_{\mathbf{y}}$ is a subspace of $(s_{\mathbf{y}}^n)^{\pi} \cap W_{\mathbf{y}} = H^{\mathbf{y}}_{z_{\mathbf{s}^n}, y_{\mathbf{s}^n}}$ it follows that $s_{\mathbf{y}}^{n+1} \subseteq (s_{\mathbf{y}}^n)^{\pi} \cap W_{\mathbf{y}} = H^{\mathbf{y}}_{z_{\mathbf{s}^n}, y_{\mathbf{s}^n}}$. Therefore the vertex $\mathbf{s}^{n+1}$ is adjacent to $\mathbf{z}$. ■

Lemma 3.6.4 *Let $n \in \{7, 8\}$ and let $\mathbf{z} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$ be a path in Γ such that*

- *the subspace $\langle z_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi}, x_{\mathbf{y}} \rangle$ is of dimension six and of rank five, or*
- *the subspace $\langle z_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi}, x_{\mathbf{y}} \rangle$ is a non-degenerate subspace of dimension five and $\langle z_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi}, x_{\mathbf{y}} \rangle \cap w_{\mathbf{y}}^{\pi}$ of rank at least two.*

Then there is a vertex $\mathbf{h} \in \{\mathbf{z}, \mathbf{y}, \mathbf{w}\}^{\perp}$ in the same connected component as $\mathbf{x}$ in $\{\mathbf{y}, \mathbf{z}\}^{\perp}$.

Proof: We will prove this statement in a way similar to the proof of lemma 3.6.3, using the same notation.

First we assume that the subspace $W_{\mathrm{y}} = \langle H_{z_{\mathrm{x}}, y_{\mathrm{x}}}^{\mathrm{y}}, x_{\mathrm{y}} \rangle$ is of dimension six and of rank five, which implies that $H_{z_{\mathrm{x}}, y_{\mathrm{x}}}^{\mathrm{y}}$ is a four-dimensional subspace in $\mathcal{G}_{\mathrm{y}}$ of rank three. The radical of $H_{z_{\mathrm{x}}, y_{\mathrm{x}}}^{\mathrm{y}}$ coincides with the radical of W_{y}. Furthermore $W_{\mathrm{y}} \cap w_{\mathrm{y}}^{\pi}$ is at least four-dimensional of rank at least two as w_{y}^{π} is a $(n-2)$-dimensional non-degenerate subspace of $\mathcal{G}_{\mathrm{y}}$. Thus we can fix a hyperbolic line h_{y} in $W_{\mathrm{y}} \cap w_{\mathrm{y}}^{\pi}$. In the case that h_{y} can be chosen to lie inside the subspace $H_{z_{\mathrm{x}}, y_{\mathrm{x}}}^{\mathrm{y}}$, then there is nothing else to prove, so we may assume for the rest of this proof that $h_{\mathrm{y}} \nsubseteq H_{z_{\mathrm{x}}, y_{\mathrm{x}}}^{\mathrm{y}}$. Next we choose a non-radical point s_{y} of $H_{z_{\mathrm{x}}, y_{\mathrm{x}}}^{\mathrm{y}}$ in the subspace $h_{\mathrm{y}}^{\pi} \cap H_{z_{\mathrm{x}}, y_{\mathrm{x}}}^{\mathrm{y}}$, which is of dimension at least two. If possible, we choose s_{y} to be singular and fix a hyperbolic line l_{y} in $H_{z_{\mathrm{x}}, y_{\mathrm{x}}}^{\mathrm{y}}$ going through s_{y}. This construction implies directly that the hyperbolic lines h_{y} and l_{y} span a regular four-dimensional space inside the subspace W_{y}, which is contained in some five-dimensional non-degenerate subspace V_{y} of W_{y}.

If s_{y} has to be chosen regular, then we pick a hyperbolic line l_{y} incident to s_{y} and not intersecting the line h_{y} in $H_{z_{\mathrm{x}}, y_{\mathrm{x}}}^{\mathrm{y}}$ in such a way that the radical of $\langle l_{\mathrm{y}}, h_{\mathrm{y}} \rangle$ is different from the radical of W_{y}. We can satisfy this requirement by the following argument. Let l_{y} and $\tilde{l}_{\mathrm{y}}$ be distinct hyperbolic lines in $H_{z_{\mathrm{x}}, y_{\mathrm{x}}}^{\mathrm{y}}$ containing the point s_{y} such that $\langle h_{\mathrm{y}}, l_{\mathrm{y}} \rangle \neq \langle h_{\mathrm{y}}, \tilde{l}_{\mathrm{y}} \rangle$. Since the regular plane $\langle h_{\mathrm{y}}, s_{\mathrm{y}} \rangle$ is contained in both, we have $\mathrm{rad}(\langle h_{\mathrm{y}}, l_{\mathrm{y}} \rangle) \neq \mathrm{rad}(\langle h_{\mathrm{y}}, \tilde{l}_{\mathrm{y}} \rangle)$. Now, $h_{\mathrm{y}}^{\pi} \cap l_{\mathrm{y}}^{\pi} \cap W_{\mathrm{y}} = \langle \mathrm{rad}(W_{\mathrm{y}}), \mathrm{rad}(\langle h_{\mathrm{y}}, l_{\mathrm{y}} \rangle) \rangle$, whence there is a point $r_{\mathrm{y}} \in W_{\mathrm{y}}$ not contained in $\langle h_{\mathrm{y}}, l_{\mathrm{y}} \rangle$ and not contained in $\langle \mathrm{rad}(W_{\mathrm{y}}), \mathrm{rad}(\langle h_{\mathrm{y}}, l_{\mathrm{y}} \rangle) \rangle$. Hence $V_{\mathrm{y}} = \langle r_{\mathrm{y}}, h_{\mathrm{y}}, l_{\mathrm{y}} \rangle$ is a five-dimensional regular space of W_{y} containing both hyperbolic lines h_{y} and l_{y}.

The local hyperbolic line h_{y} yields a vertex $\mathbf{h} \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}^{\perp}$ and the local line l_{y} a vertex $\mathbf{l} \in \{\mathbf{y}, \mathbf{w}\}^{\perp}$. By proposition 3.3.3 there exists a path from $\mathbf{h}$ to $\mathbf{l}$ inside $\mathbf{G}(V_{\mathrm{y}})$, so that $\mathbf{h}$ lies in the same connected component of $\mathbf{y}^{\perp}$ as the vertex $\mathbf{x}$. The vertex $\mathbf{h}$ is also adjacent to the vertex $\mathbf{z}$ by the same argument as in the proof of lemma 3.6.3.

Alternatively let $W_{\mathrm{y}} = \langle H_{z_{\mathrm{x}}, y_{\mathrm{x}}}^{\mathrm{y}} x_{\mathrm{y}} \rangle$ be a non-degenerate five-dimensional subspace of $\mathcal{G}_{\mathrm{y}}$ and $W_{\mathrm{y}} \cap w_{y}^{\pi}$ be a subspace of rank at least two. Then $H_{z_{\mathrm{x}}, y_{\mathrm{x}}}^{\mathrm{y}}$ is a regular plane and $n = 7$. We choose a hyperbolic line $h_{\mathrm{y}} \in H_{z_{\mathrm{x}} y_{\mathrm{x}}}^{\mathrm{y}}$ and a regular two-dimensional subspace l_{y} in the plane $W_{\mathrm{y}} \cap w_{\mathrm{y}}^{\pi}$. Again, the local line h_{y} yields a vertex $\mathbf{h} \in \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}^{\perp}$

and the local line $l_{\mathbf{y}}$ belongs to a vertex $\mathbf{l} \in \{\mathbf{y}, \mathbf{w}\}^{\perp}$. Now the proof is identical to the first part with $V_{\mathbf{y}}$ replaced by $W_{\mathbf{y}}$. ■

For the next few lemmata let $\mathbf{z}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{w}$ be vertices of Γ with $\mathbf{z} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$. In the interior space $\mathcal{G}_{\mathbf{x}}$ the vertices $\mathbf{z}$ and $\mathbf{y}$ belong to hyperbolic lines $z_{\mathbf{x}}$ and $y_{\mathbf{x}}$, and $x_{\mathbf{y}}$ and $w_{\mathbf{y}}$ are the unique regular lines in $\mathcal{G}_{\mathbf{y}}$ of the vertices $\mathbf{x}$ and $\mathbf{w}$. Moreover, $H^{\mathbf{x}}_{z_{\mathbf{x}},y_{\mathbf{x}}} = z^{\pi}_{\mathbf{x}} \cap y^{\pi}_{\mathbf{x}}$ is a subspace of dimension $n-4$ or $n-3$ in $\mathcal{G}_{\mathbf{x}}$. Since $H^{\mathbf{x}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ is contained in $y^{\pi}_{\mathbf{x}}$. This subspace can also be identified with a unique subspace of $\mathcal{G}_{\mathbf{y}}$, denoted by $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$. Similarly, $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}} = x^{\pi}_{\mathbf{y}} \cap w^{\pi}_{\mathbf{y}}$ is an $(n-4)$- or an $(n-3)$-dimensional subspace of $\mathcal{G}_{\mathbf{y}}$, corresponding to the subspace $H^{\mathbf{x}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ in $\mathcal{G}_{\mathbf{x}}$.

Lemma 3.6.5 *Let $n \geq 10$. Then the graph Γ has diameter two.*

Proof: The space $W_{\mathbf{y}} = \langle x_{\mathbf{y}}, H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \rangle$ is of dimension at least $n-2$ and of rank at least $n-4 \geq 6$. Thus, by lemma 3.6.3, the space $W_{\mathbf{y}}$ contains a hyperbolic line $h_{\mathbf{y}}$, which corresponds to a vertex $\mathbf{h} \in \{\mathbf{z}, \mathbf{y}, \mathbf{w}\}^{\perp}$. It follows that $\mathbf{z}$ and $\mathbf{w}$ have distance two. Hence by induction each connected component of Γ has diameter two, and the claim results from the connectedness of Γ. ■

Lemma 3.6.6 *Let $n = 9$. Then the graph Γ has diameter two.*

Proof: If the subspace $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ is either of dimension six and of rank at least five or of dimension five and of rank at least four, then $W_{\mathbf{y}} = \langle H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}, x_{\mathbf{y}} \rangle$ is an eight-dimensional subspace of rank at least seven or a seven-dimensional subspace of rank at least six. In both cases by lemma 3.6.3 the subspace $W_{\mathbf{y}}$ contains a hyperbolic line $h_{\mathbf{y}}$, such that the corresponding vertex $\mathbf{h}$ is an element of $\{\mathbf{z}, \mathbf{y}, \mathbf{w}\}^{\perp}$, yielding diameter two by induction.

The remaining possibility is that $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ is a five-dimensional subspace of rank three in $\mathcal{G}_{\mathbf{y}}$. In this case we choose a hyperbolic line $h_{\mathbf{y}}$ in $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ intersecting $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ in a one-dimensional subspace. This choice is possible, because the subspaces $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ and $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ are both contained in $x^{\pi}_{\mathbf{y}}$, which implies that $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}} \cap H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ has dimension at least three and so this intersection subspace contains a one-dimensional space which is not contained in the radical of $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$. This hyperbolic line $h_{\mathbf{y}}$ yields a vertex $\mathbf{h} \in \{\mathbf{x}, \mathbf{y}, \mathbf{w}\}^{\perp}$. Furthermore the subspace $\langle h_{\mathbf{x}}, z_{\mathbf{x}} \rangle$ in $\mathcal{G}_{\mathbf{x}}$ is four-dimensional and of rank at least three. Hence $H^{\mathbf{y}}_{z_{\mathbf{x}},h_{\mathbf{x}}}$ is a five-dimensional subspace of rank five or four. Applying the argumentation from above to the path $\mathbf{z} \perp \mathbf{x} \perp \mathbf{h} \perp \mathbf{w}$, it follows that the vertices $\mathbf{z}$ and $\mathbf{w}$ have distance two in Γ, again yielding diameter two by induction. ■

Lemma 3.6.7 *Let $n = 8$. Then the graph Γ has diameter two.*

Proof: We will prove the statement by induction. Therefore let $\mathbf{z}, \mathbf{x}, \mathbf{y}$ and $\mathbf{w}$ be four different vertices of Γ such that $\mathbf{z} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$. The subspaces $H^{\mathbf{x}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ and $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$

are four- or five-dimensional and of rank at least four, so we can distinguish the following cases:

case	$\dim(H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}})$	$\dim(H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}})$	$\dim(H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \cap H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}})$
one	5	5	≥ 4
two	5	4	≥ 3
three	4	5	≥ 3
four	4	4	≥ 2

Suppose we are in case one or two, i.e., $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$ is a five-dimensional subspace of rank at least four and the subspace $W_{\mathrm{y}} = \langle H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}, x_{\mathrm{y}} \rangle$ is of dimension seven and of rank at least six. Using lemma 3.6.3 we obtain a vertex $\mathbf{h}$ in Γ adjacent to the vertices $\mathbf{z}$, $\mathbf{y}$, $\mathbf{w}$, whence the distance between the vertices $\mathbf{z}$ and $\mathbf{w}$ is at most two in Γ. Symmetry handles case three.

Assume we are in the final case, i.e., $\dim(H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}) = \dim(H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}) = 4$. We will proceed by another case distinction depending on the rank of $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$ and the rank $H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$.

case	4-4	4-3	4-2	3-3	3-2	2-2
$\mathrm{rank}(H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}})$	4	4	4	3	3	2
$\mathrm{rank}(H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}})$	4	3	2	3	2	2

cases 4-∗: If $\mathrm{rank}(H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}) = 4$ then $W_{\mathrm{y}} = \langle x_{\mathrm{y}}, H^{\mathrm{y}}_{y_{\mathrm{x}},z_{\mathrm{x}}} \rangle$ is a regular subspace of dimension six. By lemma 3.6.3 the subspace W_{y} contains a hyperbolic line h_{y} yielding a unique vertex $\mathbf{h} \in \{\mathbf{z}, \mathbf{y}, \mathbf{w}\}^{\perp}$, so the vertices $\mathbf{z}$ and $\mathbf{w}$ are at most at distance two in Γ.

cases 3-∗: In these two cases the subspace $W_{\mathrm{y}} = \langle x_{\mathrm{y}}, H^{\mathrm{y}}_{y_{\mathrm{x}},z_{\mathrm{x}}} \rangle$ has dimension six and rank five. By lemma 3.6.4 there exists again a vertex $\mathbf{h} \in \{\mathbf{z}, \mathbf{y}, \mathbf{w}\}^{\perp}$. Thus $\mathbf{z}$ and $\mathbf{w}$ are at most at distance two in Γ.

case 2-2: Finally we assume that the subspaces $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$ and $H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$ are of dimension four and of rank two. Note that in this case the hyperbolic line w_{y} does not intersect the subspace x^{π}_{y}. The intersection $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \cap H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$ may have rank zero, one, or two.

If $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \cap H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$ has rank two, then $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \cap H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$ equals some hyperbolic line h_{y} and we are done, because the corresponding vertex $\mathbf{h}$ is adjacent to the vertices $\mathbf{z}$, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{w}$ in Γ implying that the distance between $\mathbf{z}$ and $\mathbf{w}$ is at most two in Γ.

Suppose $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \cap H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$ has rank one. Then we can find a hyperbolic line l_{y} in $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$, which intersects the subspace $H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$ in a one-dimensional subspace. The four-dimensional space $\langle l_{\mathrm{y}}, w_{\mathrm{y}} \rangle$ has rank three or four, thus the path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ from $\mathbf{z}$ to $\mathbf{w}$ in Γ belongs either to case 4-2 or to case 3-2, and we are done.

If $H^{\mathbf{y}}_{z_\mathbf{x},y_\mathbf{x}} \cap H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}}$ is a totally singular subspace then we define the two set of points
$S^{\mathbf{y}}_{z_\mathbf{x},y_\mathbf{x}} := \{p_\mathbf{y} \in H^{\mathbf{y}}_{z_\mathbf{x},y_\mathbf{x}} \cap H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}} \mid p_\mathbf{y} \notin \mathrm{rad}(H^{\mathbf{y}}_{z_\mathbf{x},y_\mathbf{x}}), p_\mathbf{y} \text{ a singular point}\}$
and
$S^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}} := \{p_\mathbf{y} \in H^{\mathbf{y}}_{z_\mathbf{x},y_\mathbf{x}} \cap H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}} \mid p_\mathbf{y} \notin \mathrm{rad}(H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}}), p_\mathbf{y} \text{ a singular point}\}$.

If either of $S^{\mathbf{y}}_{z_\mathbf{x},y_\mathbf{x}}$ and $S^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}}$ is not empty, then with out loss of generality we assume after relabelling that $S^{\mathbf{y}}_{z_\mathbf{x},y_\mathbf{x}} \neq \varnothing$ and choose a point $p_\mathbf{y} \in S^{\mathbf{y}}_{z_\mathbf{x},y_\mathbf{x}}$ as well as a hyperbolic line $l_\mathbf{y}$ in $H^{\mathbf{y}}_{z_\mathbf{x},y_\mathbf{x}}$ containing the point $p_\mathbf{y}$. The subspace $\langle w_\mathbf{y}, l_\mathbf{y}\rangle$ is non-degenerate and of dimension four, moreover the hyperbolic line $l_\mathbf{y}$ corresponds to a vertex $\mathbf{l} \in \{\mathbf{x},\mathbf{y},\mathbf{z}\}^\perp$. The resulting path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ belongs either to the case 4-3 or to the case 4-2, and again we are done.

In the final step we assume $S^{\mathbf{y}}_{z_\mathbf{x},y_\mathbf{x}} = \varnothing = S^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}}$, which implies $\mathrm{rad}(H^{\mathbf{x}}_{z_\mathbf{x},y_\mathbf{x}}) = H^{\mathbf{y}}_{z_\mathbf{x},y_\mathbf{x}} \cap H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}} = \mathrm{rad}(H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}})$. In other words the intersection $H^{\mathbf{y}}_{z_\mathbf{x},y_\mathbf{x}} \cap H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}}$ is a totally singular radical two-dimensional subspace of $H^{\mathbf{x}}_{z_\mathbf{x},y_\mathbf{x}}$ and of $H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}}$. For an arbitrary hyperbolic line $l_\mathbf{y}$ in $H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}}$ the subspace $l^\pi_\mathbf{y} \cap H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}}$ coincides with $\mathrm{rad}(H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}})$ and, therefore, $\mathrm{rad}(H^{\mathbf{x}}_{x_\mathbf{y},w_\mathbf{y}}) = \mathrm{rad}(H^{\mathbf{x}}_{z_\mathbf{x},y_\mathbf{x}}) \subseteq l^\pi_\mathbf{x} \cap z^\pi_\mathbf{x} = H^{\mathbf{x}}_{z_\mathbf{x},l_\mathbf{x}}$. Furthermore $\langle x_\mathbf{l}, w_\mathbf{l}\rangle = \langle x_\mathbf{y}, w_\mathbf{y}\rangle$ by lemma 3.6.2, which implies that $\mathrm{rad}(H^{\mathbf{x}}_{x_\mathbf{y},w_\mathbf{y}}) = \mathrm{rad}(H^{\mathbf{x}}_{x_\mathbf{l},w_\mathbf{l}})$ and so every point of $\mathrm{rad}(H^{\mathbf{x}}_{z_\mathbf{x},y_\mathbf{x}})$ is contained in $H^{\mathbf{x}}_{x_\mathbf{l},w_\mathbf{l}} \cap H^{\mathbf{x}}_{z_\mathbf{x},l_\mathbf{x}}$. As $\langle z_\mathbf{x}, y_\mathbf{x}\rangle \cap \langle z_\mathbf{x}, l_\mathbf{x}\rangle = z_\mathbf{x}$ it follows that $\mathrm{rad}(\langle z_\mathbf{x}, y_\mathbf{x}\rangle) \neq \mathrm{rad}(\langle z_\mathbf{x}, l_\mathbf{x}\rangle)$ and therefore not every point of $\mathrm{rad}(H^{\mathbf{x}}_{z_\mathbf{x},y_\mathbf{x}})$ is also a point of $\mathrm{rad}(H^{\mathbf{x}}_{z_\mathbf{x},l_\mathbf{x}})$. Consequently the path $\mathbf{z} \perp \mathbf{x} \perp \mathbf{l} \perp \mathbf{w}$ belongs to some case already dealt with, because $S^{\mathbf{l}}_{z_\mathbf{x},l_\mathbf{x}}$ is not empty.

Since the vertices $\mathbf{z}$ and $\mathbf{w}$ have at most distance two in Γ, by induction the graph Γ has diameter two. ■

Lemma 3.6.8 *Let $n = 7$. Then the graph Γ has diameter two.*

Proof: As before we will use induction to prove the claim, therefore let $\mathbf{z}, \mathbf{x}, \mathbf{y}$ and $\mathbf{w}$ be four different vertices of Γ forming the path $\mathbf{z} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$. The subspaces $H^{\mathbf{x}}_{z_\mathbf{x},y_\mathbf{x}}$ and $H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}}$ of $\mathcal{G}_\mathbf{x}$ resp. of $\mathcal{G}_\mathbf{y}$ have dimension three or four. We will distinguish the following four cases:

case	$\dim(H^{\mathbf{x}}_{z_\mathbf{x},y_\mathbf{x}})$	$\dim(H^{\mathbf{y}}_{x_\mathbf{y},w_\mathbf{y}})$	$\dim(H^{\mathbf{x}}_{z_\mathbf{x},y_\mathbf{x}} \cap H^{\mathbf{x}}_{x_\mathbf{y},w_\mathbf{y}})$
one	4	4	≥ 3
two	4	3	≥ 2
three	3	4	≥ 2
four	3	3	≥ 1

First we consider case one and two, and also case three by symmetry. Since $n = 7$ and the dimension of $H^{\mathbf{x}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ is four, the hyperbolic lines $y_{\mathbf{x}}$ and $z_{\mathbf{x}}$ span a three-dimensional space, whence $H^{\mathbf{x}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ has a radical of dimension at most one. Thus the subspace $W_{\mathbf{y}} = \langle H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}, x_{\mathbf{y}}\rangle$ is of dimension six and rank at least five. By lemma 3.6.3 and lemma 3.6.4 there exists a vertex $\mathbf{h} \in \{\mathbf{w},\mathbf{y},\mathbf{z}\}^{\perp}$, yielding distance two between $\mathbf{z}$ and $\mathbf{w}$ in Γ.

It remains to prove the claim in the case that $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ and $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ are planes. We split up this setting into six different cases depending on the rank of the planes $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ and $H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$:

case	3-3	3-2	3-1	2-2	2-1	1-1
$\operatorname{rank}(H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}})$	3	3	3	2	2	1
$\operatorname{rank}(H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}})$	3	2	1	2	1	1

case 3-3: If $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ is a three-dimensional subspace of the interior space $\mathcal{G}_{\mathbf{y}}$, then $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}} = H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$, thus $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ is a regular plane, which of course contains some hyperbolic line $h_{\mathbf{y}}$. The hyperbolic line corresponds to a vertex $\mathbf{h}$ in the subgraph $\{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}\}^{\perp}$, finishing the proof.

Therefore we assume that the intersection $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} \cap H^{\mathbf{y}}_{x_{\mathbf{y}},w_{\mathbf{y}}}$ is of dimension one or two. Under this condition we regard the five-dimensional non-degenerate space $W_{\mathbf{y}} = \langle x_{\mathbf{y}}, H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}\rangle$, which intersects $w^{\pi}_{\mathbf{y}}$ in a three-dimensional space of rank at least one. Moreover, for each hyperbolic line $l_{\mathbf{y}}$ in the regular plane $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$, the non-degenerate plane $H^{\mathbf{y}}_{z_{\mathbf{l}},y_{\mathbf{l}}} = \langle x_{\mathbf{y}}, l^{\pi}_{\mathbf{y}} \cap H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}\rangle$ is a subspace of $W_{\mathbf{y}}$ and intersects $w^{\pi}_{\mathbf{y}}$ in a one- or two-dimensional subspace. As $x_{\mathbf{y}} \cap H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}} = \{0\}$, for different hyperbolic lines $h_{\mathbf{y}}$ and $l_{\mathbf{y}}$ in $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$, the subspaces $H^{\mathbf{y}}_{z_{\mathbf{l}},y_{\mathbf{l}}} \cap w^{\pi}_{\mathbf{y}}$ and $H^{\mathbf{y}}_{z_{\mathbf{h}},y_{\mathbf{h}}} \cap w^{\pi}_{\mathbf{y}}$ are different.

By the formulas B.3 and B.4 on page 255 the regular plane $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ contains $q^4-q^3+q^2$ hyperbolic lines, while the plane $W_{\mathbf{y}} \cap w^{\pi}_{\mathbf{y}}$ contains at most q^3+q^2+1 different singular points. Hence we find a hyperbolic line $l_{\mathbf{y}}$ in the plane $H^{\mathbf{y}}_{z_{\mathbf{x}},y_{\mathbf{x}}}$ such that $H^{\mathbf{y}}_{z_{\mathbf{l}},y_{\mathbf{l}}} \cap w^{\pi}_{\mathbf{y}}$ contains some regular point $p_{\mathbf{y}}$. The hyperbolic line $l_{\mathbf{y}}$ determines a vertex $\mathbf{l}$ in Γ adjacent to the vertices $\mathbf{x},\mathbf{y},\mathbf{z}$. Furthermore, we choose a hyperbolic line $n_{\mathbf{y}}$ in the regular plane $H^{\mathbf{y}}_{z_{\mathbf{l}},y_{\mathbf{l}}}$ incident to the regular point $p_{\mathbf{y}}$. The hyperbolic line $n_{\mathbf{y}}$ yields a vertex $\mathbf{n}$ of $\{\mathbf{z},\mathbf{y},\mathbf{l}\}^{\perp}$. If the subspace $H^{\mathbf{n}}_{n_{\mathbf{y}},w_{\mathbf{y}}}$ is of dimension four, then the path $\mathbf{z} \perp \mathbf{n} \perp \mathbf{y} \perp \mathbf{w}$ of Γ belongs to one of the cases one, two, or three, so we may assume that $H^{\mathbf{n}}_{n_{\mathbf{y}},w_{\mathbf{y}}}$ is a three-dimensional subspace. Since the regular point $p_{\mathbf{y}}$ is perpendicular to $w_{\mathbf{y}}$, the four-dimensional subspace $\langle n_{\mathbf{y}}, w_{\mathbf{y}}\rangle$ has rank at least three and we conclude that $H^{\mathbf{n}}_{n_{\mathbf{y}},w_{\mathbf{y}}}$ has rank at least two. Thus, there exists a hyperbolic line $h_{\mathbf{y}}$ in the plane $H^{\mathbf{n}}_{n_{\mathbf{y}},w_{\mathbf{y}}}$ in such that $H^{\mathbf{n}}_{n_{\mathbf{h}},w_{\mathbf{h}}} \cap z^{\pi}_{\mathbf{n}}$ contains a regular point $d_{\mathbf{n}}$, which is possible by the argumentation above. Certainly, if $H^{\mathbf{n}}_{n_{\mathbf{y}},w_{\mathbf{y}}}$ happens to have rank two instead of rank three, then this subspace contains q^4 hyperbolic lines by the formulas B.3 and B.4on pages 255, and the above argument is still applicable. Moreover the vertex $\mathbf{h}$ corresponding to $h_\mathbf{y}$ is contained in the induced subgraph $\{\mathbf{w}, \mathbf{n}, \mathbf{y}\}^\perp$. The interior space $\mathcal{G}_\mathbf{h}$ contains the regular point $d_\mathbf{h}$ and the hyperbolic line $n_\mathbf{h}$, which in turn contains the regular point $p_\mathbf{h}$. Since the point $d_\mathbf{h}$ is contained in the subspace $n_\mathbf{h}^\pi$, the two regular points $p_\mathbf{h}$ and $d_\mathbf{h}$ span a hyperbolic line $k_\mathbf{h}$ in the space $\mathcal{G}_\mathbf{h}$, in particular $k_\mathbf{h}$ is a hyperbolic line of the subspace $w_\mathbf{h}^\pi$. Indeed the hyperbolic line $n_\mathbf{h}$ intersects the subspace $w_\mathbf{h}^\pi$ in the regular point $p_\mathbf{h}$, while the regular point $d_\mathbf{h}$ is a point of $w_\mathbf{h}^\pi$ by construction. Thus we have determined a vertex $\mathbf{k}$ adjacent to $\mathbf{w}$ and $\mathbf{h}$. Furthermore, the two hyperbolic lines $n_\mathbf{h}$ and $k_\mathbf{h}$ generate a plane in $\mathcal{G}_\mathbf{h}$ implying that $\dim(H^\mathbf{h}_{n_\mathbf{h}, k_\mathbf{h}}) = 4$. By these facts the path $\mathbf{z} \perp \mathbf{n} \perp \mathbf{h} \perp \mathbf{k}$ of Γ belongs to case two or three of this proof, so there exists a vertex $\mathbf{m} \in \{\mathbf{n}, \mathbf{k}, \mathbf{z}\}^\perp$ in the same connected component of the subgraph $\{\mathbf{n}, \mathbf{k}\}^\perp$ as the vertex $\mathbf{h}$. Local analysis of the interior space $\mathcal{G}_\mathbf{m}$ shows that the orthogonal space $z_\mathbf{m}^\pi$ of $z_\mathbf{m}$ contains the two points $p_\mathbf{m}$ and $d_\mathbf{m}$, whence the hyperbolic line $k_\mathbf{m}$ spanned by $p_\mathbf{m}$ and $d_\mathbf{m}$. Consequently, the vertex $\mathbf{k}$ is adjacent to the vertices $\mathbf{z}$ and $\mathbf{w}$, so $\mathbf{z}$ and $\mathbf{w}$ have at most distance two in Γ.

case 3-2 **and case** 3-1**:** As before we study the intersection of the subspaces $H^\mathbf{y}_{z_\mathbf{x}, y_\mathbf{x}}$ and $H^\mathbf{y}_{w_\mathbf{y}, x_\mathbf{y}}$. If the subspace $H^\mathbf{y}_{z_\mathbf{x}, y_\mathbf{x}} \cap H^\mathbf{y}_{w_\mathbf{y}, x_\mathbf{y}}$ has rank at least two, then it contains a hyperbolic line and we are done. Otherwise define $S^\mathbf{y}_{z_\mathbf{x}, y_\mathbf{x}, w_\mathbf{y}, x_\mathbf{y}}$ to be the set of all singular points incident to $H^\mathbf{y}_{z_\mathbf{x}, y_\mathbf{x}} \cap H^\mathbf{y}_{w_\mathbf{y}, x_\mathbf{y}}$. If $S^\mathbf{y}_{z_\mathbf{x}, y_\mathbf{x}, w_\mathbf{y}, x_\mathbf{y}} \neq \varnothing$, then let $p_\mathbf{y}$ be a point of $S^\mathbf{y}_{z_\mathbf{x}, y_\mathbf{x}, w_\mathbf{y}, x_\mathbf{y}}$ and we choose a hyperbolic line $l_\mathbf{y}$ in the regular plane $H^\mathbf{y}_{z_\mathbf{x}, y_\mathbf{x}}$ going through the singular point $p_\mathbf{y}$. The vertex $\mathbf{l}$ corresponding to $l_\mathbf{y}$ is contained in $\{\mathbf{z}, \mathbf{x}, \mathbf{y}\}^\perp$. Since the hyperbolic lines $l_\mathbf{y}$ and $w_\mathbf{y}$ span either a three-dimensional or a non-degenerate four-dimensional space, the path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ belongs to case four 3-3 or to case two. On the other hand, if $S^\mathbf{y}_{z_\mathbf{x}, y_\mathbf{x}, w_\mathbf{y}, x_\mathbf{y}} = \varnothing$, then we choose a regular point $r_\mathbf{y}$ in $H^\mathbf{y}_{z_\mathbf{x}, y_\mathbf{x}} \cap H^\mathbf{y}_{w_\mathbf{y}, x_\mathbf{y}}$ and a hyperbolic line $l_\mathbf{y}$ incident to the point $r_\mathbf{y}$ in the regular plane $H^\mathbf{y}_{z_\mathbf{x}, y_\mathbf{x}}$, yielding the path $\gamma : \mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ in Γ between $\mathbf{z}$ and $\mathbf{w}$. The subspace $\langle l_\mathbf{y}, w_\mathbf{y} \rangle$ is either of dimension three, in which case the path γ belongs to case two, or of dimension four. If this four-dimensional subspace is of rank four, then the path γ belongs to case 3-3. If the rank of $\langle l_\mathbf{y}, w_\mathbf{y} \rangle$ strictly less than four the we obtain the point set $S^\mathbf{y}_{z_1, y_1, w_\mathbf{y}, l_\mathbf{y}} = \{s_\mathbf{y} \mid s_\mathbf{y} \in H^\mathbf{y}_{z_1, y_1} \cap H^\mathbf{y}_{w_\mathbf{y}, l_\mathbf{y}}, s_\mathbf{y} \text{ a singular point}\}$. If $S^\mathbf{y}_{z_1, y_1, w_\mathbf{y}, l_\mathbf{y}} \neq \varnothing$, then the path γ satisfies the conditions of the previous paragraph, which leads to the fact that the path γ can be transformed to a path between the vertices $\mathbf{z}$ and $\mathbf{w}$ of length three belonging to case two or case four 3-3. If $S^\mathbf{y}_{z_1, y_1, w_\mathbf{y}, l_\mathbf{y}}$ is also empty, then we choose a regular point $d_\mathbf{y}$ in $H^\mathbf{y}_{z_1, y_1} \cap H^\mathbf{y}_{w_\mathbf{y}, l_\mathbf{y}}$ and consider the two-dimensional space $h_\mathbf{y}$ spanned by the two different points $r_\mathbf{y}$ and $d_\mathbf{y}$. Since $d_\mathbf{y}$ is a regular point in $l_\mathbf{y}^\pi \subseteq r_\mathbf{y}^\pi$, the space

h_{y} is a hyperbolic line, contained in $w_{\mathbf{y}}^{\pi}$. Thus the corresponding vertex $\mathbf{h}$ is adjacent to the vertices $\mathbf{y}$ and $\mathbf{w}$. The hyperbolic lines l_{y} and h_{y} span a plane implying $\dim(H^{\mathrm{y}}_{l_{\mathrm{y}},h_{\mathrm{y}}}) = 4$. Hence there exists a vertex $\mathbf{m} \in \{\mathbf{z}, \mathbf{l}, \mathbf{h}\}^{\perp}$ in the same connected component of $\{\mathbf{l}, \mathbf{h}\}^{\perp}$ as $\mathbf{y}$, because the path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{h}$ in Γ belongs to case two or three. Hence, by local analysis of the space $\mathcal{G}_{\mathbf{m}}$, the vertex $\mathbf{h}$ is also adjacent to $\mathbf{z}$, as the hyperbolic line $h_{\mathbf{m}}$ is contained in the subspace $z_{\mathbf{m}}^{\pi}$ by construction. Therefore $\mathbf{w}$ and $\mathbf{z}$ have a common neighbour $\mathbf{h}$ and therefore at most distance two in Γ.

case 2-2 **and case** 2-1**:** Again we will analyse the subspace $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \cap H^{\mathrm{y}}_{w_{\mathrm{y}},x_{\mathrm{y}}}$ and the set of singular points $S^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}},w_{\mathrm{y}},x_{\mathrm{y}}} = \{s_{\mathrm{y}} \mid s_{\mathrm{y}} \in H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \cap H^{\mathrm{y}}_{w_{\mathrm{y}},x_{\mathrm{y}}}, s_{\mathrm{y}} \text{ a singular point}\}$. Suppose p_{y} is an element of $S^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}},w_{\mathrm{y}},x_{\mathrm{y}}}$ not contained in the radical of $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$. In this case we choose a hyperbolic line l_{y} in $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$ incident to the point p_{y} and obtain the subspace $\langle w_y, l_y \rangle$, which is of dimension three or four and of rank at least three. As before the path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ belongs to case two or to case four (3-2).

If on the other hand $S^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}},w_{\mathrm{y}},x_{\mathrm{y}}} = \varnothing$, then every point of $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \cap H^{\mathrm{y}}_{w_{\mathrm{y}},x_{\mathrm{y}}}$ is regular. Note that $d_{\mathrm{y}} = H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \cap H^{\mathrm{y}}_{w_{\mathrm{y}},x_{\mathrm{y}}}$ is a unique point, because anisotropic two-dimensional unitary spaces over a finite field do not exist. Recall also that the regular point d_{y} is contained in q^2 hyperbolic lines and one singular line of $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$ by lemma B.1.4. Therefore the hyperbolic line w_{y} contains a singular point s_{y} such that $s_{\mathrm{y}}^{\pi} \cap H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$ is a hyperbolic line l_{y}, containing d_{y}. The subspace $\langle w_{\mathrm{y}}, l_{\mathrm{y}} \rangle$ is non-degenerate of dimension three or four, so the path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ belongs either to case two or to case four (3-2).

It remains to deal with the case that each point of $S^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}},w_{\mathrm{y}},x_{\mathrm{y}}}$ is contained in the radical of $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$. Since $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$ is a rank two plane, the point set $S^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}},w_{\mathrm{y}},x_{\mathrm{y}}}$ consists of a unique singular point. If the intersection $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \cap H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$ is a one-dimensional subspace, then it equals the radical of $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$. In this situation $x_{\mathrm{y}}^{\pi} = \langle H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}, H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \rangle$, so that $S^{\mathrm{y}} z_{\mathbf{x}}, y_{\mathbf{x}}, w_{\mathrm{y}}, x_{\mathrm{y}}$ cannot be contained in the radical of $H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$, as otherwise $S^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}},w_{\mathrm{y}},x_{\mathrm{y}}}$ is contained in the radical of x_{y}^{π}, a contradiction. Since every singular point of a rank one plane is contained in the radical of that plane, the plane $H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$ necessarily has rank two. By symmetry, working with the singular points $S^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}},w_{\mathrm{y}},x_{\mathrm{y}}}$ not in the radical of $H^{\mathrm{y}}_{w_{\mathrm{y}},x_{\mathrm{y}}}$, we are done. Now we assume that the intersection $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \cap H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$ is a two-dimensional subspace. Any hyperbolic line l_{y} of $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$ has the property that $l_{\mathrm{y}}^{\pi} \cap H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$ equals the radical of $H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$. Hence the subspace $\langle l_{\mathrm{y}}, w_{\mathrm{y}} \rangle^{\pi} \cap H^{\mathrm{y}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$ is of dimension one. Since $\mathbf{l} \perp \mathbf{x}$, the path $\mathbf{z} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$ by lemma 3.6.2 belongs to the situation that $H^{\mathrm{y}}_{z_{\mathrm{l}},y_{\mathrm{l}}} \cap H^{\mathrm{y}}_{l_{\mathrm{y}},w_{\mathrm{y}}}$ is of dimension one, that we have just dealt with.

case 1-1**:** In this final case we assume that $H^{\mathbf{x}}_{z_{\mathrm{x}},y_{\mathrm{x}}}$ and $H^{\mathrm{y}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$ are planes of rank one. Let $p_{\mathbf{x}}$ be some point in $H^{\mathbf{x}}_{z_{\mathrm{x}},y_{\mathrm{x}}} \cap H^{\mathbf{x}}_{x_{\mathrm{y}},w_{\mathrm{y}}}$. Since $z_{\mathbf{x}}^{\pi}$ is a five-dimensional non-degenerate subspace, there exists a hyperbolic line $m_{\mathbf{x}}$ in $z_{\mathbf{x}}^{\pi}$ incident to $p_{\mathbf{x}}$. The hyperbolic line $m_{\mathbf{x}}$ corresponds to a vertex $\mathbf{m} \in \{\mathbf{z}, \mathbf{x}\}^{\perp}$. Moreover, the subspace $\langle m_{\mathbf{x}}, y_{\mathbf{x}} \rangle$ is either three-dimensional or four-dimensional and of rank at least three. Hence the path $\mathbf{m} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$ belongs to one of the above cases. Thus the graph Γ contains a vertex $\mathbf{n} \in \{\mathbf{m}, \mathbf{w}\}^{\perp}$. The resulting path $\mathbf{z} \perp \mathbf{m} \perp \mathbf{n} \perp \mathbf{w}$ from $\mathbf{z}$ to $\mathbf{w}$ has the property that $\langle m_{\mathbf{n}}, w_{\mathbf{n}} \rangle$ is a plane or a four-dimensional subspace of rank at least three, because the hyperbolic line $m_{\mathbf{n}}$ intersects $w_{\mathbf{n}}^{\pi}$ the point in $p_{\mathbf{n}}$. Thus this path belongs to one of the cases above.

Thus $\mathbf{z}$ and $\mathbf{w}$ have distance at most two in Γ, and the claim follows by induction. ■

Altogether, we have proved the following.

Proposition 3.6.9 *The graph Γ has diameter two.*

In fact, the proofs of lemma 3.6.5 to lemma 3.6.8 also imply that Γ is simply connected.

Next we want to construct a global point-line geometry on the graph Γ that will allow us to determine the isomorphism type of Γ. Recall the notation introduced for local objects in the beginning of this section. The following observation will play an important role for the definition of global points.

Lemma 3.6.10 *Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be three different vertices of Γ satisfying $\mathbf{x} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x}$ and let $p_{\mathbf{x}}$ be a local point of $\mathcal{G}_{\mathbf{x}}$ such that $y_{\mathbf{x}}, z_{\mathbf{x}} \in p_{\mathbf{x}}^{\pi}$. Then the unique local point $p_{\mathbf{y}} \in \mathcal{G}_{\mathbf{y}}$ induced by the point $p_{\mathbf{y}_{\mathbf{x}}} \in \mathcal{G}_{\mathbf{y}_{\mathbf{x}}}$ and the unique local point $p_{\mathbf{z}} \in \mathcal{G}_{\mathbf{z}}$ induced by the point $p_{\mathbf{z}_{\mathbf{x}}} \in \mathcal{G}_{\mathbf{z}_{\mathbf{x}}}$ satisfy $z_{\mathbf{y}} \in p_{\mathbf{y}}^{\pi}$ and $y_{\mathbf{z}} \in p_{\mathbf{z}}^{\pi}$. Moreover, the unique local point in $\mathcal{G}_{\mathbf{z}}$ induced by $p_{\mathbf{z}_{\mathbf{y}}}$ is equal to the local point $p_{\mathbf{z}}$.*

Proof: This lemma is proved using the results from section 3.5. As by assumption $y_{\mathbf{x}}, z_{\mathbf{x}} \in p_{\mathbf{x}}^{\pi}$, the local point $p_{\mathbf{x}} \in \mathcal{G}_{\mathbf{x}}$ gives rise to a point $p_{\mathbf{x}} \cap \mathbf{y}^{\perp} = p_{\mathbf{y}_{\mathbf{x}}} = p_{\mathbf{x}_{\mathbf{y}}}$ of $\mathcal{G}_{\mathbf{y}_{\mathbf{x}}} = \mathcal{G}_{\mathbf{x}_{\mathbf{y}}}$ and to a point $p_{\mathbf{x}} \cap \mathbf{z}^{\perp} = p_{\mathbf{z}_{\mathbf{x}}} = p_{\mathbf{x}_{\mathbf{z}}}$ of $\mathcal{G}_{\mathbf{z}_{\mathbf{x}}} = \mathcal{G}_{\mathbf{x}_{\mathbf{z}}}$. Consider the unique local point $p_{\mathbf{y}}$ of $\mathcal{G}_{\mathbf{y}}$ which contains the point $p_{\mathbf{y}_{\mathbf{x}}}$ and the unique local point $p_{\mathbf{z}}$ of $\mathcal{G}_{\mathbf{z}}$ which contains the point $p_{\mathbf{z}_{\mathbf{x}}}$. Since $y_{\mathbf{x}}^{\pi} \cap z_{\mathbf{x}}^{\pi}$ is a regular subspace of dimension at least $n-4$ incident to the point $p_{\mathbf{x}}$, it also contains two hyperbolic lines $g_{\mathbf{x}}^{1}$ and $g_{\mathbf{x}}^{2}$, which are elements of $p_{\mathbf{x}}$. By construction the vertices $\mathbf{g}^{1}$, $\mathbf{g}^{2}$ belong to unique interior lines of the local points $p_{\mathbf{x}}$, $p_{\mathbf{y}}$, $p_{\mathbf{z}}$, $p_{\mathbf{x}_{\mathbf{y}}} = p_{\mathbf{y}_{\mathbf{x}}}$, $p_{\mathbf{x}_{\mathbf{z}}} = p_{\mathbf{z}_{\mathbf{x}}}$, $p_{\mathbf{y}_{\mathbf{z}}} = p_{\mathbf{z}_{\mathbf{y}}}$. Hence $z_{\mathbf{y}} \in p_{\mathbf{y}}^{\pi}$ and $y_{\mathbf{z}} \in p_{\mathbf{z}}^{\pi}$ and, by partial linearity, cf. proposition 3.4.6, the unique local point in $\mathcal{G}_{\mathbf{z}}$ induced by $p_{\mathbf{z}_{\mathbf{y}}}$ is equal to the local point $p_{\mathbf{z}}$. ■

For the construction of global points let $p_{\mathbf{x}}$ be a local singular point in the interior space $\mathcal{G}_{\mathbf{x}}$ for some vertex $\mathbf{x}$ of the graph Γ and consider the set of vertices $p_{0} = p_{\mathbf{x}} \cup \bigcup_{\mathbf{h} \in L(p_{\mathbf{x}}^{\pi})} \{p_{\mathbf{h}} \mid p_{\mathbf{x}_{\mathbf{h}}} \subseteq p_{\mathbf{h}}\}$ in Γ. Furthermore we define inductively the set of vertices $p_i = \bigcup_{p_{\mathbf{h}} \in p_{i-1}} \left(\bigcup_{\mathbf{k} \in \mathbf{L}(p_{\mathbf{h}}^{\pi})} \{ p_{\mathbf{k}} \mid p_{\mathbf{h}_{\mathbf{k}}} \subseteq p_{\mathbf{k}} \} \right)$ for $i \in \mathbb{N}$. Certainly $p_0 \subseteq p_1$ using the fact that for each local point $p_{\mathbf{h}}$ of p_0, which is different from the local point p_x, the local hyperbolic line $x_{\mathbf{h}}$ is an element of $\mathbf{L}(p_{\mathbf{h}}^{\pi})$. Thus $p_{\mathbf{x}} \subseteq p_1$. Moreover, note that $p_{\mathbf{x}} \subseteq p_0$, so by construction $\bigcup_{\mathbf{h} \in \mathbf{L}(p_{\mathbf{x}}^{\pi})} \{ p_{\mathbf{h}} \mid p_{\mathbf{x}_{\mathbf{h}}} \subseteq p_{\mathbf{h}} \} \subseteq p_1$.

Suppose there exists a vertex $\mathbf{k}$ in $p_1 \backslash p_0$. Then again by construction of the set p_1 there is a path $\mathbf{x} \perp \mathbf{y} \perp \mathbf{w} \perp \mathbf{k}$ in Γ from $\mathbf{x}$ to $\mathbf{k}$ such that $y_{\mathbf{x}}$ is a hyperbolic line contained in $p_{\mathbf{x}}^{\pi}$ and $w_{\mathbf{y}}$ is a hyperbolic line in the subspace $p_{\mathbf{y}}^{\pi}$ and $k_{\mathbf{w}}$ is a hyperbolic line going through the local point $p_{\mathbf{w}}$. Without loss of generality we may assume that $w_{\mathbf{y}}$ is a hyperbolic line of the subspace $p_{\mathbf{y}}^{\pi}$ which is not contained in $x_{\mathbf{y}}^{\pi}$ and that $k_{\mathbf{w}}$ is a hyperbolic line of the local point $p_{\mathbf{w}}$ but not of the local point $p_{\mathbf{y}_{\mathbf{w}}}$, as otherwise $\mathbf{k}$ is a vertex of p_0. Because of these assumptions $k_{\mathbf{w}}$ is not a hyperbolic line of the orthogonal subspace $y_{\mathbf{w}}^{\pi}$, but intersects the $y_{\mathbf{w}}^{\pi}$ in the singular point $p_{\mathbf{w}}$. We conclude that $\langle k_{\mathbf{w}}, y_{\mathbf{w}} \rangle$ is a four-dimensional non-degenerate space.

Due to the properties of Γ discussed in the series of lemmata in the beginning of this section there exists a vertex $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}, \mathbf{k}\}^{\perp}$ and a path $\mathbf{w} \perp \mathbf{c}^1 \perp \cdots \perp \mathbf{c}^n \perp \mathbf{z}$ in $\{\mathbf{y}, \mathbf{k}\}^{\perp}$. Since the local hyperbolic line $c_{\mathbf{w}}^1$ is incident to the subspace $k_{\mathbf{w}}^{\pi}$, it is also contained in $p_{\mathbf{w}}^{\pi}$, whence there is a local point $p_{\mathbf{c}^1} \supseteq p_{\mathbf{w}_{\mathbf{c}^1}}$ containing the local hyperbolic line $k_{\mathbf{c}_1}$. By lemma 3.6.10 we have $p_{\mathbf{y}_{\mathbf{c}^1}} \subseteq p_{\mathbf{c}^1}$ and $c_{\mathbf{y}}^1 \in p_{\mathbf{y}}^{\pi}$. Repeating this argument along the path $\mathbf{w} \perp \mathbf{c}^1 \perp \ldots \perp \mathbf{c}^n \perp \mathbf{z}$, we end up with $p_{\mathbf{y}_{\mathbf{z}}} \subseteq p_{\mathbf{z}}$, that $k_{\mathbf{z}}$ is a hyperbolic line of the local point $p_{\mathbf{z}}$ and also that $z_{\mathbf{y}}$ is contained in the subspace $p_{\mathbf{y}}^{\pi}$. This implies that the hyperbolic line $z_{\mathbf{y}}$ is incident to the subspace $p_{\mathbf{x}_{\mathbf{y}}}^{\pi}$, and, thus, a subspace of $p_{\mathbf{x}}^{\pi}$, in particular $p_{\mathbf{x}_{\mathbf{z}}} \subseteq p_{\mathbf{z}}$. Whence the global line $\mathbf{k}$ is an element of the vertex set p_0, implying that $p_0 = p_i$ for each $i \in \mathbb{N}$. This construction leads to a well-behaved set of vertices $p := p_0 = p_{\mathbf{x}} \cup \bigcup_{\mathbf{l} \in \mathbf{L}(p_{\mathbf{x}}^{\pi})} \{ p_{\mathbf{l}} : p_{\mathbf{x}_{\mathbf{l}}} \subseteq p_{\mathbf{l}} \}$ such that the local singular point $p_{\mathbf{x}} \subseteq p$.

Definition 3.6.11 A **global point** p of Γ equals $p_{\mathbf{x}} \cup \bigcup_{\mathbf{l} \in \mathbf{L}(p_{\mathbf{x}}^{\pi})} \{ p_{\mathbf{l}} \mid p_{\mathbf{x}_{\mathbf{l}}} \subseteq p_{\mathbf{l}} \}$ for some vertex $\mathbf{x} \in \Gamma$ and some local singular point $p_{\mathbf{x}}$ of the interior space $\mathcal{G}_{\mathbf{x}}$. The set of all global points of Γ is denoted by $\mathcal{P}_{\Gamma}$.

Notice that the definition of a global point p does not depend on the starting local point $p_{\mathbf{x}} \subseteq p$ because $p = p_0 = p_i$ for all $i \in \mathbb{N}$. The next property follows directly from the construction of a global point p.

Proposition 3.6.12 *Let p be a global point and $\mathbf{x}$ be vertex of Γ. Then $p \cap \mathcal{L}_{\mathbf{x}}$ is either empty or a local singular point of $\mathcal{G}_{\mathbf{x}}$.*

The pair $\mathcal{G}_{\Gamma} = (\mathcal{P}_{\Gamma}, \mathcal{L}_{\Gamma})$ with symmetrised inclusion as incidence is a point-line geometry called the **global space** on Γ.

Lemma 3.6.13 *The point-line geometry $\mathcal{G}_\Gamma$ is a connected partially linear space.*

Proof: Let p and d be two different global points of $\mathcal{P}_\Gamma$ and suppose the vertex set $p \cap d$ contains two distinct vertices $\mathbf{x}$ and $\mathbf{y}$. Since the graph Γ has diameter two by proposition 3.6.9, there exists a vertex $\mathbf{z}$ in the induced subgraph $\{\mathbf{x},\mathbf{y}\}^\perp$. It follows that the two different local points $p_\mathbf{z} = p \cap \mathcal{L}_\mathbf{z}$ and $d_\mathbf{z} = d \cap \mathcal{L}_\mathbf{z}$ are incident to the two local lines $x_\mathbf{z}$ and $y_\mathbf{z}$ in $\mathcal{G}_\mathbf{z}$, thus $p_\mathbf{z} = p_\mathbf{z}$ by lemma 3.4.7, whence $p = d$. Hence $\mathcal{G}_\Gamma$ is partially linear.

In order to prove connectedness of $\mathcal{G}_\Gamma$ let again p and d be two different global points. Choose $\mathbf{l} \in p$ and $\mathbf{m} \in d$. Using once again that the diameter of Γ is two, there is a vertex $\mathbf{k} \in \{\mathbf{m},\mathbf{l}\}^\perp$. The interior space $G_\mathbf{k}$ contains the interior points $p_\mathbf{k} = p \cap \mathcal{L}_\mathbf{k}$ and $d_\mathbf{k} = d \cap \mathcal{L}_\mathbf{k}$. Hence connectedness of $\mathcal{G}_\Gamma$ follows from the connectedness of $\mathcal{G}_\mathbf{k}$, cf. lemma 3.4.6. ∎

We intend to use theorem 3.4.1 to identify the geometry $\mathcal{G}_\Gamma$. Therefore we need to define and study planes of $\mathcal{G}_\Gamma$.

Definition 3.6.14 Two global lines $\mathbf{k}$ and $\mathbf{l}$ are defined to span a **global plane** $\langle \mathbf{k},\mathbf{l}\rangle_g$ with respect to $\mathbf{z} \in \{\mathbf{k},\mathbf{l}\}^\perp$, if $\langle k_\mathbf{z}, l_\mathbf{z}\rangle^g_{\mathcal{G}_\mathbf{z}}$ is a local geometric plane of $\mathcal{G}_\mathbf{z}$. The global plane $\langle \mathbf{k},\mathbf{l}\rangle_g$ consists of all global lines $\mathbf{m}$ such that $\mathbf{m} \in \mathbf{z}^\perp$ and $m_\mathbf{z}$ is an interior line of the local geometric plane $\langle x_\mathbf{z}, y_\mathbf{z}\rangle^g_{\mathcal{G}_\mathbf{z}}$ and contains all global points p with the property that $p_\mathbf{z} = p \cap \mathcal{L}_\mathbf{z}$ is an interior singular point of the local geometric plane $\langle x_\mathbf{z}, y_\mathbf{z}\rangle^g_{\mathcal{G}_\mathbf{z}}$.

The next step is to prove that the definition of a global plane is independent of the vertex $\mathbf{z}$ used in the definition. To this end let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$ be vertices of Γ such that $\mathbf{z}\perp\mathbf{x}\perp\mathbf{w}\perp\mathbf{y}\perp\mathbf{z}$. Since $x_\mathbf{z}$ and $y_\mathbf{z}$ are interior lines of the space $\mathcal{G}_\mathbf{z}$, the span of $x_\mathbf{z}$ and $y_\mathbf{z}$ is either a three-dimensional or a four-dimensional subspace in $\mathcal{G}_\mathbf{z}$. We want to prove that $\mathbf{x}$ and $\mathbf{y}$ span a global plane with respect to $\mathbf{z}$ if and only if they span a global plane with respect to $\mathbf{w}$. In view of lemma 3.6.2 it suffices to show that $\mathbf{w}$ and $\mathbf{z}$ can be connected via a path in $\{\mathbf{x},\mathbf{y}\}^\perp$.

Lemma 3.6.15 *Let $n \geq 7$ and let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$ be four vertices of Γ satisfying $\mathbf{z}\perp\mathbf{x}\perp\mathbf{w}\perp\mathbf{y}\perp\mathbf{z}$. If $\dim(\langle x_\mathbf{z}, y_\mathbf{z}\rangle) = 3$, then there exists a path from $\mathbf{z}$ to $\mathbf{w}$ in $\{\mathbf{x},\mathbf{y}\}^\perp$. In particular, the global lines $\mathbf{x}$ and $\mathbf{y}$ span a global plane with respect to $\mathbf{z}$ if and only if they span a global plane with respect to $\mathbf{w}$ and those two global planes are equal.*

Proof: The subspace $H^\mathbf{z}_{x_\mathbf{z},y_\mathbf{z}} = x_\mathbf{z}^\pi \cap y_\mathbf{z}^\pi$ has dimension $n-3$ and rank at least $n-4$ implying that the subspace $W_\mathbf{y} := \langle H^\mathbf{y}_{x_\mathbf{z},y_\mathbf{z}}, z_\mathbf{y}\rangle$ has dimension $n-1$ and rank at least $n-2$. This setting satisfies the assumption of lemma 3.6.3, if $n \geq 8$, and the assumption of lemma 3.6.4, if $n = 7$. Thus the graph Γ contains a vertex $\mathbf{h} \in \{\mathbf{x},\mathbf{y},\mathbf{w}\}^\perp$ in the same connected component of $\{\mathbf{x},\mathbf{y}\}^\perp$ as the vertex $\mathbf{z}$. The claim follows now from lemma 3.6.2. ∎

Proposition 3.6.16 *Any global plane of $\mathcal{G}_\Gamma$ is finite and isomorphic to a linear plane or a symplectic plane.*

Proof: Let E_g be a global plane of $\mathcal{G}_\Gamma$, thus $E_g = \langle \mathbf{x}, \mathbf{y} \rangle_g$ for some global line $\mathbf{x}, \mathbf{y}$ of Γ. By definition 3.6.14, the global plane E_g consists of all global lines $\mathbf{m}$ and all global points p such that the interior lines $m_\mathbf{z}$ and the interior points $p_\mathbf{z} = p \cap \mathcal{L}_\mathbf{z}$ are incident to the geometric plane $\langle x_\mathbf{z}, y_\mathbf{z} \rangle^g_{\mathcal{G}_\mathbf{z}}$ for some $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}^\perp$. We use lemma 3.4.9, lemma 3.4.16 and the facts that for each vertex $\mathbf{w}$ of Γ the interior space $\mathcal{G}_\mathbf{w}$ is isomorphic to a subspace of $\mathcal{G}_\Gamma$, in particular for each $\mathbf{z} \in \{\mathbf{x}, \mathbf{y}\}^\perp$, and that the global plane $E_g = \langle \mathbf{x}, \mathbf{y} \rangle_g$ is isomorphic to the local geometric plane $\langle x_\mathbf{z}, y_\mathbf{z} \rangle^g_{\mathcal{G}_\mathbf{z}}$, to determine that E_g is a finite plane of order q of the geometry $\mathcal{G}_\Gamma$. The complete statement follows now from corollary 3.4.9 and lemma 3.4.16. ∎

Corollary 3.6.17 *The point-line geometry $\mathcal{G}_\Gamma$ is a non-linear space, thus the geometry $\mathcal{G}_\Gamma$ contains two distinct global points not incident to a common global line.*

Proof: Let $\mathbf{z}$ be a vertex of Γ. Then the interior space $\mathcal{G}_\mathbf{z}$ contains some local geometric plane isomorphic to a symplectic plane, which is a non-linear subspace of $\mathcal{G}_\mathbf{z}$. It follows that the geometry $\mathcal{G}$ yields some global plane, which is isomorphic to a symplectic plane by proposition 3.6.16. ∎

Lemma 3.6.18 *The point-line geometry $\mathcal{G}_\Gamma = (\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ is a planar space, any two distinct intersecting global lines are contained in a unique plane.*

Proof: Let $\mathbf{k}$ and $\mathbf{l}$ be two global lines contained in the global planes P_g and E_g. By definition 3.6.14 we obtain that $P_g = \langle \mathbf{m}, \mathbf{n} \rangle_g = \langle m_\mathbf{z}, n_\mathbf{z} \rangle^g_{\mathcal{G}_\mathbf{z}}$ for some vertices $\mathbf{m}, \mathbf{z}, \mathbf{n}$ satisfying $\mathbf{m} \perp \mathbf{z} \perp \mathbf{n}$ and that $E_g = \langle \mathbf{s}, \mathbf{t} \rangle_g = \langle t_\mathbf{x}, t_\mathbf{x} \rangle^g_{\mathcal{G}_\mathbf{x}}$ for some vertices $\mathbf{s}, \mathbf{x}, \mathbf{t}$ of Γ such that $\mathbf{s} \perp \mathbf{x} \perp \mathbf{t}$. As the global line $\mathbf{k}$ and $\mathbf{l}$ are elements of P_g as well as elements of E_g it follows that $k_\mathbf{z}$ and $l_\mathbf{z}$ are two different interior lines of the geometric plane $\langle m_\mathbf{z}, n_\mathbf{z} \rangle^g_{\mathcal{G}_\mathbf{z}}$, thus $\langle k_\mathbf{z}, l_\mathbf{z} \rangle^g_{\mathcal{G}_\mathbf{z}} = \langle m_\mathbf{z}, n_\mathbf{z} \rangle^g_{\mathcal{G}_\mathbf{z}}$. Also $k_\mathbf{x}$ and $l_\mathbf{x}$ are different interior lines of $\langle s_\mathbf{x}, t_\mathbf{x} \rangle^g_{\mathcal{G}_\mathbf{x}}$ implying $\langle k_\mathbf{x}, l_\mathbf{x} \rangle^g_{\mathcal{G}_\mathbf{x}} = \langle s_\mathbf{x}, t_\mathbf{z} \rangle^g_{\mathcal{G}_\mathbf{x}}$. We conclude that $P_g = \langle \mathbf{m}, \mathbf{n} \rangle_g = \langle m_\mathbf{z}, n_\mathbf{z} \rangle^g_{\mathcal{G}_\mathbf{z}} = \langle k_\mathbf{z}, l_\mathbf{z} \rangle^g_{\mathcal{G}_\mathbf{z}} = \langle \mathbf{k}, \mathbf{l} \rangle_g = \langle k_\mathbf{x}, l_\mathbf{x} \rangle^g_{\mathcal{G}_\mathbf{x}} = \langle s_\mathbf{x}, t_\mathbf{z} \rangle^g_{\mathcal{G}_\mathbf{x}} = \langle \mathbf{s}, \mathbf{t} \rangle_g = E_g$. ∎

We will need the following notation for the last part of this section.

Definition 3.6.19 Let p and d be two global points of $\mathcal{G}_\Gamma$. We say that p is **orthogonal** to d, in symbols $p \perp d$, if there is a global line $\mathbf{k}$ of p and a global line $\mathbf{m}$ of d satisfying $\mathbf{k} \perp \mathbf{m}$. We denote all global points orthogonal to a global point p by $\mathcal{P}^p$ and we define $p^\sim = \mathcal{P}^p \cup p$.

Note that the point set $p^\sim$ contains all global points of $\mathcal{G}_\Gamma$ not collinear to the global point p.

Lemma 3.6.20 *Let p and d be distinct global points of $\mathcal{G}_\Gamma$. Then $p^\sim \notin d^\sim$.*

Proof: Let $\mathbf{l}$ be global line of p and $\mathbf{m}$ be an element of d. By proposition 3.6.9 there exists a vertex $\mathbf{z} \in \{\mathbf{l}, \mathbf{m}\}^\perp$. Since $p \cap \mathcal{L}_\mathbf{z} = p_\mathbf{z}$ is a local point of $\mathcal{G}_\mathbf{z}$ distinct from the local point $d \cap \mathcal{L}_\mathbf{z} = d_\mathbf{z}$, by lemma 3.4.18 we obtain that $p_\mathbf{z}^\sim \notin d_\mathbf{z}^\sim$. Since $\mathcal{G}_\mathbf{z}$ is isomorphic to a subspace of $\mathcal{G}_\Gamma$, the unique global point b containing the local point $b_\mathbf{z} \in p_\mathbf{z}^\sim$ is an element of $p^\sim$. This implies $p^\sim \notin d^\sim$. ∎

Lemma 3.6.21 *Let E_g be a linear global plane and let x be a global point. Then E_g and $x^\sim$ have a global point in common, so $E_g \cap x^\sim \neq \emptyset$.*

Proof: If x is incident to E_g, then the property that $x \in x^\sim$ implies $x^\sim \cap E_g \neq \emptyset$. Hence we consider the setup that x is not contained in the plane E_g. The plane E_g is by definition spanned by two different intersecting global lines $\mathbf{k}$ and $\mathbf{l}$, i.e., $E_g = \langle \mathbf{k}, \mathbf{l} \rangle_g$. Let $\mathbf{h}$ be a global line of the x, proposition 3.6.9 implies the existence of vertices $\mathbf{m}$, $\mathbf{n}$ and $\mathbf{z}$ such that $\mathbf{m} \in \{\mathbf{k}, \mathbf{h}\}^\perp$, of $\mathbf{n} \in \{\mathbf{h}, \mathbf{l}\}^\perp$, and of $\mathbf{z} \in \{\mathbf{l}, \mathbf{k}\}^\perp$. In the interior space $G_\mathbf{m}$, the subspace $k_\mathbf{m} \cap x_\mathbf{m}^\pi$ is of dimension at least one, so there local point $i_\mathbf{m}$ in the intersection $k_\mathbf{m} \cap x_\mathbf{m}^\pi$. We remark here that the interior point $i_\mathbf{m}$ is not necessarily singular. If the local point $i_\mathbf{m}$ is indeed singular, then $i_\mathbf{m} \in x_\mathbf{m}^\sim$, and therefore $i \in x^\sim \cap \mathbf{k} \subseteq x^\sim \cap E_\Gamma^\mathbf{g}$, where i is the unique global point containing $i_\mathbf{m}$, and we are done. Alternatively, we consider in the interior space $G_\mathbf{n}$ a local point $j_\mathbf{n}$ incident to the subspace $l_\mathbf{n} \cap x_\mathbf{n}^\pi$. Again, if $j_\mathbf{n}$ is a singular interior point, we are done.

Hence we may assume that both subspaces $l_\mathbf{n} \cap x_\mathbf{n}^\pi = j_\mathbf{n}$ and $k_\mathbf{m} \cap x_\mathbf{m}^\pi = i_\mathbf{m}$ are regular interior points. By definition of a global plane, the global lines $\mathbf{k}$ and $\mathbf{l}$ intersect in a global point p, so $p_\mathbf{m} = p \cap \mathcal{L}_\mathbf{m}$ is a singular interior point in $\mathcal{G}_\mathbf{m}$ as well as the interior point $p_\mathbf{n} = p \cap \mathcal{L}_\mathbf{n}$ of the space $\mathcal{G}_\mathbf{n}$ is singular. We may assume that neither $p_\mathbf{m}$ is incident to $x_\mathbf{m}^\pi$ nor $p_\mathbf{n}$ is incident to $x_\mathbf{n}^\pi$, as otherwise there is nothing to prove. It follows that the interior singular points $x_\mathbf{m}$ and $p_\mathbf{m}$ span an interior line $g_\mathbf{m}$, which corresponds to a vertex $\mathbf{g} \in \Gamma$. Moreover, as $\mathbf{g}$ and $\mathbf{k}$ intersect in the global point p, the lines $\mathbf{k}$ and $\mathbf{g}$ span the global plane $P_g = \langle \mathbf{k}, \mathbf{g} \rangle_g \subseteq \mathcal{G}_\mathbf{m}$. By construction of the interior line $g_\mathbf{m}$, the span $\langle k_\mathbf{m}, g_\mathbf{m} \rangle$ is a regular three-dimensional subspace of $\mathcal{G}_\mathbf{m}$, so $P_{\mathcal{G}_\mathbf{m}}^g = \langle k_\mathbf{m}, g_\mathbf{m} \rangle_{\mathcal{G}_\mathbf{m}}^g$ is a linear geometric plane.

Next we consider the path $\mathbf{g} \perp \mathbf{m} \perp \mathbf{k} \perp \mathbf{z} \perp \mathbf{l}$ between the vertices $\mathbf{g}$ and $\mathbf{l}$ in Γ. By assumption the global plane E_g is linear, thus $H_{k_\mathbf{z}, l_\mathbf{z}}^\mathbf{z} = k_\mathbf{z}^\pi \cap l_\mathbf{z}^\pi$ is an $(n - 3)$-dimensional non-degenerate subspace in $\mathcal{G}_\mathbf{z}$. As P_g is also a linear plane the subspace $H_{k_\mathbf{m}, g_\mathbf{m}}^\mathbf{m} = k_\mathbf{m}^\pi \cap g_\mathbf{m}^\pi$ of $\mathcal{G}_\mathbf{m}$ is regular and of dimension $n - 3$.

We will analyse the unique induced subspace $H_{k_\mathbf{z}, l_\mathbf{z}}^\mathbf{k}$ and $H_{k_\mathbf{m}, g_\mathbf{m}}^\mathbf{k}$ inside $\mathcal{G}_\mathbf{m}$ and claim the existence of a vertex $\mathbf{t} \in \{\mathbf{k}, \mathbf{l}, \mathbf{g}, \mathbf{z}\}^\perp$. Since $V_\mathbf{k} := \langle m_\mathbf{k}, H_{k_\mathbf{m}, g_\mathbf{m}}^\mathbf{k} \rangle$ is a regular $(n-1)$-dimensional subspace we obtain that $W_\mathbf{k} := V_\mathbf{k} \cap z_\mathbf{k}^\pi$ is of dimension at least $(n-3)$. Since the subspace $H_{k_\mathbf{z}, l_\mathbf{z}}^\mathbf{k} \subseteq z_\mathbf{k}^\pi$ is $(n-3)$-dimensional and non-degenerate, the intersection $W_\mathbf{k} \cap H_{k_\mathbf{z}, l_\mathbf{z}}^\mathbf{k}$ is at least $(n-4)$-dimensional of rank at least $n - 5 \geq 2$.

Therefore there is an interior line $t_{\mathbf{k}}$ in $W_{\mathbf{k}} \cap H^{\mathbf{k}}_{k_{\mathbf{z}},l_{\mathbf{z}}}$. This interior line $t_{\mathbf{k}}$ corresponds to a vertex $\mathbf{t} \in \{\mathbf{k},\mathbf{l},\mathbf{g},\mathbf{z}\}^{\perp}$, as claimed.

In the interior space $\mathcal{G}_{\mathbf{t}}$ the interior lines $k_{\mathbf{t}}$ and $l_{\mathbf{t}}$ span the linear geometric plane $E^{g}_{\mathcal{G}_{\mathbf{t}}} = \langle k_{\mathbf{t}}, l_{\mathbf{t}} \rangle^{g}_{\mathcal{G}_{\mathbf{t}}}$. Since $\mathbf{g}$ is a vertex of the set $x \cap \mathcal{L}_{\mathbf{t}}$, the intersection $x_{\mathbf{t}} = x \cap \mathcal{L}_{\mathbf{t}}$ is an interior singular point of $\mathcal{G}_{\mathbf{t}}$. Therefore the $(n-1)$-dimensional subspace $x^{\pi}_{\mathbf{t}}$ intersects the graphical plane $E_{\mathbf{G}(\mathcal{G}_{\mathbf{t}})}$ at least in a two-dimensional subspace, which contains an interior singular point $s_{\mathbf{t}}$. Certainly $s_{\mathbf{t}} \in x^{\sim}_{\mathbf{t}}$, which implies $s \in x^{\sim} \cap E_g$, where s is the unique global point containing $s_{\mathbf{t}}$. The claim is proved. ∎

We have now reached our goal.

Proposition 3.6.22 *The point-line geometry $\mathcal{G}_\Gamma = (\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ is isomorphic to the geometry of singular points and hyperbolic lines of an m-dimensional non-degenerate unitary polar space over $\mathbb{F}_{q^2}$.*

Proof: By lemma 3.6.13 the geometry $\mathcal{G}_\Gamma$ is a connected partially linear space. By corollary 3.6.17 it is non-linear and by lemma 3.6.18 planar. Since for every vertex $\mathbf{x}$ of Γ the interior space $\mathcal{G}_{\mathbf{x}}$ is of order q using that $\mathcal{G}_{\mathbf{z}} \cong \mathbb{H}(U_n)$ it follows by lemma 3.6.12 and the property that the geometry $\mathcal{G}_{\mathbf{z}}$ is isomorphic to a subspace of $\mathcal{G}_\Gamma$ that the space $\mathcal{G}_\Gamma$ has order q. By lemma 3.6.16, the space $\mathcal{G}_\Gamma$ satisfies hypothesis 1 of theorem 3.4.1. The validity of hypothesis 2 has been discussed in proposition 3.4.20, hypothesis 3 follows from lemma 3.6.20, hypothesis 4 from lemma 3.6.21. Hence by theorem 3.4.1 the geometry $\mathcal{G}_\Gamma$ is isomorphic to the geometry of singular points and hyperbolic lines of a non-degenerate symplectic or unitary polar space over the field $\mathbb{F}_q$ respectively $\mathbb{F}_{q^2}$. Since $\mathcal{G}_\Gamma$ contains linear planes, it is isomorphic to the geometry of hyperbolic lines of some non-degenerate unitary polar space over the field $\mathbb{F}_{q^2}$. ∎

Corollary 3.6.23 *The graph Γ is isomorphic to the hyperbolic line graph of an m-dimensional non-degenerate unitary vector space over the field $\mathbb{F}_{q^2}$.*

Theorem 3.1.2 *Let $n \geq 7$, let $q \geq 3$ be a prime power, and let Γ be a connected graph that is locally $\mathbf{G}(U_n)$. Then Γ is isomorphic to $\mathbf{G}(U_{n+2})$.*

Proof: By corollary 3.6.23, we have $\Gamma \cong \mathbf{G}(U_m)$ for some $m \in \mathbb{N}$. Since the connected graph $\mathbf{G}(U_m)$ is locally $\mathbf{G}(U_n)$ if and only if $m = n+2$, cf. proposition 3.3.3, necessarily $\Gamma \cong \mathbf{G}(U_{n+2})$ ∎

Theorem 3.1.3 *Let $n \geq 7$ and let q be an odd prime power. Let G be a group with subgroups A and B isomorphic to $\mathrm{SU}_2(\mathbb{F}_{q^2})$, and denote the central involution of A by x and the central involution of B by y. Furthermore, assume the following holds:*

- $C_G(x) = X \times K$ *with* $K \cong \mathrm{GU}_n(\mathbb{F}_{q^2})$ *and* $A \leq X$;

- *$C_G(y) = Y \times J$ with $J \cong \mathrm{GU}_n(\mathbb{F}_{q^2})$ and $Y \leq B$;*
- *A is a fundamental $\mathrm{SU}_2(\mathbb{F}_{q^2})$ subgroup of J;*
- *B is a fundamental $\mathrm{SU}_2(\mathbb{F}_{q^2})$ subgroup of K;*
- *there exists an involution in $J \cap K$ that is the central involution of a fundamental $\mathrm{SU}_2(\mathbb{F}_{q^2})$ subgroup of both J and K.*

If $G = \langle J, K \rangle$, then $\mathrm{PSU}_{n+2}(\mathbb{F}_{q^2}) \leq G/Z(G) \leq \mathrm{PGU}_{n+2}(\mathbb{F}_{q^2})$.

Proof: [22, section 6] and [35] provide a standard method how to derive the claim from theorem 3.1.2. ∎

CHAPTER FOUR

On locally complex unitary geometries

4.1 Local recognition of the line graph of complex unitary space for $n = 6$

The compact Lie groups $G = G(E_6) = {}^2E_6(\mathbb{C}) := E_{6,-78}$ resp. $G = G(A_7) = \mathrm{SU}_8(\mathbb{C})$ of type E_6 and A_7 allow to define graphs with vertex set $\{G_{\alpha,-\alpha} \mid \alpha$ a root with respect to a maximal torus of$G\}$ in which two vertices $G_{\alpha,-\alpha}$ and $G_{\beta,-\beta}$ are adjacent if and only if $[G_{\alpha,-\alpha}, G_{\beta,-\beta}] = 1$. Both of these commuting fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroup graphs are locally isomorphic to the commuting fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroup graph of the group $\mathrm{SU}_6(\mathbb{C})$, as can be read off the extended Dynkin diagrams of type $\tilde{A}_7$ and $\tilde{E}_6$.

In this chapter we will study abstract connected graphs that are locally isomorphic to the commuting fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroup graph of the group $\mathrm{SU}_6(\mathbb{C})$, which is isomorphic to the line graph $\mathbf{S}(V_6)$, satisfying that

$$|\{\mathbf{z},\mathbf{w}\}^\perp \cap \mathbf{x}^\perp| = 1 \text{ if and only if } |\{\mathbf{z},\mathbf{w}\}^\perp \cap \mathbf{y}^\perp| = 1$$

for any chain $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ in Γ of four different vertices $\mathbf{x}, \mathbf{w}, \mathbf{y}$ and $\mathbf{z}$, achieving the following results:

- each such graph contains an induced subgraph isomorphic to the commuting reflection graph of either the Weyl group of type A_7 or the Weyl group of type E_6,
- each such graph admits an abstract group of automorphisms isomorphic to the group $\mathrm{SU}_8(\mathbb{C})$ or to the group $E_{6,-78}(\mathbb{C})$,
- there exists a canonical topology turning the abstract groups of automorphisms

into compact Lie groups, by [28]
• each connected graph that is locally the commuting fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups graph of the group $\mathrm{SU}_6(\mathbb{C})$ satisfying that

$$|\{\mathbf{z},\mathbf{w}\}^\perp \cap \mathbf{x}^\perp| = 1 \text{ if and only if } |\{\mathbf{z},\mathbf{w}\}^\perp \cap \mathbf{y}^\perp| = 1$$

for any chain $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ in Γ of four different vertices $\mathbf{x}, \mathbf{w}, \mathbf{y}$ and $\mathbf{z}$, is isomorphic to the commuting fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups graph of either $\mathrm{SU}_8(\mathbb{C})$, denoted by $\mathbf{F}(SU_8(\mathbb{C})) \cong \mathbf{S}(V_8)$, or $E_{6,-78}$, denoted by $\mathbf{F}(E_{6,-78})$.

First we focus again on the line graph of an n-dimensional unitary vector space over $\mathbb{C}$. Therefore we recall the definition of $\mathbf{S}(V_n)$ for $n \in \mathbb{N}$, and state some results from chapter 2.

Definition 4.1.1 Let $n \in \mathbb{N}$, let $V = V_n$ be an n-dimensional vector space over the complex numbers and let $(\cdot,\cdot)$ be an anisotropic form (the scalar product or the negative of the scalar product) on $V \times V$. For a subspace $U \subseteq V$ the polar of U is $U^\pi = \{x \in V : (x,u) = 0 \text{ for all } u \in U\}$. The **line graph** $\mathbf{S}(V_n)$ of the complex vector space V_n is the graph on the two-dimensional subspaces of V_n, where two distinct lines l and k of V_n are adjacent (in symbols $k \perp l$) if and only if $l \subseteq k^\pi$ or, equivalently, if $k \subseteq l^\pi$.

For a vertex x of $\mathbf{S}(V_n)$, the neighbourhood graph $\mathbf{S}(V_n)_x = x^\perp$ is the subgraph induced by $\mathbf{S}(V_n)$ on the set of vertices $\{y \in \mathbf{S}(V_n) \mid x \perp y\}$. For a set of vertices X of $\mathbf{S}(V_n)$ the graph $X^\perp$ is defined as $\bigcap_{x \in X} x^\perp$.

Our main result is the classification of all connected locally $\mathbf{S}(V_6)$ graphs.

Theorem 4.1.2 *Let Γ be a connected locally $\mathbf{S}(V_6)$ graph satisfying that*

$$|\{\mathbf{z},\mathbf{w}\}^\perp \cap \mathbf{x}^\perp| = 1 \textit{ if and only if } |\{\mathbf{z},\mathbf{w}\}^\perp \cap \mathbf{y}^\perp| = 1$$

for any chain $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ in Γ of four different vertices $\mathbf{x}, \mathbf{w}, \mathbf{y}$ and $\mathbf{z}$. Then the universal cover $\widehat{\Gamma}$ of Γ is isomorphic to $\mathbf{S}(V_8)$ or to $\mathbf{F}(E_{6,-78})$.

Translating this statement into group theoretical language, we obtain the following characterization.

Theorem 4.1.3 *Let $n = 6$ and G be a group with subgroups A and B isomorphic to $\mathrm{SU}_2(\mathbb{C})$, and denote the central involution of A by x and the central involution of B by y. Moreover, we assume that the following is satisfied:*

- *$C_G(x) = X \times K$ with $K \cong \mathrm{GU}_n(\mathbb{C})$ and $A \leq X$;*

- $C_G(y) = Y \times J$ *with* $J \cong \mathrm{GU}_n(\mathbb{C})$ *and* $B \leq Y$;
- *A is a fundamental* $\mathrm{SU}_2(\mathbb{C})$ *subgroup of J;*
- *B is a fundamental* $\mathrm{SU}_2(\mathbb{C})$ *subgroup of K;*
- *the subgroup* $J \cap K$ *contains a central involution z of a fundamental* $\mathrm{SU}_2(\mathbb{C})$ *of both J and K.*

If $G = \langle J, K \rangle$ *then (up to isomorphism) either* $\mathrm{PSU}_{n+2}(\mathbb{C}) \leq G/Z(G) \leq \mathrm{PGU}_{n+2}(\mathbb{C})$ *or* $E_{6,-78}/Z(E_{6,-78}) \leq G/Z(G) \leq \mathrm{Aut}\,(E_{6,-78})/Z$.

Again this theorem 4.1.3 is directly derived from theorem 4.1.2 in a way similar to the proof of analogue in [22] and [38].

4.2 Basis systems and closed cycles in the line graph $\mathbf{S}(V_n)$

In this part let Γ be isomorphic to the line graph $\mathbf{S}(V_6)$. We state some important results from chapter 2 section 2.2 and 2.3. Recall also the notation from these sections, in particular $\mathcal{G}_\Gamma$ is the interior space of Γ.

Proposition 4.2.1 *Let* $n \geq 4$. *Then the graph* $\mathbf{S}(V_n)$ *is locally* $\mathbf{S}(V_{n-2})$.

Proof: This follows from 2.2.5 and 2.3.1. ∎

Proposition 4.2.2 *Let* $n \geq 5$. *Then the graph* $\mathbf{S}(V_n)$ *is connected. More precisely, the graph* $\mathbf{S}(V_n)$ *has diameter four, if* $n = 5$, *and diameter two, if* $n \geq 6$.

Proof: See 2.2.5 and 2.3.1. ∎

Proposition 4.2.3 *Let* $n \geq 5$ *and let* $\Gamma \cong \mathbf{S}(V_n)$. *Then reconstruction of the vector space* V_n *over* $\mathbb{C}$ *and the non-degenerate unitary form* $(\cdot,\cdot)$ *is possible from the graph* Γ.

Proof: See 2.2.14 and 2.3.7. ∎

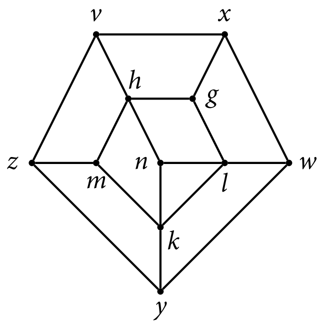
We consider closed cycles of vertices in the graph $\Gamma \cong \mathbf{S}(V_6)$ and claim that each closed cycle can be decomposed into triangles and quadrangles. Thus let γ be a closed path $y_0 \perp y_1 \perp \ldots \perp y_k \perp y_0$ in Γ. Using proposition 4.2.2 to observe that each closed path in Γ can be decomposed into pentagons, quadrangles and triangles. Therefore we will show how to decompose a pentagon in Γ into quadrangles and triangles.

Lemma 4.2.4 *Let x, w, y, z, v be five different vertices of Γ which form a closed path in the following way: $x \perp w \perp y \perp z \perp v \perp x$. Then the pentagon $x \perp w \perp y \perp z \perp v \perp x$ can be decomposed into triangles and quadrangles in Γ.*

Proof: Inside the space $\mathcal{G}_\Gamma$, let z_p be a point on the line z. Then $z_p^\pi \cap \langle z, x\rangle$ is a subspace of dimension two or three and contains a point x_p incident to the line x. We define the line h to be the span of the points x_p and z_p. In particular, the vertex h is adjacent to v as $h \subseteq \langle z, x\rangle \subseteq v^\pi$. Furthermore we fix a point y_p on y and a point w_p on the line w. Let l be the line spanned by y_p and x_p, so $l = \langle y_p, x_p\rangle \subseteq w^\pi$ and the line k to be the two-dimensional space $\langle z_p, w_p\rangle \subseteq y^\pi$, implying directly that $k \perp y \perp w \perp l$. By construction the point x_p is orthogonal to the point z_p and since $x \perp w$ the point w_p is a subspace of x_p^π, so k is contained in x_p^π. The lines w and z are incident to the subspace y^π, thus w and z are also contained in y_p^π. We conclude that the orthogonal space $l^\pi = y_p^\pi \cap x_p^\pi$ of the line l contains the line k, therefore $k \perp l$ in Γ.

Since $\langle h, k, l\rangle = \langle x_p, z_p, y_p, w_p\rangle = \langle k, l\rangle$ is a four-dimensional subspace of $\mathcal{G}_\Gamma$, the space $n = \langle h, k, l\rangle^\pi$ is a line of $\mathcal{G}_\Gamma$. As the spaces $\langle h, l, x\rangle = \langle x, y_p, z_p\rangle$ and $\langle h, k, z\rangle = \langle z, w_p, x_p\rangle$ are of dimension four, we have that $m = \langle h, k, z\rangle^\pi$ and $g = \langle h, l, x\rangle^\pi$ are two-dimensional subspaces of $\mathcal{G}_\Gamma$. Directly from the equations above we obtain that g is a vertex of $\{h, x, l\}^\perp$, the vertex m is an element of $\{h, k, z\}$ and the vertex n is adjacent to the vertices h, k and l. Thus the closed path $x \perp w \perp y \perp z \perp v \perp x$ is decomposed into the triangle $n \perp k \perp l \perp n$ and the quadrangles

$x \perp w \perp l \perp g \perp x$
$x \perp g \perp h \perp v \perp x$
$w \perp y \perp k \perp l \perp w$
$z \perp m \perp k \perp y \perp z$
$v \perp h \perp m \perp z \perp v$
$g \perp l \perp n \perp h \perp g$
$h \perp n \perp k \perp m \perp h$.



■

We will apply the following constructions and remarks very often in section 4.3.

Remark 4.2.5 Let x be a line in the complex vector space V_6 and $\alpha : v_1, v_2, v_3, v_4, v_5, v_6$ be an orthonormal basis of V_6 such that v_1 and v_2 are vectors of the line x and the vectors v_i with $3 \leq i \leq 6$ are contained in the subspace x^π. In particular the vectors v_1 and v_2 form a basis of the line x, which we denote with α^x and $\alpha^{x^\pi} : v_3, \ldots, v_6$ is a basis of the subspace x^π.

An x-$\mathrm{SU}_2(\mathbb{C})$-**action** on the vector space V_6 is the faithful action

$$x\text{-}\mathrm{SU}_2(\mathbb{C})\text{-}V_6 : \mathrm{SU}_2(\mathbb{C}) \times V_6 \to V_6 \text{ with } (\varphi, y) \mapsto \begin{pmatrix} [\varphi]_{\alpha^x} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} [y]_\alpha \,,$$

where $[\varphi]_{\alpha^x}$ is the matrix representation of the automorphism φ with respect to the basis α^x and $[y]_\alpha$ is the coordinate vector of $y \in V_6$ w.r.t. the basis α. We denote the matrix $\begin{pmatrix} [\varphi]_{\alpha^x} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$ with $[\varphi_{V_6}]_\alpha = [x - \varphi_{V_6}]_\alpha$ and the corresponding automorphism with $\varphi_{V_6} = x - \varphi_{V_6}$ for every $\varphi \in \mathrm{SU}_2(\mathbb{C})$ and any orthonormal basis α of V_6, which satisfies the conditions from above for the line x. Notice that φ_{V_6} is an element of $\mathrm{SU}_6(\mathbb{C})$ as $[\varphi_{V_6}]_\alpha$ is a unitary matrix with determinant one.

Let β be another basis of V_6 then we can determine the matrix $[\varphi_{V_6}]_\beta$ via basis transformation with the transformation matrix ${}_\beta[id_{V_6}]_\alpha$, thus $[\varphi_{V_6}]_\beta = {}_\beta[id_{V_6}]_\alpha \cdot [\varphi_{V_6}]_\alpha \cdot {}_\beta[id_{V_6}]_\alpha^{-1}$. This x-$\mathrm{SU}_2(\mathbb{C})$-$V_6$ action induces a **natural** x-$\mathrm{SU}_2(\mathbb{C})$ **action on the projective space** $\mathbb{P}(V_6)$ which is defined as follows

$$x\text{-}\mathrm{SU}_2(\mathbb{C})\text{-}\mathbb{P}(V_6) : \mathrm{SU}_2(\mathbb{C}) \times \mathbb{P}(V_6) \to \mathbb{P}(V_6)$$
$$(\varphi, p) \mapsto \langle x - \mathrm{SU}_2(\mathbb{C}) - V_6(\varphi, p_v)\rangle = \langle [\varphi_\delta[p_v]_\delta \rangle \,,$$

where p is a point of the projective space $\mathbb{P}(V_6)$, p_v is a non-zero vector of V_6 spanning the one-dimensional subspace p and δ is a basis of V_6. We denote the automorphism of $\mathbb{P}(V_6)$ induced by the vector space automorphism φ_{V_6} with $\varphi_{\mathbb{P}(V_6)}$.

Here is the main definition of this part.

Definition 4.2.6 Let Γ be a graph isomorphic to $\mathbf{S}(V_6)$ and let x be a vertex of the graph Γ. An x-$\mathrm{SU}_2(\mathbb{C})$ **action on the graph** Γ, denoted by x-$\mathrm{SU}_2(\mathbb{C})$-$\Gamma$, is the faithful action $\mathrm{SU}_2(\mathbb{C}) \to \mathrm{Aut}(\Gamma)$ induced by the natural x-$\mathrm{SU}_2(\mathbb{C})$-$\mathcal{G}_\Gamma$-action of $\mathcal{G}_\Gamma \cong \mathbb{P}(V_6)$ with respect to the decomposition $x \oplus x^\pi$ of $\mathcal{G}_\Gamma$ given by the vertex x.

In the last part of this section we will construct some special bases of the vector space V_6 under the use of a line x. This construction of some basis and the corresponding matrices with respect to these bases of the x-$\mathrm{SU}_2(\mathbb{C})$-$V_6$ action will play an important role in the next section.

Remark 4.2.7 Thus let k and l be two lines, which are orthogonal to each other so $k \in l^\pi$ and $l \in k^\pi$, in the complex vector space V_6. We fix an orthonormal basis $\alpha : w_{1_v}, \ldots, w_{6_v}$ of V_6 such that the following conditions are satisfied:

C1 the vectors w_{1_v} and w_{2_v} span the line k

C2 the vectors w_{3_v} and w_{4_v} span the line l

C3 $\langle w_{5_v}, w_{6_v}\rangle = k^\pi \cap l^\pi$.

In particular the vectors w_{1_v} and w_{2_v} form a basis α_k for the line k and $\alpha_l : w_{3_v}, w_{4_v}$ is a basis for l.

We regard the vectors w_i for $i = 1\ldots,6$ and $w_{ii+1} := w_i + w_{i+1}$ with $1 \le i \le 5$. Let $p_j = \langle w_j\rangle$ for the indices $j \in J = \{1,\ldots,6,12,23,34,45,56\}$ be the points spanned by the vectors w_j in V_6 then the points p_1 and p_2 as well as the points p_1 and p_{12} span the line k, $\langle p_3, p_4\rangle = l = \langle p_3, p_{34}\rangle$ and $\langle p_5, p_6\rangle = \langle p_5, p_{56}\rangle = k^\pi \cap l^\pi$.

Each point p_j, $j \in J$, contains infinitely many vectors of length one. Every vector of length one of the point p_j is an element of $\{e^{i\varphi}w_j : \varphi \in [0,2\pi[\}$ for $1 \le j \le 6$ and for $j \in \{12,23,34,45,56\}$ every vector of length one of the point p_j is of the form $\frac{1}{\sqrt{2}}e^{i\varphi}w_j$ in V_6. Thus let x be an element of $\{1,\ldots,6\}$ and u_x be an vector of length one contained in the point p_x. Then we determine a orthonormal basis $\alpha_{u_x} : u_1, u_2, u_3, u_4, u_5, u_6$ of V_6 and vectors $u_{12}, u_{23}, u_{34}, u_{45}, u_{56}$ of V_6 such that

B1 each vector u_j has length one for $j \in J$,

B2 u_j is a vector of the point p_j for $j \in J$,

B3 $(u_j, u_{jj+1}) = \frac{1}{\sqrt{2}}$ for $1 \le j \le 5$ and $(u_{j-1j}, u_j) = \frac{1}{\sqrt{2}}$ for $2 \le j \le 6$

We call the basis α_{u_x} together with the vectors $u_{12}, u_{23}, u_{34}, u_{45}, u_{56}$ of V the **basis system α_{u_x} with respect to the lines k and l and the vector u_x**, or shorter the **basis system α_{u_x} w.r.t. u_x**.

Proposition 4.2.8 *The basis system α_{u_x} is uniquely determined by the vector u_x.*

Proof: The vector u_x equals $e^{i\varphi}w_x$ for some fixed $\varphi \in [0,2\pi[$. For $x \le 5$ the vector u_{xx+1} is of the form $\frac{1}{\sqrt{2}}e^{i\sigma}w_{xx+1}$ with $\sigma \in [0,2\pi[$. We show that the value σ is uniquely determined by the vector u_x. Since $\frac{1}{\sqrt{2}} = (u_x, u_{xx+1}) = e^{i\varphi}\cdot\frac{1}{\sqrt{2}}\cdot e^{-i\sigma}(w_x, w_{xx+1}) = e^{i\varphi}\cdot\frac{1}{\sqrt{2}}\cdot e^{-i\sigma} = \frac{1}{\sqrt{2}}\cdot e^{i(\varphi-\sigma)} = \frac{1}{\sqrt{2}}(\cos\varphi - \sigma + i\sin\varphi - \sigma)$ implying the equations $\cos\varphi - \sigma = 1$ and $\sin\varphi - \sigma = 0$. The equation $\cos\varphi - \sigma = 1$ indicates that either $\varphi = \sigma$ or $\varphi = 2\pi + \sigma$ and the second equation $\sin\varphi - \sigma = 0$ implies that $\varphi = \sigma$ or $\varphi = \pi + \sigma$, indeed the only possible solution is $\varphi = \sigma$, which proves that the vector u_x uniquely determines the vector u_{xx+1}. With similar arguments and calculation we obtain that the vector u_{xx+1} fixes the vector u_{x+1} and so on. Also this argumentation implies that the vector u_x fixes u_{x-1x} for $x \ge 2$. Indeed the basis system α_{u_x} is uniquely determined by the vector u_x, which proves our claim. ∎

For the next step fix an index $j \in \{1, \dots, 6\}$ and let α_{u_j} and α_{z_j} be two different basis systems of V_6 obtained by the specified construction above, which means that u_j and z_j are two different vectors of length one of the point p_j, thus $u_j = e^{i\varphi} w_j = e^{i\lambda} e^{i\mu} w_j = e^{i\lambda} z_j$ for some $\varphi, \lambda, \mu \in [0, 2\pi[$. The transformation matrices between the bases α_{u_j} and α_{z_j} in V_6 are ${}_{\alpha_{z_j}}[id]_{\alpha_{u_j}} = e^{i\lambda} I$ and ${}_{\alpha_{z_j}}[id]^{-1}_{\alpha_{u_j}} = e^{-i\lambda} I$. Since α_{u_j} and α_{z_j} are bases of the vector space V_6 we can consider for each endomorphism $\mu : V_6 \to V_6$ the matrix representation of μ with respect to the basis α_{u_j}, denoted by $[\mu]_{\alpha_{u_j}}$, resp. relative to the basis α_{z_j}, denoted by $[\mu]_{\alpha_{z_j}}$. Using the special shape of the transformation matrix ${}_{\alpha_{z_j}}[id]_{\alpha_{u_j}}$, we see that $[\mu]_{\alpha_{u_j}} = ({}_{\alpha_{z_j}}[id]_{\alpha_{u_j}})^{-1} [\mu]_{\alpha_{z_j} \alpha_{z_j}} [id]_{\alpha_{u_j}} = ({}_{\alpha_{z_j}}[id]_{\alpha_{u_j}})^{-1} {}_{\alpha_{z_j}}[id]_{\alpha_{u_j}} [\mu]_{\alpha_{z_j}} = [\mu]_{\alpha_{z_j}}$. In particular, with regard to an x-$\mathrm{SU}_2(\mathbb{C})$-$V_6$ action of V_6 for some two-dimensional linear subspace x of V_6 we have that $[\varphi_{V_6}]_{\alpha_{u_j}} = [\varphi_{V_6}]_{\alpha_{z_j}}$ for each $\varphi \in \mathrm{SU}_2(\mathbb{C})$.

4.3 Automorphisms of the graph $\widehat{\Gamma}$

Let Γ be a connected locally $\mathbf{S}(V_6)$ graph satisfying the condition that

$$|\{\mathbf{z}, \mathbf{w}\}^{\perp} \cap \mathbf{x}^{\perp}| = 1 \text{ if and only if } |\{\mathbf{z}, \mathbf{w}\}^{\perp} \cap \mathbf{y}^{\perp}| = 1 \tag{4.1}$$

for any chain $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ in Γ of four different vertices $\mathbf{x}, \mathbf{w}, \mathbf{y}$ and $\mathbf{z}$. Let $\widehat{\Gamma}$ be a 2-simply connected cover of Γ (as a 2-dimensional simplicial complex), then $\widehat{\Gamma}$ is locally $\mathbf{S}(V_6)$ and satisfies the axiom from above, as coverings of 2-dimensional simplicial complexes preserve triangles. A good reference for the covering theory of simplicial complexes is [78].

Directly from this condition we can derive some basic properties of the graph Γ, which will be important later in this section.

Notation: As the graph Γ is locally $\mathbf{S}(V_6)$, for every vertex $\mathbf{x}$ of Γ, we can construct the interior space $\mathcal{G}_{\mathbf{x}} = \mathcal{G}_{\mathbf{x}^{\perp}}$ on the induced subgraph $\mathbf{x}^{\perp}$, which is isomorphic to $\mathbb{P}(V_6)$, see proposition 4.2.3. Recall that any local object (point, line, plane, etc.) only exists in an interior space $\mathcal{G}_{\mathbf{x}}$ for some vertex $\mathbf{x}$ in the graph Γ , thus to avoid confusion, we will index every local object by the vertex $\mathbf{x}$ whose interior space it belongs to. For example, if $\mathbf{x} \perp \mathbf{y}$ in the graph Γ, then $\mathbf{y}$ is a vertex of the subgraph $\mathbf{x}^{\perp}$ corresponding to the local object $y_{\mathbf{x}}$, an interior line, in the space $\mathcal{G}_{\mathbf{x}}$. By $\mathbf{y}_{\mathbf{x}}$ we denote the vertex $\mathbf{y}$ considered as a vertex of the subgraph $\mathbf{x}^{\perp}$. With the symbol $\mathbf{y}_{\mathbf{x}}^{\perp}$ we denote the subgraph $\{\mathbf{x}, \mathbf{y}\}^{\perp}$ which is of course an induced subgraph of $\mathbf{x}^{\perp}$. The interior space obtained from the graph $\mathbf{y}_{\mathbf{x}}^{\perp}$ will be denoted with $\mathcal{G}_{\mathbf{y}_{\mathbf{x}}}$.

Lemma 4.3.1 *Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and $\mathbf{w}$ be four different pairwise adjacent vertices in the graph Γ. Then the interior lines $z_{\mathbf{x}}$ and $w_{\mathbf{x}}$ span a four-dimensional subspace in $\mathcal{G}_{\mathbf{x}}$ and $\{\mathbf{z}, \mathbf{w}\}_{\mathbf{x}}^{\perp\perp} = \{\mathbf{z}, \mathbf{w}\}_{\mathbf{y}}^{\perp\perp}$.*

Proof: First of all $\{\mathbf{z},\mathbf{w}\}_{\mathbf{x}}^{\perp\perp} = \mathbf{L}(\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle)$ by lemma 2.3.2 and each element of $\mathbf{L}(\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle)$ is a two-dimensional subspace of $y_{\mathbf{x}}^{\pi}$. It follows that each vertex $\mathbf{h}$ of $\{\mathbf{z},\mathbf{w}\}_{\mathbf{x}}^{\perp\perp}$ is adjacent to $\mathbf{x}$ and $\mathbf{y}$ in Γ. Since the three lines $w_{\mathbf{y}}, z_{\mathbf{y}}$ and $x_{\mathbf{y}}$ are pairwise orthogonal we obtain that $x_{\mathbf{y}} = w_{\mathbf{y}}^{\pi} \cap z_{\mathbf{y}}^{\pi}$ implying the following identities $\{\mathbf{z},\mathbf{w}\}_{\mathbf{y}}^{\perp\perp} = \mathbf{L}(\langle z_{\mathbf{y}}, w_{\mathbf{y}}\rangle) = \mathbf{L}(x_{\mathbf{y}}^{\pi}) = \mathbf{x}_{\mathbf{y}}^{\perp}$. Hence the vertex $\mathbf{h}$ is contained in $\{\mathbf{z},\mathbf{w}\}_{\mathbf{y}}^{\perp\perp}$ proving $\{\mathbf{z},\mathbf{w}\}_{\mathbf{x}}^{\perp\perp} \subseteq \{\mathbf{z},\mathbf{w}\}_{\mathbf{y}}^{\perp\perp}$. By symmetry, we also get $\{\mathbf{z},\mathbf{w}\}_{\mathbf{y}}^{\perp\perp} \subseteq \{\mathbf{z},\mathbf{w}\}_{\mathbf{x}}^{\perp\perp}$, so $\{\mathbf{z},\mathbf{w}\}_{\mathbf{x}}^{\perp\perp} = \{\mathbf{z},\mathbf{w}\}_{\mathbf{y}}^{\perp\perp}$. ∎

Lemma 4.3.2 *Let* $\mathbf{x},\mathbf{y},\mathbf{z}$ *and* $\mathbf{w}$ *be four different vertices in the graph* Γ *such that* $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ *then* $\{\mathbf{z},\mathbf{w}\}_{\mathbf{x}}^{\perp\perp} = \{\mathbf{z},\mathbf{w}\}_{\mathbf{y}}^{\perp\perp}$.

Proof: In the special case that $\mathbf{x},\mathbf{y},\mathbf{z}$ and $\mathbf{w}$ are four different pairwise adjacent vertices of Γ lemma 4.3.1 implies the statement.

Suppose $\mathbf{z}$ and $\mathbf{w}$ are not adjacent in Γ then the subspace spanned by the lines $z_{\mathbf{x}}$ and $w_{\mathbf{x}}$ is either of dimension four implying $z_{\mathbf{x}}^{\pi} \cap w_{\mathbf{x}}^{\pi}$ is a single line $h_{\mathbf{x}}$ (and, thus, $\{\mathbf{z},\mathbf{w}\}^{\perp} \cap \mathbf{x}^{\perp} = 1$) or of dimension three; then $\{\mathbf{z},\mathbf{w}\}^{\perp} \cap \mathbf{x}^{\perp} > 1$ as $\{\mathbf{z},\mathbf{w}\}^{\perp} \cap \mathbf{x}^{\perp} = \mathbf{L}(z_{\mathbf{x}}^{\pi} \cap w_{\mathbf{x}}^{\pi})$.

If $\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$ is a four-dimensional subspace of $\mathcal{G}_{\mathbf{x}}$ then $z_{\mathbf{x}}^{\pi} \cap w_{\mathbf{x}}^{\pi} = y_{\mathbf{x}}^{\pi}$ and by lemma 2.3.2, $\{\mathbf{z},\mathbf{w}\}_{\mathbf{x}}^{\perp\perp} = \mathbf{L}(\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle) = \mathbf{L}(y_{\mathbf{x}}^{\pi}) = \mathbf{y}_{\mathbf{x}}^{\perp}$. From the facts that $|\{\mathbf{z},\mathbf{w}\}^{\perp} \cap \mathbf{y}^{\perp}| = 1$ and $\mathbf{w} \perp \mathbf{x} \perp \mathbf{z}$ we conclude that the lines $z_{\mathbf{y}}$ and $w_{\mathbf{y}}$ span a subspace of dimension four in $\mathcal{G}_{\mathbf{y}}$ and that $z_{\mathbf{y}}^{\pi} \cap w_{\mathbf{y}}^{\pi} = x_{\mathbf{y}}^{\pi}$, thus $\{\mathbf{z},\mathbf{w}\}_{\mathbf{y}}^{\perp\perp} = \mathbf{L}(\langle z_{\mathbf{y}}, w_{\mathbf{y}}\rangle) = \mathbf{L}(x_{\mathbf{y}}^{\pi}) = \mathbf{x}_{\mathbf{y}}^{\perp} = \mathbf{x}^{\perp} \cap \mathbf{y}^{\perp} = \mathbf{y}_{\mathbf{x}}^{\perp} = \{\mathbf{z},\mathbf{w}\}_{\mathbf{x}}^{\perp\perp}$ and we are done.

If otherwise the subspace $\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$ is a plane of $\mathcal{G}_{\mathbf{x}}$ then $|\{\mathbf{z},\mathbf{w}\}^{\perp} \cap \mathbf{x}^{\perp}| > 1$ implying that $|\{\mathbf{z},\mathbf{w}\}^{\perp} \cap \mathbf{y}^{\perp}| > 1$ and $\dim(\langle z_{\mathbf{y}}, w_{\mathbf{y}}\rangle) = 3$ in $\mathcal{G}_{\mathbf{y}}$. To prove the statement let $h_{\mathbf{x}}$ be a line of the plane $\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$ different from the line $z_{\mathbf{x}}$ and from the line $w_{\mathbf{x}}$. Then the subgraphs $\{\mathbf{z},\mathbf{h}\}^{\perp} \cap \mathbf{x}^{\perp}$ and $\{\mathbf{h},\mathbf{w}\}^{\perp} \cap \mathbf{x}^{\perp}$ of Γ contain more than one vertex. Thus $\dim(\langle h_{\mathbf{y}}, w_{\mathbf{y}}\rangle) = \dim(\langle h_{\mathbf{y}}, z_{\mathbf{y}}\rangle) = 3$ as $|\{\mathbf{z},\mathbf{h}\}^{\perp} \cap \mathbf{y}^{\perp}| > 1$ and $|\{\mathbf{h},\mathbf{w}\}^{\perp} \cap \mathbf{y}^{\perp}| > 1$.

Suppose the intersection point $z_{\mathbf{x}} \cap w_{\mathbf{x}}$ is a subspace of the line $h_{\mathbf{x}}$ and we assume that the line $h_{\mathbf{y}}$ is not a subspace of $\langle z_{\mathbf{y}}, w_{\mathbf{y}}\rangle$. From the previous paragraph we see that $h_{\mathbf{y}}$ intersects $z_{\mathbf{y}}$ and also $w_{\mathbf{y}}$ in the point $z_{\mathbf{y}} \cap w_{\mathbf{y}}$. Next we choose a line $g_{\mathbf{y}}$ of the plane $\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$ not going through the intersection point $z_{\mathbf{x}} \cap w_{\mathbf{x}}$. Therefore $\dim(\langle h_{\mathbf{y}}, g_{\mathbf{y}}\rangle) = 4$ and $|\{\mathbf{h},\mathbf{g}\}^{\perp} \cap \mathbf{y}^{\perp}| = 1 = |\{\mathbf{h},\mathbf{g}\}^{\perp} \cap \mathbf{x}^{\perp}|$. Considering the constellation back in the space $\mathcal{G}_{\mathbf{x}}$ we obtain that the line $g_{\mathbf{x}}$ is either contained in the plane $\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$ or intersects the subspace $\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$ in the point $z_{\mathbf{x}} \cap w_{\mathbf{x}}$. In both cases $g_{\mathbf{x}}$ intersects the line $h_{\mathbf{x}}$, contradiction. Thus $h_{\mathbf{y}}$ is a line incident to the plane $\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$.

If the line $h_{\mathbf{x}}$ is not going through the point $z_{\mathbf{x}} \cap w_{\mathbf{x}}$, then we choose a point $p_{\mathbf{x}} \in \langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$ in such a way that $p_{\mathbf{x}}$ is not a point of the three lines $z_{\mathbf{x}}, w_{\mathbf{x}}$ and $h_{\mathbf{x}}$ and fix the three different lines $g_{\mathbf{x}}^{z,h} = \langle p_{\mathbf{x}}, z_{\mathbf{x}} \cap h_{\mathbf{x}}\rangle$, $g_{\mathbf{x}}^{w,h} = \langle p_{\mathbf{x}}, w_{\mathbf{x}} \cap h_{\mathbf{x}}\rangle$ and $g_{\mathbf{x}}^{z,w} = \langle p_{\mathbf{x}}, w_{\mathbf{x}} \cap z_{\mathbf{x}}\rangle$. Certainly any two distinct elements $k_{\mathbf{x}}, l_{\mathbf{x}}$ of the line set $\{z_{\mathbf{x}}, w_{\mathbf{x}}, h_{\mathbf{x}}, g_{\mathbf{x}}^{z,w}, g_{\mathbf{x}}^{z,h}, g_{\mathbf{x}}^{w,h}\}$ span the plane $\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$ thus $|\{\mathbf{k},\mathbf{l}\}^{\perp} \cap \mathbf{x}^{\perp}| > 1$ implying $|\{\mathbf{k},\mathbf{l}\}^{\perp} \cap \mathbf{y}^{\perp}| > 1$ and $\dim(\langle k_{\mathbf{y}}, l_{\mathbf{y}}\rangle) = 3$ in $\mathcal{G}_{\mathbf{y}}$. Suppose the line $h_{\mathbf{y}}$ is not contained in the plane $\langle z_{\mathbf{y}}, w_{\mathbf{y}}\rangle$, then the six lines $z_{\mathbf{y}}, w_{\mathbf{y}}, h_{\mathbf{y}}, g_{\mathbf{y}}^{z,w}, g_{\mathbf{y}}^{z,h}$ and $g_{\mathbf{y}}^{w,h}$ intersect pairwise in the point $z_{\mathbf{y}} \cap w_{\mathbf{y}}$ and, by the previous part, $g_{\mathbf{y}}^{z,w} \subseteq \langle z_{\mathbf{y}}, w_{\mathbf{y}}\rangle$ and $g_{\mathbf{y}}^{z,h} \subseteq \langle z_{\mathbf{y}}, h_{\mathbf{y}}\rangle$ and $g_{\mathbf{y}}^{w,h} \subseteq \langle h_{\mathbf{y}}, w_{\mathbf{y}}\rangle$. In particular we can fix a line $v_{\mathbf{y}}$ in the three-dimensional subspace $\langle g_{\mathbf{y}}^{w,h}, g_{\mathbf{y}}^{z,h}\rangle$, which does not intersect the line $g_{\mathbf{y}}^{z,w}$. Thus we conclude that $\dim(\langle v_{\mathbf{y}}, g_{\mathbf{y}}^{z,w}\rangle) = 4 = \dim(\langle v_{\mathbf{y}}, z_{\mathbf{y}}\rangle) = \dim(\langle v_{\mathbf{y}}, w_{\mathbf{y}}\rangle)$ as well as that $g_{\mathbf{y}}^{w,h}, g_{\mathbf{y}}^{z,h}$ and $v_{\mathbf{y}}$ are three mutually intersecting lines in $x_{\mathbf{y}}^{\pi}$. Therefore $v_{\mathbf{x}}$ intersects the plane $\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$ either in the point $p_{\mathbf{x}}$ (a contradiction as $v_{\mathbf{x}} \cap g_{\mathbf{x}}^{z,w} = \{0\}$) or in the two different points $g_{\mathbf{x}}^{z,h} \cap v_{\mathbf{x}}$ and $g_{\mathbf{x}}^{z,h} \cap v_{\mathbf{x}}$ implying that $v_{\mathbf{x}}$ is a line of the plane $\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$, contradiction. In fact $h_{\mathbf{y}}$ is a line of the plane $\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$.

It follows that $\{\mathbf{z}, \mathbf{w}\}_{\mathbf{x}}^{\perp\perp} = \mathbf{L}(\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle) \subseteq \mathbf{L}(\langle z_{\mathbf{y}}, w_{\mathbf{y}}\rangle) = \{\mathbf{z}, \mathbf{w}\}_{\mathbf{y}}^{\perp\perp}$ and by symmetry we also obtain that $\{\mathbf{z}, \mathbf{w}\}_{\mathbf{y}}^{\perp\perp} = \mathbf{L}(\langle z_{\mathbf{y}}, w_{\mathbf{y}}\rangle) \subseteq \mathbf{L}(\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle) = \{\mathbf{z}, \mathbf{w}\}_{\mathbf{x}}^{\perp\perp}$, which shows that $\{\mathbf{z}, \mathbf{w}\}_{\mathbf{x}}^{\perp\perp} = \{\mathbf{z}, \mathbf{w}\}_{\mathbf{y}}^{\perp\perp}$. ∎

We will often use this basic result in the following form.

Lemma 4.3.3 *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *and* $\mathbf{w}$ *be four different pairwise adjacent vertices in the graph* Γ. *The interior lines* $z_{\mathbf{x}}$ *and* $w_{\mathbf{x}}$ *span a plane in* $\mathcal{G}_{\mathbf{x}}$ *if and only if the interior lines* $z_{\mathbf{y}}$ *and* $w_{\mathbf{y}}$ *span a plane in* $\mathcal{G}_{\mathbf{y}}$.

Lemma 4.3.4 *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *and* $\mathbf{w}$ *be four different vertices in the graph* Γ *with* $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ *and* $\dim(\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle) = 3$. *The line* $h_{\mathbf{x}} \subseteq \langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$ *is incident to the point* $z_{\mathbf{x}} \cap w_{\mathbf{x}}$ *if and only if the line* $h_{\mathbf{y}} \subseteq \langle z_{\mathbf{y}}, w_{\mathbf{y}}\rangle$ *is incident to the point* $z_{\mathbf{y}} \cap w_{\mathbf{y}}$.

Proof: By lemma 4.3.2 the subspace $h_{\mathbf{y}}$ is a line of the plane $\langle z_{\mathbf{y}}, w_{\mathbf{y}}\rangle$. By way of contradiction we assume that the point $z_{\mathbf{y}} \cap w_{\mathbf{y}}$ is not contained in $h_{\mathbf{y}}$. Then we fix an interior line $g_{\mathbf{y}}$ in $x_{\mathbf{y}}^{\pi}$ running through the point $z_{\mathbf{y}} \cap w_{\mathbf{y}}$, which is not contained in the plane $\langle z_{\mathbf{y}}, w_{\mathbf{y}}\rangle$. The vertex $\mathbf{g}$ is adjacent to $\mathbf{x}$ and $|\{\mathbf{g}, \mathbf{z}\}^{\perp} \cap \mathbf{y}^{\perp}| > 1$ as well as $|\{\mathbf{g}, \mathbf{w}\}^{\perp} \cap \mathbf{y}^{\perp}| > 1$. Thus $g_{\mathbf{x}}$ is not a line of the plane $\langle z_{\mathbf{x}}, w_{\mathbf{x}}\rangle$ but the two-dimensional subspace $g_{\mathbf{x}}$ intersects the two lines $z_{\mathbf{x}}$ and $w_{\mathbf{x}}$, so $g_{\mathbf{x}} \cap z_{\mathbf{x}} = g_{\mathbf{x}} \cap w_{\mathbf{x}} = z_{\mathbf{x}} \cap w_{\mathbf{x}} = z_{\mathbf{x}} \cap w_{\mathbf{x}} \cap h_{\mathbf{x}}$, contradiction. ∎

Lemma 4.3.5 *Let* $\mathbf{z} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$ *be chain in* Γ *such that* $z_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi}$ *and* $x_{\mathbf{y}}^{\pi} \cap w_{\mathbf{y}}^{\pi}$ *are planes. Then there is a vertex* $\mathbf{h} \in \{\mathbf{z}, \mathbf{x}, \mathbf{y}, \mathbf{w}\}^{\perp}$.

Proof: By lemma 4.3.2 the plane $H_{z_{\mathbf{x}}, y_{\mathbf{x}}}^{\mathbf{x}} = z_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi}$ can be identified with a unique subspace of dimension three in $\mathcal{G}_{\mathbf{y}}$; we denote it by $H_{z_{\mathbf{x}}, y_{\mathbf{x}}}^{\mathbf{y}}$. Since $H_{z_{\mathbf{x}}, y_{\mathbf{x}}}^{\mathbf{y}} \subseteq x_{\mathbf{y}}^{\pi}$, $\dim(x_{\mathbf{y}}^{\pi}) = 4$ and $\dim(x_{\mathbf{y}}^{\pi} \cap w_{\mathbf{y}}^{\pi}) = 3$ the two planes $H_{z_{\mathbf{x}}, y_{\mathbf{x}}}^{\mathbf{y}}$ and $x_{\mathbf{y}}^{\pi} \cap w_{\mathbf{y}}^{\pi}$ intersect in the line $h_{\mathbf{y}}$. Thus the vertex $\mathbf{h}$ belonging to the line $h_{\mathbf{y}}$ is an element of $\{\mathbf{z}, \mathbf{x}, \mathbf{y}, \mathbf{w}\}^{\perp}$ in the graph Γ. ∎

In lemma 4.3.2 and lemma 4.3.3 we have concluded that we can preserve in some special configurations planes in a connected locally $\mathbf{S}(V_6)$ graph Γ. In the following we show that for a vertex $\mathbf{x} \in \Gamma$ we can also identify points of the line $x_{\mathbf{y}}$ in $\mathcal{G}_{\mathbf{y}}$ with unique points on the line $x_{\mathbf{z}}$ in $\mathcal{G}_{\mathbf{z}}$ if $\mathbf{z}, \mathbf{y} \in \mathbf{x}^{\perp}$.

Lemma 4.3.6 *Let* $\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}$ *be distinct pairwise adjacent vertices of* Γ. *Then the graph* Γ *contains vertices* $\mathbf{xy} \in \{\mathbf{x}, \mathbf{y}\}^{\perp}$ *and* $\mathbf{xz} \in \{\mathbf{x}, \mathbf{z}\}^{\perp}$ *and* $\mathbf{yz} \in \{\mathbf{y}, \mathbf{z}\}^{\perp}$ *such that*

(i) *the interior lines* $xz_{\mathbf{z}}$, $yz_{\mathbf{z}}$ *and* $w_{\mathbf{z}}$ *intersect in one point of* $\mathcal{G}_{\mathbf{z}}$,

(ii) *the interior lines* $xy_{\mathbf{y}}$ *and* $yz_{\mathbf{y}}$ *and* $w_{\mathbf{y}}$ *intersect in one point of* $\mathcal{G}_{\mathbf{y}}$, *and*

(iii) *the graph* Γ *contains a vertex* $\mathbf{u}$ *adjacent to* $\mathbf{z}, \mathbf{y}, \mathbf{xy}, \mathbf{xz}, \mathbf{yz}$ *in the same connected component of* $\mathbf{xy}^{\perp} \cap \mathbf{xz}^{\perp}$ *as the vertex* $\mathbf{x}$.

Proof: Inside the interior space $\mathcal{G}_{\mathbf{y}}$ we choose an interior point $p_{\mathbf{y}}$ on $w_{\mathbf{y}}$ and consider the orthogonal space $p_{\mathbf{y}}^{\pi}$ of $p_{\mathbf{y}}$. The space $p_{\mathbf{y}}^{\pi} \cap x_{\mathbf{y}}^{\pi}$ is three-dimensional and contains the interior line $z_{\mathbf{y}}$, because $z_{\mathbf{y}} \subseteq w_{\mathbf{y}}^{\pi} \cap x_{\mathbf{y}}^{\pi}$. Choose a point $a_{\mathbf{y}}$ on the line $z_{\mathbf{y}}$ and define $xy_{\mathbf{y}} := \langle a_{\mathbf{y}}, p_{\mathbf{y}} \rangle$. Furthermore choose a point $b_{\mathbf{y}}$ on the line $x_{\mathbf{y}}$ and define $yz_{\mathbf{y}} := \langle b_{\mathbf{y}}, p_{\mathbf{y}} \rangle$. The interior lines $yz_{\mathbf{y}}$ and $xy_{\mathbf{y}}$ correspond to vertices $\mathbf{xy}$ and $\mathbf{yz}$ in the graph Γ satisfying $\mathbf{xy} \in \{\mathbf{x}, \mathbf{y}\}^{\perp}$ and that $\mathbf{yz} \in \{\mathbf{y}, \mathbf{z}\}^{\perp}$. By construction $\mathbf{xy}$, $\mathbf{yz}$ and $\mathbf{w}$ satisfy condition (ii). Let $s_{\mathbf{y}} \subseteq z_{\mathbf{y}}^{\pi}$ be a line containing $p_{\mathbf{y}}$ distinct from $w_{\mathbf{y}}$. Analysing the constellation in $\mathcal{G}_{\mathbf{z}}$ by lemma 4.3.4 the interior line $yz_{\mathbf{z}}$ intersects $w_{\mathbf{z}}$ in the point $p_{\mathbf{z}} = s_{\mathbf{z}} \cap w_{\mathbf{z}}$. Define $xz_{\mathbf{z}} := \langle p_{\mathbf{z}}, c_{\mathbf{z}} \rangle$ for some point $c_{\mathbf{z}}$ on $y_{\mathbf{z}}$. The vertex $\mathbf{xz}$ obtained from the interior line $xz_{\mathbf{z}}$ satisfies $\mathbf{xz} \in \{\mathbf{x}, \mathbf{z}\}^{\perp}$ and condition (i) holds for $\mathbf{w}, \mathbf{yz}, \mathbf{xy}$.

Next, we consider the line $u_{\mathbf{y}} = z_{\mathbf{y}}^{\pi} \cap yz_{\mathbf{y}}^{\pi} = z_{\mathbf{y}}^{\pi} \cap \langle p_{\mathbf{y}}, b_{\mathbf{y}} \rangle^{\pi} = z_{\mathbf{y}}^{\pi} \cap p_{\mathbf{y}}^{\pi} \cap b_{\mathbf{y}}^{\pi}$ in $\mathcal{G}_{\mathbf{y}}$. The subspace $u_{\mathbf{y}}$ intersects the line $w_{\mathbf{y}}$ in the point $p_{\mathbf{y}}^{\pi} \cap w_{\mathbf{y}}$ and the line $x_{\mathbf{y}}$ in the point $b_{\mathbf{y}}^{\pi} \cap x_{\mathbf{y}}$, as $w_{\mathbf{y}} \subseteq b_{\mathbf{y}}^{\pi} \cap z_{\mathbf{y}}^{\pi}$ and $x_{\mathbf{y}} \subseteq p_{\mathbf{y}}^{\pi} \cap z_{\mathbf{y}}^{\pi}$. Hence $u_{\mathbf{y}} = \langle p_{\mathbf{y}}^{\pi} \cap w_{\mathbf{y}}, b_{\mathbf{y}}^{\pi} \cap x_{\mathbf{y}} \rangle$. Since $u_{\mathbf{y}} \subseteq z_{\mathbf{y}}^{\pi} \subseteq a_{\mathbf{y}}^{\pi}$, the interior line $u_{\mathbf{y}}$ is incident to the subspace $xy_{\mathbf{y}}^{\pi} = p_{\mathbf{y}}^{\pi} \cap a_{\mathbf{y}}^{\pi}$, proving that $\mathbf{u} \in \{\mathbf{y}, \mathbf{z}, \mathbf{xy}, \mathbf{yz}\}^{\perp}$. The vertex $\mathbf{u}$ corresponds to the interior line $u_{\mathbf{z}} = \langle w_{\mathbf{z}} \cap p_{\mathbf{z}}^{\pi}, b_{\mathbf{z}}^{\pi} \cap x_{\mathbf{z}} \rangle$, where $b_{\mathbf{z}} = x_{\mathbf{z}} \cap yz_{\mathbf{z}}$ by lemma 4.3.4. As $u_{\mathbf{z}} \subseteq y_{\mathbf{z}}^{\pi} \subseteq c_{\mathbf{z}}^{\pi}$ and $u_{\mathbf{z}} \subseteq p_{\mathbf{z}}^{\pi}$, we have $u_{\mathbf{z}} \subseteq c_{\mathbf{z}}^{\pi} \cap p_{\mathbf{z}}^{\pi} = xz_{\mathbf{z}}^{\pi}$, proving $\mathbf{u} \perp \mathbf{xz}$.

To complete the proof of the statement we have to construct a path from $\mathbf{x}$ to $\mathbf{u}$ in the induced subgraph $\{\mathbf{xy}, \mathbf{xz}\}^{\perp}$. Let $v_{\mathbf{y}}$ be a line through $b_{\mathbf{y}}$ in $z_{\mathbf{y}}^{\pi} \cap xy_{\mathbf{y}}^{\pi} \supseteq x_{\mathbf{y}} \supseteq b_{\mathbf{y}}$. We have $\mathbf{v} \in \{\mathbf{y}, \mathbf{z}, \mathbf{xy}, \mathbf{xz}\}^{\perp}$. By construction $b_{\mathbf{z}} = x_{\mathbf{z}} \cap yz_{\mathbf{z}} \in u_{\mathbf{z}}^{\pi} \cap xz_{\mathbf{z}}^{\pi}$ and, moreover, by lemma 4.3.4 the line $v_{\mathbf{z}}$ passes through the point $b_{\mathbf{z}} = x_{\mathbf{z}} \cap yz_{\mathbf{z}}$, whence the plane $H^{\mathbf{z}}_{z_{\mathbf{y}}, xy_{\mathbf{y}}}$ induced by the plane $H^{\mathbf{y}}_{z_{\mathbf{y}}, xy_{\mathbf{y}}} = z_{\mathbf{y}}^{\pi} \cap xy_{\mathbf{y}}^{\pi}$ via lemma 4.3.3 contains the point $b_{\mathbf{z}}$. It follows that the interior space $\mathcal{G}_{\mathbf{u}}$ contains a unique point $b_{\mathbf{u}}$ on the line $yz_{\mathbf{y}}$ contained in $xy_{\mathbf{u}}^{\pi} \cap xz_{\mathbf{u}}^{\pi}$. By lemma 4.3.3 the space $\langle xy_{\mathbf{u}}, yz_{\mathbf{u}} \rangle$ is three-dimensional and also $\langle xz_{\mathbf{u}}, yz_{\mathbf{u}} \rangle \subseteq z_{\mathbf{u}}^{\pi}$ is a plane. Finally the two interior lines $xy_{\mathbf{u}}$ and $xz_{\mathbf{u}}$ either intersect in a point of $\mathcal{G}_{\mathbf{u}}$ or not. Assuming the latter, the space $\langle xy_{\mathbf{u}}, xz_{\mathbf{u}} \rangle$ has to be four-dimensional and has to contain $yz_{\mathbf{u}}$. We deduce $\{0\} = \langle xy_{\mathbf{u}}, xz_{\mathbf{u}} \rangle^{\pi} \cap yz_{\mathbf{u}} = xy_{\mathbf{u}}^{\pi} \cap xz_{\mathbf{u}}^{\pi} \cap yz_{\mathbf{u}} \supseteq b_{\mathbf{u}}$, a contradiction. So $xy_{\mathbf{u}}$ and $xz_{\mathbf{u}}$ intersect in a common point, therefore $xy_{\mathbf{u}}^{\pi} \cap xz_{\mathbf{u}}^{\pi}$ is three-dimensional. Moreover $u_{\mathbf{xy}}^{\pi} \cap x_{\mathbf{xy}}^{\pi}$ is also a plane by lemma 4.3.3 and the facts that $u_{\mathbf{y}}$ intersects the line $x_{\mathbf{y}}$ in a point as well as $\mathbf{y} \perp \mathbf{xy}$. Thus lemma 4.3.5 applied to the path $\mathbf{xy} \perp \mathbf{u} \perp \mathbf{xz} \perp \mathbf{x}$ yields a vertex $\mathbf{h} \in \{\mathbf{xy}, \mathbf{xz}, \mathbf{x}, \mathbf{u}\}^{\perp}$, finishing the proof. ∎

Lemma 4.3.7 *Let* $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{v}$ *be five different vertices of* Γ *such that* $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{v}$ *are elements of* $\mathbf{w}^\perp$ *and* $\mathbf{x} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{v} \perp \mathbf{x}$. *Then to each point* $p_\mathbf{x}$ *on* $w_\mathbf{x}$ *the graph* Γ *contains a vertex* $\mathbf{xvy} \in \{\mathbf{x}, \mathbf{y}, \mathbf{v}\}^\perp$ *and a vertex* $\mathbf{vyz} \in \{\mathbf{y}, \mathbf{z}, \mathbf{v}\}^\perp$ *such that*

(i) *the interior lines* $xvy_\mathbf{x}$ *and* $w_\mathbf{x}$ *intersect in* $p_\mathbf{x}$,

(ii) *the interior lines* $xvy_\mathbf{y}$, $vyz_\mathbf{y}$ *and* $w_\mathbf{y}$ *intersect in a common point of* $\mathcal{G}_\mathbf{y}$, *and*

(iii) *the graph* Γ *contains a vertex* $\mathbf{u} \in \{\mathbf{y}, \mathbf{v}, \mathbf{xvy}, \mathbf{vyz}\}^\perp$.

Proof: Inside $\mathcal{G}_\mathbf{w}$ we analyse the combination of the four lines $y_\mathbf{w}$, $x_\mathbf{w}$, $z_\mathbf{w}$ and $v_\mathbf{w}$. Since $y_\mathbf{w}$ and $v_\mathbf{w}$ are incident to the subspace $x_\mathbf{w}^\pi \cap z_\mathbf{w}^\pi$ the lines $x_\mathbf{w}$ and $z_\mathbf{w}$ span a plane in $\mathcal{G}_\mathbf{w}$, as otherwise $y_\mathbf{w} = v_\mathbf{w}$, contradiction. By symmetry also $\langle y_\mathbf{w}, v_\mathbf{w}\rangle$ is a plane.

Next we consider this configuration inside $\mathcal{G}_\mathbf{x}$. Due to lemma 4.3.3 the space $\langle y_\mathbf{x}, v_\mathbf{x}\rangle$ is of dimension three. Also $w_\mathbf{x} \subseteq y_\mathbf{x}^\pi \cap v_\mathbf{x}^\pi$, because $\mathbf{v} \perp \mathbf{w} \perp \mathbf{y}$. Let $p_\mathbf{x}$ be a point on the interior line $w_\mathbf{x}$ and choose $xvy_\mathbf{x}$ to be a two-dimensional subspace in $y_\mathbf{x}^\pi \cap v_\mathbf{x}^\pi$ through $p_\mathbf{x}$. The line $xvy_\mathbf{x}$ corresponds to a vertex $\mathbf{xvy} \in \{\mathbf{y}, \mathbf{v}, \mathbf{x}\}^\perp$. In particular $\mathbf{xvy}$ satisfies condition (i).

By lemma 4.3.3 the subspace $\langle w_\mathbf{y}, xvy_\mathbf{y}\rangle$ is of dimension three and incident to $x_\mathbf{y}^\pi$ in $\mathcal{G}_\mathbf{y}$. The same corollary implies that the space $H^\mathbf{y}_{v_\mathbf{z}, y_\mathbf{z}}$ induced by the three-dimensional space $H^\mathbf{z}_{v_\mathbf{z}, y_\mathbf{z}} = v_\mathbf{z}^\pi \cap y_\mathbf{z}^\pi$, is a plane of $z_\mathbf{y}^\pi$, which contains the line $w_\mathbf{y}$. We pick a line $vyz_\mathbf{y}$ of $H^\mathbf{y}_{v_\mathbf{z}, y_\mathbf{z}}$ incident to the point $p_\mathbf{y} = w_\mathbf{y} \cap xvy_\mathbf{y}$. From this construction it follows that the lines $xvy_\mathbf{y}$, $vyz_\mathbf{y}$ and $w_\mathbf{y}$ intersect in a common point and that $\mathbf{vyz} \in \{\mathbf{v}, \mathbf{z}, \mathbf{y}\}^\perp$. In particular $\mathbf{xvy}$ and $\mathbf{vyz}$ satisfy condition (ii). Furthermore $vyz_\mathbf{y}$ and $xvy_\mathbf{y}$ intersect in the point $p_\mathbf{y}$, so $H^\mathbf{y}_{yvz_\mathbf{y}, xyv_\mathbf{y}}$ is also a plane. In fact the four-dimensional subspace $y^\pi_\mathbf{yvz}$ contains the planes $H^\mathbf{yvz}_{yvz_\mathbf{y}, xyv_\mathbf{y}}$ (which is induced by $H^\mathbf{y}_{yvz_\mathbf{y}, xyv_\mathbf{y}}$) and $H^\mathbf{yvz}_{v_\mathbf{yvz}, y_\mathbf{yvz}} = v_\mathbf{yvz} \cap y_\mathbf{yvz}$. Analysing dimensions we conclude that $H^\mathbf{yvz}_{yvz_\mathbf{y}, xyv_\mathbf{y}} \cap H^\mathbf{yvz}_{v_\mathbf{yvz}, y_\mathbf{yvz}}$ contains some line $u_\mathbf{yvz}$. The corresponding vertex $\mathbf{u}$ satisfies $\mathbf{u} \in \{\mathbf{y}, \mathbf{v}, \mathbf{xyv}, \mathbf{yvz}\}^\perp$. ■

Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be three pairwise adjacent vertices of Γ. By proposition 4.2.3 we can consider the projective spaces $\mathcal{G}_\mathbf{x}$, $\mathcal{G}_\mathbf{y}$ and $\mathcal{G}_\mathbf{z}$ for the respective induced subgraphs $\mathbf{x}^\perp, \mathbf{y}^\perp, \mathbf{z}^\perp$. Moreover, the intersections $\mathbf{x}^\perp \cap \mathbf{y}^\perp, \mathbf{x}^\perp \cap \mathbf{z}^\perp, \mathbf{y}^\perp \cap \mathbf{z}^\perp$ are isomorphic to subspaces of the respective projective spaces of codimension two. In particular, we can translate a point p of the projective space on $\mathbf{x}^\perp$ that lies in the subspace on $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ to the projective space on $\mathbf{y}^\perp$, because we can identify each point as the intersection of two suitable lines. However, if p is a point contained in the subspace $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$, then it is not immediately clear that the point in $\mathcal{G}_\mathbf{z}$ obtained by translating it directly from $\mathcal{G}_\mathbf{x}$ coincides with the point in $\mathcal{G}_\mathbf{z}$ obtained by translation from $\mathcal{G}_\mathbf{x}$ to $\mathcal{G}_\mathbf{y}$ and then to $\mathcal{G}_\mathbf{z}$, because for $n = 6$ the induced subgraph $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$ is a unique vertex, so it is not possible to identify a point as the intersection of two lines. The purpose of the next lemma is to establish that these two points coincide.

Lemma 4.3.8 *Let* $\mathbf{w} \perp \mathbf{y}$ *be vertices of* Γ *and let* $\gamma = \mathbf{y}_0 \perp \mathbf{y}_1 \perp \ldots \perp \mathbf{y}_t \perp \mathbf{y}_0$ *be a cycle in the induced subgraph* $\mathbf{w}^\perp$ *based at the vertex* $\mathbf{y}_0 = \mathbf{y}$*. Furthermore let* $p_\mathbf{y} = p_{\mathbf{y}_0}$ *be a point on the interior line* $w_\mathbf{y}$ *of the projective space* $\mathcal{G}_\mathbf{y}$ *and let inductively* $p_{\mathbf{y}_{i+1}}$ *be the point on the interior line* $w_{\mathbf{y}_{i+1}}$ *in* $\mathcal{G}_{\mathbf{y}_{i+1}}$ *induced by the point* $p_{\mathbf{y}_i}$ *on the line* $w_{\mathbf{y}_i}$ *in* $\mathcal{G}_{\mathbf{y}_i}$*. Then the point on the interior line* $w_\mathbf{y}$ *in* $\mathcal{G}_\mathbf{y}$ *induced by the point* $p_{\mathbf{y}_t}$ *on* $w_{\mathbf{y}_t}$ *in the interior space* $\mathcal{G}_{\mathbf{y}_t}$ *is equal to* $p_\mathbf{y}$*.*

Proof: By proposition 4.2.2 and lemma 4.2.4 the graph $\mathbf{w}^\perp \cong \mathbf{G}(V_6)$ has diameter two and any cycle $\gamma \in \mathbf{w}^\perp$ can be decomposed into triangles and quadrangles. The triangle situation of the statement is studied in lemma 4.3.6. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be a triangle in $\mathbf{w}^\perp$ and let $\mathbf{xy}$, $\mathbf{xz}$, $\mathbf{yz}$ be vertices of Γ satisfying the hypotheses and, thus, the conclusion of lemma 4.3.6. Then the lines $w_\mathbf{x}$ and $xy_\mathbf{x}$ intersect in a point $p^w_\mathbf{x} = w_\mathbf{x} \cap xy_\mathbf{x}$ of $\mathcal{G}_\mathbf{x}$ and induce the point $p^w_\mathbf{y} = w_\mathbf{y} \cap xy_\mathbf{y} = w_\mathbf{y} \cap yz_\mathbf{y}$ inside $\mathcal{G}_\mathbf{y}$. Furthermore the interior lines $w_\mathbf{z}$ and $yz_\mathbf{z}$ define the point $p^w_\mathbf{z} = w_\mathbf{z} \cap yz_\mathbf{z} = w_\mathbf{z} \cap xz_\mathbf{z}$ of the interior space $\mathcal{G}_\mathbf{z}$. We claim that the two points $p^w_\mathbf{x} = w_\mathbf{x} \cap xy_\mathbf{x}$ and $w_\mathbf{x} \cap xz_\mathbf{x}$ of the space $\mathcal{G}_\mathbf{x}$ coincide. Indeed, by lemma 4.3.6 there is a vertex $\mathbf{u}$ in $\{\mathbf{y}, \mathbf{z}, \mathbf{xy}, \mathbf{xz}, \mathbf{yz}\}^\perp$, which implies that the three interior lines $xy_\mathbf{u}$, $xz_\mathbf{u}$ and $yz_\mathbf{u}$ intersect in one point $c_\mathbf{u} = xy_\mathbf{u} \cap xz_\mathbf{u} \cap yz_\mathbf{u}$ of $\mathcal{G}_\mathbf{u}$. Moreover by lemma 4.3.6 the vertices $\mathbf{u}$ and $\mathbf{x}$ lie in the same connected component of the subgraph $\{\mathbf{xy}, \mathbf{xz}\}^\perp$, thus by lemma 4.3.3 the lines $xy_\mathbf{x}$ and $xz_\mathbf{x}$ have to intersect inside $\mathcal{G}_\mathbf{x}$, as $\dim(\langle xy_\mathbf{u}, xz_\mathbf{u}\rangle) = 3$. Since $xy_\mathbf{x}$ is incident to $y^\pi_\mathbf{x}$ and $xz_\mathbf{x}$ is a subspace of $z^\pi_\mathbf{x}$ the only possible intersection point of $xy_\mathbf{x}$ and $xz_\mathbf{x}$ is $p^w_\mathbf{x} = xy_\mathbf{x} \cap w_\mathbf{x}$. Thus we see that $p^w_\mathbf{x} = xy_\mathbf{x} \cap w_\mathbf{x} = xy_\mathbf{x} \cap xz_\mathbf{x} = xz_\mathbf{x} \cap w_\mathbf{x}$, and we have proved the claim for triangles.

It remains to study quadrangles. Let $\mathbf{x} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{v} \perp \mathbf{x}$ be a cycle in $\mathbf{w}^\perp$. Then there exist $\mathbf{xyv}, \mathbf{yvz} \in \Gamma$ and a vertex $\mathbf{u} \in \{\mathbf{y}, \mathbf{v}, \mathbf{xyv}, \mathbf{yvz}\}^\perp$ as in lemma 4.3.7 such that the interior lines $xyv_\mathbf{v}$ and $w_\mathbf{v}$ intersect in the point $p^w_\mathbf{v} = xyv_\mathbf{v} \cap w_\mathbf{v}$, which induces the point $p^w_\mathbf{x} = xyv_\mathbf{x} \cap w_\mathbf{x}$, as the intersection of $xyv_\mathbf{x}$ and $w_\mathbf{x}$. Therefore the lines $xyv_\mathbf{y}$ and $w_\mathbf{y}$ define the point $p^w_\mathbf{y} = w_\mathbf{y} \cap xyv_\mathbf{y} = w_\mathbf{y} \cap yvz_\mathbf{y}$ (cf. lemma 4.3.7(ii)), which implies that $w_\mathbf{z}$ and $yvz_\mathbf{z}$ intersect in the point $p^w_\mathbf{z} = w_\mathbf{z} \cap yvz_\mathbf{z}$. Finally, the interior lines $w_\mathbf{v}$ and $yvz_\mathbf{v}$ induce the point $w_\mathbf{v} \cap yvz_\mathbf{v}$. We claim that the points $p_\mathbf{v}$ and $w_\mathbf{v} \cap yvz_\mathbf{v}$ coincide. As the lines $xyv_\mathbf{y}$ and $yvz_\mathbf{y}$ intersect in a point, the two-dimensional subspaces $xyv_\mathbf{u}$ and $yvz_\mathbf{u}$ span a plane in $\mathcal{G}_\mathbf{u}$ by lemma 4.3.3. Hence by lemma 4.3.3 again also $xyv_\mathbf{v}$ and $yvz_\mathbf{v}$ intersect. Furthermore $xyv_\mathbf{v}$ is incident to the subspace $x^\pi_\mathbf{v}$ and not contained in $z^\pi_\mathbf{v}$. On the other hand the line $xyv_\mathbf{v}$ lies inside $z^\pi_\mathbf{v}$ and is not incident to $x^\pi_\mathbf{v}$. It follows that the only possible point, which is incident to both lines $xyv_\mathbf{v}$ and $yvz_\mathbf{v}$, is $p^w_\mathbf{v} = xyv_\mathbf{v} \cap w_\mathbf{v} = xyv_\mathbf{v} \cap xyv_\mathbf{x} = yvz_\mathbf{v} \cap w_\mathbf{v}$, which proves the claim. ∎

Lemma 4.3.9 *Let* $\mathbf{y} \perp \mathbf{w} \perp \mathbf{z}$ *be vertices of* Γ *and let* $p_\mathbf{y}$ *and* $q_\mathbf{y}$ *be two points on the line* $w_\mathbf{y}$*. The points* $p_\mathbf{y}$ *and* $q_\mathbf{y}$ *are orthogonal in* $\mathcal{G}_\mathbf{y}$ *if and only if the induced points* $p_\mathbf{z}$ *and* $q_\mathbf{z}$ *are orthogonal in* $\mathcal{G}_\mathbf{z}$*. Conversely,* $p_\mathbf{y} \notin q^\pi_\mathbf{y}$ *in* $\mathcal{G}_\mathbf{y}$ *if and only if* $p_\mathbf{z} \notin q^\pi_\mathbf{z}$ *in* $\mathcal{G}_\mathbf{z}$*.*

Proof: By lemma 4.3.8 we may assume without loss of generality that $\mathbf{y} \perp \mathbf{z}$. Let $p_\mathbf{y}$

and $q_{\mathbf{y}}$ be two orthogonal points on the line $w_{\mathbf{y}}$ in $\mathcal{G}_{\mathbf{y}}$, i.e., $p_{\mathbf{y}} \subseteq q_{\mathbf{y}}^{\pi}$. Since $\mathbf{w} \perp \mathbf{z}$, we have the line $p_{\mathbf{y}}, q_{\mathbf{y}} \subseteq w_{\mathbf{y}} \subseteq z_{\mathbf{y}}^{\pi}$ and, thus, $q_{\mathbf{y}} \subseteq p_{\mathbf{y}}^{\pi} \cap z_{\mathbf{y}}^{\pi}$. Hence there exists a line $l_{\mathbf{y}}$ in $p_{\mathbf{y}}^{\pi} \cap z_{\mathbf{y}}^{\pi}$ through $q_{\mathbf{y}}$. Furthermore, the line $l_{\mathbf{y}}^{\pi} \cap z_{\mathbf{y}}^{\pi} = m_{\mathbf{y}}$ is orthogonal to the line $l_{\mathbf{y}}$ and, in particular, contains the point $p_{\mathbf{y}}$. The orthogonal lines $l_{\mathbf{y}}$ and $m_{\mathbf{y}}$ correspond to adjacent vertices $\mathbf{l}, \mathbf{m} \in \{\mathbf{y}, \mathbf{z}\}^{\perp}$. By lemma 4.3.8 the induced point $p_{\mathbf{z}}$ is incident to $w_{\mathbf{z}}$ and $m_{\mathbf{z}}$ and the induced point $q_{\mathbf{z}}$ is incident to $w_{\mathbf{z}}$ and $l_{\mathbf{z}}$. The lines $m_{\mathbf{z}}$ and $l_{\mathbf{z}}$ are orthogonal in $\mathcal{G}_{\mathbf{z}}$, as $\mathbf{l} \perp \mathbf{m}$, implying that each point of the line $l_{\mathbf{z}}$ is orthogonal to each point of the line $m_{\mathbf{z}}$. Thus $p_{\mathbf{z}} \subseteq q_{\mathbf{z}}^{\pi}$, proving the claim. ∎

The identification of points in different local projective spaces implies some interesting adjacency relations in the graph Γ. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and $\mathbf{w}$ be four different vertices in the graph Γ such that $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$. We claim that any vertex $\mathbf{m} \in \mathbf{z}^{\perp}$ which belongs to a line $m_{\mathbf{z}}$ intersecting the orthogonal lines $x_{\mathbf{z}}$ and $y_{\mathbf{z}}$ in $\mathcal{G}_{\mathbf{z}}$, is adjacent to the vertex $\mathbf{w}$.

We start with the setting that $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and $\mathbf{w}$ be four different vertices in the graph Γ such that $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ and $\langle z_{\mathbf{x}}, w_{\mathbf{x}} \rangle$ is a plane in the projective space $\mathcal{G}_{\mathbf{x}}$. Let $m_{\mathbf{z}}$ be some line of the subspace $\langle x_{\mathbf{z}}, y_{\mathbf{z}} \rangle$ such that $x_{\mathbf{z}} \cap m_{\mathbf{z}}$ is the point $p_{\mathbf{z}}^{x}$ and $y_{\mathbf{z}} \cap m_{\mathbf{z}}$ is the point $p_{\mathbf{z}}^{y}$. Furthermore we fix also the two points $s_{\mathbf{z}}^{x} = (p_{\mathbf{z}}^{x})^{\pi} \cap x_{\mathbf{z}}$ and $s_{\mathbf{z}}^{y} = (p_{\mathbf{z}}^{y})^{\pi} \cap y_{\mathbf{z}}$. By lemma 4.3.8 the points $s_{\mathbf{z}}^{y}$ and $p_{\mathbf{z}}^{y}$ induce unique orthogonal points $s_{\mathbf{x}}^{y}$ and $p_{\mathbf{x}}^{y}$ on the line $y_{\mathbf{x}}$ in the projective space $\mathcal{G}_{\mathbf{x}}$.

We consider the path $\mathbf{m} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{w}$ in Γ and by lemma 4.3.5 we find a vertex $\mathbf{h} \in \{\mathbf{m}, \mathbf{z}, \mathbf{x}, \mathbf{w}\}^{\perp}$. Moreover, we identify unique points $p_{\mathbf{h}}^{x}$, $p_{\mathbf{h}}^{y}$ and $s_{\mathbf{h}}^{x}$ on the lines $m_{\mathbf{h}}$ and $x_{\mathbf{h}}$ induced from the points $p_{\mathbf{z}}^{x}$, $p_{\mathbf{z}}^{y}$ and $s_{\mathbf{z}}^{x}$ by lemma 4.3.8. As $\mathbf{x} \perp \mathbf{y} \perp \mathbf{w}$ and the points $s_{\mathbf{x}}^{y}$ and $p_{\mathbf{x}}^{y}$ are pairwise orthogonal, we obtain that $H_{x_{\mathbf{z}}, m_{\mathbf{z}}}^{\mathbf{x}} \cap w_{\mathbf{x}}^{\pi}$ contains the point $s_{\mathbf{x}}^{y}$. Thus we choose the vertex $\mathbf{h}$ in such manner that $h_{\mathbf{x}}$ is a line in $H_{x_{\mathbf{z}}, m_{\mathbf{z}}}^{\mathbf{x}} \cap w_{\mathbf{x}}^{\pi}$ going through the point $s_{\mathbf{x}}^{y}$.

Again by lemma 4.3.5 we find a vertex $\mathbf{l} \in \{\mathbf{h}, \mathbf{m}, \mathbf{x}, \mathbf{y}\}^{\perp}$ as $\mathbf{m} \perp \mathbf{h} \perp \mathbf{x} \perp \mathbf{y}$ and $\dim(\langle h_{\mathbf{x}}, y_{\mathbf{x}} \rangle) = 3$. Using lemma 4.3.8, the projective space $\mathcal{G}_{\mathbf{l}}$ contains unique pairwise orthogonal points $p_{\mathbf{l}}^{x}$, $p_{\mathbf{l}}^{y}$, $s_{\mathbf{l}}^{x}$ and $s_{\mathbf{l}}^{y}$ on the lines $m_{\mathbf{l}}$, $x_{\mathbf{l}}$, $y_{\mathbf{l}}$ and $h_{\mathbf{l}}$ induced from the one-dimensional subspaces $p_{\mathbf{h}}^{x}$, $p_{\mathbf{x}}^{y}$, $s_{\mathbf{h}}^{x}$ and $s_{\mathbf{x}}^{y}$. In particular, the point $p_{\mathbf{l}}^{y}$ of $\mathcal{G}_{\mathbf{l}}$ and the point $p_{\mathbf{h}}^{y}$ of $\mathcal{G}_{\mathbf{h}}$ induce each other and the two lines $h_{\mathbf{l}}$ and $y_{\mathbf{l}}$ intersect in the point $s_{\mathbf{l}}^{y}$. Furthermore $l_{\mathbf{x}} \subseteq H_{x_{\mathbf{h}}, m_{\mathbf{h}}}^{\mathbf{x}} \cap h_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi}$, where $H_{x_{\mathbf{h}}, m_{\mathbf{h}}}^{\mathbf{x}}$ is the unique plane in the subspace $h_{\mathbf{x}}^{\pi}$ induced from the plane $H_{x_{\mathbf{h}}, m_{\mathbf{h}}}^{\mathbf{h}} = \langle x_{\mathbf{h}}^{\pi}, m_{\mathbf{h}}^{\pi} \rangle$, and $w_{\mathbf{x}} \subseteq h_{\mathbf{x}}^{\pi} \cap y_{\mathbf{x}}^{\pi}$ implying that the lines $l_{\mathbf{x}}$ and $w_{\mathbf{x}}$ intersect each other.

As $\dim(\langle h_{\mathbf{l}}, y_{\mathbf{l}} \rangle) = 3$ by lemma 4.3.3 we find a vertex $\mathbf{n} \in \{\mathbf{h}, \mathbf{l}, \mathbf{y}, \mathbf{w}\}^{\perp}$ by application of lemma 4.3.5 to the path $\mathbf{h} \perp \mathbf{l} \perp \mathbf{y} \perp \mathbf{w}$. Certainly the points $p_{\mathbf{l}}^{y}$ and $s_{\mathbf{l}}^{y}$ induce unique orthogonal points $p_{\mathbf{n}}^{y}$ and $s_{\mathbf{n}}^{y}$ on the line $y_{\mathbf{n}}$. Since $\mathbf{y} \perp \mathbf{w}$ and $p_{\mathbf{n}}^{y} \subseteq h_{\mathbf{n}}^{\pi}$, it follows that $p_{\mathbf{n}}^{y} \subseteq h_{\mathbf{n}}^{\pi} \cap w_{\mathbf{n}}^{\pi}$. Furthermore $p_{\mathbf{h}}^{y} \subseteq w_{\mathbf{h}}^{\pi}$, as the three points $p_{\mathbf{l}}^{y}$ of $\mathcal{G}_{\mathbf{l}}$ and $p_{\mathbf{h}}^{y}$ of $\mathcal{G}_{\mathbf{h}}$ and $p_{\mathbf{n}}^{y}$ of $\mathcal{G}_{\mathbf{n}}$ induce each other. Thus the orthogonal space $w_{\mathbf{h}}^{\pi}$ contains the two intersecting lines $x_{\mathbf{h}}$ and $m_{\mathbf{x}}$ in $\mathcal{G}_{\mathbf{h}}$ implying that $\mathbf{m} \perp \mathbf{w}$, which verifies the claim for this special choosen setting.

Lemma 4.3.10 *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *and* $\mathbf{w}$ *be four different vertices in the graph* Γ *such that* $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ *and* $\langle z_{\mathbf{x}}, w_{\mathbf{x}} \rangle$ *is a plane in the projective space* $\mathcal{G}_{\mathbf{x}}$*. Then any vertex* $\mathbf{m} \in \mathbf{z}^{\perp}$*, which belongs to a line* $m_{\mathbf{z}}$ *intersecting the orthogonal lines* $x_{\mathbf{z}}$ *and* $y_{\mathbf{z}}$ *in* $\mathcal{G}_{\mathbf{z}}$*, is adjacent to the vertex* $\mathbf{w}$*.*

If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and $\mathbf{w}$ are four different vertices in the graph Γ such that $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ and $\langle z_{\mathbf{x}}, w_{\mathbf{x}} \rangle$ is of dimension four in $\mathcal{G}_{\mathbf{x}}$. Then $\langle z_{\mathbf{x}}, w_{\mathbf{x}} \rangle = y_{\mathbf{x}}^{\pi}$ and we choose a line $k_{\mathbf{x}}$ in the subspace $y_{\mathbf{x}}^{\pi}$ intersecting $z_{\mathbf{x}}$ and $w_{\mathbf{x}}$ in some point. Thus $\mathbf{k}$ is a vertex of the graph Γ adjacent to $\mathbf{x}$ and $\mathbf{y}$ and $\dim(\langle z_{\mathbf{x}}, k_{\mathbf{x}} \rangle) = 3 = \dim(\langle w_{\mathbf{x}}, k_{\mathbf{x}} \rangle)$. It follows by lemma 4.3.10 that any vertex $\mathbf{m} \in \mathbf{z}^{\perp}$, which belongs to a line $m_{\mathbf{z}}$ intersecting $x_{\mathbf{z}}$ and $y_{\mathbf{z}}$ in some point is adjacent to the vertices $\mathbf{k}$ and $\mathbf{w}$.

Lemma 4.3.11 *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *and* $\mathbf{w}$ *be four different vertices in the graph* Γ *such that* $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$*. Then any vertex* $\mathbf{m} \in \mathbf{z}^{\perp}$ *which belongs to a line* $m_{\mathbf{z}}$ *intersecting the orthogonal lines* $x_{\mathbf{z}}$ *and* $y_{\mathbf{z}}$ *in* $\mathcal{G}_{\mathbf{z}}$*, is adjacent to the vertex* $\mathbf{w}$*.*

We now turn to the construction of automorphisms of the graph $\widehat{\Gamma}$. Thus let Γ be a connected locally $\mathbf{S}(V_6)$ graph satisfying the condition that

$$|\{\mathbf{z}, \mathbf{w}\}^{\perp} \cap \mathbf{x}^{\perp}| = 1 \text{ if and only if } |\{\mathbf{z}, \mathbf{w}\}^{\perp} \cap \mathbf{y}^{\perp}| = 1$$

for any chain $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ in Γ of four different vertices $\mathbf{x}, \mathbf{w}, \mathbf{y}$ and $\mathbf{z}$ and $\widehat{\Gamma}$ be a 2-simply connected cover of Γ. The complex vector space of the projective space $\mathcal{G}_{\mathbf{x}}$ for some vertex $\mathbf{x}$ will be denoted by $V(\mathcal{G}_{\mathbf{x}})$.

Lemma 4.3.12 *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ *be four vertices of* $\widehat{\Gamma}$ *such that* $\mathbf{x} \perp \mathbf{z} \perp \mathbf{y} \perp \mathbf{w} \perp \mathbf{x}$*. Then we can determine a unitary isomorphism between the vector subspaces* $\langle w_{\mathbf{x}}, z_{\mathbf{x}} \rangle$ *of* $V(\mathcal{G}_{\mathbf{x}})$ *and* $\langle w_{\mathbf{y}}, z_{\mathbf{y}} \rangle$ *of* $V(\mathcal{G}_{\mathbf{y}})$ *such that each point* $p_{\mathbf{x}} \in \langle w_{\mathbf{x}}, z_{\mathbf{x}} \rangle$ *is mapped to the induced point* $p_{\mathbf{z}} \in \langle w_{\mathbf{y}}, z_{\mathbf{y}} \rangle$*.*

Proof: We choose an orthonormal basis $\alpha : u_1, \ldots, u_4$ of the subspace $\langle w_{\mathbf{x}}, z_{\mathbf{x}} \rangle$ such that $\langle u_1, u_2 \rangle = w_{\mathbf{x}}$ and $\langle u_3, u_4 \rangle = z_{\mathbf{x}}$. Let v_4 be a normal vector contained in the one-dimensional space $\langle u_4 \rangle$ and let $\alpha_{v_4} : v_1, \ldots, v_4$ together with v_{ii+1} for $1 \le i \le 3$ be the basis system w.r.t. v_4 of $\langle w_{\mathbf{x}}, z_{\mathbf{x}} \rangle$. Next we consider the points $p_{\mathbf{x}, v_j} = \langle v_j \rangle$ for $j \in J = \{1, \ldots, 4, 12, 23, 34\}$ of the projective space $\mathcal{G}_{\mathbf{x}}$, which induce unique points $p_{\mathbf{y}, v_j}$ for $j \in J$ by lemma 4.3.8. Any two point $p_{\mathbf{y}, v_j}$ and $p_{\mathbf{y}, v_t}$ of the subspace $\langle w_{\mathbf{y}}, z_{\mathbf{y}} \rangle$ have the same relation to each other as the points $p_{\mathbf{x}, v_j}$ and $p_{\mathbf{x}, v_t}$ by lemma 4.3.9 for any $j, t \in J$. Hence we fix a normal vector g_4 of the point $p_{\mathbf{y}, v_4}$ and compute the basis system $\beta_{g_4} : g_1, \ldots, g_4; g_{ii+1}$ for $1 \le i \le 3$ of $\langle w_{\mathbf{y}}, z_{\mathbf{y}} \rangle$ w.r.t. the vector g_4. The map $\gamma : \langle w_{\mathbf{x}}, z_{\mathbf{x}} \rangle \to \langle w_{\mathbf{y}}, z_{\mathbf{y}} \rangle$ with $v_j \mapsto g_j$ for $1 \le j \le 4$ is an isomorphism between the vector subspaces $\langle w_{\mathbf{x}}, z_{\mathbf{x}} \rangle$ and $\langle w_{\mathbf{y}}, z_{\mathbf{y}} \rangle$. Directly from the construction it follows that the isomorphism γ preserves the scalar product and the points, thus γ is a unitary isomorphism. ∎

Remark 4.3.13 The isomorphism γ between the subspaces $\langle w_{\mathbf{x}}, z_{\mathbf{x}}\rangle$ and $\langle w_{\mathbf{y}}, z_{\mathbf{y}}\rangle$ in lemma 4.3.12 is not unique. However, we claim that any two isomorphisms γ and $\hat{\gamma}$ differ only by a scalar $e^{i\lambda}$ in the sense that ${}_{\delta}[\gamma]_{\mu} = e^{i\lambda}{}_{\delta}[\hat{\gamma}]_{\mu}$ with δ a basis of $\langle w_{\mathbf{x}}, z_{\mathbf{x}}\rangle$ and μ a basis of $\langle w_{\mathbf{y}}, z_{\mathbf{y}}\rangle$.

We will prove this statement in two steps. Keeping the notation used in the proof of lemma 4.3.12, let $\alpha : u_1, \dots, u_4$ be an orthonormal basis of $\langle w_{\mathbf{x}}, z_{\mathbf{x}}\rangle$ such that $\langle u_1, u_2\rangle = w_{\mathbf{x}}$ and $\langle u_3, u_4\rangle = z_{\mathbf{x}}$. Then, as in the proof, we obtain the basis system α_{v_4} w.r.t. a normal vector v_4 of $p_{\mathbf{x},u_4}$ and the basis system β_{g_4} w.r.t. a normal normal vector g_4 of $p_{\mathbf{y},u_4} = p_{\mathbf{y},v_4}$. Suppose we choose the normal vector $\hat{v}_4$ instead of v_4 and the normal vector $\hat{g}_4$ in place of g_4. Then $\hat{v}_4 = e^{i\lambda}v_4$ and $\hat{g}_4 = e^{i\rho}g_4$ for some $\lambda, \rho \in [0, 2\pi[$. Furthermore ${}_{\alpha_{\hat{v}_4}}[id]_{\alpha_{v_4}} = e^{i\lambda}I$ and ${}_{\beta_{\hat{g}_4}}[id]_{\beta_{g_4}} = e^{i\rho}I$, where $\alpha_{\hat{v}_4}$ is the basis system of $\langle w_{\mathbf{x}}, z_{\mathbf{x}}\rangle$ w.r.t. $\hat{v}_4$ and $\beta_{\hat{g}_4}$ is the basis system of $\langle w_{\mathbf{y}}, z_{\mathbf{y}}\rangle$ w.r.t. $\hat{g}_4$. As the two unitary isomorphisms γ and $\hat{\gamma}$ from $\langle w_{\mathbf{x}}, z_{\mathbf{x}}\rangle$ to $\langle w_{\mathbf{y}}, z_{\mathbf{y}}\rangle$ have the matrix representation ${}_{\beta_{g_4}}[\gamma]_{\alpha_{v_4}} = I$ with respect to the bases α_{v_4} and β_{g_4}, respectively ${}_{\beta_{\hat{g}_4}}[\hat{\gamma}]_{\alpha_{\hat{v}_4}} = I$ with respect to bases $\alpha_{\hat{v}_4}$ and $\beta_{\hat{g}_4}$, we get that ${}_{\beta_{\hat{g}_4}}[\gamma]_{\alpha_{\hat{v}_4}} = {}_{\beta_{\hat{g}_4}}[id]_{\beta_{g_4}} \cdot {}_{\beta_{g_4}}[\gamma]_{\alpha_{v_4}} \cdot {}_{\alpha_{v_4}}[id]_{\alpha_{\hat{v}_4}} = e^{i\rho} \cdot e^{i\lambda}I = e^{i\rho} \cdot e^{i\lambda} \cdot {}_{\beta_{\hat{g}_4}}[\hat{\gamma}]_{\alpha_{\hat{v}_4}}$.

For the next step let $\alpha_u : u_1, \dots, u_4$ and $\alpha_r : r_1, \dots, r_4$ be two orthonormal bases of $\langle w_{\mathbf{x}}, z_{\mathbf{x}}\rangle$ such that $\langle u_1, u_2\rangle = \langle r_1, r_2\rangle = w_{\mathbf{x}}$ and $\langle u_3, u_4\rangle = \langle r_4, r_4\rangle = z_{\mathbf{x}}$. Again we construct two basis systems α_{v_4} and α_{r_4} w.r.t. some normal vectors v_4 and r_4 and obtain the basis transformation matrix ${}_{\hat{\alpha}_{r_4}}[id]_{\alpha_{v_4}}$.

As described in the proof of lemma 4.3.12 we also obtain the two basis systems β_{g_4} and β_{s_4} w.r.t. some normal vectors g_4 of the point $p_{\mathbf{y},v_4}$ and s_4 of the point $p_{\mathbf{y},r_4}$ and the resulting unitary isomorphisms $\gamma : \langle w_{\mathbf{x}}, z_{\mathbf{x}}\rangle \to \langle w_{\mathbf{y}}, z_{\mathbf{y}}\rangle$ with $v_j \mapsto g_j$ and $\hat{\gamma} : \langle w_{\mathbf{x}}, z_{\mathbf{x}}\rangle \to \langle w_{\mathbf{y}}, z_{\mathbf{y}}\rangle : r_j \mapsto s_j$, for $1 \le j \le 4$. If $(v_4, r_4) = c \neq 0$, then the vectors g_4 and s_4 are not perpendicular, in fact $e^{i\lambda}c = (g_4, s_4)$ for some $\lambda \in [0, 2\pi[$. In case $(v_4, r_4) = 0$, then we consider the normal vector $\frac{1}{\sqrt{2}}(v_4 + r_4) = v_{44}$, so that $(v_{44}, v_4) = \frac{1}{\sqrt{2}} = (v_{44}, r_4)$. Define $p_{\mathbf{x},v_{44}} = \langle v_{44}\rangle$ in $\mathcal{G}_{\mathbf{x}}$, which induces the unique point $p_{\mathbf{y},v_{44}} \subseteq \langle w_{\mathbf{y}}, z_{\mathbf{y}}\rangle$. The one-dimensional space $p_{\mathbf{y},v_{44}}$ contains a unique normal vector g_{44} such that $(g_{44}, g_4) = \frac{1}{\sqrt{2}}$, thus $(g_{44}, s_4) = \frac{1}{\sqrt{2}}e^{i\lambda}$ for some $\lambda \in [0, 2\pi[$. Therefore in both cases it follows that the basis transformation matrix between β_{g_4} and β_{s_4} is ${}_{\beta_{s_4}}[id]_{\beta_{g_4}} = e^{i\lambda} \cdot {}_{\alpha_{r_4}}[id]_{\alpha_{v_4}}$, hence ${}_{\beta_{s_4}}[\gamma]_{\alpha_{r_4}} = {}_{\beta_{s_4}}[id]_{\beta_{g_4}} \cdot {}_{\beta_{g_4}}[\gamma]_{\alpha_{v_4}} \cdot {}_{\alpha_{v_4}}[id]_{\alpha_{r_4}} = e^{i\lambda}{}_{\alpha_{r_4}}[id]_{\alpha_{v_4}} \cdot {}_{\alpha_{v_4}}[id]_{\alpha_{r_4}} = e^{i\lambda} \cdot {}_{\beta_{s_4}}[\hat{\gamma}]_{\alpha_{r_4}}$.

Since all these isomorphisms differ only by a scalar, we can fix one isomorphism and denote it by $\theta = \theta^{\langle w_{\mathbf{y}}, z_{\mathbf{y}}\rangle}_{\langle w_{\mathbf{x}}, z_{\mathbf{x}}\rangle} : \langle w_{\mathbf{x}}, z_{\mathbf{x}}\rangle \to \langle w_{\mathbf{y}}, z_{\mathbf{y}}\rangle$. A vector $v \in \langle w_{\mathbf{x}}, z_{\mathbf{x}}\rangle$ has the image $\theta^{\langle w_{\mathbf{y}}, z_{\mathbf{y}}\rangle}_{\langle w_{\mathbf{x}}, z_{\mathbf{x}}\rangle}(v) = \theta(v) = v_{\theta}$.

Lemma 4.3.14 *Let* $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *be mutually adjacent vertices of* $\widehat{\Gamma}$. *An* $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$\mathcal{G}_{\mathbf{y}}$ *action on the projective space* $\mathcal{G}_{\mathbf{y}}$ *induced by the* $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$V(\mathcal{G}_{\mathbf{y}})$ *action can be uniquely* *extended to an action* $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$(\mathcal{G}_\mathbf{y} \cup \mathcal{G}_\mathbf{z}) : \mathrm{SU}_2(\mathbb{C}) \times \mathcal{G}_\mathbf{y} \cup \mathcal{G}_\mathbf{z} \to \mathcal{G}_\mathbf{y} \cup \mathcal{G}_\mathbf{z}$ *such that* $(\varphi_{\mathcal{G}_\mathbf{y} \cup \mathcal{G}_\mathbf{z}})_{|\mathcal{G}_\mathbf{x}} = \mathrm{id}_{|(\mathcal{G}_\mathbf{y} \cup \mathcal{G}_\mathbf{z}) \cap \mathcal{G}_\mathbf{x}}$ *for each* $\varphi \in \mathrm{SU}_2(\mathbb{C})$ *and the action* $(\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$(\mathcal{G}_\mathbf{y} \cup \mathcal{G}_\mathbf{z}))_{|\mathcal{G}_\mathbf{z}}$ *is the* $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$\mathcal{G}_\mathbf{z}$ *action on the projective space* $\mathcal{G}_\mathbf{z}$.

In particular, if a vector space automorphism $\varphi_{V(\mathcal{G}_\mathbf{y})}$ *has a matrix representation of the shape given in 4.2.5 with respect to a basis system* $\alpha_{u_6} : u_1, \cdots, u_6; u_{jj+1}$ *with* $1 \leq j \leq 5$ *of* $V(\mathcal{G}_\mathbf{y})$ *w.r.t.* u_6 *satisfying* $\langle u_5, u_6 \rangle = x_\mathbf{y}$ *, then the corresponding vector space automorphism* $\varphi_{V(\mathcal{G}_\mathbf{z})^\mathbf{y}}$ *also has a matrix representation of the shape given in 4.2.5 with respect to a basis system* $\beta_{v_6} : v_1, \cdots, v_6; v_{jj+1}$ *with* $1 \leq j \leq 5$ *of* $V(\mathcal{G}_\mathbf{y})$ *satisfying* $\langle v_5, v_6 \rangle = x_\mathbf{z}$ *such that the points* $\langle u_j \rangle \subseteq \mathcal{G}_\mathbf{y}$ *and* $\langle v_j \rangle \subseteq \mathcal{G}_\mathbf{z}$ *correspond to each other for* $3 \leq i \leq 6$.

Proof: In view of 4.2.5 and 4.2.6, we can study the $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$V(\mathcal{G}_\mathbf{y})$ action, where $V(\mathcal{G}_\mathbf{y})$ is the complex unitary vector space corresponding to $\mathcal{G}_\mathbf{y}$. By the construction described in remark 4.2.7, we can assume that $\alpha_{u_6} : u_1, u_2, u_3, u_4, u_5, u_6; u_{jj+1}$ for $1 \leq j \leq 5$ is a basis system of the vector space $V(\mathcal{G}_\mathbf{y})$ with $\langle u_1, u_2 \rangle = z_\mathbf{y}$, $\langle u_3, u_4 \rangle = z^\pi_\mathbf{y} \cap x^\pi_\mathbf{y}$ and $\langle u_5, u_6 \rangle = x_\mathbf{y}$. The matrix representation $[\varphi_{V(\mathcal{G}_\mathbf{y})}]_{\alpha_{u_6}}$ of the automorphism $\varphi_{V(\mathcal{G}_\mathbf{y})}$ relative to the basis α_{u_6} for every $\varphi \in \mathrm{SU}_2(\mathbb{C})$ is $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & [\varphi]_{\alpha_{u_6}^{x_\mathbf{y}}} \end{pmatrix}$. Therefore the restriction $(\varphi_{V(\mathcal{G}_\mathbf{y})})_{|z^\pi_\mathbf{y}}$ of the automorphism $\varphi_{V(\mathcal{G}_\mathbf{y})}$ to the subspace $z^\pi_\mathbf{y}$ has the matrix representation $[(\varphi_{V(\mathcal{G}_\mathbf{y})})_{|z^\pi_\mathbf{y}}]_{\alpha_{u_6}^{z^\pi_\mathbf{y}}} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & [\varphi]_{(\alpha_{u_6})^{x_\mathbf{y}}} \end{pmatrix}$ with respect to the basis $\alpha_{u_6}^{z^\pi_\mathbf{y}} : u_3, u_4, u_5, u_6$ of $z^\pi_\mathbf{y}$. Thus the restriction of the $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$V(\mathcal{G}_\mathbf{y})$- action to the four-dimensional subspace $z^\pi_\mathbf{y}$ is a $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-action on $z^\pi_\mathbf{y}$ which we denote by $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$z^\pi_\mathbf{y} : \mathrm{SU}_2(\mathbb{C}) \times z^\pi_\mathbf{y} \to z^\pi_\mathbf{y}$ with $(\varphi, w_\mathbf{y}) \mapsto [(\varphi_{V(\mathcal{G}_\mathbf{y})})_{|z^\pi_\mathbf{y}}]_{\alpha_{u_6}^{z^\pi_\mathbf{y}}} \cdot [w_\mathbf{y}]_{\alpha_{u_6}^{z^\pi_\mathbf{y}}}$.

The four-dimensional subspace $z^\pi_\mathbf{y}$ of $\mathcal{G}_\mathbf{y}$ corresponds to the subspace $y^\pi_\mathbf{z}$ of $\mathcal{G}_\mathbf{z}$, every interior line $k_\mathbf{y}$ of $z^\pi_\mathbf{y}$ corresponds to the interior line $k_\mathbf{z}$ of $y^\pi_\mathbf{z}$ and every point $p_\mathbf{y}$ of $z^\pi_\mathbf{y}$ corresponds to the point $p_\mathbf{z}$ incident to $y^\pi_\mathbf{z}$ in $\mathcal{G}_\mathbf{z}$ by lemma 4.3.8. Therefore the points $p_{j,\mathbf{y}} = \langle u_j \rangle$, $j \in \{3, 4, 5, 6, 34, 45, 56\} = J$, inside $\mathcal{G}_\mathbf{y}$ induce points $p_{j,\mathbf{z}}$ in $\mathcal{G}_\mathbf{z}$ for each $j \in J$ having identical properties. In particular $p_{3,\mathbf{z}}$, $p_{4,\mathbf{z}}$, $p_{5,\mathbf{z}}$, $p_{6,\mathbf{z}}$ are pairwise orthogonal with $\langle p_{5,\mathbf{z}}, p_{6,\mathbf{z}} \rangle = x_\mathbf{z}$ and $\langle p_{3,\mathbf{z}}, p_{4,\mathbf{z}} \rangle = x^\pi_\mathbf{z} \cap y^\pi_\mathbf{z}$. Choosing a normal vector v_6 of the point $p_{6,\mathbf{z}}$ we determine the basis system $\beta_{v_6}^{y^\pi_\mathbf{z}} : v_3, v_4, v_5, v_6; v_{jj+1}$ for $3 \leq j \leq 6$ of $y^\pi_\mathbf{z}$ w.r.t. v_6. Hence each automorphism $(\varphi_{V(\mathcal{G}_\mathbf{y})})_{|z^\pi_\mathbf{y}}$ with $\varphi \in \mathrm{SU}_2(\mathbb{C})$ induces a unique automorphism $\varphi_{y^\pi_\mathbf{z}}$ of $y^\pi_\mathbf{z}$ with matrix representation $[\varphi_{y^\pi_\mathbf{z}}]_{\beta_{v_6}^{y^\pi_\mathbf{z}}} = [(\varphi_{V(\mathcal{G}_\mathbf{y})})_{|z^\pi_\mathbf{y}}]_{\alpha_{u_6}^{z^\pi_\mathbf{y}}}$ w.r.t. the basis $\beta_{v_6}^{y^\pi_\mathbf{z}}$, so that the $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$z^\pi_\mathbf{y}$ action determines an $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$y^\pi_\mathbf{z}$ action. Notice that by remark 4.3.13 the $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$V(y^\pi_\mathbf{z})$ action is only determined up to multiplication with $e^{i\lambda}$, $\lambda \in [0, 2\pi[$. All these $\mathbf{x}$-$SU_2(\mathbb{C})$-$V(y_{\mathbf{z}}^{\pi})$ actions, however, induce the same $\mathbf{x}$-$SU_2(\mathbb{C})$-$y_{\mathbf{z}}^{\pi}$ action, which therefore is unique.

Moreover each automorphism $\varphi_{y_{\mathbf{z}}^{\pi}}$ of the subspace $y_{\mathbf{z}}^{\pi}$ can be extended to the automorphism $\varphi_{V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}}$ such that $(\varphi_{V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}})_{|y_{\mathbf{z}}} = id_{|y_{\mathbf{z}}}$. Indeed we complete the basis $\beta_{v_6}^{y_{\mathbf{z}}^{\pi}}$ of the subspace $y_{\mathbf{z}}^{\pi}$ to a basis $\beta : v_1, v_2, v_3, v_4, v_5, v_6$ such that vectors v_1 and v_2 have length one and span the line $y_{\mathbf{z}}$ of $V(\mathcal{G}_{\mathbf{z}})$. We define $\varphi_{V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}}$ to be the automorphism

$$\begin{array}{rcl} \varphi_{V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}} : V(\mathcal{G}_{\mathbf{z}}) & \to & V(\mathcal{G}_{\mathbf{z}}) \\ w & \mapsto & \begin{pmatrix} 1 & & \\ & 1 & \\ & & [(\varphi_{V(\mathcal{G}_{\mathbf{y}})^{\mathbf{y}}})_{|z_{\mathbf{y}}^{\pi}}]_{\alpha_{u_6}^{z_{\mathbf{y}}^{\pi}}} \end{pmatrix} \cdot [w]_{\beta} \end{array}$$

and the desired $\mathbf{x}$-$SU_2(\mathbb{C})$-$V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}$ action on $V(\mathcal{G}_{\mathbf{z}})$ has the description:

$$\begin{array}{rcl} \mathbf{x}\text{-}SU_2(\mathbb{C})\text{-}V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}} : SU_2(\mathbb{C}) \times V(\mathcal{G}_{\mathbf{z}}) & \to & V(\mathcal{G}_{\mathbf{z}}) \\ (\varphi, w) & \mapsto & \begin{pmatrix} 1 & & \\ & 1 & \\ & & [(\varphi_{V(\mathcal{G}_{\mathbf{y}})^{\mathbf{y}}})_{|z_{\mathbf{y}}^{\pi}}]_{\alpha_{u_6}^{z_{\mathbf{y}}^{\pi}}} \end{pmatrix} \cdot [w]_{\beta}\,. \end{array}$$

By definition 4.2.6 this $\mathbf{x}$-$SU_2(\mathbb{C})$-$V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}$-action on the vector space $V(\mathcal{G}_{\mathbf{z}})$ defines a unique action $\mathbf{x}$-$SU_2(\mathbb{C})$-$\mathcal{G}_{\mathbf{z}}^{\mathbf{y}}$ on the projective space $\mathcal{G}_{\mathbf{z}}$. Since the subspace $x_{\mathbf{z}}^{\pi}$ is spanned by the basis vectors $v_1, \ldots, v_4$ we obtain that

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & [(\varphi_{V(\mathcal{G}_{\mathbf{y}})^{\mathbf{y}}})_{|z_{\mathbf{y}}^{\pi}}]_{\alpha_{u_6}^{z_{\mathbf{y}}^{\pi}}} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & [\varphi]_{\alpha_{u_6}^{x_{\mathbf{y}}}} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & [\varphi]_{\beta^{x_{\mathbf{y}}}} \end{pmatrix} = [\varphi_{V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}}]_{\beta}$$

is the matrix representation of the automorphism $\varphi_{V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}}$ for some $\varphi \in SU_2(\mathbb{C})$. Certainly the restriction map $(\varphi_{V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}})_{|x_{\mathbf{z}}^{\pi}}$ is the identity of the subspace $x_{\mathbf{z}}^{\pi}$ for each $\varphi \in SU_2(\mathbb{C})$. Notice also that $[\varphi_{V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}}]_{\beta} \in SU_6(\mathbb{C})$, so $\varphi_{V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}}$ is a unitary automorphism.

The above construction of the automorphism $\varphi_{V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}}$ for each $\varphi \in SU_2(\mathbb{C})$ relies on the orthonormal basis β of the vector space $V(\mathcal{G}_{\mathbf{z}})$ induced from the basis α_{u_6} of $V(\mathcal{G}_{\mathbf{y}})$. It remains to prove the independence of this construction from the choice of the bases. To this end let $\alpha : a_1, \ldots, a_6$ and $\hat{\alpha} : \hat{a}_1, \ldots, \hat{a}_6$ be two orthonormal bases of $V(\mathcal{G}_{\mathbf{y}})$ with $\langle a_1, a_2\rangle = z_{\mathbf{y}} = \langle \hat{a}_1, \hat{a}_2\rangle$ and $\langle a_3, a_4\rangle = z_{\mathbf{y}}^{\pi} \cap x_{\mathbf{y}}^{\pi} = \langle \hat{a}_3, \hat{a}_4\rangle$ and $\langle a_5, a_6\rangle = x_{\mathbf{y}} = \langle \hat{a}_5, \hat{a}_6\rangle$.

Remark 4.2.7 yields two basis systems $\alpha_{u_6} : u_1, \ldots, u_6; u_{jj+1}$, $1 \le j \le 5$ w.r.t. a normal vector u_6 of $\langle a_6\rangle$ and $\hat{\alpha}_{\hat{u}_6} : \hat{u}_1, \ldots, \hat{u}_6; \hat{u}_{jj+1}$, $1 \le j \le 5$ w.r.t. a normal vector $\hat{u}_6$ of $\langle \hat{a}_6\rangle$ of $V(\mathcal{G}_{\mathbf{y}})$ and a transformation matrix ${}_{\hat{\alpha}_{\hat{u}_6}}[id]_{\alpha_{u_6}}$ from the basis α_{u_6} to the basis $\hat{\alpha}_{\hat{u}_6}$. Moreover $\alpha_{u_6}^{z_{\mathbf{y}}^{\pi}} : u_3, \ldots, u_6$ and $\hat{\alpha}_{\hat{u}_6}^{z_{\mathbf{y}}^{\pi}} : \hat{u}_3, \ldots, \hat{u}_6$ are orthonormal bases of $z_{\mathbf{y}}^{\pi}$ with transformation matrix ${}_{\hat{\alpha}_{\hat{u}_6}^{z_{\mathbf{y}}^{\pi}}}[id]_{\alpha_{u_6}^{z_{\mathbf{y}}^{\pi}}}$. Define $p_{j,\mathbf{y}} = \langle u_j \rangle \subseteq z_{\mathbf{y}}^{\pi}$ and $q_{j,\mathbf{y}} = \langle \hat{u}_j \rangle \subseteq z_{\mathbf{y}}^{\pi}$ for every $j \in J = \{3,4,5,6,34,45,56\}$. The points $p_{j,\mathbf{y}}$ induce unique points $p_{j,\mathbf{z}}$ in $y_{\mathbf{z}}^{\pi}$, for each $j \in J$; similarly, the points $q_{j,\mathbf{y}}$ induce points $q_{j,\mathbf{z}}$. Fix normal vectors $v_6 \in p_{6,\mathbf{z}}$ and $\hat{v}_6 \in q_{6,\mathbf{z}}$, so that remark 4.2.7 again determines basis systems $\beta_{v_6}^{y_{\mathbf{z}}^{\pi}} : v_3, \ldots, v_6; v_{jj+1}$ for $3 \leq j \leq 6$ w.r.t. v_6 and $\hat{\beta}_{\hat{v}_6}^{y_{\mathbf{z}}^{\pi}} : \hat{v}_3, \ldots, \hat{v}_6; \hat{v}_{jj+1}$ for $3 \leq j \leq 6$ w.r.t. $\hat{v}_6$. As the subspaces $z_{\mathbf{y}}^{\pi}$ and $y_{\mathbf{z}}^{\pi}$ are isomorphic to each other we can fix the isomorphism $\theta = \theta_{z_{\mathbf{y}}^{\pi}}^{y_{\mathbf{z}}^{\pi}} : z_{\mathbf{y}}^{\pi} \to y_{\mathbf{z}}^{\pi}$ of remark 4.3.13 which maps each vector w of $z_{\mathbf{y}}^{\pi}$ to the corresponding vector $\theta(w)$ of $y_{\mathbf{z}}^{\pi}$, in particular u_j maps to $\theta(u_j)$ and $\hat{u}_j$ to $\theta(\hat{u}_j)$ for each $j \in J$. As θ is an isomorphism between $z_{\mathbf{y}}^{\pi}$ and $y_{\mathbf{z}}^{\pi}$ both $\beta_{\theta(u_6)} : \theta(u_3), \ldots, \theta(u_6)$ and $\hat{\beta}_{\theta(\hat{u}_6)} : \theta(\hat{u}_3), \ldots, \theta(\hat{u}_6)$ are orthonormal bases of $y_{\mathbf{z}}^{\pi}$ in $V(\mathcal{G}_{\mathbf{z}})$ with the basis transformation matrix ${}_{\hat{\beta}_{\theta(\hat{u}_6)}}[id]_{\beta_{\theta(u_6)}} = {}_{\hat{\alpha}_{\hat{u}_6}^{z_{\mathbf{y}}^{\pi}}}[id]_{\alpha_{u_6}^{z_{\mathbf{y}}^{\pi}}}$. Furthermore $\theta(u_6) = e^{i\lambda} v_6$ and $\theta(\hat{u}_6) = e^{i\mu} \hat{v}_6$ for some $\lambda, \mu \in [0, 2\pi[$, which implies that ${}_{\beta_{v_6}^{y_{\mathbf{z}}^{\pi}}}[id]_{\beta_{\theta(u_6)}} = e^{-i\lambda} \cdot I$ and ${}_{\hat{\beta}_{\hat{v}_6}^{y_{\mathbf{z}}^{\pi}}}[id]_{\hat{\beta}_{\theta(\hat{u}_6)}} = e^{-i\mu} \cdot I$, in particular ${}_{\hat{\beta}_{\hat{v}_6}^{y_{\mathbf{z}}^{\pi}}}[id]_{\beta_{v_6}^{y_{\mathbf{z}}^{\pi}}} = {}_{\hat{\beta}_{\hat{v}_6}^{y_{\mathbf{z}}^{\pi}}}[id]_{\hat{\beta}_{\theta(\hat{u}_6)}} \cdot {}_{\hat{\beta}_{\theta(\hat{u}_6)}}[id]_{\beta_{\theta(u_6)}} \cdot {}_{\beta_{\theta(u_6)}}[id]_{\beta_{v_6}^{y_{\mathbf{z}}^{\pi}}} = e^{i\lambda} \cdot e^{-i\mu} \cdot {}_{\hat{\alpha}_{\hat{u}_6}^{z_{\mathbf{y}}^{\pi}}}[id]_{\alpha_{u_6}^{z_{\mathbf{y}}^{\pi}}}$ describe the basis transformation between the bases $\beta_{v_6}^{y_{\mathbf{z}}^{\pi}}$ and $\hat{\beta}_{\hat{v}_6}^{y_{\mathbf{z}}^{\pi}}$.

The independence is proved if we can show that $[\varphi_{V(\mathcal{G}_z)^{\mathbf{y}}}]_{\beta} = {}_{\beta}[id]_{\hat{\beta}} \cdot [\varphi_{V(\mathcal{G}_z)^{\mathbf{y}}}]_{\hat{\beta}} \cdot {}_{\hat{\beta}}[id]_{\beta}$ for any $\varphi \in \mathrm{SU}_2(\mathbb{C})$, any basis completion $\beta : v_1, \ldots, v_6$ from $\beta_{v_6}^{y_z^{\pi}}$ such that the vectors v_1 and v_2 span the two-dimensional subspace y_z and any basis completion $\hat{\beta} : \hat{v}_1, \ldots, \hat{v}_6$ from $\hat{\beta}_{\hat{v}_6}^{y_z^{\pi}}$ with $\langle \hat{v}_1, \hat{v}_2 \rangle = y_z$. Using the special shape of the basis transformation matrix ${}_{\beta}[id]_{\hat{\beta}} = \begin{pmatrix} P & 0 \\ 0 & {}_{\beta_{v_6}^{y_{\mathbf{z}}^{\pi}}}[id]^{-1}_{\hat{\beta}_{\hat{v}_6}^{y_{\mathbf{z}}^{\pi}}} \end{pmatrix}$, where $P = {}_{v_1,v_2}[id]_{\hat{v}_1,\hat{v}_2}$, to obtain that

$$
\begin{aligned}
& {}_{\beta}[id]_{\hat{\beta}} \cdot [\varphi_{V(\mathcal{G}_z)}]_{\hat{\beta}} \cdot {}_{\hat{\beta}}[id]_{\beta} \\
= \quad & \begin{pmatrix} P & 0 \\ 0 & {}_{\beta_{v_6}^{y_{\mathbf{z}}^{\pi}}}[id]_{\hat{\beta}_{\hat{v}_6}^{y_{\mathbf{z}}^{\pi}}} \end{pmatrix} \cdot \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & [\varphi]_{(\hat{\alpha}_{\hat{u}_6})_{x_{\mathbf{y}}}} \end{pmatrix} \cdot \begin{pmatrix} P^{-1} & 0 \\ 0 & {}_{\hat{\beta}_{\hat{v}_6}^{y_{\mathbf{z}}^{\pi}}}[id]_{\beta_{v_6}^{y_{\mathbf{z}}^{\pi}}} \end{pmatrix} \\
= \quad & \begin{pmatrix} P \cdot P^{-1} & \\ & {}_{\beta_{v_6}^{y_{\mathbf{z}}^{\pi}}}[id]_{\hat{\beta}_{\hat{v}_6}^{y_{\mathbf{z}}^{\pi}}} \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & & [\varphi]_{(\hat{\alpha}_{\hat{u}_6})_{x_{\mathbf{y}}}} \end{pmatrix} \cdot {}_{\hat{\beta}_{\hat{v}_6}^{y_{\mathbf{z}}^{\pi}}}[id]_{\beta_{v_6}^{y_{\mathbf{z}}^{\pi}}} \end{pmatrix} \\
= \quad & \begin{pmatrix} 1 & & \\ & 1 & \\ & & (e^{i\lambda})^{-1} \cdot e^{i\mu} \cdot {}_{\alpha_{u_6}^{z_{\mathbf{y}}^{\pi}}}[id]_{\hat{\alpha}_{\hat{u}_6}^{z_{\mathbf{y}}^{\pi}}} \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & & [\varphi]_{(\hat{\alpha}_{\hat{u}_6})_{x_{\mathbf{y}}}} \end{pmatrix} \cdot e^{i\lambda} \cdot (e^{i\mu})^{-1} \cdot {}_{\hat{\alpha}_{\hat{u}_6}^{z_{\mathbf{y}}^{\pi}}}[id]_{\alpha_{u_6}^{z_{\mathbf{y}}^{\pi}}} \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & {}_{\alpha_{u_6}^{z_{\mathbf{y}}^{\pi}}}[id]_{\hat{\alpha}_{\hat{u}_6}^{z_{\mathbf{y}}^{\pi}}} \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & & [\varphi]_{(\hat{\alpha}_{\hat{u}_6})_{x_{\mathbf{y}}}} \end{pmatrix} \cdot {}_{\hat{\alpha}_{\hat{u}_6}^{z_{\mathbf{y}}^{\pi}}}[id]_{\alpha_{u_6}^{z_{\mathbf{y}}^{\pi}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & [\varphi]_{(\alpha_{u_6})_{x_{\mathbf{y}}}} \end{pmatrix} = [\varphi_{V(\mathcal{G}_{\mathbf{z}})^{\mathbf{y}}}]_{\beta} ,$$

and we are done. ■

Proposition 4.3.15 *Let $\mathbf{x} \perp \mathbf{y}$ be vertices of $\widehat{\Gamma}$. An $\mathbf{x}$-$SU_2(\mathbb{C})$-$\mathcal{G}_{\mathbf{y}}$ action on $\mathcal{G}_{\mathbf{y}}$ induced by the $\mathbf{x}$-$SU_2(\mathbb{C})$-$V(\mathcal{G}_{\mathbf{y}})$ action can be uniquely extended to an action $\mathbf{x}$-$SU_2(\mathbb{C})$-$\bigcup_{\mathbf{a}\in\mathbf{x}^\perp} \mathcal{G}_{\mathbf{a}} : SU_2(\mathbb{C}) \times \bigcup_{\mathbf{a}\in\mathbf{x}^\perp} \mathcal{G}_{\mathbf{a}} \to \bigcup_{\mathbf{a}\in\mathbf{x}^\perp} \mathcal{G}_{\mathbf{a}}$ such that $(\varphi_{\bigcup_{\mathbf{a}\in\mathbf{x}^\perp} \mathcal{G}_{\mathbf{a}}})_{|\mathcal{G}_{\mathbf{x}}} = \mathrm{id}_{|\mathcal{G}_{\mathbf{x}}}$ for each $\varphi \in SU_2(\mathbb{C})$ and that for each vertex $\mathbf{w} \in \mathbf{x}^\perp$ the action $(\mathbf{x}$-$SU_2(\mathbb{C})$-$\bigcup_{\mathbf{a}\in\mathbf{x}^\perp} \mathcal{G}_{\mathbf{a}})_{|\mathcal{G}_{\mathbf{w}}}$ is an $\mathbf{x}$-$SU_2(\mathbb{C})$-$\mathcal{G}_{\mathbf{w}}$ action on the projective space $\mathcal{G}_{\mathbf{w}}$.*

Proof: Since the induced subgraph $\mathbf{x}^\perp$ is connected by lemma 4.2.2, we can use lemma 4.3.14 inductively to define the automorphism $\varphi_{V(\mathcal{G}_{\mathbf{a}})^{\mathbf{c}_n}}$ uniquely up to multiplication with $e^{i\lambda}, \lambda \in [0, 2\Pi[$ for $\varphi \in SU_2(\mathbb{C})$ and arbitrary $\mathbf{a} \in \mathbf{x}^\perp$ along a path $\gamma : \mathbf{y} \perp \mathbf{c}_1 \perp \ldots \perp \mathbf{c}_n \perp \mathbf{a}$ from $\mathbf{y}$ to $\mathbf{a}$ in $\mathbf{x}^\perp$. The automorphism $\varphi_{V(\mathcal{G}_{\mathbf{a}})^{\mathbf{c}_n}}$ and the resulting $\mathbf{x}$-$SU_2(\mathbb{C})$-$V(\mathcal{G}_{\mathbf{a}})^{\mathbf{c}_n}$ and $\mathbf{x}$-$SU_2(\mathbb{C})$-$\mathcal{G}_{\mathbf{a}}^{\mathbf{c}_n}$ actions may depend on the path γ. Thus it remains to prove path independence. In order to achieve this, it suffices to prove that this induced $\mathbf{x}$-$SU_2(\mathbb{C})$-$\mathcal{G}_{\mathbf{a}}^{\mathbf{c}_n}$ action coincides with the natural $\mathbf{x}$-$SU_2(\mathbb{C})$-$\mathcal{G}_{\mathbf{a}}$ action on $\mathcal{G}_{\mathbf{a}}$ with respect to the direct decomposition $V(\mathcal{G}_{\mathbf{a}}) = x_{\mathbf{a}} \oplus x_{\mathbf{a}}^{\pi}$, cf. by remark 4.2.5.

The matrix representation of the automorphism $\varphi_{V(\mathcal{G}_{\mathbf{c}_1})^{\mathbf{y}}}$ w.r.t. some orthonormal basis $\alpha : a_1, \ldots a_6$ such that $\langle a_5, a_6 \rangle = x_{\mathbf{c}_1}$ is $[\varphi_{V(\mathcal{G}_{\mathbf{c}_1})^{\mathbf{y}}}]_{\alpha} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & [\varphi]_{\alpha^{x_{\mathbf{c}_1}}} \end{pmatrix}$.
The automorphism $\varphi_{V(\mathcal{G}_{\mathbf{c}_1})}$ from the natural $\mathbf{x}$-$SU_2(\mathbb{C})$-$V(\mathcal{G}_{\mathbf{c}_1})$ action on the vector space $V(\mathcal{G}_{\mathbf{c}_1})$ has the matrix representation $[\varphi_{V(\mathcal{G}_{\mathbf{c}_1})}]_{\alpha} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & [\varphi]_{\alpha^{x_{\mathbf{c}_1}}} \end{pmatrix}$ by remark 4.2.5, thus $\varphi_{V(\mathcal{G}_{\mathbf{c}_1})} = \varphi_{V(\mathcal{G}_{\mathbf{c}_1})^{\mathbf{y}}}$ for each $\varphi \in SU_2(\mathbb{C})$ and the $\mathbf{x}$-$SU_2(\mathbb{C})$ actions $\mathbf{x}$-$SU_2(\mathbb{C})$-$V(\mathcal{G}_{\mathbf{c}_1}^{\mathbf{y}})$ and $\mathbf{x}$-$SU_2(\mathbb{C})$-$V(\mathcal{G}_{\mathbf{c}_1})$ on the vector space $V(\mathcal{G}_{\mathbf{c}_1})$ coincide.

Using this argument inductively we see that for every path $\gamma : \mathbf{y} \perp \mathbf{c}_1 \perp \ldots \perp \mathbf{c}_n \perp \mathbf{a}$ from the vertex $\mathbf{y}$ to the vertex $\mathbf{a}$ in the subgraph $\mathbf{x}^\perp$ the induced $\mathbf{x}$-$SU_2(\mathbb{C})$

action $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$V(\mathcal{G}_{\mathbf{a}}^{\mathbf{c}_n})$ and the natural $\mathbf{x}$-$\mathrm{SU}_2(\mathbb{C})$-$V(\mathcal{G}_{\mathbf{a}})$ action on the vector space $V(\mathcal{G}_{\mathbf{a}})$ coincide. ∎

Lemma 4.3.16 *Let $\mathbf{x} \perp \mathbf{y}$ be vertices of $\widehat{\Gamma}$ and let α and β be automorphisms of $\widehat{\Gamma}$ fixing the subgraph $\mathbf{x}^\perp$ and the complex vector space $V(\mathcal{G}_{\mathbf{x}})$ elementwise and $\alpha_{|x_{\mathbf{y}}} = \beta_{|x_{\mathbf{y}}}$ inside the vector space $V(\mathcal{G}_{\mathbf{y}})$. Then the automorphisms α and β coincide.*

Proof: We consider the vector space $V(\mathcal{G}_{\mathbf{y}})$ decomposed into the direct sum $x_{\mathbf{y}} \oplus x_{\mathbf{y}}^\pi$, where $x_{\mathbf{y}}$ is the line corresponding to the vertex $\mathbf{x} \in \widehat{\Gamma}$ and $x_{\mathbf{y}}^\pi$ is the orthogonal space of the line $x_{\mathbf{y}}$, a four-dimensional unitary complex subspace of $V(\mathcal{G}_{\mathbf{y}})$. Since α and β are automorphisms of the graph $\widehat{\Gamma}$ fixing the vector space $V(\mathcal{G}_{\mathbf{x}})$ elementwise, it follows that both α and β act trivially on the subspace $x_{\mathbf{y}}^\pi$, so the action of α and β on the vector space $V(\mathcal{G}_{\mathbf{y}})$ is determined by the respective action of α and β on the line $x_{\mathbf{y}}$ in $V(\mathcal{G}_{\mathbf{y}})$. By assumption we have $\alpha_{|x_{\mathbf{y}}} = \beta_{|x_{\mathbf{y}}}$ and hence $\alpha^{-1}_{|x_{\mathbf{y}}}\beta_{|x_{\mathbf{y}}} = \mathrm{id}_{|x_{\mathbf{y}}}$ and, thus, $\alpha^{-1}_{|V(\mathcal{G}_{\mathbf{y}})}\beta_{|V(\mathcal{G}_{\mathbf{y}})} = \mathrm{id}_{V(\mathcal{G}_{\mathbf{y}})}$, so the automorphism $\alpha^{-1}\beta$ acts trivially on $V(\mathcal{G}_{\mathbf{y}})$. Certainly, $\alpha^{-1}\beta$ acts trivially on $V(\mathcal{G}_{\mathbf{x}})$, in particular $\alpha_{|y_{\mathbf{x}}} = \beta_{|y_{\mathbf{x}}}$ inside the vector space $V(\mathcal{G}_{\mathbf{x}})$. If $\mathbf{z}$ is a neighbour of $\mathbf{y}$ in $\widehat{\Gamma}$, connectedness of the induced subgraph $\mathbf{y}^\perp$, by proposition 4.2.2, implies that $\alpha_{|y_{\mathbf{z}}} = \beta_{|y_{\mathbf{z}}}$ inside $V(\mathcal{G}_{\mathbf{z}})$. Indeed, by lemma 4.2.2 either $z_{\mathbf{y}}$ is incident to $x_{\mathbf{y}}^\pi$, so we conclude that $\alpha_{|x_{\mathbf{z}}^\pi} = \alpha_{|z_{\mathbf{x}}^\pi} = \alpha_{z_{\mathbf{x}}^\pi}$ and $\beta_{|x_{\mathbf{z}}^\pi} = \beta_{z_{\mathbf{x}}^\pi}$ and since the automorphism $\alpha_{|z_{\mathbf{x}}^\pi}$ and $\beta_{|z_{\mathbf{x}}^\pi}$ fix the vector space $z_{\mathbf{x}}^\pi$ elementwise, we see by the fact that $y_{\mathbf{z}} \subseteq x_{\mathbf{z}}^\pi$ that $\alpha_{|y_{\mathbf{z}}} = \beta_{|y_{\mathbf{z}}}$ inside $\mathcal{G}_{\mathbf{z}}$ or the vertices $\mathbf{x}$ and $\mathbf{z}$ have distance two in the induced subgraph $\mathbf{y}^\perp$, thus $\mathbf{y}^\perp$ contains a vertex $\mathbf{k}$ such that $\mathbf{x} \perp \mathbf{k} \perp \mathbf{z}$. In that case we obtain that $\alpha^{-1}_{|V(\mathcal{G}_{\mathbf{k}})}\beta_{|V(\mathcal{G}_{\mathbf{k}})} = \mathrm{id}_{|V(\mathcal{G}_{\mathbf{k}})}$ and $\alpha_{|y_{\mathbf{k}}} = \beta_{|y_{\mathbf{k}}}$ inside $V(\mathcal{G}_{\mathbf{k}})$ by the argumentation above. Moreover $\alpha_{|k_{\mathbf{z}}^\pi} = \alpha_{|z_{\mathbf{k}}^\pi}$ and $\beta_{|k_{\mathbf{z}}^\pi} = \beta_{|z_{\mathbf{k}}^\pi}$. Using the fact that $y_{\mathbf{z}} \subseteq k_{\mathbf{z}}^\pi$ and $y_{\mathbf{k}} \subseteq z_{\mathbf{k}}^\pi \cap x_{\mathbf{k}}^\pi$ we also get that $\alpha_{|y_{\mathbf{k}}} = \alpha_{|y_{\mathbf{z}}}$ and $\beta_{|y_{\mathbf{k}}} = \beta_{|y_{\mathbf{z}}}$, therefore $\alpha_{|y_{\mathbf{z}}} = \beta_{|y_{\mathbf{z}}}$. Hence $\alpha^{-1}_{|V(\mathcal{G}_{\mathbf{z}})}\beta_{|V(\mathcal{G}_{\mathbf{z}})} = \mathrm{id}_{|V(\mathcal{G}_{\mathbf{z}})}$ by the above, thus $\alpha = \beta$ follows by connectedness of the graph $\widehat{\Gamma}$. ∎

Lemma 4.3.17 *Let $\mathbf{x}$ be a vertex of the graph $\widehat{\Gamma}$ and let $\mathbf{y}$ be a neighbour of $\mathbf{x}$. Let φ be an element of $\mathrm{SU}_2(\mathbb{C})$ and $\varphi_{V(\mathcal{G}_{\mathbf{y}})}$ be the automorphism of $V(\mathcal{G}_{\mathbf{y}})$ such that $(\varphi_{V(\mathcal{G}_{\mathbf{y}})})_{|x_{\mathbf{y}}} = \varphi_{x_{\mathbf{y}}} = \varphi$ and $(\varphi_{V(\mathcal{G}_{\mathbf{y}})})_{|x_{\mathbf{y}}^\pi} = id_{x_{\mathbf{y}}^\pi}$. Then there exists a unique automorphism $\alpha_{\mathbf{x},\varphi} = \alpha$ of $\widehat{\Gamma}$ with $\alpha_{|\mathbf{x}^\perp} = id$ and $\alpha_{|\mathbf{y}^\perp}$ is the graph automorphism induced by the vector space automorphism $\varphi_{V(\mathcal{G}_{\mathbf{y}})}$.*

Proof: This proof has several steps.

Definition of $\alpha_{\mathbf{x}}$: The action on the vector space $V(\mathcal{G}_{\mathbf{x}})$ equals the identity map $\varphi_{V(\mathcal{G}_{\mathbf{x}})} := \mathrm{id}_{V(\mathcal{G}_{\mathbf{x}})}$, thus the automorphism $\varphi_{\mathcal{G}_{\mathbf{x}}}$ induced by $\varphi_{V(\mathcal{G}_{\mathbf{x}})}$ on the projective space $\mathcal{G}_{\mathbf{x}}$ is the identity map $\mathrm{id}_{\mathcal{G}_{\mathbf{x}}}$. Hence we define the graph automorphism $\alpha_{\mathbf{x}} : \mathbf{x}^\perp \cup \mathbf{x} \to \mathbf{x}^\perp \cup \mathbf{x}$ via $\mathbf{w} \mapsto \mathbf{w}$.

Definition of $\alpha_{\mathbf{x}\perp\mathbf{y}}$: Recall from lemma 4.3.14 and proposition 4.3.15 the definition of the vector space automorphism $\varphi_{V(\mathcal{G}_{\mathbf{w}})}$ for each neighbour $\mathbf{w}$ of the vertex $\mathbf{x}$ in $\widehat{\Gamma}$. For any neighbour $\mathbf{w}$ of the vertex $\mathbf{x}$ in $\widehat{\Gamma}$, we have the decomposition $x_{\mathbf{w}} \oplus x_{\mathbf{w}}^{\pi}$ of the vector space on $V(\mathcal{G}_{\mathbf{w}})$, the unique vector space automorphism $\varphi_{V(\mathcal{G}_{\mathbf{w}})}$ of $V(\mathcal{G}_{\mathbf{w}})$ with the properties that $(\varphi_{V(\mathcal{G}_{\mathbf{w}})})_{|x_{\mathbf{w}}} = \varphi_{x_{\mathbf{w}}} = \varphi$ and $(\varphi_{V(\mathcal{G}_{\mathbf{w}})})_{|x_{\mathbf{w}^{\pi}}} = \mathrm{id}_{|x_{\mathbf{w}^{\pi}}}$ and the unique automorphism $\varphi_{\mathcal{G}_{\mathbf{w}}}$ of the projective space $\mathcal{G}_{\mathbf{w}}$ induced from $\varphi_{V(\mathcal{G}_{\mathbf{w}})}$ by proposition 4.3.15, see also definition 4.2.5 and lemma 4.3.14. Hence we can uniquely extend the automorphism $\varphi_{V(\mathcal{G}_{\mathbf{x}})}$ to the set $V(\mathcal{G}_{\mathbf{x}}) \cup V(\mathcal{G}_{\mathbf{y}})$ and the graph automorphism $\alpha_{\mathbf{x}}$ to the neighbours of the vertex $\mathbf{y}$ by proposition 4.3.15, where $\mathbf{y} \perp \mathbf{x}$ in $\widehat{\Gamma}$, if we accept for a moment that our definition of the image of the line $z_{\mathbf{y}}$ and of the vertex $\mathbf{z}$ depends on the chosen path $\mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$. We set

$$\begin{array}{rcl} \varphi_{V(\mathcal{G}_{\mathbf{x}})\cup V(\mathcal{G}_{\mathbf{y}})} : V(\mathcal{G}_{\mathbf{x}}) \cup V(\mathcal{G}_{\mathbf{y}}) & \to & V(\mathcal{G}_{\mathbf{x}}) \cup V(\mathcal{G}_{\mathbf{y}}) \\ v & \mapsto & \varphi_{V(\mathcal{G}_{\mathbf{x}})}(v) \text{ if } v \in V(\mathcal{G}_{\mathbf{x}}) \\ v & \mapsto & \varphi_{V(\mathcal{G}_{\mathbf{y}})}(v) \text{ if } v \in V(\mathcal{G}_{\mathbf{y}}) \end{array}$$

and we define the automorphism

$$\begin{array}{rcl} \alpha_{\mathbf{x}\perp\mathbf{y}} : \mathbf{x}^{\perp} \cup \mathbf{y}^{\perp} & \to & \mathbf{x}^{\perp} \cup \mathbf{y}^{\perp} \\ \mathbf{w} & \mapsto & \alpha_{\mathbf{x}}(\mathbf{w}) \text{ for } \mathbf{w} \in \mathbf{x}^{\perp} \\ \mathbf{w} & \mapsto & \mathbf{d} \text{ if } d_{\mathbf{y}} = \varphi_{V(\mathcal{G}_{\mathbf{y}})}(w_{\mathbf{y}}) \text{ for } \mathbf{w} \in \mathbf{y}^{\perp}\,. \end{array}$$

Notice that the images of two adjacent vertices $\mathbf{z}$ and $\mathbf{w}$ under the automorphism $\alpha_{\mathbf{x}\perp\mathbf{y}}$ are adjacent in the induced subgraph $\mathbf{x}^{\perp} \cup \mathbf{y}^{\perp}$ and, thus, also in the graph $\widehat{\Gamma}$.

Extension of $\alpha_{\mathbf{x}\perp\ldots\perp\mathbf{v}\perp\mathbf{y}}$ to a neighbour $\mathbf{z}$ of $\mathbf{y}$ via the map $\varphi_{\mathcal{G}_{\mathbf{z}}}$: Next we consider the vector space $V(\mathcal{G}_{\mathbf{z}})$ on the subgraph $\mathbf{z}^{\perp}$ and describe how the map $\varphi_{V(\mathcal{G}_{\mathbf{v}})\cup V(\mathcal{G}_{\mathbf{y}})}$ determines an isomorphism between the vector spaces $V(\mathcal{G}_{\mathbf{z}})$ and $V(\mathcal{G}_{\alpha_{\gamma}(\mathbf{z})})$, which is unique up to a scalar $e^{i\lambda}$, $\lambda \in \mathbb{R}$, where γ is the path $\mathbf{x} \perp \ldots \perp \mathbf{v} \perp \mathbf{y}$ of length at least one. To avoid confusion in the notation we will leave out the index γ if possible, so the line in the space $V(\mathcal{G}_{\alpha_{\gamma}(\mathbf{y})})$ belonging to a vertex $\alpha_{\gamma}(\mathbf{w})$ if $\alpha_{\gamma}(\mathbf{y}) \perp \alpha_{\gamma}(\mathbf{w})$ will be denoted with $\alpha(w)_{\alpha_{\gamma}(\mathbf{y})}$.

Let $k_{\mathbf{y}}$ be a line of $v_{\mathbf{y}}^{\pi} \cap v_{\mathbf{y}}^{\pi}$ in $\mathcal{G}_{\mathbf{y}}$ and $\beta : u_1, \ldots, u_6$ be an orthonormal basis of the complex vector space $V(\mathcal{G}_{\mathbf{y}})$ such that $\langle u_1, u_2 \rangle = z_{\mathbf{y}}$ and $\langle u_3, u_4 \rangle = k_{\mathbf{y}}^{\pi} \cap z_{\mathbf{b}}^{\pi}$ and $\langle u_5, u_6 \rangle = k_{\mathbf{y}}$. The construction in remark 4.2.7 yields the basis system $\beta_{b_6} : b_1, \ldots, b_6; b_{jj+1}$ for $1 \leq j \leq 5$ w.r.t. some normal vector $b_6 \in \langle v_6 \rangle$ from the basis β. The automorphism $\varphi_{V(\mathcal{G}_{\mathbf{y}})} : V(\mathcal{G}_{\mathbf{y}}) \to V(\mathcal{G}_{\alpha_{\gamma}(\mathbf{y})})$ maps each vector b_j, $j = 3, \ldots, 6$, to $d_j = \varphi_{V(\mathcal{G}_{\alpha_{\gamma}(\mathbf{y})})}(b_j) \in {\alpha(z)_{\alpha_{\gamma}(\mathbf{y})}}^{\pi}$. Since $\varphi_{V(\mathcal{G}_{\mathbf{y}})}$ is a unitary automorphism, the vectors $d_3, \ldots, d_6, d_{jj+1}$ for $j = 3, 4, 5$ constitute this basis system with respect to d_6 of $(\alpha(z)_{\alpha_{\gamma}(\mathbf{y})})^{\pi}$, where $d_{jj+1} = \varphi_{V(\mathcal{G}_{\mathbf{y}})}(b_{jj+1})$.

Furthermore, we consider the points $p_{\mathbf{y},b_j} = \langle b_j \rangle \subseteq z_{\mathbf{y}}^{\pi}$ and the points $p_{\alpha_{\gamma}(\mathbf{y}),d_j} = \langle d_j \rangle \subseteq (\alpha(z)_{\alpha_{\gamma}(\mathbf{y})})^{\pi}$ for every $j \in J = \{3, \ldots, 6, 34, 45, 56\}$. The points $p_{\mathbf{y},b_j}$ correspond to points $p_{\mathbf{z},b_j}$ of $y_{\mathbf{z}}^{\pi}$ for every $j \in J$ by lemma 4.3.8, which by lemma 4.3.9 satisfy the same orthogonality relations. Hence a normal vector c_6 of the point $p_{\mathbf{z},b_6}$ determines via remark 4.2.7 the basis system $\psi_{c_6}^{y_{\mathbf{z}}^{\pi}} : c_3, \ldots, c_6; c_{34}, c_{45}, c_{56}$ with respect to c_6 of $y_{\mathbf{z}}^{\pi}$. Analogously, the points $p_{\alpha_y(\mathbf{y}),d_j}$ induce points $p_{\alpha_y(\mathbf{z}),d_j}$ in $\alpha(y)_{\alpha_y(\mathbf{z})}{}^{\pi}$ and a normal vector $g_6 \in p_{\alpha_y(\mathbf{z}),d_6}$ determines the basis system $\kappa_{g_6}^{\alpha(y)_{\alpha_y(\mathbf{z})}{}^{\pi}} : g_3, \ldots, g_6; g_{34}, g_{45}, g_{56}$ with respect to g_6 of the subspace $(\alpha(y)_{\alpha_y(\mathbf{z})})^{\pi}$.

We have $p_{\mathbf{y},b_5}, p_{\mathbf{y},b_6}, p_{\mathbf{y},b_{56}} \subseteq v_{\mathbf{y}}^{\pi}$, because $p_{\mathbf{y},b_{56}} \subseteq \langle p_{\mathbf{y},b_5}, p_{\mathbf{y},b_6} \rangle = k_{\mathbf{y}}$, and therefore $p_{\alpha_y(\mathbf{y}),d_5}, p_{\alpha_y(\mathbf{y}),d_6}, p_{\alpha_y(\mathbf{y}),d_{56}} \subseteq \alpha(v)_{\alpha_y(\mathbf{y})}^{\pi}$. By lemma 4.3.8 the projective space $\mathcal{G}_{\mathbf{v}}$ contains points $p_{\mathbf{v},b_5}$, $p_{\mathbf{v},b_6}$ and $p_{\mathbf{v},b_{56}}$ induced by the points $p_{\mathbf{y},b_5}, p_{\mathbf{y},b_6}, p_{\mathbf{y},b_{56}}$ of $\mathcal{G}_{\mathbf{y}}$. Again for a normal vector a_6 of $p_{\mathbf{v},b_6}$ we obtain the basis system $\nu_{a_6}^{k_{\mathbf{v}}} : a_5, a_6; a_{56}$ with respect to a normal vector $a_5 \in p_{\mathbf{v},b_5}$. Fix two orthogonal vectors v_1 and v_2 of length one spanning the line $y_{\mathbf{v}}$ to get the orthonormal basis v_1, v_2, a_5, a_6 of the subspace $\langle y_{\mathbf{v}}, k_{\mathbf{v}} \rangle$. Remark 4.2.7 gives the basis system $\nu_{a_6}^{\langle k_{\mathbf{v}}, y_{\mathbf{v}} \rangle} : a_1, a_2, a_5, a_6; a_{12}, a_{25}, a_{56}$ with respect to a_6 of $\langle y_{\mathbf{v}}, k_{\mathbf{v}} \rangle$ and we fix also the line $m_{\mathbf{v}} = \langle v_2, a_2 \rangle$, which contains the point $\langle a_{25} \rangle$ and belongs to a vertex **m** adjacent to **v** and **z** by lemma 4.3.11.

We denote by r_j the image under the map $\varphi_{V(\mathcal{G}_{\mathbf{v}})}$ of the vector a_j, for each $j \in \{1, 2, 5, 6, 12, 25, 56\}$, and obtain the points $p_{\alpha_y(\mathbf{v}),r_j} = \langle r_j \rangle$ contained in $\langle \alpha(k)_{\alpha_y(\mathbf{v})}, \alpha(y)_{\alpha_y(\mathbf{v})} \rangle$. Since $\mathbf{v} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{k} \perp \mathbf{v} \perp \mathbf{m} \perp \mathbf{z}$, also $\alpha_y(\mathbf{v}) \perp \alpha_y(\mathbf{y}) \perp \alpha_y(\mathbf{z}) \perp \alpha_y(\mathbf{k}) \perp \alpha_y(\mathbf{v}) \perp \alpha_y(\mathbf{m}) \perp \alpha_y(\mathbf{z})$, so the point $p_{\alpha_y(\mathbf{v}),r_j}$ induces a unique point $p_{\alpha_y(\mathbf{z}),r_j}$, for every $j \in \{1, 2, 5, 6, 12, 25, 56\}$, contained in $\langle \alpha(k)_{\alpha_y(\mathbf{z})}, \alpha(y)_{\alpha_y(\mathbf{z})} \rangle$, by lemma 4.3.8. Using this lemma again we conclude that $p_{\alpha_y(\mathbf{z}),r_j} = p_{\alpha_y(\mathbf{z}),d_j}$ and $p_{\mathbf{z},a_j} = p_{\mathbf{z},b_j}$ for $j \in \{5, 6, 56\}$. Thus we complete the basis $\psi_{c_6}^{y_{\mathbf{z}}^{\pi}}$ to the orthonormal basis $\psi_{c_6} : c_1, \ldots, c_6$ of the vector space $V(\mathcal{G}_{\mathbf{z}})$ such that the vectors c_i are normal vectors of $p_{\mathbf{z},a_j}$ for $j \in \{1, 2, 12, 25\}$ and $(c_1, c_{12}) = (c_{12}, c_2) = (c_2, c_{25}) = (c_{25}, c_5) = \frac{1}{\sqrt{2}}$. We also make up the basis $\kappa_{g_6}^{(\alpha(y)_{\alpha_y(\mathbf{z})})^{\pi}}$ to the orthonormal basis $\kappa_{g_6} : g_1, \ldots, g_6$ with the properties that g_j is a vector of length one of the point $p_{\alpha_y(\mathbf{z}),r_j}$ for $j \in \{1, 2, 12, 25\}$ and $(g_1, g_{12}) = (g_{12}, g_2) = (g_2, g_{25}) = (g_{25}, g_5) = \frac{1}{\sqrt{2}}$.

Since the bases κ_{g_6} and ψ_{c_6} are uniquely determined by c_6, respectively g_6, we can define the map $\varphi_{V(\mathcal{G}_{\mathbf{z}})}^{g_6,c_6}$ between the vector spaces $V(\mathcal{G}_{\mathbf{z}})$ and $V(\mathcal{G}_{\alpha_y(\mathbf{z})})$ as follows: $\mathbf{x}\text{-}\varphi_{V(\mathcal{G}_{\mathbf{z}})} = \varphi_{V(\mathcal{G}_{\mathbf{z}})} = \varphi_{V(\mathcal{G}_{\mathbf{z}})}^{g_6,c_6} : V(\mathcal{G}_{\mathbf{z}}) \to V(\mathcal{G}_{\alpha_y(\mathbf{z})})$ with $c_j \mapsto g_j$ for $1 \le j \le 6$. Certainly this map $\varphi_{V(\mathcal{G}_{\mathbf{z}})}^{g_6,c_6}$ is an isomorphism between $V(\mathcal{G}_{\mathbf{z}})$ and $V(\mathcal{G}_{\alpha_y(\mathbf{z})})$ preserving the scalar product, so that $\varphi_{V(\mathcal{G}_{\mathbf{z}})}^{g_6,c_6}$ is a unitary isomorphism.

Furthermore this isomorphism is unique up to a scalar $e^{i\lambda}$, $\lambda \in \mathbb{R}$ in the sense that, if we choose $\hat{c}_6 = e^{i\mu} c_6$ instead of c_6 and $\hat{g}_6 = e^{i\delta} g_6$ instead of g_6, then the isomorphism $\varphi_{V(\mathcal{G}_{\mathbf{z}})}^{\hat{g}_6,\hat{c}_6}$ equals $e^{i(\delta-\mu)} \varphi_{V(\mathcal{G}_{\mathbf{z}})}^{g_6,c_6}$ with $\delta, \mu \in \mathbb{R}$. Indeed, for the bases $\kappa_{\hat{g}_6}$ and $\psi_{\hat{c}_6}$ instead of κ_{g_6} and ψ_{c_6}, we compute

$$\begin{aligned} {}_{\kappa_{\hat{g}_6}}[\varphi^{g_6,c_6}_{V(\mathcal{G}_c)}]_{\psi_{\hat{c}_6}} &= {}_{\kappa_{\hat{g}_6}}[id]_{\kappa_{g_6}} \cdot {}_{\kappa_{g_6}}[\varphi^{g_6,c_6}_{V(\mathcal{G}_c)}]_{\psi_{c_6}} \cdot {}_{\psi_{c_6}}[id]_{\psi_{\hat{c}_6}} \\ &= e^{i\delta} \cdot e^{-i\mu} I \\ &= e^{i(\delta-\mu)} I, \end{aligned}$$

because ${}_{\psi_{\hat{c}_6}}[id]_{\psi_{c_6}} = e^{i\mu} I$ and ${}_{\kappa_{\hat{g}_6}}[id]_{\kappa_{g_6}} = e^{i\delta} I$ as well as ${}_{\psi_{c_6}}[\varphi^{g_6,c_6}_{V(\mathcal{G}_c)}]_{\kappa_{g_6}} = I = {}_{\psi_{\hat{c}_6}}[\varphi^{\hat{g}_6,\hat{c}_6}_{V(\mathcal{G}_c)}]_{\kappa_{\hat{g}_6}}$.

Thus we define the unique isomorphism $\varphi_{\mathcal{G}_z} = \mathbf{x}\text{-}\varphi_{\mathcal{G}_z}$ between the projective spaces $\mathcal{G}_\mathbf{z}$ and $\mathcal{G}_{\alpha_\gamma(\mathbf{z})}$ from the vector space isomorphism $\varphi_{V(\mathcal{G}_z)}$ by $\varphi_{\mathcal{G}_z} : \mathcal{G}_\mathbf{z} \to \mathcal{G}_{\alpha_\gamma(\mathbf{z})}$ such that the image of a point $p_\mathbf{z} = \langle p^v_\mathbf{z} \rangle$ is $\langle \varphi^{g_6,c_6}_{V(\mathcal{G}_z)}(p^v_\mathbf{z}) \rangle$ for some vector $p^v_\mathbf{z} \in p_\mathbf{z}$.

Of course, we have to prove that the isomorphism $\varphi_{\mathcal{G}_z}$ is independent from all the choices we made. However, before doing that we will indicate how to continue the construction in order to obtain a graph automorphism α.

Extension of $\alpha_{\mathbf{x}\perp\mathbf{y}}$ to $\alpha_{\mathbf{x}\perp\mathbf{y}\perp\mathbf{z}}$ and of $\alpha_{\mathbf{x}\perp\ldots\perp\mathbf{v}\perp\mathbf{y}}$ to $\alpha_{\mathbf{x}\perp\ldots\perp\mathbf{v}\perp\mathbf{y}\perp\mathbf{z}}$: Using the isomorphism $\varphi_{\mathcal{G}_z}$ we can extend the map $\alpha_{\mathbf{x}\perp\mathbf{y}}$ to the map

$$\begin{aligned} \alpha_{\mathbf{x}\perp\mathbf{y}\perp\mathbf{z}} : \mathbf{x}^\perp \cup \mathbf{y}^\perp \cup \mathbf{z}^\perp &\to \mathbf{x}^\perp \cup \mathbf{y}^\perp \cup \alpha_{\mathbf{x}\perp\mathbf{y}}(\mathbf{z})^\perp \\ \mathbf{w} &\mapsto \begin{cases} \mathbf{w} & \text{for } \mathbf{w} \in \mathbf{x}^\perp \\ \mathbf{d} & \text{if } d_\mathrm{y} = \varphi_{V(\mathcal{G}_y)}(w_\mathrm{y}) \text{ for } \mathbf{w} \in \mathbf{y}^\perp \\ \mathbf{h} & \text{if } h_{\alpha_{\mathbf{x}\perp\mathbf{y}}(\mathbf{z})} = \varphi_{\mathcal{G}_z}(w_\mathbf{z}) \text{ for } \mathbf{w} \in \mathbf{z}^\perp \end{cases}. \end{aligned}$$

In general, for $\gamma = \mathbf{x} \perp \ldots \perp \mathbf{v} \perp \mathbf{y}$ we extend α_γ to

$$\begin{aligned} \alpha_{\gamma\perp\mathbf{z}} : \mathbf{x}^\perp \cup \cdots \cup \mathbf{v}^\perp \cup \mathbf{y}^\perp \cup \mathbf{z}^\perp &\to \mathbf{x}^\perp \cup \cdots \cup \alpha_\gamma(\mathbf{v})^\perp \cup \alpha_\gamma(\mathbf{y})^\perp \cup \alpha_\gamma(\mathbf{z})^\perp \\ \mathbf{w} &\mapsto \begin{cases} \mathbf{h} & \text{if } h_{\alpha_\gamma(\mathbf{z})} = \varphi_{\mathcal{G}_z}(w_\mathbf{z}) \text{ for } \mathbf{w} \in \mathbf{z}^\perp \\ \alpha_\gamma(\mathbf{w}) & \text{else} \end{cases}. \end{aligned}$$

The connectedness of the graph $\widehat{\Gamma}$ implies that we will end up with a unique automorphism $\alpha_{\mathbf{x},\varphi} = \alpha : \widehat{\Gamma} \to \widehat{\Gamma}$ satisfying the hypotheses.
It remains to check that α is well-defined.

Independence of $\varphi_{\mathcal{G}_z}$ from β, v_1, v_2: Let k_y be a two-dimensional subspace of $v^\pi_\mathbf{y} \cap z^\pi_\mathbf{y}$. We choose two orthonormal bases $\beta : u_1, \ldots, u_6$ and $\hat{\beta} : \hat{u}_1, \ldots, \hat{u}_6$ of the vector space $V(\mathcal{G}_\mathrm{y})$ such that $\langle u_1, u_2 \rangle = z_\mathrm{y} = \langle \hat{u}_1, \hat{u}_2 \rangle$ and $\langle u_3, u_4 \rangle = k^\pi_\mathbf{y} \cap z^\pi_\mathbf{y} = \langle \hat{u}_3, \hat{u}_4 \rangle$ and $\langle u_5, u_6 \rangle = k_\mathrm{y} = \langle \hat{u}_5, \hat{u}_6 \rangle$. Following the described construction we determine the two orthonormal bases $\beta_{b_6} : b_1, \ldots, b_6$ and $\hat{\beta}_{\hat{b}_6} : \hat{b}_1, \ldots, \hat{b}_6$ of the vector space $V(\mathcal{G}_\mathrm{y})$. The next step in the construction is to get the images of the vectors $b_3, \ldots, b_6$ which are $d_j = \varphi_{V(\mathcal{G}_y)}(b_j)$ for $3 \le j \le 6$ and the vectors $\hat{d}_j = \varphi_{V(\mathcal{G}_y)}(\hat{b}_j)$ for $3 \le j \le 6$. Notice that $d_3, \ldots, d_6$ and $\hat{d}_3, \ldots, \hat{d}_6$ are two different orthonormal bases of the subspace $\alpha(v)^\pi_{\alpha_\gamma(\mathbf{y})}$, in particular we have the equality

$$ {}_{\hat{d}_3,\dots,\hat{d}_6}[id]_{d_3,\dots,d_6} = {}_{\hat{\beta}^{z^\pi_{\mathbf{y}}}_{\hat{b}_6}}[id]_{\beta^{z^\pi_{\mathbf{y}}}_{b_6}} = {}_{\hat{b}_3,\hat{b}_4,\hat{b}_5,\hat{b}_6}[id]_{b_3,b_4,b_5,b_6} $$

of the basis transformation matrices by the fact that $\varphi_{V(\mathcal{G}_{\mathbf{y}})}$ is an unitary automorphism of $V(\mathcal{G}_{\mathbf{y}})$.

Following the plan, we get the bases $\psi^{y^\pi_{\mathbf{z}}}_{c_6} : c_3,\dots,c_6$ and $\hat{\psi}^{y^\pi_{\mathbf{z}}}_{\hat{c}_6} : \hat{c}_3,\dots,\hat{c}_6$ of $y^\pi_{\mathbf{z}}$. Our next stopover is to get control over the basis transformation matrix ${}_{\hat{\psi}_{\hat{c}_6}}[id]_{\psi_{c_6}}$. Since $z^\pi_{\mathbf{y}}$ and $y^\pi_{\mathbf{z}}$ are isomorphic and since each point of one subspace induces a unique point in the other subspace, by lemma 4.3.8, we can choose the isomorphism $\theta = \theta^{y^\pi_{\mathbf{z}}}_{z^\pi_{\mathbf{y}}} : z^\pi_{\mathbf{y}} \to y^\pi_{\mathbf{z}}$ of remark 4.3.13 which maps each vector v of the space $z^\pi_{\mathbf{y}}$ to its corresponding vector $\theta(v)$ of $y^\pi_{\mathbf{z}}$, in particular it maps b_j to $\theta(b_j)$ and $\hat{b}_j$ to $\theta(\hat{b}_j)$ for $3 \le j \le 6$ implying that $\psi_{\theta(b_6)} : \theta(b_3),\dots,\theta(b_6)$ and $\hat{\psi}_{\theta(\hat{b}_6)} : \theta(\hat{b}_3),\dots,\theta(\hat{b}_6)$ are orthonormal bases of $y^\pi_{\mathbf{z}}$ with the property that ${}_{\hat{\psi}_{\theta(\hat{b}_6)}}[id]_{\psi_{\theta(b_6)}} = {}_{\hat{\beta}_{\hat{b}_6}}[id]_{\beta_{b_6}}$. Since the vectors $\theta(b_6)$ and c_6 have length one with $\langle\theta(b_6)\rangle = p_{\mathbf{z},b_6} = \langle c_6\rangle$, we have $c_6 = e^{i\lambda}\theta(b_6)$ for some $\lambda \in [0,2\pi[$. Hence the basis transformation matrix between $\psi_{\theta(b_6)}$ and $\psi^{y^\pi_{\mathbf{z}}}_{c_6}$ is ${}_{\psi^{y^\pi_{\mathbf{z}}}_{c_6}}[id]_{\psi_{\theta(b_6)}} = e^{i\lambda}\cdot I$. Also by the same argument $\hat{c}_6 = e^{i\rho}\theta(\hat{b}_6)$ for some $\rho \in [0,2\pi[$ and ${}_{\hat{\psi}^{y^\pi_{\mathbf{z}}}_{\hat{c}_6}}[id]_{\hat{\psi}_{\theta(\hat{b}_6)}} = e^{i\rho}\cdot I$. Finally, we get the equality

$$
\begin{aligned}
{}_{\hat{\psi}_{\hat{c}_6}}[id]_{\psi_{c_6}} &= {}_{\hat{\psi}^{y^\pi_{\mathbf{z}}}_{\hat{c}_6}}[id]_{\hat{\psi}_{\theta(\hat{b}_6)}} \cdot {}_{\hat{\psi}_{\theta(\hat{b}_6)}}[id]_{\psi_{\theta(b_6)}} \cdot {}_{\psi_{\theta(b_6)}}[id]_{\psi^{y^\pi_{\mathbf{z}}}_{c_6}} \\
&= (e^{i\lambda})^{-1}\cdot e^{i\rho}\cdot {}_{\hat{\beta}^{y^\pi_{\mathbf{z}}}_{\hat{b}_6}}[id]_{\beta^{y^\pi_{\mathbf{z}}}_{b_6}} \\
&= e^{i(\rho-\lambda)}\cdot {}_{\hat{\beta}^{y^\pi_{\mathbf{z}}}_{\hat{b}_6}}[id]_{\beta^{y^\pi_{\mathbf{z}}}_{b_6}}.
\end{aligned}
$$

Next we construct the two orthonormal bases $\kappa^{\alpha(y)_{\alpha_y(\mathbf{z})}{}^\pi}_{g_6} : g_3,\dots,g_6$ and $\hat{\kappa}^{\alpha(y)_{\alpha_y(\mathbf{z})}{}^\pi}_{\hat{g}_6} : \hat{g}_3,\dots,\hat{g}_6$ for the subspace $\alpha(y)_{\alpha_y(\mathbf{z})}{}^\pi$ of $V(\mathcal{G}_{\alpha_y(\mathbf{z})})$. With a similar argument as above we see that the basis transformation matrix between $\kappa^{\alpha(y)_{\alpha_y(\mathbf{z})}{}^\pi}_{g_6}$ and $\hat{\kappa}^{\alpha(y)_{\alpha_y(\mathbf{z})}{}^\pi}_{\hat{g}_6}$ is of the form

$$
\begin{aligned}
{}_{\hat{\kappa}^{\alpha(y)_{\alpha_y(\mathbf{z})}{}^\pi}_{\hat{g}_6}}[id]_{\kappa^{\alpha(y)_{\alpha_y(\mathbf{z})}{}^\pi}_{g_6}} &= (e^{i\rho})^{-1}\cdot e^{io}\cdot {}_{\hat{\beta}^{y^\pi_{\mathbf{z}}}_{\hat{b}_6}}[id]_{\beta^{y^\pi_{\mathbf{z}}}_{b_6}} \\
&= e^{i(o-\rho)}\cdot {}_{\hat{\beta}^{y^\pi_{\mathbf{z}}}_{\hat{b}_6}}[id]_{\beta^{y^\pi_{\mathbf{z}}}_{b_6}}
\end{aligned}
$$

for $\rho, o \in [0,2\pi[$.

The next step of the construction is to determine the two bases $v_{a_6}^{k_v} : a_5, a_6$ and $\hat{v}_{\hat{a}_6}^{k_v} : \hat{a}_5, \hat{a}_6$. As above we have

$$\begin{aligned} {}_{\hat{v}_{\hat{a}_6}^{k_v}}[id]_{v_{a_6}^{k_v}} &= (e^{i\varepsilon})^{-1} \cdot e^{i\sigma} \cdot {}_{\hat{\beta}_{\hat{b}_6}^{y_z^\pi}}[id]_{\beta_{b_6}^{y_z^\pi}} \\ &= e^{i(\sigma-\varepsilon)} \cdot {}_{\hat{\beta}_{\hat{b}_6}^{y_z^\pi}}[id]_{\beta_{b_6}^{y_z^\pi}} \end{aligned}$$

for some $\sigma, \varepsilon \in [0, 2\pi[$. We now choose four normal vectors v_1, v_2, $\hat{v}_1$ and $\hat{v}_2$ of the line y_v such that $\langle v_1, v_2 \rangle = y_v = \langle \hat{v}_1, \hat{v}_2 \rangle$ and $(v_1, v_2) = 0 = (\hat{v}_1, \hat{v}_2)$. This leads by construction to the bases $v_{a_6}^{(k_v, y_v)} : a_1, a_2, a_5, a_6$ (with $a_i \in \langle v_i \rangle$ for $i \in \{1, 2\}$), $\hat{v}_{\hat{a}_6}^{(k_v, y_v)} : \hat{a}_1, \hat{a}_2, \hat{a}_5, \hat{a}_6$ (with $\hat{a}_i \in \langle \hat{v}_i \rangle$ for $i \in \{1, 2\}$), and to the images of these vectors under the isomorphism $\varphi_{V(\mathcal{G}_v)}$, which are $r_j = \varphi_{V(\mathcal{G}_v)}(a_j)$ and $\hat{r}_j = \varphi_{V(\mathcal{G}_v)}(\hat{a}_j)$ for $j = 1, 2, 5, 6$.

As $\langle a_1, a_2 \rangle = y_v = \langle \hat{a}_1, \hat{a}_2 \rangle$ we have

$$\begin{aligned} {}_{\hat{v}_{\hat{a}_6}^{(k_v, y_v)}}[id]_{v_{a_6}^{(k_v, y_v)}} &= \left({}_{\hat{v}_{\hat{a}_6}^{k_v}}[id]_{v_{a_6}^{k_v}} \right) \\ &= \begin{pmatrix} {}_{\hat{a}_1, \hat{a}_2}[id]_{a_1, a_2} & 0 \\ 0 & e^{i(\sigma-\varepsilon)} \cdot {}_{\hat{\beta}_{\hat{b}_6}^{k_y}}[id]_{\beta_{b_6}^{k_y}} \end{pmatrix} \end{aligned}$$

and ${}_{\hat{r}_1, \hat{r}_2, \hat{r}_5, \hat{r}_6}[id]_{r_1, r_2, r_5, r_6} = {}_{\hat{v}_{\hat{a}_6}^{(k_v, y_v)}}[id]_{v_{a_6}^{(k_v, y_v)}}$. Following the road map of the construction we complete the two bases $\psi_{c_6}^{y_z^\pi}$ and $\hat{\psi}_{\hat{c}_6}^{y_z^\pi}$ of the subspace $y_{\mathbf{z}}^\pi$ to the orthonormal bases $\psi_{c_6} : c_1, \ldots, c_6$ and $\hat{\psi}_{\hat{c}_6} : \hat{c}_1, \ldots, \hat{c}_6$ and determine the basis transformation matrix

$$\begin{aligned} {}_{\hat{\psi}_{\hat{c}_6}^{(k_z, y_z)}}[id]_{\psi_{c_6}^{(k_z, y_z)}} &= e^{i(\vartheta-\eta)} \cdot {}_{\hat{v}_{\hat{a}_6}^{(k_v, y_v)}}[id]_{v_{a_6}^{(k_v, y_v)}} \\ &= e^{i(\vartheta-\eta)} \cdot \begin{pmatrix} {}_{\hat{a}_1, \hat{a}_2}[id]_{a_1, a_2} & 0 \\ 0 & e^{i(\sigma-\varepsilon)} \cdot {}_{\hat{\beta}_{\hat{b}_6}^{k_y}}[id]_{\beta_{b_6}^{k_y}} \end{pmatrix} \end{aligned}$$

for $\eta, \vartheta \in \mathbb{R}$ by a similar argument as above. With these arguments we also compute the basis transformation matrix

$$\begin{aligned} &{}_{\hat{\kappa}_{\hat{g}_6}^{((\alpha(k)_{\alpha_y(\mathbf{z})}, \alpha(y)_{\alpha_y(\mathbf{z})})}}[id]_{\kappa_{g_6}^{(\alpha(k)_{\alpha_y(\mathbf{z})}, \alpha(y)_{\alpha_y(\mathbf{z})})}} \\ = \; & e^{i(\xi-\iota)} \cdot {}_{\hat{v}_{\hat{a}_6}^{(k_v, y_v)}}[id]_{v_{a_6}^{(k_v, y_v)}} \\ = \; & e^{i(\xi-\iota)} \cdot \begin{pmatrix} {}_{\hat{a}_1, \hat{a}_2}[id]_{a_1, a_2} & 0 \\ 0 & e^{i(\sigma-\varepsilon)} \cdot {}_{\hat{\beta}_{\hat{b}_6}^{k_y}}[id]_{\beta_{b_6}^{k_y}} \end{pmatrix}, \end{aligned}$$

with $\iota, \xi \in \mathbb{R}$, where $\kappa_{g_6} : g_1, \dots, g_6$ is the basis completion of $\kappa_{g_6}^{\alpha(y)^{\pi}_{\alpha_y(\mathbf{z})}}$ and $\hat\kappa_{\hat g_6} : \hat g_1, \dots, \hat g_6$, is the completion of the basis $\hat\kappa_{\hat g_6}^{\alpha(y)^{\pi}_{\alpha_y(\mathbf{z})}}$.

Finally putting all this information together we conclude that

$$\begin{aligned} e^{i(\rho-\lambda)} \cdot {}_{\hat\beta^{k_{\mathbf{y}}}_{\hat b_6}}[id]_{\beta^{k_{\mathbf{y}}}_{b_6}} &= {}_{\hat\psi^{k_{\mathbf{z}}}_{\hat c_6}}[id]_{\psi^{k_{\mathbf{z}}}_{c_6}} \\ &= e^{i(\vartheta-\eta)} \cdot e^{i(\sigma-\varepsilon)} \cdot {}_{\hat\beta^{k_{\mathbf{y}}}_{\hat b_6}}[id]_{\beta^{k_{\mathbf{y}}}_{b_6}} \\ e^{i(o-\rho)} \cdot {}_{\hat\beta^{k_{\mathbf{y}}}_{\hat b_6}}[id]_{\beta^{k_{\mathbf{y}}}_{b_6}} &= {}_{\hat\kappa_{\hat g_6}^{\alpha_y(\mathbf{k})_{\alpha_y(\mathbf{z})}}}[id]_{\kappa_{g_6}^{\alpha(k)_{\alpha_y(\mathbf{z})}}} \\ &= e^{i(\xi-\iota)} \cdot e^{i(\sigma-\varepsilon)} \cdot {}_{\hat\beta^{k_{\mathbf{y}}}_{\hat b_6}}[id]_{\beta^{k_{\mathbf{y}}}_{b_6}} \end{aligned}$$

thus $-\eta + \vartheta = -\lambda + \rho + \varepsilon - \sigma$ and $-\iota + \xi = -\rho + o + \varepsilon - \sigma$. Hence we obtain the following basis transformation matrices.

$$\begin{aligned} {}_{\hat\psi_{\hat c_6}}[id]_{\psi_{c_6}} &= \begin{pmatrix} e^{i(\vartheta-\eta)} \cdot {}_{\hat a_1,\hat a_2}[id]_{a_1,a_2} & 0 \\ 0 & e^{i(\rho-\lambda)} \cdot {}_{\hat\beta^{z^{\pi}_{\mathbf{y}}}_{\hat b_6}}[id]_{\beta^{z^{\pi}_{\mathbf{y}}}_{b_6}} \end{pmatrix} \\ {}_{\hat\kappa_{\hat g_6}}[id]_{\kappa_{g_6}} &= \begin{pmatrix} e^{i(\xi-\iota)} \cdot {}_{\hat a_1,\hat a_2}[id]_{a_1,a_2} & 0 \\ 0 & e^{i(o-\rho)} \cdot {}_{\hat\beta^{z^{\pi}_{\mathbf{y}}}_{\hat b_6}}[id]_{\beta^{z^{\pi}_{\mathbf{y}}}_{b_6}} \end{pmatrix} \end{aligned}$$

implying that

$$\begin{aligned} & {}_{\hat\kappa_{\hat g_6}}[id]_{\kappa_{g_6}} \cdot {}_{\kappa_{g_6}}[\varphi^{c_6,g_6}_{V(\mathcal{G}_{\mathbf{z}})}]_{\psi_{c_6}} \cdot {}_{\psi_{c_6}}[id]_{\hat\psi_{c_6}} \\ = \; & {}_{\hat\kappa_{\hat g_6}}[id]_{\kappa_{g_6}} \cdot {}_{\psi_{c_6}}[id]_{\hat\psi_{c_6}} \\ = \; & \begin{pmatrix} e^{i(\eta-\vartheta-\iota+\xi)} \cdot {}_{\hat a_1,\hat a_2}[id]_{a_1,a_2} \cdot ({}_{\hat a_1,\hat a_2}[id]_{a_1,a_2})^{-1} & 0 \\ 0 & e^{i(\lambda-\rho-\rho+o)} \cdot {}_{\hat\beta^{z^{\pi}_{\mathbf{y}}}_{\hat b_6}}[id]_{\beta^{z^{\pi}_{\mathbf{y}}}_{b_6}} \cdot {}_{\hat\beta^{z^{\pi}_{\mathbf{y}}}_{\hat b_6}}[id]^{-1}_{\beta^{z^{\pi}_{\mathbf{y}}}_{b_6}} \end{pmatrix} \\ = \; & \begin{pmatrix} e^{i(\eta-\vartheta-\iota+\xi)} \cdot I & 0 \\ 0 & e^{i(\lambda-\rho-\rho+o)} \cdot I \end{pmatrix} \\ = \; & e^{i(\lambda-\rho-\rho+o)} \cdot I \\ = \; & {}_{\hat\kappa_{\hat g_6}}[\varphi^{c_6,g_6}_{V(\mathcal{G}_{\mathbf{z}})}]_{\hat\psi_{\hat c_6}} \\ = \; & e^{ic}\, {}_{\hat\kappa_{\hat g_6}}[\varphi^{\hat c_6,\hat g_6}_{V(\mathcal{G}_{\mathbf{z}})}]_{\hat\psi_{\hat c_6}} \end{aligned}$$

for $\lambda - \rho - \rho + o = c \in \mathbb{R}$, which finishes the proof of the claim that the construction of the isomorphism $\varphi_{\mathcal{G}_\mathbf{z}}$ is independent from the choice of the basis β and the vectors v_1 and v_2.

Independence of $\varphi_{\mathcal{G}_\mathbf{z}}$ from $k_\mathbf{y}$: Now we show that the construction of the isomorphism $\varphi_{\mathcal{G}_\mathbf{z}}$ is also independent from the choice of the two-dimensional subspace $k_\mathbf{y}$ inside the vector subspace $v_\mathbf{y}^\pi \cap z_\mathbf{y}^\pi$. If the intersection of $v_\mathbf{y}^\pi$ and $z_\mathbf{y}^\pi$ is of dimension two, then $k_\mathbf{y} = v_\mathbf{y}^\pi \cap z_\mathbf{y}^\pi$ and there is nothing to prove. If otherwise $v_\mathbf{y}^\pi \cap z_\mathbf{y}^\pi$ is a plane, then any two different lines $k_\mathbf{y}$ and $l_\mathbf{y}$ of $v_\mathbf{y}^\pi \cap z_\mathbf{y}^\pi$ intersect in a one-dimensional subspace of $V(\mathcal{G}_\mathbf{y})$. We start with two orthonormal bases $\beta^k : u_1^k, \dots, u_6^k$ and $\beta^l : u_1^l, \dots, u_6^l$ such that $\langle u_1^k, u_2^k \rangle = z_\mathbf{y} = \langle u_1^l, u_2^l \rangle$ and $\langle u_3^k, u_4^k \rangle = z_\mathbf{y}^\pi \cap k_\mathbf{y}^\pi$ and $\langle u_3^l, u_4^l \rangle = z_\mathbf{y}^\pi \cap l_\mathbf{y}^\pi$ and $\langle u_5^k, u_6^k \rangle = k_\mathbf{y}$ and $\langle u_5^l, u_6^l \rangle = l_\mathbf{y}$. Since the construction of $\varphi_{V(\mathcal{G}_\mathbf{z})}$ is independent from the choice of the basis, we may assume that $u_4^k = u_4^l$ and $u_5^k = u_5^l$, so $\langle u_5^k \rangle = \langle u_5^l \rangle = k_\mathbf{y} \cap l_\mathbf{y}$ and $\langle u_4^k \rangle = \langle u_4^l \rangle = k_\mathbf{y}^\pi \cap l_\mathbf{y}^\pi \cap z_\mathbf{y}^\pi$. From the bases β^k and β^l we obtain the orthonormal bases $\beta_{b_6^k} : b_1^k, \dots, b_6^k$ and $\beta_{b_6^l} : b_1^l, \dots, b_6^l$ via the construction in remark 4.2.7 together with the extra condition that we choose the normal vector b_6^l in such a manner that $b_4^k = b_4^l$ and $b_5^k = b_5^l$. Since both the set of vectors $b_3^k, \dots, b_6^k$ and the set of vectors $b_3^l, \dots, b_6^l$ constitute a basis of $z_\mathbf{y}^\pi$, there exists a basis transformation matrix between the two bases, which is of the form

$$
{}_{\beta_{b_6^l}^{z_\mathbf{y}^\pi}}[id]_{\beta_{b_6^k}^{z_\mathbf{y}^\pi}} = \begin{pmatrix} * & * & 0 & * \\ * & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ * & 0 & * & * \end{pmatrix}.
$$

Next we consider the images of the vectors $b_3^k, \dots, b_6^k$ and $b_3^l, \dots, b_6^l$ under the isomorphism $\varphi_{V(\mathcal{G}_\mathbf{y})}$ and get the vectors $d_j^k = \varphi_{V(\mathcal{G}_\mathbf{y})}(b_j^k)$ as well as the vectors $d_j^l = \varphi_{V(\mathcal{G}_\mathbf{y})}(b_j^l)$ for $3 \leq j \leq 6$. Certainly both $d_3^k, \dots, d_6^k$ and $d_3^l, \dots, d_6^l$ are orthonormal bases of the subspace $\alpha(z)^\pi_{\alpha_y(\mathbf{y})}$, as $\varphi_{V(\mathcal{G}_\mathbf{y})}$ is a unitary isomorphism and we get that ${}_{d_3^l,\dots,d_6^l}[id]_{d_3^k,\dots,d_6^k} = {}_{\beta_{b_6^l}^{z_\mathbf{y}^\pi}}[id]_{\beta_{b_6^k}^{z_\mathbf{y}^\pi}}$.

Following the concept of the construction of the isomorphism $\varphi_{\mathcal{G}_\mathbf{z}}$ we compute the bases $\psi_{c_6^k}^{y_\mathbf{z}^\pi} : c_3^k, \dots, c_6^k$ and $\psi_{c_6^l}^{y_\mathbf{z}^\pi} : c_3^l, \dots, c_6^l$ under the extra condition that $c_5^k = c_5^l$ implying that $c_4^k = c_4^l$ and by previous arguments that

$$
{}_{\psi_{c_6^l}^{y_\mathbf{z}^\pi}}[id]_{\psi_{c_6^l}^{y_\mathbf{z}^\pi}} = (e^{i\lambda})^{-1} \cdot e^{i\mu} \cdot {}_{\beta_{b_6^l}^{z_\mathbf{y}^\pi}}[id]_{\beta_{b_6^k}^{z_\mathbf{y}^\pi}}
$$

for some $\lambda, \mu \in [0, 2\pi[$. As $b_5^k = b_5^l$ and $c_5^k = c_5^l$ it follows that $\lambda = \mu$ thus ${}_{\psi_{c_6^l}^{y_\mathbf{z}^\pi}}[id]_{\psi_{c_6^k}^{y_\mathbf{z}^\pi}} = {}_{\beta_{b_6^l}^{z_\mathbf{y}^\pi}}[id]_{\beta_{b_6^k}^{z_\mathbf{y}^\pi}}$. Applying this argumentation also to the bases $\kappa_{g_6^k}^{\alpha(y)^\pi_{\alpha_y(\mathbf{z})}} : g_3^k, \dots, g_6^k$ and $\kappa_{g_6^l}^{\alpha(y)^\pi_{\alpha_y(\mathbf{z})}} : g_3^l, \dots, g_6^l$, where the vectors g_5^k and g_5^l

are equal we have also

$$
{}_{\kappa_{g_6^l}^{\alpha(y)^{\pi}_{\alpha_y(\mathbf{z})}}}[id]_{\kappa_{g_6^k}^{\alpha(y)^{\pi}_{\alpha_y(\mathbf{z})}}} = {}_{\beta^{z_{\mathbf{y}}^{\pi}}_{b_6^l}}[id]_{\beta^{z_{\mathbf{y}}^{\pi}}_{b_6^k}}.
$$

In the next step we determine the bases $v_{a_6^l}^{\langle k_{\mathbf{v}}, y_{\mathbf{v}}\rangle} : a_1^k, a_2^k, a_5^k, a_6^k$ and $v_{a_6^l}^{\langle l_{\mathbf{v}}, y_{\mathbf{v}}\rangle} : a_1^l, a_2^l, a_5^l, a_6^l$. Since the complete construction is independent of the choice of the basis β and the vectors v_1 and v_2, we require that $a_5^k = a_5^l$ implying directly that $a_1^k = a_1^l$ and $a_2^k = a_2^l$. Furthermore we consider also the images of the vectors of the bases $v_{a_6^l}^{\langle k_{\mathbf{v}}, y_{\mathbf{v}}\rangle}$ and $v_{a_6^l}^{\langle l_{\mathbf{v}}, y_{\mathbf{v}}\rangle}$ under the isomorphism $\varphi_{V(\mathcal{G}_{\mathbf{v}})}$ and identify the vectors $r_j^k = \varphi_{V(\mathcal{G}_{\mathbf{v}})}(a_j^k) = \varphi_{V(\mathcal{G}_{\mathbf{v}})}(a_j^l) = r_j^l$ for $j = 1, 2, 5$ and $r_6^k = \varphi_{V(\mathcal{G}_{\mathbf{v}})}(a_6^k)$ as well as $r_6^l = \varphi_{V(\mathcal{G}_{\mathbf{v}})}(a_6^l)$. Next we complete the basis $\psi_{c_6^k}^{y_{\mathbf{z}}^{\pi}}$ to the orthonormal basis $\psi_{c_6^k} : c_1^k, \dots, c_6^k$ of the vector space $V(\mathcal{G}_{\mathbf{z}})$ and the basis $\psi_{c_6^l}^{y_{\mathbf{z}}^{\pi}}$ to $\psi_{c_6^l} : c_1^l, \dots, c_6^l$. The fact that $c_5^k = c_5^l$ indicates also that $c_2^k = c_2^l$ as well as $c_1^k = c_1^l$ and implies that

$$
{}_{\psi_{c_6^l}}[id]_{\psi_{c_6^k}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}_{\beta^{z_{\mathbf{y}}^{\pi}}_{b_6^l}}[id]_{\beta^{z_{\mathbf{y}}^{\pi}}_{b_6^k}} \end{pmatrix}.
$$

We determine also the bases $\kappa_{g_6^k} : g_1^k, \dots, g_6^k$ and $\kappa_{g_6^l} : g_1^l, \dots, g_6^l$ of $V(\mathcal{G}_{\mathbf{z}})$. Again from $g_5^k = g_5^l$ we obtain that $g_2^k = g_2^l$ and $g_1^k = g_1^l$, thus

$$
{}_{\kappa_{g_6^l}}[id]_{\kappa_{g_6^k}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & {}_{\beta^{z_{\mathbf{y}}^{\pi}}_{b_6^l}}[id]_{\beta^{z_{\mathbf{y}}^{\pi}}_{b_6^k}} \end{pmatrix}.
$$

We verify that

$$
\begin{aligned}
& {}_{\kappa_{g_6^l}}[id]_{\kappa_{g_6^k}} \cdot {}_{\kappa_{g_6^k}}[\varphi_{V(\mathcal{G}_{\mathbf{z}})}^{c_6^k, g_6^k}]_{\psi_{c_6^k}} \cdot {}_{\psi_{c_6^k}}[id]_{\psi_{c_6^l}} \\
= \quad & \kappa_{g_6^l}[id]_{\kappa_{g_6^k}} \cdot {}_{\psi_{c_6^k}}[id]_{\psi_{c_6^l}} \\
= \quad & I \\
= \quad & {}_{\kappa_{g_6^l}}[\varphi_{V(\mathcal{G}_{\mathbf{z}})}^{c_6^k, g_6^k}]_{\psi_{c_6^l}} \\
= \quad & e^{i\mu}\, {}_{\kappa_{g_6^l}}[\varphi_{V(\mathcal{G}_{\mathbf{z}})}^{\hat{c}_6^k, \hat{g}_6^k}]_{\psi_{c_6^l}}
\end{aligned}
$$

for some $\mu \in [0, 2\pi[$. Hence the construction of the isomorphism $\varphi_{\mathcal{G}_{\mathbf{z}}}$ is independent from the choice of the line $k_{\mathbf{y}}$ inside the space $v_{\mathbf{y}}^{\pi} \cap z_{\mathbf{y}}^{\pi}$.

Independence of $\alpha_{\mathbf{x}\perp\ldots\perp\mathbf{v}\perp\mathbf{y}\perp\mathbf{z}}$ from $\mathbf{x}\perp\ldots\perp\mathbf{v}\perp\mathbf{y}\perp\mathbf{z}$: In order to prove that α is a graph automorphism of $\widehat{\Gamma}$, we still have to show that it is well-defined, i.e., we have to prove that the image of a vertex $\mathbf{a}$ is independent from the path between $\mathbf{x}$ and $\mathbf{a}$ used to define that image. This again is equivalent to proving that the definition of α is consistent on a set of generators of the 1-fundamental group of $\widehat{\Gamma}$, i.e., the fundamental group of $\widehat{\Gamma}$ considered as a two-dimensional simplicial complex. Since $\widehat{\Gamma}$ is 2-simply connected i.e., simply connected as a two-dimensional simplicial complex, its 1-fundamental group is generated by its triangles.

The consistency of α along triangles is proved, if we can show that for pairwise adjacent vertices $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w}$ in $\widehat{\Gamma}$ the isomorphism $\varphi_{\mathcal{G}_\mathbf{d}}$ for $\mathbf{d}\in\{\mathbf{z},\mathbf{w}\}^\perp$ is uniquely determined, regardless whether the isomorphism is defined using the path $\mathbf{y}\perp\mathbf{z}\perp\mathbf{d}$ or the path $\mathbf{y}\perp\mathbf{w}\perp\mathbf{d}$. This in turn follows, if we can show that $(\varphi_{\mathcal{G}_\mathbf{z}})_{|w_\mathbf{z}^\pi}$ coincides with $(\varphi_{\mathcal{G}_\mathbf{w}})_{|z_\mathbf{w}^\pi}$, which is equivalent to $(\varphi_{V(\mathcal{G}_\mathbf{z})})_{|w_\mathbf{z}^\pi} = e^{i\lambda}\cdot(\varphi_{V(\mathcal{G}_\mathbf{w})})_{|z_\mathbf{w}^\pi}$ for some $\lambda\in\mathbb{R}$, where the isomorphisms $\varphi_{\mathcal{G}_\mathbf{z}}$ and $\varphi_{\mathcal{G}_\mathbf{w}}$ are defined using the path $\gamma\perp\mathbf{z}$ resp. the path $\gamma\perp\mathbf{w}$ where $\gamma:\mathbf{x}\perp\ldots\perp\mathbf{v}\perp\mathbf{y}$ is the path from $\mathbf{x}$ to $\mathbf{y}$ that we used to define the isomorphism $\varphi_{\mathcal{G}_\mathbf{y}}$.

We consider the two-dimensional subspaces $k_\mathbf{y}\subseteq z_\mathbf{y}^\pi\cap v_\mathbf{y}^\pi$ and $l_\mathbf{y}\subseteq w_\mathbf{y}^\pi\cap v_\mathbf{y}^\pi$ and $u_\mathbf{y}=z_\mathbf{y}^\pi\cap w_\mathbf{y}^\pi$. Choose an orthonormal basis u_1, u_2 of $u_\mathbf{y}$ and obtain a basis system $\omega_{h_2}: h_1,h_2;h_{12}$. Now let $\beta^z: v_1^z,\ldots,v_6^z$ and $\beta^w: v_1^w,\ldots,v_6^w$ be two orthonormal bases of the vector space $V(\mathcal{G}_\mathbf{y})$ such that $\langle v_1^z,v_2^z\rangle = z_\mathbf{y}$, $\langle v_1^w,v_2^w\rangle = w_\mathbf{y}$, $\langle v_3^z,v_3^z\rangle = z_\mathbf{y}^\pi\cap k_\mathbf{y}^\pi$, $\langle v_3^w,v_4^w\rangle = w_\mathbf{y}^\pi\cap l_\mathbf{y}^\pi$, $\langle v_5^z,v_6^z\rangle = k_\mathbf{y}$ and $\langle v_5^w,v_6^w\rangle = l_\mathbf{y}$. Suppose the vector v_6^z is not orthogonal to the vector h_1, i.e., $c := (v_n,h_1)\neq 0$ for some normal vector $v_n\in\langle v_6^z\rangle$. For $v\in[0,2\pi[$ and $s^\mathbf{z}=e^{iv}c$ we fix a normal vector $b_6^z\in\langle v_6^z\rangle$ such that $(b_6^z,h_1)=s^\mathbf{z}$ and determine the basis system $\beta_{b_6^z}: b_1^z,\ldots,b_6^z$ of $\mathcal{G}_\mathbf{y}$. Otherwise, if $(v_6^z,h_1)=0$, then the normal vector $\frac{1}{\sqrt{2}}(u_1+v_6^z)$ is not orthogonal to h_1 and v_6^z. Thus we can identify a unique normal vector $h_{v_6^z}\in\langle\frac{1}{\sqrt{2}}(u_1+v_6^z)\rangle$ with $\frac{1}{\sqrt{2}}=(h_1,h_{v_6^z})$ and from the vector $h_{v_6^z}$ a unique normal vector b_6^z satisfying the equation $\frac{1}{\sqrt{2}}=(b_6^z,h_{v_6^z})$. Again we use the normal vector b_6^z and the construction of a basis systems to obtain the basis $\beta_{b_6^z}: b_1^z,\ldots,b_6^z$ of $\mathcal{G}_\mathbf{y}$. Hence we obtain an orthonormal basis $\beta_{b_6^z}: b_1^z,\ldots,b_6^z$ and, analogously, an orthonormal basis $\beta_{b_6^w}: b_1^w,\ldots,b_6^w$ of $\mathcal{G}_\mathbf{y}$.

Define $p_{\mathbf{y},h_j}=\langle h_j\rangle$ for $j=1,2,12$ and $p_{\mathbf{y},h_{v_6^z}}=\langle h_{v_6^z}\rangle$ and $p_{\mathbf{y},h_{v_6^w}}=\langle h_{v_6^w}\rangle$. By lemma 4.3.8 these points induce points $p_{\mathbf{z},h_j}$ for each $j\in\{1,2,12\}$, $p_{\mathbf{z},h_{v_6^z}}$, $p_{\mathbf{z},h_{v_6^w}}$ of $y_\mathbf{z}^\pi\subseteq\mathcal{G}_\mathbf{z}$ and points $p_{\mathbf{w},h_i}$, $i=1,2,12$, $p_{\mathbf{w},h_{v_6^z}}$, $p_{\mathbf{w},h_{v_6^w}}$ of $y_\mathbf{w}^\pi\subseteq\mathcal{G}_\mathbf{w}$. Using basis systems as before, we obtain the orthonormal basis $\tau_{q_2^z}: q_1^z,q_2^z$ of the two-dimensional space $u_\mathbf{z}$ for some normal vector $q_2^z\in p_{\mathbf{z},h_2}$ and the orthonormal basis $\tau_{q_2^w}: q_1^w,q_2^w$ of $u_\mathbf{w}$ for a normal vector $q_2^w\in p_{\mathbf{w},h_2}$. From the vector q_2^z we get a normal vector $c_6^z\in p_{\mathbf{z},v_6^z}$ — in complete analogy to the construction of b_6^z above — and, again using basis systems, we obtain the basis $\psi_{c_6^z}^{y_{\mathbf{z}}^\pi} : c_3^z, \dots c_6^z$ of $y_{\mathbf{z}}^\pi$. Similarly, we get a normal vector $c_6^w \in p_{\mathbf{w},v_6^w}$ and the basis $\psi_{c_6^w}^{y_{\mathbf{w}}^\pi} : c_3^w, \dots c_6^w$ of $y_{\mathbf{w}}^\pi$.

Since the vector spaces $z_{\mathbf{y}}^\pi$ and $y_{\mathbf{z}}^\pi$ are isomorphic, we fix the isomorphism $\theta = \theta_{z_{\mathbf{y}}^\pi}^{y_{\mathbf{z}}^\pi} : z_{\mathbf{y}}^\pi \to y_{\mathbf{z}}^\pi$ of remark 4.3.13 and determine the orthonormal basis $\beta_{\theta(b_6^z)}^{y_{\mathbf{z}}^\pi} : \theta(b_3^z), \dots, \theta(b_6^z)$ and the corresponding basis transformation matrix

$$\psi_{c_6^z}^{y_{\mathbf{z}}^\pi}[id]_{\beta_{b_6^z}^{z_{\mathbf{y}}^\pi}} = \psi_{c_6^z}^{y_{\mathbf{z}}^\pi}[id]_{\beta_{\theta(b_6^z)}^{y_{\mathbf{z}}^\pi}} = e^{i\lambda} I$$

for some $\lambda \in [0, 2\pi[$. Likewise, define $\beta_{\theta(b_6^w)}^{y_{\mathbf{w}}^\pi}$, where $\theta = \theta_{w_{\mathbf{y}}^\pi}^{y_{\mathbf{w}}^\pi}$ with basis transformation matrix

$$\psi_{c_6^w}^{y_{\mathbf{w}}^\pi}[id]_{\beta_{b_6^w}^{w_{\mathbf{y}}^\pi}} = \psi_{c_6^w}^{y_{\mathbf{w}}^\pi}[id]_{\beta_{\theta(b_6^w)}^{y_{\mathbf{w}}^\pi}} = e^{i\sigma} I$$

for some $\sigma \in [0, 2\pi[$.

The isomorphism $\theta = \theta_{\langle u_{\mathbf{z}}, y_{\mathbf{z}}\rangle}^{\langle u_{\mathbf{w}}, y_{\mathbf{w}}\rangle} : \langle u_{\mathbf{z}}, y_{\mathbf{z}}\rangle \to \langle u_{\mathbf{w}}, y_{\mathbf{w}}\rangle$ from remark 4.3.13 with image $\theta(a)$ for every vector $a \in \langle u_{\mathbf{z}}, y_{\mathbf{z}}\rangle$ maps the orthonormal basis $\tau_{(q_2^z)}$ of $u_{\mathbf{z}}$ to the orthonormal basis $\tau_{\theta(q_2^z)} : \theta(q_1^z), \theta(q_2^z)$ of $u_{\mathbf{w}}$. Since

$$\tau_{q_2^z}[id]_{\tau_{\theta(q_2^z)}} = \psi_{c_6^w}^{y_{\mathbf{w}}^\pi}[id]_{\beta_{b_6^w}^{w_{\mathbf{y}}^\pi}} \cdot \beta_{b_6^w}^{w_{\mathbf{y}}^\pi}[id]_{\beta_{b_6^z}^{z_{\mathbf{y}}^\pi}} \cdot \left(\psi_{c_6^z}^{y_{\mathbf{z}}^\pi}[id]_{\beta_{b_6^z}^{z_{\mathbf{y}}^\pi}} \right)^{-1}$$

the transformation matrix between the bases $\tau_{q_2^z}$ and $\tau_{q_2^w}$ is

$$\tau_{q_2^w}[id]_{\tau_{q_2^z}} = \tau_{q_2^w}[id]_{\tau_{\theta(q_2^z)}} = \psi_{c_6^w}^{y_{\mathbf{w}}^\pi}[id]_{\beta_{b_6^w}^{w_{\mathbf{y}}^\pi}} \cdot I \cdot \left(\psi_{c_6^z}^{y_{\mathbf{z}}^\pi}[id]_{\beta_{b_6^z}^{z_{\mathbf{y}}^\pi}} \right)^{-1} = e^{i(\sigma-\lambda)} I.$$

Following the idea of the construction of the isomorphisms $\varphi_{V(\mathcal{G}_{\mathbf{z}})}$ and $\varphi_{V(\mathcal{G}_{\mathbf{w}})}$ we obtain the images $f_j = \varphi_{V(\mathcal{G}_{\mathbf{y}})}(h_j)$ of the vectors h_j, $j = 1, 2, 12$, under the isomorphism $\varphi_{V(\mathcal{G}_{\mathbf{y}})}$ and the images $f_{v_6^z} = \varphi_{V(\mathcal{G}_{\mathbf{y}})}(h_{v_6^z})$ and $f_{v_6^w} = \varphi_{V(\mathcal{G}_{\mathbf{y}})}(h_{v_6^w})$ of $h_{v_6^z}$ and $h_{v_6^w}$. Moreover, let $p_{\alpha_{\mathbf{y}}, f_j} = \langle f_j \rangle$ for $j \in \{1, 2, 12\}$ and let $p_{\alpha_{\mathbf{y}}, f_{v_6^z}} = \langle f_{v_6^z} \rangle$, resp. $p_{\alpha_{\mathbf{y}}, f_{v_6^w}} = \langle f_{v_6^w} \rangle$. Using lemma 4.3.8 we obtain the points $p_{\alpha_{\mathbf{z}}, f_j} = \langle f_j \rangle$ and $p_{\alpha_{\mathbf{z}}, f_{v_6^z}}$ in $\mathcal{G}_{\alpha(\mathbf{z})}$ and the points $p_{\alpha_{\mathbf{w}}, f_j}$ and $p_{\alpha_{\mathbf{w}}, f_{v_6^w}}$ in $\mathcal{G}_{\alpha(\mathbf{w})}$ induced by the points $p_{\alpha_{\mathbf{y}}, f_j}$, for $j \in \{1, 2, 12\}$, and $p_{\alpha_{\mathbf{y}}, f_{v_6^z}}$, resp. $p_{\alpha_{\mathbf{y}}, f_{v_6^w}}$. Now let $t_2^z \in p_{\alpha_{\mathbf{z}}, f_2}$ and $t_2^w \in p_{\alpha_{\mathbf{w}}, f_2}$ be normal vectors. Then we determine the basis system $\eta_{t_2^z} : t_1^z, t_2^z; t_{f_6^z}$ of $\alpha(u)_{\alpha(\mathbf{z})}$ and the basis system $\eta_{t_2^w} : t_1^w, t_2^w; t_{f_6^w}$ of $\alpha(u)_{\alpha(\mathbf{w})}$.

Again in analogy to the construction of b_6^z, the vector t_2^z determines a normal vector $g_6^z \in p_{\alpha_{\mathbf{z}}, d_6^z}$, where $p_{\alpha_{\mathbf{z}}, d_6^z}$ is induced by the point $p_{\alpha_{\mathbf{y}}, d_6^z} = \langle d_6^z \rangle$

with $d_6^z = \varphi_{V(\mathcal{G}_y)}(v_6^z)$ and, similarly, the vector t_2^w induces a normal vector $g_6^w \in p_{\alpha_z, d_6^w}$. Via basis systems the vectors g_6^z and g_6^w determine the bases

$$\kappa_{g_6^z}^{\alpha(y)^{\pi}_{\alpha(\mathbf{z})}} : g_3^z, \ldots, g_6^z$$

of $\alpha(y)^{\pi}_{\alpha(\mathbf{z})}$ and

$$\kappa_{g_6^w}^{\alpha(y)^{\pi}_{\alpha(\mathbf{w})}} : g_3^w, \ldots, g_6^w$$

of $\alpha(y)^{\pi}_{\alpha(\mathbf{w})}$.

As above, the isomorphisms $\theta = \theta^{\alpha(y)^{\pi}_{\alpha(\mathbf{z})}}_{\alpha(z)^{\pi}_{\alpha(\mathbf{y})}}$ and $\vartheta = \theta^{\alpha(y)^{\pi}_{\alpha(\mathbf{w})}}_{\alpha(w)^{\pi}_{\alpha(\mathbf{y})}}$ defined in remark 4.3.13 imply

$$_{\kappa_{g_6^z}^{\alpha(y)^{\pi}_{\alpha(\mathbf{z})}}}[id]_{d_3^z,\ldots,d_6^z} = {}_{\kappa_{g_6^z}^{\alpha(y)^{\pi}_{\alpha(\mathbf{z})}}}[id]_{\theta(d_3^z),\ldots,\theta(d_6^z)} = e^{i\mu} I$$

for $\mu \in [0, 2\pi[$ and

$$_{\kappa_{g_6^w}^{\alpha(y)^{\pi}_{\alpha(\mathbf{w})}}}[id]_{d_3^w,\ldots,d_6^w} = {}_{\kappa_{g_6^w}^{\alpha(y)^{\pi}_{\alpha(\mathbf{w})}}}[id]_{\vartheta(d_3^w),\ldots,\vartheta(d_6^w)} = e^{i\rho} I$$

for $\rho \in [0, 2\pi[$, where $d_j^z = \varphi_{V(\mathcal{G}_y)}(v_j^z)$ and $d_j^w = \varphi_{V(\mathcal{G}_y)}(v_j^w)$ for $3 \le j \le 6$. Hence the basis transformation matrix between the bases $\eta_{t_2^z}$ of $\alpha(u)_{\alpha(\mathbf{z})}$ and $\eta_{t_2^w}$ of $\alpha(u)_{\alpha(\mathbf{w})}$ is

$$\begin{aligned}
_{\eta_{t_2^w}}[id]_{\eta_{t_2^z}} &= {}_{\eta_{t_2^w}}[id]_{\eta_{\theta(t_2^z)}} \\
&= \left({}_{\kappa_{g_6^w}^{\alpha(y)^{\pi}_{\alpha(\mathbf{w})}}}[id]_{d_3^w,\ldots,d_6^w} \right)^{-1} \cdot {}_{\kappa_{g_6^z}^{\alpha(y)^{\pi}_{\alpha(\mathbf{z})}}}[id]_{d_3^z,\ldots,d_6^z} \\
&= e^{i(\rho-\mu)} I,
\end{aligned}$$

where θ is the isomorphism between the subspaces $\langle \alpha(y)_{\alpha(\mathbf{z})}, \alpha(u)_{\alpha(\mathbf{z})} \rangle$ and $\langle \alpha(y)_{\alpha(\mathbf{w})}, \alpha(u)_{\alpha(\mathbf{w})} \rangle$ from remark 4.3.13.

Suppose the vectors v_6^z and v_6^w are not orthogonal, i.e., $m := (v_6^z, v_6^w) \neq 0$. Then we choose a normal vector a_6^z of the point $p_{\mathbf{v}, v_6^z}$ (induced by $p_{\mathbf{y}, v_6^z}$) and obtain via basis systems the orthonormal basis $v_{a_6^z}^{k_{\mathbf{v}}} : a_5^z, a_6^z$. Let a_6^w be the normal vector of the $p_{\mathbf{v}, v_6^w}$ such that $(a_6^z, a_6^w) = m$, which yields the othonormal basis $v_{a_6^w}^{l_{\mathbf{v}}} : a_5^w, a_6^w$. Moreover, since $_{v_{a_6^z}^{k_{\mathbf{v}}}}[id]_{\beta_{b_6^z}^{k_{\mathbf{y}}}} = e^{i\delta} I$ for some $\delta \in [0, 2\pi[$, we see that $_{v_{a_6^w}^{l_{\mathbf{v}}}}[id]_{\beta_{b_6^w}^{l_{\mathbf{y}}}} = e^{i\delta} I$. For the next step let v_1 and v_2 be an orthonormal basis of $y_{\mathbf{v}}$ and let $v_{a_6^z}^{\langle k_{\mathbf{v}}, y_{\mathbf{v}} \rangle} : a_1^z, a_2^z, a_5^z, a_6^z$ be the orthonormal basis of $\langle k_{\mathbf{v}}, y_{\mathbf{v}} \rangle$ resulting from v_1, v_2, a_5^z, a_6^z via basis systems. Furthermore, let $v^{\langle l_{\mathbf{v}},y_{\mathbf{v}}\rangle}_{a^w_6} : a^w_1, a^w_2, a^w_5, a^w_6$ be the orthonormal basis of $\langle l_{\mathbf{v}}, y_{\mathbf{v}}\rangle$ determined analogously. Of course the basis transformation between a^z_1, a^z_2 and a^w_1, a^w_2 is a scalar multiple of the identity matrix, i.e., ${}_{a^w_1,a^w_2}[id]_{a^z_1,a^z_2} = e^{i\varepsilon} I$ for some $\varepsilon \in [0, 2\pi[$. Notice that ${}_{r^w_1,r^w_2}[id]_{r^z_1,r^z_2} = e^{i\varepsilon} I$ as well, where $r^z_j = \varphi_{V(\mathcal{G}_{\mathbf{v}})}(a^z_j)$ and $r^w_j = \varphi_{V(\mathcal{G}_{\mathbf{v}})}(a^w_j)$ for $i = 1, 2, 5, 6$.

By the plan of the construction for the isomorphisms $\varphi_{V(\mathcal{G}_{\mathbf{z}})}$ and $\varphi_{V(\mathcal{G}_{\mathbf{w}})}$ we complete the basis $\psi^{y^\pi_{\mathbf{z}}}_{c^z_6}$ to the orthonormal basis $\psi_{c^z_6} : c^z_1, \ldots c^z_6$, the basis $\psi^{y^\pi_{\mathbf{w}}}_{c^w_6}$ to the orthonormal basis $\psi_{c^w_6} : c^w_1, \ldots c^w_6$, the basis $\kappa^{\alpha(y)^\pi_{\alpha(\mathbf{z})}}_{g^z_6}$ to $\kappa_{g^z_6} : g^z_1, \ldots g^z_6$, and the basis $\kappa^{\alpha(y)^\pi_{\alpha(\mathbf{w})}}_{g^w_6}$ to $\kappa_{g^w_6} : g^w_1, \ldots g^w_6$, in each case using basis systems. We compute the basis transformation matrices

$$\begin{aligned}
{}_{v^{y_{\mathbf{v}}}_{a^z_6}}[id]_{\psi^{y_{\mathbf{z}}}_{c^z_6}} &= {}_{v^{\langle k_{\mathbf{v}},y_{\mathbf{v}}\rangle}_{a^z_6}}[id]_{\psi^{\langle k_{\mathbf{z}},v_{\mathbf{z}}\rangle}_{c^z_6}} \\
&= {}_{v^{k_{\mathbf{v}}}_{a^z_6}}[id]_{\psi^{k_{\mathbf{z}}}_{c^z_6}} \\
&= {}_{v^{k_{\mathbf{v}}}_{a^z_6}}[id]_{\beta^{k_{\mathbf{y}}}_{b^z_6}} \cdot {}_{\beta^{k_{\mathbf{y}}}_{b^z_6}}[id]_{\psi^{k_{\mathbf{z}}}_{c^z_6}} \\
&= e^{i(\delta-\lambda)},
\end{aligned}$$

$$\begin{aligned}
{}_{v^{y_{\mathbf{v}}}_{a^w_6}}[id]_{\psi^{y_{\mathbf{w}}}_{c^w_6}} &= {}_{v^{\langle l_{\mathbf{v}},y_{\mathbf{v}}\rangle}_{a^w_6}}[id]_{\psi^{\langle l_{\mathbf{w}},y_{\mathbf{w}}\rangle}_{c^w_6}} \\
&= {}_{v^{l_{\mathbf{v}}}_{a^w_6}}[id]_{\psi^{l_{\mathbf{w}}}_{c^w_6}} \\
&= {}_{v^{l_{\mathbf{v}}}_{a^w_6}}[id]_{\beta^{l_{\mathbf{y}}}_{b^w_6}} \cdot {}_{\beta^{l_{\mathbf{y}}}_{b^w_6}}[id]_{\psi^{l_{\mathbf{z}}}_{c^z_6}} \\
&= e^{i(\delta-\sigma)} I,
\end{aligned}$$

$$\begin{aligned}
{}_{r^z_1,r^z_2}[id]_{\kappa^{\alpha(y)_{\alpha(\mathbf{z})}}_{g^z_6}} &= {}_{r^z_1,r^z_2,r^z_5,r^z_6}[id]_{\kappa^{\langle \alpha(k)_{\alpha(\mathbf{z})},\alpha(y)_{\alpha(\mathbf{z})}\rangle}_{g^z_6}} \\
&= {}_{r^w_5,r^w_6}[id]_{d^w_5,d^w_6} \cdot {}_{d^w_5,d^w_6}[id]_{\kappa^{\alpha(k)_{\alpha(\mathbf{z})}}_{g^z_6}} \\
&= {}_{v^{k_{\mathbf{y}}}_{a^z_6}}[id]_{\beta^{k_{\mathbf{v}}}_{b^z_6}} \cdot {}_{d^z_5,d^z_6}[id]_{\kappa^{\alpha(k)_{\alpha(\mathbf{z})}}_{g^z_6}} \\
&= e^{i(\delta-\mu)} I,
\end{aligned}$$

$$\begin{aligned}
{}_{r^w_1,r^w_2}[id]_{\kappa^{\alpha(v)_{\alpha(\mathbf{w})}}_{g^w_6}} &= {}_{r^w_1,r^w_2,r^w_5,r^w_6}[id]_{\kappa^{\langle \alpha(l)_{\alpha(\mathbf{w})},\alpha(v)_{\alpha(\mathbf{w})}\rangle}_{g^w_6}} \\
&= {}_{r^z_5,r^z_6}[id]_{d^z_5,d^z_6} \cdot {}_{d^w_5,d^w_6}[id]_{\kappa^{\alpha(l)_{\alpha(\mathbf{w})}}_{g^w_6}} \\
&= {}_{v^{l_{\mathbf{y}}}_{a^w_6}}[id]_{\beta^{l_{\mathbf{v}}}_{b^w_6}} \cdot {}_{d^z_5,d^z_6}[id]_{\kappa^{\alpha(k)_{\alpha(\mathbf{z})}}_{g^z_6}} \\
&= e^{i(\delta-\rho)} I.
\end{aligned}$$

It follows that

$$
\begin{aligned}
& {}_{\kappa_{g_6^z}^{\alpha(w)^{\pi}_{\alpha(\mathbf{z})}}}\left[\left(\varphi_{V(\mathcal{G}_{\mathbf{z}})}\right)_{|w_{\mathbf{z}}^{\pi}}\right]_{\psi_{c_6^z}^{w_{\mathbf{z}}^{\pi}}} \\
= \quad & {}_{\kappa_{g_6^z}^{\alpha(w)^{\pi}_{\alpha(\mathbf{z})}}}[id]_{\kappa_{g_6^w}^{\alpha(z)^{\pi}_{\alpha(\mathbf{w})}}} \cdot {}_{\kappa_{g_6^w}^{\alpha(z)^{\pi}_{\alpha(\mathbf{w})}}}\left[\left(\varphi_{V(\mathcal{G}_{\mathbf{w}})}\right)_{|z_{\mathbf{w}}^{\pi}}\right]_{\psi_{c_6^w}^{z_{\mathbf{w}}^{\pi}}} \cdot {}_{\psi_{c_6^w}^{z_{\mathbf{w}}^{\pi}}}[id]_{\psi_{c_6^z}^{w_{\mathbf{z}}^{\pi}}} \\
= \quad & \begin{pmatrix} e^{i(\mu-\rho-\lambda+\sigma)}I & 0 \\ 0 & e^{i(\mu-\rho-\lambda+\sigma)}I \end{pmatrix} \cdot {}_{\kappa_{g_6^w}^{\alpha(z)^{\pi}_{\alpha(\mathbf{w})}}}\left[\left(\varphi_{V(\mathcal{G}_{\mathbf{w}})}\right)_{|z_{\mathbf{w}}^{\pi}}\right]_{\psi_{c_6^w}^{z_{\mathbf{w}}^{\pi}}} \\
= \quad & e^{i\iota} \cdot {}_{\kappa_{g_6^w}^{\alpha(z)^{\pi}_{\alpha(\mathbf{w})}}}\left[\left(\varphi_{V(\mathcal{G}_{\mathbf{w}})}\right)_{|z_{\mathbf{w}}^{\pi}}\right]_{\psi_{c_6^w}^{z_{\mathbf{w}}^{\pi}}}
\end{aligned}
$$

for some $\iota = \mu - \rho - \lambda + \sigma \in \mathbb{R}$ as the basis transformation matrix

$$
\begin{aligned}
& {}_{\kappa_{g_6^z}^{\alpha(w)^{\pi}_{\alpha(\mathbf{z})}}}[id]_{\kappa_{g_6^w}^{\alpha(z)^{\pi}_{\alpha(\mathbf{w})}}} \\
= \quad & \begin{pmatrix} {}_{\kappa_{g_6^z}^{\alpha(y)_{\alpha(\mathbf{z})}}}[id]_{r_1^z, r_2^z} & 0 \\ 0 & {}_{\kappa_{g_6^z}^{\alpha(u)_{\alpha(\mathbf{z})}}}[id]_{\eta_{t_2^z}} \end{pmatrix} \cdot {}_{\tilde{\eta}}[id]_{\tilde{\nu}} \cdot \begin{pmatrix} {}_{r_1^w, r_2^w}[id]_{\kappa_{g_6^w}^{\alpha(y)_{\alpha(\mathbf{w})}}} & 0 \\ 0 & {}_{\eta_{t_2^w}}[id]_{\kappa_{g_6^w}^{\alpha(u)_{\alpha(\mathbf{w})}}} \end{pmatrix} \\
= \quad & \begin{pmatrix} e^{i(\mu-\delta)}I & 0 \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} {}_{r_1^z, r_2^z}[id]_{r_1^w, r_2^w} & 0 \\ 0 & {}_{\eta_{t_2^z}}[id]_{\eta_{t_2^w}} \end{pmatrix} \cdot \begin{pmatrix} e^{i(-\rho+\delta)}I & 0 \\ 0 & I \end{pmatrix} \\
= \quad & \begin{pmatrix} e^{i(\mu-\delta)}I & 0 \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} e^{-i\varepsilon}I & 0 \\ 0 & e^{i(\mu-\rho)} \cdot I \end{pmatrix} \cdot \begin{pmatrix} e^{i(-\rho+\delta)}I & 0 \\ 0 & I \end{pmatrix} \\
= \quad & \begin{pmatrix} e^{i(\mu-\varepsilon-\rho)}I & 0 \\ 0 & e^{i(\mu-\rho)}I \end{pmatrix},
\end{aligned}
$$

with $\tilde{\eta}$ and $\tilde{\nu}$ are the bases $\tilde{\eta} : r_1^z, r_2^z, t_1^z, t_2^z$ and $\tilde{\nu} : r_1^w, r_2^w, t_1^w, t_2^w$, and

$$
\begin{aligned}
& {}_{\psi_{c_6^w}^{z_{\mathbf{w}}^{\pi}}}[id]_{\psi_{c_6^z}^{w_{\mathbf{z}}^{\pi}}} \\
= \quad & \begin{pmatrix} {}_{\psi_{c_6^w}^{y_{\mathbf{w}}}}[id]_{\nu_{a_6^w}^{y_{\mathbf{v}}}} & 0 \\ 0 & {}_{\psi_{c_6^w}^{u_{\mathbf{w}}}}[id]_{\tau_{q_2^w}} \end{pmatrix} \cdot \begin{pmatrix} {}_{\nu_{a_6^w}^{y_{\mathbf{v}}}}[id]_{\nu_{a_6^z}^{y_{\mathbf{v}}}} & 0 \\ 0 & {}_{\tau_{q_2^w}}[id]_{\tau_{q_2^z}} \end{pmatrix} \cdot \begin{pmatrix} {}_{\nu_{a_6^z}^{y_{\mathbf{v}}}}[id]_{\psi_{c_6^z}^{y_{\mathbf{z}}}} & 0 \\ 0 & {}_{\tau_{q_2^z}}[id]_{\psi_{c_6^z}^{u_{\mathbf{z}}}} \end{pmatrix} \\
= \quad & \begin{pmatrix} e^{i(\sigma-\delta)} \cdot I & 0 \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} e^{i\varepsilon} & 0 \\ 0 & e^{i(-\lambda+\sigma)}I \end{pmatrix} \cdot \begin{pmatrix} e^{i(-\lambda+\delta)}I & 0 \\ 0 & I \end{pmatrix} \\
= \quad & \begin{pmatrix} e^{i(\sigma-\lambda+\varepsilon)}I & 0 \\ 0 & e^{i(\sigma-\lambda)}I \end{pmatrix},
\end{aligned}
$$

which proves that the isomorphisms $(\varphi_{\mathcal{G}_{\mathbf{w}}})_{z_{\mathbf{w}}^{\pi}}$ and $(\varphi_{\mathcal{G}_{\mathbf{z}}})_{w_{\mathbf{z}}^{\pi}}$ coincide. Thus α is a graph isomorphism, the uniqueness of α follows from lemma 4.3.16.

■

Remark 4.3.18 The method to define a graph automorphism along each path and then check that it is well-defined for each generator of the 1-fundamental group of that graph has been pointed out to us by Sergey Shpectorov [79].

The above lemmas allow us, for each vertex $\mathbf{x}$ of $\widehat{\Gamma}$, to construct a faithful action $\mathrm{SU}_2(\mathbb{C}) \to \mathrm{Aut}(\widehat{\Gamma})$ fixing $\mathbf{x}^\perp$ elementwise and acting naturally on the two-dimensional vector space $x_{\mathbf{y}}$ inside $\mathcal{G}_{\mathbf{y}}$ for a neighbour $\mathbf{y}$ of $\mathbf{x}$. The subgroup of $\mathrm{Aut}(\widehat{\Gamma})$ isomorphic to $\mathrm{SU}_2(\mathbb{C})$ obtained in this way is denoted by $\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}}$. The group $\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}}$ induces an action of $\mathrm{SU}_2(\mathbb{C})$ as a fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroup on the vector space structure $V(\mathcal{G}_{\mathbf{y}})$ for any neighbour $\mathbf{y}$ of the vertex $\mathbf{x}$ in $\widehat{\Gamma}$, the centraliser $\{v \in V(\mathcal{G}_{\mathbf{y}}) \mid vf = v \text{ for all } f \in \mathrm{SU}_2(\mathbb{C})_{\mathbf{x}}\}$ being $x_{\mathbf{y}}^\pi$ and the commutator $\{vf - v \in V(\mathcal{G}_{\mathbf{y}}) \mid f \in \mathrm{SU}_2(\mathbb{C})_{\mathbf{x}}, v \in V(\mathcal{G}_{\mathbf{y}})\}$ being $x_{\mathbf{y}}$.

Definition 4.3.19 Let $G_{\widehat{\Gamma}}$ be the subgroup of $\mathrm{Aut}(\widehat{\Gamma})$ generated by the subgroups $(\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}})_{\mathbf{x}\in\widehat{\Gamma}}$ of $\mathrm{Aut}(\widehat{\Gamma})$, in symbols $G_{\widehat{\Gamma}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{x}} \mid \mathbf{x} \in \widehat{\Gamma} \rangle$.

In the sequel we will prove that the group $G_{\widehat{\Gamma}}$ admits a weak Phan system of type A_7 or of type E_6 over $\mathbb{C}$. It then follows from [42] that $G_{\widehat{\Gamma}}$ is a central quotient of the compact group $\mathrm{SU}_8(\mathbb{C})$ or of the compact group ${}^2E_6(\mathbb{C}) = E_{6,-78}$.

4.4 A reflection graph inside the graph Γ

Let V be an euclidean space, so V is a finite dimensional vector space over $\mathbb{R}$ endowed with a symmetric bilinear form $\omega : V \times V \to \mathbb{R}$, which is positive definite. A **reflection** in V is an invertible linear transformation fixing some hyperplane H of V pointwise and mapping any vector u perpendicular to H with respect to ω to its negative $-u$. Furthermore for any non-zero vector $u \in V$ we determine the reflection ρ_u with reflecting hyperplane $H_u = \{w \in V \mid \omega(u,w) = 0\}$. The explicit formula for the reflection ρ_u is

$$\begin{array}{rcl} \rho_u : V & \to & V \\ w & \mapsto & w - \frac{2\omega(w,u)}{\omega(u,u)} u \,. \end{array}$$

A **root system** of V is a finite set Φ of non-zero vectors, called **roots**, that satisfies the following properties:

- The roots span the euclidean space V.
- The only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are α and $-\alpha$.
- For every root $\alpha \in \Phi$, the reflection ρ_α leaves Φ invariant.

- (Integrality condition)

  If α and β are roots in Φ, then the projection of β onto the line through α is a half-integral multiple of α, that is, $\frac{2\omega(\beta,\alpha)}{\omega(\alpha,\alpha)} \in \mathbb{Z}$.

The group of **isometries** of V generated by the reflections ρ_α with α a root of the root system Φ of V is called the **Weyl group** $\mathcal{W}(\Phi)$ of Φ. The Weyl group $\mathcal{W}(\Phi)$ acts faithfully on the finite set Φ and is always finite. Directly from the definition we know that $\omega(\alpha,\beta) = 0$ implies that the reflections ρ_α and ρ_β commute for any pair of roots α,β in Φ. The length of a root $\alpha \in \Phi$ is just the length of the vector α w.r.t. to the form ω. Furthermore we call a root system Φ **reduced** whenever for two proportional roots α and β we have either $\beta = \alpha$ or $\beta = -\alpha$. A root system Φ is called **irreducible** if Φ admits no nontrivial disjoint decomposition $\Phi = \Phi_1 \cup \Phi_2$ where every member of Φ_1 is orthogonal to every member of Φ_2.

Root systems Φ_1 of the euclidean space E_1 w.r.t. ω_1 and Φ_2 of the euclidean space E_2 w.r.t. ω_2 are **isomorphic** if the vector spaces E_1 and E_2 are isomorphic via an isomorphism φ, which is not necessarily an isometry, for which $\varphi(\Phi_1) = \Phi_2$ and $\frac{2\omega_1(\beta,\alpha)}{\omega_1(\alpha,\alpha)} = \frac{2\omega_2(\varphi(\beta),\varphi(\alpha))}{\omega_2(\varphi(\alpha),\varphi(\alpha))}$ for any two roots $\alpha,\beta \in \Phi_1$. It is well know that for two non-proportional roots α,β of a root system Φ of V the element $\alpha - \beta$ is a root in Φ if $\omega(\alpha,\beta) > 0$ and otherwise if $\omega(\alpha,\beta) < 0$ then $\alpha + \beta \in \Phi$. Thus for two non-proportional roots α and β in Φ we look for all roots of the form $\beta + \lambda\alpha$ with $\lambda \in \mathbb{Z}$, denoted by the α-**string through** β. For a root system Φ of V we call $n = \dim(V)$ the **rank** of Φ.

Definition 4.4.1 A subset Δ of Φ is called a **basis of the root stem** Φ of V if Δ is a basis of the vector space V and each root $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha\in\Delta} \lambda_\alpha \alpha$ where all integral scalars k_α are either non-negative or non-positive.

Once a basis Δ has been chosen, the root β for which all $k_\alpha \geq 0$ for all $\alpha \in \Delta$ is called **positive root** with respect to the basis Δ. Certainly a root β is called a **negative root** if all integral scalars k_α are non-positive.

The elements of a basis Δ are called **simple roots**. Furthermore the **height of a root** β w.r.t. the basis Δ is just the integral scalar $\sum_{\alpha\in\Delta} \lambda_\alpha = ht(\beta)$ if $\beta = \sum_{\alpha\in\Delta} \lambda_\alpha \alpha$.

Important properties are that the Weyl group $\mathcal{W}$ act transitively on the collection of all bases of Φ, that for a basis Δ of Φ the Weyl group $\mathcal{W}$ is generated by the reflections ρ_α with $\alpha \in \Delta$ and that for any root $\alpha \in \Phi$ there is an element $w \in \mathcal{W}$ such that $w(\alpha) \in \Delta$.

A consequence for root systems, which will be used later in section 4.6 is the next result.

Lemma 4.4.2 (lemma C and lemma A in 10.4 of [55]) *Let Φ be an irreducible root system with basis Δ. Relative to the definition of a height of a root, there is a unique maximal root $\tilde{\alpha}$ with respect to the basis Δ in Φ, in particular for each root $\alpha \in \Phi$ different from $\tilde{\alpha}$ the equations $ht(\alpha) < ht(\tilde{\alpha})$ and $\omega(\tilde{\alpha}, \alpha) \geq 0$ hold. If $\tilde{\alpha} = \sum_{\alpha \in \Delta} \lambda_\alpha \alpha$ for some basis Δ of Φ then all integral scalars λ_α are positive. Furthermore at most two root lengths occur in the root system Φ, and all roots of a given length are conjugate under $\mathcal{W}$.*

Notation: In an irreducible root system Φ, with two distinct root lengths, we speak of **long and short roots**. If all roots are of equal length, it is convention to call all roots **long**.

Corollary 4.4.3 *Let Φ be an irreducible root system with basis Δ and let $\tilde{\alpha}$ be the maximal root w.r.t. Δ in Φ. Then any root $\alpha \in \Phi$ is conjugate under the Weyl group $\mathcal{W}$ to $\tilde{\alpha}$ if α and $\tilde{\alpha}$ have the same length. In particular every root $\alpha \in \Phi$ is a maximal root with respect to some basis of Φ if all roots are long roots.*

For the next part let Δ be a basis, with a fixed ordering $(\alpha_1, \ldots, \alpha_l)$ of a root system Φ of V. The matrix $\mathcal{C}_\Phi = (\frac{2\omega(\alpha_i,\alpha_j)}{\omega(\alpha_j,\alpha_j)})_{1\leq i,j\leq l}$ is called the **Cartan matrix** . This matrix is independent of the choice of Δ as $\mathcal{W}$ acts transitively on all bases of Φ. It turns out that the Cartan matrix characterise a root system Φ completely by the following theorem.

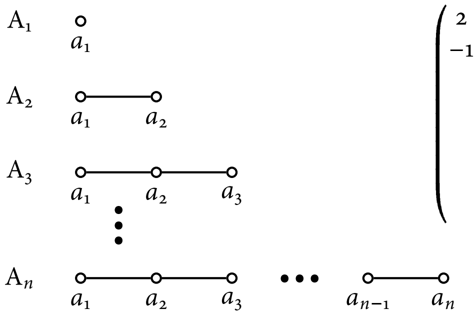
Proposition 4.4.4 (chapter III 11.1 of [55]) *Let E and E' be two euclidean vector spaces over $\mathbb{R}$ with root system Φ respectively Φ' and basis $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ respectively $\Delta' = \{\alpha_1', \ldots, \alpha_l'\}$. If $\frac{2\omega(\alpha_i,\alpha_j)}{\omega(\alpha_j,\alpha_j)} = \frac{2\omega'(\alpha_i',\alpha_j')}{\omega'(\alpha_j',\alpha_j')}$ for all $1 \leq i, j \leq l$ then the bijection $\alpha_i \to \alpha_i'$ extends uniquely to an isomorphism $\varphi : E \to E'$ mapping Φ onto Φ' and satisfying $\frac{2\omega(\alpha,\beta)}{\omega(\beta,\beta)} = \frac{2\omega'(\varphi(\alpha),\varphi(\beta))}{\omega'(\varphi(\beta),\varphi(\beta))}$ for all $\alpha, \beta \in \Phi$. Thus the Cartan matrix determines the root system Φ up to isomomorphism.*

We define the **Coxeter graph** of a root system Φ w.r.t. a basis $\Delta = \alpha_1, \ldots, \alpha_n$, with a fixed ordering, to be the graph having n vertices, where the ith vertex is joined with the jth vertex by $\frac{2\omega(\alpha_i,\alpha_j)}{\omega(\alpha_j,\alpha_j)} \cdot \frac{2\omega(\alpha_j,\alpha_i)}{\omega(\alpha_i,\alpha_i)}$ different edges. If two or more edges between two different vertices occur in the Coxeter graph of Φ then we use an arrow pointing to the shorter root of the two roots, to add the information which is the short root and which is the long root of both. This figure is called the **Dynkin diagram** of Φ.

There exists a classification of irreducible root systems or equivalently of connected Dynkin diagrams.

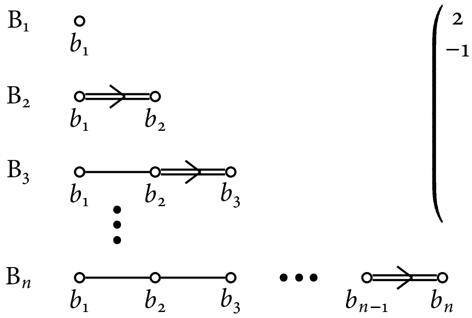
Theorem 4.4.5 (chapter III 11.4 of [55]) *If Φ is an irreducible root system of rank n, its Dynkin diagram and Cartan matrix are one of the following:*

- A_n, *for* $n \geq 1$



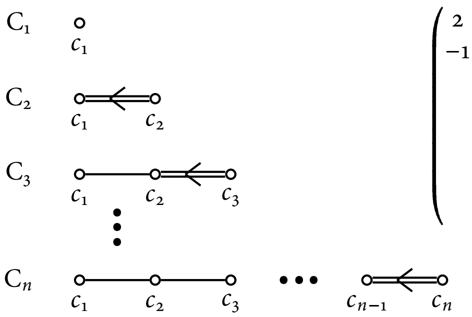
$$\begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix}$$

- B_n, *for* $n \geq 1$



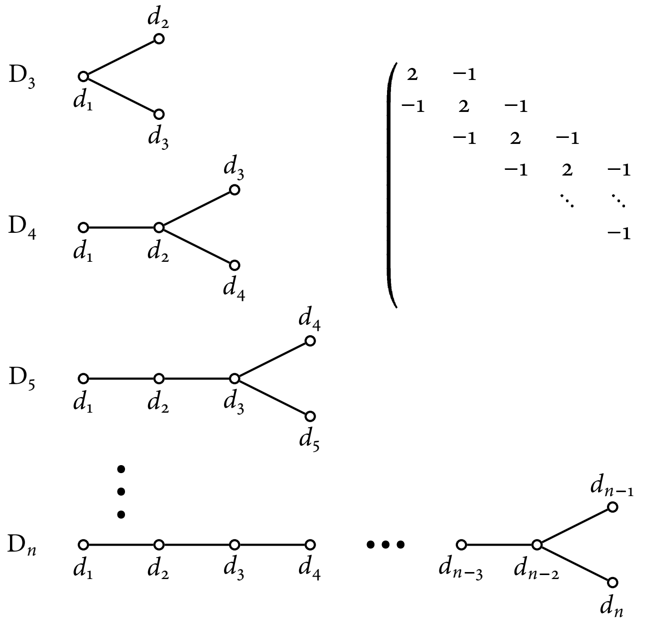
$$\begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -2 \\ & & & & & -1 & 2 \end{pmatrix}$$

- C_n, *for* $n \geq 1$



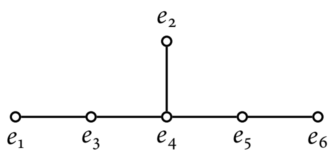
$$\begin{pmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -2 & 2 \end{pmatrix}$$

- D_n, *for* $n \geq 3$



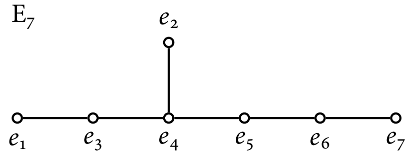
$$\begin{pmatrix}
2 & -1 & & & & & & \\
-1 & 2 & -1 & & & & & \\
 & -1 & 2 & -1 & & & & \\
 & & -1 & 2 & -1 & & & \\
 & & & \ddots & \ddots & \ddots & & \\
 & & & & -1 & 2 & -1 & -1 \\
 & & & & & -1 & 2 & 0 \\
 & & & & & -1 & 0 & 2
\end{pmatrix}$$

- E_6



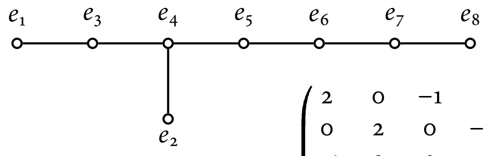
$$\begin{pmatrix}
2 & 0 & -1 & & & \\
0 & 2 & 0 & -1 & & \\
-1 & 0 & 2 & -1 & & \\
 & -1 & -1 & 2 & -1 & \\
 & & & -1 & 2 & -1 \\
 & & & & -1 & 2
\end{pmatrix}$$

- E_7



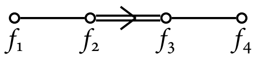
$$\begin{pmatrix}
2 & 0 & -1 & & & & \\
0 & 2 & 0 & -1 & & & \\
-1 & 0 & 2 & -1 & & & \\
 & -1 & -1 & 2 & -1 & & \\
 & & & -1 & 2 & -1 & \\
 & & & & -1 & 2 & -1 \\
 & & & & & -1 & 2
\end{pmatrix}$$

- E_8



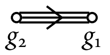
$$\begin{pmatrix} 2 & 0 & -1 & & & & & \\ 0 & 2 & 0 & -1 & & & & \\ -1 & 0 & 2 & -1 & & & & \\ & -1 & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}$$

- F_4



$$\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -2 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$$

- G_2



$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Theorem 4.4.6 *For each Dynkin diagram Δ (or Cartan matrix $\mathcal{C}_\Delta = \mathcal{C}_{\Phi_\Delta}$) of type* A *to* G *there exists an irreducible root system Φ having the given diagram.*

We then call Φ a (irreducible) root system of type Δ and set $\Phi = \Phi_\Delta$. All irreducible root systems of type A to G are listed in [10] or [56]. The (irreducible) root systems Φ_{A_n} and Φ_{E_6} are also listed in appendix A.9. Moreover let $\Phi = \Phi_\Delta$ be a root system of type Δ. We call $(\mathcal{W}, S)$ **a spherical Coxeter system of type** Δ, where $\mathcal{W} = \mathcal{W}(\Phi)$ is the Weyl group of Φ, $S = \{\rho_\alpha \mid \alpha \in \Phi\}$. The notation of a spherical Coxeter system is defined in a more general context, see [10] or [56] or appendix A.6.

Let $(\mathcal{W}, S)$ be a spherical Coxeter system of type Δ and let $\mathbf{W}(\Delta)$ **be the graph on the reflections of** $\mathcal{W}$, i.e. the graph on the conjugates of the elements of S, in which distinct reflections are adjacent if and only if they commute in the group $\mathcal{W}$. Our first aim is to find an induced subgraph Σ in the graph Γ, which is locally $\mathbf{W}(A_5)$ (i.e., the **Kneser graph** $K(6,2)$, defined in [48]). Using theorem 2 of [17] or theorem 2 of [49] we know that a connected locally $\mathbf{W}(A_5)$ graph Σ is isomorphic to

* $\mathbf{W}(A_7)$ (i.e., the Kneser graph $K(8,2)$) with 28 elements,
* $\mathbf{W}(E_6)$ with 36 elements, or
* a graph related to D_6 (but not isomorphic to $\mathbf{W}(D_6)$) with 32 elements.

In this section we construct a locally $\mathbf{W}(A_5)$ induced subgraph Σ of Γ, which will enable us to construct a weak Phan system for the group G_Γ, see [42] which we studied in section 4.5. The philosophy is to construct a Coxeter system from local data that is naturally associated to Γ. Recall also the definition of the induced subgraph $\Psi_{\mathbf{z}}$ of a graph Ψ and a vertex $\mathbf{z} \in \Psi$ from the introduction.

Let $\mathbf{x}$ be a vertex of the graph Γ. We consider the interior space $\mathcal{G}_{\mathbf{x}}$ from the induced subgraph $\mathbf{x}^\perp$, which is a unitary projective space, and the unitary vector space $V(\mathcal{G}_{\mathbf{x}})$, which is of dimension six by proposition 4.2.3. Next we fix a orthonormal basis $\alpha : v_1, \dots, v_6$ of $V(\mathcal{G}_{\mathbf{x}})$ and obtain the 15 different two-dimensional subspaces $y_{\mathbf{x}}^{ij} = \langle v_i, v_j \rangle$ for $1 \leq i < j \leq 6$. Certainly each line $y_{\mathbf{x}}^{ij}$ corresponds to a unique vertex $\mathbf{y}^{ij}$ of the induced subgraph $\mathbf{x}^\perp$ for $1 \leq i < j \leq 6$. These 15 different vertices $\mathbf{y}^{ij}$ for $1 \leq i < j \leq 6$ together with the vertex $\mathbf{x}$ form the vertex set $\mathcal{V}(\Upsilon)$ for the graph Υ. We define the graph Υ to be the induced subgraph of Γ on the vertex set $\{\mathbf{x}, \mathbf{y}^{ij} \mid 1 \leq i < j \leq 6\} = \mathcal{V}(\Upsilon)$.

Regard that the indices i, j for the vertices $\mathbf{y}^{ij} \in \mathcal{V}(\Upsilon)$ are ordered. This ordering is not necessary as the line $y_{\mathbf{x}}^{ij}$ equals the line $y_{\mathbf{x}}^{ji}$ in $\mathcal{G}_{\mathbf{x}}$ for $i, j \in \{1, \dots, 6\}$, $i \neq j$. We still use this ordering for the next parts and we will mention if we disregard the ordering.

Lemma 4.4.7 *The induced subgraph $\Upsilon_{\mathbf{x}}$ of the connected graph Υ is isomorphic to the graph $\mathbf{W}(A_5) \cong K(6,2)$.*

Proof: Two different vertices $\mathbf{y}^{ij}$ and $\mathbf{y}^{kl}$ of $\mathcal{V}(\Upsilon)$ are adjacent, so $\mathbf{y}^{ij} \perp \mathbf{y}^{kl}$, if and only if $y_{\mathbf{x}}^{ij} \subseteq (y_{\mathbf{x}}^{kl})^\pi$ by definition of Γ. Since $y_{\mathbf{x}}^{ij} = \langle v_i, v_j \rangle$ and $y_{\mathbf{x}}^{kl} = \langle v_k, v_l \rangle$, we get $\langle v_i, v_j \rangle = y_{\mathbf{x}}^{ij} \subseteq (y_{\mathbf{x}}^{kl})^\pi = \langle v_k, v_l \rangle^\pi$ if and only if $\{i, j\} \cap \{k, l\} = \varnothing$, thus the map $\Upsilon_{\mathbf{x}} \to K(6,2) \cong \mathbf{W}(A_5) : \mathbf{y}^{ij} \mapsto \{i, j\}$ is a graph isomorphism. ∎

Next we will specify the construction of the graph Σ. We consider again the vector space $V(\mathcal{G}_{\mathbf{x}})$ of the vertex $\mathbf{x}$ and in this vector space we obtain the one-dimensional subspaces $p_{\mathbf{x},v_i} = \langle v_i \rangle$ for $1 \leq i \leq 6$. These are the intersection points of the lines $y_{\mathbf{x}}^{ij}$ and $y_{\mathbf{x}}^{mn}$ with $|\{i, j, m, n\}| = 3$, where $i, j, m, n \in \{1, \dots, 6\}$, $i < j$, $m < n$. More precisely two different lines $y_{\mathbf{x}}^{ij}$ and $y_{\mathbf{x}}^{mn}$ intersect in the point $p_{\mathbf{x},v_k}$ if and only if $|\{i, j, m, n\}| = 3$ and $\{k\} = \{i, j\} \cap \{m, n\}$. These unique interior points of the projective space $\mathcal{G}_{\mathbf{x}}$ induce explicit points $p_{\mathbf{y}^{st},v_k}$ in the projective space $\mathcal{G}_{\mathbf{y}^{st}}$ for $1 \leq s < t \leq 6$, $k \in \{1 \dots 6\} \backslash \{s, t\}$. Indeed each pair of indices s and t with $1 \leq s < t \leq 6$ the projective space $\mathcal{G}_{\mathbf{y}^{st}}$ contains the interior lines $y_{\mathbf{y}^{st}}^{ij}$ with $1 \leq i < j \leq 6$, $\{i, j\} \cap \{s, t\} = \varnothing$ and any two different lines $y_{\mathbf{y}^{st}}^{ij}$ and $y_{\mathbf{y}^{st}}^{mn}$ with $|\{i, j, n, m\}| = 3$ and $\{k\} = \{i, j\} \cap \{m, n\}$ define the point $p_{\mathbf{y}^{st},v_k}$. Directly from the construction we get that each line $y_{\mathbf{y}^{st}}^{ij}$ contains the points $p_{\mathbf{y}^{st},v_i}$ and $p_{\mathbf{y}^{st},v_j}$, which are orthogonal to each other by similar arguments as in lemma 4.3.9 applied to the vertices $\mathcal{V}(\Upsilon)$.

Remark 4.4.8 We would like to point out that we can not use lemma 4.3.8 for the construction above if we restrict ourselves to use only the 16 vertices of the graph Υ without additional information about Γ. Indeed let k be a fixed element of the index set $\{1,\dots,6\}$ then it is not possible to conclude that the points $p_{\mathbf{y}^{ij},v_k}$ induced by the three different vertices $\mathbf{y}^{nm},\mathbf{y}^{st},\mathbf{y}^{pq}$ for $1 \le i < j \le 6$, $\{n,m,s,t,p,q\} = \{1,\dots,6\}\backslash\{i,j\}$, $\{n,m\}\cap\{s,t\}\cap\{p,q\} = \{k\}$ induce each other. As an example we consider the line $y^{34}_{\mathbf{x}}$ in the projective space $\mathcal{G}_{\mathbf{x}}$. Certainly the vertices $\mathbf{y}^{34}$, $\mathbf{y}^{35}$ and $\mathbf{y}^{36}$ define the explicit point $p_{\mathbf{y}^{12},v_3}$ in the projective space $\mathcal{G}_{\mathbf{y}^{12}}$. Also $\mathbf{y}^{34}$, $\mathbf{y}^{31}$ and $\mathbf{y}^{32}$ identify the point $p_{\mathbf{y}^{56},v_3}$ as the intersection point of the lines $y^{34}_{\mathbf{y}^{56}}$, $y^{31}_{\mathbf{y}^{56}}$ and $y^{32}_{\mathbf{y}^{56}}$ in the space $\mathcal{G}_{\mathbf{y}^{56}}$. But it is not possible to conclude that the point $p_{\mathbf{y}^{12},v_3}$ induces the point $p_{\mathbf{y}^{56},v_3}$ using only the vertices of Υ, since only the vertex $\mathbf{y}^{34}$ belongs in both projective spaces, $\mathcal{G}_{\mathbf{y}^{12}}$ and $\mathcal{G}_{\mathbf{y}^{56}}$, to a two-dimensional subspace. Hence the point $p_{\mathbf{y}^{12},v_3}$ correlates either to the point $p_{\mathbf{y}^{56},v_4}$ of the line $y^{34}_{\mathbf{y}^{56}}$ or the point $p_{\mathbf{y}^{56},v_3}$ of $y^{34}_{\mathbf{y}^{56}}$. Thus in general if $\mathbf{y}^{ij}$ and $\mathbf{y}^{mn}$ are two different adjacent vertices with $i,j,m,n \in \{1,\dots,6\}$, $i<j, m<n$ in Υ then the point $p_{\mathbf{y}^{ij},v_k}$ on the line $y^{st}_{\mathbf{y}^{ij}}$ of $\mathcal{G}_{\mathbf{y}^{ij}}$ correlates either to the point $p_{\mathbf{y}^{mn},v_s}$ of the line $y^{st}_{\mathbf{y}^{mn}}$ or the point $p_{\mathbf{y}^{mn},v_t}$ of $y^{st}_{\mathbf{y}^{mn}}$, where $\{s,t,i,j,m,n\} = \{1,\dots,6\}$, $k\in\{s,t\}$. In particular these two possibilities will turn out to distinguish $\mathbf{W}(A_7)$ from $\mathbf{W}(E_6)$.

We consider the complex vector space $V(\mathcal{G}_{\mathbf{y}^{12}})$. Let $\beta : d_1,d_2,v_3,v_4,v_5,v_6$ be an orthonormal basis of $V(\mathcal{G}_{\mathbf{y}^{12}})$ such that $v_i \in p_{\mathbf{y}^{12},v_i}$ for $3\le i\le 6$ and the orthogonal vectors d_1 and d_2 span the line $x_{\mathbf{y}^{12}}$. By construction the two vectors v_i and v_j constitute a basis for the line $y^{ij}_{\mathbf{y}^{12}}$ in $\mathcal{G}_{\mathbf{y}^{12}}$ for $3\le i<j\le 6$. We fix the lines $(z^{ij}_{12})_{\mathbf{y}^{12}} = \langle d_i, v_j\rangle$ for $3 \le i \le 6$, $j\in\{1,2\}$, which belong to the vertices $\mathbf{z}^{ij}_{12}$ of the graph Γ, and the two points $p_{\mathbf{y}^{12},d_1} = \langle d_1\rangle$ and $p_{\mathbf{y}^{12},d_2} = \langle d_2\rangle$. Hence this projective space $\mathcal{G}_{\mathbf{y}^{12}}$ contains the line $x_{\mathbf{y}^{12}}$, the six lines $y^{34}_{\mathbf{y}^{12}}$, $y^{35}_{\mathbf{y}^{12}}$, $y^{36}_{\mathbf{y}^{12}}$, $y^{45}_{\mathbf{y}^{12}}$, $y^{46}_{\mathbf{y}^{12}}$, $y^{56}_{\mathbf{y}^{12}}$ and the eight two-dimensional subspaces $(z^{13}_{12})_{\mathbf{y}^{12}}$, $(z^{14}_{12})_{\mathbf{y}^{12}}$, $(z^{15}_{12})_{\mathbf{y}^{12}}$, $(z^{16}_{12})_{\mathbf{y}^{12}}$, $(z^{23}_{12})_{\mathbf{y}^{12}}$, $(z^{24}_{12})_{\mathbf{y}^{12}}$, $(z^{25}_{12})_{\mathbf{y}^{12}}$, $(z^{26}_{12})_{\mathbf{y}^{12}}$. Analysing the configuration of these 15 different lines it follows that for indices $i,m\in\{1,2\}$, $j,k,l,n\in\{3\dots,6\}$, $k<l$, we know $(z^{ij}_{12})_{\mathbf{y}^{12}} \subseteq (y^{kl}_{\mathbf{y}^{12}})^\pi$ if and only if $j\notin\{k,l\}$ and $(z^{ij}_{12})_{\mathbf{y}^{12}} \subseteq (z^{mn}_{12})^\pi_{\mathbf{y}^{12}}$ if and only if $i\ne m$ and $j\ne n$. Thus for indices $i,m\in\{1,2\}$, $j,k,l,n\in\{3\dots,6\}$, $k<l$, we obtain

$$\begin{aligned} &\mathbf{z}^{ij}_{12} \perp \mathbf{y}^{kl} \text{ in } \Gamma \quad \text{if and only if} \quad j\notin\{k,l\}, \text{ and}\\ &\mathbf{z}^{ij}_{12} \perp \mathbf{z}^{mn}_{12} \text{ in } \Gamma \quad \text{if and only if} \quad i\ne m \text{ and } j\ne n. \end{aligned}$$

Furthermore for $k\in\{1,2\}$, $l\in\{3,\dots,6\}$ every line $(z^{kl}_{12})_{\mathbf{y}^{12}}$ intersects the line $x_{\mathbf{y}^{12}}$ in the point $p_{\mathbf{y}^{12},d_k}$ and the two lines $(z^{1l}_{12})_{\mathbf{y}^{12}}$ and $(z^{2l}_{12})_{\mathbf{y}^{12}}$ intersect each other in the point $p_{\mathbf{y}^{12},v_l}$. The three lines $y^{mn}_{\mathbf{y}^{12}}$, $y^{pq}_{\mathbf{y}^{12}}$ and $y^{st}_{\mathbf{y}^{12}}$ with $\{p,q,m,n,s,t\} = \{3,\dots,6\}$ intersect in the point $p_{\mathbf{y}^{12},v_l}$ for $l\in\{m,n\}\cap\{s,t\}\cap\{p,q\}$. Also any two lines

$(z_{12}^{kl})_{\mathbf{y}^{12}}$ and $(z_{12}^{mn})_{\mathbf{y}^{12}}$ with $k, m \in \{1,2\}$, $n, l \in \{3,\dots,6\}$, $n \neq l$ either intersects in the point $p_{\mathbf{y}^{12},d_k}$ of the line $x_{\mathbf{y}^{12}}$ if $k = m$ or do not intersect in the projective space $\mathcal{G}_{\mathbf{y}^{12}}$.

Using the discussion from above the unique interior points $p_{\mathbf{y}^{12},d_k}$ for $k \in \{1,2\}$ of the projective space $\mathcal{G}_{\mathbf{y}^{12}}$ induce explicit points $p_{\mathbf{y}^{ij},d_k}$ on the line $x_{\mathbf{y}^{ij}}$ in the projective space $\mathcal{G}_{\mathbf{y}^{ij}}$ for $3 \leq i < j \leq 6$. As before we define the point $p_{\mathbf{y}^{ij},d_k}$ of the projective space $\mathcal{G}_{\mathbf{y}^{ij}}$ to be the intersection point of the three lines $x_{\mathbf{y}^{ij}}$, $(z_{12}^{kl})_{\mathbf{y}^{kj}}$ and $(z_{12}^{km})_{\mathbf{y}^{ij}}$ with $l, m \in \{3,\dots,6\}\backslash\{i,j\}$, $l \neq m$. Certainly the two points $p_{\mathbf{y}^{ij},d_1}$ and $p_{\mathbf{y}^{ij},d_2}$ are pairwise orthogonal and orthogonal to the points $p_{\mathbf{y}^{ij},v_q}$ for $q \in \{1,\dots,6\}\backslash\{i,j\}$ by a similar argumentation as in lemma 4.3.9 applied to the vertices $\mathcal{V}(\Upsilon) \cup \{\mathbf{z}_{12}^{ij} \mid i \in \{1,2\}, 3 \leq j \leq 6\}$.

Therefore let $4 \leq k < l \leq 6$, then we obtain in the projective space $\mathcal{G}_{\mathbf{y}^{kl}}$ the six pairwise orthogonal points $p_{\mathbf{y}^{kl},d_1}, p_{\mathbf{y}^{kl},d_2}$ and $p_{\mathbf{y}^{kl},v_j}$ for $j \in \{1,\dots,6\}\backslash\{k,l\}$. We determine the eight different two-dimensional subspaces $(z_{kl}^{ij})_{\mathbf{y}^{kl}} = \langle p_{\mathbf{y}^{kl},d_i}, p_{\mathbf{y}^{kl},v_j}\rangle$ for $i \in \{1,2\}$, $j \in \{1,\dots,6\}\backslash\{k,l\}$ in $\mathcal{G}_{\mathbf{y}^{kl}}$, which again belong to the vertices $\mathbf{z}_{kl}^{ij}$ in the graph Γ. A similar analysis for the projective space $\mathcal{G}_{\mathbf{y}^{kl}}$ as was done for the space $\mathcal{G}_{\mathbf{y}^{12}}$ leads to the facts that the vertices $\mathbf{z}_{kl}^{1i}, \mathbf{z}_{kl}^{1j}, \mathbf{z}_{kl}^{2i}$ and $\mathbf{z}_{kl}^{2j}$ are elements of the induced subgraph $\{\mathbf{y}^{st}\}^{\perp}$ in Γ for $\{i,j,s,t\} = \{1,\dots,6\}\backslash\{k,l\}$, that the two different lines $x_{\mathbf{y}^{kl}}$ and $(z_{kl}^{mn})_{\mathbf{y}^{kl}}$ intersect in the point $p_{\mathbf{y}^{kl},d_m}$ for $m \in \{1,2\}$, $n \in \{1,\dots,6\}\backslash\{k,l\}$ and that any two different lines $(z_{kl}^{st})_{\mathbf{y}^{kl}}$ and $(z_{kl}^{mn})_{\mathbf{y}^{kl}}$ either intersect in the point $p_{\mathbf{y}^{kl},d_m}$ of the line $x_{\mathbf{y}^{kl}}$ if $s = m$ or do not intersect at all in the projective space $\mathcal{G}_{\mathbf{y}^{kl}}$ for $s, m \in \{1,2\}$, $n, t \in \{1,\dots,6\}\backslash\{k,l\}$, $n \neq t$.

The points $p_{\mathbf{y}^{kl},d_1}$ and $p_{\mathbf{y}^{kl},d_2}$ induce non-ambiguous points $p_{\mathbf{y}^{ij},d_1}$ and $p_{\mathbf{y}^{ij},d_2}$ in the spaces $\mathcal{G}_{\mathbf{y}^{ij}}$ where $i \in \{1,2\}$, $j \in \{3,\dots,6\}\backslash\{k,l\}$. As before the point $p_{\mathbf{y}^{ij},d_s}$ is defined to be the intersection point of the three different lines $x_{\mathbf{y}^{ij}}$, $(z_{kl}^{st})_{\mathbf{y}^{ij}}$, $(z_{kl}^{sr})_{\mathbf{y}^{ij}}$ with $\{r,t\} = \{1,\dots,6\}\backslash\{i,j,k,l\}$, $s \in \{1,2\}$. Certainly with an argument similar to the one used in lemma 4.3.9 applied to the vertices $\mathcal{V}(\Upsilon) \cup \{\mathbf{z}_{12}^{mn} \mid m \in \{1,2\}, 3 \leq n \leq 6\} \cup \{\mathbf{z}_{oq}^{mn} \mid m \in \{1,2\}, 4 \leq o < q \leq 6, n \in \{3,\dots,6\}\backslash\{o,q\}\}$ it follows that the two points $p_{\mathbf{y}^{ij},d_1}$ and $p_{\mathbf{y}^{ij},d_2}$ are pairwise perpendicular and also orthogonal to the points $p_{\mathbf{y}^{ij},v_c}$ for any $c \in \{1,\dots,6\}\backslash\{i,j\}$.

At this point, in each projective space $\mathcal{G}_{\mathbf{y}^{ij}}$ for $1 \leq i < j \leq 6$ we have identified six pairwise orthogonal points $p_{\mathbf{y}^{ij},d_k}$, $p_{\mathbf{y}^{ij},v_l}$ with $k \in \{1,2\}$, $l \in \{1,\dots,6\}\backslash\{i,j\}$. Notice, as before in remark 4.4.8, we can not conclude that the point $p_{\mathbf{y}^{ij},d_k}$ correlates to the point $p_{\mathbf{y}^{st},d_k}$ for any two vertices $\mathbf{y}^{ij} \perp \mathbf{y}^{st}$ in Γ for $k \in \{1,2\}$, $i,j,s,t \in \{1,\dots,6\}$, $i < j$, $s < t$ without any additional information. Next we consider in each projective space $\mathcal{G}_{\mathbf{y}^{ij}}$ for $1 \leq i < j \leq 6$, the eight different two-dimensional subspace $(z_{ij}^{kl})_{\mathbf{y}^{ij}}$ for $k \in \{1,2\}$, $l \in \{1,\dots,6\}\backslash\{i,j\}$ which belong to the vertices $\mathbf{z}_{kl}^{ij}$ of the graph Γ. Here we have an ordering on the indices i,j and k,l for a vertex $\mathbf{z}_{kl}^{ij}$. The ordering on the pair k, l is not essential. Certainly the two vertices $\mathbf{z}_{kl}^{ij}$ and $\mathbf{z}_{lk}^{ij}$ coincide and later we will disregard this ordering. But we would like to point out that we can not interchange the position of the indices i and j. As an example $\mathbf{z}_{56}^{12} \neq \mathbf{z}_{56}^{21}$ in Γ.

We define the graph Σ to be the induced subgraph of Γ on the vertex set

$$\mathcal{V}(\Sigma) = \{\mathbf{x}, \mathbf{y}^{ij}, \mathbf{z}_{mn}^{kl} \mid k \in \{1,2\},\ i,j,l,m,n \in \{1,\ldots,6\},\ i < j,\ m < n,\ m \neq l \neq n\}.$$

Since $\mathcal{V}(\Sigma)$ possesses 15 different vertices $\mathbf{y}^{ij}$, at most $15 \cdot 8 = 120$ vertices of type $\mathbf{z}_{mn}^{kl}$ and the vertex $\mathbf{x}$, it follows that the set $\mathcal{V}(\Sigma)$ contains at most 136 different elements. We denote with $\mathcal{V}(z)$ the vertices of type $\mathbf{z}_{mn}^{kl}$ of the graph Σ. Thus

$$\mathcal{V}(z) = \{\mathbf{z}_{ij}^{kl} \mid k \in \{1,2\},\ l,m,n \in \{1,\ldots,6\},\ m < n,\ m \neq l \neq n\}.$$

In order to show that the induced subgraph Σ of Γ is locally $\mathbf{W}(A_5)$, we need to establish the isomorphism type of the induced subgraphs $\Sigma_{\mathbf{w}}$ for each vertex $\mathbf{w}$ in $\mathcal{V}(\Sigma)$.

Corollary 4.4.9 *Every vertex $\mathbf{z}_{ij}^{kl}$ of $\mathcal{V}(z)$ is different from $\mathbf{x}$ and not adjacent to $\mathbf{x}$ in the graph Σ.*

Proof: Since each line $(z_{ij}^{kl})_{\mathbf{y}^{ij}}$ intersects the line $x_{\mathbf{y}^{ij}}$ in the unique point $p_{\mathbf{y}^{ij},d_k}$ by construction, the vertices $\mathbf{z}_{ij}^{kl}$ and $\mathbf{x}$ are neither adjacent nor equal. ■

Lemma 4.4.10 *The induced subgraph $\Sigma_{\mathbf{x}}$ of the connected graph Σ is isomorphic to the graph $\mathbf{W}(A_5) \cong K(6,2)$.*

Proof: This statement follows from corollary 4.4.9 and lemma 4.4.7. ■

Study now the projective space $\mathcal{G}_{\mathbf{y}^{12}}$ with the four lines $(z_{12}^{13})_{\mathbf{y}^{12}}$, $(z_{12}^{14})_{\mathbf{y}^{12}}$, $(z_{12}^{23})_{\mathbf{y}^{12}}$, $(z_{12}^{24})_{\mathbf{y}^{12}}$, which are orthogonal to the line $y_{\mathbf{y}^{12}}^{56}$. Therefore we can consider these lines in the projective space $\mathcal{G}_{\mathbf{y}^{56}}$. Recall that the two lines $(z_{12}^{i3})_{\mathbf{y}^{56}}$ and $(z_{12}^{i4})_{\mathbf{y}^{56}}$ intersect the line $x_{\mathbf{y}^{56}}$ in the point $p_{\mathbf{y}^{56},d_i}$ by construction for $i \in \{1,2\}$. Also recall that $(z_{12}^{13})_{\mathbf{y}^{56}}$ and $(z_{12}^{23})_{\mathbf{y}^{56}}$ intersect the line $y_{\mathbf{y}^{56}}^{34}$ either in the point $p_{\mathbf{y}^{56},v_3}$ or in the point $p_{\mathbf{y}^{56},v_4}$, cf. remark 4.4.8. Thus the first case is that $(z_{12}^{13})_{\mathbf{y}^{56}} \cap (z_{12}^{23})_{\mathbf{y}^{56}} \cap y_{\mathbf{y}^{56}}^{34} = p_{\mathbf{y}^{56},v_3}$ implying that $(z_{12}^{13})_{\mathbf{y}^{56}} = \langle p_{\mathbf{y}^{56},d_1}, p_{\mathbf{y}^{56},v_3}\rangle = (z_{56}^{13})_{\mathbf{y}^{56}}$ and $(z_{12}^{23})_{\mathbf{y}^{56}} = \langle p_{\mathbf{y}^{56},d_2}, p_{\mathbf{y}^{56},v_3}\rangle = (z_{56}^{23})_{\mathbf{y}^{56}}$, so we conclude the equality $\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i3}$ in the graph Σ for $i \in \{1,2\}$. Alternatively, we get that $(z_{12}^{13})_{\mathbf{y}^{56}} \cap (z_{12}^{23})_{\mathbf{y}^{56}} \cap y_{\mathbf{y}^{56}}^{34} = p_{\mathbf{y}^{56},v_4}$, which leads to $(z_{12}^{13})_{\mathbf{y}^{56}} = \langle p_{\mathbf{y}^{56},d_1}, p_{\mathbf{y}^{56},v_4}\rangle = (z_{56}^{14})_{\mathbf{y}^{56}}$ and $(z_{12}^{23})_{\mathbf{y}^{56}} = \langle p_{\mathbf{y}^{56},d_2}, p_{\mathbf{y}^{56},v_4}\rangle = (z_{56}^{24})_{\mathbf{y}^{56}}$, thus $\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i4}$ in Σ for $i \in \{1,2\}$.

Case 1 : $\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i3}$ for $i \in \{1,2\}$

Proposition 4.4.11 *Suppose* $\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i3}$ *for* $i \in \{1,2\}$. *Then the following identities for* $k \in \{1,2\}$ *hold in* Σ *:*

$$\mathbf{z}^{k1} := \mathbf{z}_{23}^{k1} = \mathbf{z}_{24}^{k1} = \mathbf{z}_{25}^{k1} = \mathbf{z}_{26}^{k1} = \mathbf{z}_{34}^{k1} = \mathbf{z}_{35}^{k1} = \mathbf{z}_{36}^{k1} = \mathbf{z}_{45}^{k1} = \mathbf{z}_{46}^{k1} = \mathbf{z}_{56}^{k1},$$

$$\mathbf{z}^{k2} := \mathbf{z}_{13}^{k2} = \mathbf{z}_{14}^{k2} = \mathbf{z}_{15}^{k2} = \mathbf{z}_{16}^{k2} = \mathbf{z}_{34}^{k2} = \mathbf{z}_{35}^{k2} = \mathbf{z}_{36}^{k2} = \mathbf{z}_{45}^{k2} = \mathbf{z}_{46}^{k2} = \mathbf{z}_{56}^{k2},$$

$$\mathbf{z}^{k3} := \mathbf{z}_{12}^{k3} = \mathbf{z}_{14}^{k3} = \mathbf{z}_{15}^{k3} = \mathbf{z}_{16}^{k3} = \mathbf{z}_{24}^{k3} = \mathbf{z}_{25}^{k3} = \mathbf{z}_{26}^{k3} = \mathbf{z}_{45}^{k3} = \mathbf{z}_{46}^{k3} = \mathbf{z}_{56}^{k3},$$

$$\mathbf{z}^{k4} := \mathbf{z}_{12}^{k4} = \mathbf{z}_{13}^{k4} = \mathbf{z}_{15}^{k4} = \mathbf{z}_{16}^{k4} = \mathbf{z}_{23}^{k4} = \mathbf{z}_{25}^{k4} = \mathbf{z}_{26}^{k4} = \mathbf{z}_{35}^{k4} = \mathbf{z}_{36}^{k4} = \mathbf{z}_{56}^{k4},$$

$$\mathbf{z}^{k5} := \mathbf{z}_{12}^{k5} = \mathbf{z}_{13}^{k5} = \mathbf{z}_{14}^{k5} = \mathbf{z}_{16}^{k5} = \mathbf{z}_{23}^{k5} = \mathbf{z}_{24}^{k5} = \mathbf{z}_{26}^{k5} = \mathbf{z}_{34}^{k5} = \mathbf{z}_{36}^{k5} = \mathbf{z}_{46}^{k5},$$

$$\mathbf{z}^{k6} := \mathbf{z}_{12}^{k6} = \mathbf{z}_{13}^{k6} = \mathbf{z}_{14}^{k6} = \mathbf{z}_{15}^{k6} = \mathbf{z}_{23}^{k6} = \mathbf{z}_{24}^{k6} = \mathbf{z}_{25}^{k6} = \mathbf{z}_{34}^{k6} = \mathbf{z}_{35}^{k6} = \mathbf{z}_{45}^{k6}.$$

Proof: The proof of this statement is the technical part of case 1. By assumption $(z_{12}^{13})_{\mathbf{y}^{56}} \cap (z_{12}^{23})_{\mathbf{y}^{56}} \cap y_{\mathbf{y}^{56}}^{34} = p_{\mathbf{y}^{56},v_3}$ and $\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i3}$ in Σ for $i \in \{1,2\}$. Then the intersection point of the three lines $(z_{12}^{14})_{\mathbf{y}^{56}}$, $(z_{12}^{24})_{\mathbf{y}^{56}}$ and $y_{\mathbf{y}^{56}}^{34}$, which is orthogonal to the intersection point of $(z_{12}^{13})_{\mathbf{y}^{56}}$, $(z_{12}^{23})_{\mathbf{y}^{56}}$ and $y_{\mathbf{y}^{56}}^{34}$ is the point $p_{\mathbf{y}^{56},v_4}$ implying directly that $(z_{12}^{i4})_{\mathbf{y}^{56}} = \langle p_{\mathbf{y}^{56},d_i}, p_{\mathbf{y}^{56},v_4}\rangle = (z_{56}^{i4})_{\mathbf{y}^{56}}$ and that $\mathbf{z}_{12}^{i4} = \mathbf{z}_{56}^{i4}$ for $i \in \{1,2\}$. This configuration yields also that the line $(z_{12}^{i3})_{\mathbf{y}^{56}} = (z_{56}^{i3})_{\mathbf{y}^{56}}$ intersects the lines $y_{\mathbf{y}^{56}}^{j3}$ in the projective space $\mathcal{G}_{\mathbf{y}^{56}}$ for $i,j \in \{1,2\}$. Hence the vertices $\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i3}$ and $\mathbf{y}^{j3}$ are not adjacent in the graph Σ for $i,j \in \{1,2\}$.

Switching to the projective space $\mathcal{G}_{\mathbf{y}^{45}}$, certainly the two lines $(z_{12}^{i3})_{\mathbf{y}^{45}}$ for $i \in \{1,2\}$ intersect in the line $y_{\mathbf{y}^{45}}^{36}$ either in the point $p_{\mathbf{y}^{45},v_3}$ or in the point $p_{\mathbf{y}^{45},v_6}$ and the two lines $(z_{12}^{1j})_{\mathbf{y}^{45}}$ for $j \in \{3,6\}$ intersect the line $x_{\mathbf{y}^{45}}$ in the point $p_{\mathbf{y}^{45},d_1}$. Thus $(z_{12}^{13})_{\mathbf{y}^{45}}$ equals either the line $(z_{45}^{13})_{\mathbf{y}^{45}}$ or the line $(z_{45}^{16})_{\mathbf{y}^{45}}$. Suppose $(z_{12}^{13})_{\mathbf{y}^{45}}$ intersects the subspace $y_{\mathbf{y}^{45}}^{36}$ in the point $p_{\mathbf{y}^{45},v_6}$, then $(z_{12}^{13})_{\mathbf{y}^{45}} = \langle p_{\mathbf{y}^{45},d_1}, p_{\mathbf{y}^{45},v_6}\rangle = (z_{45}^{16})_{\mathbf{y}^{45}}$, which is contained in the orthogonal space $(y_{\mathbf{y}^{45}}^{23})^{\pi}$ of the line $y_{\mathbf{y}^{45}}^{23}$, contradicting the fact that $\mathbf{z}_{12}^{13}$ is not adjacent to the vertex $\mathbf{y}^{23}$ in Γ. Therefore the two-dimensional subspaces $(z_{12}^{13})_{\mathbf{y}^{45}}$ and $y_{\mathbf{y}^{45}}^{36}$ intersect in the point $p_{\mathbf{y}^{45},v_3}$ implying that $(z_{12}^{1i})_{\mathbf{y}^{45}} = \langle p_{\mathbf{y}^{45},d_1}, p_{\mathbf{y}^{45},v_i}\rangle = (z_{45}^{1i})_{\mathbf{y}^{45}}$ and $(z_{12}^{2i})_{\mathbf{y}^{45}} = \langle p_{\mathbf{y}^{45},d_2}, p_{\mathbf{y}^{45},v_i}\rangle = (z_{45}^{2i})_{\mathbf{y}^{45}}$ for $i \in \{3,6\}$, hence $\mathbf{z}_{12}^{13} = \mathbf{z}_{45}^{13}$ and $\mathbf{z}_{12}^{16} = \mathbf{z}_{45}^{16}$ and $\mathbf{z}_{12}^{23} = \mathbf{z}_{45}^{23}$ and $\mathbf{z}_{12}^{26} = \mathbf{z}_{45}^{26}$.

The vertices $\mathbf{z}_{12}^{1i}$ and $\mathbf{z}_{12}^{2i}$ are adjacent to the vertex $\mathbf{y}^{46}$ for $i \in \{3,5\}$, therefore the intersection of the lines $(z_{12}^{i3})_{\mathbf{y}^{45}}$ for $i \in \{1,2\}$ and $y_{\mathbf{y}^{46}}^{35}$ is either the point $p_{\mathbf{y}^{46},v_3}$ or the point $p_{\mathbf{y}^{46},v_5}$ and the two lines $(z_{12}^{1j})_{\mathbf{y}^{45}}$ for $j \in \{3,5\}$ intersect the line $x_{\mathbf{y}^{46}}$ in the point $p_{\mathbf{y}^{46},d_1}$ in $\mathcal{G}_{\mathbf{y}^{46}}$. Again the line $(z_{12}^{13})_{\mathbf{y}^{46}}$ is either equal to the subspace $(z_{46}^{13})_{\mathbf{y}^{46}}$ or the line $(z_{46}^{15})_{\mathbf{y}^{46}}$. If $(z_{12}^{13})_{\mathbf{y}^{46}} = \langle p_{\mathbf{y}^{46},d_1}, p_{\mathbf{y}^{46},v_5}\rangle = (z_{46}^{15})_{\mathbf{y}^{46}}$, then the two vertices $\mathbf{z}_{12}^{13} = \mathbf{z}_{46}^{15}$ and $\mathbf{y}^{23}$ are adjacent in Γ, contradiction to the fact that the lines $(z_{12}^{13})_{\mathbf{y}^{56}} = (z_{56}^{13})_{\mathbf{y}^{56}}$

and $y^{23}_{\mathbf{y}^{56}}$ intersect in $\mathcal{G}_{\mathbf{y}^{56}}$. This implies that $(z^{ij}_{12})_{\mathbf{y}^{46}} = \langle p_{\mathbf{y}^{46},d_i}, p_{\mathbf{y}^{46},v_j}\rangle = (z^{ij}_{46})_{\mathbf{y}^{46}}$ for $i \in \{1,2\}$, $j \in \{3,5\}$. Summarising the identities, we know that:

$$\begin{aligned} \mathbf{z}^{i3}_{12} = \mathbf{z}^{i3}_{56} = \mathbf{z}^{i3}_{45} = \mathbf{z}^{i3}_{46} \text{ for } i \in \{1,2\} &\quad \mathbf{z}^{i5}_{12} = \mathbf{z}^{i5}_{46} \text{ for } i \in \{1,2\} \\ \mathbf{z}^{i4}_{12} = \mathbf{z}^{i4}_{56} \text{ for } i \in \{1,2\} &\quad \mathbf{z}^{i6}_{12} = \mathbf{z}^{i6}_{45} \text{ for } i \in \{1,2\}. \end{aligned}$$

Next we consider the vertices $\mathbf{z}^{i3}_{12} = \mathbf{z}^{i3}_{kl}$ and $\mathbf{z}^{ij}_{kl}$ for $i, j \in \{1,2\}$, $4 \le k < l \le 6$, certainly for these indices we know that $\mathbf{z}^{i3}_{12} = \mathbf{z}^{i3}_{kl}$ and $\mathbf{z}^{ij}_{kl}$ are elements of the induced graphs $\{\mathbf{y}^{mn}\}^\perp$ for $m \in \{1,2\}\backslash\{j\}$, $n \in \{4,5,6\}\backslash\{k,l\}$. The line $(z^{i3}_{kl})_{\mathbf{y}^{mn}}$ is orthogonal to the line $y^{kl}_{\mathbf{y}^{mn}}$, intersects the line $x_{\mathbf{y}^{mn}}$ in the point $p_{\mathbf{y}^{mn},d_i}$ by construction and the line $y^{s3}_{\mathbf{y}^{mn}}$ either in the point $p_{\mathbf{y}^{mn},v_s}$ or in the point $p_{\mathbf{y}^{mn},v_3}$ in the projective space $\mathcal{G}_{\mathbf{y}^{mn}}$ for $m, s \in \{1,2\}$, $n \in \{4,\dots,6\}\backslash\{k,l\}$. Suppose $(z^{i3}_{kl})_{\mathbf{y}^{mn}} \cap y^{s3}_{\mathbf{y}^{mn}}$ is the point $p_{\mathbf{y}^{mn},v_s}$ in $\mathcal{G}_{\mathbf{y}^{mn}}$ then of course $(z^{i3}_{kl})_{\mathbf{y}^{mn}} = \langle p_{\mathbf{y}^{mn},d_i}, p_{\mathbf{y}^{mn},v_s}\rangle = (z^{is}_{mn})_{\mathbf{y}^{mn}}$, which is contained in the polar space $(y^{3k}_{\mathbf{y}^{mn}})^\pi$ of the line $y^{3k}_{\mathbf{y}^{mn}}$, contradiction as the vertices $\mathbf{z}^{i3}_{kl}$ and $\mathbf{y}^{3k}$ are not adjacent in the graph Γ. Therefore $(z^{i3}_{kl})_{\mathbf{y}^{mn}} = \langle p_{\mathbf{y}^{mn},d_i}, p_{\mathbf{y}^{mn},v_3}\rangle = (z^{i3}_{mn})_{\mathbf{y}^{mn}}$ and $(z^{ij}_{kl})_{\mathbf{y}^{mn}} = \langle p_{\mathbf{y}^{mn},d_i}, p_{\mathbf{y}^{kl},v_j}\rangle = (z^{ij}_{mn})_{\mathbf{y}^{mn}}$ for the indices $i, m, j \in \{1,2\}$, $m \ne j$, $4 \le k < l \le 6$, $n \in \{4,\dots,6\}\backslash\{k,l\}$. Thus we obtain the relations

$$\begin{aligned} &\mathbf{z}^{i3}_{12} = \mathbf{z}^{i3}_{56} = \mathbf{z}^{i3}_{45} = \mathbf{z}^{i3}_{46} = \mathbf{z}^{i3}_{14} = \mathbf{z}^{i3}_{15} = \mathbf{z}^{i3}_{16} = \mathbf{z}^{i3}_{24} = \mathbf{z}^{i3}_{25} = \mathbf{z}^{i3}_{26} \text{ for } i \in \{1,2\} \\ &\mathbf{z}^{i4}_{12} = \mathbf{z}^{i4}_{56} \text{ for } i \in \{1,2\} \qquad \mathbf{z}^{i1}_{26} = \mathbf{z}^{i1}_{45} \text{ for } i \in \{1,2\} \\ &\mathbf{z}^{i5}_{12} = \mathbf{z}^{i5}_{46} \text{ for } i \in \{1,2\} \qquad \mathbf{z}^{i2}_{14} = \mathbf{z}^{i2}_{56} \text{ for } i \in \{1,2\} \\ &\mathbf{z}^{i6}_{12} = \mathbf{z}^{i6}_{45} \text{ for } i \in \{1,2\} \qquad \mathbf{z}^{i2}_{15} = \mathbf{z}^{i2}_{46} \text{ for } i \in \{1,2\} \\ &\mathbf{z}^{i1}_{24} = \mathbf{z}^{i1}_{56} \text{ for } i \in \{1,2\} \qquad \mathbf{z}^{i2}_{16} = \mathbf{z}^{i2}_{45} \text{ for } i \in \{1,2\} \\ &\mathbf{z}^{i1}_{25} = \mathbf{z}^{i1}_{46} \text{ for } i \in \{1,2\}. \end{aligned}$$

Furthermore the fact that the six different points $p_{\mathbf{y}^{ij},v_k}, p_{\mathbf{y}^{ij},d_l}$ for $i, j, k \in \{1,\dots,6\}$, $i < j$, $i \ne k \ne j$, $l \in \{1,2\}$ are mutually orthogonal in the projective space $\mathcal{G}_{\mathbf{y}^{ij}}$ indicates that $\mathbf{z}^{mn}_{st} \perp \mathbf{y}^{cd}$ for three pairwise different indices d, n, t under the condition that $\{d,n,t\} = \{4,5,6\}$, $c, m, s \in \{1,2\}$ and $c \ne m$. Certainly, in $\mathcal{G}_{\mathbf{y}^{cd}}$, the line $(z^{mn}_{st})_{\mathbf{y}^{cd}}$ intersects the subspace $x_{\mathbf{y}^{cd}}$ in the unique point perpendicular to $p_{\mathbf{y}^{cd},d_w} = (z^{w3}_{st})_{\mathbf{y}^{cd}} \cap x_{\mathbf{y}^{cd}} = (z^{w3}_{cd})_{\mathbf{y}^{cd}} \cap x_{\mathbf{y}^{cd}}$ for $w \in \{1,2\}\backslash\{m\}$. Also the intersection space of $(z^{mn}_{st})_{\mathbf{y}^{cd}}$ and $y^{3n}_{\mathbf{y}^{cd}}$ is of dimension one and perpendicular to the point $(z^{w3}_{st})_{\mathbf{y}^{cd}} \cap y^{3n}_{\mathbf{y}^{cd}} = (z^{w3}_{cd})_{\mathbf{y}^{cd}} \cap y^{3n}_{\mathbf{y}^{cd}} = p_{\mathbf{y}^{cd},v_3}$ for $w \in \{1,2\}$. Thus we determine that $(z^{mn}_{st})_{\mathbf{y}^{cd}} = \langle p_{\mathbf{y}^{cd},d_m}, p_{\mathbf{y}^{cd},v_n}\rangle = (z^{mn}_{cd})_{\mathbf{y}^{cd}}$ in the projective space $\mathcal{G}_{\mathbf{y}^{cd}}$. Thus the vertices $\mathbf{z}^{mn}_{st}$ and $\mathbf{z}^{mn}_{cd}$ coincide in the graphs Γ and Σ for three pairwise different indices d, n, t under the condition that $\{d,n,t\} = \{4,5,6\}$, $c, m, s \in \{1,2\}$ and $c \ne m$. Combining all known relations, we observe for Σ that

$$\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i3} = \mathbf{z}_{45}^{i3} = \mathbf{z}_{46}^{i3} = \mathbf{z}_{14}^{i3} = \mathbf{z}_{15}^{i3} = \mathbf{z}_{16}^{i3} = \mathbf{z}_{24}^{i3} = \mathbf{z}_{25}^{i3} = \mathbf{z}_{26}^{i3} \text{ for } i \in \{1,2\}$$

$$\begin{array}{ll}
\mathbf{z}_{12}^{i4} = \mathbf{z}_{56}^{i4} \text{ for } i \in \{1,2\} & \mathbf{z}_{24}^{i6} = \mathbf{z}_{15}^{i6} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{26}^{i4} = \mathbf{z}_{15}^{i4} \text{ for } i \in \{1,2\} & \mathbf{z}_{24}^{i1} = \mathbf{z}_{56}^{i1} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{16}^{i4} = \mathbf{z}_{25}^{i4} \text{ for } i \in \{1,2\} & \mathbf{z}_{25}^{i1} = \mathbf{z}_{46}^{i1} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{12}^{i5} = \mathbf{z}_{46}^{i5} \text{ for } i \in \{1,2\} & \mathbf{z}_{26}^{i1} = \mathbf{z}_{45}^{i1} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{24}^{i5} = \mathbf{z}_{16}^{i5} \text{ for } i \in \{1,2\} & \mathbf{z}_{14}^{i2} = \mathbf{z}_{56}^{i2} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{14}^{i5} = \mathbf{z}_{26}^{i5} \text{ for } i \in \{1,2\} & \mathbf{z}_{15}^{i2} = \mathbf{z}_{46}^{i2} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{14}^{i6} = \mathbf{z}_{25}^{i6} \text{ for } i \in \{1,2\} & \mathbf{z}_{16}^{i2} = \mathbf{z}_{45}^{i2} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{15}^{i6} = \mathbf{z}_{24}^{i6} \text{ for } i \in \{1,2\}. &
\end{array}$$

Since the vertex $\mathbf{z}_{12}^{i3}$ is equal to the vertex $\mathbf{z}_{kl}^{i3}$ for $i \in \{1,2\}$, $k,l \in \{1,\dots,6\}\backslash\{3\}$ and $k < l$ it follows that $\mathbf{z}_{12}^{i3} \in \{\mathbf{y}^{kl}\}^{\perp}$ and $\mathbf{z}_{12}^{i3} \perp \mathbf{z}_{kl}^{mn}$ in the graph Γ if $m \in \{1,2\}\backslash\{i\}$ and $n \in \{1,\dots,6\}\backslash\{3,k,l\}$. Moreover, by lemma 4.3.2 and remark 4.3.3 the vertices $\mathbf{y}^{kl}$ correspond to the lines $y_{\mathbf{z}_{12}^{i3}}^{kl}$ in the projective space $\mathcal{G}_{\mathbf{z}_{12}^{i3}}$, which induces the five pairwise orthogonal points $p_{\mathbf{z}_{12}^{i3}}^{g} = y_{\mathbf{z}_{12}^{i3}}^{op} \cap y_{\mathbf{z}_{12}^{i3}}^{qr} \cap y_{\mathbf{z}_{12}^{i3}}^{st} \cap y_{\mathbf{z}_{12}^{i3}}^{un}$ for $\{g\} = \{o,p\} \cap \{q,r\} \cap \{s,t\} \cap \{u,n\}$ and $\{n,o,p,q,r,s,t,u\} = \{1,\dots,6\}\backslash\{3\}$ in $\mathcal{G}_{\mathbf{z}_{12}^{i3}}$. Furthermore the line $(z_{12}^{m4})_{\mathbf{z}_{12}^{i3}} = (z_{56}^{m4})_{\mathbf{z}_{12}^{i3}}$, which is orthogonal to the two lines $y_{\mathbf{z}_{12}^{i3}}^{12}$ and $y_{\mathbf{z}_{12}^{i3}}^{56}$, intersects the subspaces $y_{\mathbf{z}_{12}^{i3}}^{46}$ and $y_{\mathbf{z}_{12}^{i3}}^{24}$. Therefore, we obtain that $(z_{12}^{m4})_{\mathbf{z}_{12}^{i3}} = \langle p_{\mathbf{z}_{12}^{i3}}^{4}, p_{\mathbf{z}_{12}^{i3}}^{3} \rangle$, where $p_{\mathbf{z}_{12}^{i3}}^{3} = \langle p_{\mathbf{z}_{12}^{i3}}^{g} \mid g \in \{1,2,4,5,6\} \rangle^{\pi}$. Using now stepwise lemma 4.3.1, lemma 4.3.2 and corollary 4.3.3 we regard that $(z_{kl}^{mn})_{\mathbf{z}_{12}^{i3}} = \langle p_{\mathbf{z}_{12}^{i3}}^{n}, p_{\mathbf{z}_{12}^{i3}}^{3} \rangle$. Thus we get the relations $(z_{kl}^{mn})_{\mathbf{z}_{12}^{i3}} = (z_{cd}^{mn})_{\mathbf{z}_{12}^{i3}}$ in the space $\mathcal{G}_{\mathbf{z}_{12}^{i3}}$ for $m \in \{1,2\}\backslash\{i\}$, $n \in \{1,\dots,6\}\backslash\{3\}$, $c,d,k,l \in \{1,\dots,6\}\backslash\{3,n\}$, $k < l$, $c < d$, which imply directly the identities

$$\begin{array}{l}
\mathbf{z}_{12}^{i4} = \mathbf{z}_{56}^{i4} = \mathbf{z}_{26}^{i4} = \mathbf{z}_{15}^{i4} = \mathbf{z}_{16}^{i4} = \mathbf{z}_{25}^{i4} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{12}^{i5} = \mathbf{z}_{46}^{i5} = \mathbf{z}_{24}^{i5} = \mathbf{z}_{16}^{i5} = \mathbf{z}_{14}^{i5} = \mathbf{z}_{26}^{i5} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{14}^{i2} = \mathbf{z}_{56}^{i2} = \mathbf{z}_{15}^{i2} = \mathbf{z}_{46}^{i2} = \mathbf{z}_{16}^{i2} = \mathbf{z}_{45}^{i2} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{24}^{i1} = \mathbf{z}_{56}^{i1} = \mathbf{z}_{25}^{i1} = \mathbf{z}_{46}^{i1} = \mathbf{z}_{26}^{i1} = \mathbf{z}_{45}^{i1} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{14}^{i6} = \mathbf{z}_{25}^{i6} = \mathbf{z}_{15}^{i6} = \mathbf{z}_{24}^{i6} = \mathbf{z}_{24}^{i6} = \mathbf{z}_{15}^{i6} \text{ for } i \in \{1,2\}
\end{array}$$

in the graph Σ.

Next we analyse the situation for the line $(z_{12}^{i4})_{\mathbf{y}^{12}}$ in the projective space $\mathcal{G}_{\mathbf{y}^{12}}$ for $i \in \{1,2\}$. This line is orthogonal to the lines $y_{\mathbf{y}^{12}}^{56}, y_{\mathbf{y}^{12}}^{35}$ and $y_{\mathbf{y}^{12}}^{36}$, intersects the lines $y_{\mathbf{y}^{12}}^{45}, y_{\mathbf{y}^{12}}^{46}$ and $y_{\mathbf{y}^{12}}^{34}$ in the point $p_{\mathbf{y}^{12},v_4}$ and the line $x_{\mathbf{y}^{12}}$ in $p_{\mathbf{y}^{12},d_i}$. Therefore in the projective spaces $\mathcal{G}_{\mathbf{y}^{3k}}$ with $k \in \{5,6\}$, we determine that the two-dimensional subspace $(z_{12}^{i4})_{\mathbf{y}^{3k}}$ intersects the line $x_{\mathbf{y}^{12}}$ also in the point $p_{\mathbf{y}^{3k},d_i}$ by construction and the line $y_{\mathbf{y}^{3k}}^{4j}$ with $j \in \{5,6\}\backslash\{k\}$ either in the point $p_{\mathbf{y}^{3k},v_4}$ or in the point $p_{\mathbf{y}^{3k},v_j}$. If $(z_{12}^{i4})_{\mathbf{y}^{3k}} \cap y_{\mathbf{y}^{3k}}^{4j} = p_{\mathbf{y}^{3k},v_j}$ then $(z_{12}^{i4})_{\mathbf{y}^{3k}} = \langle p_{\mathbf{y}^{3k},d_i}, p_{\mathbf{y}^{3k},v_j} \rangle = (z_{3k}^{ij})_{\mathbf{y}^{3k}} \subseteq (y_{\mathbf{y}^{3k}}^{24})^{\pi}$, contradiction. Thus $(z^{i4}_{12})_{\mathbf{y}^{3k}} = \langle p_{\mathbf{y}^{3k},d_i}, p_{\mathbf{y}^{3k},v_4}\rangle = (z^{i4}_{3k})_{\mathbf{y}^{3k}}$ implying that $(z^{ij}_{12})_{\mathbf{y}^{3k}} = (z^{ij}_{3k})_{\mathbf{y}^{3k}}$ for $j \in \{5,6\}\backslash\{k\}$ and furthermore the line $(z^{i4}_{12})_{\mathbf{y}^{3k}}$ is orthogonal to the line $y^{jm}_{\mathbf{y}^{3k}}$ with $j \in \{1,2\}$ and $m \in \{5,6\}\backslash\{k\}$.

The projective space $\mathcal{G}_{\mathbf{y}^{kl}}$ with $k \in \{1,2\}$, $l \in \{5,6\}$ contains the four different lines $(z^{i4}_{3m})_{\mathbf{y}^{jk}}$, $(z^{ij}_{3m})_{\mathbf{y}^{jk}}$ for $i,j \in \{1,2\}$, $j \neq l$, $m \in \{5,6\}\backslash\{k\}$. Since $\mathbf{z}^{i4}_{12} = \mathbf{z}^{i4}_{56} = \mathbf{z}^{i4}_{26} = \mathbf{z}^{i4}_{15} = \mathbf{z}^{i4}_{16} = \mathbf{z}^{i4}_{25} = \mathbf{z}^{i4}_{35} = \mathbf{z}^{i4}_{36}$, it follows that the intersecting of the lines $(z^{ij}_{3m})_{\mathbf{y}^{kl}}$ and $y^{j4}_{\mathbf{y}^{kl}}$ is the one-dimensional subspace $p_{\mathbf{y}^{jk},v_j}$ implying that $(z^{ij}_{3m})_{\mathbf{y}^{kl}} = (z^{ij}_{1l})_{\mathbf{y}^{kl}}$.

Switching now back to the projective space $\mathcal{G}_{\mathbf{y}^{56}}$, we know that $(z^{i4}_{12})_{\mathbf{y}^{56}} = (z^{i4}_{56})_{\mathbf{y}^{56}}$ for $i \in \{1,2\}$. This line $(z^{i4}_{56})_{\mathbf{y}^{56}}$ is orthogonal to the subspace $y^{j3}_{\mathbf{y}^{56}}$ for $j \in \{1,2\}$, intersects $x_{\mathbf{y}^{56}}$ in the point $p_{\mathbf{y}^{56},d_i}$ and the three lines $y^{14}_{\mathbf{y}^{56}}$, $y^{24}_{\mathbf{y}^{56}}$ and $y^{34}_{\mathbf{y}^{56}}$. With a similar argumentation as we used several times before we obtain the relations $(z^{i4}_{56})_{\mathbf{y}^{n3}} = (z^{i4}_{n3})_{\mathbf{y}^{n3}}$ and $(z^{ij}_{56})_{\mathbf{y}^{n3}} = (z^{il}_{n3})_{\mathbf{y}^{n3}}$ for $i,j,n \in \{1,2\}$, $j \neq n$. Thus for the moment we worked out the following list of relations for the graph Σ:

$$
\begin{aligned}
&\mathbf{z}^{i3}_{12} = \mathbf{z}^{i3}_{56} = \mathbf{z}^{i3}_{45} = \mathbf{z}^{i3}_{46} = \mathbf{z}^{i3}_{14} = \mathbf{z}^{i3}_{15} = \mathbf{z}^{i3}_{16} = \mathbf{z}^{i3}_{24} = \mathbf{z}^{i3}_{25} = \mathbf{z}^{i3}_{26} \text{ for } i \in \{1,2\}\\
&\mathbf{z}^{i4}_{12} = \mathbf{z}^{i4}_{56} = \mathbf{z}^{i4}_{26} = \mathbf{z}^{i4}_{15} = \mathbf{z}^{i4}_{16} = \mathbf{z}^{i4}_{25} = \mathbf{z}^{i4}_{13} = \mathbf{z}^{i4}_{23} = \mathbf{z}^{i4}_{35} = \mathbf{z}^{i4}_{36} \text{ for } i \in \{1,2\}\\
&\mathbf{z}^{i5}_{12} = \mathbf{z}^{i5}_{46} = \mathbf{z}^{i5}_{24} = \mathbf{z}^{i5}_{16} = \mathbf{z}^{i5}_{14} = \mathbf{z}^{i5}_{26} = \mathbf{z}^{i5}_{36} \text{ for } i \in \{1,2\}\\
&\mathbf{z}^{i6}_{14} = \mathbf{z}^{i6}_{25} = \mathbf{z}^{i6}_{15} = \mathbf{z}^{i6}_{24} = \mathbf{z}^{i6}_{24} = \mathbf{z}^{i6}_{15} = \mathbf{z}^{i6}_{35} \text{ for } i \in \{1,2\}\\
&\mathbf{z}^{i1}_{24} = \mathbf{z}^{i1}_{56} = \mathbf{z}^{i1}_{25} = \mathbf{z}^{i1}_{46} = \mathbf{z}^{i1}_{26} = \mathbf{z}^{i1}_{45} = \mathbf{z}^{i1}_{35} = \mathbf{z}^{i1}_{36} \text{ for } i \in \{1,2\}\\
&\mathbf{z}^{i2}_{14} = \mathbf{z}^{i2}_{56} = \mathbf{z}^{i2}_{15} = \mathbf{z}^{i2}_{46} = \mathbf{z}^{i2}_{16} = \mathbf{z}^{i2}_{45} = \mathbf{z}^{i2}_{35} = \mathbf{z}^{i2}_{36} \text{ for } i \in \{1,2\}.
\end{aligned}
$$

The last step in the identification of all relations between the vertices of $\mathcal{V}(z)$ is to analyse the situation for the lines $(z^{i5}_{12})_{\mathbf{y}^{12}}$ in the projective space $\mathcal{G}_{\mathbf{y}^{12}}$ for $i \in \{1,2\}$. Obviously these lines are orthogonal to $y^{34}_{\mathbf{y}^{12}}$ and using a similar argument as above we get that $(z^{i5}_{12})_{\mathbf{y}^{34}} = (z^{i5}_{34})_{\mathbf{y}^{34}}$ and $(z^{i6}_{12})_{\mathbf{y}^{34}} = (z^{i6}_{34})_{\mathbf{y}^{34}}$ in $\mathcal{G}_{\mathbf{y}^{34}}$. From the facts that $(z^{i5}_{26})_{\mathbf{y}^{34}} = (z^{i5}_{12})_{\mathbf{y}^{34}} = (z^{i5}_{34})_{\mathbf{y}^{34}}$, $(z^{i6}_{15})_{\mathbf{y}^{34}} = (z^{i6}_{12})_{\mathbf{y}^{34}} = (z^{i6}_{34})_{\mathbf{y}^{34}}$ and that $\mathbf{z}^{i5}_{26}, \mathbf{z}^{i6}_{15} \in \{\mathbf{y}^{34}\}^{\perp}$ we obtain equality between the lines $(z^{i1}_{26})_{\mathbf{y}^{34}}$ and $(z^{i1}_{34})_{\mathbf{y}^{34}}$ and between the lines $(z^{i2}_{15})_{\mathbf{y}^{34}}$ and $(z^{i2}_{34})_{\mathbf{y}^{34}}$ for $i \in \{1,2\}$. Next we consider the lines $(z^{i5}_{12})_{\mathbf{y}^{46}} = (z^{i5}_{46})_{\mathbf{y}^{46}}$ in the space $\mathcal{G}_{\mathbf{y}^{46}}$ for $i \in \{1,2\}$ and the two-dimensional subspaces $(z^{i6}_{12})_{\mathbf{y}^{45}} = (z^{i6}_{45})_{\mathbf{y}^{45}}$ inside $\mathcal{G}_{\mathbf{y}^{45}}$ for $i \in \{1,2\}$. In both cases the lines $(z^{ij}_{4n})_{\mathbf{y}^{4n}}$ are orthogonal to the lines $y^{m3}_{\mathbf{y}^{4n}}$ for $\{j,n\} = \{5,6\}$, $i,m \in \{1,2\}$. Therefore we also detect the lines $(z^{ij}_{4n})_{\mathbf{y}^{m3}}$ in the projective space $\mathcal{G}_{\mathbf{y}^{m3}}$. Certainly $(z^{ij}_{4n})_{\mathbf{y}^{m3}}$ intersects $x_{\mathbf{y}^{m3}}$ in a unique point and the subspace $(z^{ij}_{4n})_{\mathbf{y}^{m3}} \cap x_{\mathbf{y}^{m3}} = (z^{ij}_{12})_{\mathbf{y}^{m3}} \cap x_{\mathbf{y}^{m3}}$ is orthogonal to the point $(z^{kj}_{12})_{\mathbf{y}^{m3}} \cap x_{\mathbf{y}^{m3}} = (z^{k4}_{12})_{\mathbf{y}^{m3}} \cap x_{\mathbf{y}^{m3}} = (z^{k4}_{56})_{\mathbf{y}^{m3}} \cap x_{\mathbf{y}^{m3}} = p_{\mathbf{y}^{m3},d_k}$ for $k \in \{1,2\}\backslash\{i\}$ implying $(z^{ij}_{4n})_{\mathbf{y}^{m3}} \cap x_{\mathbf{y}^{m3}} = p_{\mathbf{y}^{m3},d_i}$. Of course the intersection point of $(z^{ij}_{4n})_{\mathbf{y}^{m3}}$ and $y^{kj}_{\mathbf{y}^{m3}}$ is the point $p_{\mathbf{y}^{m3},v_j}$ for $k \in \{1,2\}\backslash\{i\}$, as otherwise $(z^{ij}_{4n})_{\mathbf{y}^{m3}} = (z^{ik}_{m3})_{\mathbf{y}^{m3}} \subseteq (y^{4n}_{\mathbf{y}^{m3}})^{\pi}$, contradiction. Finally, we get that $(z^{ij}_{4n})_{\mathbf{y}^{m3}} = (z^{ij}_{m3})_{\mathbf{y}^{m3}}$ and $(z^{ip}_{4n})_{\mathbf{y}^{m3}} = (z^{ip}_{m3})_{\mathbf{y}^{m3}}$ for $\{j,n\} = \{5,6\}$, $i,m,p \in \{1,2\}$, $m \neq p$. ∎

Corollary 4.4.12 *The graph Σ consists of the* 28 *pairwise distinct vertices* $\mathbf{x}$, $\mathbf{y}^{ij}$, $\mathbf{z}^{kl}$ *with* $1 \le i < j \le 6$, $1 \le l \le 6$, $k \in \{1,2\}$.

Proof: The vertex $\mathbf{x}$ is distinct from each vertex $\mathbf{y}^{ij}$, $1 \le i < j \le 6$, because $\mathbf{y}^{ij} \in \Sigma_{\mathbf{x}}$. As $\mathbf{z}^{kl} = \mathbf{z}^{kl}_{ij}$ for $1 \le i < j \le 6$, $i \ne l \ne j$ by lemma 4.4.9, we get $\mathbf{x} \ne \mathbf{z}^{kl}_{ij} = \mathbf{z}^{kl}$. Furthermore two vertices $\mathbf{y}^{ij}$ with $1 \le i < j \le 6$ are pairwise distinct by definition. Because $\mathbf{x} \perp \mathbf{y}^{ij}$ and $\mathbf{x} \not\perp \mathbf{z}^{kl}$, cf. lemma 4.4.9, we have $\mathbf{y}^{ij} \ne \mathbf{z}^{kl}$ for any indices $1 \le i < j \le 6$, $1 \le l \le 6$, $k \in \{1,2\}$.

Finally, the vertices $\mathbf{z}^{kl}$ for $1 \le l \le 6$, $k \in \{1,2\}$ are pairwise distinct. Indeed for each pair of vertices $\mathbf{z}^{kl}$ and $\mathbf{z}^{mn}$ with $l, n \in \{1,\dots,6\}$, $k, m \in \{1,2\}$ there exists a set $\{i,j\} \subseteq \{1,\dots,6\} \backslash \{l,n\}$ with $\mathbf{z}^{kl} = \mathbf{z}^{kl}_{ij} \perp \mathbf{y}^{ij} \perp \mathbf{z}^{mn}_{ij} = \mathbf{z}^{mn}$, whence $\mathbf{z}^{kl} = \mathbf{z}^{kl}_{ij} = \mathbf{z}^{mn}_{ij} = \mathbf{z}^{mn}$ if and only if $k = m$ and $n = l$. Therefore Σ contains $1 + 15 + 12 = 28$ pairwise different vertices. ∎

Lemma 4.4.13 *Any two vertices* $\mathbf{z}^{kl}$ *and* $\mathbf{y}^{ij}$ *are adjacent in the graph* Σ *for some indices* $1 \le i < j \le 6$, $1 \le l \le 6$, $k \in \{1,2\}$ *if and only if* $\mathbf{z}^{kl} = \mathbf{z}^{kl}_{ij}$.

Proof: By proposition 4.4.11 we have $\mathbf{z}^{kl} = \mathbf{z}^{kl}_{mn}$ for indices $m, n \in \{1,\dots,6\} \backslash \{l\}$, $n < m$ and $\mathbf{y}^{ij} \perp \mathbf{y}^{cd}$ for any indices $c, d \in \{1,\dots,6\} \backslash \{i,j\}$, $c < d$. Thus the vertices $\mathbf{z}^{kl}$ and $\mathbf{y}^{ij}$ are elements of the induced subgraph $\{\mathbf{y}^{st}\}^{\perp}$ for $1 \le s < t \le 6$ if $\{s,t\} \cap \{l,i,j\} = \varnothing$. The line $(z^{kl}_{mn})_{\mathbf{y}^{st}}$ is contained in the polar space $(y^{ij}_{\mathbf{y}^{st}})^{\pi}$ if and only if $l \notin \{i,j\}$ by construction, which is equivalent to $\mathbf{z}^{kl} \perp \mathbf{y}^{ij}$ if and only if $l \notin \{i,j\}$ which is equal to $\mathbf{z}^{kl} = \mathbf{z}^{kl}_{ij}$ by proposition 4.4.11. ∎

Lemma 4.4.14 *Any two vertices* $\mathbf{z}^{ij}$ *and* $\mathbf{z}^{kl}$ *are adjacent in the graph* Σ *for some indices* $j, l \in \{1,\dots,6\}$, $i, k \in \{1,2\}$ *if and only if* $i \ne k$ *and* $l \ne j$.

Proof: By lemma 4.4.13 the vertex $\mathbf{y}^{cd}$ with $1 \le c < d \le 6$ is adjacent to $\mathbf{z}^{ij}$ and $\mathbf{z}^{kl}$ in Σ if and only if $\{j,l\} \cap \{c,d\} = \varnothing$. In the induced subgraph $\{\mathbf{y}^{cd}\}^{\perp}$, the vertices $\mathbf{z}^{kl}$ and $\mathbf{z}^{ij}$ are adjacent if and only if $(z^{kl}_{cd})_{\mathbf{y}^{cd}} \subseteq (z^{ij}_{cd})^{\pi}_{\mathbf{y}^{cd}}$ in $\mathcal{G}_{\mathbf{y}^{cd}}$ which is equivalent to $i \ne k$ and $j \ne l$ by construction. ∎

Corollary 4.4.15 *Let* $\mathbf{w}$ *be a vertex of* Σ *then the induced subgraph* $\Sigma_{\mathbf{w}}$ *has exactly* 15 *vertices.*

Proof: If $\mathbf{w} = \mathbf{x}$ then the statement follows from lemma 4.4.7. For $1 \le i < j \le 6$ the induced subgraph $\Sigma_{\mathbf{y}^{ij}}$ has the vertex set

$$\mathcal{V}(\Sigma_{\mathbf{y}^{ij}}) = \{\mathbf{x}, \mathbf{y}^{mn}, \mathbf{z}^{kl} \mid m, n, l \in \{1,\dots,6\} \backslash \{i,j\},\ m < n,\ 1 \le k \le 2\}$$

by lemma 4.4.13 and lemma 4.4.14. As we have two possibilities to choose the index k, four possibilities to choose the index l and $\binom{4}{2}$ options for the pair m, n it follows that $|\mathcal{V}(\Sigma_{\mathbf{y}^{ij}})| = 1 + 2 \cdot 4 + \binom{4}{2} = 15$.

Alternatively if $\mathbf{w} = \mathbf{z}^{ij}$ then

$$\mathcal{V}(\Sigma_{\mathbf{z}^{ij}}) = \{\mathbf{y}^{mn}, \mathbf{z}^{kl} \mid m, n, l \in \{1, \ldots, 6\} \backslash \{j\},\ m < n,\ k = \{1,2\} \backslash \{i\}\}$$

by corollary 4.4.9, lemma 4.4.13 and lemma 4.4.14. Since we have no choice for the index k, five possibilities to choose the index l and $\binom{5}{2}$ options for the pair m, n it follows that $|\mathcal{V}(\Sigma_{\mathbf{y}^{ij}})| = 1 \cdot 5 + \binom{5}{2} = 15$. ■

Lemma 4.4.16 *For any vertex* $\mathbf{w} \in \Sigma$ *the induced subgraph* $\Sigma_{\mathbf{w}}$ *is isomorphic to* $\mathbf{W}(A_5)$.

Proof: The statement is proved in lemma 4.4.10 for the vertex $\mathbf{x} \in \Sigma$, so $\Sigma_{\mathbf{x}} \cong \mathbf{W}(A_5)$. For a vertex $\mathbf{y}^{ij}$, the statement follows from the construction of Σ together with lemma 4.4.13 and corollary 4.4.15 for any indices $1 \le i < j \le 6$. Finally,

$$\mathcal{V}(\Sigma_{\mathbf{z}^{ij}}) = \{\mathbf{y}^{mn}, \mathbf{z}^{kl} \mid m, n, l \in \{1, \ldots, 6\} \backslash \{j\},\ m < n,\ k = \{1,2\} \backslash \{i\}\},$$

by corollary 4.4.15 for a vertex $\mathbf{z}^{ij}$ with $i \in \{1,2\}$, $j \in \{1, \ldots, 6\}$. In this last part of the proof we will disregard the ordering of the indices k, l for the vertices $\mathbf{y}^{kl}$, therefore $\mathbf{y}^{kl} = \mathbf{y}^{lk}$. Let $\gamma_{\mathbf{z}^{ij}} : \Sigma_{\mathbf{z}^{ij}} \to \Sigma_{\mathbf{x}}$ be the map such that the image of $\mathbf{y}^{mn}$ is $\mathbf{y}^{mn}$, $\gamma_{\mathbf{z}^{ij}}(\mathbf{z}^{kl}) = \mathbf{y}^{jl}$. To verify that $\gamma_{\mathbf{z}^{ij}}$ is a graph isomorphism, let $\mathbf{u}$ and $\mathbf{v}$ be two adjacent vertices of the graph $\Sigma_{\mathbf{z}^{ij}}$. Certainly if $\mathbf{u} = \mathbf{y}^{mn}$ and $\mathbf{v} = \mathbf{y}^{st}$ for suitable indices $m, n, s, t \in \{1, \ldots, 6\} \backslash \{j\}$, then by construction $\{m, n\} \cap \{s, t\} = \emptyset$ and of course $\gamma_{\mathbf{z}^{ij}}(\mathbf{u}) = \gamma_{\mathbf{z}^{ij}}(\mathbf{y}^{mn}) = \mathbf{y}^{mn} \perp \mathbf{y}^{st} = \gamma_{\mathbf{z}^{ij}}(\mathbf{y}^{st}) = \gamma_{\mathbf{z}^{ij}}(\mathbf{v})$ in $\Sigma_{\mathbf{x}}$. If alternatively $\mathbf{y}^{mn} = \mathbf{u} \perp \mathbf{v} = \mathbf{z}^{kl}$ with $k = \{1,2\} \backslash \{i\}$, $l, m, n \in \{1, \ldots, 6\} \backslash \{j\}$, $m \neq n$ in $\Sigma_{\mathbf{z}^{ij}}$ then the indices l, m, n are pairwise different, the index j does not belongs to the set $\{l, m, n\}$ and $k \neq i$. Furthermore $\gamma_{\mathbf{z}^{ij}}(\mathbf{u}) = \gamma_{\mathbf{z}^{ij}}(\mathbf{y}^{mn}) = \mathbf{y}^{mn}$ and $\gamma_{\mathbf{z}^{ij}}(\mathbf{v}) = \gamma_{\mathbf{z}^{ij}}(\mathbf{z}^{kl}) = \mathbf{y}^{jl}$. In particular the four indices l, m, n and j are pairwise different, which implies that $\gamma_{\mathbf{y}}(\mathbf{u})$ and $\gamma_{\mathbf{y}}(\mathbf{v})$ are two adjacent vertices in $\Sigma_{\mathbf{x}}$. Hence $\gamma_{\mathbf{z}^{ij}}$ is a graph isomorphism between $\Sigma_{\mathbf{z}^{ij}}$ and $\Sigma_{\mathbf{x}}$ implying $\Sigma_{\mathbf{z}^{ij}} \cong \Sigma_{\mathbf{x}} \cong \mathbf{W}(A_5)$ by lemma 4.4.7. ■

Proposition 4.4.17 *Suppose* $\mathbf{z}^{i3}_{12} = \mathbf{z}^{i3}_{56}$ *for* $i \in \{1,2\}$. *Then* $\Sigma \cong \mathbf{W}(A_7)$.

Proof: By lemma 4.4.16 the graph Σ is locally $\mathbf{W}(A_5)$. By corollary 4.4.12 the graph Σ has 28 vertices. Hence by theorem 2 in [17] or theorem 2 in [49] the graph Σ is isomorphic to $\mathbf{W}(A_7)$. ■

Case 2 : $\mathbf{z}^{i3}_{12} = \mathbf{z}^{i4}_{56}$ for $i \in \{1,2\}$

Proposition 4.4.18 *Suppose* $\mathbf{z}^{i3}_{12} = \mathbf{z}^{i3}_{56}$ *for* $i \in \{1,2\}$. *Then the following identities for* $\{k, l\} = \{1,2\}$ *hold in* Σ *:*

$$\mathbf{z}^{k3}_{12} = \mathbf{z}^{k2}_{13} = \mathbf{z}^{k1}_{23} = \mathbf{z}^{k6}_{45} = \mathbf{z}^{k5}_{46} = \mathbf{z}^{k4}_{56}$$

$$\mathbf{z}_{12}^{k4} = \mathbf{z}_{14}^{k2} = \mathbf{z}_{24}^{k1} = \mathbf{z}_{35}^{k6} = \mathbf{z}_{36}^{k5} = \mathbf{z}_{56}^{k3}$$

$$\mathbf{z}_{12}^{k5} = \mathbf{z}_{15}^{k2} = \mathbf{z}_{25}^{k1} = \mathbf{z}_{34}^{k6} = \mathbf{z}_{36}^{k4} = \mathbf{z}_{46}^{k3}$$

$$\mathbf{z}_{12}^{k6} = \mathbf{z}_{16}^{k2} = \mathbf{z}_{26}^{k1} = \mathbf{z}_{34}^{k5} = \mathbf{z}_{35}^{k4} = \mathbf{z}_{45}^{k3}$$

$$\mathbf{z}_{56}^{k1} = \mathbf{z}_{23}^{k4} = \mathbf{z}_{24}^{k3} = \mathbf{z}_{34}^{l2} = \mathbf{z}_{15}^{l6} = \mathbf{z}_{16}^{l5}$$

$$\mathbf{z}_{45}^{k1} = \mathbf{z}_{23}^{k6} = \mathbf{z}_{26}^{k3} = \mathbf{z}_{36}^{l2} = \mathbf{z}_{14}^{l5} = \mathbf{z}_{15}^{l4}$$

$$\mathbf{z}_{46}^{k1} = \mathbf{z}_{23}^{k5} = \mathbf{z}_{25}^{k3} = \mathbf{z}_{35}^{l2} = \mathbf{z}_{14}^{l6} = \mathbf{z}_{16}^{l4}$$

$$\mathbf{z}_{56}^{k2} = \mathbf{z}_{13}^{k4} = \mathbf{z}_{14}^{k3} = \mathbf{z}_{34}^{l1} = \mathbf{z}_{25}^{l6} = \mathbf{z}_{26}^{l5}$$

$$\mathbf{z}_{45}^{k2} = \mathbf{z}_{13}^{k6} = \mathbf{z}_{16}^{k3} = \mathbf{z}_{36}^{l1} = \mathbf{z}_{24}^{l5} = \mathbf{z}_{25}^{l4}$$

$$\mathbf{z}_{46}^{k2} = \mathbf{z}_{13}^{k5} = \mathbf{z}_{15}^{k3} = \mathbf{z}_{35}^{l1} = \mathbf{z}_{24}^{l6} = \mathbf{z}_{26}^{l4}$$

Proof: As in the proof of proposition 4.4.11 we will work out a list of relation of the vertices $\mathbf{z}_{kl}^{ij} \in \mathcal{V}(z)$ in the graph Σ. By assumption we have the relation $\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i4}$ for $i \in \{1,2\}$, in Σ.

Looking at the unitary space $\mathcal{G}_{\mathbf{y}^{56}}$ the intersection point of the three lines $y_{\mathbf{y}^{56}}^{34}$, $(z_{12}^{14})_{\mathbf{y}^{56}}$ and $(z_{12}^{24})_{\mathbf{y}^{56}}$, which is orthogonal to the point $(z_{12}^{13})_{\mathbf{y}^{56}} \cap (z_{12}^{23})_{\mathbf{y}^{56}} \cap y_{\mathbf{y}^{56}}^{34}$, is the point $p_{\mathbf{y}^{56},v_3}$. Thus we determine that $(z_{12}^{14})_{\mathbf{y}^{56}} \cap (z_{12}^{24})_{\mathbf{y}^{56}} \cap y_{\mathbf{y}^{56}}^{34} = p_{\mathbf{y}^{56},v_3}$ and $(z_{12}^{i4})_{\mathbf{y}^{56}} = (z_{56}^{i3})_{\mathbf{y}^{56}}$ for $i \in \{1,2\}$ in $\mathcal{G}_{\mathbf{y}^{56}}$. Furthermore the line $(z_{12}^{i3})_{\mathbf{y}^{56}} = (z_{56}^{i4})_{\mathbf{y}^{56}}$ is contained in the polar space $(y_{\mathbf{y}^{56}}^{13})^{\pi}$ of $y_{\mathbf{y}^{56}}^{13}$ and in the subspace $(y_{\mathbf{y}^{56}}^{23})^{\pi}$ and the line $(z_{12}^{i4})_{\mathbf{y}^{56}} = (z_{56}^{i3})_{\mathbf{y}^{56}}$ is a subspace of $(y_{\mathbf{y}^{56}}^{14})^{\pi} \cap (y_{\mathbf{y}^{56}}^{24})^{\pi}$. Therefore we conclude the connectivity of $\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i4}$ with the vertex $\mathbf{y}^{j3}$ and the adjacency of $\mathbf{z}_{12}^{i4} = \mathbf{z}_{56}^{i3}$ with the vertex $\mathbf{y}^{j4}$ in the graph Γ for $j, i \in \{1,2\}$.

Using now the complete analysis of the space $\mathcal{G}_{\mathbf{y}^{12}}$, we recall that the vertices $\mathbf{z}_{12}^{i3}$, $\mathbf{z}_{12}^{i6}$ are adjacent to $\mathbf{y}^{45}$ and $\mathbf{z}_{12}^{i3}, \mathbf{z}_{12}^{i5} \in (\mathbf{y}^{46})^{\perp}$ for $i \in \{1,2\}$. Thus the two-dimensional subspace $(z_{12}^{ik})_{\mathbf{y}^{45}}$ intersects the line $x_{\mathbf{y}^{45}}$ in the point $p_{\mathbf{y}^{45},d_i}$ and the line $y_{\mathbf{y}^{45}}^{36}$ either in the point $p_{\mathbf{y}^{45},v_3}$ or in $p_{\mathbf{y}^{45},v_6}$ for $k \in \{3,6\}$. Since the lines $(z_{12}^{i3})_{\mathbf{y}^{45}}$ and $y_{\mathbf{y}^{45}}^{j3}$ are orthogonal for $i, j \in \{1,2\}$ and the point $(z_{12}^{i3})_{\mathbf{y}^{45}} \cap y_{\mathbf{y}^{45}}^{36}$ is perpendicular to the point $(z_{12}^{i5})_{\mathbf{y}^{45}} \cap y_{\mathbf{y}^{45}}^{36}$ it follows that $(z_{12}^{i3})_{\mathbf{y}^{45}} = \langle p_{\mathbf{y}^{45},d_i} p_{\mathbf{y}^{45},v_6} \rangle = (z_{45}^{i6})_{\mathbf{y}^{45}}$ and $(z_{12}^{i6})_{\mathbf{y}^{45}} = \langle p_{\mathbf{y}^{45},d_i}, p_{\mathbf{y}^{45},v_3} \rangle = (z_{45}^{i3})_{\mathbf{y}^{45}}$ for $i \in \{1,2\}$. Consequently, the vertex $\mathbf{z}_{12}^{i3}$ coincides with the vertices $\mathbf{z}_{45}^{i5}$ and $\mathbf{z}_{12}^{i6}$ equals $\mathbf{z}_{45}^{i3}$ in Γ for $i \in \{1,2\}$.

Next we consider the lines which correspond to the vertices $\mathbf{z}_{12}^{i3}$ and $\mathbf{z}_{12}^{i5}$ in the projective space $\mathcal{G}_{\mathbf{y}^{46}}$ for $i \in \{1,2\}$. Each subspace $(z_{12}^{ik})_{\mathbf{y}^{46}}$ intersects the subspace $x_{\mathbf{y}^{46}}$ in the point $p_{\mathbf{y}^{46},d_i}$ for $i \in \{1,2\}$, $k \in \{3,5\}$ by construction of Σ.

Moreover the line $(z_{12}^{i3})_{\mathbf{y}^{46}}$ is orthogonal to the lines $y_{\mathbf{y}^{46}}^{13}$ and $y_{\mathbf{y}^{46}}^{23}$ in $\mathcal{G}_{\mathbf{y}^{46}}$ for $i \in \{1,2\}$ and the intersection point of the three lines $(z_{12}^{13})_{\mathbf{y}^{46}}$, $(z_{12}^{23})_{\mathbf{y}^{46}}$ and $y_{\mathbf{y}^{46}}^{35}$

is orthogonal to the point $(z_{12}^{15})_{\mathbf{y}^{46}} \cap (z_{12}^{25})_{\mathbf{y}^{46}} \cap y_{\mathbf{y}^{46}}^{35}$. So $(z_{12}^{i3})_{\mathbf{y}^{46}} = \langle p_{\mathbf{y}^{46},d_i}, p_{\mathbf{y}^{46},v_5} \rangle = (z_{46}^{i5})_{\mathbf{y}^{46}}$ and $(z_{12}^{i5})_{\mathbf{y}^{46}} = \langle p_{\mathbf{y}^{46},d_i}, p_{\mathbf{y}^{46},v_3} \rangle = (z_{46}^{i3})_{\mathbf{y}^{46}}$.

We now switch to the projective space $\mathcal{G}_{\mathbf{y}^{m3}}$ for $m \in \{1,2\}$. This space contains the lines corresponding to the vertices $\mathbf{z}_{12}^{i3} = \mathbf{z}_{46}^{i5} = \mathbf{z}_{45}^{i6} = \mathbf{z}_{56}^{i4}$ and $\mathbf{z}_{kl}^{ij}$ for $i, j \in \{1,2\}$, $j \neq m$, $4 \leq k < l \leq 6$. By construction of Σ the lines $(z_{56}^{i4})_{\mathbf{y}^{m3}}$ intersect the subspace $x_{\mathbf{y}^{m3}}$ in the point $p_{\mathbf{y}^{m3},d_i}$, thus $(z_{kl}^{ij})_{\mathbf{y}^{m3}} \cap x_{\mathbf{y}^{m3}} = p_{\mathbf{y}^{m3},d_i}$. The fact that $\mathbf{z}_{12}^{i3} \in \{\mathbf{y}^{45}\}^{\perp}$ for implies $(z_{56}^{i4})_{\mathbf{y}^{m3}} \subseteq (y_{\mathbf{y}^{m3}}^{45})^{\pi}$. So we conclude that the line $(z_{56}^{i4})_{\mathbf{y}^{m3}}$ intersects the subspace $y_{\mathbf{y}^{m3}}^{j4}$ in the point $p_{\mathbf{y}^{m3},v_j}$. In particular $(z_{56}^{i4})_{\mathbf{y}^{j3}} = \langle p_{\mathbf{y}^{3j},d_i}, p_{\mathbf{y}^{j3},v_j} \rangle = (z_{m3}^{ij})_{\mathbf{y}^{m3}}$ and $(z_{kl}^{ij})_{\mathbf{y}^{m3}} \cap y_{\mathbf{y}^{m3}}^{jh} = p_{\mathbf{y}^{m3},v_h}$ for $h \in \{4,\dots,6\} \backslash \{k,l\}$. Therefore we verified that $(z_{kl}^{ij})_{\mathbf{y}^{m3}} = (z_{m3}^{ih})_{\mathbf{y}^{m3}}$ in the projective space $\mathcal{G}_{\mathbf{y}^{m3}}$. Hence we achieved the following relation in the graph Γ.

$\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i4} = \mathbf{z}_{45}^{i6} = \mathbf{z}_{46}^{i5} = \mathbf{z}_{13}^{i2} = \mathbf{z}_{23}^{i1}$ for $i \in \{1,2\}$	$\mathbf{z}_{45}^{i2} = \mathbf{z}_{13}^{i6}$ for $i \in \{1,2\}$
$\mathbf{z}_{12}^{i4} = \mathbf{z}_{56}^{i3}$ for $i \in \{1,2\}$	$\mathbf{z}_{46}^{i2} = \mathbf{z}_{13}^{i5}$ for $i \in \{1,2\}$
$\mathbf{z}_{12}^{i5} = \mathbf{z}_{46}^{i3}$ for $i \in \{1,2\}$	$\mathbf{z}_{56}^{i1} = \mathbf{z}_{23}^{i4}$ for $i \in \{1,2\}$
$\mathbf{z}_{12}^{i6} = \mathbf{z}_{45}^{i3}$ for $i \in \{1,2\}$	$\mathbf{z}_{45}^{i1} = \mathbf{z}_{23}^{i6}$ for $i \in \{1,2\}$
$\mathbf{z}_{56}^{i2} = \mathbf{z}_{13}^{i4}$ for $i \in \{1,2\}$	$\mathbf{z}_{46}^{i1} = \mathbf{z}_{23}^{i5}$ for $i \in \{1,2\}$

Next we perform the same procedure for the vertices $\mathbf{z}_{12}^{i4}$ for $i \in \{1,2\}$. Thus we will consider the line $(z_{12}^{i4})_{\mathbf{y}^{35}}$ in $\mathcal{G}_{\mathbf{y}^{35}}$, the line $(z_{12}^{i4})_{\mathbf{y}^{36}}$ inside $\mathcal{G}_{\mathbf{y}^{36}}$ and the subspace $(z_{56}^{i4})_{\mathbf{y}^{j4}}$ in $\mathcal{G}_{\mathbf{y}^{j4}}$ for $j \in \{1,2\}$.

We start with the lines $(z_{12}^{i4})_{\mathbf{y}^{3l}}$ and $(z_{12}^{ik})_{\mathbf{y}^{3l}}$ in the unitary space $\mathcal{G}_{\mathbf{y}^{3l}}$ for $i \in \{1,2\}$, $\{k,l\} = \{5,6\}$. As before we have $(z_{12}^{i4})_{\mathbf{y}^{3l}} \cap x_{\mathbf{y}^{3l}} = p_{\mathbf{y}^{3l},d_i} = (z_{12}^{ik})_{\mathbf{y}^{3l}} \cap x_{\mathbf{y}^{3l}}$ by the construction of Σ. Since $\mathbf{z}_{12}^{i4} \perp \mathbf{y}^{14}$ in Γ and the two-dimensional subspace $(z_{12}^{i4})_{\mathbf{y}^{3l}}$ intersects the line $y_{\mathbf{y}^{3l}}^{4k}$, it follows that $(z_{12}^{i4})_{\mathbf{y}^{35}} \cap y_{\mathbf{y}^{3l}}^{4k} = p_{\mathbf{y}^{3l},v_k}$, $(z_{12}^{ik})_{\mathbf{y}^{3l}} \cap y_{\mathbf{y}^{3l}}^{4k} = p_{\mathbf{y}^{3l},v_4}$, $(z_{12}^{i4})_{\mathbf{y}^{3l}} = \langle p_{\mathbf{y}^{3l},d_i}, p_{\mathbf{y}^{3l},v_k} \rangle = (z_{3l}^{ik})_{\mathbf{y}^{3l}}$ and $(z_{12}^{ik})_{\mathbf{y}^{3l}} = \langle p_{\mathbf{y}^{3l},d_i}, p_{\mathbf{y}^{3l},v_4} \rangle = (z_{3l}^{i4})_{\mathbf{y}^{3l}}$ for $i \in \{1,2\}$ and $\{k,l\} = \{5,6\}$.

Next we consider the line $(z_{56}^{i4})_{\mathbf{y}^{m4}}$ in the unitary projective space $\mathcal{G}_{\mathbf{y}^{m4}}$ for the indices $i, k \in \{1,2\}$. Looking back to the projective spaces $\mathcal{G}_{\mathbf{y}^{56}}, \mathcal{G}_{\mathbf{y}^{35}}$ and $\mathcal{G}_{\mathbf{y}^{36}}$ we obtain that $(z_{kl}^{ij})_{\mathbf{y}^{kl}} \subseteq (y_{\mathbf{y}^{kl}}^{j4})^{\pi}$ and that the lines $(z_{12}^{i4})_{\mathbf{y}^{kl}}$ intersect the subspace $y_{\mathbf{y}^{kl}}^{jh}$ for $j \in \{1,2\} \backslash \{m\}$, $\{h,k,l\} = \{3,5,6\}, k < l$. Thus the intersection of $(z_{56}^{i4})_{\mathbf{y}^{m4}}$ and $x_{\mathbf{y}^{m4}}$ coincides with the point $p_{\mathbf{y}^{m4},d_i}$, implying $(z_{56}^{i4})_{\mathbf{y}^{m4}} \cap y_{\mathbf{y}^{m4}}^{jh} = p_{\mathbf{y}^{m4},v_j}$ and $(z_{kl}^{ij})_{\mathbf{y}^{m4}} \cap y_{\mathbf{y}^{m4}}^{jh} = p_{\mathbf{y}^{m4},v_h}$. Certainly we have equality between the subspaces $(z_{56}^{i4})_{\mathbf{y}^{m4}}$ and $(z_{m4}^{ij})_{\mathbf{y}^{m4}}$ as well as $(z_{kl}^{ij})_{\mathbf{y}^{m4}} = (z_{m4}^{ih})_{\mathbf{y}^{m4}}$ in the space $\mathcal{G}_{\mathbf{y}^{m4}}$.

Summarising all known relations of the vertices $\mathbf{z}_{kl}^{ij} \in \mathcal{V}(z)$ with in the graph Γ at this point in a list, we see that

$$
\begin{array}{ll}
\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i4} = \mathbf{z}_{45}^{i6} = \mathbf{z}_{46}^{i5} = \mathbf{z}_{13}^{i2} = \mathbf{z}_{23}^{i1} \text{ for } i \in \{1,2\} & \\
\mathbf{z}_{12}^{i4} = \mathbf{z}_{56}^{i3} = \mathbf{z}_{35}^{i6} = \mathbf{z}_{36}^{i5} = \mathbf{z}_{14}^{i2} = \mathbf{z}_{24}^{i1} \text{ for } i \in \{1,2\} & \\
\mathbf{z}_{12}^{i5} = \mathbf{z}_{46}^{i3} = \mathbf{z}_{36}^{i4} \text{ for } i \in \{1,2\} & \mathbf{z}_{12}^{i6} = \mathbf{z}_{45}^{i3} = \mathbf{z}_{35}^{i4} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{56}^{i2} = \mathbf{z}_{13}^{i4} = \mathbf{z}_{14}^{i3} \text{ for } i \in \{1,2\} & \mathbf{z}_{56}^{i1} = \mathbf{z}_{23}^{i4} = \mathbf{z}_{24}^{i3} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{45}^{i2} = \mathbf{z}_{13}^{i6} \text{ for } i \in \{1,2\} & \mathbf{z}_{46}^{i2} = \mathbf{z}_{13}^{i5} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{35}^{i2} = \mathbf{z}_{14}^{i6} \text{ for } i \in \{1,2\} & \mathbf{z}_{36}^{i2} = \mathbf{z}_{14}^{i5} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{45}^{i1} = \mathbf{z}_{23}^{i6} \text{ for } i \in \{1,2\} & \mathbf{z}_{46}^{i1} = \mathbf{z}_{23}^{i5} \text{ for } i \in \{1,2\} \\
\mathbf{z}_{35}^{i1} = \mathbf{z}_{24}^{i6} \text{ for } i \in \{1,2\} & \mathbf{z}_{36}^{i1} = \mathbf{z}_{24}^{i5} \text{ for } i \in \{1,2\}
\end{array}
$$

From the analysis of the spaces $\mathcal{G}_{\mathbf{y}^{12}}$ and $\mathcal{G}_{\mathbf{y}^{j6}}$ for $j \in \{3,4\}$ we know already that the vertex $\mathbf{z}_{12}^{i5} = \mathbf{z}_{46}^{i3} = \mathbf{z}_{36}^{i4}$ is adjacent to the vertices $\mathbf{y}^{34}$ and $\mathbf{y}^{k5}$ for $i, k \in \{1,2\}$ and that the line $(z_{12}^{i5})_{\mathbf{y}^{12}}$ intersects the lines $y_{\mathbf{y}^{12}}^{35}$ and $y_{\mathbf{y}^{12}}^{56}$. Moreover the subspace $(z_{46}^{i3})_{\mathbf{y}^{46}}$ intersects the subspace $y_{\mathbf{y}^{46}}^{m3}$ in the projective space $\mathcal{G}_{\mathbf{y}^{46}}$ for $m \in \{1,2\}$ as well as the subspace $(z_{36}^{i4})_{\mathbf{y}^{36}} \cap y_{\mathbf{y}^{36}}^{n4}$ is not trivial in $\mathcal{G}_{\mathbf{y}^{36}}$ for $n \in \{1,2\}$.

Therefore in the unitary space $\mathcal{G}_{\mathbf{y}^{34}}$ the line $(z_{12}^{il})_{\mathbf{y}^{34}}$ intersects the subspace $x_{\mathbf{y}^{34}}$ in the point $p_{\mathbf{y}^{34},d_i}$ for $l \in \{5,6\}$. Furthermore two-dimensional subspace $(z_{12}^{i5})_{\mathbf{y}^{34}}$ intersects the line $y_{\mathbf{y}^{34}}^{56}$ either in the point $p_{\mathbf{y}^{34},v_5}$ or in the point $p_{\mathbf{y}^{34},v_6}$ and is contained in the subspace $(y_{\mathbf{y}^{34}}^{k5})^{\pi}$, thus $(z_{12}^{i5})_{\mathbf{y}^{34}} = \langle p_{\mathbf{y}^{34},d_i}, p_{\mathbf{y}^{34},v_6} \rangle = (z_{34}^{i6})_{\mathbf{y}^{34}}$. Using now the fact that the two points $(z_{12}^{i5})_{\mathbf{y}^{34}} \cap y_{\mathbf{y}^{34}}^{56}$ and $(z_{12}^{i6})_{\mathbf{y}^{34}} \cap y_{\mathbf{y}^{34}}^{56}$ are perpendicular, we get $(z_{12}^{i6})_{\mathbf{y}^{34}} = \langle p_{\mathbf{y}^{34},d_i}, p_{\mathbf{y}^{34},v_5} \rangle = (z_{34}^{i5})_{\mathbf{y}^{34}}$ for $i \in \{1,2\}$. Notice that $(z_{34}^{i6})_{\mathbf{y}^{34}} \subseteq (y_{\mathbf{y}^{34}}^{k5})^{\pi}$, the line $(z_{34}^{ij})_{\mathbf{y}^{34}}$ is a subspace of the polar space $(y_{\mathbf{y}^{34}}^{k5})^{\pi}$ of $y_{\mathbf{y}^{34}}^{k5}$ if $j \in \{1,2\} \backslash \{k\}$ and that each of the lines $(z_{34}^{ij})_{\mathbf{y}^{34}}$ and $(z_{34}^{ij})_{\mathbf{y}^{34}}$ intersects the two-dimensional subspace $y_{\mathbf{y}^{34}}^{j6}$.

The analysis of the projective space $\mathcal{G}_{\mathbf{y}^{k5}}$ for $k \in \{1,2\}$ implies that the intersection of the subspaces $(z_{46}^{im})_{\mathbf{y}^{k5}}$ and $x_{\mathbf{y}^{k5}}$ is the point $p_{\mathbf{y}^{k5},d_i}$ for $i \in \{1,2\}$, $m \in \{1,2,3\} \backslash \{k\}$. Since $(z_{46}^{i3})_{\mathbf{y}^{k5}}$ intersects the line $y_{\mathbf{y}^{k5}}^{l3}$ for $l \in \{1,2\} \backslash \{k\}$ and is orthogonal to the lines $y_{\mathbf{y}^{k5}}^{st}$ for $s, t \in \{3,4,6\}$, $s < t$, the point $p_{\mathbf{y}^{k5},v_l}$ is the intersection point of the subspaces $(z_{46}^{i3})_{\mathbf{y}^{k5}}$ and $y_{\mathbf{y}^{k5}}^{l3}$. Therefore the orthogonal point $p_{\mathbf{y}^{k5},v_3}$ of $p_{\mathbf{y}^{k5},v_l}$ on the two-dimensional subspace $y_{\mathbf{y}^{k5}}^{l3}$ is the intersection point of the lines $(z_{46}^{il})_{\mathbf{y}^{k5}}$ and $y_{\mathbf{y}^{k5}}^{l3}$, implying $(z_{46}^{il})_{\mathbf{y}^{k5}} = (z_{k5}^{i3})_{\mathbf{y}^{k5}}$. Taking this argumentation again under the use of the lines $y_{\mathbf{y}^{k5}}^{cd}$ for $c \in \{1,2\} \backslash \{k\}$, $\{d,p\} = \{4,6\}$ we obtain that $(z_{3p}^{ic})_{\mathbf{y}^{k5}} \cap y_{\mathbf{y}^{k5}}^{cd} = p_{\mathbf{y}^{k5},v_d}$ implying $(z_{3p}^{ic})_{\mathbf{y}^{k5}} = (z_{k5}^{id})_{\mathbf{y}^{k5}}$. We extend the relation list accordingly now have:

$$\begin{array}{ll}
\multicolumn{2}{l}{\mathbf{z}^{i3}_{12} = \mathbf{z}^{i4}_{56} = \mathbf{z}^{i6}_{45} = \mathbf{z}^{i5}_{46} = \mathbf{z}^{i2}_{13} = \mathbf{z}^{i1}_{23} \text{ for } i \in \{1,2\}} \\
\multicolumn{2}{l}{\mathbf{z}^{i4}_{12} = \mathbf{z}^{i3}_{56} = \mathbf{z}^{i6}_{35} = \mathbf{z}^{i5}_{36} = \mathbf{z}^{i2}_{14} = \mathbf{z}^{i1}_{24} \text{ for } i \in \{1,2\}} \\
\multicolumn{2}{l}{\mathbf{z}^{i5}_{12} = \mathbf{z}^{i3}_{46} = \mathbf{z}^{i4}_{36} = \mathbf{z}^{i6}_{34} = \mathbf{z}^{i2}_{15} = \mathbf{z}^{i1}_{25} \text{ for } i \in \{1,2\}} \\
\mathbf{z}^{i2}_{36} = \mathbf{z}^{i5}_{14} = \mathbf{z}^{i3}_{15} \text{ for } i \in \{1,2\} & \mathbf{z}^{i2}_{56} = \mathbf{z}^{i4}_{13} = \mathbf{z}^{i3}_{14} \text{ for } i \in \{1,2\} \\
\mathbf{z}^{i2}_{46} = \mathbf{z}^{i5}_{13} = \mathbf{z}^{i4}_{15} \text{ for } i \in \{1,2\} & \mathbf{z}^{i1}_{56} = \mathbf{z}^{i4}_{23} = \mathbf{z}^{i3}_{24} \text{ for } i \in \{1,2\} \\
\mathbf{z}^{i1}_{46} = \mathbf{z}^{i5}_{23} = \mathbf{z}^{i4}_{25} \text{ for } i \in \{1,2\} & \mathbf{z}^{i1}_{36} = \mathbf{z}^{i5}_{24} = \mathbf{z}^{i3}_{25} \text{ for } i \in \{1,2\} \\
\mathbf{z}^{i2}_{45} = \mathbf{z}^{i6}_{13} \text{ for } i \in \{1,2\} & \mathbf{z}^{i2}_{35} = \mathbf{z}^{i6}_{14} \text{ for } i \in \{1,2\} \\
\mathbf{z}^{i2}_{34} = \mathbf{z}^{i6}_{15} \text{ for } i \in \{1,2\} & \mathbf{z}^{i1}_{45} = \mathbf{z}^{i6}_{23} \text{ for } i \in \{1,2\} \\
\mathbf{z}^{i1}_{35} = \mathbf{z}^{i6}_{24} \text{ for } i \in \{1,2\} & \mathbf{z}^{i1}_{34} = \mathbf{z}^{i6}_{25} \text{ for } i \in \{1,2\} \\
\multicolumn{2}{l}{\mathbf{z}^{i6}_{12} = \mathbf{z}^{i3}_{45} = \mathbf{z}^{i4}_{35} = \mathbf{z}^{i5}_{34} \text{ for } i \in \{1,2\} \text{ in the graph } \Gamma.}
\end{array}$$

A similar argumentation in the projective spaces $\mathcal{G}_{\mathbf{y}^{j6}}$ for $j \in \{1,2\}$ about the lines $(z^{i6}_{12})_{\mathbf{y}^{j6}} = (z^{i3}_{45})_{\mathbf{y}^{j6}}$ shows that $(z^{i3}_{45})_{\mathbf{y}^{j6}} = (z^{il}_{j6})_{\mathbf{y}^{j6}}$ for $i, l \in \{1,2\}$, $l \neq j$. Indeed each of the lines $(z^{ik}_{mn})_{\mathbf{y}^{j6}}$ for $k \in \{1,2,3\}\backslash\{j\}$, $m, n \in \{3,\dots,5\}\backslash\{k\}$ intersect the subspace $x_{\mathbf{y}^{j6}}$ in the point $p_{\mathbf{y}^{j6},d_i}$ and $(z^{i3}_{45})_{\mathbf{y}^{j6}} \cap y^{l3}_{\mathbf{y}^{j6}} = p_{\mathbf{y}^{j6},v_l}$ by the containment of $(z^{il}_{mn})_{\mathbf{y}^{j6}}$ in $(y^{34}_{\mathbf{y}^{j6}})^{\pi} \cap (y^{35}_{\mathbf{y}^{j6}})^{\pi}$. Furthermore $p_{\mathbf{y}^{j6},v_t}$ is the intersection of $(z^{il}_{mn})_{\mathbf{y}^{j6}}$ and $y^{lt}_{\mathbf{y}^{j6}}$ for $t \in \{3,\dots,5\}\backslash\{m,n\}$ due to the facts that $(z^{il}_{mn})_{\mathbf{y}^{j6}} \cap y^{lt}_{\mathbf{y}^{j6}}$ is orthogonal to the point $(z^{i3}_{45})_{\mathbf{y}^{j6}} \cap y^{l3}_{\mathbf{y}^{j6}} = (z^{i4}_{35})_{\mathbf{y}^{j6}} \cap y^{l4}_{\mathbf{y}^{j6}} = (z^{i5}_{34})_{\mathbf{y}^{j6}} \cap y^{l5}_{\mathbf{y}^{j6}}$, which show that $(z^{i3}_{45})_{\mathbf{y}^{j6}} = (z^{il}_{j6})_{\mathbf{y}^{j6}}$ and $(z^{il}_{mn})_{\mathbf{y}^{j6}} = (z^{it}_{j6})_{\mathbf{y}^{j6}}$. We end up with the list

$$\begin{array}{ll}
\multicolumn{2}{l}{\mathbf{z}^{i3}_{12} = \mathbf{z}^{i4}_{56} = \mathbf{z}^{i6}_{45} = \mathbf{z}^{i5}_{46} = \mathbf{z}^{i2}_{13} = \mathbf{z}^{i1}_{23} \text{ for } i \in \{1,2\}} \\
\multicolumn{2}{l}{\mathbf{z}^{i4}_{12} = \mathbf{z}^{i3}_{56} = \mathbf{z}^{i6}_{35} = \mathbf{z}^{i5}_{36} = \mathbf{z}^{i2}_{14} = \mathbf{z}^{i1}_{24} \text{ for } i \in \{1,2\}} \\
\multicolumn{2}{l}{\mathbf{z}^{i5}_{12} = \mathbf{z}^{i3}_{46} = \mathbf{z}^{i4}_{36} = \mathbf{z}^{i6}_{34} = \mathbf{z}^{i2}_{15} = \mathbf{z}^{i1}_{25} \text{ for } i \in \{1,2\}} \\
\multicolumn{2}{l}{\mathbf{z}^{i6}_{12} = \mathbf{z}^{i3}_{45} = \mathbf{z}^{i4}_{35} = \mathbf{z}^{i5}_{34} = \mathbf{z}^{i2}_{16} = \mathbf{z}^{i1}_{26} \text{ for } i \in \{1,2\}} \\
\mathbf{z}^{i2}_{56} = \mathbf{z}^{i4}_{13} = \mathbf{z}^{i3}_{14} \text{ for } i \in \{1,2\} & \mathbf{z}^{i2}_{45} = \mathbf{z}^{i6}_{13} = \mathbf{z}^{i3}_{16} \text{ for } i \in \{1,2\} \\
\mathbf{z}^{i2}_{46} = \mathbf{z}^{i5}_{13} = \mathbf{z}^{i3}_{15} \text{ for } i \in \{1,2\} & \mathbf{z}^{i2}_{35} = \mathbf{z}^{i6}_{14} = \mathbf{z}^{i4}_{16} \text{ for } i \in \{1,2\} \\
\mathbf{z}^{i2}_{36} = \mathbf{z}^{i5}_{14} = \mathbf{z}^{i4}_{15} \text{ for } i \in \{1,2\} & \mathbf{z}^{i2}_{34} = \mathbf{z}^{i6}_{15} = \mathbf{z}^{i5}_{16} \text{ for } i \in \{1,2\} \\
\mathbf{z}^{i1}_{56} = \mathbf{z}^{i4}_{23} = \mathbf{z}^{i3}_{24} \text{ for } i \in \{1,2\} & \mathbf{z}^{i1}_{45} = \mathbf{z}^{i6}_{23} = \mathbf{z}^{i3}_{26} \text{ for } i \in \{1,2\} \\
\mathbf{z}^{i1}_{46} = \mathbf{z}^{i5}_{23} = \mathbf{z}^{i3}_{25} \text{ for } i \in \{1,2\} & \mathbf{z}^{i1}_{35} = \mathbf{z}^{i6}_{24} = \mathbf{z}^{i4}_{26} \text{ for } i \in \{1,2\} \\
\mathbf{z}^{i1}_{36} = \mathbf{z}^{i5}_{24} = \mathbf{z}^{i4}_{25} \text{ for } i \in \{1,2\} & \mathbf{z}^{i1}_{34} = \mathbf{z}^{i6}_{25} = \mathbf{z}^{i5}_{26} \text{ for } i \in \{1,2\}
\end{array}$$

for the vertices $\mathbf{z}^{ij}_{kl} \in \mathcal{V}(z)$ in Σ for this moment.

In the last step in the identification of all relations between the vertices of $\in \mathcal{V}(z)$ in Γ, we consider simultaneously the unitary projective spaces $\mathcal{G}_{\mathbf{z}^{15}_{12}}$ and $\mathcal{G}_{\mathbf{z}^{16}_{12}}$. So for the next part we fix the indices h and g such that $\{h, g\} = \{5, 6\}$ and we remind the reader of the adjacency in the graph Γ between the vertex $\mathbf{z}^{1g}_{12}$ and the three vertices $\mathbf{z}^{2c}_{12} = \mathbf{z}^{2g}_{mn}$ with $c \in \{3,4,h\}$, $m, n \in \{3,4,h\}\backslash\{c\}$, $m < n$, between $\mathbf{z}^{1g}_{12}$ and $\mathbf{y}^{mn}$ for either $m, n \in \{1, 2, g\}$, $m < n$ or $m, n \in \{3, 4, h\}$, $m < n$ and of the connectivity between $\mathbf{z}_{12}^{1g}$ and the six vertices $\mathbf{z}_{mn}^{2k} = \mathbf{z}_{fg}^{2d}$ for $\{k, f\} = \{1, 2\}$, $m, n \in \{3, 4, h\}$, $m < n$, $d \in \{3, 4, h\} \backslash \{m, n\}$ in Γ.

First of all the three lines $y_{\mathbf{z}_{12}^{1g}}^{12}$, $y_{\mathbf{z}_{12}^{1g}}^{1g}$ and $y_{\mathbf{z}_{12}^{1g}}^{2g}$ and also the three two-dimensional subspaces $y_{\mathbf{z}_{12}^{1g}}^{34}$, $y_{\mathbf{z}_{12}^{1g}}^{3h}$ and $y_{\mathbf{z}_{12}^{i3}}^{4h}$ intersect pairwise in the space $\mathcal{G}_{\mathbf{z}_{12}^{1g}}$ for $i \in \{1, 2\}$ by corollary 4.3.3. Thus let $p_{\mathbf{z}_{12}^{1g}}^{a}$ be the intersection point of the two lines $y_{\mathbf{z}_{12}^{1g}}^{mn}$ and $y_{\mathbf{z}_{12}^{1g}}^{kl}$ for $k, l, m, n \in \{1, 2, g\}$, $k < l$, $m < n$ and for $k, l, m, n \in \{3, 4, h\}$, $k < l$, $m < n$ where $a = \{m, n\} \cap \{k, l\}$. Moreover, any two points $p_{\mathbf{z}_{12}^{1g}}^{a}$ and $p_{\mathbf{z}_{12}^{1g}}^{b}$ are orthogonal in $\mathcal{G}_{\mathbf{z}_{12}^{1g}}$ if $a \neq b$. Indeed this follows from the argumentation of lemma 4.3.9 and the statements that

- each vertex of the set $\{\mathbf{y}^{mn} \mid m, n \in \{1, 2, g\}, m < n\}$ is adjacent to every vertex of $\{\mathbf{y}^{mn} \mid m, n \in \{3, 4, h\}, m < n\}$,
- the vertex $\mathbf{z}_{12}^{2c} = \mathbf{z}_{mn}^{2g}$ is adjacent to the vertices $\mathbf{y}^{12}$ and $\mathbf{y}^{mn}$ for $c \in \{3, 4, h\}$, $m, n \in \{3, 4, h\} \backslash \{c\}$, $m < n$,
- the two vertices $\mathbf{y}^{mn}$ and $\mathbf{y}^{fg}$ are connected with the vertex $\mathbf{z}_{mn}^{2k} = \mathbf{z}_{fg}^{2d}$ in the graph Γ if $\{k, f\} = \{1, 2\}$, $\{m, n, d\} = \{3, 4, h\}$, $m < n$,
- the intersection of the three lines $(z_{12}^{2c})_{\mathbf{y}^{mn}} = (z_{mn}^{2g})_{\mathbf{y}^{mn}}$, $y_{\mathbf{y}^{mn}}^{1g}$ and $y_{\mathbf{y}^{mn}}^{2g}$ in the space $\mathcal{G}_{\mathbf{y}^{mn}}$ is the point $p_{\mathbf{y}^{mn}, v_g}$ for $c \in \{3, 4, h\}$, $m, n \in \{3, 4, h\} \backslash \{c\}$, $m < n$, and
- the three lines $(z_{mn}^{2k})_{\mathbf{y}^{fg}} = (z_{fg}^{2d})_{\mathbf{y}^{fg}}$, $y_{\mathbf{y}^{fg}}^{st}$ and $y_{\mathbf{y}^{fg}}^{pq}$ intersect in the point $p_{\mathbf{y}^{fg}, v_d}$ in the space $\mathcal{G}_{\mathbf{y}^{fg}}$ for $\{k, f\} = \{1, 2\}$, $\{d, m, n\} = \{3, 4, h\}$, $m < n$, $\{s, t, p, q\} = \{3, 4, h\}$, $s < t$, $p < q$, $d = \{s, t\} \cap \{p, q\}$.

So we conclude that $y_{\mathbf{z}_{12}^{1g}}^{mn}$ equals $\langle p_{\mathbf{z}_{12}^{1g}}^{m}, p_{\mathbf{z}_{12}^{1g}}^{n} \rangle$ for either $m, n \in \{1, 2, g\}$, $m < n$ or $m, n \in \{3, 4, h\}$, $m < n$. Next we consider the lines $(z_{12}^{2c})_{\mathbf{z}_{12}^{1g}} = (z_{mn}^{2g})_{\mathbf{z}_{12}^{1g}}$ in the space $\mathcal{G}_{\mathbf{z}_{12}^{1g}}$ for the indices $\{c, m, n\} = \{3, 4, h\}$, $n < m$ and obtain that the two-dimensional subspace $(y_{\mathbf{z}_{12}^{1g}}^{12})^{\pi} \cap (y_{\mathbf{z}_{12}^{1g}}^{mn})^{\pi} = \langle p_{\mathbf{z}_{12}^{1g}}^{c}, p_{\mathbf{z}_{12}^{1g}}^{g} \rangle$ contains the line $(z_{12}^{2c})_{\mathbf{z}_{12}^{1g}} = (z_{mn}^{2g})_{\mathbf{z}_{12}^{1g}}$ implying $(z_{12}^{2c})_{\mathbf{z}_{12}^{1g}} = (z_{mn}^{2g})_{\mathbf{z}_{12}^{1g}} = (y_{\mathbf{z}_{12}^{1g}}^{12})^{\pi} \cap (y_{\mathbf{z}_{12}^{1g}}^{mn})^{\pi}$. Using the same argumentation again for the lines $(z_{mn}^{2k})_{\mathbf{z}_{12}^{1g}} = (z_{fg}^{2d})_{\mathbf{z}_{12}^{1g}}$ under the conditions that $\{k, f\} = \{1, 2\}$, $\{d, m, n\} = \{3, 4, h\}$ and $m < n$ we get that $(z_{mn}^{2k})_{\mathbf{z}_{12}^{1g}} = (z_{fg}^{2d})_{\mathbf{z}_{12}^{1g}} = (y_{\mathbf{z}_{12}^{1g}}^{mn})^{\pi} \cap (y_{\mathbf{z}_{12}^{1g}}^{fg})^{\pi} = \langle p_{\mathbf{z}_{12}^{1g}}^{k}, p_{\mathbf{z}_{12}^{1g}}^{d} \rangle$. Thus the polar space $(z_{12}^{2c})_{\mathbf{z}_{12}^{1g}}^{\pi}$ of $(z_{12}^{2c})_{\mathbf{z}_{12}^{1g}}$ for $c \in \{3, 4, h\}$ contains every line of the set $\{(z_{mn}^{2k})_{\mathbf{z}_{12}^{1g}} \mid k \in \{1, 2\}, m < n, m, n \in \{3, 4, h\}, c \in \{m, n\}\}$. In particular the vertex $\mathbf{z}_{12}^{23}$ is adjacent to the vertices $\mathbf{z}_{36}^{21}$, $\mathbf{z}_{36}^{22}$, $\mathbf{z}_{34}^{21}$, $\mathbf{z}_{34}^{22}$, $\mathbf{z}_{35}^{21}$, $\mathbf{z}_{35}^{22}$ in Γ.

Finally we study the projective spaces $\mathcal{G}_{\mathbf{y}^{45}}$, $\mathcal{G}_{\mathbf{y}^{46}}$ and $\mathcal{G}_{\mathbf{y}^{56}}$ again. We fix two indices $4 \leq k < l \leq 6$. Notices that the line $(z_{12}^{ij})_{\mathbf{y}^{kl}}$ coincides with the two-dimensional subspace $(z_{kl}^{iu})_{\mathbf{y}^{kl}}$ in $\mathcal{G}_{\mathbf{y}^{kl}}$ if $j, u \in \{3, 4, 5, 6\} \backslash \{k, l\}$, $i \in \{1, 2\}$. Moreover this projective space $\mathcal{G}_{\mathbf{y}^{kl}}$ contains the four lines $(z_{12}^{im})_{\mathbf{y}^{kl}}$ and $(z_{3n}^{im})_{\mathbf{y}^{kl}}$ for $m \in \{1, 2\}$ and $n \in \{4, 5, 6\} \backslash \{k, l\}$. Due to the facts that $\mathbf{z}_{34}^{im} = \mathbf{z}_{p5}^{i6} = \mathbf{z}_{p6}^{i5}$, $\mathbf{z}_{35}^{im} = \mathbf{z}_{p4}^{i6} = \mathbf{z}_{p6}^{i4}$ and

$\mathbf{z}^{im}_{36} = \mathbf{z}^{i5}_{p4} = \mathbf{z}^{i4}_{p5}$ for an index $p \in \{1,2\}\backslash\{m\}$, we get that the line $(z^{im}_{3n})_{\mathbf{y}^{kl}}$ is contained in the polar space $(y^{ph}_{\mathbf{y}^{kl}})^{\pi}$ of $y^{ph}_{\mathbf{y}^{kl}}$ if $h \in \{3, n\}$, thus $(z^{im}_{3n})_{\mathbf{y}^{kl}} \cap y^{12}_{\mathbf{y}^{kl}}$ is the point $p_{\mathbf{y}^{kl},v_q}$ for $q \in \{1,2\}\backslash\{m\}$. Furthermore using the results from above we obtain that $(z^{2m}_{3n})_{\mathbf{y}^{kl}}$ is orthogonal to the line $(z^{23}_{12})_{\mathbf{y}^{kl}} = (z^{2n}_{kl})_{\mathbf{y}^{kl}}$, hence we conclude that the point $(z^{2m}_{3n})_{\mathbf{y}^{kl}} \cap x_{\mathbf{y}^{kl}}$ is orthogonal to $(z^{23}_{12})_{\mathbf{y}^{kl}} \cap x_{\mathbf{y}^{kl}} = (z^{2n}_{kl})_{\mathbf{y}^{kl}} \cap x_{\mathbf{y}^{kl}} = p_{\mathbf{y}^{kl},d_2}$. All together we know now that $(z^{2m}_{3n})_{\mathbf{y}^{kl}} = (z^{1p}_{kl})_{\mathbf{y}^{kl}}$ and $(z^{1m}_{3n})_{\mathbf{y}^{kl}} = (z^{2p}_{kl})_{\mathbf{y}^{kl}}$ for $\{m,p\} = \{1,2\}$, $n \in \{4,5,6\}\backslash\{k,l\}$. ∎

Observation 4.4.19 Let $\mathbf{z}^{ab}_{cd} \in \mathcal{V}(z)$ be a vertex of the graph Σ. Then for each index set $\{m,n\} \subseteq \{b,c,d\}$ with $m < n$ there exists an index $i \in \{1,2\}$ such that $\mathbf{z}^{ab}_{cd} = \mathbf{z}^{ij}_{mn}$ where $\{j,m,n\} = \{b,c,d\}$. Also, for each index set $\{m,n\} \subseteq \{1,\dots,6\}\backslash\{b,c,d\}$ with $m < n$ there exists an index $i \in \{1,2\}$ such that $\mathbf{z}^{ab}_{cd} = \mathbf{z}^{ij}_{mn}$ where $\{j,m,n\} = \{1,\dots,6\}\backslash\{b,c,d\}$.

For a better understanding we will divide this observation up into two parts.

Observation 4.4.20 Let $\mathbf{z}^{ab}_{12}$ be a vertex of Σ with indices $a \in \{1,2\}$, $3 \le b \le 6$. Then $\mathbf{z}^{ab}_{12} = \mathbf{z}^{aj}_{mn}$ for either $\{j,m,n\} = \{1,2,b\}$, $m < n$ or $\{j,m,n\} = \{1,\dots,6\}\backslash\{1,2,b\}$.

Observation 4.4.21 Let $\mathbf{z}^{ab}_{cd}$ be a vertex of Σ such that $a,b \in \{1,2\}$, $3 \le c < d \le 6$. Then $\mathbf{z}^{ab}_{cd} = \mathbf{z}^{ij}_{mn}$ for $\{m,n\} = \{3,\dots,6\}\backslash\{c,d\}$, $m < n$, $i,j \in \{1,2\}$, $i \ne a$, $j \ne b$. Furthermore $\mathbf{z}^{ab}_{cd} = \mathbf{z}^{ij}_{bn}$ for $\{j,n\} \subseteq \{c,d\}$, $i \in \{1,2\}\backslash\{a\}$ and also $\mathbf{z}^{ab}_{cd} = \mathbf{z}^{aj}_{mn}$ for $\{j,n\} = \{4,\dots,6\}\backslash\{c,d\}$, $i = \{1,2\}\backslash\{b\}$.

Corollary 4.4.22 (of proposition 4.4.18) *The graph Σ consists of the 36 pairwise distinct vertices $\mathbf{x}$, $\mathbf{y}^{ij}$, $\mathbf{z}^{kl}_{cd}$ for indices $1 \le i < j \le 6$, $k \in \{1,2\}$, $1 \le c < d \le 6$, $l \in \{1,\dots,6\}\backslash\{c,d\}$.*

Proof: The vertex $\mathbf{x}$ is distinct from each vertex $\mathbf{y}^{ij}$ for $1 \le i < j \le 6$, because $\mathbf{y}^{ij} \in \Sigma_{\mathbf{x}}$ and $\mathbf{x}$ is not an element of $\mathcal{V}(z)$ by lemma 4.4.9. The vertices $\mathbf{y}^{ij}$ for $1 \le i < j \le 6$ are pairwise distinct by the construction of the graph Σ. We also have $\mathbf{y}^{ij} \ne \mathbf{z}^{kl}_{cd}$ for $1 \le i < j \le 6$ and $\mathbf{z}^{kl}_{cd} \in \mathcal{V}(z)$ because $\mathbf{x} \perp \mathbf{y}^{ij}$ and $\mathbf{x} \not\perp \mathbf{z}^{kl}_{cd}$, cf. lemma 4.4.9.

Finally we have to prove that the vertices $\mathbf{z}^{kl}_{cd}$ for $k \in \{1,2\}$, $1 \le c < d \le 6$ and $l \in \{1,\dots,6\}\backslash\{c,d\}$ are pairwise distinct. For each pair of vertices $\mathbf{z}^{kl}_{cd}$ and $\mathbf{z}^{mn}_{st}$ such that the indices satisfy $m,k \in \{1,2\}$, $1 \le c < d \le 6$, $l \in \{1,\dots,6\}\backslash\{c,d\}$, $1 \le s < t \le 6$, $n \in \{1,\dots,6\}\backslash\{s,t\}$ there exists a set $\{i,j\}$ of cardinality two contained either in $\{c,d,l\}\backslash\{n,s,t\}$ or in $\{1,\dots,6\}\backslash(\{c,d,l\} \cup \{n,s,t\})$. Indeed if $|\{c,d,l\}\backslash\{n,s,t\}| \le 1$ then we find two different indices g and h in the index set $\{c,d,l\} \cap \{n,s,t\}$ thus $3 \le |\{c,d,l\} \cup \{n,s,t\}| \le 4$ implying that the index set $\{1,\dots,6\}\backslash(\{c,d,l\} \cup \{n,s,t\})$ contains at least two different elements. Therefore, by observation 4.4.19, there exist indices a,b,p,q with $\mathbf{z}^{kl}_{cd} = \mathbf{z}^{ab}_{ij} \perp \mathbf{y}^{ij} \perp \mathbf{z}^{pq}_{ij} = \mathbf{z}^{mn}_{st}$, whence, by local analysis of the subgraph $\Sigma_{\mathbf{y}^{ij}}$, we have $\mathbf{z}_{cd}^{kl} = \mathbf{z}_{ij}^{ab} = \mathbf{z}_{ij}^{pq} = \mathbf{z}_{st}^{mn}$ if and only if $a = p$ and $b = q$. Hence proposition 4.4.18 describes all identities that exist between vertices of Σ. Therefore Σ contains $1 + 15 + 20 = 36$ vertices. ■

Lemma 4.4.23 *Any two vertices $\mathbf{z}_{mn}^{ij} \in \mathcal{V}(z)$ and $\mathbf{y}^{kl}$ of the graph Σ are adjacent in Σ if and only if $\mathbf{z}_{mn}^{ij} = \mathbf{z}_{kl}^{ab}$ for some indices $a \in \{1,2\}$, $b \in \{1,\dots,6\}\backslash\{k,l\}$. More precisely, the vertex $\mathbf{z}_{mn}^{ij}$ is adjacent to $\mathbf{y}^{kl}$ if and only if either $k,l \in \{m,n,j\}$ or $k,l \in \{1,\dots,6\}\backslash\{m,n,j\}$.*

Proof: One of the sets $\{j,m,n\}\backslash\{k,l\}$ or $\{1,\dots,6\}\backslash(\{j,m,n\}\cup\{k,l\})$ necessarily contains two elements c and d. Truly if $|\{j,m,n\}\backslash\{k,l\}| \leq 1$ then $\{k,l\} \subseteq \{j,m,n\}$ and therefore we count $|\{1,\dots,6\}\backslash(\{j,m,n\}\cup\{k,l\})| = |\{1,\dots,6\}\backslash\{j,m,n\}| = 3$. By observation 4.4.19 we have $\mathbf{z}_{mn}^{ij} = \mathbf{z}_{cd}^{st}$ for suitable indices s, t, whence $\mathbf{z}_{mn}^{ij} = \mathbf{z}_{cd}^{st}$ and $\mathbf{z}_{cd}^{st} \perp \mathbf{y}^{cd} \perp \mathbf{y}^{kl}$. The vertices $\mathbf{z}_{mn}^{ij}$ and $\mathbf{y}^{kl}$ are adjacent in Σ if and only if $(z_{mn}^{ij})_{\mathbf{y}^{cd}} = (z_{cd}^{st})_{\mathbf{y}^{cd}} \subseteq (y_{\mathbf{y}^{cd}}^{kl})^{\pi}$ which is equal to the condition that $k \neq t \neq l$. Thus $\{k,l\} \subseteq \{1,\dots,6\}\backslash\{c,d,t\}$ and by observation 4.4.19 there exists a suitable index $p \in \{1,2\}$ such that $z_{cd}^{st} = z_{kl}^{pq}$ with $q = \{1,\dots,6\}\backslash\{c,d,t,k,l\}$. The second claim now follows immediately from observation 4.4.19. ■

Notation: Let $\mathbf{z}_{12}^{im}$ be a vertex of Σ with $i \in \{1,2\}$ and $3 \leq m \leq 6$. We denote with $\mathcal{V}_{12}^{im}$ the vertex set

$$\{\mathbf{z}_{12}^{st}, \mathbf{z}_{cd}^{sv} \mid s \in \{1,2\}\backslash\{i\},\ v \in \{1,2\}, t,c,d \in \{3,\dots,6\}\backslash\{m\},\ c < d\} =$$
$$\{\mathbf{z}_{12}^{st}, \mathbf{z}_{1m}^{st}, \mathbf{z}_{2m}^{st} \mid s \in \{1,2\}\backslash\{i\}, t \in \{3,\dots,6\}\backslash\{m\}\}.$$

On the other hand let $\mathbf{z}_{gh}^{mn}$ be a vertex of the graph Σ for some indices $m,n \in \{1,2\}$, $g,h \in \{4,\dots,6\}$ and $g < h$, then we set

$$\mathcal{V}_{gh}^{mn} := \{\mathbf{z}_{gh}^{ab}, \mathbf{z}_{ng}^{mb}, \mathbf{z}_{nh}^{mb} \mid a \in \{1,2\}\backslash\{m\}, b \in \{1,\dots,6\}\backslash\{n,g,h\}\} =$$
$$\left\{ \mathbf{z}_{gh}^{ab}, \mathbf{z}_{cd}^{ag}, \mathbf{z}_{cd}^{ah}, \mathbf{z}_{3k}^{mg}, \mathbf{z}_{3k}^{mh} \;\middle|\; \begin{array}{l} c,a,b \in \{1,2\},\ a \neq m,\ c \neq n,\ b \neq n \\ d,k,b \in \{3,\dots,6\}\backslash\{g,h\},\ k \neq 3 \end{array} \right\}.$$

In general for a vertex $\mathbf{z}_{cd}^{ij}$ of Σ with the indices $i \in \{1,2\}$, $c,d,j \in \{1,\dots,6\}$, $c < d$, $c \neq j \neq d$, we denote with $\mathcal{V}_{cd}^{ij}$ the vertex set

$$\mathcal{V}_{cd}^{ij} = \begin{cases} \mathcal{V}_{12}^{im} \text{ with } 3 \leq m \leq 6 & : \text{ if } \mathbf{z}_{cd}^{ij} = \mathbf{z}_{12}^{im} \\ \mathcal{V}_{gh}^{mn} \text{ with } n,m \in \{1,2\}, 4 \leq g < h \leq 6 & : \text{ if } \mathbf{z}_{cd}^{ij} = \mathbf{z}_{gh}^{mn} \end{cases}.$$

It will turn out that the elements of $\mathcal{V}_{cd}^{ij}$ are just the neighbours of $\mathbf{z}_{cd}^{ij}$ in Σ, which are contained in the vertex set $\mathcal{V}(z)$. More precisely, we will prove that $\mathcal{V}_{cd}^{ij} = \Sigma_{\mathbf{z}_{cd}^{ij}} \cap \mathcal{V}(z)$.

Lemma 4.4.24 *Let $\mathbf{z}_{kl}^{ij} \in \mathcal{V}(z)$ and $\mathbf{z}_{st}^{uw} \in \mathcal{V}(z)$ be two different vertices of Σ. These two vertices $\mathbf{z}_{kl}^{ij}$ and $\mathbf{z}_{st}^{uw}$ are adjacent in Σ if and only if $\mathbf{z}_{st}^{uw} \in \mathcal{V}_{kl}^{ij}$.*

Proof: Let $\mathbf{z}_{kl}^{ij}$ and $\mathbf{z}_{st}^{uw}$ be two adjacent vertices in Γ. For the first part we assume that $\mathbf{z}_{kl}^{ij} = \mathbf{z}_{12}^{im}$ for some $m \in \{3, \dots, 6\}$.

If also the vertex $\mathbf{z}_{st}^{uw}$ equals a vertex $\mathbf{z}_{12}^{uv}$ for some $v \in \{3, \dots, 6\}$, then the assumption implies that $(z_{kl}^{ij})_{\mathbf{y}^{12}} = (z_{12}^{im})_{\mathbf{y}^{12}} \subseteq (z_{12}^{uv})_{\mathbf{y}^{12}}^{\pi} = (z_{st}^{uw})_{\mathbf{y}^{12}}^{\pi}$ in $\mathcal{G}_{\mathbf{y}^{12}}$. Thus $u \neq i$ and $v \neq m$ and due to the relation list we have that $\mathbf{z}_{st}^{uw} = \mathbf{z}_{12}^{uv} \in \mathcal{V}_{12}^{im}$. Suppose $\mathbf{z}_{st}^{uw} = \mathbf{z}_{gh}^{ab}$ for some $a, b \in \{1, 2\}$, $g, h \in \{4, \dots, 6\}$, $g < h$, then we fix the vertex $\mathbf{y}^{nm}$ with $n \in \{1, 2\} \backslash \{b\}$ if $m \notin \{g, h\}$ and $n = b$ if $m \in \{g, h\}$. Due to lemma 4.4.23 the vertices $\mathbf{z}_{kl}^{ij} = \mathbf{z}_{12}^{im}$ and $\mathbf{z}_{st}^{uw} = \mathbf{z}_{gh}^{ab}$ are adjacent with the vertex $\mathbf{y}^{nm}$ in the graph Σ, moreover $\mathbf{z}_{kl}^{ij} = \mathbf{z}_{12}^{im} = \mathbf{z}_{nm}^{ir}$ for $r \in \{1, 2\} \backslash \{b\}$ and $\mathbf{z}_{st}^{uw} = \mathbf{z}_{gh}^{ab} = \mathbf{z}_{nm}^{cd}$ with $c = a$, $d \in \{3, \dots, 6\} \backslash \{g, h, m\}$ if $m \notin \{g, h\}$ and $c \in \{1, 2\} \backslash \{a\}$, $d \in \{g, h\} \backslash \{m\}$ if $m \in \{g, h\}$. The fact that $(z_{st}^{uw})_{\mathbf{y}^{nm}} = (z_{nm}^{cd})_{\mathbf{y}^{nm}} \subseteq (z_{kl}^{ij})_{\mathbf{y}^{nm}}^{\pi} = (z_{nm}^{ir})_{\mathbf{y}^{nm}}^{\pi}$ in $\mathcal{G}_{\mathbf{y}^{nm}}$ implies that $c \neq i$ and using the relation list again we get that $\mathbf{z}_{st}^{uw} = \mathbf{z}_{gh}^{ab} = \mathbf{z}_{nm}^{cd} \in \mathcal{V}_{12}^{im}$.

For the next part of this proof let $\mathbf{z}_{kl}^{ij} = \mathbf{z}_{gh}^{mn}$ for some indices $m, n \in \{1, 2\}$, $g, h \in \{4, \dots, 6\}$, $g < h$. Again we have to consider the two different cases that either $\mathbf{z}_{st}^{uw} = \mathbf{z}_{12}^{uv}$ for some $v \in \{3, \dots, 6\}$ or $\mathbf{z}_{st}^{uw} = \mathbf{z}_{cd}^{ab}$ for $a, b \in \{1, 2\}$, $c, d \in \{4, \dots, 6\}$, $c < d$.

We start with the possibility that $\mathbf{z}_{st}^{uw} = \mathbf{z}_{12}^{uv}$ for some $v \in \{3, \dots, 6\}$. Then by the argumentation above $\mathbf{z}_{kl}^{ij} \in \mathcal{V}_{12}^{uv}$ and it follows that $\mathbf{z}_{st}^{uw} = \mathbf{z}_{12}^{uv} \in \mathcal{V}_{gh}^{mn}$. Next let $\mathbf{z}_{st}^{uw} = \mathbf{z}_{cd}^{ab}$ for $a, b \in \{1, 2\}$, $c, d \in \{4, \dots, 6\}$, $c < d$. Suppose we have equality between the sets $\{g, h\}$ and $\{c, d\}$ then the adjacency of the vertices $\mathbf{z}_{kl}^{ij}$ and $\mathbf{z}_{st}^{uw}$ in Γ leads to the containment of the line $(z_{kl}^{ij})_{\mathbf{y}^{gh}} = (z_{gh}^{mn})_{\mathbf{y}^{gh}}$ in the polar space $(z_{st}^{uw})_{\mathbf{y}^{gh}}^{\pi} = (z_{gh}^{ab})_{\mathbf{y}^{gh}}^{\pi}$ of the line $(z_{st}^{uw})_{\mathbf{y}^{gh}} = (z_{gh}^{ab})_{\mathbf{y}^{gh}}$ inside $\mathcal{G}_{\mathbf{y}^{gh}}$ implying that $m \neq a$ and $n \neq b$ so $\mathbf{z}_{st}^{uw} = \mathbf{z}_{gh}^{ab} \in \mathcal{V}_{gh}^{mn}$. Alternatively $\{g, h\} \cap \{c, d\} = \{r\}$, then the vertex $\mathbf{y}^{ef}$ is adjacent to both, the vertex $\mathbf{z}_{kl}^{ij} = \mathbf{z}_{gh}^{mn}$ and the vertex $\mathbf{z}_{st}^{uw} = \mathbf{z}_{cd}^{ab}$ in Γ for either $e \in \{1, 2\} \backslash \{n\}$, $f = 3$ if $n = b$ or $e = n$, $f \in \{g, h\} \backslash \{r\}$ if $n \neq b$. Using the relation list of proposition 4.4.18 we obtain that $\mathbf{z}_{kl}^{ij} = \mathbf{z}_{e3}^{mp}$ for $p \in \{4, 5, 6\} \backslash \{g, h\}$ and $\mathbf{z}_{st}^{uw} = \mathbf{z}_{e3}^{uv}$ for $v \in \{4, 5, 6\} \backslash \{c, d\}$ if $n = b$. In the projective space $\mathcal{G}_{\mathbf{y}^{e3}}$ the two different lines $(z_{kl}^{ij})_{\mathbf{y}^{e3}} = (z_{e3}^{mp})_{\mathbf{y}^{e3}}$ and $(z_{st}^{uw})_{\mathbf{y}^{e3}} = (z_{e3}^{uv})_{\mathbf{y}^{e3}}$ are orthogonal, as $\mathbf{z}_{kl}^{ij} \perp \mathbf{z}_{st}^{uw}$, thus $m \neq u$, implying $\mathbf{z}_{st}^{uw} = \mathbf{z}_{cd}^{ab} = \mathbf{z}_{e3}^{uv} \in \mathcal{V}_{gh}^{mn}$. In the other case, if $n \neq b$ then we get equality between the vertices $\mathbf{z}_{kl}^{ij}$ and $\mathbf{z}_{nf}^{pr}$ with $p \in \{1, 2\} \backslash \{m\}$ and $\mathbf{z}_{st}^{uw} = \mathbf{z}_{nf}^{u3}$. Due to the assumption that $\mathbf{z}_{nf}^{pr} = \mathbf{z}_{kl}^{ij} \perp \mathbf{z}_{st}^{uw} = \mathbf{z}_{nf}^{u3}$ we obtain that the indices p and u are different and that $\mathbf{z}_{st}^{uw} = \mathbf{z}_{nf}^{u3}$ is an element of the vertex set $\mathcal{V}_{gh}^{mn}$.

The other direction of the statement is obviously true. ■

Again we split up this statement into two parts.

Corollary 4.4.25 *Let $\mathbf{z}_{12}^{im}$ be a vertex of Σ with $i \in \{1,2\}$, $3 \leq m \leq 6$. Then $\mathbf{z}_{12}^{im}$ is adjacent to each element of $\mathcal{V}_{12}^{im}$. Moreover two different elements* $\mathbf{h}$ *and* $\mathbf{q}$ *of $\mathcal{V}_{12}^{im}$ are adjacent in Σ if and only if*

- *either* $\mathbf{h} = \mathbf{z}_{12}^{rg}$ *and* $\mathbf{q} = \mathbf{z}_{jm}^{rh}$ *with $g \neq h$ for $j, r \in \{1,2\}$, $i \neq r$, $g, h \in \{3,\dots,6\} \backslash \{m\}$*
- *or* $\mathbf{h} = \mathbf{z}_{1m}^{rg}$ *and* $\mathbf{q} = \mathbf{z}_{2m}^{rh}$ *such that $g \neq h$ for $r \in \{1,2\} \backslash i$, $g, h \in \{3,\dots,6\} \backslash \{m\}$.*

Proof: Let g be an element of the index set $\{3,\dots,6\} \backslash \{m\}$ then $|\{1,2,g,m\}| = 4$ and we set $\{v,w\} = \{1,\dots,6\} \backslash \{1,2,g,m\}$ with $v < w$. By observation 4.4.21 and observation 4.4.20 we get that $\mathbf{z}_{12}^{rg} = \mathbf{z}_{vw}^{rm}$ and $\mathbf{z}_{jm}^{rg} = \mathbf{z}_{vw}^{rt}$ for $t \in \{1,2\} \backslash \{j\}$. The vertices $\mathbf{z}_{vw}^{rm}$ and $\mathbf{z}_{vw}^{rt}$ are not adjacent in Σ as $(z_{vw}^{rm})_{\mathbf{y}^{vw}} \notin (z_{vw}^{rt})_{\mathbf{y}^{vw}}^{\pi}$ and $\{r,m\} \cap \{r,t\} \neq \varnothing$. Also $\mathbf{z}_{2m}^{rg} = \mathbf{z}_{vw}^{r1} \not\perp \mathbf{z}_{vw}^{r2} = \mathbf{z}_{1m}^{rg}$ due to the fact that $(z_{vw}^{r1})_{\mathbf{y}^{vw}} \notin (z_{vw}^{r2})_{\mathbf{y}^{vw}}^{\pi}$ in the space $\mathcal{G}_{\mathbf{y}^{vw}}$.

Next let g and h be two different elements of the index set $\{3,\dots,6\} \backslash \{m\}$. Since $\{1,2,g\} \backslash \{j,m,h\} = \{t,g\}$ for $\{j,t\} = \{1,2\}$ we know that $\mathbf{z}_{12}^{rg} = \mathbf{z}_{tg}^{rj}$ and $\mathbf{z}_{jm}^{rh} = \mathbf{z}_{tg}^{is}$ for $s = \{3,\dots,6\} \backslash \{m,g,h\}$ by observation 4.4.21 and observation 4.4.20. Certainly the lines $(z_{tg}^{rj})_{\mathbf{y}^{tg}}$ and $(z_{tg}^{is})_{\mathbf{y}^{tg}}$ are orthogonal in $\mathcal{G}_{\mathbf{y}^{tg}}$ thus $\mathbf{z}_{12}^{rg} = \mathbf{z}_{tg}^{rj} \perp \mathbf{z}_{tg}^{is} = \mathbf{z}_{jm}^{rh}$ in Σ.

By the same argumentation we get that $\mathbf{z}_{2m}^{rg} = \mathbf{z}_{2g}^{im} \perp \mathbf{z}_{2g}^{rs} = \mathbf{z}_{1m}^{rh}$ in Σ. Indeed $\{2,g,m\} \backslash \{1,m,h\} = \{2,g\}$, so due to observation 4.4.21, we get $\mathbf{z}_{2m}^{rg} = \mathbf{z}_{2g}^{im}$ and $\mathbf{z}_{1m}^{rh} = \mathbf{z}_{2g}^{is}$ for $s \in \{3,\dots,6\} \backslash \{m,g,h\}$. Moreover $(z_{2g}^{is})_{\mathbf{y}^{2g}} \subseteq (z_{2g}^{rm})_{\mathbf{y}^{2g}}^{\pi}$ in $\mathcal{G}_{\mathbf{y}^{2g}}$ as $\{i,s\} \cap \{r,m\} = \varnothing$. ∎

For the last statements of this section we will disregard the ordering of the two different indices i, j for the vertices $\mathbf{y}^{ij}$, thus we make no difference between the vertices $\mathbf{y}^{ij}$ and $\mathbf{y}^{ji}$ for $i, j \in \{1,\dots,6\}$. We are also allowed to interchange the position of k and l for any vertices $\mathbf{z}_{kl}^{st} \in \mathcal{V}(z)$, so $\mathbf{z}_{kl}^{st} = \mathbf{z}_{lk}^{st}$, but we can not approved to interchange the position of the two indices s and t for the vertices $\mathbf{z}_{kl}^{st}$.

Corollary 4.4.26 *Let z_{gh}^{mn} be a vertex of Σ with $m, n \in \{1,2\}$, $g, h \in \{4\dots,6\}$, $g \neq h$. Then z_{gh}^{mn} is adjacent to each element of $\mathcal{V}_{gh}^{mn}$. Furthermore two different elements* $\mathbf{h}$ *and* $\mathbf{q}$ *of $\mathcal{V}_{gh}^{mn}$ are adjacent in Σ if and only if*

- *either* $\mathbf{h} = \mathbf{z}_{gh}^{rv}$ *and* $\mathbf{q} = \mathbf{z}_{nl}^{mk}$ *with $k \neq v$ for $l \in \{g,h\}$, $k, v \in (\{1,\dots,6\} \backslash \{g,h,n\}$*
- *or* $\mathbf{h} = z_{ng}^{mv}$ *and* $\mathbf{q} = z_{nh}^{mk}$ *such that $v \neq k$ for $v, k \in \{1,\dots,6\} \backslash \{g,h,n\}$.*

Proof: For a fixed index $v \in \{1,\dots,6\} \backslash \{g,h,n\}$, the index set $\{g,h,n,v\}$ has cardinality four. We fix two different indices $p, q \in \{1,\dots,6\} \backslash \{g,h,n,v\}$ and let $w \in \{g,h\} \backslash \{l\}$. By observation 4.4.21 and observation 4.4.20, we get

$$\mathbf{z}_{gh}^{rv} = \begin{cases} \mathbf{z}_{pq}^{mn} & \text{if } v \in \{1,2\} \backslash \{n\} \\ \mathbf{z}_{pq}^{rn} & \text{if } v \in \{3,\dots,6\} \backslash \{g,h\} \end{cases} \text{ and } \mathbf{z}_{nl}^{mv} = \begin{cases} \mathbf{z}_{pq}^{mw} & \text{if } v \in \{1,2\} \backslash \{n\} \\ \mathbf{z}_{pq}^{rw} & \text{if } v \in \{3,\dots,6\} \backslash \{g,h\} \end{cases}.$$

Thus the lines $(z^{rv}_{gh})_{\mathbf{y}^{pq}}$ and $(z^{mv}_{nl})_{\mathbf{y}^{pq}}$ intersect in the projective space $\mathcal{G}_{\mathbf{y}^{pq}}$ and $(z^{mv}_{ng})^{\pi}_{\mathbf{y}^{pq}} \cap (z^{mv}_{nh})^{\pi}_{\mathbf{y}^{pq}}$ is a subspace of dimension one. Therefore in the graph Σ we obtain the relations that $\mathbf{z}^{rv}_{gh} \not\perp \mathbf{z}^{mv}_{nl}$ and $z^{mv}_{ng} \notin \{z^{mv}_{nh}\}^{\perp}$.

On the other hand let k and v be two different indices of $\{1,\ldots,6\}\backslash\{g,h,n\}$ and let $w \in \{g,h\}\backslash\{l\}$. Using observation 4.4.21 and observation 4.4.20 we determine that

$$\mathbf{z}^{rv}_{gh} = \begin{cases} \mathbf{z}^{ml}_{vw} & \text{if } v \in \{1,2\}\backslash\{n\} \\ \mathbf{z}^{rl}_{vw} & \text{if } v \in \{3,\ldots,6\}\backslash\{g,h\} \end{cases} \text{ and } \mathbf{z}^{mk}_{nl} = \begin{cases} \mathbf{z}^{mt}_{vw} & \text{if } k \in \{1,2\}\backslash\{n\} \\ & \text{or if } k,v \in \{3,\ldots,6\}\backslash\{g,h\} \\ \mathbf{z}^{rt}_{vw} & \text{if } v \in \{1,2\}\backslash\{n\} \end{cases}$$

for $t \in \{1,\ldots,6\}\backslash\{k,l,n,v,w\}$ as $\{g,h,v\}\backslash\{k,l,n\} = \{v,w\}$. Also

$$\mathbf{z}^{mv}_{ng} = \begin{cases} \mathbf{z}^{mn}_{vg} & \text{if } v \in \{1,2\}\backslash\{n\} \\ \mathbf{z}^{rn}_{vg} & \text{if } v \in \{3,\ldots,6\}\backslash\{g,h\} \end{cases} \text{ and } \mathbf{z}^{mk}_{nh} = \begin{cases} \mathbf{z}^{mt}_{gv} & \text{if } k \in \{1,2\}\backslash\{n\} \\ & \text{or if } k,v \in \{3,\ldots,6\}\backslash\{g,h\} \\ \mathbf{z}^{rt}_{gv} & \text{if } v \in \{1,2\}\backslash\{n\} \end{cases}$$

for $t \in \{1,\cdots,6\}\backslash\{g,h,k,n,v\}$ as $\{g,n,v\}\backslash\{k,n,h\} = \{g,v\}$. This implies that $\mathbf{z}^{rv}_{gh} \in \{\mathbf{z}^{mk}_{nl}\}^{\perp}$ and $z^{mv}_{ng} \perp z^{mk}_{nh}$ in Σ. ∎

Lemma 4.4.27 *Let* $\mathbf{w}$ *be a vertex of* Σ*. Then the induced subgraph* $\Sigma_{\mathbf{w}}$ *has exactly* 15 *vertices.*

Proof: The statement is proved in lemma 4.4.10 for the vertex $\mathbf{x}$ of the graph Σ.

If $\mathbf{w} = \mathbf{z}^{ij}_{kl} \in \mathcal{V}(z)$ then each element of the vertex set $\{\mathbf{y}^{ab}, \mathbf{y}^{cd} \mid a,b \in \{j,k,l\}, c,d \in \{1,\ldots,6\}\backslash\{j,k,l\}\}$ and every vertex of the set $\mathcal{V}^{ij}_{kl}$ is adjacent to the vertex $\mathbf{z}^{ij}_{kl}$ in the graph Γ. Since $|\{\mathbf{y}^{ab}, \mathbf{y}^{cd} \mid a,b \in \{j,k,l\}, c,d \in \{1,\ldots,6\}\backslash\{j,k,l\}\}| = 6$ and $|\mathcal{V}^{ij}_{kl}| = 9$ the graph $\Sigma_{\mathbf{z}^{ij}_{kl}}$ contains exactly 15 vertices by lemma 4.4.23 and 4.4.24.

The last possibility is the case that $\mathbf{w} = \mathbf{y}^{kl}$ with $1 \leq k < l \leq 6$. Certainly the vertex $\mathbf{y}^{kl}$ is adjacent to the six different vertices $\{\mathbf{y}^{cd} \mid c,d \in \{1,\ldots,6\}\backslash\{k,l\}\}$ and to the vertex $\mathbf{x}$. Furthermore the elements of the set $\{\mathbf{z}^{ij}_{kl} \mid i \in \{1,2\}, j \in \{1,\ldots,6\}\backslash\{k,l\}\}$ are vertices of the induced graph $\Sigma_{\mathbf{y}^{kl}}$ and due to lemma 4.4.23 and lemma 4.4.9 we obtain that the set $\{\mathbf{x}, \mathbf{y}^{cd}, \mathbf{z}^{ij}_{kl} \mid c,d,j \in \{1,\ldots,6\}\backslash\{k,l\}, i \in \{1,2\}\} = \mathcal{V}^{kl}$ contains exactly 15 different vertices of the graph $\Sigma_{\mathbf{y}^{kl}}$. Suppose $\mathbf{p}$ is a vertex of Σ not contained in $\mathcal{V}^{kl}$ then either $\mathbf{p} = \mathbf{y}^{ef}$ with $\{e,f\} \cap \{k,l\} \neq \emptyset$ implying that $\mathbf{p} = \mathbf{y}^{ef} \not\perp \mathbf{y}^{kl}$ in Γ or $\mathbf{p} = \mathbf{z}^{st}_{mn}$ for some indices $s \in \{1,2\}$, $1 \leq m < n, t \leq 6$, $m \neq t \neq n$ with $\mathbf{p} = \mathbf{z}^{st}_{mn} \notin \{\mathbf{z}^{ij}_{kl} \mid i \in \{1,2\}; j \in \{1,\ldots,6\}\backslash\{k,l\}\}$. By lemma 4.4.23 we observe that the vertex $\mathbf{p} = \mathbf{z}^{st}_{mn}$ is adjacent to the vertex $\mathbf{y}^{kl}$ if and only if $\mathbf{p} = \mathbf{z}^{st}_{mn} = \mathbf{z}^{ef}_{kl}$ for some indices $e \in \{1,2\}, f \in \{1,\ldots,6\}$ implying that $\mathbf{p} = \mathbf{z}^{st}_{mn} \not\perp \mathbf{y}^{kl}$ in Γ, which finishes the proof of the statement. ∎

Lemma 4.4.28 *For any vertex* $\mathbf{w} \in \Sigma$ *the graph* $\Sigma_{\mathbf{w}}$ *is isomorphic to* $\mathbf{W}(A_5)$.

Proof: The statement is proved in lemma 4.4.10 for the vertex $\mathbf{x} \in \Sigma$, thus we know that $\Sigma_{\mathbf{x}} \cong \mathbf{W}(A_5)$. For a vertex $\mathbf{y}^{ij}$ with $i, j \in \{1, \dots, 6\}$, $i \neq j$ the statement follows from the construction of Σ together with lemma 4.4.23 and lemma 4.4.27. Finally, a vertex $\mathbf{z}_{cd}^{ij} \in \mathcal{V}(z)$ has the six neighbours $\mathbf{y}^{kl}$ such that either $\{k, l\} \subseteq \{c, d, j\}$ or $\{k, l\} \subseteq \{1, \dots, 6\} \backslash \{c, d, j\}$ by lemma 4.4.23 and is adjacent to each element of the vertex set $\mathcal{V}_{cd}^{ij}$ of nine elements by lemma 4.4.27. The map

$$\begin{array}{rcl} \gamma_{\mathbf{z}_{cd}^{ij}} : \Sigma_{\mathbf{z}_{cd}^{ij}} & \to & \Sigma_{\mathbf{x}} \\ \mathbf{y}^{kl} & \mapsto & \mathbf{y}^{kl} \\ \mathbf{z}_{st}^{kl} & \mapsto & \mathbf{y}^{uj} \end{array}$$

with $u = \{1, 2, j\} \backslash \{s, t\}$ if $\mathbf{z}_{cd}^{ij} = \mathbf{z}_{12}^{im}$ for $m \in \{3, \dots, 6\}$ and $u = \{g, h, n\} \backslash \{s, t\}$ if $\mathbf{z}_{cd}^{ij} = \mathbf{z}_{gh}^{mn}$ for $m, n \in \{1, 2\}$, $1 \leq g < h \leq 6$ is a bijection.

In order to prove $\Sigma_{\mathbf{z}_{cd}^{ij}} \cong \mathbf{W}(A_5)$, we will verify that $\gamma_{\mathbf{z}_{cd}^{ij}}$ is a graph homomorphism. We will illustrate the map $\gamma_{\mathbf{z}_{cd}^{ij}} : \Sigma_{\mathbf{z}_{cd}^{ij}} \to \Sigma_{\mathbf{x}}$ for the vertex $\mathbf{z}_{cd}^{ij} = \mathbf{z}_{12}^{13} = \mathbf{z}_{13}^{12} = \mathbf{z}_{23}^{11}$. Its $\mathbf{y}^{ij}$-neighbours are $\mathbf{y}^{12}, \mathbf{y}^{13}, \mathbf{y}^{23}, \mathbf{y}^{45}, \mathbf{y}^{46}, \mathbf{y}^{56}$ and its $\mathbf{z}_{st}^{kl}$-neighbours plus their images are

$$\begin{array}{cccc} \mathbf{z}_{12}^{24} \mapsto \mathbf{y}^{34} & \mathbf{z}_{13}^{24} \mapsto \mathbf{y}^{24} & \mathbf{z}_{23}^{24} \mapsto \mathbf{y}^{14} & \in \{\mathbf{y}^{56}\}^{\perp} \\ \mathbf{z}_{12}^{25} \mapsto \mathbf{y}^{35} & \mathbf{z}_{13}^{25} \mapsto \mathbf{y}^{25} & \mathbf{z}_{23}^{25} \mapsto \mathbf{y}^{15} & \in \{\mathbf{y}^{46}\}^{\perp} \\ \mathbf{z}_{12}^{26} \mapsto \mathbf{y}^{36} & \mathbf{z}_{13}^{26} \mapsto \mathbf{y}^{26} & \mathbf{z}_{23}^{26} \mapsto \mathbf{y}^{16} & \in \{\mathbf{y}^{45}\}^{\perp} \\ \in \{\mathbf{y}^{12}\}^{\perp} & \in \{\mathbf{y}^{13}\}^{\perp} & \in \{\mathbf{y}^{23}\}^{\perp} & \end{array} .$$

In general, the map $\Sigma_{\mathbf{z}_{cd}^{ij}} \to \Sigma_{\mathbf{x}}$ looks as follows:
if $\mathbf{z}_{cd}^{ij} = \mathbf{z}_{12}^{im} = \mathbf{z}_{2m}^{i1} = \mathbf{z}_{1m}^{i2}$ for some $m \in \{3, \dots, 6\}$ then

$$\begin{array}{cccc} \mathbf{z}_{12}^{ru} \mapsto \mathbf{y}^{um} & \mathbf{z}_{1m}^{ru} \mapsto \mathbf{y}^{2u} & \mathbf{z}_{2m}^{ru} \mapsto \mathbf{y}^{1u} & \in \{\mathbf{y}^{vw}\}^{\perp} \\ \mathbf{z}_{12}^{rv} \mapsto \mathbf{y}^{vm} & \mathbf{z}_{1m}^{rv} \mapsto \mathbf{y}^{2v} & \mathbf{z}_{2m}^{rv} \mapsto \mathbf{y}^{1v} & \in \{\mathbf{y}^{uw}\}^{\perp} \\ \mathbf{z}_{12}^{rw} \mapsto \mathbf{y}^{wm} & \mathbf{z}_{1m}^{rw} \mapsto \mathbf{y}^{2w} & \mathbf{z}_{2m}^{rw} \mapsto \mathbf{y}^{1w} & \in \{\mathbf{y}^{uv}\}^{\perp} \\ \in \{\mathbf{y}^{12}\}^{\perp} & \in \{\mathbf{y}^{1m}\}^{\perp} & \in \{\mathbf{y}^{2m}\}^{\perp} & \end{array} ,$$

where $r \in \{1, 2\} \backslash \{i\}$, $\{u, v, w\} = \{3, \dots, 6\} \backslash \{m\}$.
If otherwise $\mathbf{z}_{cd}^{ij} = \mathbf{z}_{gh}^{mn} = \mathbf{z}_{nh}^{rg} = \mathbf{z}_{ng}^{rh}$ for some indices $g, h \in \{4, 5, 6\}$, $m, n \in \{1, 2\}$ then

$$\begin{array}{cccc} \mathbf{z}_{gh}^{rw} \mapsto \mathbf{y}^{nw} & \mathbf{z}_{ng}^{iw} \mapsto \mathbf{y}^{wh} & \mathbf{z}_{nh}^{iw} \mapsto \mathbf{y}^{wg} & \in \{\mathbf{y}^{3t}\}^{\perp} \\ \mathbf{z}_{gh}^{r3} \mapsto \mathbf{y}^{n3} & \mathbf{z}_{ng}^{i3} \mapsto \mathbf{y}^{3h} & \mathbf{z}_{nh}^{i3} \mapsto \mathbf{y}^{3g} & \in \{\mathbf{y}^{wt}\}^{\perp} \\ \mathbf{z}_{gh}^{rt} \mapsto \mathbf{y}^{nt} & \mathbf{z}_{ng}^{it} \mapsto \mathbf{y}^{th} & \mathbf{z}_{nh}^{it} \mapsto \mathbf{y}^{tg} & \in \{\mathbf{y}^{w3}\}^{\perp} \\ \in \{\mathbf{y}^{uv}\}^{\perp} & \in \{\mathbf{y}^{ng}\}^{\perp} & \in \{\mathbf{y}^{nh}\}^{\perp} & \end{array} ,$$

where r, w and t are indices satisfying the conditions $r \in \{1, 2\} \backslash \{m\}$, $\{w, t\} = \{1, \dots, 6\} \backslash \{n, g, h, 3\}$.

To verify that these maps are graph homomorphisms first we consider the case that $\mathbf{z}^{ij}_{cd} = \mathbf{z}^{im}_{12}$ for some index $m \in \{3,\dots,6\}$. Let $\mathbf{h}$ and $\mathbf{q}$ be two adjacent vertices of $\Sigma_{\mathbf{z}^{im}_{12}}$ such that $\mathbf{h} = \mathbf{y}^{ab}$ for either $\{a,b\} \subseteq \{1,2,m\}$ or $\{a,b\} \subseteq \{3,\dots,6\}\backslash\{m\}$ and $\mathbf{q} = \mathbf{z}^{rn}_{ef} \in \mathcal{V}^{im}_{12}$. As $\mathbf{y}^{ab} = \mathbf{h} \perp \mathbf{q} = \mathbf{z}^{rn}_{ef}$ by assumption lemma 4.4.13 yields that either $\{a,b\} = \{e,f\}$ or $\{a,b\} \subseteq \{3,\dots,6\}\backslash\{m,n\}$ implying $n \notin \{a,b\}$. Furthermore $\gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{z}^{rn}_{ef}) = \mathbf{y}^{np}$ such that $p = \{1,2,m\}\backslash\{e,f\}$ and $\mathbf{y}^{np} \perp \mathbf{y}^{ab}$. Therefore $\{a,b\} \cap \{n,p\} = \varnothing$ implying $\gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{z}^{rn}_{ef}) \perp \mathbf{y}^{ab} = \gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{y}^{ab})$. Next let $\mathbf{h}$ and $\mathbf{q}$ be two adjacent vertices of $\mathcal{V}^{im}_{12} = \mathcal{V}^{ij}_{cd}$ in $\Sigma_{\mathbf{z}^{im}_{12}} = \Sigma_{\mathbf{z}^{ij}_{cd}}$, then by lemma 4.4.25 either $\mathbf{h} = \mathbf{z}^{rg}_{12}$ and $\mathbf{q} = \mathbf{z}^{rh}_{km}$ or $\mathbf{h} = \mathbf{z}^{rg}_{1m}$ and $\mathbf{q} = \mathbf{z}^{rh}_{2m}$ for the some indices $r \in \{1,2\}\backslash\{i\}$, $k \in \{1,2\}$, $g,h \in \{3,\dots,6\}\backslash\{m\}$, $g \neq h$. Thus $\gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{z}^{rg}_{12}) = \mathbf{y}^{mg} \perp \mathbf{y}^{qh} = \gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{z}^{rh}_{km})$ for $q \in \{1,2\}\backslash\{k\}$ and $\gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{z}^{rg}_{1m}) = \mathbf{y}^{2g} \perp \mathbf{y}^{1h} = \gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{z}^{rh}_{2m})$, which proves that $\gamma_{\mathbf{z}^{ij}_{cd}}$ is a graph homomorphism if $\mathbf{z}^{ij}_{cd} = \mathbf{z}^{im}_{12}$.

On the other hand if $\mathbf{z}^{ij}_{cd} = \mathbf{z}^{mn}_{gh}$ for some indices $m,n \in \{1,2\}$, $g,h \in \{g,h\}$ then first we consider two adjacent vertices $\mathbf{h}$ and $\mathbf{q}$ of the subgraph $\Sigma_{\mathbf{z}^{kl}_{cd}} = \Sigma_{\mathbf{z}^{mn}_{gh}}$ such that $\mathbf{h} = \mathbf{y}^{ab}$ with $a \neq b$ and either $\{a,b\} \subseteq \{n,g,h\}$ or $\{a,b\} \subseteq \{1,\dots,6\}\backslash\{n,g,h\}$ and $\mathbf{q} = \mathbf{z}^{uv}_{ef} \in \mathcal{V}^{mn}_{gh}$. By assumption $\mathbf{h} \perp \mathbf{q}$ in the subgraph $\Sigma_{\mathbf{z}^{kl}_{cd}}$ thus due to lemma4.4.26 $\{a,b\} = \{e,f\}$ or $\{a,b\} \subseteq \{1,\dots,6\}\backslash\{n,g,h,v\}$ implying that $v \in \{a,b\}$. As $\gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{z}^{uv}_{ef}) = \mathbf{y}^{vw}$ for $w \in \{n,g,h\}\backslash\{e,f\}$ we have $\{a,b\} \cap \{v,w\} = \varnothing$, hence $\gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{z}^{vu}_{ef}) \perp \mathbf{y}^{ab} = \gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{y}^{ab})$.

The other case is that $\mathbf{h}$ and $\mathbf{q}$ are two adjacent vertices of $\mathcal{V}^{ij}_{cd} = \mathcal{V}^{mn}_{gh}$. Then either $\mathbf{h} = \mathbf{z}^{rv}_{gh}$ and $\mathbf{q} = \mathbf{z}^{nl}_{mk}$ or $\mathbf{h} = \mathbf{z}^{mv}_{ng}$ and $\mathbf{q} = \mathbf{z}^{mk}_{nh}$ for $r \in \{1,2\}\backslash\{m\}$, $l \in \{g,h\}$, $k,v \in \{1,\dots,6\}\backslash\{n,g,h\}$, $k \neq v$ by lemma 4.4.26. It follows that $\gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{z}^{rv}_{gh}) = \mathbf{y}^{mv} \perp \mathbf{y}^{kw} = \gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{z}^{mk}_{nl})$ for $w \in \{g,h\}\backslash\{l\}$ and $\gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{z}^{mv}_{ng}) = \mathbf{y}^{vh} \perp \mathbf{y}^{gk} = \gamma_{\mathbf{z}^{ij}_{cd}}(\mathbf{z}^{mk}_{nh})$, thus $\gamma_{\mathbf{z}^{ij}_{cd}}$ is also a graph homomorphism if $\mathbf{z}^{ij}_{cd} = \mathbf{z}^{mn}_{gh}$. ■

To have better control over this proposition and its proof in the appendix C all graph isomorphism $\gamma_{\mathbf{z}^{kl}_{cd}}$ are listed for a vertex $\mathbf{z}^{kl}_{cd} \in \mathcal{V}(z)$.

Proposition 4.4.29 *Suppose* $\mathbf{z}^{i3}_{12} = \mathbf{z}^{i4}_{56}$ *for* $i \in \{1,2\}$. *Then* $\Sigma \cong \mathbf{W}(E_6)$.

Proof: By lemma 4.4.28 the graph Σ is locally $\mathbf{W}(A_5)$. By corollary 4.4.22 the graph Σ has 36 vertices. Hence by [17, theorem 2] or [49, theorem 2] the graph Σ is isomorphic to $\mathbf{W}(E_6)$. ■

Altogether we have proved the following in this section.

Proposition 4.4.30 *The graph* Σ *is isomorphic to either* $\mathbf{W}(A_7)$ *or* $\mathbf{W}(E_6)$.

4.5 Properties of the group $G_{\widehat{\Gamma}}$

In section 4.3 we have defined the big subgroup $G_{\widehat{\Gamma}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{x}} \mid \mathbf{x} \in \widehat{\Gamma} \rangle$ of the automorphism group $\mathrm{Aut}(\widehat{\Gamma})$, where for each vertex $\mathbf{x}$ of the graph $\widehat{\Gamma}$ the group $\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}} \leq \mathrm{Aut}(\widehat{\Gamma})$ has the following properties:

- $\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}} \cong \mathrm{SU}_2(\mathbb{C})$
- $\alpha(\mathbf{x}) = \mathbf{x}$ for each $\alpha \in \mathrm{SU}_2(\mathbb{C})_{\mathbf{x}}$
- every α in $\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}}$ fixes the vector space structure $V(\mathcal{G}_{\mathbf{x}})$ constructed from the subgraph $\mathbf{x}^{\perp}$ elementwise
- $\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}}$ induces the natural action of a fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroup on the two-dimensional subspace $x_{\mathbf{y}}$ of the complex vector space $V(\mathcal{G}_{\mathbf{y}})$ for any vertex $\mathbf{y} \in \mathbf{x}^{\perp}$ of $\widehat{\Gamma}$.

In this section we collect some properties of the group $G_{\widehat{\Gamma}}$. In particular, we show that the group $G_{\widehat{\Gamma}}$ acts transitively on the set of vertices of $\widehat{\Gamma}$ and on the set of ordered edges of $\widehat{\Gamma}$. Before we start to prove these properties we observe some local features of the group $G_{\widehat{\Gamma}}$.

We denote by $\mathrm{GL}_n(\mathbb{C})$ the **complex general linear group** consisting of all complex non-singular $n \times n$-matrices, which is a subset of $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$. Hence $\mathrm{GL}_n(\mathbb{C})$ is a topological real group in a natural way. The group

$$\mathrm{SU}_8(\mathbb{C}) = \{X \in \mathrm{GL}_n(\mathbb{C}) \mid X \cdot X^* = I, \det(X) = 1\}$$

is a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ with respect to this topology, where $X^* = \overline{X}^t$.

If A is a complex $n \times n$-matrix, then we define

$$\exp(A) = e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n,$$

the **exponential of the matrix** A. It is well know that the exponential function of matrices has the following algebraic properties.

Proposition 4.5.1 (Proposition 0.11 of [63]) *For any complex $n \times n$ matrices X and Y:*

- $e^X \cdot e^Y = e^{X+Y}$ *if X and Y commute,*
- e^X *is a nonsingular matrix,*

- $t \mapsto e^{tX}$ *is a smooth curve into* $\mathrm{GL}_n(\mathbb{C})$ *with* $e^{0 \cdot X} = I$,
- $\frac{d}{dt} e^{tX} = X e^{tX}$,
- $\det(e^X) = e^{\mathrm{tr}(X)}$, *where* $\mathrm{tr}(X) = \sum_{i=1}^n x_{ii}$, *is the trace of* X,
- $X \mapsto e^X$ *is a* $\mathcal{C}^\infty$ *mapping from the matrix space into itself.*

Let G be a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$. Then we relate to G the set of matrices $\mathfrak{g} = L(G) = \{X \in M_n(\mathbb{C}) \mid \exp(\mathbb{R}X) \subseteq G\}$, where $M_n(\mathbb{C})$ is the set of all complex $n \times n$ matrices. The set $\mathfrak{g}$ is a vector space over $\mathbb{C}$ and we define $[X, Y] = XY - YX$ for any two $X, Y \in \mathfrak{g}$. Now we turn to the notion of a Lie algebra.

A vector space V over $\mathbb{F}$ with an operation $[\cdot,\cdot] : V \times V \to V$, called **Lie bracket**, that is linear in each variable and satisfies the properties

- $[x, x] = 0$ for all $x \in V$, hence $[x, y] = -[y, x]$,
- $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in V$, the Jacobi identity,

is called an $\mathbb{F}$-**Lie algebra**.

Here are some very elementary definitions and notations of a Lie algebra, which we will need later. A **homomorphism** of Lie algebras is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ such that $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in \mathfrak{g}$. Of course an **isomorphism** is a one-to-one and onto homomorphism. If $\mathfrak{a}$ and $\mathfrak{b}$ are subsets of a Lie algebra $\mathfrak{g}$ then we write $[\mathfrak{a}, \mathfrak{b}] = \{[x, y] \mid x \in \mathfrak{a}, y \in \mathfrak{b}\}$. A **Lie subalgebra** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ satisfying $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, in particular a Lie subalgebra is itself a Lie algebra. An **ideal** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace of $\mathfrak{g}$ with the property that $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. We call a Lie algebra $\mathfrak{g}$ **abelian** if $[\mathfrak{g}, \mathfrak{g}] = 0$. A Lie algebra $\mathfrak{g}$ is defined to be **nilpotent** if there exists a decreasing finite sequence of ideals $(\mathfrak{g}_i)_{0 \le i \le l}$ with $\mathfrak{g} = \mathfrak{g}_0$, $\mathfrak{g}_l = 0$ such that $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+1}$ for $1 \le i \le l$. We say a Lie algebra $\mathfrak{g}$ is **solvable** if there exits a finite chain of subalgebras $\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_l = 0$ such that $[\mathfrak{g}_i, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+1}$. Certainly a nilpotent Lie algebra is solvable. Let $\mathfrak{g}$ be a Lie algebra, then for a subset $\mathfrak{s}$ of $\mathfrak{g}$ we call $Z_{\mathfrak{g}}(\mathfrak{s}) = \{x \in \mathfrak{g} \mid [x, s] = 0 \text{ for all } s \in \mathfrak{s}\}$ the **centraliser of** $\mathfrak{s}$ **in** $\mathfrak{g}$. Moreover for a Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ we denote with $N_{\mathfrak{g}}(\mathfrak{s}) = \{x \in \mathfrak{g} \mid [x, s] \in \mathfrak{s} \text{ for all } s \in \mathfrak{s}\}$ the **normaliser of** $\mathfrak{s}$ **in** $\mathfrak{g}$.

Accordingly for any closed linear subgroup G of $\mathrm{GL}_n(\mathbb{C})$ we call $\mathfrak{g} = L(G)$ the **Lie algebra of** G .

The complex Lie algebra of $\mathrm{GL}_n(\mathbb{C})$ is the vector space $\mathfrak{gl}_n(\mathbb{C})$ of all complex $n \times n$-matrices and the real Lie algebra of $\mathrm{SU}_n(\mathbb{C})$ is the space $\mathfrak{su}_n(\mathbb{C}) = \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid X + X^* = 0, tr(X) = 0\}$, where $X^* = \overline{X}^t$ for every complex $n \times n$-matrix. From this description, it is clear that $\dim_{\mathbb{R}}(\mathfrak{su}_n(\mathbb{C})) = n^2 - 1$.

One can verify that the three matrices

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

generate the real Lie algebra $\mathfrak{su}_2(\mathbb{C})$. A similar set of generators can be given for the real Lie algebra $\mathfrak{su}_n(\mathbb{C})$. If $X = (x_{st})_{1\le s,t\le n} \in \mathfrak{su}_n(\mathbb{C})$, then for each $1 \le s \le n$ we have $x_{ss} = \lambda i$ for some $\lambda \in \mathbb{R}$, while for any $1 \le s < t \le n$ we have $x_{st} = -\overline{x}_{st}$. By dimension, the following $n^2 - 1$ matrices, $1 \le k < l \le n$, $1 \le m \le n-1$, form a basis of the vector space $\mathfrak{su}_n(\mathbb{C})$, if they are $\mathbb{R}$-linearly independent:

$$A_{kl} = (a_{st})_{1\le s,t\le n} \text{ with } a_{st} = \begin{cases} 1 & \text{if } s = k, t = l \\ -1 & \text{if } s = l, t = k \\ 0 & \text{otherwise} \end{cases},$$

$$B_{kl} = (b_{st})_{1\le s,t\le n} \text{ with } b_{st} = \begin{cases} i & \text{if } s = k, t = l \\ i & \text{if } s = l, t = k \\ 0 & \text{otherwise} \end{cases} \text{ and}$$

$$C_{mm+1} = \operatorname{diag}(c_1, \dots, c_n) \text{ with } c_j = \begin{cases} i & \text{if } j = m \\ -i & \text{if } j = m+1 \\ 0 & \text{if } j \notin \{m, m+1\} \end{cases}.$$

To check that these matrices are linearly independent over $\mathbb{R}$, suppose

$$\sum_{1\le k<l\le n} \lambda_{kl} A_{kl} + \sum_{1\le d<e\le n} \mu_{de} B_{de} + \sum_{1=r}^{n-1} \delta_r C_{rr+1} = H = (h_{st})_{1\le s,t\le n} = 0$$

for some scalars $\lambda_{kl}, \mu_{de}, \delta_r \in \mathbb{C}$. Then for each pair of indices s, t with $1 \le s, t \le n$ we get

$$\sum_{1\le k<l\le n} \lambda_{kl} (a_{st})_{kl} + \sum_{1\le d<e\le n} \mu_{de} (b_{st})_{de} + \sum_{1=r}^{n-1} \delta_r (c_{st})_{rr+1} = h_{st},$$

so that

$$0 = h_{st} = \begin{cases} \lambda_{kl}(a_{st})_{kl} + \mu_{kl}(b_{st})_{kl} & \text{if } s \neq t, \{s,t\} = \{k,l\} \\ \delta_r (c_{st})_{rr+1} & \text{if } s = t \end{cases}.$$

Thus, $\delta_r = 0$ for $1 \le r \le n-1$. Moreover, $\lambda_{kl} + \mu_{kl} i = 0$ for $\lambda_{kl}, \mu_{kl} \in \mathbb{R}$ implies $\lambda_{kl} = \mu_{kl} = 0$ for all $1 \le k < l \le n$ and we have proved $\mathbb{R}$-linear independence.

Proposition 4.5.2 (Corollary 0.20 of [63]) *If G is a closed linear subgroup of a Lie group and $\mathfrak{g}$ is its Lie algebra, then $\exp(\mathfrak{g})$ generates the identity component G° of G.*

Proposition 4.5.3 *The group $\mathrm{SU}_n(\mathbb{C})$ is generated by the set of matrices*

$$\{e^{\lambda A_{k,l}}, e^{\lambda B_{k,l}}, e^{\lambda C_r} \mid 1 \leq k, l \leq n; 1 \leq r \leq n-1; \lambda \in \mathbb{R}\}.$$

Coming back to the graph $\widehat{\Gamma}$ and the group $G_{\widehat{\Gamma}}$, let $\mathbf{y}$ be a vertex of $\widehat{\Gamma}$. We define $U_{\mathbf{y}}$ to be the subgroup $U_{\mathbf{y}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{y}^\perp \rangle$ of $G_{\widehat{\Gamma}}$. For each vertex $\mathbf{z}$ of $\mathbf{y}^\perp$ the subgroup $\mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \leq U_{\mathbf{y}}$ acts as the identity on the subgraph $\mathbf{z}^\perp$. Furthermore, $(\mathrm{SU}_2(\mathbb{C})_{\mathbf{z}})_{|\mathbf{y}^\perp} = \mathbf{z}\text{-}\mathrm{SU}_2(\mathbb{C})\text{-}\mathbf{y}^\perp$ by lemma 4.3.17. Notice that, in particular, the action $\mathbf{z}\text{-}\mathrm{SU}_2(\mathbb{C})\text{-}\mathbf{y}^\perp$ on the graph $\mathbf{y}^\perp$ is induced by the natural $\mathbf{z}\text{-}\mathrm{SU}_2(\mathbb{C})\text{-}\mathcal{G}_{\mathbf{y}}$ action on the projective space $\mathcal{G}_{\mathbf{y}}$ with respect to the line $z_{\mathbf{y}}$.

Lemma 4.5.4 *Let $\mathbf{y}$ be a vertex of $\widehat{\Gamma}$ and let $\delta : v_1, \ldots, v_6$ be an orthonormal basis of the unitary vector space $V(\mathcal{G}_{\mathbf{y}})$. Furthermore, let $l_{\mathbf{y}}^{st} = \langle v_s, v_t \rangle$ for $1 \leq s < t \leq 6$. Each line $l_{\mathbf{y}}^{st}$ belongs to a vertex of the subgraph $\mathbf{y}^\perp$ of $\widehat{\Gamma}$, thus $\langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{l}_{st}} \mid 1 \leq s < t \leq 6 \rangle$ is a subgroup of $\langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{y}^\perp \rangle$.*

The following groups are isomorphic:

$$\begin{aligned} \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{l}_{st}} \mid 1 \leq s < t \leq 6 \rangle &\cong \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{y}^\perp \rangle \\ &\cong \langle (\mathrm{SU}_2(\mathbb{C})_{\mathbf{z}})_{|\mathbf{y}^\perp} \mid \mathbf{z} \in \mathbf{y}^\perp \rangle \\ &\cong \langle (\mathrm{SU}_2(\mathbb{C})_{\mathbf{l}_{st}})_{|\mathbf{y}^\perp} \mid 1 \leq s < t \leq 6 \rangle \\ &\cong \mathrm{SU}_6(\mathbb{C}). \end{aligned}$$

Proof: Let γ be the restriction map $\langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{y}^\perp \rangle \to \langle (\mathrm{SU}_2(\mathbb{C})_{\mathbf{z}})_{|\mathbf{y}^\perp} \mid \mathbf{z} \in \mathbf{y}^\perp \rangle$ with $\gamma(\varphi) = \varphi_{|\mathbf{y}^\perp}$, which certainly is a surjective group homomorphism. The kernel of the restriction map γ contains all elements $\varphi \in \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{y}^\perp \rangle$ such that $\varphi_{|\mathbf{y}^\perp} = \mathrm{id}_{\mathbf{y}^\perp}$. If $\varphi \in \ker(\gamma)$, then $\varphi_{\mathbf{y}^\perp} = \mathrm{id}_{|\mathbf{y}^\perp}$, so that φ fixes the vector space structure of the vertex $\mathbf{y}$ elementwise, whence $\varphi = \mathrm{id}_{\widehat{\Gamma}}$ by lemma 4.3.16. We conclude that γ is a group isomorphism, thus

$$\langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{y}^\perp \rangle \cong \langle (\mathrm{SU}_2(\mathbb{C})_{\mathbf{z}})_{|\mathbf{y}^\perp} \mid \mathbf{z} \in \mathbf{y}^\perp \rangle.$$

With the same argument

$$\langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{l}_{st}} \mid 1 \leq s < t \leq 6 \rangle \cong \langle (\mathrm{SU}_2(\mathbb{C})_{\mathbf{l}_{st}})_{|\mathbf{y}^\perp} \mid 1 \leq s < t \leq 6 \rangle.$$

Hence it suffices to study the situation in $V(\mathcal{G}_{\mathbf{y}})$. For each line $l_{\mathbf{y}}^{st}$, we consider the $\mathbf{l}_{ij}\text{-}\mathrm{SU}_2(\mathbb{C})\text{-}V(\mathcal{G}_{\mathbf{y}})$ action. The generators $e^{\lambda A_{l_{st}}}, e^{\mu B_{l_{st}}}, e^{\varepsilon C_{l_{st}}}$ with

$$A_{l_{st}} = ((a_{l_{st}})_{nm})_{1\le n,m\le 6} \text{ with } a_{l_{st}} = \begin{cases} 1 & \text{if } m = s, n = t \\ -1 & \text{if } n = t, m = s \\ 0 & \text{otherwise} \end{cases},$$

$$B_{l_{st}} = ((b_{l_{st}})_{nm})_{1\le n,m\le 6} \text{ with } b_{l_{st}} = \begin{cases} i & \text{if } m = s, n = t \\ i & \text{if } n = t, m = s \\ 0 & \text{otherwise} \end{cases},$$

$$C_{l_{st}} = \operatorname{diag}((c_{l_{st}})_1, \dots, (c_{l_{st}})_6) \text{ with } (c_{l_{st}})_j = \begin{cases} 1 & \text{if } j = s \\ -1 & \text{if } t = s \\ 0 & \text{if } j \notin \{s,t\} \end{cases}$$

and $\lambda, \mu, \varepsilon \in \mathbb{R}$ — as defined in the beginning of this section — generate the group $\langle (\mathrm{SU}_2(\mathbb{C})_{\mathbf{l}_{st}})_{|\mathbf{y}^\perp} \mid 1 \le s < t \le 6\rangle$, so by proposition 4.5.3 this group is isomorphic to $\mathrm{SU}_6(\mathbb{C})$. The isomorphism $\langle (\mathrm{SU}_2(\mathbb{C})_{\mathbf{z}})_{|\mathbf{y}^\perp} \mid \mathbf{z} \in \mathbf{y}^\perp\rangle \cong \langle (\mathrm{SU}_2(\mathbb{C})_{\mathbf{l}_{st}})_{|\mathbf{y}^\perp} \mid 1 \le s < t \le 6\rangle$ finally follows from the observation that each $(\mathrm{SU}_2(\mathbb{C})_{\mathbf{z}})_{|\mathbf{y}^\perp}$ is already contained in $\langle (\mathrm{SU}_2(\mathbb{C})_{\mathbf{l}_{st}})_{|\mathbf{y}^\perp} \mid 1 \le s < t \le 6\rangle$. ∎

Since the subgroup $U_{\mathbf{y}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{y}^\perp\rangle$ of $G_{\widehat{\Gamma}}$ acts transitively on the ordered edges of the graph $\mathbf{y}^\perp$, we have the following result.

Lemma 4.5.5 *Let* $\mathbf{y}$ *be a vertex of the graph* $\widehat{\Gamma}$*. Then both the stabiliser* $G_{\mathbf{y}} := \mathrm{Stab}_{G_{\widehat{\Gamma}}}(\mathbf{y})$ *and the subgroup* $U_{\mathbf{y}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{y}^\perp\rangle$ *act transitively on the ordered edges of the induced subgraph* $\mathbf{y}^\perp$.

Lemma 4.5.6 *The group* $G_{\widehat{\Gamma}}$ *acts transitively on the ordered edges of the graph* $\widehat{\Gamma}$.

Proof: Since the graph $\widehat{\Gamma}$ is connected, the statement follows from the fact that the group $G_{\mathbf{y}}$ acts transitively on the set of ordered edges of the subgraph $\mathbf{y}^\perp$ for every $\mathbf{y} \in \widehat{\Gamma}$, cf. lemma 4.5.5. ∎

Lemma 4.5.7 *The group* $G_{\widehat{\Gamma}}$ *acts transitively on the vertices of* $\widehat{\Gamma}$.

Proof: Let $\mathbf{x}$ and $\mathbf{y}$ be two vertices of $\widehat{\Gamma}$ and let $(\mathbf{x}, \mathbf{x}')$ and $(\mathbf{y}, \mathbf{y}')$ be ordered edges of $\widehat{\Gamma}$. By lemma 4.5.6 there exists an element of $G_{\widehat{\Gamma}}$ mapping $(\mathbf{x}, \mathbf{x}')$ to $(\mathbf{y}, \mathbf{y}')$ and, hence $\mathbf{x}$ to $\mathbf{y}$. ∎

In the next part we study some special subgroups of $G_{\widehat{\Gamma}}$.

Lemma 4.5.8 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two adjacent vertices of* $\widehat{\Gamma}$*. Then*

$$\langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{x}}, \mathrm{SU}_2(\mathbb{C})_{\mathbf{y}}\rangle = \mathrm{SU}_2(\mathbb{C})_{\mathbf{x}} \times \mathrm{SU}_2(\mathbb{C})_{\mathbf{y}} \cong \mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C}).$$

Proof: Since $\widehat{\Gamma}$ is locally $\mathbf{S}(V_6)$, we find a vertex $\mathbf{z} \in \mathbf{x}^\perp \cap \mathbf{y}^\perp$. By lemma 4.5.4, we can study $(\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}})_{|\mathbf{z}^\perp}$ and $(\mathrm{SU}_2(\mathbb{C})_{\mathbf{y}})_{|\mathbf{z}^\perp}$ instead. There the claim is obviously true. ∎

Lemma 4.5.9 *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be two different vertices of the induced subgraph* $\mathbf{z}^{\perp}$ *for some* $\mathbf{z} \in \widehat{\Gamma}$. *If the lines* $x_{\mathbf{z}}$ *and* $y_{\mathbf{z}}$ *intersect in a point and* $x_{\mathbf{z}} \cap y_{\mathbf{z}}^{\pi} = p_{\mathbf{z}}$ *is one-dimensional, then* $G_{xy}^{\mathbf{z}} := \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{x}}, \mathrm{SU}_2(\mathbb{C})_{\mathbf{y}} \rangle \cong SU_3(\mathbb{C})$ *and* $Z_{G_{xy}^{\mathbf{z}}}(v) = \mathrm{SU}_2(\mathbb{C})_{\mathbf{y}}$, *where* $v \in p_{\mathbf{z}} \subseteq V(\mathcal{G}_{\mathbf{z}})$ *is normal.*

Proof: By the proof of lemma 4.5.4 we have $\mathrm{SU}_2(\mathbb{C}) \cong \mathrm{SU}_2(\mathbb{C})_{\mathbf{x}} \cong (\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}})_{|\mathbf{z}^{\perp}}$ and $\mathrm{SU}_2(\mathbb{C}) \cong \mathrm{SU}_2(\mathbb{C})_{\mathbf{y}} \cong (\mathrm{SU}_2(\mathbb{C})_{\mathbf{y}})_{|\mathbf{z}^{\perp}}$, so that we can study the situation in $V(\mathcal{G}_{\mathbf{z}})$. Fix an orthonormal basis $\delta : v_1, \ldots, v_6$ of $V(\mathcal{G}_{\mathbf{z}})$ with $\langle v_1, v_2 \rangle = x_{\mathbf{z}}$ and $\langle v_2, v_3 \rangle = y_{\mathbf{z}}$ and $\langle v_2 \rangle = x_{\mathbf{z}} \cap y_{\mathbf{z}}$, which is possible by our assumption. By proposition 4.5.3 every element of $(\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}})_{|\mathbf{z}^{\perp}} \cong \mathrm{SU}_2(\mathbb{C})$ is generated by the elements $\{e^{\lambda A_{12}}, e^{\lambda B_{12}}, e^{\lambda C_{12}} \mid \lambda \in \mathbb{R}\}$ and any element of $(\mathrm{SU}_2(\mathbb{C})_{\mathbf{y}})_{|\mathbf{z}^{\perp}}$ is generated by the elements $\{e^{\lambda A_{23}}, e^{\lambda B_{23}}, e^{\lambda C_{23}} \mid \lambda \in \mathbb{R}\}$ with A_{kk+1}, B_{kk+1} and C_{kk+1} as in the beginning of this section for $n = 6$ and $k = 1, 2$. It follows that each element of $\langle (\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}})_{|\mathbf{z}^{\perp}}, (\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}})_{|\mathbf{z}^{\perp}} \rangle$ is a finite product of elements of the set $\{e^{\lambda A_{12}}, e^{\lambda B_{12}}, e^{\lambda C_{12}}, e^{\lambda A_{23}}, e^{\lambda B_{23}}, e^{\lambda C_{23}} \mid \lambda \in \mathbb{R}\}$. Fix isomorphisms $\varphi, \mu \in \mathrm{SU}_2(\mathbb{C})_{\mathbf{y}}$ such that

$$ {}_{v_1,v_2,v_3}[\varphi]_{v_1,v_2,v_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} =: T $$

and

$$ {}_{v_1,v_2,v_3}[\mu]_{v_1,v_2,v_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} =: S. $$

Since $Te^{\lambda A_{12}} S = e^{\lambda T A_{12} S} = e^{\lambda A_{13}}$ and $Te^{\mu B_{12}} S = e^{\mu T B_{12} S} = e^{\mu B_{13}}$, it follows that

$$ \begin{aligned} & \langle (\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}})_{|\mathbf{z}^{\perp}}, (\mathrm{SU}_2(\mathbb{C})_{\mathbf{y}})_{|\mathbf{z}^{\perp}} \rangle \\ = \; & \langle e^{\lambda A_{12}}, e^{\mu B_{12}}, e^{\varepsilon C_{12}}, e^{\lambda A_{23}}, e^{\mu B_{23}}, e^{\varepsilon C_{23}} \mid \lambda, \mu, \varepsilon \in \mathbb{R} \rangle \\ = \; & \langle e^{\lambda A_{12}}, e^{\mu B_{12}}, e^{\varepsilon C_{12}}, e^{\lambda A_{23}}, e^{\mu B_{23}}, e^{\varepsilon C_{23}}, e^{\lambda A_{13}}, e^{\mu B_{13}} \mid \lambda, \mu, \varepsilon \in \mathbb{R} \rangle \\ \cong \; & \mathrm{SU}_3(\mathbb{C}) \end{aligned} $$

by proposition 4.5.3.

So it remains to study $Z_{G_{xy}^{\mathbf{z}}}(v)$, which by lemma 4.5.4 we can do via the action of $G_{xy}^{\mathbf{z}}$ on $\langle x_{\mathbf{z}}, y_{\mathbf{z}} \rangle \subseteq V(\mathcal{G}_{\mathbf{z}})$. Choose a normal vector $w \in x_{\mathbf{z}} \cap y_{\mathbf{z}}$ and a normal vector $u \in w^{\pi} \cap y_{\mathbf{z}}$. Then v, w, u is an orthonormal basis of $\langle x_{\mathbf{z}}, y_{\mathbf{z}} \rangle$. The claim now follows from the observation that the centraliser of v consists of matrices of the form

$$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} $$

with respect to the basis v, w, u. ∎

The group $G_{\widehat{\Gamma}}$ is a subgroup of Aut $(\widehat{\Gamma})$. Therefore we obtain the group homomorphism $\upsilon_g : G_{\widehat{\Gamma}} \to G_{\widehat{\Gamma}}$ with $\upsilon_g(h) = ghg^{-1}$ for some $g \in G_{\widehat{\Gamma}}$. Let $\gamma \in G_{\widehat{\Gamma}}$ then $\gamma(\mathbf{y}^{\perp}) = \gamma(\mathbf{y})^{\perp}$ which gives arise to the isomorphism $\gamma : \mathcal{G}_{\mathbf{y}} \to \mathcal{G}_{\gamma(\mathbf{y})}$ by proposition 4.2.3 for every vertex $\mathbf{y} \in \widehat{\Gamma}$.

Lemma 4.5.10 *Let* $\mathbf{x}$ *be a vertex of* $\in \widehat{\Gamma}$ *and* γ *be an element of group* $G_{\widehat{\Gamma}}$, *then* $\upsilon_\gamma(\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}}) = \gamma(\mathrm{SU}_2(\mathbb{C})_{\mathbf{x}})\gamma^{-1} = \mathrm{SU}_2(\mathbb{C})_{\gamma(\mathbf{x})}$.

Proof: Let $\alpha_{\mathbf{x},\varphi} \in \mathrm{SU}_2(\mathbb{C})_{\mathbf{x}}$ be the automorphism induced by the map $\varphi_{\mathcal{G}_{\mathbf{y}}} = \mathbf{x}\text{-}\varphi_{\mathcal{G}_{\mathbf{y}}}$ for some $\mathbf{y} \in \mathbf{x}^{\perp}$ and an element $\varphi \in \mathrm{SU}_2(\mathbb{C})$. We set $\gamma(\mathbf{x}) = \mathbf{z}$ and regard that $\upsilon_\gamma(\alpha_{\varphi,\mathbf{x}})(\mathbf{z}) = (\gamma\alpha_{\varphi,\mathbf{x}}\gamma^{-1})(\mathbf{z}) = (\gamma\alpha_{\varphi,\mathbf{x}})(\gamma^{-1}\mathbf{z}) = \gamma\alpha_{\varphi,\mathbf{x}}\mathbf{x} = \mathbf{z}$. Furthermore for every vertex $\mathbf{w} \in \mathbf{z}^{\perp}$ we obtain that $\upsilon_\gamma(\alpha_{\varphi,\mathbf{x}})(\mathbf{w}) = (\gamma\alpha_{\varphi,\mathbf{x}}\gamma^{-1})(\mathbf{w}) = (\gamma\alpha_{\varphi,\mathbf{x}})(\gamma^{-1}\mathbf{w}) = \gamma\gamma^{-1}\mathbf{w} = \mathbf{w}$ as $\gamma^{-1}(\mathbf{w}) = \upsilon_{\gamma^{-1}}(\mathbf{w}) \in \mathbf{x}^{\perp}$ and $\alpha_{\varphi,\mathbf{x}}(\mathbf{a}) = \mathbf{a}$ for each $\mathbf{a} \in \mathbf{x}^{\perp}$.

The next step is to prove that $\upsilon_\gamma(\alpha_{\varphi,\mathbf{x}}) = \alpha_{\varphi,\mathbf{z}}$, where $\alpha_{\varphi,\mathbf{z}}$ is the element of $\mathrm{SU}_2(\mathbb{C})_{\mathbf{z}}$ induced by $\varphi_{\mathcal{G}_{\mathbf{y}}} = \mathbf{x}\text{-}\varphi_{\mathcal{G}_{\mathbf{y}}}$ for some $\mathbf{y} \in \mathbf{z}^{\perp}$ and an element $\varphi \in \mathrm{SU}_2(\mathbb{C})$. From above we verify that $\upsilon_\gamma(\alpha_{\varphi,\mathbf{x}})_{|\mathcal{G}_{\mathbf{z}}} = \gamma\alpha_{\varphi,\mathbf{x}}\gamma^{-1}_{|\mathcal{G}_{\mathbf{z}}} = \gamma(\alpha_{\varphi,\mathbf{x}})_{|\mathcal{G}_{\mathbf{x}}} = \gamma(\mathbf{x}\text{-}\varphi_{\mathcal{G}_{\mathbf{x}}}) = \gamma(\mathrm{id}_{\mathcal{G}_{\mathbf{x}}}) = \mathrm{id}_{\mathcal{G}_{\mathbf{z}}} = \mathbf{z} - \varphi_{\mathcal{G}_{\mathbf{z}}}$ and $\upsilon_\gamma(\alpha_{\varphi,\mathbf{x}})_{|\mathcal{G}_{\mathbf{w}}} = \gamma\alpha_{\varphi,\mathbf{x}}\gamma^{-1}_{|\mathcal{G}_{\mathbf{w}}} = \gamma(\alpha_{\varphi,\mathbf{x}})_{|\mathcal{G}_{\gamma^{-1}(\mathbf{w})}} = \gamma(\mathbf{x}\text{-}\varphi_{\mathcal{G}_{\gamma^{-1}(\mathbf{w})}}) = \mathbf{z} - \varphi_{\mathcal{G}_{\mathbf{w}}}$ for some $\mathbf{w} \in \mathbf{x}^{\perp}$. Finally let $\mathbf{v}$ be a vertex of $\widehat{\Gamma}$, then we conclude that $\upsilon_\gamma(\alpha_{\varphi,\mathbf{x}})_{|\mathcal{G}_{\mathbf{v}}} = \gamma\alpha_{\varphi,\mathbf{x}}\gamma^{-1}_{|\mathcal{G}_{\mathbf{v}}} = \gamma(\alpha_{\varphi,\mathbf{x}})_{|\mathcal{G}_{\gamma^{-1}(\mathbf{u})}} = \gamma(\mathbf{x}\text{-}\varphi_{\mathcal{G}_{\gamma^{-1}(\mathbf{u})}}) = \mathbf{z} - \varphi_{\mathcal{G}_{\mathbf{v}}}$. ■

4.6 The identification of the group $G_{\widehat{\Gamma}} \leq \mathrm{Aut}(\widehat{\Gamma})$

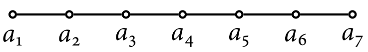
Recall the construction of the induced subgraph Σ of $\widehat{\Gamma}$ from section 4.4. By proposition 4.4.30 the graph Σ is isomorphic to either $\mathbf{W}(A_7)$ or $\mathbf{W}(E_6)$. The goal of this section is to identify the group $G_{\widehat{\Gamma}}$ from the graph Σ, which we will achieve by the case distinction $\Sigma \cong \mathbf{W}(A_7)$ or $\Sigma \cong \mathbf{W}(E_6)$.

For that task we will consider a certain induced subgraph of Σ.

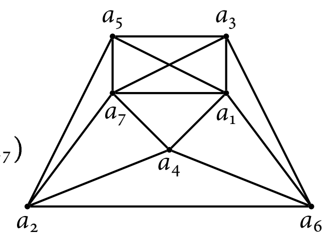
Definition 4.6.1 Let Φ be a root system with basis $\psi : \alpha_1, \dots, \alpha_l$ and Dynkin diagram (resp. Coxeter graph) of type Δ. Then we call the graph $\mathbf{H}(\Delta)$ on the vertex set $\{\alpha_1, \dots, \alpha_l\}$ where two elements α_i and α_j are joint by an edge if and only α_i and α_j are not connected by an edge in the Dynkin diagram to be the **graph associated to the Dynkin diagram** (resp. to the Coxeter graph) of type Δ.

The graph $\mathbf{H}(A_7)$ has the seven vertices $a_1, \dots a_7$ such that the vertex a_1 is adjacent to the five vertices $a_3, \dots, a_7$, the five vertices $a_1, \dots, a_5$ are neighbours of the vertex a_7 and for $i \in \{2, \dots, 6\}$ the vertex a_i is connected to each element of thevertex set $\{a_j \mid 1 \leq j \leq 7; i-1 \neq j \neq i+1\}$, i.e., $\mathbf{H}(A_7)$ is the complement of the following Coxeter graph of type A_7:

A_7



$\mathbf{H}(A_7)$

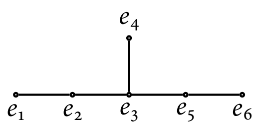


In contrast the graph $\mathbf{H}(E_6)$ has six vertices $e_1, \ldots, e_6$ and the vertices satisfy the following neighbourhood properties:

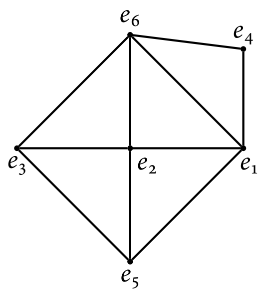
- e_1 is adjacent to the vertices e_3, e_4, e_5, e_6,
- e_6 is adjacent to the vertices e_1, e_2, e_3, e_4,
- e_2 is adjacent to the vertices e_4, e_5, e_6,
- e_5 is adjacent to the vertices e_1, e_2, e_4,
- e_3 is adjacent to the vertices e_1 and e_6
- e_4 is adjacent to the vertices e_1, e_2, e_5, e_6,

i.e., $\mathbf{H}(E_6)$ is the complement of the following Coxeter graph of type E_6.

E_6



$\mathbf{H}(E_6)$



Since the graph Σ is isomorphic to one of the reflection graphs $\mathbf{W}(A_7)$ or $\mathbf{W}(E_6)$, it contains an induced subgraph Λ isomorphic $\mathbf{H}(A_7)$, resp. $\mathbf{H}(E_6)$.

Case 1: $\Sigma \cong \mathbf{W}(A_7)$

In this case we want to construct an induced subgraph Λ of Σ isomorphic to $\mathbf{H}(A_7)$. Fix the vertices $\mathbf{w}_1 = \mathbf{x}$, $\mathbf{w}_2 = \mathbf{z}^{11}$, $\mathbf{w}_3 = \mathbf{y}^{12}$, $\mathbf{w}_4 = \mathbf{y}^{23}$, $\mathbf{w}_5 = \mathbf{y}^{34}$, $\mathbf{w}_6 = \mathbf{y}^{45}$, $\mathbf{w}_7 = \mathbf{y}^{56}$ of Σ, cf. section 4.4, and let Λ be the induced subgraph of Σ on these vertices.

Thus by section 4.4 we conclude that

$\Lambda_{\mathbf{x}}$ is the induced subgraph of Λ on the vertex set $\mathbf{y}^{12}, \mathbf{y}^{23}, \mathbf{y}^{34}, \mathbf{y}^{45}, \mathbf{y}^{56}$,

$\Lambda_{\mathbf{z}^{11}}$ is the induced subgraph of Λ on the vertex set $\mathbf{y}^{23}, \mathbf{y}^{34}, \mathbf{y}^{45}, \mathbf{y}^{56}$,

$\Lambda_{\mathbf{y}^{12}}$ is the induced subgraph of Λ on the vertex set $\mathbf{x}, \mathbf{y}^{34}, \mathbf{y}^{45}, \mathbf{y}^{56}$,

$\Lambda_{\mathbf{y}^{23}}$ is the induced subgraph of Λ on the vertex set $\mathbf{x}, \mathbf{y}^{45}, \mathbf{y}^{56}, \mathbf{z}^{11}$,

$\Lambda_{\mathbf{y}^{34}}$ is the induced subgraph of Λ on the vertex set $\mathbf{x}, \mathbf{y}^{12}, \mathbf{y}^{56}, \mathbf{z}^{11}$,

$\Lambda_{\mathbf{y}^{45}}$ is the induced subgraph of Λ on the vertex set $\mathbf{x}, \mathbf{y}^{12}, \mathbf{y}^{23}, \mathbf{z}^{11}$,

$\Lambda_{\mathbf{y}^{56}}$ is the induced subgraph of Λ on the vertex set $\mathbf{x}, \mathbf{y}^{12}, \mathbf{y}^{23}, \mathbf{y}^{34}, \mathbf{z}^{11}$.

Lemma 4.6.2 *We have* $\Lambda \cong \mathbf{H}(A_7)$.

Proof: The map $\varphi : \Lambda \to \mathbf{H}(A_7) : \varphi(\mathbf{w}_i) = a_i$, $1 \le i \le 7$, is a graph isomorphism. Indeed, the list

$$\begin{array}{rcl}
\varphi(\{\mathbf{y}^{12}, \mathbf{y}^{23}, \mathbf{y}^{34}, \mathbf{y}^{45}, \mathbf{y}^{56}\}) & = & \{a_3, a_4, a_5, a_6, a_7\}, \\
\varphi(\{\mathbf{y}^{23}, \mathbf{y}^{34}, \mathbf{y}^{45}, \mathbf{y}^{56}\}) & = & \{a_4, a_5, a_6, a_7\}, \\
\varphi(\{\mathbf{x}, \mathbf{y}^{34}, \mathbf{y}^{45}, \mathbf{y}^{56}\}) & = & \{a_1, a_5, a_6, a_7\}, \\
\varphi(\{\mathbf{x}, \mathbf{y}^{45}, \mathbf{y}^{56}, \mathbf{z}^{11}\}) & = & \{a_1, a_6, a_7, a_2\}, \\
\varphi(\{\mathbf{x}, \mathbf{y}^{12}, \mathbf{y}^{56}, \mathbf{z}^{11}\}) & = & \{a_1, a_3, a_7, a_2\}, \\
\varphi(\{\mathbf{x}, \mathbf{y}^{12}, \mathbf{y}^{23}, \mathbf{z}^{11}\}) & = & \{a_1, a_3, a_4, a_2\}, \\
\varphi(\{\mathbf{x}, \mathbf{y}^{12}, \mathbf{y}^{23}, \mathbf{y}^{34}, \mathbf{z}^{11}\}) & = & \{a_1, a_3, a_4, a_5, a_2\}
\end{array}$$

shows that neighbours of $\mathbf{w}_i$ in the graph Λ are mapped to neighbours of a_i in $\mathbf{H}(A_7)$, so that the bijection φ indeed is a graph isomorphism. ∎

Let A be the subgroup of $G_{\widehat{\Gamma}}$ generated by the groups $\mathrm{SU}_2(\mathbb{C})_{\mathbf{w}_i} \cong \mathrm{SU}_2(\mathbb{C})$ for $1 \le i \le 7$, i.e.

$$A = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{w}_i} \mid 1 \le i \le 7 \rangle.$$

We want to identify the isomorphism type of A via the main theorem of [42]. In order to apply this theorem, we need to check its hypotheses for which we need the following definitions.

For $n \ge 2$, let Δ be a Dynkin diagram of rank n and $I = \{1, \ldots, n\}$ be the set of labels of Δ. A group H admits a **weak Phan system of type Δ over the complex numbers**, if H is generated by subgroups U_i, $i \in I$, that are central quotients of simply connected compact semisimple Lie groups of rank one, i.e. $U_i \cong \mathrm{SU}_2(\mathbb{C})$ or $U_i \cong \mathrm{SO}_3(\mathbb{R}) \cong \mathrm{PSU}_2(\mathbb{C})$ for all $1 \le i \le n$, and if the groups U_i are embedded as rank one groups with respect to a fundamental system of roots of the groups $U_{ij} = \langle U_i, U_j \rangle$, which have the following isomorphism types:

$$\langle U_i, U_j\rangle \cong \begin{cases} (U_i \times U_j)/Z & \text{in case } \underset{i}{\circ}\qquad\underset{j}{\circ} \quad \text{where} \\ & Z \text{ is a central subgroup of } U_i \times U_j \\ \mathrm{SU}_3(\mathbb{C}) \text{ or } \mathrm{PSU}_3(\mathbb{C}) & \text{in case } \underset{i}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{j}{\circ} \\ \mathrm{U}_2(\mathbb{H}) \text{ or } \mathrm{SO}_5(\mathbb{R}) & \text{in case } \underset{i}{\circ}\!\Rightarrow\!\underset{j}{\circ} \quad \text{or} \quad \underset{i}{\circ}\!\Leftarrow\!\underset{j}{\circ} \\ G_{2,-14} & \text{in case } \underset{i}{\circ}\!\Lleftarrow\!\underset{j}{\circ} \quad \text{or} \quad \underset{i}{\circ}\!\Rrightarrow\!\underset{j}{\circ} \end{cases}.$$

By lemma 4.5.8 and lemma 4.5.9, the group A admits a weak Phan system over $\mathbb{C}$ of type A_7, so that by the main theorem of [42], we have that A is isomorphic to a central quotient of $\mathrm{SU}_8(\mathbb{C})$.

Next we consider the induced subgraphs $\Lambda_{\mathbf{x}}$ and $\Lambda_{\mathbf{y}^{56}}$ of Σ, which are isomorphic to the graph $\mathbf{H}(A_5)$, and the corresponding subgroups $A_{\mathbf{x}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{w}} \mid \mathbf{w} \in \mathcal{V}(\Lambda_x)\rangle$ and $A_{\mathbf{y}^{56}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{w}} \mid \mathbf{w} \in \mathcal{V}(\Lambda_{\mathbf{y}^{56}})\rangle$ of A. By the same argument as above, the subgroups $A_{\mathbf{x}}$ and $A_{\mathbf{y}^{56}}$ admit weak Phan systems of type A_5 over $\mathbb{C}$, thus the subgroup $A_{\mathbf{x}}$ is isomorphic to a central quotient of $\mathrm{SU}_6(\mathbb{C})$, say $A_{\mathbf{x}} \cong \mathrm{SU}_6(\mathbb{C})/Z_{\mathbf{x}}$, where $Z_{\mathbf{x}} \leq Z(\mathrm{SU}_6(\mathbb{C}))$ and also $A_{\mathbf{y}^{56}}$ is isomorphic to a central quotient of $\mathrm{SU}_6(\mathbb{C})$, say $A_{\mathbf{y}^{56}} \cong \mathrm{SU}_6(\mathbb{C})/Z_{\mathbf{y}^{56}}$, where $Z_{\mathbf{y}^{56}} \leq Z(\mathrm{SU}_6(\mathbb{C}))$. Thus we can link the two groups A and $G_{\widehat{\Gamma}}$ by the next proposition.

Proposition 4.6.3 *The group $G_{\widehat{\Gamma}}$ is isomorphic to a central quotient of* $\mathrm{SU}_8(\mathbb{C})$.

Proof: Since A is isomorphic to $\mathrm{SU}_8(\mathbb{C})/Z$ for some subgroup $Z \leq Z(SU_8(\mathbb{C}))$ and as $A \leq G_{\widehat{\Gamma}}$, it suffices to show that $G_{\widehat{\Gamma}}$ is a subgroup of A. To this end, for each vertex $\mathbf{w} \in \widehat{\Gamma}$, we will show that $\mathrm{SU}_2(\mathbb{C})_{\mathbf{w}}$ is a subgroup of A. This in turn is equivalent to showing that for each vertex $\mathbf{w} \in \widehat{\Gamma}$ there is a group element $h \in A$ such that $h\mathbf{x} = \mathbf{w}$ for the vertex $\mathbf{x} \in \Lambda$, by lemma 4.5.10. Recall that $A_{\mathbf{y}^{56}} \cong \mathrm{SU}_6(\mathbb{C})/Z_{\mathbf{y}^{56}}$ and $\mathbf{x}, \mathbf{y}^{12} \in \{\mathbf{y}^{56}\}^{\perp}$. The group $A_{\mathbf{y}^{56}}$ contains an element $\varphi_{\mathbf{x}}$ with $\varphi_{\mathbf{x}}(\mathbf{x}) = \mathbf{y}^{12}$, because the group $A_{\mathbf{y}^{56}} \cong \mathrm{SU}_6(\mathbb{C})/Z_{\mathbf{y}^{56}}$ acts transitively on the vertices of the graph $\{\mathbf{y}^{56}\}^{\perp} = \mathbf{S}(V(\mathcal{G}_{\mathbf{y}^{56}})) \cong \mathbf{S}(V_6)$ by lemma 4.5.5. Now let $\mathbf{u}$ be a vertex of $\mathbf{x}^{\perp}$. Then there exists a graph automorphism $\mu_{\mathbf{x},\mathbf{u}} \in A_{\mathbf{x}} \cong \mathrm{SU}_6(\mathbb{C})/Z$ such that $\mu_{\mathbf{x},\mathbf{u}}(\mathbf{y}^{12}) = \mathbf{u}$, as $\mathrm{SU}_6(\mathbb{C})/Z_{\mathbf{x}}$ acts transitively on all lines of $\mathcal{G}_{\mathbf{x}}$. We conclude that A contains the element $\delta_{\mathbf{x},\mathbf{u}} = \mu_{\mathbf{x},\mathbf{u}} \circ \varphi_{\mathbf{x}}$ which maps $\mathbf{x}$ to the vertex $\mathbf{u} \in \mathbf{x}^{\perp}$. Since $\delta_{\mathbf{x},\mathbf{u}}$ is a graph automorphism of $\widehat{\Gamma}$, the image of Λ under $\delta_{\mathbf{x},\mathbf{u}}$, denoted by $\Lambda_{\delta_{\mathbf{x},\mathbf{u}}}$, is also isomorphic to $\mathbf{H}(A_7)$. Furthermore, $\delta_{\mathbf{x},\mathbf{u}}(\mathrm{SU}_2(\mathbb{C})_{\mathbf{z}})\delta_{\mathbf{x},\mathbf{u}}^{-1} = \mathrm{SU}_2(\mathbb{C})_{\delta_{\mathbf{x},\mathbf{u}}(\mathbf{z})}$, cf. lemma 4.5.10, is a subgroup of A for each $\mathbf{z} \in \Lambda$. Hence, for each neighbour $\mathbf{u}$ of $\mathbf{x}$, the group $\mathrm{SU}_2(\mathbb{C})_{\mathbf{u}}$ is a subgroup of A.

An iteration of this argument, using the connectedness of $\widehat{\Gamma}$, shows that the group $G_{\widehat{\Gamma}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \widehat{\Gamma}\rangle$ is a subgroup of A. ∎

Case 2: $\Sigma \cong \mathbf{W}(E_6)$

As before we will construct an induced subgraph Λ of Σ isomorphic to $\mathbf{H}(E_6)$ on the six vertices $\mathbf{w}_1 = \mathbf{y}^{12}$, $\mathbf{w}_2 = \mathbf{y}^{23}$, $\mathbf{w}_3 = \mathbf{z}^{14}_{12} = \mathbf{z}^{13}_{56}$, $\mathbf{w}_4 = \mathbf{x}$, $\mathbf{w}_5 = \mathbf{y}^{45}$, $\mathbf{w}_6 = \mathbf{y}^{56}$.

These six vertices satisfy the following neighbourhood properties in the graph Λ by section 4.4:

e_1 is adjacent to the vertices e_3, e_4, e_5 and e_6 in Λ,

e_6 is adjacent to the vertices e_1, e_2, e_3 and e_4 in Λ,

e_2 is adjacent to the vertices e_4, e_5 and e_6 in Λ,

e_5 is adjacent to the vertices e_1, e_2 and e_4 in Λ,

e_3 is adjacent to the vertices e_1 and e_6 in Λ,

e_4 is adjacent to the vertices e_1, e_2, e_5 and e_6 in Λ.

Lemma 4.6.4 *We have* $\Lambda \cong \mathbf{H}(E_6)$.

Proof: The map $\varphi : \Psi \to \mathbf{H}(E_6)$ with $\varphi(\mathbf{w}_i) = e_i$, $1 \le i \le 6$ is a graph isomorphism. Indeed,

$$\begin{array}{rcl}
\varphi(\{\mathbf{x}, \mathbf{y}^{45}, \mathbf{y}^{56}, \mathbf{z}^{14}_{12}\}) & = & \{e_4, e_5, e_6, e_3\}, \\
\varphi(\{\mathbf{x}, \mathbf{y}^{45}, \mathbf{y}^{56}\}) & = & \{e_4, e_5, e_6\}, \\
\varphi(\{\mathbf{y}^{12}, \mathbf{y}^{56}\}) & = & \{e_1, e_6\}, \\
\varphi(\{\mathbf{y}^{12}, \mathbf{y}^{23}, \mathbf{y}^{45}, \mathbf{y}^{56}\}) & = & \{e_1, e_2, e_5, e_6\}, \\
\varphi(\{\mathbf{x}, \mathbf{y}^{12}, \mathbf{y}^{23}\}) & = & \{e_4, e_1, e_2\}, \\
\varphi(\{\mathbf{x}, \mathbf{y}^{12}, \mathbf{y}^{23}, \mathbf{z}^{13}_{56}\}) & = & \{e_4, e_1, e_2, e_3\},
\end{array}$$

implying that the neighbours of $\mathbf{w}_i$ in Λ are mapped onto the neighbours of e_i in $\mathbf{H}(E_6)$. It follows that the map φ is a graph homomorphism, thus $\Lambda \cong \mathbf{H}(E_6)$. ∎

Again due to the main theorem of [42], the group $E = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{w}_i} \mid 1 \le i \le 6 \rangle$ is isomorphic to a central quotient of the simply connected compact semisimple Lie group ${}^2E_6(\mathbb{C})$, whose complexification is the simply connected complex semisimple Lie group of type E_6, because E admits a weak Phan system of type E_6 over the complex numbers by lemma 4.5.8 and lemma 4.5.9. Following the strategy from above, we want to identify the group $G_{\widehat{\Gamma}}$ by proving that $G_{\widehat{\Gamma}}$ is a subgroup of E.

The next lemma will turn out to be very useful to reach this result.

Lemma 4.6.5 *The group E contains the subgroup* $\langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{x}^{\perp} \rangle$.

Proof: The group E contains the subgroup $E_{\mathbf{y}^{12}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{w}} \mid \mathbf{w} \in \mathcal{V}(\Lambda_{\mathbf{y}^{12}}) \rangle$, which is isomorphic to $\mathrm{SU}_5(\mathbb{C})/Z_{\mathbf{y}^{12}}$, where $Z_{\mathbf{y}^{12}} \le Z(\mathrm{SU}_5(\mathbb{C}))$, as $E_{\mathbf{y}^{12}}$ admits a weak Phan system of type A_4. This group act transitively on the set of lines of $\langle w_{\mathbf{y}^{12}} \mid \mathbf{w} \in \mathcal{V}(\Lambda_{\mathbf{y}^{12}}) \rangle$, which contains the lines $y^{46}_{\mathbf{y}^{12}}$, $(z^{j5}_{12})_{\mathbf{y}^{12}}$ and $(z^{j6}_{12})_{\mathbf{y}^{12}}$ for

$j \in \{1,2\}$ implying that $\mathrm{SU}_2(\mathbb{C})_{\mathbf{v}} \leq E$ for $\mathbf{v} \in \{\mathbf{y}^{46}, \mathbf{z}_{12}^{j5}, \mathbf{z}_{12}^{j6} \mid 1 \leq j \leq 2\}$. We consider the subgroup $E_{\mathbf{y}^{56}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{w}} \mid \mathbf{w} \in \mathcal{V}(\Lambda_{\mathbf{y}^{56}})\rangle$ of E. Also the subgroup $E_{\mathbf{y}^{56}}$ admits a weak Phan system of type A_4, hence $E_{\mathbf{y}^{56}} \cong \mathrm{SU}_5(\mathbb{C})/Z_{\mathbf{y}^{56}}$, where $Z_{\mathbf{y}^{56}} \leq Z(\mathrm{SU}_5(\mathbb{C}))$ and therefore the subgroup $E_{\mathbf{y}^{56}}$ acts transitively on the set of lines of $\langle w_{\mathbf{y}^{56}} \mid \mathbf{w} \in \mathcal{V}(\Lambda_{\mathbf{y}^{56}})\rangle$ and this five-dimensional subspace of $\mathcal{G}_{\mathbf{y}^{56}}$ contains the line $(z_{56}^{22})_{\mathbf{y}^{56}}$, thus $\mathrm{SU}_2(\mathbb{C})_{\mathbf{z}_{56}^{22}} \leq E$. We switch now to the projective space $\mathcal{G}_{\mathbf{y}^{45}}$ and to the subgroup subgroup $E_{\mathbf{y}^{45}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{v}}, \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}_{12}^{16}} \mid \mathbf{v} \in \mathcal{V}(\Lambda_{\mathbf{y}^{45}})\}$ of E, which is isomorphic to a central quotient of $\mathrm{SU}_4(\mathbb{C})$ and acts transitively on the lines of the subspace $\langle (z_{12}^{16})_{\mathbf{y}^{45}}, x_{\mathbf{y}^{45}}, y_{\mathbf{y}^{45}}^{12}, y_{\mathbf{y}^{45}}^{23}\rangle$. Indeed, the subgroup $E_{\mathbf{y}^{45}}$ admits a weak Phan system of type A_3 over $\mathbb{C}$. Since the two-dimensional subspace $(z_{45}^{11})_{\mathbf{y}^{45}}$ is a line of the subspace $\langle (z_{12}^{16})_{\mathbf{y}^{45}}, x_{\mathbf{y}^{45}}, y_{\mathbf{y}^{45}}^{12}, y_{\mathbf{y}^{45}}^{23}\rangle$ in $\mathcal{G}_{\mathbf{y}^{45}}$, we conclude that $\mathrm{SU}_2(\mathbb{C})_{\mathbf{z}_{45}^{11}} \leq E$. Furthermore, the subgroup $E_{\mathbf{y}^{46}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{v}} \mid \mathbf{v} \in \{\mathbf{x}, \mathbf{y}^{12}, \mathbf{y}^{23}, \mathbf{z}_{12}^{15}\}\rangle$ of $G_{\widehat{\Gamma}}$ admits a weak Phan system of type A_4 over $\mathbb{C}$, so $E_{\mathbf{y}^{46}} \cong \mathrm{SU}_5(\mathbb{C})/Z_{\mathbf{y}^{46}}$ with $Z_{\mathbf{y}^{46}} \leq Z(\mathrm{SU}_5(\mathbb{C})$. Since $\{\mathbf{x}, \mathbf{y}^{12}, \mathbf{y}^{23}, \mathbf{z}_{12}^{15}\} \subseteq \{\mathbf{y}^{46}\}^{\perp}$, the group $E_{\mathbf{y}^{46}}$ acts transitively on the lines of the subspace $\langle x_{\mathbf{y}^{46}}, y_{\mathbf{y}^{46}}^{12}, y_{\mathbf{y}^{46}}^{23}, (z_{12}^{15})_{\mathbf{y}^{46}}\rangle$ of $\mathcal{G}_{\mathbf{y}^{46}}$ containing the line $(z_{46}^{12})_{\mathbf{y}^{46}}$. Therefore $\mathrm{SU}_2(\mathbb{C})_{\mathbf{z}_{46}^{12}} \leq E$.

Finally, the vertices $\mathbf{z}_{12}^{25}$, $\mathbf{z}_{12}^{26}$, $\mathbf{z}_{12}^{26}$, $\mathbf{z}_{12}^{26}$ and $\mathbf{z}_{12}^{26}$ are all adjacent to $\mathbf{z}_{12}^{14}$ and $y_{\mathbf{z}_{12}^{14}}^{36}$ and are lines of the space $\langle (z_{12}^{25})_{\mathbf{z}_{12}^{14}}, (z_{12}^{26})_{\mathbf{z}_{12}^{14}}, (z_{12}^{26})_{\mathbf{z}_{12}^{14}}, (z_{12}^{26})_{\mathbf{z}_{12}^{14}}, (z_{12}^{26})_{\mathbf{z}_{12}^{14}}\rangle = \mathcal{G}_{\mathbf{z}_{12}^{14}}$. Moreover, the group $E_{\mathbf{z}_{12}^{14}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}_{12}^{25}}, \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}_{12}^{26}}, \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}_{12}^{26}}, \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}_{12}^{26}}, \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}_{12}^{26}}\rangle$ is isomorphic to a central quotient of the $\mathrm{SU}_6(\mathbb{C})$, as $E_{\mathbf{z}_{12}^{14}}$ admits a weak Phan system of type A_5 over $\mathbb{C}$, implying that $\mathrm{SU}_2(\mathbb{C})_{\mathbf{y}^{36}} \leq E$.

In the following we verify that the group $E_{\mathbf{x}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{v}} \mid \mathbf{v} \in \{\mathbf{y}^{12}, \mathbf{y}^{23}, \mathbf{y}^{36}, \mathbf{y}^{56}, \mathbf{y}^{45}\}\rangle$ is isomorphic to $\mathrm{SU}_6(\mathbb{C})/Z_{\mathbf{x}} \cong \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{x}^{\perp}\rangle$, where $Z_{\mathbf{x}} \leq Z(\mathrm{SU}_6(\mathbb{C}))$. If this is the case, then $E_{\mathbf{x}} \leq \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{x}^{\perp}\rangle \cong E_{\mathbf{x}}$ implies $\langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{x}^{\perp}\rangle \leq E$. Indeed, the group $E_{\mathbf{x}} \cong \mathrm{SU}_6(\mathbb{C})/Z_{\mathbf{x}}$ as $E_{\mathbf{x}}$ admits by construction a weak Phan system of type A_5 over $\mathbb{C}$. ∎

Proposition 4.6.6 *The group $G_{\widehat{\Gamma}}$ is isomorphic to the group $E \cong {}^2E_6(\mathbb{C})/Z$.*

Proof: Recall that the subgroup $E_{\mathbf{y}^{56}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{w}} \mid \mathbf{w} \in \mathcal{V}(\Lambda_{\mathbf{y}^{56}})\rangle \cong \mathrm{SU}_5(\mathbb{C})/Z_{\mathbf{y}^{56}}$ of E acts transitively on the set of lines of the subspace $\langle w_{\mathbf{y}^{56}} \mid \mathbf{w} \in \mathcal{V}(\Lambda_{\mathbf{y}^{56}})\rangle$ containing $x_{\mathbf{y}^{56}}, y_{\mathbf{y}^{56}}^{12}$. Thus E contains a graph automorphism $\varphi_{\mathbf{x}}$ such that $\varphi_{\mathbf{x}}(\mathbf{x}) = \mathbf{y}^{12}$. Furthermore the group E contains an element $\mu_{\mathbf{x},\mathbf{u}} \in \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \mathbf{x}^{\perp}\rangle$ mapping $\mathbf{y}^{12}$ to the vertex $\mathbf{u}$ for some vertex $\mathbf{u} \in \mathbf{x}^{\perp}$. Certainly the automorphism $\delta_{\mathbf{x},\mathbf{u}} = \mu_{\mathbf{x},\mathbf{u}} \circ \varphi_{\mathbf{x}}$ is an element of E satisfying the property that $\delta_{\mathbf{x},\mathbf{u}}(\mathbf{x}) = \mathbf{u}$ for some $\mathbf{u} \in \mathbf{x}^{\perp}$. Furthermore $\delta_{\mathbf{x},\mathbf{u}}(E_{\mathbf{x}}) = E_{\mathbf{u}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{v}} \mid \mathbf{v} \in \mathbf{u}^{\perp}\}$ and $\delta_{\mathbf{x},\mathbf{u}}(E_{\mathbf{y}^{56}}) = E_{\delta_{\mathbf{x},\mathbf{u}}(\mathbf{y}^{56})} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{v}} \mid \mathbf{v} \in \delta_{\mathbf{x},\mathbf{u}}(\mathcal{V}(\Lambda_{\mathbf{y}^{56}}))\rangle \cong \mathrm{SU}_5(\mathbb{C})/Z_{\mathbf{y}^{56}}$ acts transitively on all lines of $\langle u_{\delta_{\mathbf{x},\mathbf{u}}(\mathbf{y}^{56})}, \delta_{\mathbf{x},\mathbf{u}}(y^{12})_{\delta_{\mathbf{x},\mathbf{u}}(\mathbf{y}^{56})}, \delta_{\mathbf{x},\mathbf{u}}(y^{23})_{\delta_{\mathbf{x},\mathbf{u}}(\mathbf{y}^{56})}, \delta_{\mathbf{x},\mathbf{u}}(z_{56}^{13})_{\delta_{\mathbf{x},\mathbf{u}}(\mathbf{y}^{56})}\rangle$. Thus we can fix an automorphism $\varphi_{\mathbf{u}} \in E_{\delta_{\mathbf{x},\mathbf{u}}(\mathbf{y}^{56})} \leq E$ with $\varphi_{\mathbf{u}}(\mathbf{u}) = \delta_{\mathbf{x},\mathbf{u}}(\mathbf{y}^{12})$ and $\mu_{\mathbf{u},\mathbf{v}} \in E_{\mathbf{u}} \leq E$ such that $\mu_{\mathbf{u},\mathbf{v}}(\delta_{\mathbf{x},\mathbf{u}}(\mathbf{y}^{12})) = \mathbf{v}$ for a vertex $\mathbf{v} \in \mathbf{u}^{\perp}$ implying that E contains the graph automorphism $\delta_{\mathbf{u},\mathbf{v}} = \mu_{\mathbf{u},\mathbf{v}} \circ \varphi_{\mathbf{u}}$ mapping $\mathbf{u}$ to $\mathbf{v}$ and the element $\delta_{\mathbf{u},\mathbf{v}} \circ \delta_{\mathbf{x},\mathbf{u}}$ sending $\mathbf{x}$ to $\mathbf{v}$.

By the connectivity of the graph $\widehat{\Gamma}$ we can fix some path $\gamma_{\mathbf{w}} : \mathbf{x} \perp \mathbf{c}_1 \perp \cdots \perp \mathbf{c}_n \perp \mathbf{w}$ between the vertices $\mathbf{x}$ and $\mathbf{w}$ for each $\mathbf{w} \in \widehat{\Gamma}$. Applying the construction from above in each step on the path $\gamma_{\mathbf{w}}$ we determine the graph automorphism $\delta_{\mathbf{c}_n,\mathbf{w}} \circ \cdots \circ \delta_{\mathbf{c}_1,\mathbf{c}_2} \circ \delta_{\mathbf{x},\mathbf{c}_1}$ which maps the vertex $\mathbf{x}$ to $\mathbf{w}$. Certainly $\delta_{\mathbf{c}_n,\mathbf{w}} \circ \cdots \circ \delta_{\mathbf{c}_1,\mathbf{c}_2} \circ \delta_{\mathbf{x},\mathbf{c}_1}$ is an element of E, hence $\mathrm{SU}_2(\mathbb{C})_{\mathbf{w}}$ is a subgroup of E.

It follows that E contains the group $\langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{z}} \mid \mathbf{z} \in \widehat{\Gamma} \rangle = G_{\widehat{\Gamma}}$. ∎

Altogether we have proved the following:

Theorem 4.6.7 *The group $G_{\widehat{\Gamma}}$ is isomorphic to a central quotient of* $\mathrm{SU}_8(\mathbb{C})$ *if and only if $\Sigma \cong \mathbf{H}(A_7)$ and isomorphic to a central quotient of ${}^2E_6(\mathbb{C})$ if and only if $\Sigma \cong \mathbf{H}(E_6)$.*

4.7 The fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups graph of $E_{6,-78}$ and $\mathrm{SU}_8(\mathbb{C})$

Here we study the fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups graph $\mathbf{F}(\mathrm{SU}(V_8)) \cong \mathbf{S}(V_8)$, where V_8 is an eight-dimensional vector space over $\mathbb{C}$ with the scalar product $(\cdot,\cdot)$ and the graph $\mathbf{F}(E_{6,-78})$, the fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups graph of the compact Lie group $E_{6,-78}$. We will prove that both graphs, $\mathbf{S}(V_8)$ and $\mathbf{F}(E_{6,-78})$ are locally $\mathbf{S}(V_6)$ and simply connected. Moreover we show that $\mathbf{S}(V_8)$ contains the reflection graph $\mathbf{W}(A_7)$ and $\mathbf{W}(E_6)$ is an induced subgraph of $\mathbf{F}(E_{6,-78})$. Therefore the universal cover $\widehat{\Gamma}$ of a graph Γ, which is locally $\mathbf{S}(V_6)$ satisfying certain technical condition 4.1 is either isomorphic to $\mathbf{S}(V_8)$ or to $\mathbf{F}(E_{6,-78})$.

The graph $\mathbf{S}(V_8)$ is a connected locally $\mathbf{S}(V_6)$ graph of diameter two by proposition 4.2.1 and proposition 4.2.2.

Lemma 4.7.1 *The graph $\mathbf{S}(V_8)$ contains an induced subgraph $\Sigma \cong \mathbf{W}(A_7)$.*

Proof: Let $\beta : v_1, \ldots, v_8$ be an orthonormal basis of V_8. We consider the 28 different two-dimensional subspaces $l_{ij} = \langle v_i, v_j \rangle$ such that $1 \leq i < j \leq 8$ and we define $\mathcal{V}(\Sigma) = \{l_{ij} \mid 1 \leq i < j \leq 8\}$ to be the vertex set of Σ. Hence Σ is the induced subgraph of $\mathbf{S}(V_8)$ on 28 different vertices.

By definition of the graph $\mathbf{S}(V_8)$, see definition 1.1.10, two different vertices l_{mn} and l_{st} of $\mathcal{V}(\Sigma)$ are adjacent if and only if $l_{mn} \subseteq l_{st}^{\pi}$ or equivalent if and only if $\{m,n\} \cap \{s,t\} = \emptyset$. Thus the map $\Sigma \to K(8,2) \cong W(A_7) : l_{st} \mapsto \{s,t\}$ is a graph isomorphism. ∎

In order to state and prove the next result, we have to introduce some notation and to recall some known facts. We turn back to the situation that $\mathfrak{g}$ is a Lie algebra over the field $\mathbb{K}$.

Definition 4.7.2 Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and x be an element of $\mathfrak{g}$. The linear mapping $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ with $y \mapsto [x, y]$ is called the **adjoint linear mapping of** x.

The **Killing form** is the symmetric bilinear form on the Lie algebra $\mathfrak{g}$ defined by $B(x, y) = \mathrm{tr}(\mathrm{ad}_x \, \mathrm{ad}_y)$. It is invariant in the sense that $B([x, y], z) = B(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$.

Furthermore a Lie algebra homomorphism of $\mathfrak{g}$ into the Lie algebra $\mathrm{GL}(M)$, where M is a $\mathbb{K}$-module, is called a **representation of** $\mathfrak{g}$ **on the module** M. An injective representation is called **faithful**. The representation $\mathfrak{g} \to \mathrm{GL}(\mathfrak{g})$ with $x \mapsto \mathrm{ad}_x$ of $\mathfrak{g}$ on the $\mathbb{K}$ module $\mathfrak{g}$ is called the **adjoint representation of** $\mathfrak{g}$. A representation of $\mathfrak{g}$ on M is called **semi-simple** if this representation is similar to a direct sum of simple representations.

A Lie algebra $\mathfrak{g}$ is **semi-simple** if and only if the only commutative ideal of $\mathfrak{g}$ is $\{0\}$. Also we call a Lie algebra $\mathfrak{g}$ **reductive** if its adjoint representation is semi-simple.

Here is a connection between semi-simple Lie algebras and reductive Lie algebras.

Lemma 4.7.3 (lemma 1, chapter I 6.2 of [9]) *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then the adjoint representation of $\mathfrak{g}$ is semi-simple.*

In particular, any semi-simple Lie algebra is reductive.

Definition 4.7.4 Let $\mathfrak{g}$ be a Lie algebra over an infinite field. A **Cartan subalgebra** of $\mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{g}$ equals to its own normaliser.

A Cartan subalgebra $\mathfrak{h}$ of a semi-simple Lie algebra $\mathfrak{g}$ is called **splittable** if for all $x \in \mathfrak{h}$ the adjoint linear map ad_x of x is triangularizable. A semi-simple Lie algebra $\mathfrak{g}$ is called **splittable** if it has a splitting Cartan subalgebra and a **split semi-simple Lie algebra** is a pair $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g}$ is a semi-simple Lie algebra and $\mathfrak{h}$ is a splittable Cartan subalgebra of $\mathfrak{g}$.

Suppose that $\mathbb{K}$ is an infinite algebraically closed field of characteristic zero. Then a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ has Cartan algebras, all of the same dimension and conjugate to each other under the group of elementary automorphisms of $\mathfrak{g}$. It turns out that a Cartan subalgebra of a Lie group $\mathfrak{g}$ over an infinite field is a maximal nilpotent subalgebra of $\mathfrak{g}$, by proposition 2, chapter VII 2.2 by [11]. Moreover every semi-simple Lie algebra $\mathfrak{g}$ over an algebraically closed field is splittable and every Cartan subalgebra of $\mathfrak{g}$ splits. In this case it follows also that a Cartan subalgebra is commutative, due to theorem 2, chapter VII 2.4 of [11].

Next we shall define the notion of a **root** of a split semi-simple Lie algebra $(\mathfrak{g}, \mathfrak{h})$ over $\mathbb{K}$. We consider the dual space $\mathfrak{h}^*$ of a Cartan subalgebra $\mathfrak{h}$ and denote for each

$\lambda \in \mathfrak{h}^*$ the subspace $\{y \in \mathfrak{g} \mid [h, y] = \lambda(h)y$ for all $h \in \mathfrak{h}\}$ with $\mathfrak{g}_\lambda(\mathfrak{h})$ or just $\mathfrak{g}_\lambda$. Notice that $\mathfrak{g}_0 = \{y \in \mathfrak{g} \mid [h, y] = 0$ for all $h \in \mathfrak{h}\} = \mathfrak{h}$ and that $\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\alpha$.

Definition 4.7.5 A **root** of a semi-simple split Lie algebra $(\mathfrak{g}, \mathfrak{h})$ is a non-zero element $\lambda \in \mathfrak{h}^*$ such that $\mathfrak{g}_\lambda \neq \{0\}$. We denote by $R(\mathfrak{g}, \mathfrak{h}) = R$ the set of all roots of $(\mathfrak{g}, \mathfrak{h})$, the **root system** of $\mathfrak{g}$ relative to $\mathfrak{h}$.

It is known that $R(\mathfrak{g}, \mathfrak{h})$ is a reduced root system in $\mathfrak{h}^*$ and we can form the root-space decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in R} \mathfrak{g}_\alpha.$$

Theorem 4.7.6 (theorem 1, chapter VIII 2.2 of [11]) *Let α be a root of a semi-simple split Lie algebra $(\mathfrak{g}, \mathfrak{h})$ over $\mathbb{F}$. Then*

- *$\mathfrak{g}_\alpha$ is a one-dimensional vector subspace of $\mathfrak{g}$.*
- *The vector space $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathfrak{h}_\alpha$ of the Cartan subalgebra $\mathfrak{h}$ is of dimension one.*
- *The vector space $\mathfrak{g}_{\alpha,-\alpha} = \langle \mathfrak{h}_\alpha, \mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha} \rangle$ is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{F})$, also called a fundamental $\mathfrak{sl}_2(\mathbb{F})$ Lie subalgebra of $\mathfrak{g}$. Furthermore, $\mathfrak{g}_{\alpha,-\alpha}$ is generated as a Lie subalgebra by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.*

Proposition 4.7.7 (section 14.2 of [55]) *Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, $\Phi = R(\mathfrak{g}, \mathfrak{h})$ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$ and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ a basis of Φ. Then $\mathfrak{g}$ is generated as a Lie algebra by the root spaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta$ or equivalently $\mathfrak{g}$ is generated by arbitrary non-zero root vectors $x_\alpha \in \mathfrak{g}_\alpha$, $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta$.*

We shall call the set $\{x_\alpha, x_{-\alpha} \mid \alpha \in \Delta\}$ or $\{x_\alpha, x_{-\alpha}\}$ with $[x_\alpha, x_{-\alpha}] = h_\alpha$ a standard set of generators for the semi-simple Lie algebra $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$, the corresponding root system $\Phi = R(\mathfrak{g}, \mathfrak{h})$ and the basis $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. Then these generators satisfy at least the following relations:

$(S1)$ $[h_{\alpha_i}, h_{\alpha_j}] = 0$ for $\alpha_i, \alpha_j \in \Delta$

$(S2)$ $[x_{\alpha_i}, x_{-\alpha_i}] = h_{\alpha_i}$ and $[x_{\alpha_i}, x_{-\alpha_j}] = 0$ if $\alpha_i \neq \alpha_j$

$(S3)$ $[h_{\alpha_i}, x_{\alpha_j}] = \frac{2\sigma(\alpha_j,\alpha_i)}{\sigma(\alpha_i,\alpha_i)} x_{\alpha_j}$ and $[h_{\alpha_i}, x_{-\alpha_j}] = \frac{-2\sigma(\alpha_j,\alpha_i)}{\sigma(\alpha_i,\alpha_i)} x_{-\alpha_j}$

(S_{ij}^+) $(\mathrm{ad}_{x_{\alpha_i}})^{\frac{-2\sigma(\alpha_j,\alpha_i)}{\sigma(\alpha_i,\alpha_i)}+1}(x_{\alpha_j}) = 0$ for $\alpha_i \neq \alpha_j$

(S_{ij}^-) $(\mathrm{ad}_{x_{-\alpha_i}})^{\frac{-2\sigma(\alpha_j,\alpha_i)}{\sigma(\alpha_i,\alpha_i)}+1}(x_{-\alpha_j}) = 0$ for $\alpha_i \neq \alpha_j$

Theorem 4.7.8 (Serre, section 18.3 of [55]) *Let Φ be a fixed root system with basis $\Delta = \{\alpha_1, \dots, \alpha_l\}$ and let $\mathfrak{g}$ be the unique Lie algebra generated by $3l$ elements $\{x_i, y_i, h_i = [x_i, y_i] \mid 1 \leq i \leq l\}$ with $0 \neq x_i \in \mathfrak{g}_{\alpha_i}, 0 \neq y_i \in \mathfrak{g}_{-\alpha_i}$, subject to the relations S1 to S3, S^+_{ij} and S^-_{ij}. Then $\mathfrak{g}$ is a finite dimensional semi-simple Lie algebra, with Cartan subalgebra spanned by the h_i and with corresponding root system Φ.*

Theorem 4.7.9 (section 18.4 of [55]) *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be semi-simple Lie algebras with respective Cartan subalgebras $\mathfrak{h}, \mathfrak{h}'$ and root systems Φ, Φ'. Let an isomorphism between the root systems $\Phi \to \Phi'$ be given sending a given basis Δ of Φ to a basis Δ' of Φ' and denote by $\pi : \mathfrak{h} \to \mathfrak{h}'$ the associated isomorphism between the Cartan subalgebras. For each $\alpha \in \Delta$, ($\alpha' \in \Delta'$) select arbitrary non-zero $x_\alpha \in \mathfrak{g}_\alpha$, ($x'_{\alpha'} \in \mathfrak{g}'_{\alpha'}$). Then there exists a unique isomorphism $\pi : \mathfrak{g} \to \mathfrak{g}'$ extending $\pi : \mathfrak{h} \to \mathfrak{h}'$ and sending x_α to $x'_{\alpha'}$ for each $\alpha \in \Delta$.*

Moreover for a semi-simple Lie algebra over an algebraic closed field $\mathbb{K}$ of characteristic zero with Cartan subalgebra and root system $R(\mathfrak{g}, \mathfrak{h}) = \Phi$ with basis $\Delta : \alpha_1, \dots, \alpha_l$, we can construct a **Chevalley basis of** $\mathfrak{g}$. A Chevalley basis of $\mathfrak{g}$ is any basis $\{x_\alpha \in \mathfrak{g}_\alpha \backslash \{0\} \mid \alpha \in \Phi\} \cup \{h_i \mid 1 \leq i \leq l\}$ such that

- $[x_\alpha, x_{-\alpha}] = h_\alpha$
- if $\alpha, \beta, \alpha + \beta \in \Phi$ and $[x_\alpha, x_\beta] = c_{\alpha,\beta}$ then $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$
- $h_i = h_{\alpha_i}$ for some basis $\Delta = \{\alpha_1, \dots, \alpha_l\}$ of Φ.

Indeed by chapter VII of [55], a Chevalley basis of a semi-simple Lie algebra $\mathfrak{g}$ exists and has the following structure constants.

Theorem 4.7.10 (Chevalley, section 25.2 of [55]) *Let $\mathfrak{g}$ be a semi-simple Lie algebra and $\{x_\alpha \in \mathfrak{g}_\alpha \backslash \{0\} \mid \alpha \in \Phi\} \cup \{h_i \mid 1 \leq i \leq l\}$ be a chevalley basis of $\mathfrak{g}$. Then*

- $[h_i, h_j] = 0$ *for* $1 \leq i, j \leq l$
- $[h_i, x_\alpha] = \frac{2(\alpha, \alpha_i)}{(\alpha_i, \alpha_i)} x_\alpha$ *for* $1 \leq i \leq l$, $\alpha \in \Phi$
- $[x_\alpha, x_{-\alpha}] = h_\alpha$ *is a $\mathbb{Z}$-linear combination of* $h_1, \dots, h_l$.
- *if α and β are independent roots, $\beta - r\alpha, \dots, \beta + q\alpha$ the α-string through β, then $[x_\alpha, x_\beta] = 0$ if $q = 0$ while $[x_\alpha, x_\beta] = \pm(r+1)x_{\alpha+\beta}$ if $\beta + \alpha \in \Phi$.*

Let $\mathfrak{g}$ be semi-simple Lie algebra over $\mathbb{C}$, $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra and $\Phi = R(\mathfrak{g}, \mathfrak{h})$ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. We choose a basis Δ of Φ and a decomposition

$\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ denotes the set of positive roots with respect to Δ and Φ^- denotes the set of negative roots with respect to Δ.

Now we define
$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Phi^+} g_\alpha \text{ and } \mathfrak{n}^- = \bigoplus_{\alpha\in\Phi^-} g_\alpha.$$
The subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$ is the **Borel subalgebra** of $\mathfrak{g}$.

Theorem 4.7.11 *Let $\mathfrak{g}$ be semi-simple complex Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, $\Phi = R(\mathfrak{g},\mathfrak{h})$ be the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$ and Δ some basis of Φ. Then* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^- = \mathfrak{b} \oplus \mathfrak{n}^-$. *Moreover*

- $\mathfrak{n}^+$ *and* $\mathfrak{n}^-$ *are nilpotent subalgebras of* $\mathfrak{g}$.
- *The Borel subalgebra* $\mathfrak{b}$ *is a solvable subalgebra of* $\mathfrak{g}$.

Theorem 4.7.12 (Borel-Morozow) *Every solvable subalgebra of a semi-simple complex Lie algebra $\mathfrak{g}$ is conjugate via an inner automorphism to a subalgebra of the Borel subalgebra $\mathfrak{b}$. Inparticular, $\mathfrak{b}$ is a maximal solvable subalgebra of $\mathfrak{g}$.*

Next we want to introduce the notion of a **Lie group**. Let G be a set. A group structure and an analytical $\mathbb{K}$-manifold structure on G are called compatible if the following condition holds:

(GL) The mapping $(g,h) \mapsto gh^{-1}$ of $G \times G \to G$ is analytic.

Definition 4.7.13 (1.1, chapter III, of [9]) A **Lie group over** $\mathbb{K}$ is a set G with a group structure and an analytic $\mathbb{K}$-manifold structure such that these two structures are compatible.

A Lie group over $\mathbb{R}$ (resp. $\mathbb{C}$) is called a **real** (resp. **complex**) **Lie group** .

By [9] a Lie group is a complete metrizable topological group. Thus a Lie group is **compact** if the topological group is also a compact space. Moreover a real (resp. complex) Lie group is **locally connected** , by proposition 2, chapter III, 1.1 of [9]. For a Lie group G we denote by G° the **identity component** of G, which is the connected component of G containing the identity element e. G_0 is a normal closed subgroup of G. Next, we describe a natural construction that associates a certain Lie algebra $L(G)$ to every Lie group G.

Suppose M is a C^∞ manifold. A real-valued function $g : M \to \mathbb{R}$ belongs to $C^\infty(M,\mathbb{R}) = C^\infty(M)$ if $g \circ \varphi^{-1}$ is infinitely often differentiable for every chart $\varphi : U \to \mathbb{R}^n$. With respect to the pointwise product and sum of functions and scalar multiplication $C^\infty(M)$ is a real associative algebra. Pick a point p in M. A

derivation at p is a linear map $\mathrm{D} : C^\infty(M) \to \mathbb{R}$ which has the property that for all g, h in $C^\infty(M)$ we have the identity

$$\mathrm{D}(gh) = \mathrm{D}(g) \cdot h(p) + g(p) \cdot \mathrm{D}(h) ,$$

modelled on the product rule of calculus. These derivations form a real vector space in a natural manner for every element $p \in M$, which is the **tangent space** $T_p(M)$. Let $f : M \to N$ be a smooth map of smooth manifolds. Given some $p \in M$ the **differential** df_p of f is the linear map $\mathrm{d}f_p : T_p(M) \to T_{f(p)}(N)$ given by $\mathrm{d}f_p(\mathrm{D}(g)) = \mathrm{d}f_p(\mathrm{D})(g) = \mathrm{D}(g \circ f)$. If $\mathrm{d}f_p$ is injective, then f is said to be an **immersion at** x. If f is an immersion at every point, it is called an **immersion**. A vector field of a manifold M is a derivation of the algebra $C^\infty(M)$, therefore a linear map $X : C^\infty(M) \to C^\infty(M)$ with $X(fg) = f(X(g)) + (X(f))g$. We denote the set of all vector fields of M with $\mathcal{V}(M)$. $\mathcal{V}(M)$ is a Lie algebra with the Lie bracket $[X, Y] = X \circ Y - Y \circ X$.

Let G be a Lie group and let $\lambda_g : G \to G$ denote the diffeomorphisms corresponding to left multiplication by g for every $g \in G$. If X is a vector field on G, then let X_p be the value of X at a point $p \in G$, so $X_p \in T_p(G)$. A vector field X on G is called left-invariant if X is invariant with respect to all left multiplications, so $X(f \circ \lambda_g) = X(f) \circ \lambda_g$ for all $g \in G$ and $f \in C^\infty(G)$. The map α of the real vector space of all left-invariant vector fields X on G onto $T_e(G)$ with $X \mapsto X_e$ is a linear isomorphism, see [53].

Definition 4.7.14 The **Lie algebra of a Lie group** G, denoted by $L(G) = \mathfrak{g}$, is the tangent space $T_e(G)$ under the Lie bracket $[X, Y] = \alpha([\alpha^{-1}X, \alpha^{-1}Y])$. The linear action Ad of the Lie group G on the Lie algebra $L(G)$ mapping each $g \in G$ to the differential $\mathrm{Ad}(g) = \mathrm{d}(\mathrm{int}_g)$ of the inner automorphism $\mathrm{int}_g : G \to G$ with $x \mapsto gxg^{-1}$, is called the **adjoint representation** of the Lie group G. The adjoint representation of $L(G) = \mathfrak{g}$ is the differential of Ad at the identity, thus $\mathrm{ad} = \mathrm{d}(\mathrm{Ad}) : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. Moreover for an element $X \in L(G)$ we define $\exp(X) := \gamma_X(1)$, where γ_X is an integral curve of X with $\gamma_X(0) = 1$, see [53].

We recall some properties of the exponential function. Let G be a Lie group. There exists one and only one exponential mapping of G defined on the Lie algebra $L(G)$. This mapping has the following properties for ρ an analytical linear representation of G, $x \in L(G)$ and $g \in G$.

- $\rho(\exp(x)) = \exp(L(\rho)x)$
- $\mathrm{Ad}(\exp(x)) = \exp(\mathrm{ad}_x)$
- $g(\exp(x))g^{-1} = \exp(\mathrm{Ad}(g)x)$

Furthermore let $\alpha : G_1 \to G_2$ be an analytic homomorphism of Lie groups. Then the differential of α is the homomorphism $L(\alpha) = \mathrm{d}(\alpha)(1) : L(G_1) \to L(G_2)$ such that $\alpha(\exp_{G_1} X) = \exp_{G_2}(\mathrm{d}(\alpha)(1)X) = \exp_{G_2}(L(\alpha)X)$ for all $X \in L(G_1)$, i.e. the next diagram commutes, see [53].

$$\begin{tikzcd} L(G_1) \arrow[r, "L(\alpha)"] \arrow[d, "\exp_{G_1}"'] & L(G_2) \arrow[d, "\exp_{G_2}"] \\ G_1 \arrow[r, "\alpha"] & G_2 \end{tikzcd}$$

Corollary 4.7.15 (corollary 1, chapter IX 2.2 of [11]) *The exponential map of a compact connected Lie group G is surjective.*

An **integral subgroup of a Lie group** G is a subgroup with a connected Lie group structure such that the canonical injection from the subgroup into G is an immersion. Recall that a connected Lie subgroup of G is an integral subgroup of G. Integral subgroups have some nice properties, which we will use later.

Proposition 4.7.16 (proposition 10, III 6.4 and proposition 9, III 9.5 of [9]) *Let G be a Lie group and H be an integral subgroup of G, then $\exp_H = (\exp_G)_{|L(H)}$.*

Furthermore, if G is a finite dimensional real or complex Lie group, $\mathfrak{g}$ is Lie algebra and $\mathfrak{h} = L(H)$. Then $Z_G(H)$ is a Lie subgroup of G with Lie algebra $Z_{\mathfrak{g}}(\mathfrak{h})$.

Proposition 4.7.17 (proposition 4.7 of [27]) *If G is a Lie group and G° is the connected identity component, then G° is generated by $\exp(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.*

Theorem 4.7.18 (theorem 3 (i), chapter III 6.3 [9]) *If $\mathfrak{g}$ is a finite dimensional Lie algebra, then there exists a simply connected Lie group G such that $L(G)$ is isomorphic to $\mathfrak{g}$.*

A **torus** of a Lie group G is any closed commutative connected compact subgroup of G. Thus a torus T of G is a Lie subgroup isomorphic to $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, where $\mathbb{S}^1$ is the **circle group** $U_1(\mathbb{C}) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ identified with the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. The maximal closed commutative connected compact subgroup of G, ordered by inclusion, are called the **maximal tori** of G.

Theorem 4.7.19 (theorem 2, chapter IX 2.2 of [11]) *Let G be a connected compact Lie group. Then the Lie algebras of the maximal tori of G are the Cartan subalgebras of the Lie algebra $L(G)$ and any two maximal tori of G are conjugate. Moreover for a Cartan subalgebra $\mathfrak{t}$ in $L(G)$, the integral subgroup of G, whose Lie algebra is $\mathfrak{t}$ is a torus of G, so $\exp(\mathfrak{t}) = T$ for T a torus of G and $L(T) = \mathfrak{t}$ or vice versa.*

For a connected compact Lie group G, let T be a maximal torus of G and denote by $N_G(T) = \{g \in G \mid gTg^{-1} = T\}$ the **normaliser of** T in G. The quotient group $N_G(T)/T$ is finite and called the **Weyl group** $W_G(T)$ of G relative to the maximal torus T. Any two maximal tori are conjugate, so different choices of maximal tori in G yield isomorphic Weyl groups. Certainly the subgroup $N_G(T)$ acts on T by conjugation, so we obtain an induced action of the Weyl group $W_g(T)$ on T by $W_g(T) \times T \to T$ with $(nT, t) \mapsto ntn^{-1}$.

In the next part we obtain compact real forms of a complex Lie algebra $\mathfrak{a}$.

Let V be a vector space over $\mathbb{R}$. We call $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV$ the **complexification** of V. Certainly $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ is a vector space over $\mathbb{C}$ with the natural embedding $V \to V \otimes_{\mathbb{R}} \mathbb{C}$ by $v \mapsto v \otimes 1$.

On the other hand let W be a complex vector space, then restricting the scalars to $\mathbb{R}$ leads to a vector space $W_{\mathbb{R}}$ over $\mathbb{R}$, thus we regard W as a real vector space. Let $\{w_j \mid j \in I\}$ be a basis of W, then $\{w_j, i \cdot w_j \mid j \in I\}$ is a basis of $W_{\mathbb{R}}$ and $W_{\mathbb{R}} = V \oplus Vi = (V_{\mathbb{C}})_{\mathbb{R}}$ if V is the real span of the basis vectors $\{w_j \mid j \in I\}$.

If $\mathfrak{a}$ is a complex Lie algebra, then by $\mathfrak{a}_{\mathbb{R}}$ we denote the real Lie algebra obtained by restricting the scalars of $\mathfrak{a}$ to $\mathbb{R}$. On the other hand if $\mathfrak{g}$ is a real Lie algebra then $\mathfrak{g}_{\mathbb{C}}$ is the complex Lie algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$.

A **real form of a complex Lie algebra** $\mathfrak{a}$ is a real subalgebra $\mathfrak{g}$ of $\mathfrak{a}$ such that the subspace $\mathfrak{g}$ and $i\mathfrak{g}$ of the real vector space $\mathfrak{a}_{\mathbb{R}}$ are complementary and $\mathfrak{a}_{\mathbb{R}} = \mathfrak{g} \oplus i\mathfrak{g}$. The real form $\mathfrak{g}$ of $\mathfrak{a}$ is associated a conjugation σ of $\mathfrak{a}_{\mathbb{R}}$ relative to $\mathfrak{g}$, which is the $\mathbb{R}$ linear that is 1 on $\mathfrak{g}$ and -1 on $-i\mathfrak{g}$, so $x + iy \mapsto x - iy$ with $x, y \in \mathfrak{g}$.

Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. The map $[\cdot,\cdot] : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ given by $(x \otimes a, y \otimes b) \mapsto ([x, y] \otimes ab)$ extends the Lie bracket of $\mathfrak{g}$ in a complex bilinear way. In an analog way we have the complexification of ad_x with $x \in \mathfrak{g}$ to $\mathrm{ad}_x : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ for any $x \in \mathfrak{g}_{\mathbb{C}}$.

Observation 4.7.20 Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. If the complexification $\mathfrak{t}_{\mathbb{C}} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

Definition 4.7.21 A Lie algebra $\mathfrak{g}$ is called **compact** if $\mathfrak{g}$ is isomorphic to the Lie algebra of a compact Lie group.

By proposition 1 of [11] chapter IX 1.3 and theorem 1 of [9] chapter I 6.2, a compact Lie algebra $\mathfrak{g}$ is reductive and semi-simple.

If $\mathfrak{a}$ is a reductive complex Lie algebra then a real subalgebra $\mathfrak{g}$ is a **compact real form** if $\mathfrak{g}$ is real form of $\mathfrak{a}$ and $\mathfrak{g}$ is compact.

Theorem 4.7.22 (theorem 1 and corollary 2, chapter IX 3.3 of [11]) *Let $\mathfrak{a}$ be a complex semi-simple Lie algebra. Then $\mathfrak{a}$ has compact (resp. splittable) real forms. Furthermore there exists a compact real Lie algebra $\mathfrak{g}$ such that $\mathfrak{a} \cong \mathfrak{g}_{\mathbb{C}}$ and also there exists a compact Lie group G such that $\mathfrak{a} \cong L(G)_{\mathbb{C}}$.*

Moreover we consider the conjugation $\rho : \mathfrak{a}_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}$ with $\rho(x + iy) = x - yi$ of a complex semi-simple Lie algebra $\mathfrak{a}$ relative to a real form $\mathfrak{g}$.

Proposition 4.7.23 (proposition 1, chapter IX 3.1 of [11]) *Let $\mathfrak{a}$ be a complex semi-simple Lie algebra and $\mathfrak{g}$ a real form of $\mathfrak{a}$. Then*

$$\rho^2 = id_{\mathfrak{a}}, \quad \rho(\lambda x + \mu y) = \overline{\lambda}\rho(x) + \overline{\mu}\rho(y), \quad [\sigma(x) + \sigma(y)] = \sigma([x, y])$$

for any $\lambda, \mu \in \mathbb{C}$, $x, y \in \mathfrak{a}$. An element belongs to $\mathfrak{g}$ if and only if $\sigma(x) = x$.

On the other hand let $\rho : \mathfrak{a}_{\mathbb{R}} \to \mathfrak{a}_{\mathbb{R}}$ be a map satisfying the properties just stated, for a complex Lie algebra $\mathfrak{a}$. Then the set $\mathfrak{g}$ of fixed points of ρ is a real form of $\mathfrak{a}$ and ρ is the conjugation of $\mathfrak{a}$ relative to the real form $\mathfrak{g}$.

A useful result of H. Weyl is the following theorem about connected Lie groups with compact Lie algebras.

Theorem 4.7.24 (Weyl, theorem 1, chapter IX 1.4 of [11]) *Let G be a connected Lie group whose Lie algebra is compact and semi-simple. Then G is compact and its centre is finite.*

We now introduce the notion of roots for connected compact Lie groups. We denote with $X(G)$ the commutative group of continuous homomorphism from the compact Lie group G to the topological group $\mathbb{C}^\times$. The elements of $X(G)$ are morphisms of Lie groups, therefore for all $\alpha \in X(G)$, the differential of α is an $\mathbb{R}$-linear map $L(\alpha) = \mathrm{d}(\alpha)(1) : L(G) \to L(\mathbb{C}^\times) = \mathbb{C}$. Thus every element $\alpha \in X(G)$ is associated to an element $L(\alpha) \in \mathrm{Hom}_{\mathbb{R}}(L(G), \mathbb{C})$. We denote by $\delta(\alpha)$ the element of $\mathrm{Hom}_{\mathbb{R}}(L(G)_{\mathbb{C}}, \mathbb{C}) = L(G)^*_{\mathbb{C}}$ whose restriction to $L(G)$ coincides with $L(\alpha)$, thus we obtain a map $\delta : X(G) \to L(G)^*_{\mathbb{C}}$. Certainly for all $x \in L(G)$ and $\alpha \in X(G)$ we have $\alpha(\exp_G x) = e^{\delta(\alpha)x}$, where $z \mapsto e^z$ denotes the usual exponential function from $\mathbb{C}$ to $\mathbb{C}^\times$.

Let V be a finite dimensional vector space over $\mathbb{K}$, where $\mathbb{K}$ is either the real numbers or the complex numbers. We consider a continuous (real analytic) representation $\varphi : G \to \mathrm{GL}(V)$ of a connected compact Lie group G on V. We define $\tilde{V} = \begin{cases} V & \text{if } \mathbb{K} = \mathbb{C} \\ V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V & \text{if } \mathbb{K} = \mathbb{R} \end{cases}$ and $\tilde{\varphi} = \begin{cases} \varphi & \text{if } \mathbb{K} = \mathbb{C} \\ \mu \circ \varphi & \text{if } \mathbb{K} = \mathbb{R} \end{cases}$, where μ is the canonical homomorphism between $\mathrm{GL}(V)$ and $\mathrm{GL}(\tilde{V})$.

We consider now the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(L(G)) = \mathrm{GL}(\mathfrak{g})$ and a maximal torus T of G. For all $\lambda \in X(T)$, we denote by $\tilde{\mathfrak{g}}(T)_\lambda = \tilde{\mathfrak{g}}_\lambda$ the vector subspace $\{v \in \tilde{\mathfrak{g}} \mid \tilde{\mathrm{Ad}}(g)v = \lambda(g)v \text{ for all } g \in T\}$.

Definition 4.7.25 A **root** of G relative to T is a non-zero element $\lambda \in X(T)$ such that $\tilde{\mathfrak{g}}(T)_\lambda \neq \{0\}$ We write for the set of root of G relative to T the symbol $R(G,T)$. Certainly $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(T)_0 \oplus \bigoplus_{\lambda\in R(G,T)} \tilde{\mathfrak{g}}(T)_\lambda$.

Following chapter IX 4.4 of [11], the map $\delta : X(T) \to \mathfrak{t}_{\mathbb{C}}^*$, where $\mathfrak{t}_{\mathbb{C}}^*$ is the dual space of the complex Lie algebra $\mathfrak{t}_{\mathbb{C}}$ and $\mathfrak{t} = L(T)$ is the Lie algebra of the torus T, maps the root system $R(G,T)$ bijectively onto the set $R(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

Finally we consider the reduced irreducible root system $\Phi_{E_6} = \Phi$ of type E_6 and the basis $\Delta_{E_6} = \Delta = \alpha_1, \ldots, \alpha_6$ described in [10] Plate V. Let $L(E_6) = \mathfrak{g}(E_6) = \mathfrak{g}$ be the finite dimensional semi-simple complex Lie algebra with corresponding root system Φ and Cartan subalgebra $\mathfrak{h}$, which exists by theorem 4.7.8. Furthermore let $\{x_\alpha \in \mathfrak{g}_\alpha\backslash\{0\} \mid \alpha \in \Phi;\ h_i \mid 1 \le i \le 6\}$ be a Chevalley basis of $\mathfrak{g}(E_6)$. Then $\mathfrak{g}$ is generated by the non-zero root vectors $x_{\alpha_i}, x_{-\alpha_i}$ for $1 \le i \le 6$ by proposition 4.7.7 and for each $\alpha \in \Phi$ we know the Lie subalgebra $\mathfrak{g}_{\alpha,-\alpha}(E_6) = \mathfrak{g}_{\alpha,-\alpha} = \langle x_\alpha, x_{-\alpha}, [x_\alpha, x_{-\alpha}]\rangle \cong \mathfrak{sl}_2(\mathbb{C})$ due to theorem 4.7.6.

Let $\tilde{\alpha}$ be the maximal root of Φ with respect to Δ, so $\tilde{\alpha} = \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$ by Plate V of [10]. Then due to theorem 4.7.10, $[x_{\tilde{\alpha}}, x_{\alpha_i}] = 0 = [x_{\tilde{\alpha}}, x_{-\alpha_i}]$ and $[x_{-\tilde{\alpha}}, x_{\alpha_i}] = 0 = [x_{-\tilde{\alpha}}, x_{-\alpha_i}]$ in $\mathfrak{g}$ for $1 \le i \le 6$, $i \neq 2$ as $(\tilde{\alpha}, \alpha_i) = 0$ for $1 \le i \le 6$, $i \neq 2$. Furthermore we determine also that $[x_{\tilde{\alpha}}, x_{\alpha_2}] = 0 = [x_{-\tilde{\alpha}}, x_{-\alpha_2}]$ by the fact that $(\tilde{\alpha}, \alpha_2) = 1 = (-\tilde{\alpha}, -\alpha_2) > 0$ and thus $\tilde{\alpha} + \alpha_2$ and $-\tilde{\alpha} - \alpha_2$ are not roots of $\mathfrak{g}$. On the other hand $(\tilde{\alpha}, -\alpha_2) = -1 = (-\tilde{\alpha}, \alpha_2) < 0$ therefore the elements $\tilde{\alpha} - \alpha_2$ and $-\tilde{\alpha} + \alpha_2$ are roots of Φ. After calculation we obtain that $(\tilde{\alpha} - \alpha_2, \alpha_2) = 1 = (-\tilde{\alpha} + \alpha_2, -\alpha_2) > 0$ thus $\tilde{\alpha}, \tilde{\alpha} - \alpha_2$ is the $-\alpha_2$-string through $\tilde{\alpha}$ and $-\tilde{\alpha}, -\tilde{\alpha} + \alpha_2$ is the α_2-string through $-\tilde{\alpha}$. So $[x_{\tilde{\alpha}}, x_{-\alpha_2}] = \pm x_{\tilde{\alpha}-\alpha_2}$ and $[x_{-\tilde{\alpha}}, x_{\alpha_2}] = \pm x_{\tilde{\alpha}-\alpha_2}$ by theorem 4.7.10. Using the same statement again we get that $[x_{\tilde{\alpha}}, h_{\alpha_i}] = 0 = [x_{-\tilde{\alpha}}, h_{\alpha_i}]$ for all $i \in \{1, \ldots, 6\}\backslash\{2\}$ as $(\tilde{\alpha}, \alpha_i) = 0$ for $1 \le i \le 6; i \neq 2$ and $[x_{\tilde{\alpha}}, h_{\alpha_2}] = 2\frac{(\alpha_2,\tilde{\alpha})}{(\alpha_2\alpha_2)} x_{\tilde{\alpha}} = x_{\tilde{\alpha}}$ as well as $[x_{-\tilde{\alpha}}, h_{\alpha_2}] = 2\frac{(\alpha_2,-\tilde{\alpha})}{(\alpha_2\alpha_2)} x_{-\tilde{\alpha}} = -x_{-\tilde{\alpha}}$. Because $\mathfrak{g}_{\tilde{\alpha}} = \langle x_{\tilde{\alpha}}\rangle$ respectively $\mathfrak{g}_{-\tilde{\alpha}} = \langle x_{-\tilde{\alpha}}\rangle$ we conclude that $Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha}}) = \langle x_{\alpha_i}, x_{-\alpha_j}, h_{\alpha_j} \mid 1 \le i, j \le 6, j \neq 2\rangle$ resp. $Z_{\mathfrak{g}}(\mathfrak{g}_{-\tilde{\alpha}}) = \langle x_{\alpha_j}, x_{-\alpha_i}, h_{\alpha_j} \mid 1 \le i, j \le 6, j \neq 2\rangle$. Indeed let $y = \lambda x_{-\alpha_2} + \mu h_{\alpha_2}$ then $[y, x_{\tilde{\alpha}}] = \pm\lambda x_{\tilde{\alpha}-\alpha_2} + \mu x_{\tilde{\alpha}} \neq 0$ for $\lambda \neq 0 \neq \mu$ as $x_{\tilde{\alpha}-\alpha_2}$ and $x_{\tilde{\alpha}}$ are linearly independent.

Furthermore we obtain also that $Z_{\mathfrak{g}}(\langle h_{\tilde{\alpha}}\rangle) = \langle x_{\alpha_j}, x_{-\alpha_j}, h_{\alpha_i} \mid 1 \le i, j \le 6, j \neq 2\rangle$. Indeed by theorem 4.7.10, $[x_{\tilde{\alpha}}, x_{-\tilde{\alpha}}] = h_{\tilde{\alpha}}$, which is a $\mathbb{Z}$-linear combination of the vectors h_{α_i} with $1 \le i \le 6$ implying $[h_{\tilde{\alpha}}, h_{\alpha_j}] = 0$ for every $1 \le j \le 6$. By the Jacobi identity, $[h_{\tilde{\alpha}}, y] = [[x_{\tilde{\alpha}}, x_{-\tilde{\alpha}}], y] = -[[x_{-\tilde{\alpha}}, y], x_{\tilde{\alpha}}] - [[y, x_{\tilde{\alpha}}], x_{-\tilde{\alpha}}]$ for any $y \in \mathfrak{g}(E_6)$, therefore we determine that $[h_{\tilde{\alpha}}, x_{\alpha_j}] = 0 = [h_{\tilde{\alpha}}, x_{-\alpha_j}]$ for $1 \le j \le 6$, $j \neq 2$ and $[h_{\tilde{\alpha}}, x_{\alpha_2}] = -[\pm x_{-\tilde{\alpha}+\alpha_2}, x_{\tilde{\alpha}}] = \pm x_{\alpha_2}$ resp. $[h_{\tilde{\alpha}}, x_{-\alpha_2}] = -[\pm x_{\tilde{\alpha}-\alpha_2}, x_{-\tilde{\alpha}}] = \pm x_{-\alpha_2}$.

Since the non-zero vectors x_{α_2} and $x_{-\alpha_2}$ are linearly independent we determine that $[h_{\tilde{\alpha}}, \lambda x_{\alpha_2} + \mu x_{-\alpha_2}] = 0$ if and only if $\lambda = \mu = 0$, which confirms that $Z_{\mathfrak{g}}(\langle h_{\tilde{\alpha}} \rangle) = \langle x_{\alpha_j}, x_{-\alpha_j}, h_{\alpha_i} \mid 1 \le i, j \le 6, j \ne 2 \rangle$.

This proves

$$\begin{aligned} Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}}) &= Z_{\mathfrak{g}}(\langle x_{\tilde{\alpha}}, x_{-\tilde{\alpha}}, h_{\tilde{\alpha}} \rangle) \\ &= Z_{\mathfrak{g}}(\langle x_{\tilde{\alpha}} \rangle) \cap Z_{\mathfrak{g}}(\langle x_{-\tilde{\alpha}} \rangle) \cap Z_{\mathfrak{g}}(\langle h_{\tilde{\alpha}} \rangle) \\ &= \langle x_{\alpha_i}, x_{-\alpha_i}, h_{\alpha_i} \mid 1 \le i \le 6, i \ne 2 \rangle . \end{aligned}$$

Lemma 4.7.26 *The centraliser $Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})$ of $\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}}$ in $\mathfrak{g} = \mathfrak{g}(E_6)$ is isomorphic to the Lie algebra $\mathfrak{sl}_6(\mathbb{C})$.*

Proof: First of all $Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}}) = \langle x_{\alpha_i}, x_{-\alpha_i}, h_{\alpha_i} \mid 1 \le i \le 6; i \ne 2 \rangle$ from the argumentation above. Thus by theorem 4.7.8, $Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})$ is a finite dimensional semi-simple Lie subalgebra with root system $\{\beta \in \Phi \mid \beta = \sum_{i=1, i\ne 2}^{6} \lambda_i \alpha_i, \lambda_i \in \mathbb{Z}\} = \Phi_{Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})}$, basis $\Delta_{Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})} : \alpha_1, \alpha_3, \dots, \alpha_6$ and Cartan subalgebra $H_{Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})} = \langle h_{\alpha_i} \mid 1 \le i \le 6, i \ne 2 \rangle$. The Cartan matrix of $\Phi_{Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})}$ is

$$\mathcal{C}_{Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})} = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix},$$

thus the map $\Delta_{Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})} \to \Delta_{A_5}$ with $\alpha_1 \mapsto \alpha_1^{A_5}$ and $\alpha_i \mapsto \alpha_{i-1}^{A_5}$ for $3 \le i \le 6$ is a bijection between the bases $\Delta_{Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})}$ and Δ_{A_5} such that $\mathcal{C}_{Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})} = \mathcal{C}_{A_5}$. Proposition 4.4.4 implies an isomorphism between the root systems $\Phi_{Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})}$ and Φ_{A_5}. It follows directly from theorem 4.7.9 and chapter VIII 13.1 of [11] that $Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}}) \cong L(A_5) \cong \mathfrak{sl}_6(\mathbb{C})$. ∎

Corollary 4.7.27 *The centraliser $Z_{\mathfrak{g}}(\mathfrak{g}_{\alpha,-\alpha})$ of $\mathfrak{g}_{\alpha,-\alpha}$ in $\mathfrak{g}(E_6)$ is isomorphic to $\mathfrak{sl}_6(\mathbb{C})$ for each root $\alpha \in \Phi_{E_6}$.*

Proof: Since Φ_{E_6} is a reduced irreducible root system and all roots in Φ_{E_6} have the same length, we conclude by lemma 4.4.2 that any root $\alpha \in \Phi_{E_6}$ is conjugate to $\tilde{\alpha}$ under the Weyl group $\mathcal{W}$. The Weyl group $\mathcal{W}$ permutes the root system Φ. Thus let α be a root of Φ then there exists an element $w \in \mathcal{W}$ such that $w\tilde{\alpha}w^{-1} = \alpha$, moreover $w\Delta w^{-1} : w\alpha_1 w^{-1}, \cdots, w\alpha_6 w^{-1}$ is a basis of Φ with the property that α is the maximal root of Φ relative to $w\Delta w^{-1}$ implying by lemma 4.7.26 that $Z_{\mathfrak{g}}(\mathfrak{g}_{\alpha,-\alpha}(E_6))$ is isomorphic to $\mathfrak{sl}_6(\mathbb{C})$. ∎

The complex semi-simple Lie algebra $\mathfrak{g} = \mathfrak{g}(E_6)$ has real compact forms by theorem 4.7.22. More precisely there exists a real compact reductive semi-simple Lie algebra

$\mathfrak{a}$ such that $\mathfrak{a}_{\mathbb{C}} \cong \mathfrak{g}(E_6)$. Denote with ρ the involution of $\mathfrak{g}_{\mathbb{R}}$ with respect to $\mathfrak{a}$. Furthermore by theorem 4.7.18 we also find a simply connected Lie group G such that $L(G) \cong \mathfrak{a}$ implying that $L(G)_{\mathbb{C}} \cong \mathfrak{a}_{\mathbb{C}} \cong \mathfrak{g}(E_6)$. Certainly theorem 4.7.24 implies that the Lie group G is compact.

Let T be a torus of G and $\Phi(G,T)$ be the root system of G with respect to T. There is a bijection between the root system $\Phi(G,T)$ and $\Phi(L(G)_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, where $\mathfrak{t} = L(T)$ a Cartan subalgebra of $\mathfrak{a}$ and $\Phi(L(G)_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is the root system of $L(G)_{\mathbb{C}} \cong \mathfrak{g}(E_6)$ w.r.t. the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$, see observation 4.7.20.

Corollary 4.7.28 *The simply connected compact Lie group G has a root system of type E_6.*

Proof: By the argumentation above we have a bijection between the root systems $\Phi(G,T)$ and $\Phi(L(G)_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \cong \Phi(\mathfrak{g}(E_6), \mathfrak{h})$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}(E_6)$. By chapter IX 4.9, proposition 16, remark part(b) of [11] it follows from the choice of G that $\Phi(G,T) \cong \Phi(\mathfrak{g}(E_6), \mathfrak{h}) = \Phi$, which is the reduced irreducible root system of type E_6. ∎

Let α be an element of the root system $\Phi(G,T)$ and let δ be the isomorphism between $\Phi(L(G)_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \cong \Phi(\mathfrak{g}(E_6), \mathfrak{h})$ and $\Phi(G,T)$, thus $\delta(\alpha) \in \mathfrak{h}^*$ is a root of $\Phi(\mathfrak{g}(E_6), \mathfrak{h})$. We know that $\mathfrak{g}_{\delta(\alpha),\delta(-\alpha)} \cong \mathfrak{sl}_2(\mathbb{C})$ and denote with $\mathfrak{a}_{\delta(\alpha),-\delta(\alpha)}$ the fixed points of $\mathfrak{g}_{\delta(\alpha),-\delta(\alpha)}$ under the involution ρ. Certainly $\rho_{|\mathfrak{g}_{\delta(\alpha),-\delta(\alpha)}}$ satisfies the condition of proposition 4.7.23, thus $\mathfrak{a}_{\delta(\alpha),-\delta(\alpha)}$ is a real compact form of $\mathfrak{g}_{\delta(\alpha),-\delta(\alpha)}$, as $\mathfrak{a}$ is a compact form of $\mathfrak{g}$. By chapter IX.3.4 of [11], the real compact form of $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra $\mathfrak{su}_2(\mathbb{C})$ implying that $\mathfrak{a}_{\delta(\alpha),-\delta(\alpha)} \cong \mathfrak{su}_2(\mathbb{C})$. Recall that $\exp(\mathfrak{su}_2(\mathbb{C})) = SU_2(\mathbb{C})$. It follows that $U_{\alpha,-\alpha} := \exp(\mathfrak{a}_{\delta(\alpha),-\delta(\alpha)}) \cong SU_2(\mathbb{C})$. We call the subgroup $U_{\alpha,-\alpha}$ of G be **fundamental** $SU_2(\mathbb{C})$ **subgroup** of the Lie group G for each root $\alpha \in \Phi(G,T)$.

Lemma 4.7.29 *The identity component $Z_G(U_{\alpha,-\alpha})^\circ$ of the centraliser $Z_G(U_{\alpha,-\alpha})$ of $U_{\alpha,-\alpha}$ in G is isomorphic to $SU_6(\mathbb{C})$ for each root α of G.*

Proof: Since all maximal tori of G are conjugate and the Lie algebra of a maximal torus is a Cartan subalgebra of $L(G)$ by theorem 4.7.19, w.l.o.g. we can fix a maximal torus T of G and prove that $Z_G(U_{\alpha,-\alpha})^\circ \cong SU_6(\mathbb{C})$ for all roots $\alpha \in \Phi(G,T)$.

From corollary 4.7.27 we obtain that for each root $\alpha \in \Phi(G,T)$ the centraliser of $\mathfrak{g}_{\delta(\alpha),-\delta(\alpha)}$ in $\mathfrak{g}(E_6) = \mathfrak{g}$ is isomorphic to $\mathfrak{sl}_6(\mathbb{C})$. By $Z_{\mathfrak{a}}(\mathfrak{a}_{\delta(\alpha),-\delta(\alpha)})$ we denote the fixed points of $Z_{\mathfrak{g}}(\mathfrak{g}_{\delta(\alpha),-\delta(\alpha)})$ under the involution ρ. Thus using the argumentation from above and by chapter IX.3.4 of [11], the Lie subalgebra $Z_{\mathfrak{a}}(\mathfrak{a}_{\delta(\alpha),-\delta(\alpha)})$ is isomorphic to $\mathfrak{su}_6(\mathbb{C})$. Furthermore as $U_{\alpha,-\alpha} = \exp(\mathfrak{a}_{\delta(\alpha),-\delta(\alpha)}) \cong SU_2(\mathbb{C})$ and $SU_2(\mathbb{C})$ is a connected Lie group we conclude that $U_{\alpha,-\alpha}$ is an integral subgroup of G and also that $Z_{\mathfrak{a}}(\mathfrak{a}_{\delta(\alpha),-\delta(\alpha)})$ is the Lie algebra of the Lie subgroup $Z_G(U_{\alpha,-\alpha})$ of G by proposition 4.7.16. Due to proposition 4.7.17 it follows that

$\exp(Z_{\mathfrak{a}}(\mathfrak{a}_{\delta(\alpha),-\delta(\alpha)}))$ generates the identity component $Z_G(U_{\alpha,-\alpha})^{\circ}$ of the centraliser $Z_G(U_{\alpha,-\alpha})$. As we know $\langle \exp(Z_{\mathfrak{a}}(\mathfrak{a}_{\delta(\alpha),-\delta(\alpha)}))\rangle \cong \exp(\mathfrak{su}_6(\mathbb{C})) = \mathrm{SU}_6(\mathbb{C})$ implying $Z_G(U_{\alpha,-\alpha})^{\circ} \cong \mathrm{SU}_6(\mathbb{C})$. ∎

Finally, for some torus T of the Lie group G, we consider the set

$$\mathcal{R} = \{\alpha \in \Phi(G,T) \mid T \text{ be a torus of } G\} = \{\alpha \in \Phi(G, gTg^{-1}) \mid g \in G\},$$

by lemma 4.7.19.

The graph $\mathbf{F}(G) = \mathbf{F}(G(E_{6,-78})) = \mathbf{F}(E_{6,-78})$ has the vertex set $\{U_{\alpha,-\alpha} \mid \alpha \in \mathcal{R}\}$ and two fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups of G, say $U_{\alpha,-\alpha}$ and $U_{\beta,-\beta}$ are joined by an edge if and only if $U_{\alpha,-\alpha} \subseteq Z_G(U_{\beta,-\beta})$ or equivalently $U_{\beta,-\beta} \subseteq Z_G(U_{\alpha,-\alpha})$. Since $U_{\alpha,-\alpha} \cong \mathrm{SU}_2(\mathbb{C})$ for each $\alpha \in \mathcal{R}$, which is a connected Lie subgroup of G, we work with the weaker version, so two fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups of G, say $U_{\alpha,-\alpha}$ and $U_{\beta,-\beta}$ are joined by an edge if and only if $U_{\alpha,-\alpha} \subseteq Z_G(U_{\beta,-\beta})^{\circ}$ or equivalently $U_{\beta,-\beta} \subseteq Z_G(U_{\alpha,-\alpha})^{\circ}$.

In the next part we show that $\mathbf{F}(E_{6,-78})$ is a locally $\mathbf{S}(V_6)$ graph containing a locally $\mathbf{W}(A_5)$ subgraph $\Sigma \cong \mathbf{W}(E_6)$.

By the notation above let $\tilde{\alpha}$ be the root of maximal height of $\Phi(E_6)$ with respect to the basis Δ. Then $Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}}) = \langle x_{\alpha_i}, x_{-\alpha_i}, h_{\alpha_i} \mid 1 \le i \le 6; i \neq 2\rangle \cong \mathfrak{sl}_6(\mathbb{C})$ with irreducible reduced root system $\{\beta \in \Phi \mid \beta = \sum_{i=1,i\neq 2}^{6} \lambda_i\alpha_i, \lambda_i \in \mathbb{Z}\} = \Phi_{Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})}$ and basis $\Delta_{Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})} : \alpha_1, \alpha_3, \dots, \alpha_6$ implying $\mathfrak{g}_{\alpha_i,-\alpha_i} = \langle x_{\alpha_i}, x_{-\alpha_i}, h_{\alpha_i}\rangle \subseteq Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})$. Moreover let φ be the isomorphism between $Z_{\mathfrak{g}}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})$ and $\mathfrak{sl}_6(\mathbb{C})$, where every element μ of $\mathfrak{sl}_6(\mathbb{C})$ will be represented as the matrix $[\mu]_\delta$ with respect to some orthonormal basis $\delta : d_1, \dots, d_6$ of the complex six-dimensional unitary vector space V_6 endowed with the usual scalar product $(\cdot,\cdot)$, such that

$$x_{\alpha_i} \mapsto [\mu_{\alpha_j}]_\delta = A_j = (a_{kl})_{1\le k,l\le 6} \text{ with } \begin{cases} a_{kl} = 1 & \text{if } k = j \text{ and } l = j+1 \\ a_{kl} = 0 & \text{else} \end{cases},$$

$$x_{-\alpha_i} \mapsto [\mu_{-\alpha_j}]_\delta = B_j = (b_{kl})_{1\le k,l\le 6} \text{ with } \begin{cases} b_{kl} = 1 & \text{if } k = j+1 \text{ and } l = j \\ b_{kl} = 0 & \text{else} \end{cases} \quad \text{and}$$

$$h_{\alpha_i} \mapsto [\vartheta_{\alpha_j}]_\delta = C_j = (c_{kl})_{1\le k,l\le 6} \text{ with } \begin{cases} c_{kl} = 1 & \text{if } k = j \text{ and } l = j \\ c_{kl} = 0 & \text{else} \end{cases}$$

for $j = i$ if $i = 1$ and $j = i-1$ if $3 \le i \le 6$. Using the Cartan involution σ of $\mathfrak{g}(E_6)_{\mathbb{R}}$ relative to $\mathfrak{a}$, we obtain an induced isomorphism between $Z_{\mathfrak{a}}(\mathfrak{a}_{\tilde{\alpha},-\tilde{\alpha}})$ and $\mathfrak{su}_6(\mathbb{C})$

such that $\mathfrak{a}_{\alpha_i,-\alpha_i}$ maps to the set of matrices

$$\{[\tau_{\alpha_j}]_\delta = D = (d_{kl})_{1\le k,l\le 6} \mid \begin{cases} D_j = \begin{pmatrix} d_{jj} & d_{jj+1} \\ d_{j+1j} & d_{j+1j+1}\end{pmatrix} & D_j \in \mathfrak{su}_2(\mathbb{C}) \\ d_{kl} = 0 & \text{else} \end{cases} \}$$

for $i = 1 = j$ and $j = i - 1$ for $3 \le i \le 6$. Using this map we obtain the isomorphism $\varphi_G : Z(U_{\tilde{\alpha},-\tilde{\alpha}}) \to SU_6(\mathbb{C})$ such that

$$U_{\alpha_i,-\alpha_i} \mapsto \{[\psi_{\alpha_j}]_\delta = H = (h_{kl})_{1\le k,l\le 6} \mid \begin{cases} H_j = \begin{pmatrix} h_{jj} & h_{jj+1} \\ h_{j+1j} & h_{j+1j+1}\end{pmatrix} & H_j \in SU_2(\mathbb{C}) \\ h_{kl} = 1 & \text{for } k = l \text{ and} \\ & k \in \{1,\dots,6\}\backslash\{j, j+1\} \\ h_{kl} = 0 & \text{else} \end{cases} \}$$

for $i = 1 = j$ and $j = i - 1$ for $3 \le i \le 6$, hence $= \varphi_G(U_{\alpha_i,-\alpha_i}) \cong SU_2(\mathbb{C})$. Furthermore let γ be the action of $SU_6(\mathbb{C})$ on the vector space V_6. The commutator $[\varphi_G(U_{\alpha_i,-\alpha_i}), V_6] = \{[\psi]_\delta[v]_\delta - [v]_\delta \mid v \in V_6, \psi \in \varphi_G(U_{\alpha_i,-\alpha_i})]\}$ of $\varphi_G(U_{\alpha_i,-\alpha_i})$ is a two-dimensional subspace of V_6 and its centraliser is the four-dimensional vector subspace $C_{V_6}(\varphi_G(U_{\alpha_i,-\alpha_i})) = \{v \in V \mid [\psi]_\delta[v]_\delta = [v]_\delta \text{ for all } \psi \in \varphi(U_{\alpha_i,-\alpha_i})\}$. Therefore the subgroup $U_{\alpha_i,-\alpha_i}$ corresponds to a line in V_6 for $i \in \{1, 3, \dots, 6\}$ implying that for each root $\alpha \in \Phi_{Z_\mathfrak{g}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})}$ the fundamental $SU_2(\mathbb{C})$ subgroup $U_{\alpha,-\alpha}$ belongs to a two-dimensional subspace in V_6, as the Weyl group acts transitively on the root system $\Phi_{Z_\mathfrak{g}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})}$.

Let $g \in \mathfrak{g}(E_6)$ then $g\mathfrak{g}_\alpha g^{-1} = \{v \in \mathfrak{g} \mid [h, v] = (g\alpha g^{-1}(h))v \text{ for all } h \in g\mathfrak{t}g^{-1}\} = \mathfrak{g}_{g\alpha g^{-1}} = \mathfrak{g}_\beta$ for a root $\alpha \in \Phi(E_6)$ and a root $\beta = g\alpha g^{-1} : g\mathfrak{t}g^{-1} \to \mathbb{C}^\times$ with $w \mapsto \alpha(g^{-1}wg)$. Thus for each $g \in G$ the subgroup $gU_{\alpha,-\alpha}g^{-1}$ coincides with the fundamental $SU_2(\mathbb{C})$ subgroup $U_{g\alpha g^{-1},-g\alpha g^{-1}}$ of G. Since the group $SU_6(\mathbb{C})$ acts transitively on all lines of the vector space V_6, it follows that the fundamental $SU_2(\mathbb{C})$ subgroups of G which are contained in $Z_\mathfrak{g}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})$ are in one-to-one correspondence with lines of V_6.

As φ_G is an isomorphism we get that two fundamental $SU_2(\mathbb{C})$ subgroups of G which are contained in $Z_\mathfrak{g}(\mathfrak{g}_{\tilde{\alpha},-\tilde{\alpha}})$, say $U_{\alpha,-\alpha}$ and $U_{\beta,-\beta}$, commute if and only if their images, so $\varphi_G(U_{\alpha,-\alpha})$ and $\varphi_G(U_{\beta,-\beta})$, commute.

Furthermore we claim that $[\varphi_G(U_{\alpha,-\alpha}), V_6]^\pi = C_{V_6}(\varphi_G(U_{\alpha,-\alpha}))$ for any root α of the root sytem $\Phi(E_6)$. Thus let $w \in C_{V_6}(\varphi(U_{\alpha,-\alpha}))$, so $\tau(w) = w$ for every $\tau \in \varphi_G(U_{\alpha,-\alpha})$ then $(\mu(v) - v, w) = (\mu(v), w) - (v, w) = (\mu(v), \mu(w)) - (v, w) = (v, w) - (v, w) = 0$ for every $v \in V_6$ and any element μ in $\varphi_G(U_{\alpha,-\alpha})$ implying $C_{V_6}(\varphi_G(U_{\alpha,-\alpha})) \subseteq [\varphi_G(U_{\alpha,-\alpha}), V_6]^\pi$. On the other hand let $v \in [\varphi_G(U_{\alpha_i,-\alpha_i}), V_6]^\pi$, hence $(\mu(w) - w, v) = 0$ for $w \in V_6$ and $\mu \in \varphi_G(U_{\alpha,-\alpha})$, in particular $(\mu(v) - v, v) = 0$ for every $\mu \in \varphi_G(U_{\alpha,-\alpha})$. Therefore $\mu(v) - v = 0$, which is equivalent to $\mu(v) = v$, thus $[\varphi_G(U_{\alpha,-\alpha}), V_6]^\pi \subseteq C_{V_6}(\varphi_G(U_{\alpha,-\alpha}))$ proving the claim.

Finally we will show that two fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups of G, say $U_{\alpha,-\alpha}$ and $U_{\beta,-\beta}$ commute if and only if $[\varphi_G(U_{\alpha,-\alpha}), V_6] \subseteq C_{V_6}(\varphi_G(U_{\beta,-\beta}))$ by the claim above. Since $[\varphi_G(U_{\alpha,-\alpha}), V_6]$ is a line of V_6 and $\dim(C_{V_6}(\varphi_G(U_{\alpha,-\alpha}))) = 4$ we choose an orthonormal basis $\kappa : k_1, \dots, k_6$ such that $\langle k_1, k_2\rangle = [\varphi_G(U_{\alpha,-\alpha}), V_6]$ and $\langle k_3, \dots, k_6\rangle = [\varphi_G(U_{\alpha,-\alpha}), V_6]^\pi = C_{V_6}(\varphi_G(U_{\alpha,-\alpha}))$. Thus for each $\mu \in \varphi_G(U_{\alpha,-\alpha})$ we get

$$[\mu]_\kappa = \begin{pmatrix} M_{a,b} & 0 \\ 0 & 1 \end{pmatrix} \text{ with } M_{ab} = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \text{ and } \det(M_{ab}) = \overline{a}a + \overline{b}b = 1,$$

thus $M_{ab} \in \mathrm{SU}_2(\mathbb{C})$.

Suppose $\varphi_G(U_{\alpha,-\alpha})$ and $\varphi_G(U_{\beta,-\beta})$ commute then $\varphi_G(U_{\beta,-\beta})$ is a subgroup of $Z_{\mathrm{SU}_6(\mathbb{C})}(\varphi_G(U_{\alpha,-\alpha}))$ implying that $[\psi]_\nu = \begin{pmatrix} 1 & 0 \\ 0 & M_{ab} \end{pmatrix}$, for each $\psi \in \varphi_G(U_{\beta,-\beta})$ with respect to an orthonormal basis $\nu : k_1, k_2, n_3, \dots, n_6$. Certainly $[\varphi_G(U_{\beta,-\beta}), V_6] = \langle n_5, n_6\rangle$ and we conclude that $[\varphi_G(U_{\beta,-\beta})), V_6] \subseteq C_{V_6}(\varphi_G(U_{\alpha,-\alpha}))$.

For the other direction if $[\varphi_G(U_{\beta,-\beta}), V_6] \subseteq C_{V_6}(\varphi_G(U_{\alpha,-\alpha}))$ then we find an orthonormal basis $\nu : k_1, k_2, n_3, \dots, n_6$ such that $[\varphi_G(U_{\beta,-\beta}), V_6] = \langle n_5, n_6\rangle$ implying $[\psi]_\nu = \begin{pmatrix} 1 & 0 \\ 0 & M_{ab} \end{pmatrix}$, for each $\psi \in \varphi_G(U_{\beta,-\beta})$. As any two matrices $[\mu]_\nu = \begin{pmatrix} M_{ab} & 0 \\ 0 & 1 \end{pmatrix}$ and $[\psi]_\nu = \begin{pmatrix} 1 & 0 \\ 0 & M_{cd} \end{pmatrix}$ commute for $\psi \in \varphi_G(U_{\beta,-\beta})$ and $\mu \in \varphi_G(U_{\alpha,-\alpha})$, we also get that the subgroups $\varphi_G(U_{\beta,-\beta})$ and $\varphi_G(U_{\alpha,-\alpha})$ of $\mathrm{SU}_6(\mathbb{C})$ commute and we are done. So we have proved that the induced subgraph $\mathbf{F}(E_{6,-78})_{\tilde{\alpha},-\tilde{\alpha}}$ is isomorphic to $\mathbf{S}(V_6)$ and verified the next proposition.

Proposition 4.7.30 *The graph* $\mathbf{F}(E_{6,-78})$ *of the fundamental* $\mathrm{SU}_2(\mathbb{C})$ *subgroups of* G *is locally* $\mathbf{S}(V_6)$.

Proof: The statement is proved for the maximal root $\tilde{\alpha}$ in Φ with respect to a basis Δ by the argumentation above. Using a similar argument as in corollary 4.7.27, it follows that for each root α in Φ the induced subgraph of $\mathbf{F}(E_{6,-78})$ on the neighbours of $U_{\alpha,-\alpha}$ is isomorphic to $\mathbf{S}(V_6)$. ■

We will construct in the last step an induced subgraph Σ of $\mathbf{F}(E_{6,-78})$ containing 32 different vertices such that $\Sigma \cong \mathbf{W}(E_6)$. We start with the root system $\Phi = \Phi_{E_6}$ of G and define the vertex set of Σ to be $\mathcal{V}(\Sigma) = \{U_{\alpha,-\alpha} \mid \alpha \in \Phi\}$. We recall the definition for the reflection graph $\mathbf{W}(E_6)$. The vertices of the graph $\mathbf{W}(E_6)$ are the reflections $\{\rho_\alpha \mid \alpha \in \Phi(E_6)\}$ and two different reflections are joined by an edge if and only if they commute. Notice that two reflections ρ_α and ρ_β commute if and only if $\sigma(\alpha, \beta) = (\alpha, \beta) = 0$. Also for two fundamental $\mathfrak{sl}_2(\mathbb{C})$ Lie subalgebras $\mathfrak{g}_{\alpha,-\alpha}$ and $\mathfrak{g}_{\beta,-\beta}$ in $\mathfrak{g} = \mathfrak{g}(E_6)$ we know via a Chevalley basis of $\mathfrak{g}$ that $\mathfrak{g}_{\alpha,-\alpha} \subseteq Z_{\mathfrak{g}}(\mathfrak{g}_{\beta,-\beta})$

or equivalently $\mathfrak{g}_{\beta,-\beta} \subseteq Z_{\mathfrak{g}}(\mathfrak{g}_{\alpha,-\alpha})$, if and only if $(\alpha, \beta) = 0$. Therefore two fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups $U_{\alpha,-\alpha}$ and $U_{\beta,-\beta}$ in G commute if and only if $(\alpha, \beta) = 0$. It follows that the bijective map $\Sigma \to \mathbf{W}(E_6)$ with $U_{\alpha,-\alpha} \mapsto \rho_\alpha = \rho_{-\alpha}$ is a graph isomorphism.

Lemma 4.7.31 *The graph $\mathbf{F}(E_{6,-78})$ contains an induced subgraph $\Sigma \cong \mathbf{W}(E_6)$.*

Moreover we have to prove that $\mathbf{S}(V_8)$ and $\mathbf{F}(E_{6,-78})$ are simply connected graphs. To achieve this result we will reconstruct from the graphs $\mathbf{S}(V_8)$ and $\mathbf{F}(E_{6,-78})$ a building of type A_7 respectively of type E_6 and use that certain chamber systems are simply connected.

Let $\mathfrak{g}$ be a real semi-simple Lie algebra and B its Killing form. An involution θ of the Lie algebra $\mathfrak{g}$ (understood to respect brackets) such that the symmetric bilinear form $B_\theta(x, y) = -B(x, \theta(y))$ is positive definite, is called a Cartan involution of $\mathfrak{g}$. Correspondingly there is a Cartan decomposition of $\mathfrak{g}$ given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where the subspaces are understood to be the 1 and -1 eigenspaces of θ and B is negative on $\mathfrak{k}$, positive on $\mathfrak{p}$ and $B(\mathfrak{k}, \mathfrak{p}) = 0$.

Let $\mathfrak{a}$ be a complex semi-simple Lie algebra, let $\mathfrak{g}$ be a compact real form of $\mathfrak{a}$ and let ρ be the corresponding conjugation of $\mathfrak{a}$. If $\mathfrak{a}$ is regarded as a real Lie algebra, then ρ is a Cartan involution of $\mathfrak{a}$.

Corollary 4.7.32 (chapter VI.2 of [63])

- *Every real semi-simple Lie algebra $\mathfrak{g}$ has a Cartan involution.*
- *If $\mathfrak{a}$ is a complex semi-simple Lie algebra and is considered as real Lie algebra, then the only Cartan involutions of $\mathfrak{a}$ are the conjugations with respect to the compact real forms of $\mathfrak{a}$.*

Theorem 4.7.33 (theorem 6.31 of [63]) *Let G be connected semi-simple Lie group, let θ be a Cartan involution of its Lie algebra $\mathfrak{g}$, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition, and K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Then*

- *there exists a Lie group automorphism Θ of G with differential θ, and $\Theta^2 = \mathrm{id}$.*
- *the subgroup of G fixed by Θ is K.*
- *the map $K \times \rho \to G$ given by $(k, x) \mapsto k \exp(x)$ is a diffeomorphism onto G.*
- *K is closed.*
- *K contains the center Z of G.*
- *K is compact if and only if Z is finite.*
- *When Z is finite, K is a maximal compact subgroup of G.*

Next we will obtain the Iwasawa decomposition of a complex Lie group G, which is the complexification of a compact connected Lie group K.

Let K be a Lie group. A **complexification of** K consists of a complex Lie group G with a Lie group homomorphism $\tau : K \to G$ such that whenever $f : K \to H$ is a Lie group homomorphism into a complex Lie group, there exists an analytic homomorphism $F : G \to H$ with $f = F \circ \tau$. This is a universal property, so it characterizes the Lie group G up to isomorphism.

Theorem 4.7.34 (theorem 27.1 of [19]) *Let K be a compact connected Lie group. Then K has a complexification $\tau : K \to G$, where G is a complex Lie group. The Lie algebra $L(G)$ of G is the complexification of the Lie algebra $L(K)$ of K.*

Let G be a complexification of a compact connected Lie group K, let T be a maximal torus of K and $\mathfrak{t} = L(T)$ be the Lie algebra of T. Moreover let $T_{\mathbb{C}}$ be the complexification of T and $\mathfrak{t}_{\mathbb{C}}$ be the complexification of the Cartan subalgebra $\mathfrak{t}$. Thus $\mathfrak{t}_{\mathbb{C}}$ is a Cartan subalgebra of $L(G) = \mathfrak{g}$. We consider the root system $R(L(G), \mathfrak{t}_{\mathbb{C}}) = \Phi$ with respect to $\mathfrak{t}_{\mathbb{C}}$ and choose a basis Δ of Φ. Then $N = \{\exp(x) \mid x \in \mathfrak{n}^+\}$ and $B = T_{\mathbb{C}}N$ are closed complex Lie subgroups of G, whose Lie subalgebras are $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ and $\mathfrak{b} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^+$ by theorem 29.2 of [19]. The group B is called the **standard Borel subgroup** of G. A conjugate of B is called a Borel subgroup. The subgroups of G containing a Borel subgroup are called **parabolic subgroups**.

The Borel subgroup B has non trivial intersection with the compact Lie group K. We set $\mathfrak{a} = i\mathfrak{t}$. Then $\mathfrak{a}$ is a Lie subalgebra of some connected Lie subgroup A of the maximal torus T.

Theorem 4.7.35 (Iwasawa Decomposition, theorem 29.3 of [19]) *Let G be a complexification of a compact connected Lie group K and let T be a maximal torus of K. Then every element $g \in G$ can be factored uniquely as bk with $b \in AN$ and $k \in K$, or $g = ank$ with $a \in A, n \in N$ and $k \in K$. The multiplication map $A \times N \times K \to G$ is a diffeomorphism.*

Let Φ be a reduced irreducible root system and Δ be a basis of Φ. Let $\mathfrak{g}$ be the finite dimensional semi-simple complex Lie algebra with corresponding root system Φ and Cartan subalgebra $\mathfrak{h}$, which exists by theorem 4.7.8. By theorem 4.7.22, let $\mathfrak{a}$ be a real compact form of $\mathfrak{g}$ such that $\mathfrak{a}_{\mathbb{C}} = \mathfrak{g}$ and denote with θ the involution of $\mathfrak{g}_{\mathbb{R}}$ with respect to $\mathfrak{a}$. Thus $\mathfrak{a} = \{x \in \mathfrak{g} \mid \theta(x) = x\} = \mathfrak{g}^\theta$ and θ is a Cartan involution of $\mathfrak{g}$ by corollary 4.7.32. Furthermore, by theorem 4.7.18 and theorem 4.7.24, there exists a simply connected compact Lie group A such that $L(A) \cong \mathfrak{a}$. Thus the Lie group A has a complexification G such that $L(G) = \mathfrak{a}_{\mathbb{C}} = \mathfrak{g}$ by theorem 4.7.34. It follows now from theorem 4.7.33 that $A = \{g \in G \mid \Theta(g) = g\} = G^\Theta$, where Θ is a Lie group homomorphisms of G with differential θ. Next we consider the maximal torus t of the Lie group G^Θ such that $L(t) = \mathfrak{h}^\theta$, where $\mathfrak{h}^\theta$ is a Cartan subalgebra of $\mathfrak{g}^\theta$ with $\mathfrak{h}^\theta_{\mathbb{C}} = \mathfrak{h}$. Then $R(L(G), \mathfrak{t}_{\mathbb{C}}) = \Phi$ and $B = TN^+$ is the standard Borel subgroup of G with Lie algebra $\mathfrak{b} = \mathfrak{h}^\theta_{\mathbb{C}} \oplus \mathfrak{n}^+$ with $N^+ = \{\exp(x) \mid x \in \mathfrak{n}^+\}$, $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ and $T = t_{\mathbb{C}}$. By the Iwasawa decomposition, theorem 4.7.35, we may write $G = G^\Theta B$. Moreover, let $N = N_G(T)$ be the normaliser of $T = t_{\mathbb{C}}$ in the Lie group G and let S be the set of simple reflections in $W = N/T$ with respect to the basis Δ of the root system $R(G, T)$. Then (G, B, N, S) is a Tits system by [57]. Hence $(G/B, \delta)$ is a thick building of type (W, S) by proposition A.7.4 and by the Bruhat decomposition. Certainly, the compact Lie group G^Θ acts on the chamber system $\mathcal{C}(G/B)$ by left multiplication. Thus $\alpha : G^\Theta \to \mathrm{Aut}(\mathcal{C}(G/B))$ with $h \mapsto \alpha_h$ and $\alpha_h : \mathcal{C}(G/B) \to \mathcal{C}(G/B)$ such that $gB \to hgB$ for every $g \in G$. Furthermore, using the notation from section A.4, $gB \sim_i kB$ if and only if $g^{-1}k \in P_i$, whence $g^{-1}h^{-1}hk \in P_i$ implying that $hgB \sim_i hkB$. Since α_h is a bijective homomorphism with inverse $\alpha_{h^{-1}}$ for every $h \in G^\Theta$, the map α is a permutation representation of G^Θ in $\mathcal{C}(G/B)$. Proposition A.4.2 implies now that $\mathcal{C}(G/B) \cong \mathcal{C}(G^\Theta, G^\Theta_B, (P^B_i)_{i\in I})$.

Lemma 4.7.36 *The stabiliser G^Θ_B of B in G^Θ is the maximal torus t of G^Θ. The stabiliser P^B_i of the i-panel of $\mathcal{C}(G/B)$ containing B in G^Θ is $P_i \cap G^\Theta$ for every $i \in I$.*

Proof: $\mathrm{Stab}_{G^\Theta}(B) = G^\Theta_B = \{g \in G^\Theta \mid gB = B\} = \{g \in G^\Theta \mid g \in B\}$, therefore $G^\Theta_B = G^\Theta \cap B$. Since $\Theta(B) = B^-$ where B^- is the Lie group of the Lie algebra $\mathfrak{b}^- = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}_\alpha$ by [47], it follows that $B \cap \Theta(B) = T$. Thus we obtain that $G^\Theta \cap B = G^\Theta \cap B \cap \Theta(B) = G^\Theta \cap T = t$.

For the second statement let $\mathcal{R}_i(B)$ be the i-panel of the chamber system $\mathcal{C}(G/B)$ containing B for some $i \in I$, thus $\mathcal{R}_i(B) = \{gB \mid g \in G, gB \sim_i B\}$. Then

$$\begin{aligned}
\mathrm{Stab}_{G^\Theta}(\mathcal{R}_i(B)) &= P^B_i \\
&= \{g \in G^\Theta \mid ghB \in \mathcal{R}_i(B) \text{ for all } hB \in \mathcal{R}_i(B)\} \\
&= \{g \in G^\Theta \mid ghB \sim_i B \text{ for all } hB \in \mathcal{R}_i(B)\} \\
&= \{g \in G^\Theta \mid gh \in P_i \text{ for all } hB \in \mathcal{R}_i(B)\} = G^\Theta \cap P_i\,.
\end{aligned}$$

■

For the next part let Φ to be the reduced irreducible root system of type A_7. Thus $\Phi = \Phi(A_7) = \Phi_{A_7}$ and the basis $\Delta = \Delta_{A_7} = \alpha_1, \dots, \alpha_7$ as describe in [10] Plate I. Moreover by the notation from above we obtain that $\mathfrak{g} = \mathfrak{g}(A_7) = \mathfrak{g}_{A_7} \cong \mathfrak{sl}_8(\mathbb{C})$, the finite dimensional semi-simple complex Lie algebra with a root system of type A_7, $\mathfrak{g}(A_7)^\theta \cong \mathfrak{su}_8(\mathbb{C})$, the compact real form of $\mathfrak{sl}_8(\mathbb{C})$ and $G^\theta = SU_8(\mathbb{C})$. Furthermore, let $\{x_\alpha \in \mathfrak{sl}_8(\mathbb{C})_\alpha \backslash \{0\} \mid \alpha \in \Phi_{A_7}; h_i \mid 1 \le i \le 7\}$ be a Chevalley basis of the Lie algebra $\mathfrak{sl}_8(\mathbb{C})$.

We also consider the connected graph $\mathbf{S}(V_8) = \mathbf{F}(SU_8(\mathbb{C}))$ and choose the induced subgraph Σ with the vertex set $\mathcal{V}(\Sigma) = \{L(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha),-\delta(\alpha)}) \mid \alpha \in \Phi_{A_7}\}$, where δ is the isomorphism between the root systems $\Phi_{A_7} \cong \Phi(\mathfrak{sl}_8(\mathbb{C}), \mathfrak{h})$ and $\Phi(\mathrm{SU}_8(\mathbb{C}), t)$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}_8(\mathbb{C})$ and t is the maximal torus of $\mathrm{SU}_8(\mathbb{C})$, both with respect to the basis Δ_{A_7}. Certainly, $L(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha),-\delta(\alpha)}) \cong \mathrm{SU}_2(\mathbb{C})$ and Σ is isomorphic to $\mathbf{W}(A_7)$. Therefore Σ contains the induced subgraph Λ isomorphic to $\mathbf{H}(A_7)$, whose vertex set is $\mathcal{V}(\Lambda) = \{L(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha),-\delta(\alpha)}) \mid \alpha \in \Delta_{A_7}\}$. We fix the set of induced subgraphs $\mathcal{M} = \{g\Lambda \mid g \in \mathrm{SU}_8\mathbb{C}\}$ of $\mathbf{F}(\mathrm{SU}_8(\mathbb{C}))$ and claim that $\beta : \mathrm{SU}_8(\mathbb{C})/t \to \mathcal{M}$ with $gt \mapsto g\Lambda$ is a bijection. The map β is surjective by construction, since for any element $g\Lambda$ of $\mathcal{M}$ with $g \in \mathrm{SU}_8(\mathbb{C})$ the preimage is $gt \in \mathrm{SU}_8(\mathbb{C})/t$. The map β is injective if from the equality $gt = ht$ for some different $g, h \in \mathrm{SU}_8(\mathbb{C})$ follows that $g\Lambda = h\Lambda$. The relation $gt = ht$ for some different $g, h \in \mathrm{SU}_8(\mathbb{C})$ implies $g^{-1}h \in t$. Therefore the map β is injective if the stabiliser of Λ in $\mathrm{SU}_8(\mathbb{C})$ is the maximal torus t, thus we have to show that $\mathrm{Stab}_{\mathrm{SU}_8(\mathbb{C})}(\Lambda) = \{g \in \mathrm{SU}_8(\mathbb{C}) \mid g\Lambda = \Lambda\} = t$. Since the vertices of Λ are the Lie subgroups $L(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha),-\delta(\alpha)})$ for $\alpha \in \Delta_{A_7}$ we get that

$$\begin{aligned}\mathrm{Stab}_{\mathrm{SU}_8(\mathbb{C})}(\Lambda) \;=\; &\{g \in \mathrm{SU}_8(\mathbb{C}) \mid gL(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha),-\delta(\alpha)})g^{-1} = L(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha),-\delta(\alpha)})\\ &\text{for every } \alpha \in \Delta_{A_7}\}\,.\end{aligned}$$

Hence the Lie group $\mathrm{Stab}_{\mathrm{SU}_8(\mathbb{C})}(\Lambda)$ is therefore the Lie group of the Lie subalgebra $\bigcap_{\alpha\in\Delta_{A_7}} N_{\mathfrak{su}_8(\mathbb{C})}(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha),-\delta(\alpha)})$ implying that $\mathrm{Stab}_{\mathrm{SU}_8(\mathbb{C})}(\Lambda)$ is the Lie group of Lie subalgebra $(\bigcap_{\alpha\in\Delta_{A_7}} N_{\mathfrak{sl}_8(\mathbb{C})}(\mathfrak{sl}_8(\mathbb{C})_{\alpha,-\alpha}))^{\theta}$. Since $\mathfrak{sl}_8(\mathbb{C})$ is generated by the non zero vectors $x_{\alpha_i}, x_{-\alpha_i}$ for $1 \le i \le 7$ of the choosen Chevalley basis of $\mathfrak{sl}_8(\mathbb{C})$ by proposition 4.7.7 and $\mathfrak{sl}_8(\mathbb{C})_{\alpha_i,-\alpha_i} = \langle h_i, x_{\alpha_i}, x_{-\alpha_i} \mid 1 \le i \le 7\rangle$ we know that $N_{\mathfrak{sl}_8(\mathbb{C})}(\mathfrak{sl}_8(\mathbb{C})_{\alpha_i,-\alpha_i}) = \{g \in \mathfrak{sl}_8(\mathbb{C}) \mid [g, x_{\alpha_i}], [g, x_{-\alpha_i}], [g, h_i] \in \mathfrak{sl}_8(\mathbb{C})_{\alpha_i,-\alpha_i}\}$. By theorem 4.7.10

$$\begin{aligned}&N_{\mathfrak{sl}_8(\mathbb{C})}(\mathfrak{sl}_8(\mathbb{C})_{\alpha_1,-\alpha_1}) = \langle x_{\alpha_1}, x_{-\alpha_1}, x_{\alpha_j}, x_{-\alpha_j}, h_k \mid 1 \le k \le 7, 3 \le j \le 7\rangle\,,\\ &\text{for } 2 \le i \le 6\\ &N_{\mathfrak{sl}_8(\mathbb{C})}(\mathfrak{sl}_8(\mathbb{C})_{\alpha_i,-\alpha_i}) = \langle x_{\alpha_j}, x_{-\alpha_j}, h_k \mid 1 \le k \le 7, j \in \{1, \dots, 7\}\backslash\{i+1, i-1\}\rangle \text{ and}\\ &N_{\mathfrak{sl}_8(\mathbb{C})}(\mathfrak{sl}_8(\mathbb{C})_{\alpha_7,-\alpha_7}) = \langle x_{\alpha_7}, x_{-\alpha_7}, x_{\alpha_j}, x_{-\alpha_j}, h_k \mid 1 \le k \le 7, 1 \le j \le 5\rangle,\end{aligned}$$

implying that $\bigcap_{\alpha\in\Delta_{A_7}} N_{\mathfrak{sl}_8(\mathbb{C})}(\mathfrak{sl}_8(\mathbb{C})_{\alpha,-\alpha}) = \langle h_k \mid 1 \le k \le 7\rangle = \mathfrak{h}$ and thus we conclude that $(\bigcap_{\alpha\in\Delta_{A_7}} N_{\mathfrak{sl}_8(\mathbb{C})}(\mathfrak{sl}_8(\mathbb{C})_{\alpha,-\alpha}))^{\theta} = \mathfrak{h}^{\theta}$, which proves the statement that $\mathrm{Stab}_{\mathrm{SU}_8(\mathbb{C})}(\Lambda) = t$.

Using this fact, we define the chamber system $\mathcal{C}(\mathcal{M}) = (\mathcal{M}, (\sim_i)_{1\le i\le 7})$, where two chambers $g\Lambda$ and $h\Lambda$ are i-adjacent for $1 \le i \le 7$, in symbols $g\Lambda \sim_i h\Lambda$, if and only if $g^{-1}h \in P_i \cap \mathrm{SU}_8(\mathbb{C})$, see lemma 4.7.36 to get the following lemma by construction.

Lemma 4.7.37 *The two chamber systems $\mathcal{C}(\mathrm{SU}_8(\mathbb{C}), t, (P_i \cap \mathrm{SU}_8(\mathbb{C}))_{i\in\{1,\dots,7\}})$ and $\mathcal{C}(\mathrm{SL}_8(\mathbb{C})/B)$ are isomorphic. Furthermore via the map $\beta : \mathrm{SU}_8(\mathbb{C})/t \to \mathcal{M}$ with $gt \mapsto g\Lambda$ the two different chamber systems $\mathcal{C}(\mathrm{SU}_8(\mathbb{C}), t, (P_i \cap \mathrm{SU}_8(\mathbb{C}))_{i\in\{1,\dots,7\}})$ and $\mathcal{C}(\mathcal{M}) = (\mathcal{M}, (\sim_i)_{1\le i\le 7})$ are also isomorphic.*

By lemma 4.7.37 and the fact that $\mathcal{C}(SL_8(\mathbb{C})/B)$ is a simply connected chamber system, see proposition A.7.2, the chamber system $\mathcal{C}(\mathcal{M}) = (\mathcal{M}, (\sim_i)_{1\le i\le 7})$ is simply connected. Therefore let $\gamma : g\Lambda, g_1\Lambda, \dots, g_n\Lambda, g\Lambda$ be a closed gallery of the chamber system $\mathcal{C}(\mathcal{M}) = (\mathcal{M}, (\sim_i)_{1\le i\le 7})$ then γ is null-2-homotopic, thus there is a finite sequence $\gamma_0, \gamma_1, \dots, \gamma_l$ of galleries of $\mathcal{C}(\mathcal{M})$ such that $\gamma = \gamma_0$, $g\Lambda = \gamma_l$ and the gallery γ_{k-1} is elementary 2-homotopic to γ_k for each $1 \le k \le l$.

It follows that $\gamma_{k-1} = \varepsilon\kappa_{k-1}\psi$ and $\gamma_k = \varepsilon\kappa_k\psi$ for some galleries ε and ψ and two J-galleries κ_{k-1} and κ_k for some subset $J \subseteq \{1, \dots, 7\}$ of cardinality two for each $1 \le k \le l$. That implies that $\kappa_{k-1}\kappa_k$ is a closed gallery in $\mathcal{R}_{i,j}(h\Lambda)$ for some different indices $i, j \in \{1, \dots, 7\}$ and $h \in SU_8(\mathbb{C})$. So $\kappa_{k-1}\kappa_k$ is a combination of closed galleries $h\Lambda, h_1\Lambda, h_2\Lambda, h_3\Lambda, h\Lambda$ of length four with $h\Lambda \sim_i h_1\Lambda \sim_j h_2\Lambda \sim_i h_3\Lambda \sim_j h\Lambda$ for some different $i, j \in \{1, \dots, 7\}$ if $\mathcal{R}_{i,j}(h\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{———}}\circ$ or a combination of closed galleries $h\Lambda, h_1\Lambda, h_2\Lambda, h_3\Lambda, h_4\Lambda, h_5\Lambda, h\Lambda$ with $h\Lambda \sim_i h_1\Lambda \sim_j h_2\Lambda \sim_i h_3\Lambda \sim_j h_4\Lambda \sim_i h_5\Lambda \sim_j h\Lambda$ of length six for some different $i, j \in \{1, \dots, 7\}$ if $\mathcal{R}_{i,j}(h\Lambda)$ is a rank two building of type $\circ\overset{3}{\text{———}}\circ$.

Let $\gamma : v_{\alpha_0}, v_{\alpha_1}, \dots, v_{\alpha_{n-1}}, v_{\alpha_n} = v_{\alpha_0}$ be a closed cycle in the connected graph $\mathbf{S}(V_8) = \mathbf{F}(SU_8(\mathbb{C}))$ with $v_{\alpha_i} = L(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha_i),-\delta(\alpha_i)})$ for $\alpha_i \in \Phi_{A_7}$ and $0 \le i \le n$. Since $SU_8(\mathbb{C})$ acts transitively on the graph $\mathbf{S}(V_8)$ we find for each $i \in \{1, \dots, n\}$ an element $g_i \in SU_8(\mathbb{C})$ such that $g_i\Lambda$ is an induced subgraph of $\mathbf{S}(V_8)$ containing the vertex v_{α_i}. By the connectivity of the chamber system $\mathcal{C}(\mathcal{M})$ we find finitely many group elements $g_1^{i-1,i}, \dots, g_{l_{i-1}}^{i-1,i}$ in $SU_8(\mathbb{C})$ for $1 \le i \le n$ in such a way that $g_0\Lambda, g_1^{0,1}\Lambda, \dots, g_{l_0}^{0,1}\Lambda, g_1\Lambda, g_1^{1,2}\Lambda, \dots, g_{n-1}\Lambda, g_1^{n-1,n}\Lambda, \dots, g_{l_{n-1}}^{n-1,n}\Lambda, g_n\Lambda = g_0\Lambda$ is a closed gallery in $\mathcal{C}(\mathcal{M})$. By the last paragraph and the transitivity of the group $SU_8(\mathbb{C})$ on $\mathbf{S}(V_8)$, the graph $\mathbf{S}(V_8) = \mathbf{F}(SU_8(\mathbb{C}))$ is simply connected if each closed gallery $\Lambda, g_1\Lambda, \dots, g_{2m-1}, \Lambda$ in every rank two residue $\mathcal{R}_{i,j}(\Lambda)$, for $i, j \in \{1, \dots, 7\}$, $i \ne j$, which is a rank two building of type $\circ\overset{m}{\text{———}}\circ$ with either $m = 2$ or $m = 3$, is simply connected as graph in $\mathbf{S}(V_8)$. In other words, $\mathbf{S}(V_8)$ is simply connected if $\Lambda \cup (\bigcup_{k=1}^{2m-1} g_k\Lambda)$ is a simply connected graph.

Recall that in the chamber system $\mathcal{C}(\mathcal{M})$ two elements $g\Lambda$ and $h\Lambda$ are i-adjacent if and only if $\Lambda \sim_i g^{-1}h\Lambda$ if and only if $g^{-1}h \in P_i \cap SU_8\mathbb{C}$ for $1 \le i \le 7$. Since the Lie subgroup $P_i \cap SU_8\mathbb{C}$ is the Lie group of the Lie subalgebra $\mathfrak{p}_i \cap \mathfrak{su}_8(\mathbb{C})$ with $\mathfrak{p}_i = \mathfrak{t}_{\mathbb{C}} \bigoplus_{\alpha\in\Phi^+\cup\{-\alpha_i\}} \mathfrak{sl}_8(\mathbb{C})_\alpha$ for each $i \in \{1, \dots, 7\}$, we obtain for each index $i \in \{1, \dots, 7\}$ that $\mathfrak{p}_i \cap \mathfrak{su}_8(\mathbb{C}) = \mathfrak{t} \oplus \mathfrak{sl}_8(\mathbb{C})^\theta_{\alpha_i,-\alpha_i}$. Thus

$$
\begin{aligned}
&\mathfrak{p}_1 \cap \mathfrak{su}_8(\mathbb{C}) \le N_{\mathfrak{sl}_8(\mathbb{C})}(\mathfrak{sl}_8(\mathbb{C})_{\alpha_j,-\alpha_j}) && \text{for } 3 \le j \le 7 \text{ and } j = 1\,, \\
&\text{for } 2 \le i \le 6 && \\
&\mathfrak{p}_i \cap \mathfrak{su}_8(\mathbb{C}) \le N_{\mathfrak{sl}_8(\mathbb{C})}(\mathfrak{sl}_8(\mathbb{C})_{\alpha_j,-\alpha_j}) && \text{for } j \in \{1, \dots, 7\}\backslash\{i-1, i+1\} \text{ and} \\
&\mathfrak{p}_7 \cap \mathfrak{su}_8(\mathbb{C}) \le N_{\mathfrak{sl}_8(\mathbb{C})}(\mathfrak{sl}_8(\mathbb{C})_{\alpha_j,-\alpha_j}) && \text{for } 1 \le j \le 5 \text{ and } j = 7\,.
\end{aligned}
$$

Hence for $g \in SU_8(\mathbb{C})$ we get that

for $i=1$ and for $3 \le j \le 7$ and $j=1$
$\Lambda \sim_1 g\Lambda \quad \Leftrightarrow \quad gL(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha_j),-\delta(\alpha_j)})g^{-1} = L(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha_j),-\delta(\alpha_j)})$
for $2 \le i \le 6$ and for $j \in \{1,\dots,7\}\backslash\{i-1,i+1\}$
$\Lambda \sim_i g\Lambda \quad \Leftrightarrow \quad gL(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha_j),-\delta(\alpha_j)})g^{-1} = L(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha_j),-\delta(\alpha_j)})$
for $i=7$ and for $1 \le j \le 5$ and $j=7$
$\Lambda \sim_7 g\Lambda \quad \Leftrightarrow \quad gL(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha_j),-\delta(\alpha_j)})g^{-1} = L(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha_j),-\delta(\alpha_j)})$.

We simplify the notation, thus for $1 \le j \le 7$ we set $L(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha_j),-\delta(\alpha_j)}) = a_j$ and $gL(\mathfrak{su}_8(\mathbb{C})_{\delta(\alpha_j),-\delta(\alpha_j)})g^{-1} = a_j^g$. Therefore Λ is the induced subgraph on the vertices $a_1,\dots,a_7$ and $g\Lambda$ is the induced subgraph on the vertices $a_1^g,\dots,a_7^g$. Thus for $g \in SU_8(\mathbb{C})$ obtain that

$g\Lambda \sim_1 h\Lambda \quad \Leftrightarrow \quad a_j^g = a_j^h$ for $3 \le j \le 7$ and $j=1$
for $2 \le i \le 6$
$g\Lambda \sim_i h\Lambda \quad \Leftrightarrow \quad a_j^g = a_j^h$ for $j \in \{1,\dots,7\}\backslash\{i-1,i+1\}$
$g\Lambda \sim_7 h\Lambda \quad \Leftrightarrow \quad a_j^g = a_j^h$ for $1 \le j \le 5$ and $j=7$.

In the next table we collect some information of every $\mathcal{R}_{i,j}(\Lambda)$ residue of $\mathcal{C}(\mathcal{M})$ for $i,j \in \{1,\dots,7\}$, $i \neq j$.

$\mathcal{R}_{1,2}(\Lambda)$ 3-gon	$\Lambda \sim_1 g_1\Lambda \sim_2 g_2\Lambda \sim_1 g_3\Lambda \sim_2 g_4\Lambda \sim_1 g_5\Lambda \sim_2 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 5$, $4 \le l \le 7$
$\mathcal{R}_{1,3}(\Lambda)$ 2-gon	$\Lambda \sim_1 g_1\Lambda \sim_3 g_2\Lambda \sim_1 g_3\Lambda \sim_3 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 3$, $l \in \{1,3,5,6,7\}$
$\mathcal{R}_{1,4}(\Lambda)$ 2-gon	$\Lambda \sim_1 g_1\Lambda \sim_4 g_2\Lambda \sim_1 g_3\Lambda \sim_4 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 3$, $l \in \{1,4,6,7\}$
$\mathcal{R}_{1,5}(\Lambda)$ 2-gon	$\Lambda \sim_1 g_1\Lambda \sim_5 g_2\Lambda \sim_1 g_3\Lambda \sim_5 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 3$, $l \in \{1,3,5,7\}$
$\mathcal{R}_{1,6}(\Lambda)$ 2-gon	$\Lambda \sim_1 g_1\Lambda \sim_6 g_2\Lambda \sim_1 g_3\Lambda \sim_6 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 3$, $l \in \{1,3,4,6\}$
$\mathcal{R}_{1,7}(\Lambda)$ 2-gon	$\Lambda \sim_1 g_1\Lambda \sim_7 g_2\Lambda \sim_1 g_3\Lambda \sim_7 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 3$, $l \in \{1,3,4,5,7\}$
$\mathcal{R}_{2,3}(\Lambda)$ 3-gon	$\Lambda \sim_2 g_1\Lambda \sim_3 g_2\Lambda \sim_2 g_3\Lambda \sim_3 g_4\Lambda \sim_2 g_5\Lambda \sim_3 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 5$, $5 \le l \le 7$
$\mathcal{R}_{2,4}(\Lambda)$ 2-gon	$\Lambda \sim_2 g_1\Lambda \sim_4 g_2\Lambda \sim_2 g_3\Lambda \sim_4 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 3$, $l \in \{2,4,6,7\}$

$\mathcal{R}_{2,5}(\Lambda)$ 2-gon	$\Lambda \sim_2 g_1\Lambda \sim_5 g_2\Lambda \sim_2 g_3\Lambda \sim_5 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{2,5,7\}$
$\mathcal{R}_{2,6}(\Lambda)$ 2-gon	$\Lambda \sim_2 g_1\Lambda \sim_6 g_2\Lambda \sim_2 g_3\Lambda \sim_6 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{2,4,6\}$
$\mathcal{R}_{2,7}(\Lambda)$ 2-gon	$\Lambda \sim_2 g_1\Lambda \sim_7 g_2\Lambda \sim_2 g_3\Lambda \sim_7 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{2,4,5,7\}$
$\mathcal{R}_{3,4}(\Lambda)$ 3-gon	$\Lambda \sim_3 g_1\Lambda \sim_4 g_2\Lambda \sim_3 g_3\Lambda \sim_4 g_4\Lambda \sim_3 g_5\Lambda \sim_4 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 5$, $l \in \{1,6,7\}$
$\mathcal{R}_{3,5}(\Lambda)$ 2-gon	$\Lambda \sim_3 g_1\Lambda \sim_5 g_2\Lambda \sim_3 g_3\Lambda \sim_5 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{1,3,5,7\}$
$\mathcal{R}_{3,6}(\Lambda)$ 2-gon	$\Lambda \sim_3 g_1\Lambda \sim_6 g_2\Lambda \sim_3 g_3\Lambda \sim_6 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{1,3,6\}$
$\mathcal{R}_{3,7}(\Lambda)$ 2-gon	$\Lambda \sim_3 g_1\Lambda \sim_7 g_2\Lambda \sim_3 g_3\Lambda \sim_7 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{1,3,5,7\}$
$\mathcal{R}_{4,5}(\Lambda)$ 3-gon	$\Lambda \sim_4 g_1\Lambda \sim_5 g_2\Lambda \sim_4 g_3\Lambda \sim_5 g_4\Lambda \sim_4 g_5\Lambda \sim_5 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 5$, $l \in \{1,2,7\}$
$\mathcal{R}_{4,6}(\Lambda)$ 2-gon	$\Lambda \sim_4 g_1\Lambda \sim_6 g_2\Lambda \sim_4 g_3\Lambda \sim_6 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{1,2,4,6\}$
$\mathcal{R}_{3,7}(\Lambda)$ 2-gon	$\Lambda \sim_4 g_1\Lambda \sim_7 g_2\Lambda \sim_4 g_3\Lambda \sim_7 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{1,2,4,7\}$
$\mathcal{R}_{5,6}(\Lambda)$ 3-gon	$\Lambda \sim_5 g_1\Lambda \sim_6 g_2\Lambda \sim_5 g_3\Lambda \sim_6 g_4\Lambda \sim_5 g_5\Lambda \sim_6 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 5$, $l \in \{1,2,3\}$
$\mathcal{R}_{5,7}(\Lambda)$ 2-gon	$\Lambda \sim_5 g_1\Lambda \sim_7 g_2\Lambda \sim_5 g_3\Lambda \sim_7 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{1,2,3,5,7\}$
$\mathcal{R}_{6,7}(\Lambda)$ 3-gon	$\Lambda \sim_6 g_1\Lambda \sim_7 g_2\Lambda \sim_6 g_3\Lambda \sim_7 g_4\Lambda \sim_6 g_5\Lambda \sim_7 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 5$, $l \in \{1,2,3,4\}$

To prove that the graph $\Lambda \cup (\bigcup_{k=1}^{2m-1} g_k\Lambda)$ is simply connected for a closed gallery $\Lambda, g_1\Lambda, \ldots, g_{2m-1}, \Lambda$ in the rank two building $\mathcal{R}_{i,j}(\Lambda)$ of type $\circ\overset{m}{\text{———}}\circ$ for some $i, j \in \{1, \ldots, 7\}$, $i \neq j$, we will use the following statements.

Lemma 4.7.38 *Let $\Gamma = (\mathcal{X}, E)$ be a simply connected graph. Then the derived graph $\widetilde{\Gamma} = (\mathcal{X} \cup \{v\}, E \cup \{\{v, v_1\}, \{v, v_2\}\})$ from Γ with $v_1, v_2 \in \mathcal{X}$, $v_1 \neq v_2$ and $v \notin \mathcal{X}$ is simply connected if and only if $\{v_1, v_2\} \in E$.*

Proof: Certainly if $\{v_1, v_2\} \in E$ then $\widetilde{\Gamma}$ is simply connected. If otherwise the edge $\{v_1, v_2\}$ not an element of E then let $\gamma : v_1, x_1, \dots, x_n, v_2$ be path from v_1 to v_2 in Γ. The closed path $v, v_1, x_1, \dots, x_n, v_2, v$ in $\widetilde{\Gamma}$ can not be decomposed into triangles, thus $\widetilde{\Gamma}$ is not simply connected. ∎

Lemma 4.7.39 *Let $\Gamma = (\mathcal{X}, E)$ be a simply connected graph. Then the derived graph $\widetilde{\Gamma} = (\mathcal{X}, E \cup \{\{v_1, v_2\}\})$ from Γ with $v_1, v_2 \in \mathcal{X}$, $v_1 \neq v_2$, is simply connected if and only if $\{v_1, v_2\} \in E$ or there exists a vertex $v \in \mathcal{X}$ such that $\{v_1, v\}, \{v_2, v\} \in E$.*

Proof: If $\{v_1, v_2\} \in E$ then $\Gamma = \widetilde{\Gamma}$ and we have nothing to prove. Therefore, we assume $\{v_1, v_2\} \notin E$. If we can find a vertex $v \in \mathcal{X}$ such that $\{v_1, v\}, \{v_2, v\} \in E$, then certainly $\widetilde{\Gamma}$ is simply connected. Suppose the vertices v_1 and v_2 have distance $d \geq 3$ in Γ. Let $\gamma : v_1, x_1, \dots, x_{d-1}, v_2$ be a shortest path between v_1 to v_2 in Γ, then $v_1, x_1, \dots, x_{d-1}, v_2, v_1$ is a closed path in $\widetilde{\Gamma}$, which can not be decomposed into triangles, thus $\widetilde{\Gamma}$ is not simply connected. ∎

Let $\Lambda, g_1\Lambda, \dots, g_{2m-1}, \Lambda$ be a closed gallery in the rank two building $\mathcal{R}_{i,j}(\Lambda)$ of type $\circ\!\overset{m}{\text{———}}\!\circ$ for some different indices $i, j \in \{1, \dots, 7\}$. We colour the vertices of the graphs $\Lambda, g_1\Lambda, \dots, g_{2m-1}$ in different ways. First of all the vertices

$$\begin{aligned} \mathcal{V}(\Lambda) \cap \Big(\bigcap_{k=1}^{2m-1} \mathcal{V}(g_k\Lambda)\Big) &= \{a_1, \dots, a_7\} \cap \Big(\bigcap_{k=1}^{2m-1} \{a_1^{g_k}, \dots, a_7^{g_k}\}\Big) \\ &= \{a_i \mid 1 \leq i \leq 7, a_i = a_i^{g_k} \text{ for } 1 \leq k \leq 2m-1\} \end{aligned}$$

are coloured black. Then the vertices $\mathcal{V}(\Lambda) = \{a_1, \dots, a_7\}$ are coloured in blue and the vertices $\mathcal{V}(g_k\Lambda) = \{a_1^{g_k}, \dots, a_7^{g_k}\}$ are coloured in c_{g_k} for $1 \leq k \leq 7$. Moreover we set $\Psi = \Lambda \cup (\bigcup_{k=1}^{2m-1} g_k\Lambda)$, thus Ψ is a subgraph of $\mathbf{S}(V_8)$. We denote with Ψ^c the induced subgraph of Ψ on the vertex set $\mathcal{V}(\Psi^c)$, which contains all vertices with the colour c, for $c \in \{\text{black}, \text{blue}, c_{g_1}, \dots, c_{g_{2m-1}}\}$.

Recall that $\Lambda \cong \mathbf{H}(A_7) \cong g_k\Lambda$ for $1 \leq k \leq 7$ and that $\mathbf{H}(A_7)$ is a simply connected graph by lemma 4.7.38 and lemma 4.7.39.

Proposition 4.7.40 *Let $\Lambda, g_1\Lambda, \dots, g_{2m-1}, \Lambda$ be a closed gallery in the rank two building $\mathcal{R}_{i,j}(\Lambda)$ of type $\circ\!\overset{m}{\text{———}}\!\circ$ for some $i, j \in \{1, \dots, 7\}$, $i \neq j$. If the graph Ψ^{black} is simply connected and contains a vertex $w \in \mathcal{V}(\Psi^{\text{black}})$ such that every vertex $v \in \mathcal{V}(\Psi) \backslash \mathcal{V}(\Psi^{\text{black}})$ is adjacent to w in Ψ, then Ψ is a simply connected graph.*

Proof: Let $\gamma : z = 0, z_1, \dots, z_t, z$ be a closed path in Ψ. If the vertices of γ have a colour in common then γ is a path in the simply connected graph Ψ^c for some colour $c \in \{\text{black}, \text{blue}, c_{g_1}, \dots, c_{g_{2m-1}}\}$ and we are done. Suppose the vertices of γ have no colour in common, then let cc_1 be the first index in $\{1, \dots, t-1\}$ such that the vertices $\{z_k \mid 0 \leq k \leq cc_1\}$ have a colour $c_1 \in \{\text{black}, \text{blue}, c_{g_1}, \dots, c_{g_{2m-1}}\}$ in common and the vertex z_{cc_1+1} has not the colour c_1. Now let cc_l be the index of

$\{1, \dots, t-1\}$ such that the vertices $\{z_k \mid cc_{l-1} \leq k \leq cc_l\}$ have a colour c_l of the set $\{\text{black}, \text{blue}, c_{g_1}, \dots, c_{g_{2m-1}}\}$ in common and the vertex z_{cc_l+1} has not the colour c_l. Thus $\gamma = z, \dots, z_{cc_1}, z_{cc_1+1}, \dots, z_{cc_r}, z_{cc_r+1}, \dots, z$ for some $1 \leq r < t-1$.

For every index $cc_i \in \{1, \dots, r\}$ and $cc_i = cc_r + 1$ we provide the following changes on the path γ. If z_{cc_i} is not equal to w and has the colour black, then let $\gamma^t_{cc_i}$ be a path in Ψ^{black} from z_{cc_i} to w and $\gamma^b_{cc_i}$ be a path from w to z_{cc_i} in Ψ^{black}. If $z_{cc_i} = w$ or z_{cc_i} has not the colour black then we choose the pathes $\gamma^t_{cc_i}$ and $\gamma^b_{cc_i}$ are equal to the empty path.

By the assymption that graph Ψ^{black} is simply connected we obtain that γ is homotopically equivalent to the closed path $z, \dots, z_{cc_1}, \gamma^t_{cc_1}, w, \gamma^b_{cc_1}, z_{cc_1}, z_{cc_1+1}, \dots, z_{cc_r}, \gamma^t_{cc_r}, w, \gamma^b_{cc_r}, z_{cc_r}, z_{cc_r+1}, \dots, z$ and homotopically equivalent to the path $z, \dots, z_{cc_1}, \gamma^t_{cc_1}, w, \gamma^b_{cc_1+1}, z_{cc_1+1}, \gamma^t_{cc_1+1}, w, \gamma^b_{cc_1}, z_{cc_1}, z_{cc_1+1}, \dots, z_{cc_r}, \gamma^t_{cc_r}, w, \gamma^b_{cc_r+1}, z_{cc_r+1}, \gamma^t_{cc_r+1}, w, \gamma^b_{cc_r}, z_{cc_r}, z_{cc_r+1}, \dots, z$. Since $z_{cc_l+1}, \gamma^t_{cc_l+1}, w, \gamma^b_{cc_l}, z_{cc_l}, z_{cc_l+1}$ is either a closed path in the simply connected graph $\Psi^{c_{l+1}}$ if $\gamma^t_{cc_l+1}$ is the empty path or a closed path in the simply connected graph Ψ^{c_l} if $\gamma^b_{cc_l}$ is the empty path or a closed path in the simply connected graph Ψ^{black} if $\gamma^t_{cc_l+1}$ and $\gamma^b_{cc_l}$ are not the empty path or the triangle $z_{cc_l+1}, w, z_{cc_l}, z_{cc_l+1}$ if $\gamma^t_{cc_l+1}$ and $\gamma^b_{cc_l}$ are the empty path, for each index $1 \leq l \leq r$. Therefore we get that γ is homotopically equivalent to the closed path $z, \dots, z_{cc_1}, \gamma^t_{cc_1}, w, \gamma^b_{cc_1+1}, z_{cc_1+1}, \dots, z_{cc_2}, \gamma^t_{cc_2}, w, \gamma^b_{cc_2+1}, z_{cc_2+1}, \dots, z_{cc_r}, \gamma^t_{cc_r}, w, \gamma^b_{cc_r+1}, z_{cc_r+1}, \dots, z$. Furthermore, for each index $1 \leq l \leq r-1$ the closed path $w, \gamma^b_{cc_l+1}, z_{cc_l+1}, \dots, z_{cc_{l+1}}, \gamma^t_{cc_{l+1}}, w$ runs completely in the simply connected graph $\Psi^{c_{l+1}}$, thus γ is homotopically equivalent to the closed path $z, \dots, z_{cc_1}, \gamma^t_{cc_1}, w, \gamma^b_{cc_r+1}, z_{cc_r+1}, \dots, z$.

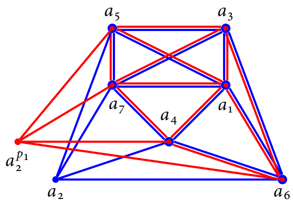
Finally is the colour c_1 equal to the colour c_{r+1}, where c_{r+1} is the common colour of the vertices $z_{cc_r+1}, \dots, z$ then $z, \dots, z_{cc_1}, \gamma^t_{cc_1}, w, \gamma^b_{cc_r+1}, z_{cc_r+1}, \dots, z$ is a closed path in the simply connected graph Ψ^{c_1} implying that Ψ is simply connected.

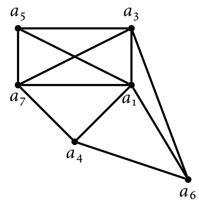
If otherwise c_1 and c_{r+1} are to different colours. Then let γ^t_z be a path from w to z and γ^b_z be a path from z to w in Ψ^{black} if $z \in \mathcal{V}(\Psi^{\text{black}})$ and if z has not the colour black then we choose γ^t_z and γ^b_z are equal to the empty path. Certainly $z, \dots, z_{cc_1}, \gamma^t_{cc_1}, w, \gamma^b_{cc_r+1}, z_{cc_r+1}, \dots, z$ is homotopically equivalent to the closed path $z, \dots, z_{cc_1}, \gamma^t_{cc_1}, w, \gamma^t_z, z, \gamma^b_z, w, \gamma^b_{cc_r+1}, z_{cc_r+1}, \dots, z$. Moreover $z, \dots, z_{cc_1}, \gamma^t_{cc_1}, w, \gamma^t_z, z$ is a closed path in Ψ^{c_1} and $z, \gamma^b_z, w, \gamma^b_{cc_r+1}, z_{cc_r+1}, \dots, z$ a is closed path in $\Psi^{c_{r+1}}$ implying that Ψ is simply connected. ∎

Next, we consider the graphs $\Lambda \cup p_i\Lambda$ and $\Lambda \cap p_i\Lambda$ for some $p_i \in P_i \cap SU_8(\mathbb{C})$ for $1 \leq i \leq 7$. Recall that the vertices of Λ are coloured blue and the we set in each case $c_{p_i} = \text{red}$. So the vertices of the graph $p_i\Lambda$ are coloured red.

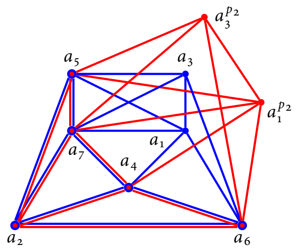
$\Lambda \cup p_i \Lambda$	$\Lambda \cap p_i \Lambda$

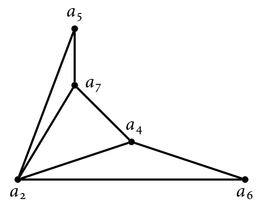
$p_1 \in P_1 \cap \mathrm{SU}_8(\mathbb{C})$



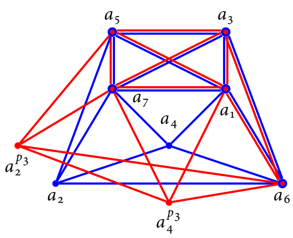


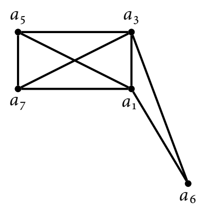
$p_2 \in P_2 \cap \mathrm{SU}_8(\mathbb{C})$



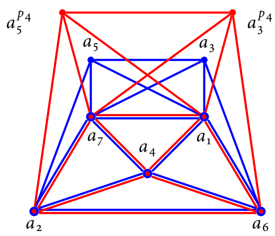


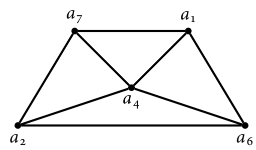
$p_3 \in P_3 \cap \mathrm{SU}_8(\mathbb{C})$





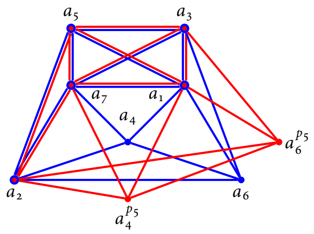
$p_4 \in P_4 \cap \mathrm{SU}_8(\mathbb{C})$

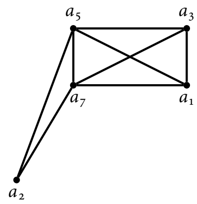




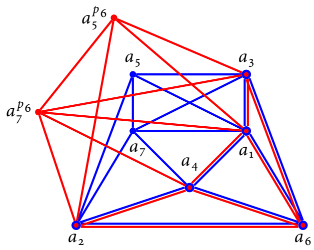
$\Lambda \cup p_i\Lambda$	$\Lambda \cap p_i\Lambda$

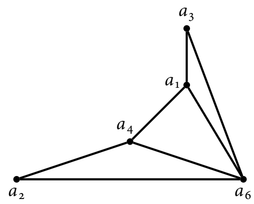
$p_5 \in P_5 \cap \mathrm{SU}_8(\mathbb{C})$



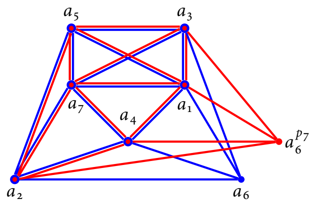


$p_6 \in P_6 \cap \mathrm{SU}_8(\mathbb{C})$





$p_7 \in P_7 \cap \mathrm{SU}_8(\mathbb{C})$



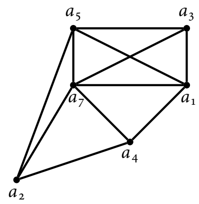
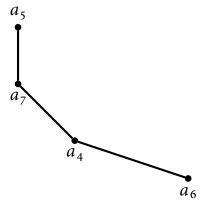
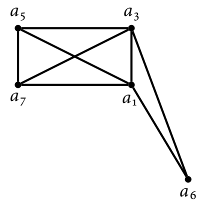
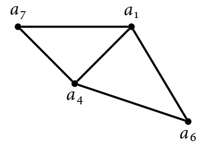
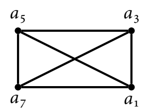
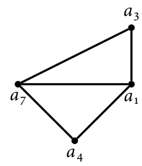


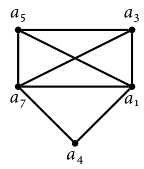
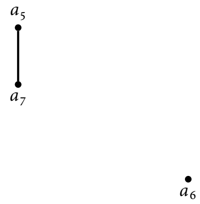
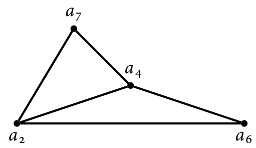
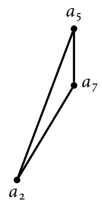
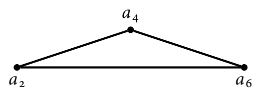
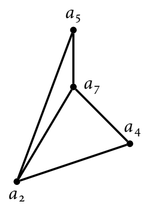
Table 4.7: the graphs $\Lambda \cup p_i\Lambda$ and $\Lambda \cap p_i\Lambda$ for $1 \leq i \leq 7$

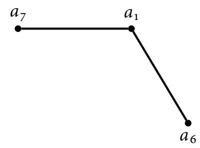
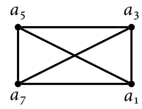
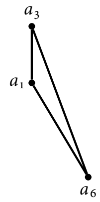
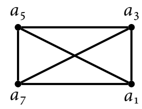
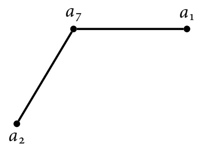
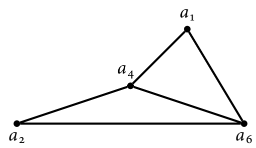
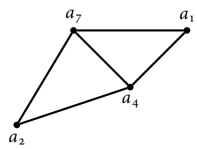
Proposition 4.7.41 *The graph $\Lambda \cup p_i\Lambda$ is simply connected for each $p_i \in P_i \cap \mathrm{SU}_8(\mathbb{C})$ and $1 \leq i \leq 7$.*

Proof: Let $p_i \in P_i \cap \mathrm{SU}_8(\mathbb{C})$ for $1 \leq i \leq 7$. By the facts that $\Lambda \cong \mathbf{H}(A_7) \cong p_i\Lambda$, $\mathbf{H}(A_7)$ is a simply connected graph and $\Lambda \cap p_i\Lambda$ is simply connected by table 4.7, lemma 4.7.38 and lemma 4.7.39, we obtain that the graph $\Lambda \cup p_i\Lambda$ is simply connected by theorem A.5.2. ∎

Let $\Lambda, g_1\Lambda, \ldots, g_{2m-1}, \Lambda$ be a closed gallery in the rank two building $\mathcal{R}_{i,j}(\Lambda)$ of type $\circ\overset{m}{\text{———}}\circ$ for some different indices $i, j \in \{1, \ldots, 7\}$. By proposition 4.7.40 and the following table 4.8, the graph $\Psi = \Lambda \cup \left(\bigcup_{k=1}^{2m-1} g_k\Lambda\right)$ is simply connected for the indices pairs $\{i, j\} \in \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,7\}, \{2,4\}, \{3,4\}, \{4,5\}, \{4,6\}, \{4,7\}, \{5,7\}, \{6,7\}\}$.

$\mathcal{R}_{i,j}(\Lambda)$	m-gon	Ψ^{black}	simply connected possible choice for $w \in \mathcal{V}(\Psi^{\text{black}})$
$\mathcal{R}_{1,2}(\Lambda)$	3-gon		yes a_5, a_6, a_7
$\mathcal{R}_{1,3}(\Lambda)$	2-gon		yes a_7
$\mathcal{R}_{1,4}(\Lambda)$	2-gon		yes a_7
$\mathcal{R}_{1,5}(\Lambda)$	2-gon		yes
$\mathcal{R}_{1,6}(\Lambda)$	2-gon		yes

$\mathcal{R}_{i,j}(\Lambda)$	m-gon	Ψ^{black}	simply connected possible choice for $w \in \mathcal{V}(\Psi^{\text{black}})$
$\mathcal{R}_{1,7}(\Lambda)$	2-gon		yes a_4
$\mathcal{R}_{2,3}(\Lambda)$	3-gon		no
$\mathcal{R}_{2,4}(\Lambda)$	2-gon		yes a_7
$\mathcal{R}_{2,5}(\Lambda)$	2-gon		yes
$\mathcal{R}_{2,6}(\Lambda)$	2-gon		yes
$\mathcal{R}_{2,7}(\Lambda)$	2-gon		yes

$\mathcal{R}_{i,j}(\Lambda)$	m-gon	Ψ^{black}	simply connected possible choice for $w \in \mathcal{V}(\Psi^{\text{black}})$
$\mathcal{R}_{3,4}(\Lambda)$	3-gon		yes a_7
$\mathcal{R}_{3,5}(\Lambda)$	2-gon		yes
$\mathcal{R}_{3,6}(\Lambda)$	2-gon		yes
$\mathcal{R}_{3,7}(\Lambda)$	2-gon		yes
$\mathcal{R}_{4,5}(\Lambda)$	3-gon		yes a_1
$\mathcal{R}_{4,6}(\Lambda)$	2-gon		yes a_1
$\mathcal{R}_{4,7}(\Lambda)$	2-gon		yes a_1

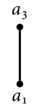
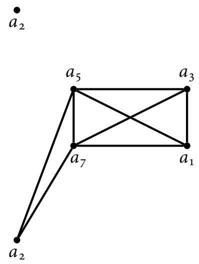
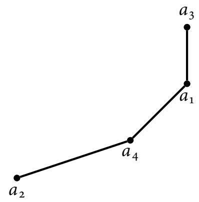
$\mathcal{R}_{i,j}(\Lambda)$	m-gon	Ψ^{black}	simply connected possible choice for $w \in \mathcal{V}(\Psi^{\text{black}})$
$\mathcal{R}_{5,6}(\Lambda)$	3-gon		no
$\mathcal{R}_{5,7}(\Lambda)$	2-gon		yes a_1, a_2
$\mathcal{R}_{6,7}(\Lambda)$	3-gon		yes a_1

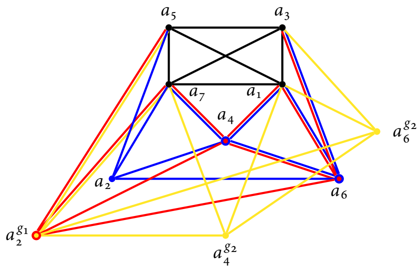
Table 4.8: the graph Ψ^{black} for $1 \leq i < j \leq 7$

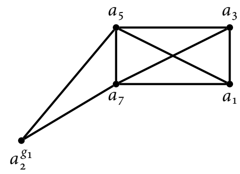
Thus to obtain that $\mathbf{S}(V_8)$ is simply connected we have to show that the connected graph $\Psi = \Lambda \cup (\bigcup_{k=1}^{2m-1} g_k \Lambda)$ is simply connected for every closed gallery $\Lambda, g_1\Lambda, \ldots, g_{2m-1}, \Lambda$ in the rank two building $\mathcal{R}_{i,j}(\Lambda)$ of type $\circ \overset{m}{\text{——}} \circ$ for $\{i, j\} \in \{\{1,5\}, \{1,6\}, \{2,3\}, \{2,5\}, \{2,6\}, \{2,7\}, \{3,5\}, \{3,6\}, \{3,7\}, \{5,6\}\}$. Using proposition 4.7.41 we know already that the graph $\Lambda \cup g_1\Lambda$ is simply connected. Futhermore by theorem A.5.2 and the facts that the graphs $\Lambda \cup g_1\Lambda$, $g_2\Lambda$ and $\Lambda \cup g_1\Lambda \cap g_2\Lambda$, see table 4.9, are simply connected, we obtain also that the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda$ is simply connected for $\{i, j\} \in \{\{1,5\}, \{1,6\}, \{2,5\}, \{2,6\}, \{2,7\}, \{3,5\}, \{3,6\}, \{3,7\}\}$. If $\{i, j\} \in \{\{2,3\}, \{5,6\}\}$ then the graph $\Lambda \cup (\bigcup_{k=1}^{4} g_i\Lambda)$ is simply connected again by theorem A.5.2, as the graph Λ, the graphs $g_l\Lambda$ for $1 \leq l \leq 4$ and the graphs $\Lambda \cup (\bigcup_{s=1}^{t} g_s\Lambda)$ and $(\Lambda \cup (\bigcup_{s=1}^{t} g_s\Lambda)) \cap g_{t+1}\Lambda$ for $1 \leq t \leq 3$ are simply connected, see table 4.10.

For the next paragraph, the vertices of Ψ^{black} are only coloured black and the vertices $\mathcal{V}(\Lambda)\backslash\mathcal{V}(\Psi^{\text{black}})$ are coloured blue. Moreover c_{g_1} = red, c_{g_2} = goldenrod, c_{g_3} = green, c_{g_4} = thistle and c_{g_4} = cornflower and the vertices $\mathcal{V}(g_i\Lambda)\backslash\mathcal{V}(\Psi^{\text{black}})$ are coloured with c_{g_i} for $1 \le i \le 4$.

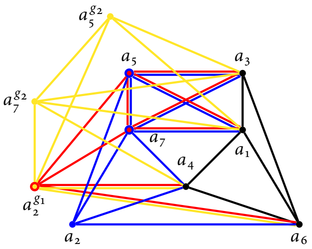
$\Lambda \cup g_1\Lambda \cup g_2\Lambda$	$(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$

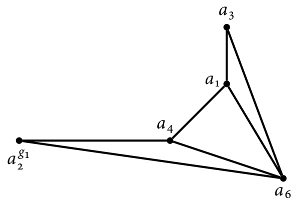
$\mathcal{R}_{1,5}$



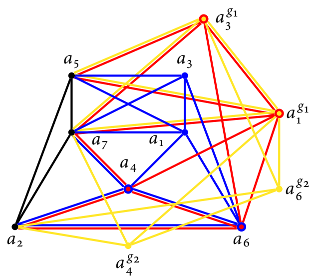


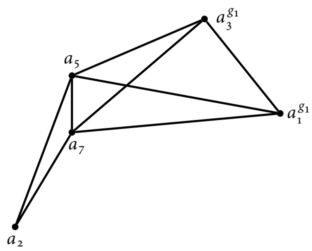
$\mathcal{R}_{1,6}$





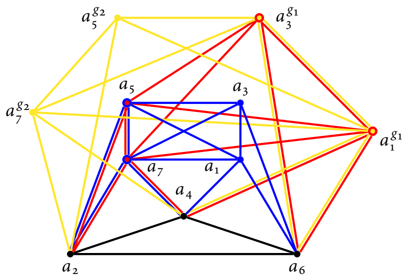
$\mathcal{R}_{2,5}$

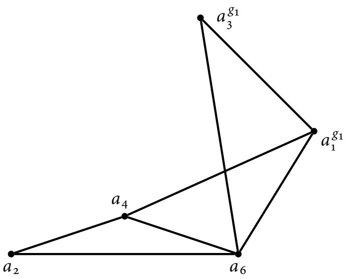




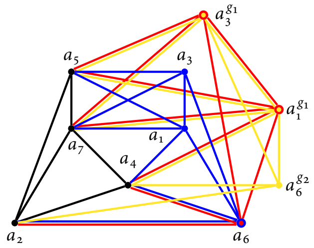
$\Lambda \cup g_1\Lambda \cup g_2\Lambda$	$(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$

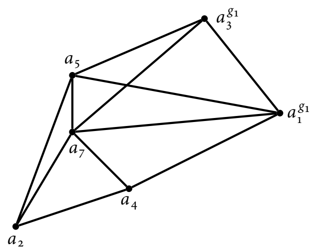
$\mathcal{R}_{2,6}$



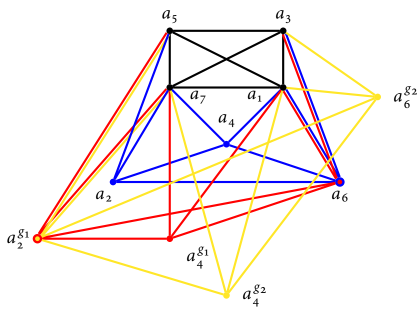


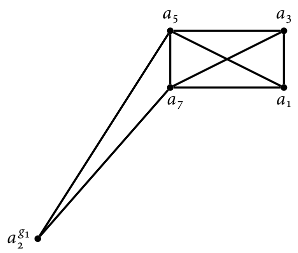
$\mathcal{R}_{2,7}$





$\mathcal{R}_{3,5}$ with $a_4 \neq a_4^{g_2}$





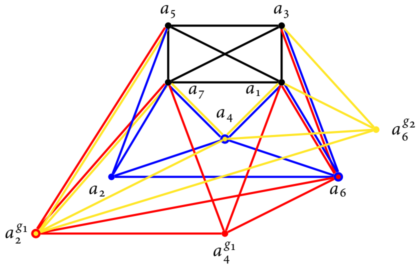
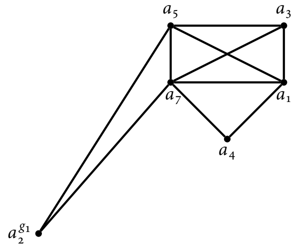
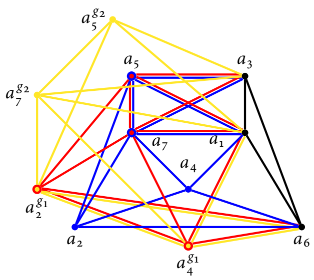
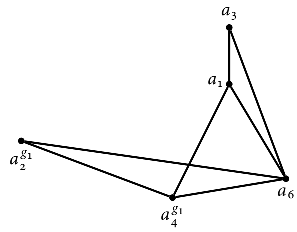
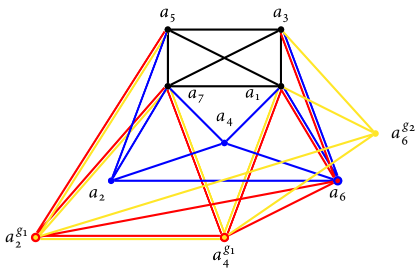
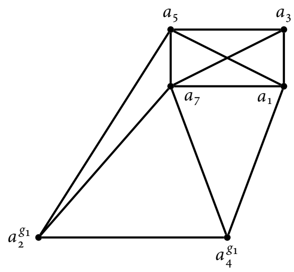
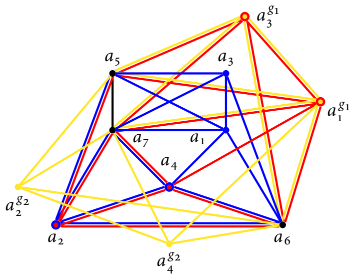
$\Lambda \cup g_1\Lambda \cup g_2\Lambda$	$(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$
$\mathcal{R}_{3,5}$ with $a_4 = a_4^{g_2}$	
	
$\mathcal{R}_{3,6}$	
	
$\mathcal{R}_{3,7}$	
	

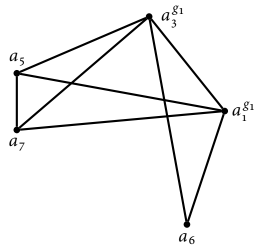
Table 4.9: the graphs $\Lambda \cup g_1\Lambda \cup g_2\Lambda$ and $(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$ in $\mathcal{R}_{i,j}(\Lambda)$ for $\{i,j\} \in \{\{1,5\},\{1,6\},\{2,5\},\{2,6\},\{2,7\},\{3,5\},\{3,6\},\{3,7\}\}$

$\mathcal{R}_{2,3}(\Lambda)$

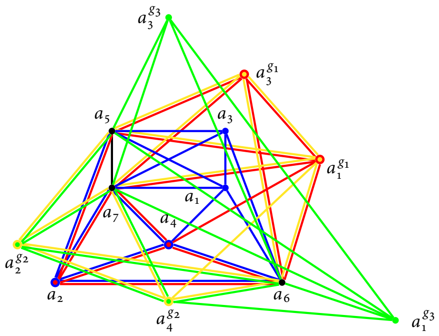
$\Lambda \cup g_1\Lambda \cup g_2\Lambda$



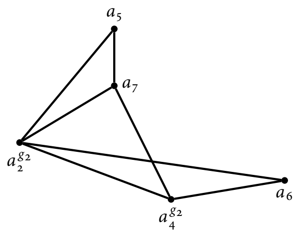
$(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$



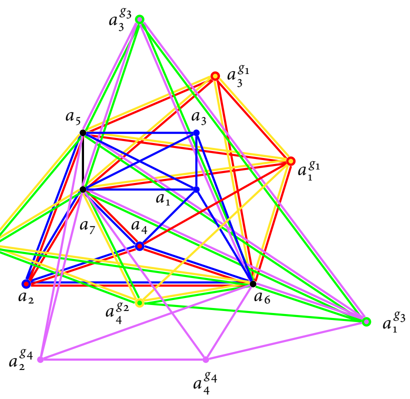
$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$



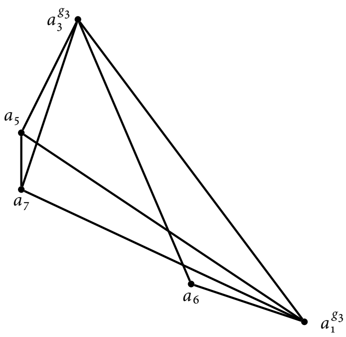
$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$



$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda \cup g_4\Lambda$

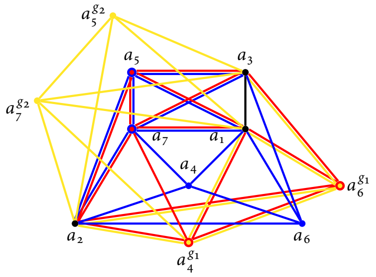


$(\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda) \cap g_4\Lambda$

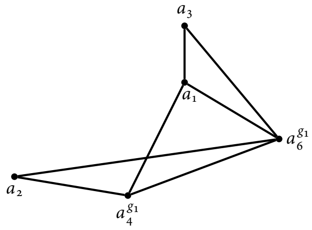


$\mathcal{R}_{5,6}(\Lambda)$

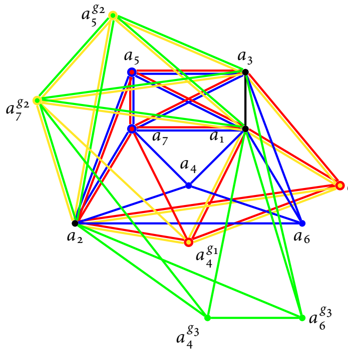
$\Lambda \cup g_1\Lambda \cup g_2\Lambda$



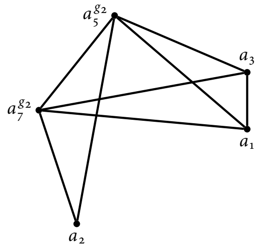
$(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$



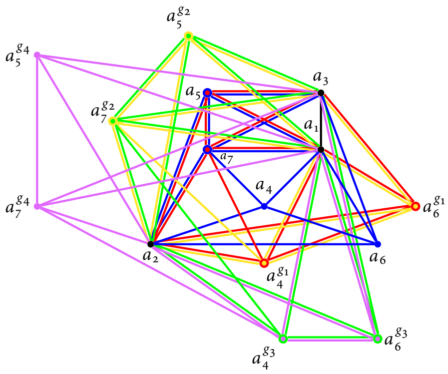
$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$



$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$



$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda \cup g_4\Lambda$



$(\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda) \cap g_4\Lambda$

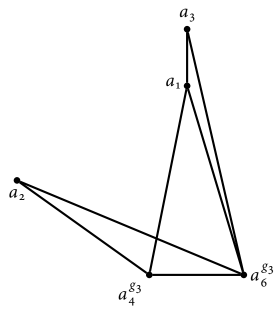
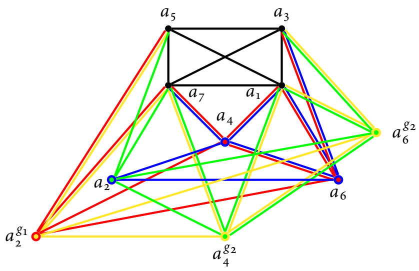


Table 4.10: the graphs $\Lambda \cup g_1\Lambda \cup \bigcup_{k=2}^{t} g_k\Lambda$ and $(\Lambda \cup g_1\Lambda \cup \bigcup_{k=2}^{l} g_k\Lambda) \cap g_{l+1}\Lambda$ for $2 \le t \le 4$ and $l \in \{2,3\}$ in $\mathcal{R}_{i,j}(\Lambda)$ for $\{i,j\} \in \{\{2,3\},\{5,6\}\}$

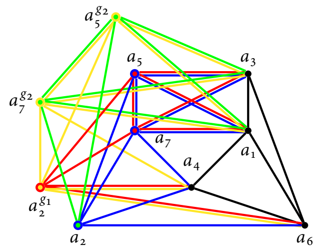
For the final statement, we study each single case.

- $\{i,j\} = \{1,5\}$: $\mathcal{R}_{1,5}(\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{———}}\circ$.
  As $a_2 = a_2^{g_3}$ we obtain that $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda = (\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_2, a_4^{g_2}\}, \{a_2, a_6^{g_2}\}\})$, which is a simply connected graph by lemma 4.7.39 as a_2 and $a_4^{g_2}$ are adjacent to a_7 in $\Lambda \cup g_1\Lambda \cup g_2\Lambda$ and a_2 and $a_6^{g_2}$ are adjacent to $a_4^{g_2}$ in the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_2, a_4^{g_2}\}\})$.



Graph 4.1: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ in $\mathcal{R}_{1,5}(\Lambda)$

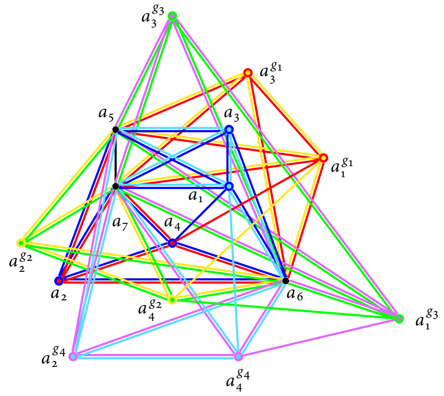
- $\{i,j\} = \{1,6\}$: $\mathcal{R}_{1,6}(\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{———}}\circ$.
  From the fact that $a_2 = a_2^{g_3}$, we conclude that $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda = (\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_2, a_5^{g_2}\}, \{a_2, a_7^{g_2}\}\})$. From lemma 4.7.39 and the facts that a_2 and $a_7^{g_2}$ are adjacent to a_4 in $\Lambda \cup g_1\Lambda \cup g_2\Lambda$ and the vertices a_2 and $a_5^{g_2}$ are adjacent to $a_7^{g_2}$ in the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_2, a_7^{g_2}\}\})$ follows that the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ is simply connected.



Graph 4.2: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ in $\mathcal{R}_{1,6}(\Lambda)$

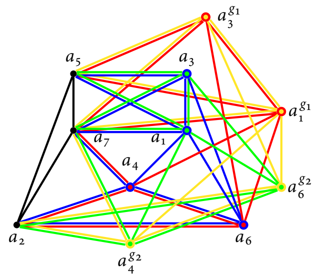
- $\{i,j\} = \{2,3\}$: $\mathcal{R}_{2,3}(\Lambda)$ is a rank two building of type $\circ\overset{3}{\text{———}}\circ$
  We know that $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda \cup g_4\Lambda$ is a simply connected graph. Since $a_1^{g_5} = a_1$ and $a_3^{g_5} = a_3$ we have $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda \cup g_4\Lambda \cup g_5\Lambda = (\mathcal{V}(\Lambda \cup g_1\Lambda g_2\Lambda \cup g_3\Lambda \cup g_4\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda \cup g_4\Lambda) \cup \{\{a_1, a_4^{g_4}\}\})$. As the

vertices a_1 and $a_4^{g_4}$ are adjacent to a_6 in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda \cup g_4\Lambda$, the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda \cup g_4\Lambda \cup g_5\Lambda$ is simply connected by lemma 4.7.39.



Graph 4.3: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ in $\mathcal{R}_{2,3}(\Lambda)$

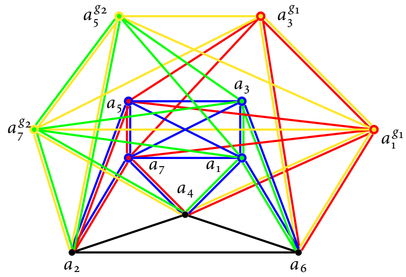
- $\{i, j\} = \{2, 5\}$: $\mathcal{R}_{2,5}(\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{———}}\circ$
  Since $a_1^{g_2} = a_1$ and $a_3^{g_2} = a_3$ we have $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda = (\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_4^{g_2}\}, \{a_1, a_6^{g_2}\}, \{a_3, a_6^{g_2}\}\})$. By lemma 4.7.39, the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ is simply connected as a_1 and $a_4^{g_2}$ are adjacent to a_7 in $\Lambda \cup g_1\Lambda \cup g_2\Lambda$, the vertices a_1 and $a_6^{g_2}$ are adjacent to $a_4^{g_2}$ in the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_4^{g_2}\}\})$ and a_3 and $a_6^{g_2}$ are adjacent to a_1 in the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_4^{g_2}\}, \{a_1, a_6^{g_2}\}\})$.



Graph 4.4: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ in $\mathcal{R}_{2,5}(\Lambda)$

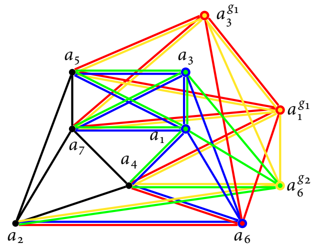
- $\{i, j\} = \{2, 6\}$: $\mathcal{R}_{2,6}(\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{———}}\circ$
  We consider the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$, which is equal to the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_7^{g_2}\}, \{a_3, a_7^{g_2}\}, \{a_3, a_5^{g_2}\}, \{a_1, a_5^{g_2}\}\})$

as $a_1^{g_3} = a_1$ and $a_3^{g_3} = a_3$. This graph is simply connected, using lemma 4.7.39 and the relations that a_1 and $a_7^{g_2}$ are adjacent to a_4 in $\Lambda \cup g_1\Lambda \cup g_2\Lambda$, the vertices a_3 and $a_7^{g_2}$ are adjacent to a_1 in the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_7^{g_2}\}\})$, the vertex $a_7^{g_2}$ is a common neighbor of a_3 and $a_5^{g_2}$ in $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_7^{g_2}\}, \{a_3, a_7^{g_2}\}\})$ and that the vertices a_1 and $a_5^{g_2}$ have the common neighbor a_3 in the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_7^{g_2}\}, \{a_3, a_7^{g_2}\}, \{a_3, a_5^{g_2}\}\})$.



Graph 4.5: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ in $\mathcal{R}_{2,6}(\Lambda)$

- $\{i,j\} = \{2,7\}$: $\mathcal{R}_{2,7}(\Lambda)$ is a rank two building of type $\circ\overset{2}{—}\circ$
  Since $a_1 = a_1^{g_3}$ and $a_3 = a_3^{g_3}$ we get that $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda = (\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_6^{g_2}\}, \{a_3, a_6^{g_2}\}\})$. Moreover the vertices a_1 and $a_6^{g_2}$ are adjacent to a_4 in $\Lambda \cup g_1\Lambda \cup g_2\Lambda$ and a_3 and $a_6^{g_2}$ are adjacent to a_1 in $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_6^{g_2}\}\})$, thus by lemma 4.7.39 the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ is simply connected.



Graph 4.6: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ in $\mathcal{R}_{2,7}(\Lambda)$

- $\{i,j\} = \{3,5\}$: $\mathcal{R}_{3,5}(\Lambda)$ is a rank two building of type $\circ\overset{2}{—}\circ$
  First we see that $a_2 = a_2^{g_3}$ in $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$, but in difference to all cases before, we obtain that $a_4 \neq a_4^{g_1}$, $a_4 \neq a_4^{g_3}$, $a_4^{g_1} \neq a_4^{g_2}$ and $a_4^{g_2} \neq a_4^{g_3}$. Therefore we have to consider four different possibilities in this case.

case one $a_4, a_4^{g_1}, a_4^{g_2}$ and $a_4^{g_3}$ are four different vertices of the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

case two $a_4 = a_4^{g_2}$ and $a_4^{g_1} \neq a_4^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

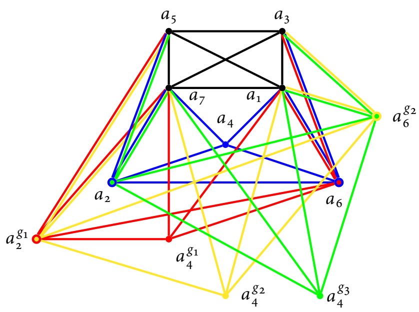
case three $a_4 = a_4^{g_2}$ and $a_4^{g_1} = a_4^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

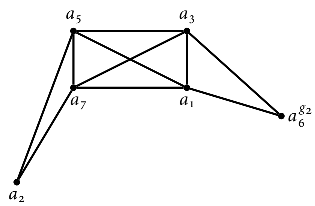
case four $a_4 \neq a_4^{g_2}$ and $a_4^{g_1} = a_4^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

In each case we obtain that the graph $(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$ is simply connected, see table 4.11. Thus by theorem A.5.2 the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ is simply connected.

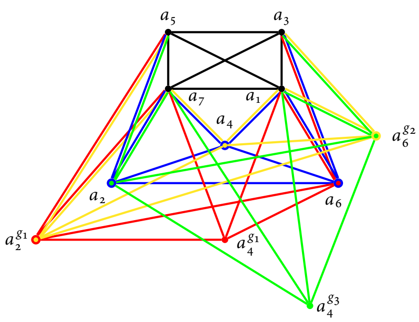
$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$	$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$

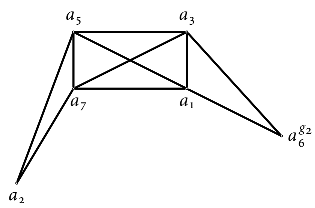
case one





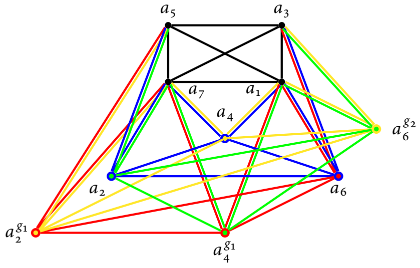
case two

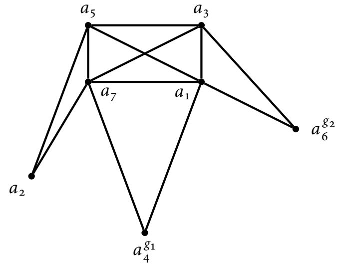




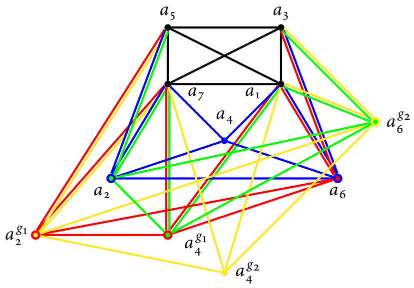
$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$	$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$

case three





case four



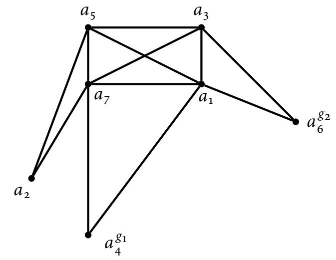
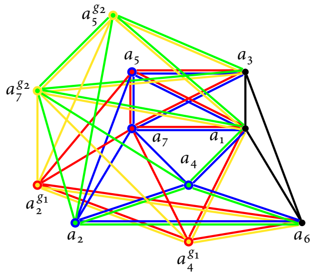


Table 4.11: the graphs $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ and $(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$ in $\mathcal{R}_{3,5}(\Lambda)$

- $\{i,j\} = \{3,6\}$: $\mathcal{R}_{3,6}(\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{———}}\circ$

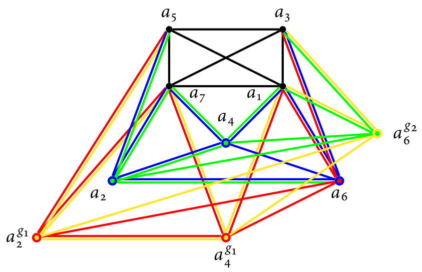
  Certainly, $a_2 = a_2^{g_3}$ and $a_4 = a_4^{g_3}$ implying that $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda = (\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_2, a_5^{g_2}\}, \{a_2, a_7^{g_2}\}, \{a_4, a_7^{g_2}\}\}$, which is a simply connected graph by lemma 4.7.39. In detail the vertices a_4 and $a_7^{g_2}$ are adjacent to a_1 in $\Lambda \cup g_1\Lambda \cup g_2\Lambda$, thus by lemma 4.7.39 the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_4, a_7^{g_2}\}\})$ is simply connected. As a_2 and $a_7^{g_2}$ have the common neighbor a_4 in $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_4, a_7^{g_2}\}\})$, again by lemma 4.7.39 the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_4, a_7^{g_2}\}, \{a_2, a_7^{g_2}\}\}$ is simply connected. Finally the vertex $a_7^{g_2}$ is adjacent to $a_5^{g_2}$ and a_2 in the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_4, a_7^{g_2}\}, \{a_2, a_7^{g_2}\}\}$. Hence $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ is simply connected by lemma 4.7.39 as stated above.



Graph 4.7: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ in $\mathcal{R}_{3,6}(\Lambda)$

- $\{i, j\} = \{3, 7\}$: $\mathcal{R}_{3,7}(\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{——}}\circ$

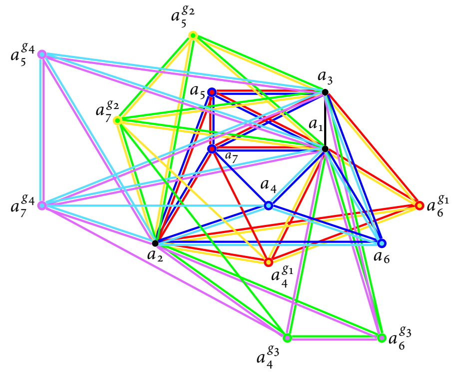
  Since $a_2 = a_2^{g_3}$, $a_4 = a_4^{g_3}$, the vertices a_4 and $a_6^{g_2}$ are adjacent to a_1 in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda$ and the vertices a_2 and $a_6^{g_2}$ are adjacent to a_4 in $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_4, a_6\}\})$ it follows that the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ is simply connected by lemma 4.7.39.



Graph 4.8: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ in $\mathcal{R}_{3,7}(\Lambda)$

- $\{i, j\} = \{5, 6\}$: $\mathcal{R}_{5,6}(\Lambda)$ is a rank two building of type $\circ\overset{3}{\text{——}}\circ$

  We obtain that the graphs $(\mathcal{V}(\Lambda \cup \bigcup_{k=1}^{4} g_k\Lambda), E(\Lambda \cup \bigcup_{k=1}^{4} g_k\Lambda) \cup \{\{a_4, a_7^{g_4}\}\})$ and $\Lambda \cup \bigcup_{k=1}^{5} g_k\Lambda$ are identical. Certainly $a_4 = a_4^{g_5}$ and $a_6 = a_6^{g_5}$, furthermore as $\Lambda \cup \bigcup_{k=1}^{4} g_k\Lambda$ is simply connected and the vertices a_4 and $a_7^{g_4}$ are adjacent to a_2 in the graph $\Lambda \cup \bigcup_{k=1}^{4} g_k\Lambda$ we see that $\Lambda \cup \bigcup_{k=1}^{5} g_k\Lambda$ is simply connected graph by lemma 4.7.39.



Graph 4.9: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda \cup g_4\Lambda \cup g_5\Lambda$ in $\mathcal{R}_{5,6}(\Lambda)$

Thus we have the following theorem.

Theorem 4.7.42 *The connected graph* $\mathbf{S}(V_8) = \mathbf{F}(SU_8(\mathbb{C}))$ *is simply connected.*

Next let Φ to be the reduced irreducible root system of type E_6. Recall the notation from above, then $\Phi = \Phi(E_6) = \Phi_{E_6}$ and $\Delta = \Delta_{E_6} = \alpha_1, \dots, \alpha_6$ as described in [10] Plate V is a basis of Φ_{E_6}. Moreover, $\mathfrak{g} = \mathfrak{g}(E_6) = \mathfrak{g}_{E_6}$ is the finite dimensional semi-simple complex Lie algebra with root system of type E_6, the Lie algebra $\mathfrak{g}(E_6)^\theta = \mathfrak{g}_{E_6}^\theta = L(E_{6,-78})$ is the compact real form of $\mathfrak{g}_{E_6}$, $G = E_6(\mathbb{C}) = L(\mathfrak{g}_{E_6})$ and $G^\theta = E_{6,-78}$. Furthermore, let $\{x_\alpha \in \mathfrak{g}(E_6)_\alpha \backslash \{0\} \mid \alpha \in \Phi_{E_6}; h_i \mid 1 \le i \le 6\}$ be a Chevalley basis of the Lie algebra $\mathfrak{g}(E_6)$. We consider the connected graph $\mathbf{F}(E_{6,-78})$ and prove in the next parts that $\mathbf{F}(E_{6,-78})$ is simply connected. Therefore, let Σ be the induced subgraph of lemma 4.7.31, thus the vertex set of Σ is $\mathcal{V}(\Sigma) = \{U_{\alpha,-\alpha} \mid \alpha \in \Phi\}$, where $U_{\beta,-\beta}$ is a fundamental $SU_2(\mathbb{C})$ subgroup of $E_{6,-78}$ for $\beta \in \Phi$ and $\Sigma \cong \mathbf{W}(E_6)$. Via the isomorphism δ between the root systems $\Phi_{E_6} \cong \Phi(\mathfrak{g}(E_6), \mathfrak{h})$ and $\Phi(E_{6,-78}, t)$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}(E_6)$ and t is the maximal torus of $E_{6,-78}$ both with respect to the basis Δ_{E_6}, we describe the vertex set of Σ in the following way, $\mathcal{V}(\Sigma) = \{L(\mathfrak{g}(E_6)^\theta_{\delta(\alpha),-\delta(\alpha)}) \mid \alpha \in \Phi\}$. Certainly, Σ contains the induced subgraph Λ isomorphic to $\mathbf{H}(E_6)$, whose vertex set is $\mathcal{V}(\Lambda) = \{L(\mathfrak{g}(E_6)^\theta_{\delta(\alpha),-\delta(\alpha)}) \mid \alpha \in \Delta_{E_6}\}$. As in the case before we fix the set of induced subgraphs $\mathcal{M} = \{g\Lambda \mid g \in E_{6,-78}\}$ of $\mathbf{F}(E_{6,-78})$ and show that $\beta : E_{6,-78}/t \to \mathcal{M}$ with $gt \mapsto g\Lambda$ is a bijection. The map β is by construction surjective, as the preimage of $g\Lambda \in \mathcal{M}$ with $g \in E_{6,-78}$ is gt. Moreover, β is injective if we can deduce from the equality of gt and ht for some different $g, h \in SU_8(\mathbb{C})$ the equality of the graphs $g\Lambda$ and $h\Lambda$. Certainly we know that $gt = ht$ if and only if $g^{-1}h \in t$ for different elements $g, h \in E_{6,-78}$. Thus the map β is injective if and only if $\mathrm{Stab}_{E_{6,-78}}(\Lambda) = \{g \in E_{6,-78} \mid g\Lambda = \Lambda\} = t$. As the vertices of Λ are the fundamental $\mathrm{SU}_2(\mathbb{C})$ subgroups $L(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha),-\delta(\alpha)}) = U_{\alpha,-\alpha}$ of $E_{6,-78}$ for $\alpha \in \Delta_{E_6}$, we obtain that

$$\begin{aligned}\mathrm{Stab}_{E_{6,-78}}(\Lambda) &= \{g \in E_{6,-78} \mid g\Lambda = \Lambda\} \\ &= \{g \in E_{6,-78} \mid gL(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha),-\delta(\alpha)})g^{-1} = L(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha),-\delta(\alpha)}) \\ &\quad \text{for every } \alpha \in \Delta_{E_6}\}\,.\end{aligned}$$

It follows that the Lie group $\mathrm{Stab}_{E_{6,-78}}(\Lambda)$ is the Lie group of the Lie subalgebra $\bigcap_{\alpha\in\Delta_{E_6}} N_{\mathfrak{g}(E_6)^{\theta}}(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha),-\delta(\alpha)})$, which is equal to the Lie group of the Lie algebra $(\bigcap_{\alpha\in\Delta_{E_6}} N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha,-\alpha}))^{\theta}$. By proposition 4.7.7 the finite dimensional semi-simple complex Lie algebra $\mathfrak{g}(E_6)$ is generated by the non-zero vectors $x_{\alpha_i}, x_{-\alpha_i}, h_i$ for $1 \le i \le 6$ of the choosen Chevalley basis of $\mathfrak{g}(E_6)$ and $\mathfrak{g}(E_6)_{\alpha_i,-\alpha_i} = \langle h_i, x_{\alpha_i}, x_{-\alpha_i}\rangle$ for $1 \le i \le 6$. Therefore, we have

$$N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_i,-\alpha_i}) = \{g \in \mathfrak{g}(E_6) \mid [g, x_{\alpha_i}], [g, x_{-\alpha_i}], [g, h_i] \in \mathfrak{g}(E_6)_{\alpha_i,-\alpha_i}\}.$$

Using theorem 4.7.10, it follows that

$$\begin{aligned}
N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_1,-\alpha_1}) &= \langle x_{\alpha_j}, x_{-\alpha_j}, h_k \mid 1 \le k \le 6, j \in \{1,2,4,5,6\}\rangle\,, \\
N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_2,-\alpha_2}) &= \langle x_{\alpha_j}, x_{-\alpha_j}, h_k \mid 1 \le k \le 6, j \in \{1,2,3,5,6\}\rangle\,, \\
N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_3,-\alpha_3}) &= \langle x_{\alpha_j}, x_{-\alpha_j}, h_k \mid 1 \le k \le 6, j \in \{2,3,5,6\}\rangle\,, \\
N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_4,-\alpha_4}) &= \langle x_{\alpha_j}, x_{-\alpha_j}, h_k \mid 1 \le k \le 6, j \in \{1,4,6\}\rangle\,, \\
N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_5,-\alpha_5}) &= \langle x_{\alpha_j}, x_{-\alpha_j}, h_k \mid 1 \le k \le 6, j \in \{1,2,3,5\}\rangle \text{ and} \\
N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_6,-\alpha_6}) &= \langle x_{\alpha_j}, x_{-\alpha_j}, h_k \mid 1 \le k \le 6, j \in \{1,2,3,4,6\}\rangle\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\bigcap_{\alpha\in\Delta_{E_6}} N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha,-\alpha}) &= \langle h_k \mid 1 \le k \le 6\rangle = \mathfrak{h} \text{ and} \\
(\bigcap_{\alpha\in\Delta_{E_6}} N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_i,-\alpha_i}))^{\theta} &= \mathfrak{h}^{\theta}\,,
\end{aligned}$$

which implies that $\mathrm{Stab}_{E_{6,-78}}(\Lambda) = t$. Using this argumentation, we define the chamber system $\mathcal{C}(\mathcal{M}) = (\mathcal{M}, (\sim_i)_{1\le i\le 6})$, where the chambers $g\Lambda$ and $h\Lambda$ are i-adjacent, in symbols $g\Lambda \sim_i h\Lambda$, if and only if $g^{-1}h \in P_i \cap E_{6,-78}$, see lemma 4.7.36. By construction we obtain the following statement.

Lemma 4.7.43 *The chamber system $\mathcal{C}(E_{6,-78}, t, (P_i \cap E_{6,-78})_{i\in\{1,\dots,6\}})$ is isomorphic to $\mathcal{C}(E_6(\mathbb{C})/B)$. Moreover the chamber systems $\mathcal{C}(E_{6,-78}, t, (P_i \cap E_{6,-78})_{i\in\{1,\dots,6\}})$ and $\mathcal{C}(\mathcal{M}) = (\mathcal{M}, (\sim_i)_{1\le i\le 6})$ are also isomorphic via the map $\beta : E_{6,-78}/t \to \mathcal{M}$ with $gt \mapsto g\Lambda$.*

By proposition A.7.2 and lemma 4.7.43 the chamber system $\mathcal{C}(E_6(\mathbb{C})/B)$ is simply connected, thus the chamber system $\mathcal{C}(\mathcal{M}) = (\mathcal{M}, (\sim_i)_{1\leq i\leq 6})$ is also simply connected implying that every closed gallery $\gamma : g\Lambda, g_1\Lambda, \ldots, g_n\Lambda, g\Lambda$ in the chamber system $\mathcal{C}(\mathcal{M}) = (\mathcal{M}, (\sim_i)_{1\leq i\leq 7})$ is null-2-homotopic. Furthermore, we find a finite sequence $\gamma_0, \gamma_1, \ldots, \gamma_l$ of galleries of $\mathcal{C}(\mathcal{M})$ such that $\gamma = \gamma_0$, $g\Lambda = \gamma_l$ and the gallery γ_{k-1} is elementary 2-homotopic to γ_k for each index $1 \leq k \leq l$. In detail, for each index $1 \leq k \leq l$ there are some galleries ε and ψ and two J-galleries κ_{k-1} and κ_k with $J \subseteq \{1, \ldots, 6\}$ and $|J| = 2$ such that $\gamma_{k-1} = \varepsilon\kappa_{k-1}\psi$ and $\gamma_k = \varepsilon\kappa_k\psi$. Moreover we regard the closed gallery $\kappa_{k-1}\kappa_k$ in the rank two residue $\mathcal{R}_{i,j}(h\Lambda)$ for $i, j \in \{1, \ldots, 6\}$, $i \neq j$ and $h \in E_{6,-78}$. The closed gallery $\kappa_{k-1}\kappa_k$ is a combination of closed galleries $h\Lambda, h_1\Lambda, h_2\Lambda, h_3\Lambda, h\Lambda$ of length four with $h\Lambda \sim_i h_1\Lambda \sim_j h_2\Lambda \sim_i h_3\Lambda \sim_j h\Lambda$ for some different $i, j \in \{1, \ldots, 6\}$ if $\mathcal{R}_{i,j}(h\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{———}}\circ$ or a combination of closed galleries $h\Lambda, h_1\Lambda, h_2\Lambda, h_3\Lambda, h_4\Lambda, h_5\Lambda, h\Lambda$ with $h\Lambda \sim_i h_1\Lambda \sim_j h_2\Lambda \sim_i h_3\Lambda \sim_j h_4\Lambda \sim_i h_5\Lambda \sim_j h\Lambda$ of length six for some different $i, j \in \{1, \ldots, 6\}$ if $\mathcal{R}_{i,j}(h\Lambda)$ is a rank two building of type $\circ\overset{3}{\text{———}}\circ$.

Let $\gamma : v_{\alpha_0}, v_{\alpha_1}, \ldots, v_{\alpha_{n-1}}, v_{\alpha_n} = v_{\alpha_0}$ be a closed cycle in the graph $\mathbf{F}(E_{6,-78})$, then $v_{\alpha_i} = L(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_i),-\delta(\alpha_i)})$ with $\alpha_i \in \Phi_{E_6}$ for each index $1 \leq i \leq n$. We use the fact that $E_{6,-78}$ acts transitive on the graph $\mathbf{F}(E_{6,-78})$ to find for each $i \in \{1, \ldots, n\}$ an element $g_i \in E_{6,-78}$ such that $g_i\Lambda$ is an induced subgraph of $\mathbf{F}(E_{6,-78})$ containing the vertex v_{α_i}. Furthermore, by the connectivity of $\mathcal{C}(\mathcal{M})$ we obtain finitely many group elements $g_1^{i-1,i}, \ldots, g_{l_{i-1}}^{i-1,i}$ in $E_{6,-78}$ for each index $1 \leq i \leq n$ in such a way that $g_0\Lambda, g_1^{0,1}\Lambda, \ldots, g_{l_0}^{0,1}\Lambda, g_1\lambda, g_1^{1,2}\Lambda, \ldots, g_{n-1}\Lambda, g_1^{n-1,n}\Lambda, \ldots, g_{l_{n-1}}^{n-1,n}\Lambda, g_n\Lambda = g_0\Lambda$ is a closed gallery in the chamber system $\mathcal{C}(\mathcal{M})$. The last paragraph together with the transitivity of $E_{6,-78}$ on $\mathbf{F}(E_{6,-78})$ implies that the graph $\mathbf{F}(E_{6,-78})$ is simply connected if each closed gallery $\Lambda, g_1\Lambda, \ldots, g_{2m-1}, \Lambda$ in the rank two residue $\mathcal{R}_{i,j}(\Lambda)$, which is a rank two building of type $\circ\overset{m}{\text{———}}\circ$ with either $m = 2$ or $m = 3$, for each pair $i, j \in \{1, \ldots, 6\}$, $i \neq j$, is as graph simply connected inside $\mathbf{F}(E_{6,-78})$. In other words, $\mathbf{F}(E_{6,-78})$ is simply connected if $\Lambda \cup \bigcup_{k=1}^{2m-1} g_k\Lambda$ is a simply connected graph.

Recall that two elements $g\Lambda$ and $h\Lambda$ are i-adjacent in the chamber system $\mathcal{C}(\mathcal{M})$ if and only if $\Lambda \sim_i g^{-1}h\Lambda$ if and only if $g^{-1}h \in P_i \cap E_{6,-78}$ for $1 \leq i \leq 6$. The Lie subgroup $P_i \cap E_{6,-78}$ is the Lie group of the Lie subalgebra $\mathfrak{p}_i \cap \mathfrak{g}(E_6)^{\theta}$ with $\mathfrak{p}_i = \mathfrak{t}_{\mathbb{C}} \bigoplus_{\alpha\in\Phi^+\cup\{-\alpha_i\}} \mathfrak{g}(E_6)_{\alpha}$ for each $i \in \{1, \ldots, 6\}$ by theorem A.7.8. Therefore $\mathfrak{p}_i \cap \mathfrak{g}(E_6)^{\theta} = \mathfrak{t} \oplus \mathfrak{g}(E_6)^{\theta}_{\alpha_i,-\alpha_i}$ for each $i \in \{1, \ldots, 6\}$. and it follows that

$$
\begin{aligned}
&\mathfrak{p}_1 \cap \mathfrak{g}(E_6)^{\theta} \leq N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_j,-\alpha_j}) \text{ for} j \in \{1,2,4,5,6\},\\
&\mathfrak{p}_2 \cap \mathfrak{g}(E_6)^{\theta} \leq N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_j,-\alpha_j}) \text{ for } j \in \{1,2,3,5,6\},\\
&\mathfrak{p}_3 \cap \mathfrak{g}(E_6)^{\theta} \leq N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_j,-\alpha_j}) \text{ for} j \in \{2,3,5,6\},\\
&\mathfrak{p}_4 \cap \mathfrak{g}(E_6)^{\theta} \leq N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_j,-\alpha_j}) \text{ for } j \in \{1,4,6\},\\
&\mathfrak{p}_5 \cap \mathfrak{g}(E_6)^{\theta} \leq N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_j,-\alpha_j}) \text{ for } j \in \{1,2,3,5\} \text{ and}\\
&\mathfrak{p}_6 \cap \mathfrak{g}(E_6)^{\theta} \leq N_{\mathfrak{g}(E_6)}(\mathfrak{g}(E_6)_{\alpha_j,-\alpha_j}) \text{ for } j \in \{1,2,3,4\}\ .
\end{aligned}
$$

Hence for $g \in E_{6,-78}$ we obtain that

$$\begin{array}{lll}
\Lambda \sim_1 g\Lambda & \Leftrightarrow gL(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)})g^{-1} = L(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)}) & \text{for } j \in \{1,2,4,5,6\} \\
\Lambda \sim_2 g\Lambda & \Leftrightarrow gL(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)})g^{-1} = L(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)}) & \text{for } j \in \{1,2,3,5,6\} \\
\Lambda \sim_3 g\Lambda & \Leftrightarrow gL(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)})g^{-1} = L(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)}) & \text{for } j \in \{2,3,5,6\} \\
\Lambda \sim_4 g\Lambda & \Leftrightarrow gL(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)})g^{-1} = L(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)}) & \text{for } j \in \{1,4,6\} \\
\Lambda \sim_5 g\Lambda & \Leftrightarrow gL(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)})g^{-1} = L(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)}) & \text{for } j \in \{1,2,3,5\} \\
\Lambda \sim_6 g\Lambda & \Leftrightarrow gL(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)})g^{-1} = L(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)}) & \text{for } j \in \{1,2,3,4,6\}
\end{array}$$

To simplify the notation thus for $1 \le j \le 6$ and $g \in E_6$ we set $L(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)}) = a_j$ and $gL(\mathfrak{g}(E_6)^{\theta}_{\delta(\alpha_j),-\delta(\alpha_j)})g^{-1} = a_j^g$. Therefore Λ is the induced subgraph on the vertices $a_1, \dots, a_6$ and $g\Lambda$ is the induced subgraph on the vertices $a_1^g, \dots, a_6^g$.

Therefore we get for $g \in E_{6,-78}$ the relations

$$\begin{array}{llll}
g\Lambda \sim_1 h\Lambda & \Leftrightarrow & a_j^g = a_j^h & \text{for } j \in \{1,2,4,5,6\} \\
g\Lambda \sim_2 h\Lambda & \Leftrightarrow & a_j^g = a_j^h & \text{for } j \in \{1,2,3,5,6\} \\
g\Lambda \sim_3 h\Lambda & \Leftrightarrow & a_j^g = a_j^h & \text{for } j \in \{2,3,5,6\} \\
g\Lambda \sim_4 h\Lambda & \Leftrightarrow & a_j^g = a_j^h & \text{for } j \in \{1,4,6\} \\
g\Lambda \sim_5 h\Lambda & \Leftrightarrow & a_j^g = a_j^h & \text{for } j \in \{1,2,3,5\} \\
g\Lambda \sim_6 h\Lambda & \Leftrightarrow & a_j^g = a_j^h & \text{for } j \in \{1,2,3,4,6\}.
\end{array}$$

In the next table we collect some information of every $\mathcal{R}_{i,j}(\Lambda)$ residue of $\mathcal{C}(\mathcal{M})$ for $i, j \in \{1, \dots, 6\}$, $i \neq j$.

$\mathcal{R}_{1,2}(\Lambda)$ 2-gon	$\Lambda \sim_1 g_1\Lambda \sim_2 g_2\Lambda \sim_1 g_3\Lambda \sim_2 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 3$, $l \in \{1,2,5,6\}$
$\mathcal{R}_{1,3}(\Lambda)$ 3-gon	$\Lambda \sim_1 g_1\Lambda \sim_3 g_2\Lambda \sim_1 g_3\Lambda \sim_3 g_4\Lambda \sim_1 g_5\Lambda \sim_3 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 5$, $l \in \{2,5,6\}$
$\mathcal{R}_{1,4}(\Lambda)$ 2-gon	$\Lambda \sim_1 g_1\Lambda \sim_4 g_2\Lambda \sim_1 g_3\Lambda \sim_4 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 3$, $l \in \{1,4,6\}$
$\mathcal{R}_{1,5}(\Lambda)$ 2-gon	$\Lambda \sim_1 g_1\Lambda \sim_5 g_2\Lambda \sim_1 g_3\Lambda \sim_5 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 3$, $l \in \{1,2,5\}$
$\mathcal{R}_{1,6}(\Lambda)$ 2-gon	$\Lambda \sim_1 g_1\Lambda \sim_6 g_2\Lambda \sim_1 g_3\Lambda \sim_6 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 3$, $l \in \{1,2,4,6\}$
$\mathcal{R}_{2,3}(\Lambda)$ 2-gon	$\Lambda \sim_2 g_1\Lambda \sim_3 g_2\Lambda \sim_2 g_3\Lambda \sim_3 \Lambda$	$a_l = a_l^{g_k}$ for $1 \le k \le 3$, $l \in \{2,3,5,6\}$

$\mathcal{R}_{2,4}(\Lambda)$ 3-gon	$\Lambda \sim_2 g_1\Lambda \sim_4 g_2\Lambda \sim_2 g_3\Lambda \sim_4 g_4\Lambda \sim_2 g_5\Lambda \sim_4 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 5$, $l \in \{1, 6\}$
$\mathcal{R}_{2,5}(\Lambda)$ 2-gon	$\Lambda \sim_2 g_1\Lambda \sim_5 g_2\Lambda \sim_2 g_3\Lambda \sim_5 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{1, 2, 3, 5\}$
$\mathcal{R}_{2,6}(\Lambda)$ 2-gon	$\Lambda \sim_2 g_1\Lambda \sim_6 g_2\Lambda \sim_2 g_3\Lambda \sim_6 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{1, 2, 3, 6\}$
$\mathcal{R}_{3,4}(\Lambda)$ 3-gon	$\Lambda \sim_3 g_1\Lambda \sim_4 g_2\Lambda \sim_3 g_3\Lambda \sim_4 g_4\Lambda \sim_3 g_5\Lambda \sim_4 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 5$, $l \in \{6\}$
$\mathcal{R}_{3,5}(\Lambda)$ 2-gon	$\Lambda \sim_3 g_1\Lambda \sim_5 g_2\Lambda \sim_3 g_3\Lambda \sim_5 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{2, 3, 5\}$
$\mathcal{R}_{3,6}(\Lambda)$ 2-gon	$\Lambda \sim_3 g_1\Lambda \sim_6 g_2\Lambda \sim_3 g_3\Lambda \sim_6 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{2, 3, 6\}$
$\mathcal{R}_{4,5}(\Lambda)$ 3-gon	$\Lambda \sim_4 g_1\Lambda \sim_5 g_2\Lambda \sim_4 g_3\Lambda \sim_5 g_4\Lambda \sim_4 g_5\Lambda \sim_5 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 5$, $l \in \{1\}$
$\mathcal{R}_{4,6}(\Lambda)$ 2-gon	$\Lambda \sim_4 g_1\Lambda \sim_6 g_2\Lambda \sim_4 g_3\Lambda \sim_6 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 3$, $l \in \{1, 4, 6\}$
$\mathcal{R}_{5,6}(\Lambda)$ 3-gon	$\Lambda \sim_5 g_1\Lambda \sim_6 g_2\Lambda \sim_5 g_3\Lambda \sim_6 g_4\Lambda \sim_5 g_5\Lambda \sim_6 \Lambda$	$a_l = a_l^{g_k}$ for $1 \leq k \leq 5$, $l \in \{1, 2, 3\}$

Let $\Lambda, g_1\Lambda, \ldots, g_{2m-1}, \Lambda$ be a closed gallery in the rank two building $\mathcal{R}_{i,j}(\Lambda)$ of type $\circ\overset{m}{\text{———}}\circ$ for some $i, j \in \{1, \ldots, 6\}$, $i \neq j$. We colour the vertices of the graphs $\Lambda, g_1\Lambda, \ldots, g_{2m-1}$ in different ways as in the case above. First of all

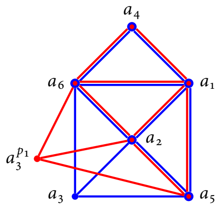
$$\begin{aligned}\mathcal{V}(\Lambda) \cap \Big(\bigcap_{k=1}^{2m-1} \mathcal{V}(g_k\Lambda)\Big) &= \{a_1, \ldots, a_6\} \cap \Big(\bigcap_{k=1}^{2m-1} \{a_1^{g_k}, \ldots, a_6^{g_k}\}\Big) \\ &= \{a_i \mid 1 \leq i \leq 6, a_i = a_i^{g_k} \text{ for } 1 \leq k \leq 2m-1\}\end{aligned}$$

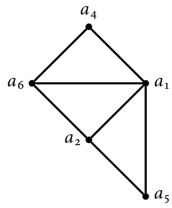
are coloured black. The vertices $\mathcal{V}(\Lambda) = \{a_1, \ldots, a_6\}$ are coloured in blue and the vertices $\mathcal{V}(g_k\Lambda) = \{a_1^{g_k}, \ldots, a_6^{g_k}\}$ are coloured in c_{g_k} for $1 \leq k \leq 2m-1$. Moreover we set $\Psi = \Lambda \cup (\bigcup_{k=1}^{2m-1} g_k\Lambda)$, thus Ψ is a subgraph of $\mathbf{F}(E_{6,-78})$. We denote with Ψ^c the induced subgraph of Ψ on the vertex set $\mathcal{V}(\Psi^c)$, which contains all vertices coloured with the colour c, for $c \in \{\text{black}, \text{blue}, c_{g_1}, \ldots, c_{g_{2m-1}}\}$.

Next, we consider the graphs $\Lambda \cup p_i\Lambda$ and $\Lambda \cap p_i\Lambda$ with $p_i \in P_i \cap E_{6,-78}$ for $1 \leq i \leq 6$. The vertices of the graph Λ are coloured in blue and for each index $1 \leq i \leq 6$ we set $c_{p_i} = \text{red}$. Thus the vertices of the graph $p_i\Lambda$ are coloured red.

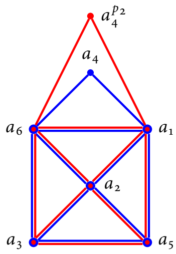
$\Lambda \cup p_i \Lambda$	$\Lambda \cap p_i \Lambda$

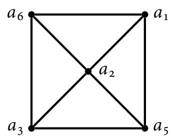
$p_1 \in P_1 \cap E_{6,-78}$



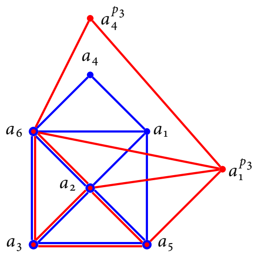


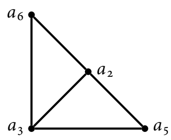
$p_2 \in P_2 \cap E_{6,-78}$





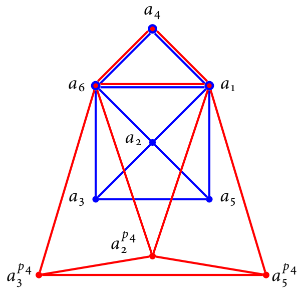
$p_3 \in P_3 \cap E_{6,-78}$

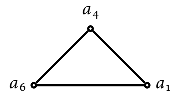




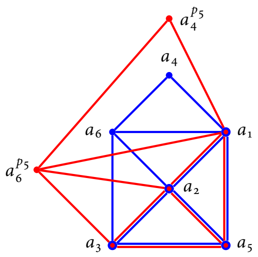
$\Lambda \cup p_i \Lambda$	$\Lambda \cap p_i \Lambda$

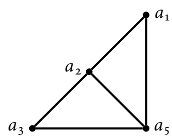
$p_4 \in P_4 \cap E_{6,-78}$



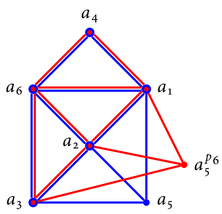


$p_5 \in P_5 \cap E_{6,-78}$





$p_6 \in P_6 \cap E_{6,-78}$



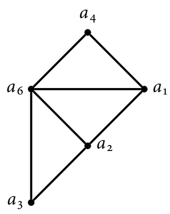
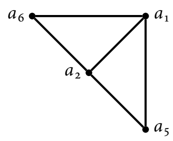
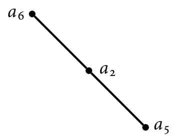
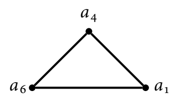
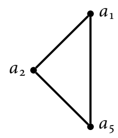


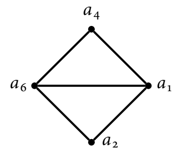
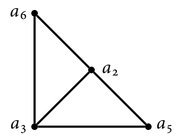
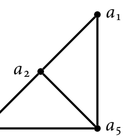
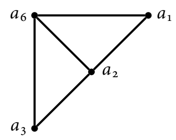
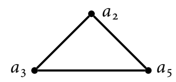
Table 4.15: the graphs $\Lambda \cup p_i \Lambda$ and $\Lambda \cap p_i \Lambda$ for $1 \leq i \leq 6$

Proposition 4.7.44 *The graph $\Lambda \cup p_i\Lambda$ is simply connected for each $p_i \in P_i \cap E_{6,-78}$ and $1 \leq i \leq 6$.*

Proof: Since $\Lambda \cong \mathbf{H}(E_6) \cong p_i\Lambda$ and the graphs $\mathbf{H}(E_6)$ and $\Lambda \cap p_i\Lambda$ are simply connected by table 4.15, lemma 4.7.38 and lemma 4.7.39 for each index $1 \leq i \leq 6$, by theorem A.5.2 we obtain that the graph $\Lambda \cup p_i\Lambda$ is simply connected for each index $1 \leq i \leq 6$. ∎

By proposition 4.7.40 and the following table 4.16, the graph $\Psi = \Lambda \cup \bigcup_{k=1}^{2m-1} g_k\Lambda$ is simply connected for $\{i,j\} \in \{\{1,2\},\{1,3\},\{1,6\},\{2,3\},\{2,5\},\{2,6\},\{5,6\}\}$.

$\mathcal{R}_{i,j}(\Lambda)$	m-gon	Ψ^{black}	simply connected possible choice for $w \in \mathcal{V}(\Psi^{\text{black}})$
$\mathcal{R}_{1,2}(\Lambda)$	2-gon		yes a_6
$\mathcal{R}_{1,3}(\Lambda)$	3-gon		yes a_6
$\mathcal{R}_{1,4}(\Lambda)$	2-gon		yes
$\mathcal{R}_{1,5}(\Lambda)$	2-gon		yes

$\mathcal{R}_{i,j}(\Lambda)$	m-gon	Ψ^{black}	simply connected possible choice for $w \in \mathcal{V}(\Psi^{\text{black}})$
$\mathcal{R}_{1,6}(\Lambda)$	2-gon		yes a_2
$\mathcal{R}_{2,3}(\Lambda)$	2-gon		yes a_6
$\mathcal{R}_{2,4}(\Lambda)$	3-gon		yes
$\mathcal{R}_{2,5}(\Lambda)$	2-gon		yes a_1
$\mathcal{R}_{2,6}(\Lambda)$	2-gon		yes a_1
$\mathcal{R}_{3,4}(\Lambda)$	3-gon		yes
$\mathcal{R}_{3,5}(\Lambda)$	2-gon		yes

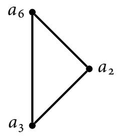
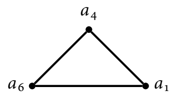
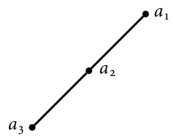
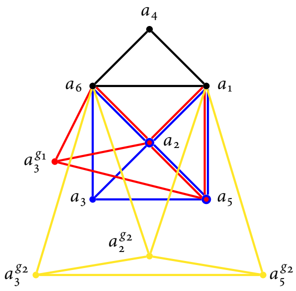
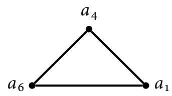
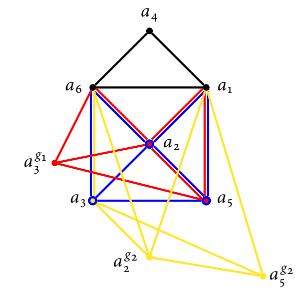
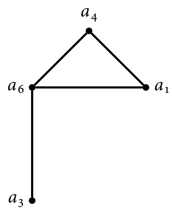
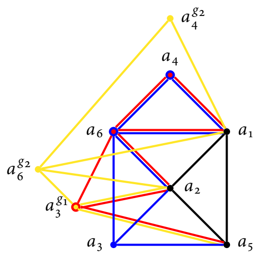
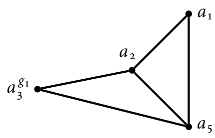
$\mathcal{R}_{i,j}(\Lambda)$	m-gon	Ψ^{black}	simply connected possible choice for $w \in \mathcal{V}(\Psi^{\text{black}})$
$\mathcal{R}_{3,6}(\Lambda)$	2-gon		yes
$\mathcal{R}_{4,5}(\Lambda)$	3-gon		yes
$\mathcal{R}_{4,6}(\Lambda)$	2-gon		yes
$\mathcal{R}_{5,6}(\Lambda)$	3-gon		yes a_1

Table 4.16: the graph Ψ^{black} for $1 \le i < j \le 6$

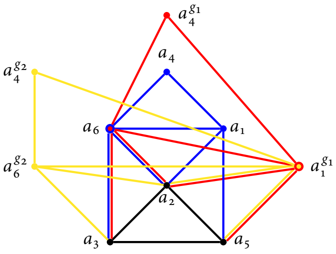
To finish the proof of the statement that the graph $\mathbf{F}(E_{6,-78})$ is simply connected we have to show that for every closed gallery $\Lambda, g_1\Lambda, \dots, g_{2m-1}, \Lambda$ in the rank two building $\mathcal{R}_{i,j}(\Lambda)$ of type $\circ\overset{m}{\text{——}}\circ$ for $\{i,j\} \in \{\{1,4\},\{1,5\},\{1,6\},\{2,4\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\}\}$ the graph $\Psi = \Lambda \cup (\bigcup_{k=1}^{2m-1} g_k\Lambda)$ is simply connected. By proposition 4.7.44 the graph $\Lambda \cup g_1\Lambda$ is simply connected for every closed gallery $\Lambda, g_1\Lambda, \dots, g_{2m-1}, \Lambda$ in $\mathcal{R}_{i,j}(\Lambda)$ for $1 \le i < j \le 6$. Furthermore, since the graphs $\Lambda \cup g_1\Lambda$, $g_2\Lambda$ and $(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$ are simply connected, see table 4.18, it follows by theorem A.5.2 that also the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda$ is simply connected for the pairs of indices $\{i,j\} \in \{\{1,4\},\{1,5\},\{1,6\},\{3,6\},\{4,6\}\}$ and for $\{i,j\} = \{3,5\}$ under the assumption that $a_4 \neq a_4^{g_2}$. If $\{i,j\} = \{3,5\}$ and $a_4 = a_4^{g_2}$ then by lemma 4.7.39 the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda), E(\Lambda \cup g_1\Lambda) \cup \{a_1^{g_1}, a_4\})$ is simply connected as the vertices $a_1^{g_2}$ and a_4 are adjacent to a_6 in the simply connected graph $\Lambda \cup g_1\Lambda$. By table 4.18, the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda), E(\Lambda \cup g_1\Lambda) \cup \{a_1^{g_1}, a_4\}) \cap g_2\Lambda$ is simply connected implying that $\Lambda \cup g_1\Lambda \cup g_2\Lambda$ is a simply connected graph by theorem A.5.2. If otherwise

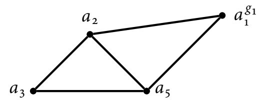
$\{i,j\} \in \{\{2,3\},\{3,4\},\{4,5\}\}$ then the graph $\Lambda \cup (\bigcup_{k=1}^{4} g_i\Lambda)$ is simply connected again by theorem A.5.2 as the graphs Λ, $g_l\Lambda$ for $1 \le l \le 4$, $\Lambda \cup (\bigcup_{s=1}^{t} g_s\Lambda)$ and $(\Lambda \cup (\bigcup_{s=1}^{t} g_s\Lambda)) \cap g_{t+1}\Lambda$ with $1 \le t \le 3$ are simply connected, see table 4.19.

$\Lambda \cup g_1\Lambda \cup g_2\Lambda$	$(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$
$\mathcal{R}_{1,4}$ with $a_3^{g_2} \neq a_3$	
	
$\mathcal{R}_{1,4}$ with $a_3^{g_2} = a_3$	
	
$\mathcal{R}_{1,5}$	
	

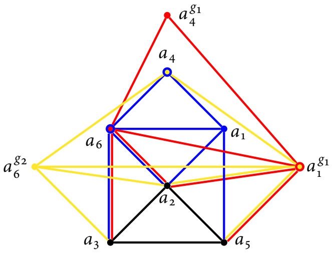
$\Lambda \cup g_1\Lambda \cup g_2\Lambda$	$(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$

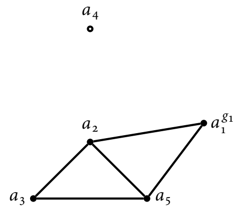
$\mathcal{R}_{3,5}$ with $a_4^{g_2} \neq a_4$



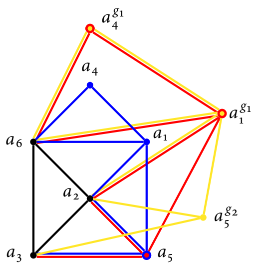


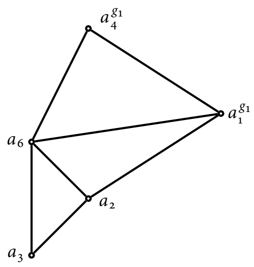
$\mathcal{R}_{3,5}$ with $a_4^{g_2} = a_4$





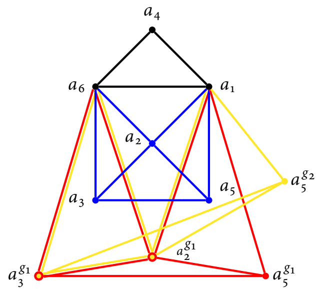
$\mathcal{R}_{3,6}$

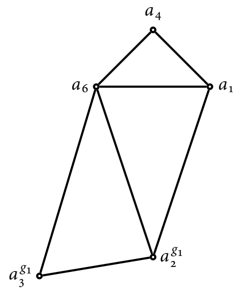




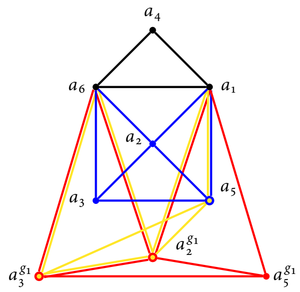
$\Lambda \cup g_1\Lambda \cup g_2\Lambda$	$(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$

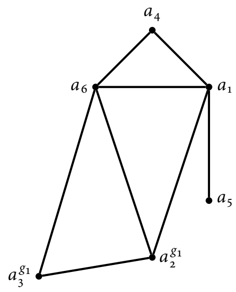
$\mathcal{R}_{4,6}$ with $a_5 \neq a_5^{g_2}$





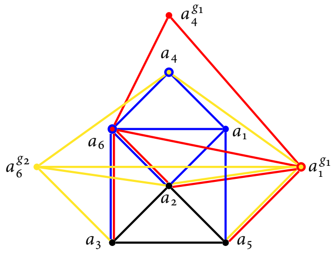
$\mathcal{R}_{4,6}$ with $a_5 = a_5^{g_2}$





$\Lambda \cup g_1\Lambda \cup g_2\Lambda$	$(\mathcal{V}(\Lambda \cup g_1\Lambda), E(\Lambda \cup g_1\Lambda) \cup \{a_1^{g_1}, a_4\}) \cap g_2\Lambda$

$\mathcal{R}_{3,5}(\Lambda)$ with $a_4^{g_2} = a_4$



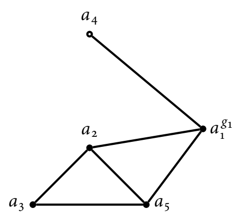
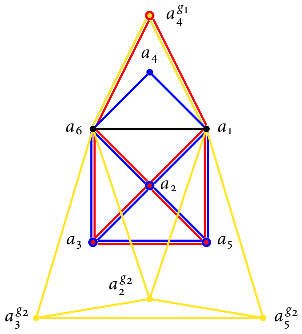


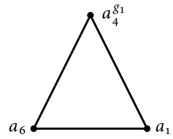
Table 4.18: the graphs $\Lambda \cup g_1\Lambda \cup g_2\Lambda$ and $\Lambda \cup g_1\Lambda \cap g_2\Lambda$ in $\mathcal{R}_{i,j}(\Lambda)$ for the indices $\{i,j\} \in \{\{1,4\},\{1,5\},\{3,5\},\{3,6\},\{4,6\}\}$

$\mathcal{R}_{2,4}(\Lambda)$

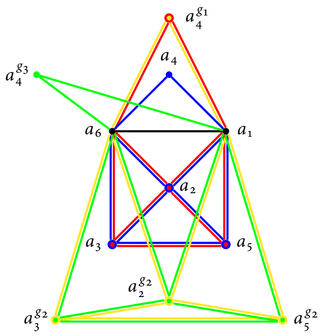
$\Lambda \cup g_1\Lambda \cup g_2\Lambda$



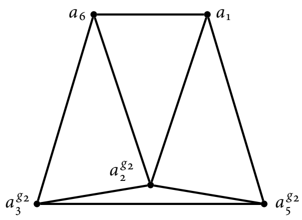
$(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$



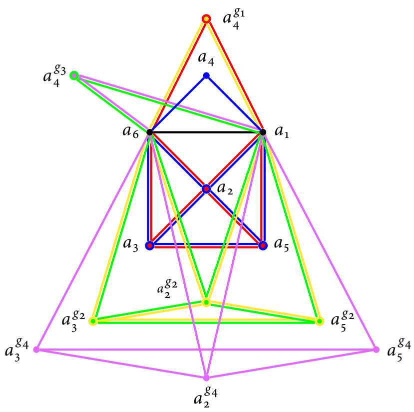
$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$



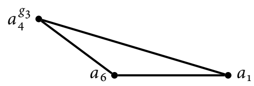
$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$



$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda \cup g_4\Lambda$

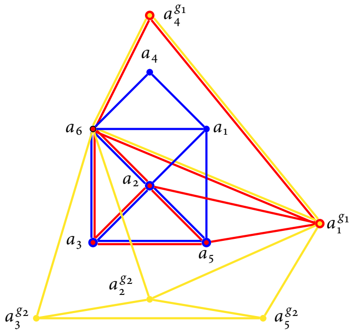


$(\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda) \cap g_4\Lambda$

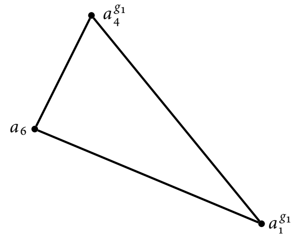


$\mathcal{R}_{3,4}(\Lambda)$

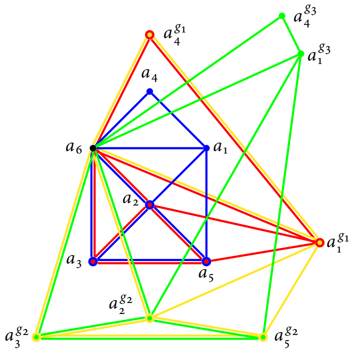
$\Lambda \cup g_1\Lambda \cup g_2\Lambda$



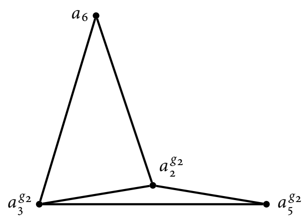
$(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$



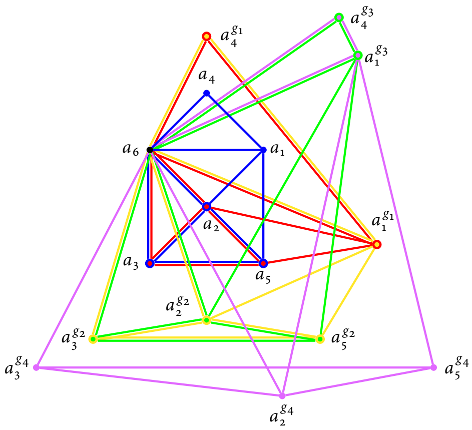
$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$



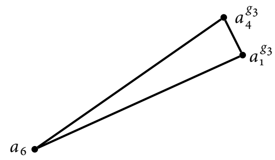
$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$



$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda \cup g_4\Lambda$

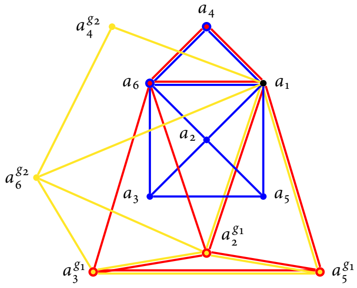


$(\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda) \cap g_4\Lambda$

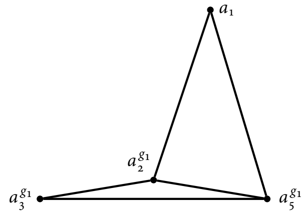


$\mathcal{R}_{4,5}(\Lambda)$

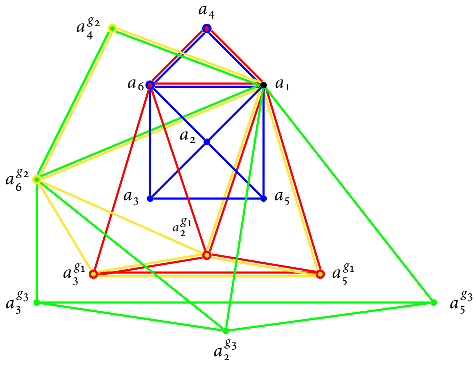
$\Lambda \cup g_1\Lambda \cup g_2\Lambda$



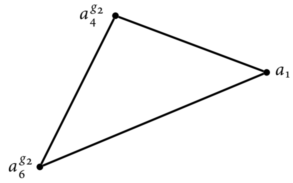
$(\Lambda \cup g_1\Lambda) \cap g_2\Lambda$



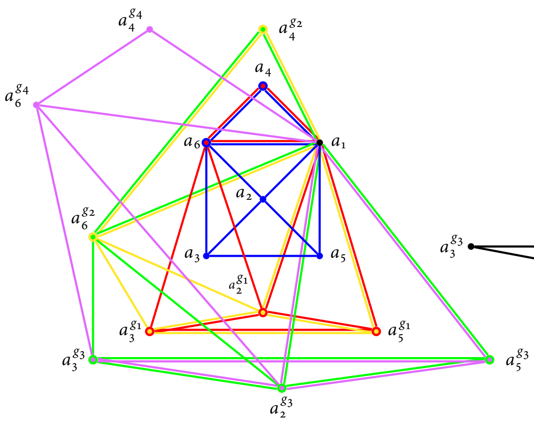
$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$



$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$



$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda \cup g_4\Lambda$



$(\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda) \cap g_4\Lambda$

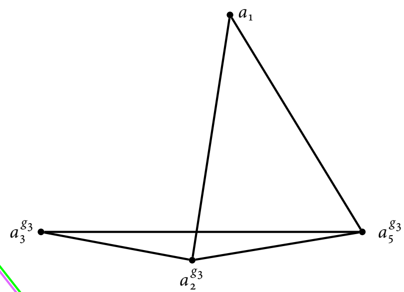


Table 4.19: the graphs $\Lambda \cup g_1\Lambda \cup (\bigcup_{k=2}^{t} g_k\Lambda)$ and $(\Lambda \cup g_1\Lambda \cup (\bigcup_{k=2}^{l} g_k\Lambda)) \cap g_{l+1}\Lambda$ for $2 \le t \le 4$ and $l \in \{2,3\}$ in $\mathcal{R}_{i,j}(\Lambda)$ for $\{i,j\} \in \{\{2,4\},\{3,4\},\{4,5\}\}$

For the final statement, we will study each single case.

- $\{i, j\} = \{1, 4\}$: $\mathcal{R}_{1,4}(\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{——}}\circ$.
  We obtain that $a_2 = a_2^{g_3}$ and $a_5 = a_5^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ but in difference, we also get that $a_3 \neq a_3^{g_1}$, $a_3^{g_1} \neq a_3^{g_2}$, $a_3^{g_2} \neq a_3^{g_3}$ and $a_3^{g_3} \neq a_3$. Therefore we have to consider four different possibilities in this case.

case one $a_3, a_3^{g_1}, a_3^{g_2}$ and $a_3^{g_3}$ are four different vertices in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

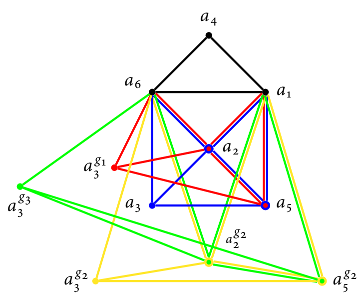
case two $a_3 = a_3^{g_2}$ and $a_3^{g_1} \neq a_3^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

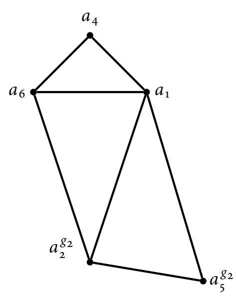
case three $a_3 = a_3^{g_2}$ and $a_3^{g_1} = a_3^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

case four $a_3 \neq a_3^{g_2}$ and $a_3^{g_1} = a_3^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

In each case, we know that the graph $(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$ is simply connected, see table 4.20. Thus by theorem A.5.2, the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ is simply connected.

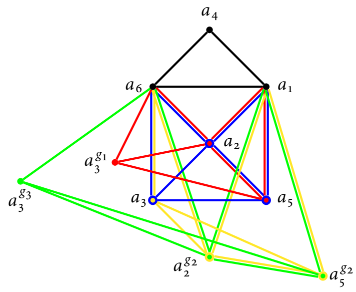
$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$	$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$
case one	

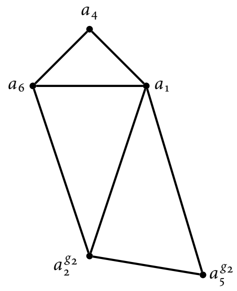




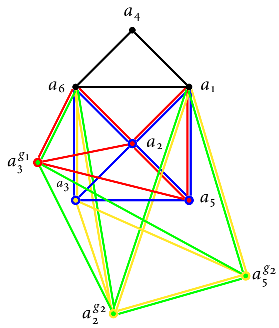
$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$	$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$

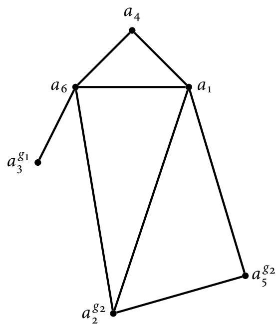
case two



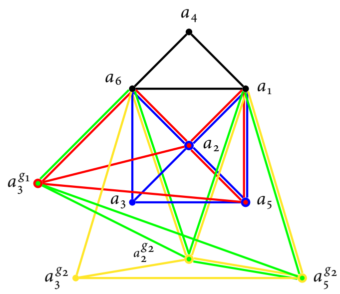


case three





case four



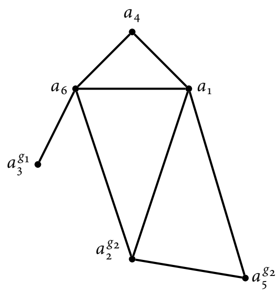
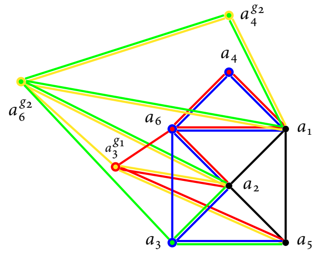


Table 4.20: the graphs $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ and $(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$ in $\mathcal{R}_{1,4}(\Lambda)$

- $\{i, j\} = \{1, 5\}$: $\mathcal{R}_{1,5}(\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{———}}\circ$.

  As $a_3 = a_3^{g_3}$, we obtain that $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda = (\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_3, a_6^{g_2}\}\})$, which is a simply connected graph by lemma 4.7.39 by the fact that a_3 and $a_6^{g_2}$ are adjacent to the vertex a_2 in $\Lambda \cup g_1\Lambda \cup g_2\Lambda$.



Graph 4.10: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ in $\mathcal{R}_{1,5}(\Lambda)$

- $\{i, j\} = \{2, 4\}$: $\mathcal{R}_{2,4}(\Lambda)$ is a rank two building of type $\circ\overset{3}{\text{———}}\circ$.

  We use theorem A.5.2 to prove that the graph $\Lambda \cup (\bigcup_{i=1}^{5} g_i\Lambda)$ is simply connected. Certainly $a_4 = a_4^{g_5}$ which implies that $(\Lambda \cup (\bigcup_{i=1}^{4} g_i\Lambda)) \cap g_5\Lambda$ is a simply connected graph, see table 4.21. Moreover the graph $\Lambda \cup (\bigcup_{i=1}^{4} g_i\Lambda)$ is simply connected and $\Lambda \cup (\bigcup_{i=1}^{5} g_i\Lambda) = (\Lambda \cup (\bigcup_{i=1}^{4} g_i\Lambda)) \cup g_5\Lambda$ thus by theorem A.5.2, $\Lambda \cup (\bigcup_{i=1}^{5} g_i\Lambda)$ is a simply connected graph.

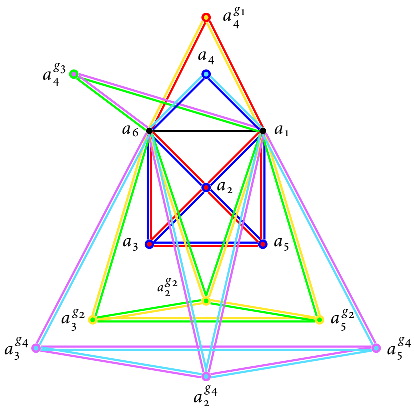
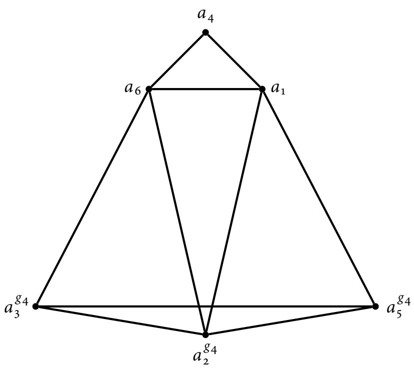
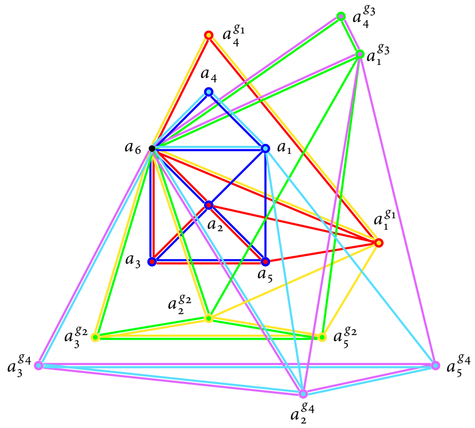
$\Lambda \cup (\bigcup_{i=1}^{5} g_i\Lambda)$	$(\Lambda \cup (\bigcup_{i=1}^{4} g_i\Lambda)) \cap g_5\Lambda$
	

Table 4.21: the graphs $\Lambda \cup (\bigcup_{i=1}^{5} g_i\Lambda)$ and $(\Lambda \cup (\bigcup_{i=1}^{4} g_i\Lambda)) \cap g_5\Lambda$ in $\mathcal{R}_{2,4}(\Lambda)$

- $\{i,j\} = \{3,4\}$: $\mathcal{R}_{3,4}(\Lambda)$ is a rank two building of type $\circ\overset{3}{\text{———}}\circ$.

  We consider the graph connected $\Lambda \cup (\bigcup_{i=1}^{5} g_i\Lambda)$. Since $a_1 = a_1^{g_5}$ and $a_4 = a_4^{g_5}$ as well as $\Lambda \cup (\bigcup_{i=1}^{5} g_i\Lambda) = (\mathcal{V}(\Lambda \cup (\bigcup_{i=1}^{4} g_i\Lambda)), E(\Lambda \cup (\bigcup_{i=1}^{4} g_i\Lambda)) \cup \{\{a_1, a_2^{g_4}\}, \{a_1, a_5^{g_4}\}\})$, we obtain that the graph $\Lambda \cup \bigcup_{i=1}^{5} g_i\Lambda$ is simply connected by lemma 4.7.39. In detail, the vertices a_1 and $a_2^{g_4}$ are adjacent to the vertex a_6 in the graph $\Lambda\cup(\bigcup_{i=1}^{4} g_i\Lambda)$ implying that $(\mathcal{V}(\Lambda\cup(\bigcup_{i=1}^{4} g_i\Lambda)), E(\Lambda\cup(\bigcup_{i=1}^{4} g_i\Lambda)) \cup \{\{a_1, a_2^{g_4}\}\})$ is simply connected by lemma 4.7.39 and the two vertices a_1 and $a_5^{g_4}$ are adjacent to the vertex $a_2^{g_4}$ in the simply connected graph $(\mathcal{V}(\Lambda\cup(\bigcup_{i=1}^{4} g_i\Lambda)), E(\Lambda\cup(\bigcup_{i=1}^{4} g_i\Lambda))\cup\{\{a_1, a_2^{g_4}\}\})$. Hence, again by lemma 4.7.39 $(\mathcal{V}(\Lambda\cup(\bigcup_{i=1}^{4} g_i\Lambda)), E(\Lambda\cup(\bigcup_{i=1}^{4} g_i\Lambda))\cup\{\{a_1, a_2^{g_4}\}, \{a_1, a_5^{g_4}\}\})$ is a simply connected graph.



Graph 4.11: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ in $\mathcal{R}_{3,4}(\Lambda)$

- $\{i,j\} = \{3,5\}$: $\mathcal{R}_{3,5}(\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{———}}\circ$.

  We have to consider four different cases as $a_4 \neq a_4^{g_1}$, $a_4^{g_1} \neq a_4^{g_2}$, $a_4^{g_2} \neq a_4^{g_3}$ and $a_4^{g_3} \neq a_4$.

case one $a_4, a_4^{g_1}, a_4^{g_2}$ and $a_4^{g_3}$ are four different vertices in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

case two $a_4 = a_4^{g_2}$ and $a_4^{g_1} \neq a_4^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

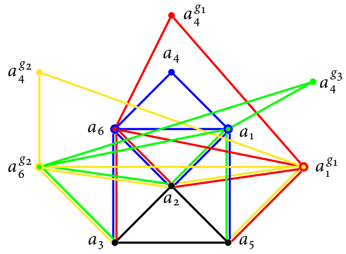
case three $a_4 = a_4^{g_2}$ and $a_4^{g_1} = a_4^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

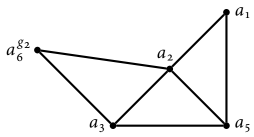
case four $a_4 \neq a_4^{g_2}$ and $a_4^{g_1} = a_4^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

On the other hand, we know that $a_1 = a_1^{g_3}$, thus in case one and two, the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ is simply connected by theorem A.5.2 as $(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$ is simply connected, see table 4.22.

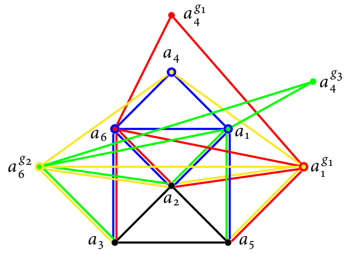
$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$	$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$

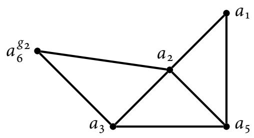
case one



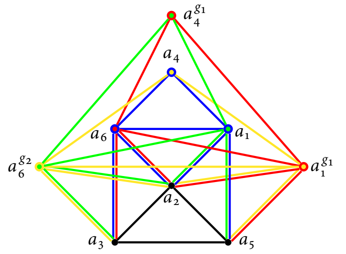


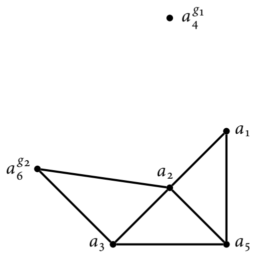
case two



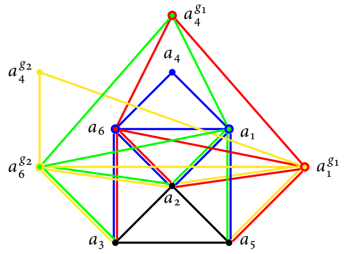


case three





$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$	$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$
case four	



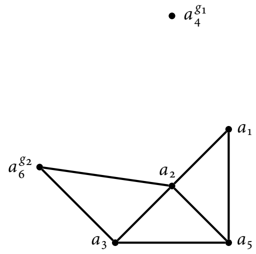
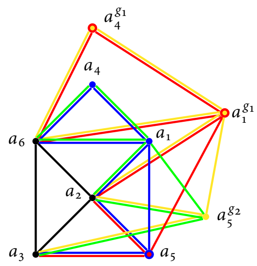


Table 4.22: the graphs $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ and $(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$ in $\mathcal{R}_{3,5}(\Lambda)$

In case three and four we will use lemma 4.7.39 to obtain that the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ is simply connected. In both cases we regard that the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ is identical to the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_4^{g_1}\}, \{a_1, a_6^{g_2}\}, \{a_4^{g_1}, a_6^{g_2}\}\})$. Since a_1 and $a_6^{g_2}$ are neighbors of the vertex a_2 as well as a_1 and $a_4^{g_1}$ are neighbors of the vertex a_6 in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda$ by lemma 4.7.39 the graph $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_4^{g_1}\}, \{a_1, a_6^{g_2}\}\})$ is simply connected. Furthermore the vertex a_1 is adjacent to the vertices $a_6^{g_2}$ and $a_4^{g_1}$ in $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_4^{g_1}\}, \{a_1, a_6^{g_2}\}\})$ implying that $(\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_4^{g_1}\}, \{a_1, a_6^{g_2}\}, \{a_4^{g_1}, a_6^{g_2}\}\})$ is a simply connected graph using lemma 4.7.39 again.

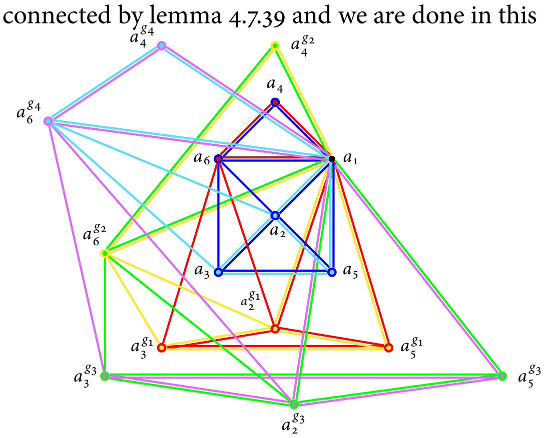
- $\{i, j\} = \{3, 6\}$: $\mathcal{R}_{3,6}(\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{———}}\circ$.
  $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda = (\mathcal{V}(\Lambda \cup g_1\Lambda \cup g_2\Lambda), E(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cup \{\{a_1, a_5^{g_2}\}\})$ as $a_1 = a_1^{g_3}$ and $a_4 = a_4^{g_3}$. Since a_2 is a common neighbor of the vertices a_1 and $a_5^{g_2}$ in the simply connected graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda$, it follows by lemma 4.7.39 that $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ is a simply connected graph, too.



Graph 4.12: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ in $\mathcal{R}_{3,6}(\Lambda)$

- $\{i,j\} = \{4,5\}$: $\mathcal{R}_{4,5}(\Lambda)$ is a rank two building of type $\circ\overset{3}{\text{—}}\circ$.

  By construction we obtain that $a_2 = a_2^{g_5}$, $a_3 = a_3^{g_5}$ and $a_5 = a_5^{g_5}$, thus $\Lambda \cup (\bigcup_{i=1}^{5} g_i\Lambda) = (\mathcal{V}(\Lambda\cup(\bigcup_{i=1}^{4} g_i\Lambda)), E(\Lambda\cup(\bigcup_{i=1}^{4} g_i\Lambda))\cup\{\{a_2, a_6^{g_4}\}, \{a_3, a_6^{g_4}\}\})$. As the vertices a_2 and $a_6^{g_4}$ are adjacent to the vertex a_1 in the simply connected graph $\Lambda \cup (\bigcup_{i=1}^{4} g_i\Lambda)$ and the vertices a_3 and $a_6^{g_4}$ are adjacent to a_2 in the graph $\mathcal{V}(\Lambda \cup (\bigcup_{i=1}^{4} g_i\Lambda)), E(\Lambda \cup (\bigcup_{i=1}^{4} g_i\Lambda)) \cup \{\{a_2, a_6^{g_4}\}\})$, the two graphs $\mathcal{V}(\Lambda\cup(\bigcup_{i=1}^{4} g_i\Lambda)), E(\Lambda\cup(\bigcup_{i=1}^{4} g_i\Lambda))\cup\{a_2, a_6^{g_4}\})$ and $\Lambda\cup(\bigcup_{i=1}^{5} g_i\Lambda)$ are simply connected by lemma 4.7.39 and we are done in this case.



Graph 4.13: the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ in $\mathcal{R}_{4,5}(\Lambda)$

- $\{i,j\} = \{4,6\}$: $\mathcal{R}_{4,6}(\Lambda)$ is a rank two building of type $\circ\overset{2}{\text{—}}\circ$.

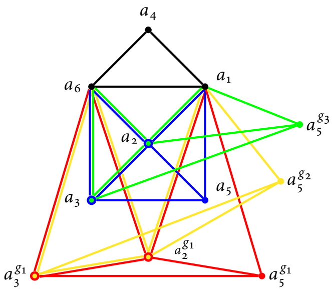
  First of all $a_2 = a_2^{g_3}$ and $a_3 = a_3^{g_3}$ in the graph $\Lambda\cup g_1\Lambda\cup g_2\Lambda\cup g_3\Lambda$. On the other hand we obtain the relations that $a_5 \neq a_5^{g_1}$, $a_5^{g_1} \neq a_5^{g_2}$, $a_5^{g_2} \neq a_5^{g_3}$ and $a_5^{g_3} \neq a_5$. Therefore we have to consider the following four different possibilities in this case.

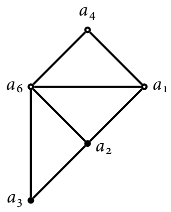
case one	$a_5, a_5^{g_1}, a_5^{g_2}$ and $a_5^{g_3}$ are four different vertices in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$
case two	$a_5 = a_5^{g_2}$ and $a_5^{g_1} \neq a_5^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$
case three	$a_5 = a_5^{g_2}$ and $a_5^{g_1} = a_5^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$
case four	$a_5 \neq a_5^{g_2}$ and $a_5^{g_1} = a_5^{g_3}$ in the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$

  Since $\Lambda\cup g_1\Lambda\cup g_2\Lambda\cup g_3\Lambda = (\Lambda\cup g_1\Lambda\cup g_2\Lambda)\cup g_3$ and the graphs $\Lambda\cup g_1\Lambda\cup g_2\Lambda$ and $(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$ are simply connected by table 4.18 in each case, we conclude via theorem A.5.2 that the graph $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cap g_3\Lambda$ is simply.

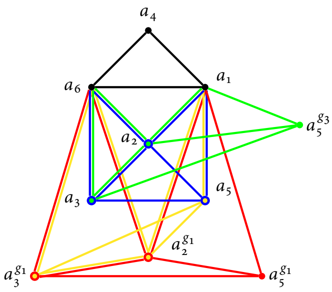
$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$	$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$

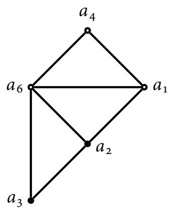
case one



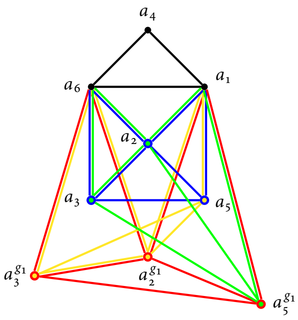


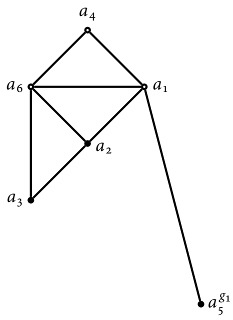
case two





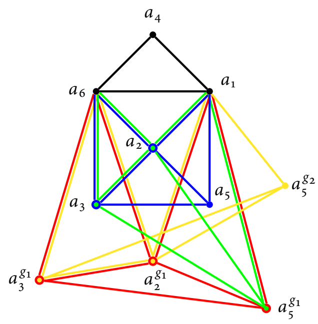
case three





$\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$	$(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$

case four



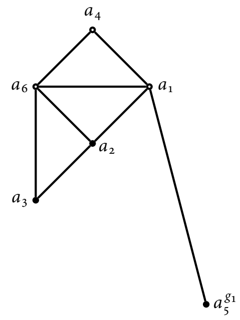


Table 4.23: the graphs $\Lambda \cup g_1\Lambda \cup g_2\Lambda \cup g_3\Lambda$ and $(\Lambda \cup g_1\Lambda \cup g_2\Lambda) \cap g_3\Lambda$ in $\mathcal{R}_{4,6}(\Lambda)$

Finally we have the following theorem.

Theorem 4.7.45 *The connected graph* $\mathbf{F}(E_{6,-78})$ *is simply connected.*

4.8 Classification of the graph $\widehat{\Gamma}$

In this final part we will show that the universal cover $\widehat{\Gamma}$ of a connected locally $\mathbf{S}(V_6)$ graph Γ is isomorphic to either $\mathbf{S}(V_8)$ or $\mathbf{F}(E_{6,-78})$. Thus let $\widehat{\Gamma}$ and $\widehat{\Upsilon}$ be universal covers of the connected locally $\mathbf{S}(V_6)$ graphs Γ and Υ such that

$$|\{\mathbf{z},\mathbf{w}\}^{\perp} \cap \mathbf{x}^{\perp}| = 1 \text{ if and only if } |\{\mathbf{z},\mathbf{w}\}^{\perp} \cap \mathbf{y}^{\perp}| = 1$$

for any chain $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ in Γ resp. Υ of four different vertices $\mathbf{x}, \mathbf{w}, \mathbf{y}$ and $\mathbf{z}$. Moreover, let $\Sigma_{\widehat{\Gamma}}$ be the induced subgraph of $\widehat{\Gamma}$ and $\Sigma_{\widehat{\Upsilon}}$ the induced subgraph of $\widehat{\Upsilon}$ as constructed in section 4.4. If $\Sigma_{\widehat{\Upsilon}} \cong \Sigma_{\widehat{\Gamma}}$, then by theorem 4.6.7 we have $G_{\widehat{\Gamma}} \cong G_{\widehat{\Upsilon}}$. We denote with $N_{G_{\widehat{\Gamma}}}(\mathbf{w}_{\widehat{\Gamma}})$ the normaliser of the subgroup $\mathrm{SU}_2(\mathbb{C})_{\mathbf{w}_{\widehat{\Gamma}}}$ in $G_{\widehat{\Gamma}}$ for a vertex $\mathbf{w}_{\widehat{\Gamma}} \in \widehat{\Gamma}$, thus

$$\begin{aligned} N_{G_{\widehat{\Gamma}}}(\mathbf{w}_{\widehat{\Gamma}}) &= N_{G_{\widehat{\Gamma}}}(\mathrm{SU}_2(\mathbb{C})_{\mathbf{w}_{\widehat{\Gamma}}}) \\ &= \{g \in G_{\widehat{\Gamma}} \mid g(\mathrm{SU}_2(\mathbb{C})_{\mathbf{w}_{\widehat{\Gamma}}})g^{-1} = \mathrm{SU}_2(\mathbb{C})_{\mathbf{w}_{\widehat{\Gamma}}}\}\,. \end{aligned}$$

Moreover we fix the induced subgraphs $\Lambda_{\widehat{\Gamma}}$ in $\Sigma_{\widehat{\Gamma}}$ and $\Lambda_{\widehat{\Upsilon}}$ in Σ_{Υ}, as in lemma 4.6.2, respectively lemma 4.6.4.

To avoid confusion we will index every vertex by the graph which it belongs to, thus

$$\mathcal{V}(\Lambda_{\widehat{\Gamma}}) = \begin{cases} \{\mathbf{x}_{\widehat{\Gamma}}, \mathbf{z}^{11}_{\widehat{\Gamma}}, \mathbf{y}^{12}_{\widehat{\Gamma}}, \mathbf{y}^{23}_{\widehat{\Gamma}}, \mathbf{y}^{34}_{\widehat{\Gamma}}, \mathbf{y}^{45}_{\widehat{\Gamma}}, \mathbf{y}^{56}_{\widehat{\Gamma}}\} & \text{if } \Sigma_{\widehat{\Gamma}} \cong \mathbf{H}(A_7), \\ \{\mathbf{y}^{12}_{\widehat{\Gamma}}, \mathbf{y}^{23}_{\widehat{\Gamma}}, (\mathbf{z}^{14}_{12})_{\widehat{\Gamma}}, \mathbf{x}_{\widehat{\Gamma}}, \mathbf{y}^{45}_{\widehat{\Gamma}}, \mathbf{y}^{56}_{\widehat{\Gamma}}\} & \text{if } \Sigma_{\widehat{\Gamma}} \cong \mathbf{H}(E_6), \end{cases}$$

and

$$\mathcal{V}(\Lambda_{\widehat{\Upsilon}}) = \begin{cases} \{\mathbf{x}_{\widehat{\Upsilon}}, \mathbf{z}^{11}_{\widehat{\Upsilon}}, \mathbf{y}^{12}_{\widehat{\Upsilon}}, \mathbf{y}^{23}_{\widehat{\Upsilon}}, \mathbf{y}^{34}_{\widehat{\Upsilon}}, \mathbf{y}^{45}_{\widehat{\Upsilon}}, \mathbf{y}^{56}_{\widehat{\Upsilon}}\} & \text{if } \Sigma_{\widehat{\Upsilon}} \cong \mathbf{H}(A_7), \\ \{\mathbf{y}^{12}_{\widehat{\Upsilon}}, \mathbf{y}^{23}_{\widehat{\Upsilon}}, (\mathbf{z}^{14}_{12})_{\widehat{\Upsilon}}, \mathbf{x}_{\widehat{\Upsilon}}, \mathbf{y}^{45}_{\widehat{\Upsilon}}, \mathbf{y}^{56}_{\widehat{\Upsilon}}\} & \text{if } \Sigma_{\widehat{\Upsilon}} \cong \mathbf{H}(E_6). \end{cases}$$

Furthermore $G_{\widehat{\Gamma}} = \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{v}_{\widehat{\Gamma}}} \mid \mathbf{v}_{\widehat{\Gamma}} \in \Lambda_{\widehat{\Gamma}} \rangle \cong \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{w}_{\widehat{\Upsilon}}} \mid \mathbf{w}_{\widehat{\Upsilon}} \in \mathcal{V}(\Lambda_{\widehat{\Upsilon}}) \rangle = G_{\widehat{\Upsilon}}$. Therefore we fix the isomorphism $\mu : G_{\widehat{\Gamma}} \to G_{\widehat{\Upsilon}}$ with $\mu(\mathrm{SU}_2(\mathbb{C})_{\mathbf{w}_{\widehat{\Gamma}}}) \mapsto \mathrm{SU}_2(\mathbb{C})_{\mathbf{w}_{\widehat{\Upsilon}}}$ and $\mu(\alpha_{\mathbf{w}_{\widehat{\Gamma}},\varphi}) = \alpha_{\mathbf{w}_{\widehat{\Upsilon}},\varphi}$ for every $\varphi \in \mathrm{SU}_2(\mathbb{C})$ and each vertex $\mathbf{w}_{\widehat{\Gamma}} \in \mathcal{V}(\Lambda_{\widehat{\Gamma}})$. Moreover we consider an element $\delta \in \langle \mathrm{SU}_2(\mathbb{C})_{\mathbf{v}_{\widehat{\Gamma}}} \mid \mathbf{v}_{\widehat{\Gamma}} \in \mathcal{V}((\Lambda_{\widehat{\Gamma}})_{\mathbf{y}^{56}_{\widehat{\Gamma}}}) \rangle \cong \begin{cases} \mathrm{SU}_6(\mathbb{C}) \text{ if } \Sigma_{\widehat{\Gamma}} \cong \mathbf{H}(A_7) \\ \mathrm{SU}_5(\mathbb{C}) \text{ if } \Sigma_{\widehat{\Gamma}} \cong \mathbf{H}(E_6) \end{cases}$ where $\delta(\mathbf{x}_{\widehat{\Gamma}}) = \mathbf{y}^{12}_{\widehat{\Gamma}}$. Of course $\delta = \Pi_{i=1}^n \Pi_{\mathbf{v}_{\widehat{\Gamma}} \in \mathcal{V}((\Lambda_{\widehat{\Gamma}})_{\mathbf{y}^{56}_{\widehat{\Gamma}}})} \alpha_{\mathbf{v}_{\widehat{\Gamma}},\varphi_i^{\mathbf{v}}}$ such that $\varphi_i^{\mathbf{v}} \in \mathrm{SU}_2(\mathbb{C})$ for $1 \leq i \leq n$. Then $\mu(\delta) = \mu(\Pi_{i=1}^n \Pi_{\mathbf{v}_{\widehat{\Gamma}} \in \mathcal{V}((\Lambda_{\widehat{\Gamma}})_{\mathbf{y}^{56}_{\widehat{\Gamma}}})} \alpha_{\mathbf{v}_{\widehat{\Gamma}},\varphi_i^{\mathbf{v}}}) = \Pi_{i=1}^n \Pi_{\mathbf{v}_{\widehat{\Upsilon}} \in \mathcal{V}((\Lambda_{\widehat{\Upsilon}})_{\mathbf{y}^{56}_{\widehat{\Upsilon}}})} \alpha_{\mathbf{v}_{\widehat{\Upsilon}},\varphi_i^{\mathbf{v}}}$ and $\mu(\delta)(\mathbf{x}_{\widehat{\Upsilon}}) = \mathbf{y}^{12}_{\widehat{\Upsilon}}$.

We claim that $\widehat{\Gamma}$ and $\widehat{\Upsilon}$ are isomorphic. In order to prove this statement, we consider the map $\gamma : \widehat{\Gamma} \to \widehat{\Upsilon}$ with $\mathbf{x}_{\widehat{\Gamma}} \mapsto \mathbf{x}_{\widehat{\Upsilon}}$ and $g\mathbf{x}_{\widehat{\Gamma}} \mapsto \mu(g)\mathbf{x}_{\widehat{\Upsilon}}$ for every $g \in G_{\widehat{\Gamma}}$. By lemma 4.5.7, $G_{\widehat{\Gamma}}$ acts vertex-transitively on $\widehat{\Gamma}$, thus $\gamma(\widehat{\Gamma})$ is a subgraph of $\widehat{\Upsilon}$. To prove injectivity of γ let $g\mathbf{x}_{\widehat{\Gamma}}$ and $h\mathbf{x}_{\widehat{\Gamma}}$ be two different vertices of $\widehat{\Gamma}$. Suppose $\gamma(g\mathbf{x}_{\widehat{\Gamma}}) = \mu(g)\mathbf{x}_{\widehat{\Upsilon}} = \mu(h)\mathbf{x}_{\widehat{\Upsilon}} = \gamma(h\mathbf{x}_{\widehat{\Gamma}})$ then $(\mu(h))^{-1}\mu(g) \in N_{G_{\widehat{\Upsilon}}}(\mathbf{x}_{\widehat{\Upsilon}})$, which is equivalent to $\mu(h^{-1}g) \in N_{G_{\widehat{\Upsilon}}}(\mathbf{x}_{\widehat{\Upsilon}})$ implying that $h^{-1}g \in N_{G_{\widehat{\Gamma}}}(\mathbf{x}_{\widehat{\Gamma}})$, contradiction. Let $\mathbf{u}_{\widehat{\Upsilon}}$ be a vertex of $\widehat{\Upsilon}$. By the vertex transitivity of $G_{\widehat{\Upsilon}}$ we find a group element $g \in G_{\widehat{\Upsilon}}$ such that $g\mathbf{x}_{\widehat{\Upsilon}} = \mathbf{u}_{\widehat{\Upsilon}}$. In particular $\mu^{-1}(g)\mathbf{x}_{\widehat{\Gamma}}$ is a preimage of $\mathbf{u}_{\widehat{\Upsilon}}$ under γ as $\gamma(\mu^{-1}(g)\mathbf{x}_{\widehat{\Gamma}}) = \mu(\mu^{-1}(g))\mathbf{x}_{\widehat{\Upsilon}} = \mathbf{u}_{\widehat{\Upsilon}}$. Hence γ is a bijective map between the graphs $\widehat{\Gamma}$ and $\widehat{\Upsilon}$. We consider the vertex $\mathbf{y}^{12}_{\widehat{\Gamma}}$ and a graph isomorphism $\delta \in (\mathrm{SU}_2(\mathbb{C}))_{\mathbf{y}^{56}_{\widehat{\Gamma}}}$ such that $\delta(\mathbf{x}_{\widehat{\Gamma}}) = \mathbf{y}^{12}_{\widehat{\Gamma}}$, thus $\mu(\delta\mathbf{x}_{\widehat{\Gamma}}) = \mu(\delta)(\mathbf{x}_{\widehat{\Upsilon}}) = \mathbf{y}^{12}_{\widehat{\Upsilon}}$. Finally let $\mathbf{w}_{\widehat{\Gamma}}$ and $\mathbf{u}_{\widehat{\Gamma}}$ be two adjacent vertices of $\widehat{\Gamma}$. Then by lemma 4.5.6 the group $G_{\widehat{\Gamma}}$ contains an element g such that $\mathbf{w}_{\widehat{\Gamma}} = g\mathbf{x}_{\widehat{\Gamma}}$ and $\mathbf{u}_{\widehat{\Gamma}} = g\mathbf{y}^{12}_{\widehat{\Gamma}}$, thus $\mathbf{u}_{\widehat{\Gamma}} = g\delta\mathbf{x}_{\widehat{\Gamma}}$ for some $\delta, g \in G_{\widehat{\Gamma}}$. From the properties of the group isomorphism μ it follows that $\gamma(g\mathbf{x}_{\widehat{\Gamma}}) = \mu(g)\mathbf{x}_{\widehat{\Upsilon}}$ and $\gamma(g\delta\mathbf{x}_{\widehat{\Gamma}}) = \mu(g)\mu(\delta)\mathbf{x}_{\widehat{\Upsilon}} = \mu(g)\mathbf{y}^{12}_{\widehat{\Upsilon}}$. Applying that $\mu(g)$ is a graph automorphisms of $\widehat{\Upsilon}$ we obtain that the vertices $\mu(g)\mathbf{x}_{\widehat{\Upsilon}}$ and $\mu(g)\mathbf{y}^{12}_{\widehat{\Upsilon}}$ are adjacent, which confirms the statement that γ is a graph isomorphism, so that we obtain the following result.

Proposition 4.8.1 *Let $\widehat{\Gamma}$ and $\widehat{\Upsilon}$ be universal covers of connected locally $\mathbf{S}(V_6)$ graphs Γ and Υ which satisfying the property that*

$$|\{\mathbf{z}, \mathbf{w}\}^\perp \cap \mathbf{x}^\perp| = 1 \textit{ if and only if } |\{\mathbf{z}, \mathbf{w}\}^\perp \cap \mathbf{y}^\perp| = 1$$

for any chain $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ *in* Γ *resp. in* Υ *of four different vertices* $\mathbf{x}, \mathbf{w}, \mathbf{y}$ *and* $\mathbf{z}$*. If the induced subgraphs* $\Sigma_{\widehat{\Gamma}}$ *and* $\Sigma_{\widehat{\Upsilon}}$ *constructed in section 4.4 are isomorphic, then the graphs* $\widehat{\Gamma}$ *and* $\widehat{\Upsilon}$ *are isomorphic as well.*

Thus the first main result is the following.

Theorem 4.8.2 *Let* Γ *be a connected graph locally* $\mathbf{S}(V_6)$ *satisfying that*

$$|\{\mathbf{z}, \mathbf{w}\}^{\perp} \cap \mathbf{x}^{\perp}| = 1 \textit{ if and only if } |\{\mathbf{z}, \mathbf{w}\}^{\perp} \cap \mathbf{y}^{\perp}| = 1$$

for any chain $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ *in* Γ *of four different vertices* $\mathbf{x}, \mathbf{w}, \mathbf{y}$ *and* $\mathbf{z}$ *and* $\widehat{\Gamma}$ *its universal cover. If* $\Sigma_{\widehat{\Gamma}} \cong W(A_7)$*, then* $\widehat{\Gamma} \cong \mathbf{S}(V_8)$*.*

Proof: The statement follows directly from lemma 4.7.1, theorem 4.7.42 and proposition 4.8.1. ■

Theorem 4.8.3 *Let* Γ *be a connected locally* $\mathbf{S}(V_6)$ *graph satisfying that*

$$|\{\mathbf{z}, \mathbf{w}\}^{\perp} \cap \mathbf{x}^{\perp}| = 1 \textit{ if and only if } |\{\mathbf{z}, \mathbf{w}\}^{\perp} \cap \mathbf{y}^{\perp}| = 1$$

for any chain $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ *in* Γ *of four different vertices* $\mathbf{x}, \mathbf{w}, \mathbf{y}$ *and* $\mathbf{z}$ *and* $\widehat{\Gamma}$ *its universal cover. If* $\Sigma_{\widehat{\Gamma}} \cong W(E_6)$*, then* $\widehat{\Gamma} \cong \mathbf{F}(E_{6,-78})$*.*

Proof: As before the statement follows from lemma 4.7.31, theorem 4.7.45 and proposition 4.8.1. ■

Now theorem 4.8.2, proposition 4.8.1 and theorem 4.8.3 imply together the mentioned result.

Theorem 4.1.2 *Let* Γ *be a connected locally* $\mathbf{S}(V_6)$ *graph satisfying that*

$$|\{\mathbf{z}, \mathbf{w}\}^{\perp} \cap \mathbf{x}^{\perp}| = 1 \textit{ if and only if } |\{\mathbf{z}, \mathbf{w}\}^{\perp} \cap \mathbf{y}^{\perp}| = 1$$

for any chain $\mathbf{x} \perp \mathbf{w} \perp \mathbf{y} \perp \mathbf{z} \perp \mathbf{x} \perp \mathbf{y}$ *in* Γ *of four different vertices* $\mathbf{x}, \mathbf{w}, \mathbf{y}$ *and* $\mathbf{z}$ *and* $\widehat{\Gamma}$ *its universal cover. Then* $\widehat{\Gamma}$ *is isomorphic to* $\mathbf{S}(V_8)$ *or to* $\mathbf{F}(E_{6,-78})$*.*

APPENDIX ONE

Synthetic geometry

Here we will collect some basic definitions, concepts and notations, that we use throughout the main part of this work. For a more systematical approach we refer the reader to the literature, for example [16], [20] or [73] and for an overview see [15].

A.1 Concept of a geometry

We use the word **geometry** or more precisely **incidence geometry** in a quite technical sense. A geometry consists of elements of different types like **points**, **lines** or **subspaces**. The **rank** of a geometry is the number of distinct types, moreover incidence is a symmetric, reflexive relation on the set of elements of a geometry. Here is the formal definition.

Definition A.1.1 Let I be a set, called the type set. An **incidence system** is a triple $\mathcal{G} = (X, *, \mathrm{typ})$, where X is a set containing the elements of $\mathcal{G}$, $*$ is a symmetric and reflexive relation defined on the set X, called the **incidence relation** of $\mathcal{G}$ and typ is a map from X to I such that the two identities $\mathrm{typ}(x) = \mathrm{typ}(y)$ and $x * y$ imply that $x = y$.

To an incidence system is associated in a natural way the graph $\Gamma_{\mathcal{G}} = (X, *)$, which we also call the **incidence graph** of $\mathcal{G}$. The vertices of $\Gamma_{\mathcal{G}}$ are just the elements of $\mathcal{G}$ and two different elements x and y of $\mathcal{G}$ are joined by an edge in $\Gamma_{\mathcal{G}}$ if and only if $x * y$ in $\mathcal{G}$. An incidence system $\mathcal{G}$ is **connected** if $\Gamma_{\mathcal{G}} = (X, *)$ is a connected graph.

With the notation above, if $A \subseteq X$ we say A is of type typ(A) and the **rank** of A is just the cardinality of typ(A). The **corank** of A is the cardinality of $I\backslash$typ(A). Furthermore for an element $x \in X$, we denote with x^* all elements of X which are incident to x. For a non empty subset A of X we define $A^* = \bigcup_{x \in A} x^*$, thus A^* contains all elements of X which are incident with every element of A.

A set of mutually incident elements of an incidence system $\mathcal{G}$ is called **flag of** $\mathcal{G}$. Certainly by Zorn's lemma, every flag is contained in at least one maximal flag, which is a flag not properly contained in any other flag. We denote flags of type I as **chambers** .

A **geometry** over I is an incidence system $\mathcal{G}$ over I in which every maximal flag is a chamber. A geometry is **firm** resp. **thick**, if every flag of type other than I is contained in at least two resp. three distinct chambers of $\mathcal{G}$.

Definition A.1.2 A **point-line geometry** or **point-line space** is a rank two geometry.

A **subspace** X of a point-line geometry $G = (P, L)$ is a subset of the point set P such that any line of L intersecting the set X in at least two points is completely contained in X. Using the observation that the intersection of subspaces again is a subspace, we define for each subset Y of the point set P the subspace $\langle Y \rangle$ generated by Y to be the intersection of all subspaces of G containing the set Y. Hence $\langle Y \rangle$ denotes the smallest subspace of G containing Y. A **plane** is a subspace of G generated by two intersecting lines. The point-line geometry G is called **planar** if any pair of intersecting lines are contained in a unique plane.

The **order** of a geometry G equals $k \in \mathbb{N}$, if all lines of G are incident with exactly $k + 1$ points.

A **partially linear space** is a point-line geometry $G = (P, L)$ with the property that each line contains at least two different points and two different points are in at most one common line, the **connecting line** of these two points. We call two different points contained in a common line **collinear**. A partial linear space is called **thick**, if all lines contain at least three points.

The **point graph** of G is the graph with vertex set P in which two different points are adjacent if and only if a, b are collinear. G is **connected** if the point graph of G is a connected graph.

A **linear space** is a partially linear space in which any two points are collinear, so any two different points admit a connecting line.

A **projective space** is a linear space in which the **Veblen-Young axiom** is satisfied: Suppose a, b, c and d are distinct points. Then the connecting line l_{ab} of a and b intersect the connecting line l_{cd} of c and d if and only if the connecting lines l_{ac} and l_{bd} of a and c resp. of b and d intersect, see also page 2.

A **polar space** is a partially linear space in which the **Buekenhout-Shult axiom**, see [18], holds: Suppose p is a point and l is a non-incident line. Then either one or all points on l are collinear to p.

We recall the definition of a graph. A **graph** is a set of vertices with a family of (unordered) pairs of distinct vertices, called edges. Two vertices are called adjacent if there is an edge to which they both belong. A graph is **bipartite** if its set of vertices can be partitioned into two disjoint subsets such that no two vertices in the same subset lie on a common edge. A **clique** or **complete graph** is a graph in which all unordered pairs of vertices are edges. A **circuit** or a **cycle** of a graph is a closed path in that graph. The **girth** of a graph is the length of the shortest cycle contained in that graph. It is simple to identify all flags of a geometry $\mathcal{G}$ in the incidence graph $\Gamma_{\mathcal{G}}$ as a flag of $\mathcal{G}$ is a clique in $\Gamma_{\mathcal{G}}$ and a clique in the incidence graph $\Gamma_{\mathcal{G}}$ is a flag of $\mathcal{G}$.

We will associate to a graph some geometries. Therefore let Γ be a graph with vertex set $\mathcal{V}$ and edge set E. We set $I = \{\text{vertex, edge}\}$,

let $X = \mathcal{V} \cup E$,

let $\operatorname{typ} : X \to I$ be the map with $\operatorname{typ}(x) = \begin{cases} \text{vertex} & \text{if } x \in \mathcal{V} \\ \text{edge} & \text{if } x \in E \end{cases}$, and

let $*$ be the symmetrised containment on the set X, so $x * y$ if and only if either $x \in y$ or $y \in x$ for any two elements $x, y \in X$.

Then $\mathcal{G}_\Gamma = (X, *, \operatorname{typ})$ is called the **vertex-edge-incidence system** of the graph Γ or the 1-**simplex incidence system** of Γ.

Next we denote with T the set of all triples $\{x, y, z\} \subseteq \mathcal{V}$ such that any pair $\{g, h\} \subseteq \{x, y, z\}$ is an edge in Γ. Therefore T is the set of all 3-cliques, or triangles, of the graph Γ. In this setup we define $I = \{\text{vertex, edge, triangle}\}$,

let $X = \mathcal{V} \cup E \cup T$,

let $\operatorname{typ} : X \to I$ be the map with $\operatorname{typ}(x) = \begin{cases} \text{vertex} & \text{if } x \in \mathcal{V} \\ \text{edge} & \text{if } x \in E \\ \text{triangle} & \text{if } x \in T \end{cases}$, and

let $*$ be the symmetrised containment on the set X, so $x * y$ if and only if either $x \in y$ or $y \in x$ for any two elements $x, y \in X$.

Then $\mathcal{G}_\Gamma = (X, *, \mathrm{typ})$ is called the **vertex-edge-triangle incidence system** of the graph Γ or the 2-**simplex incidence system** of Γ.

Let V be a vector space of finite dimension $n+1$. Then the **projective geometry** $\mathbb{P}(V) = (X, *, \mathrm{typ})$ is defined as follows.

- The elements are all non-trivial subspaces of V, so $X = \bigcup_{1 \le k \le n} G_k(V)$, where $G_k(V)$ are the Grassmannian of V of dimension k.
- For an non-trival subspace W of V of dimension $k+1$, we set $\mathrm{typ}(W) = k$, thus $I = \{0, \dots, n-1\}$.
- Subspaces U and W are incident, in symbols $U * W$, if and only if either $U \subseteq W$ or $W \subseteq U$, (this condition is also called symmetrised inclusion).

The elements of type 0 resp. 1 are called **points** resp. **lines**.

Next we give the definition of a residue. Let $\mathcal{G} = (X, *, \mathrm{typ})$ be an incidence system over the index set I and F be a flag of $\mathcal{G}$. Then the **residue of** F in $\mathcal{G}$ is the triple $\mathcal{G}_F = (X_F, *_F, \mathrm{typ}_F)$, where $X_F = F^* \backslash F$ and $*_F, \mathrm{typ}_F$ are just the restrictions of $*$ and typ to $X_F \times X_F$ and X_F, respectively.

Proposition A.1.3 (Proposition 1.5.3 of [16]) *For a geometry $\mathcal{G} = (X, *, \mathrm{typ})$ over I and a flag F of $\mathcal{G}$, the following assertions hold.*

- *The residue $\mathcal{G}_F$ is a subgeometry of $\mathcal{G}$ over type $I \backslash \mathrm{typ}(F)$.*
- *A subset A of X_F is a flag of $\mathcal{G}_F$ if and only if $F \cup A$ is a flag of $\mathcal{G}$.*
- *If A is a flag of $\mathcal{G}_F$, then $(\mathcal{G}_F)_A = \mathcal{G}_{F \cup A}$.*

An incidence system $\mathcal{G} = (X, *, \mathrm{typ})$ over I is called **residually connected** if and only if for each flag F of corank at least two of $\mathcal{G}$ the incidence system $\mathcal{G}_F$ is connected. Residual connectedness gives rise to some powerful properties.

Proposition A.1.4 (Lemma 1.6.4 of [16]) *A residually connected incidence system is a residually connected geometry if no flag of corank one is maximal.*

In the following part, we capture some characteristics of rank 2 geometries. Therefore for the next part we consider a rank 2 geometry over $\{p, l\}$. Recall, in a connected geometry $\mathcal{G} = (X, *, \mathrm{typ})$ over I, two elements x, y of $\mathcal{G}$ are said to be at distance $k = d(x, y)$ if they are at distance k in the incidence graph $\Gamma_{\mathcal{G}}$. For $j \in I$, the j-**diameter** d_j of $\mathcal{G}$ is the largest number occurring as a diameter of $\Gamma_{\mathcal{G}}$ at some element of type j. If $I = \{p, l\}$, then the difference between d_p and d_l is at most one and the larger one is equal to the diameter d of $\Gamma_{\mathcal{G}}$.

A **circuit** in a geometry $\mathcal{G}$ over $I = \{p, l\}$ is a chain $x = x_0 * x_1 * x_2 * \ldots * x_{2n} = x$ from x to x, with $x_i \notin \{x_{i-2}, x_{i-1}, x_{i+1}, x_{i+2}\}$ for $i = 0, \ldots, 2n$ (all indices taken modulo $2n$ and $n > 0$). Its length $2n$ is necessarily even. The minimal number $g > 0$ such that $\mathcal{G}$ has a circuit of length $2g$ is called the **girth** of the geometry $\mathcal{G}$. If $\mathcal{G}$ has no circuits, we put $g = \infty$.

The girth g of a geometry $\mathcal{G}$ over $I = \{p, l\}$ satisfies

$$\text{either } 2 \leq g \leq d_p \leq d_l \leq d_p + 1$$

$$\text{or } 2 \leq g \leq d_l \leq d_p \leq d_l + 1.$$

For a proof of this statement see lemma 2.3.6 of[16].

Definition A.1.5 If $\mathcal{G}$ is a $\{p, l\}$-geometry with finite diameter d and with girth g having the same diameter d_i at all elements of type i, for $i \in \{p, l\}$ then $\mathcal{G}$ is called a (g, d_p, d_l)-**gon over** (p, l) . If, in addition, $g = d_p = d_l$, then $\mathcal{G}$ is called a **generalized** g-**gon**.

Generalized 2-gons are also called **generalized digon** and in a generalized digon each element of type p is incident with each element of type l. Generalized 3-gons are also called projective planes or **generalized triangle** , generalized 4-gons are called **generalized quadrangles** and likewise, generalized 6-gons (respectively, 8-gons) are called **generalized hexagons** (respectively, **generalized octagons**). **Generalized polygons** is the name used for all generalized g-gons ($g \geq 2$).

A.2 Coverings and simple connectedness of geometries

Definition A.2.1 Let $\mathcal{G}$ be a geometry. A path or a chain of length k in the geometry is a sequence of elements $x_0, x_1, \ldots, x_n$ such that $x_i \neq x_i + 1$ (we do not allow repetitions) and $x_i * x_{i+1}$ for $0 \leq i \leq n - 1$. A cycle or a circuit based at an element $x = x_0$ is a path of length k in $\mathcal{G}$ with the property that $x_0 = x_k$. So a path (cycle) of $\mathcal{G}$ is a path (cycle) in the incidence graph $\Gamma_{\mathcal{G}}$.

Two paths of the geometry $\mathcal{G}$ are **homotopically equivalent** if one can be obtained from the other using only the following operations: inserting or deleting a cycle of length two, a **return**, or a cycle of length three, a **triangle**. These operations are called **elementary homotopies**. A cycle that is homotopically equivalent to a cycle of length 0 is called **null homotopic**, or **homotopically trivial**.

The equivalence classes of homotopically equivalent cycles based at an element x is a group with concatenation as operation. This group is called the **fundamental group** of $\mathcal{G}$ (at x) and is denoted by $\pi_1(\mathcal{G}, x)$. The fundamental group $\pi_1(\mathcal{G}, x)$ of the geometry $\mathcal{G}$ is independent of the choice of the base point x in a fixed connected component.

In geometry, as in every structure theory, the concept of different types of morphisms is essential.

Definition A.2.2 Let $\mathcal{G} = (X, *, \mathrm{typ})$ be an incidence system over the index set I and $\widehat{\mathcal{G}} = (\widehat{X}, \hat{*}, \widehat{\mathrm{typ}})$ be an incidence system over the index set $\widehat{I}$. A **morphism** $\varphi : \widehat{\mathcal{G}} \to \mathcal{G}$ is a map $\varphi : \widehat{X} \to X$ such that for all $x, y \in \widehat{X}$,

- $x \hat{*} y$ implies $\varphi(x) * \varphi(y)$,
- $\widehat{\mathrm{typ}}(x) = \widehat{\mathrm{typ}}(y)$ if and only if $\mathrm{typ}(\varphi(x)) = \mathrm{typ}(\varphi(y))$.

If also $I = \widehat{I}$ and if $\mathrm{typ}(x) = \widehat{\mathrm{typ}}(\varphi(x))$ for all $x \in X$ then the morphism φ is called a **homomorphism**.

An injective homomorphism $\varphi : \widehat{\mathcal{G}} \to \mathcal{G}$ of incidence systems is also called an **embedding** of $\widehat{\mathcal{G}}$ into $\mathcal{G}$. A bijective morphism φ is called a **correlation** if the inverse map φ^{-1} is a morphism, as well.

Suppose a morphism $\varphi : \widehat{\mathcal{G}} \to \mathcal{G}$ of incidence systems is a homomorphism and a correlation, then φ is an **isomorphism** and we write $\widehat{\Gamma} \cong \Gamma$.

A morphism $\varphi : \widehat{\mathcal{G}} \to \mathcal{G}$ of incidence systems is called **covering** if and only if φ is surjective and for every non-empty flag F in $\widehat{\mathcal{G}}$ the morphism φ induces an isomorphism between the residue $\widehat{\mathcal{G}}_F$ and the residue $\mathcal{G}_{\varphi(F)}$. We call $\widehat{\mathcal{G}}$ a **cover** of $\mathcal{G}$. Furthermore a connected incidence system $\mathcal{G}$ is called **simply connected** if any covering $\varphi : \widehat{\mathcal{G}} \to \mathcal{G}$ is in fact an isomorphism.

If I is a finite set and $k < |I|$, then a k-**covering** $\varphi : \widehat{\mathcal{G}} \to \mathcal{G}$ between connected incidence systems $\widehat{\mathcal{G}}$ and $\mathcal{G}$ over I is a surjective homomorphism such that for every flag F of $\widehat{\mathcal{G}}$ of corank at most k the morphism φ induces an isomorphism between $\widehat{\mathcal{G}}_F$ and $\mathcal{G}_{\varphi(F)}$. We call $\widehat{\mathcal{G}}$ a k-**cover** of $\mathcal{G}$.

Directly from the definitions we have that a covering between incidence systems $\widehat{\mathcal{G}}$ and $\mathcal{G}$ is a covering between geometries $\widehat{\mathcal{G}}$ and $\mathcal{G}$ if and only if either $\widehat{\mathcal{G}}$ or $\mathcal{G}$ is a geometry.

If we consider a geometry via its incidence graph, which is a simplicial complex, we can use the following result from the theory of simplicial complexes.

Proposition A.2.3 (chapter 8 of [78]) *If $\mathcal{G}$ is a connected geometry and x an element of $\mathcal{G}$, then the group $\pi_1(\mathcal{G}, x)$ is trivial if and only if all coverings of $\mathcal{G}$ are isomorphisms.*

In particular, a connected geometry with trivial fundamental group is simply connected. We are returning to graphs and want to define the notion of a covering for graphs. Let $\Gamma_i = (\mathcal{V}_i, E_i)$ for $i = 1, 2$ be two graphs. A **graph morphism** is a mapping $\delta : \Gamma_1 \to \Gamma_2$ such that if $\{x, y\}$ is an edge of E_1 then either $\delta(x) = \delta(y)$ or else $\{\delta(x), \delta(y)\}$ is an edge of E_2. A graph morphism is injective resp. surjective if and only if the map δ as a map between the vertex sets $\mathcal{V}_1$ and $\mathcal{V}_2$ is injective resp. surjective.

Definition A.2.4 A 1-**covering** $\pi : \widehat{\Gamma} \to \Gamma$ between connected graphs is a covering between the standard vertex-edge incidence systems of $\widehat{\Gamma}$ and Γ. A 2-**covering**, also called a covering $\pi : \widehat{\Gamma} \to \Gamma$ between connected graphs is a covering between the standard vertex-edge-triangle incidence systems of $\widehat{\Gamma}$ and Γ.

From a graph theoretical point a 1-covering $\pi : \widehat{\Gamma} \to \Gamma$ of connected graphs is a surjective graph morphism, such that for each vertex x of $\widehat{\Gamma}$ the map π induces a bijective graph morphism between $\widehat{\Gamma}_x$ and $\Gamma_{\pi(x)}$. On the other hand a 2-covering $\pi : \widehat{\Gamma} \to \Gamma$ of connected graphs is a surjective graph morphism π inducing a graph isomorphism between $\hat{\Gamma}_x$ and $\Gamma_{\pi(x)}$ for every vertex x of $\widehat{\Gamma}$.

Let Γ be a connected graph then Γ is 2-**simply connected**, or **simply connected**, if the vertex-edge-triangle incidence system of $\mathcal{G}$ is simply connected.

A covering $\pi : \widehat{\Gamma} \to \Gamma$ of connected graphs $\widehat{\Gamma}$ and Γ mapping the vertex $\hat{x} \in \widehat{\Gamma}$ to $x \in \Gamma$ is called **universal** if for any covering $\delta : \Delta \to \Gamma$ and any $y \in \delta^{-1}(x)$, there exists a unique covering map $\varphi : \widehat{\Gamma} \to \Delta$ with $\pi = \varphi \circ \delta$ and $\varphi(\hat{x}) = y$.

Proposition A.2.5 *Let Γ be a connected graph. Then a universal covering $\pi : \widehat{\Gamma} \to \Gamma$ for Γ always exists. Moreover this universal cover $\widehat{\Gamma}$ is simply connected and Γ is locally Σ if and only $\widehat{\Gamma}$ is locally Σ.*

Proposition A.2.6 *A connected graph Γ is simply connected if and only if every cycle of Γ can be decomposed into triangles.*

A.3 Permuatation groups

If G is a group and X a set, a group homomorphism $\alpha : G \to \mathrm{Sym}(X)$ is called a **permutation representation** of G in X. In this case, X is also referred to as a G-**set**.

A permutation representation is called **faithful** if α is injective. Let $x \in X$, the set $\{\alpha(g)(x) \mid g \in G\}$ is called the G-**orbit** of x in X. The permutation representation is called **transistive** if there is only one G-orbit in X. The **stabiliser** of x in G, denoted by G_x is the subgroup $\{g \in G \mid \alpha(g)(x) = x\}$ of G.

As usual, instead of a permutation representation of G in X, we shall often speak of an action of G on X. If H is a subgroup of G, there is a standard way of constructing a transitive permutation representation.

For H a subgroup of the group G, set $\alpha(g)(aH) = gaH$ for all $g, a \in G$. We call the representation $\alpha : G \to \mathrm{Sym}\,(G/H)$ the **permutation representation of G over H**. Any transitive permutation representation can be described as a permutation representation over a subgroup. To be more precise, we need the notion of equivalence.

Two permutation representations $\alpha : G \to \mathrm{Sym}\,(X)$ and $\beta : G \to \mathrm{Sym}\,(Y)$ are said to be **equivalent** if there is a bijection $\gamma : X \to Y$ such that $\gamma\alpha(g)\gamma^{-1} = \beta(g)$ for each $g \in G$ or equivalently, $\gamma\alpha(g) = \beta(g)\gamma$ for all $g \in G$.

For instance, if α is the permutation representation of G over a subgroup H, then, for $a \in G$, the stabiliser of aH is aHa^{-1}. So, α is equivalent to the permutation representation of G over the conjugate aHa^{-1} of H.

Theorem A.3.1 (Fundamental Theorem of Permutation Groups, theorem 8.1.5 of [16]) *Let $\alpha : G \to Sym(X)$ be a transitive permutation representation. Then, for any $x \in X$, the permutation representation of G over G_x is equivalent to α.*

This theorem has analogue for several structures whose underlying set admits a transistive permutatution. Certainly, if the structure will be transferred into group data, the group of the representation should act as a group of automorphisms.

Definition A.3.2 If Δ denotes a structure and $\mathrm{Aut}\,(\Delta)$ the group of all automorphisms of Δ then we shall say that α is a **representation** of G in Δ if it is a group homomorphism $\alpha : G \to \mathrm{Aut}\,(\Delta)$.

Two representations α, β of a group G in structures Δ, Δ' (of same kind) are called **equivalent** if there is an isomorphism $\gamma : \Delta \to \Delta'$ establishing equivalence between the associated ordinary representations $\alpha : G \to \mathrm{Sym}\,(\Delta)$ and $\beta : G \to \mathrm{Sym}\,(\Delta')$, where $\mathrm{Sym}\,(\Delta)$ is the symmetric group on the natural set underlying Δ.

A.4 Chamber system

A **chamber system** $\mathcal{C} = (C, (\sim_i)_{i \in I})$ over a type set I is a set C whose elements are called chambers together with equivalence relations $\sim_i$, $i \in I$, on the chamber set C, such that if $c \sim_i d$ and $c \sim_j d$ then either $i = j$ or $c = d$. For $i \in I$ and chambers $c, d \in C$ we say that c and d are i-**adjacent** if $c \sim_i d$. The chambers c and d are called **adjacent** if they are i-adjacent for some $i \in I$. The **rank** of the chamber system $\mathcal{C}$ is $|I|$.

A chamber system called **thick** if for every $i \in I$ and every chamber $c \in \mathcal{C}$, there are at least three chambers (c itself and at least two other chambers) i-adjacent to c. A chamber system is called **thin** if c is i-adjacent to exactly two chambers (to c itself and excatly one other chamber) for every $i \in I$ and every $c \in \mathcal{C}$.

Let I be a finite index set and $\mathcal{C} = (C, (\sim_i)_{i \in I})$ be a chamber system over I. A **gallery** γ in $\mathcal{C}$ is a finite sequence $\gamma = c_0, c_1, \ldots, c_n$ of chambers in $\mathcal{C}$ such that c_{k-1} is adjacent to c_k for $1 \leq k \leq n$. The **length** of the gallery γ is the number k. Let γ be a gallery of $\mathcal{C}$, then we set $\alpha(\gamma) = c_0$ and $\omega(\gamma) = c_n$. The chambers $c, d \in \mathcal{C}$ are **joint by a gallery** γ in $\mathcal{C}$ if there is a gallery γ of $\mathcal{C}$ such that $\alpha(\gamma) = c$ and $\omega(\gamma) = d$. We say also the gallery γ joints the chambers c and d of $\mathcal{C}$. The chamber system $\mathcal{C}$ is **connected** if for any two chambers of $\mathcal{C}$ are joint by a gallery. A gallery γ is called **closed** if $\alpha(\gamma) = \omega(\gamma)$ and we say a gallery γ is **simple** if $c_{k-1} \neq c_k$ for all $1 \leq k \leq n$.

Let $\gamma = c_0, c_1, \ldots, c_n$ be a gallery of a chamber system $\mathcal{C}$ then we denote with γ^{-1} the gallery $c_n, c_{n-1}, \ldots, c_1, c_0$ and if $\delta = d_0, d_1, \ldots, d_m$ is also a gallery of $\mathcal{C}$ with $\alpha(\delta) = \omega(\gamma)$ then $\gamma\delta$ is the gallery $c_0, c_1, \ldots, c_n, d_0, d_1, \ldots, d_m$ of $\mathcal{C}$.

Next let $J \subseteq I$, a J-**gallery** of $\mathcal{C}$ is a gallery γ such that $c_{k-1} \sim_j c_k$ with $j \in J$ for each index $1 \leq k \leq n$. Given two chamber c and d of $\mathcal{C}$, then we say c and d are J-**equivalent** if there is a J-gallery γ of $\mathcal{C}$ joining c and d and we write $c \sim_J d$. Certainly, if two chambers c and d are i-adjacent then c and d are also i-equivalent.

We fix a subset J of I and a chamber $c \in \mathcal{C}$. The set of chambers $R_J(c) = \{d \in \mathcal{C} \mid c \sim_j d \text{ for some } j \in J\}$ is the J-**residue** of c. For each $c \in \mathcal{C}$ and each subset J of I the pair $\mathcal{R}_J(c) = (R_J(c), (\sim_j)_{j \in J})$ is a connected chamber system of type J and rank $|J|$. If $J = i$, then we call the rank one residue $\mathcal{R}_i(c)$ of type i the i-**panel** of c or the i-**panel** containing c.

Throught the next part, let $\mathcal{C}$ be a chamber system of type I, where I is finite and m be some natural number, so $m \geq 1$.

Two galleries γ and δ of $\mathcal{C}$ are **elementary** m-**homotopic** if there are two galleries ε and ψ and two J-galleries γ_0 and δ_0 for some $J \subseteq I$ of cardinality at most m such that $\gamma = \varepsilon\gamma_0\psi$ and $\delta = \varepsilon\delta_0\psi$. Two galleries are m-**homotopic** if there is a finite sequence $\gamma_0, \gamma_1, \ldots, \gamma_l$ of galleries of $\mathcal{C}$ such that $\gamma = \gamma_0$, $\delta = \gamma_l$ and γ_{k-1} is elementary m-homotopic to γ_k for each $1 \leq k \leq l$. Certainly if two galleries γ and δ are m-homotopic then $\alpha(\gamma) = \alpha(\delta)$ and $\omega(\gamma) = \omega(\delta)$.

Let γ be a closed gallery of $\mathcal{C}$, then γ is called **null**-m-**homotopic** if γ is m-homotopic to the gallery $\alpha(\gamma)$. A chamber system $\mathcal{C}$ is m-**simply connected** if every closed gallery of $\mathcal{C}$ is null-m-homotopic. If $m = 2$ then we also say $\mathcal{C}$ is **simply connected** instead of 2-simply connected.

A **morphism** $\alpha : (C, (\sim_i)_{i\in I}) \to (C', (\sim_{i'})_{i'\in I})$ of chamber systems over I is a map $\alpha : C \to C'$ for which a permutation π of I can be found such that, for all $c, d \in C$, the relation $c \sim_i d$ implies $\alpha(c) \sim_{\pi(i)} \alpha(d)$. If $\pi = \mathrm{id}$, the morphism is said to be a **homomorphism**. As usual, a bijective homomorphism whose inverse is also a homomorphism is called an **isomorphism** and an isomorphism from $\mathcal{C}$ to $\mathcal{C}$ is called an **automorphism** of $\mathcal{C}$. We denote by $\mathrm{Aut}(\mathcal{C})$ the group of all automorphisms of $\mathcal{C}$.

Let G be a group of automorphisms of a chamber system $\mathcal{C}$. When G is transitive on the set of chambers of $\mathcal{C}$, we say G is **chamber transistive** on $\mathcal{C}$. We also say that a chamber system $\mathcal{C}$ is **chamber transistive** if $\mathrm{Aut}(\mathcal{C})$ is chamber transistive.

If $\mathcal{C}$ is chamber transitive, an easy description of $\mathcal{C}$ can be given in terms of G and of some its subgroups.

Let G be a group, B a subgroup, $(P_i)_{i\in I}$ a family of subgroups of G such that $B \le P_i$ for every $i \in I$. The **chamber system determined by** G **on** B **with respect to** $(P_i)_{i\in I}$, notation $\mathcal{C} = \mathcal{C}(G, B, (P_i)_{i\in I})$, is defined as the set of all cosets gB, $g \in G$, with $gB \sim_i hB$ if and only if $gP_i = hP_i$.

The group G acts as a group of automorphisms of $\mathcal{C}(G, B, (P_i)_{i\in I})$ by left multiplication and of course G is chamber transistive on $\mathcal{C}(G, B, (P_i)_{i\in I})$.

Proposition A.4.1 (proposition 3.4.5 of [16]) *Suppose that $\mathcal{C}$ is a chamber system over I with chamber transitive group of automorphisms G. Then there are subgroups B and subgroups $(P_i)_{i\in I}$ of G such that $\mathcal{C}$ is isomorphic to $\mathcal{C}(G, B, (P_i)_{i\in I})$.*

Fix a chamber c in $\mathcal{C}$ then we put $B = G_c$ and set $P_i = P_i^c$ the stabiliser in G of the i-panel in $\mathcal{C}$ containing c. Then $\mathcal{C}$ is isomorphic to $\mathcal{C}(G, B, (P_i)_{i\in I})$.

The next proposition is a variation of theorem A.3.1 concerns chamber systems, in which G is a chamber transitive group of automorphisms of a chamber system $\mathcal{C}$ over I.

A homomorphism $G \to \mathrm{Aut}(\mathcal{C})$ is called a permutation representation of G in the chamber system $\mathcal{C}$.

Proposition A.4.2 (proposition 8.6.2 of [16]) *If $\alpha : G \to \mathrm{Aut}(\mathcal{C})$ is a chamber transitive representation of G on a chamber system $\mathcal{C}$, then, for any chamber c of $\mathcal{C}$, the canonical representation of G in $\mathcal{C}(G, G_c, (P_i^c)_{i\in I})$ is equivalent to α.*

A.5 The fundamental group of a topological space and the Seifert-Van Kampen theorem

The fundamental group is a tool to study topological spaces. First we concern paths in a topological space, which leads to a description of the first homotopy group π_1.

A **path** ω in a topological space X is defined to be a continuous map $\omega : I = [0,1] \to X$. The **origin** of the path ω is the point $\omega(0)$ and the **end** of the path ω is the point $\omega(1)$. We say, that ω is a path from $\omega(0)$ to $\omega(1)$. A **closed path** or a **loop** , at $x_0 \in X$ is a path ω such that $\omega(0) = \omega(1) = x_0$. For paths ω and ω' in X with $\omega(1) = \omega'(0)$, we define the **product path** $\omega \star \omega'$ in X be the formula

$$\omega \star \omega'(t) = \begin{cases} \omega(2t) & 0 \leq t \leq \frac{1}{2} \\ \omega'(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} .$$

Two paths ω and ω' in X are briefly said to be **homotopic**, denoted by $\omega \cong \omega'$, if there exists a continuous map $F : [0,1] \times [0,1] \to X$ such that $F(t,0) = \omega(t)$ and $F(t,1) = \omega'(t)$ for every $t \in [0,1]$ and $F(t,s) = \omega(t)$ for $t \in \{0,1\}$ and every $s \in [0,1]$. For any points $x_0, x_1 \in X$ the relation $\omega \cong \omega'$ is an equivalence relation in the set of paths from x_0 to x_1. The resulting equivalence classes are called **path classes** and if ω is a path in X, the path class containing ω is denoted by $[\omega]$.

Theorem A.5.1 (theorem 1.7.7 and theorem 1.7.8 of [81]) *For each topological space X there is a category $\mathcal{P}(X)$ whose objects are the points of X, whose morphisms from x_1 to x_0 are the path classes with x_0 as origin and x_1 as end, and composite is the product of path classes. Moreover $\mathcal{P}(X)$ is a groupoid.*

Furthermore we say a topological space X is **path-connected** if any two point of X can be joined by a path. Let X be a topological space and $x_0 \in X$. The **fundamental group of** X **at** x_0, denoted by $\pi(X, x_0)$, is defined to be the group of path classes with x_0 as origin and end. Certainly, $\pi(X, x_0)$ is a group.

We define a topological space X to be **simply connected** if and only if X is path-connected and the fundamental group of X is trivial, i.e. consists only of the identity element.

The Seifert-Van Kampen theorem is a very important computational tool used for computing the fundamental group, and essentially relates a space to (smaller) portions of that space.

Theorem A.5.2 (Seifert-Van Kampen Theorem, see [81]) *Let $X = U \cup V$ be a connected topological space, where U, V and $U \cap V$ are nonempty and connected. Let* *furthermore* x_0 *be a point in* $U \cap V$ *and* $i_U : \pi(U, x_0) \to \pi(X, x_0)$ *as well as* $i_V : \pi(V, x_0) \to \pi(X, x_0)$ *be the induced homomorphisms by the inclusions maps. Then* $\pi(X, x_0) = \pi(U, x_0) \star_{\pi(U \cap V, x_0)} \pi(V, x_0)$, *that is, the fundamental group of* X *is the free product of the fundamental groups of* U *and* V *with amalgamated subgroup the fundamental group of* $U \cap V$.

In detail $G_1 \star_A G_2$ is the free product $G_1 \star G_2$ with relations given by $\varphi_1(a) = \varphi_2(a)$, where $\varphi_i : A \to G_i$ for $i = 1, 2$.

For example, if $X = U \cup V$ and $U \cap V$ is simply connected, then $\pi(X, x_0) = \pi(U, x_0) \star \pi(V, x_0)$. Likewise, if V is simply connected and $X = U \cup V$, then $\pi(X, x_0) = \pi(U, x_0)/N$, where N is a subgroup of $\pi(U, x_0)$ generated by the image of $\pi(U \cap V, x_0)$.

A.6 Coxeter systems

In this section we introduce Coxeter systems.

Definition A.6.1 A **Coxeter matrix** is a symmetric matrix $M = (m_{ij})_{i,j \in I}$ where I is some index set of arbitrary cardinaliy, such that for all $i, j \in I$ the entry m_{ij} is either a positive integer or ∞ and $m_{ij} = 1$ if and only if $i = j$. The **Coxeter diagram** or **Coxeter graph** of a Coxeter matrix M is the graph with vertex set I joining two different vertices i and j by an edge labelled m_{ij} whenever this number (including ∞) is at least three.

A Coxeter diagram is called **irreducible** if its underlying graph is connected. The rank of a Coxeter diagram is the cardinality of its vertex set.

Definition A.6.2 A **Coxeter system** is a pair of (W, S) consisting of a group W and a set of generators $S \subset W$, subject only to relations of the form $(ss')^{m_{ss'}} = 1$, where $m_{ss} = 1$ and $m_{ss'} = m_{s's} \geq 2$ for $s \neq s'$ in S. In case that no relation occurs for a pair s, s' of S, we set $m_{ss'} = \infty$.

Thus formally $W = F/N$, where F is a free group on the set S and N is the normal subgroup generated by all elements $(ss')^{m_{ss'}}$. Moreover we call W be a **Coxeter group**.

To specify a Coxeter system (W, S) is to specify a finite set S and the symmetric matrix $M_{(W,S)} = (m_{ss'})_{s,s' \in S}$, thus the Coxeter matrix $M_{(W,S)}$ of the Coxeter system (W, S).

A Coxeter system (W, S) is called spherical if the corresponding Coxeter group W is finite.

Here are some examples:

Example 1: In case all $m(s, s')$ are infinite, when $s \neq s'$, we call W a **universal** Coxeter group. If $|S| = 2$, then W is just the infinite dihedral group D_∞.

Example 2: Let $S = \{s_1, s_2, s_3\}$ with $m_{s_1,s_2} = 3$, $m_{s_1,s_3} = 2$, $m_{s_2,s_3} = \infty$, so the Coxeter graph is $\circ\!\!-\!\!\!-\!\!\!-\!\!\circ\!\!\overset{\infty}{-\!\!\!-\!\!\!-}\!\!\circ$. The resulting Coxeter group W turns out to be isomorphic to $\mathrm{PGL}_2(\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Z})/\{\pm 1\}$.

A.7 Buildings

Let $\Phi = \Phi_\Delta$ be a root system of type Δ over $I = \{1, \dots, n\}$ and (W, S) be a spherical Coxeter system of type Δ, so $S = \{\rho_{\alpha_i} \mid i \in I\}$ where $\Phi_{basis} : \alpha_1, \dots, \alpha_n$ is a basis of the root system Φ.

Definition A.7.1 A **building** $\mathcal{B}$ of type Δ is a pair $\mathcal{B} = (C, \delta)$ where C is a set and $\delta : C \times C \to W$ is a **distance function** satisfying the following axioms for $x, y \in C$ and $w = \delta(x, y)$:

(Bu 1) $w = 1$ if and only if $x = y$

(Bu 2) if $z \in C$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z) \in \{w, ws\}$, and if, furthermore, $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$ (or if no shortest representation of w ends with s, then in fact $\delta(x, z) = ws$

(Bu 3) if $s \in S$, there exists $z \in C$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

The group W is called the **Weyl group** of the building $\mathcal{B}$ and the building $\mathcal{B}$ is called **spherical** if its Weyl group W is finite.

Given a building $\mathcal{B} = (C, \delta)$, then we define from the building $\mathcal{B}$ a chamber system $\mathcal{C}(\mathcal{B}) = (C, (\sim_i)_{i\in I})$ where two chambers $x, y \in C$ are defined to be i-adjacent if $\delta(x, y) = s_i$ or $\delta(x, y) = 1$. The building $\mathcal{B}$ can be recovered from its chamber system $\mathcal{C}(\mathcal{B})$.

Proposition A.7.2 *Let M be a Coxeter diagram over a set I and let $\mathcal{B}$ be a building of type M. Then the chamber system $\mathcal{C}(\mathcal{B})$ is simply connected.*

Proof: This is theorem 4.3 in [76]. ■

Let G be a group and B a subgroup of G. Then the group $B \times B$ act on G by $(b,h)(g) = bgh^{-1}$ for $b,h \in B$ and $g \in G$. The orbits of $B \times B$ in G are the sets BgB. They form a partition of G and the corresponding quotient space is denoted by $B\backslash G/B$.

Assume G is generated by B and an other subgroup N such that $T = B \cap N$ is a normal subgroup in N. Let $W = N/T$ and suppose that W is generated by a subset S consisting of involutions, elements of order two. Then we have the following definition.

Definition A.7.3 A **Tits-system** or a BN-**pair** in a group G is a pair B, N of subgroups such that following axioms are satisfied:

(i) $G = \langle B, N \rangle$

(ii) $H := B \cap N$ is a normal subgroup of N and the factor group N/H is a Coxeter group with a set S of distinguished generators $s_1, \dots, s_n$

(iii) $BsBwB \subseteq BwB \cup BswB$ for $s \in S$ and $w \in W$

(iv) $sBs \neq B$ for $s \in S$.

Note that the double cosets like BwB are well-defined, if $n, n' \in N$ have the same image in W, then $nB = n'B$ and so wB is well defined. Furthermore axiom (i) and (ii) imply that $G = BNB$. Thus these axiom lead to the **Bruhat decompostion**, see [19] or [76], which says that G is the disjoint union $G = \bigcup_{w \in W} BwB$. The Bruhat decomposition allows to go from BN-pair to a building, as follows:

Proposition A.7.4 *Let (G, B, N, S) be a Tits system, $G/B = \{gB \mid g \in G\}$ and $\delta : G/B \times G/B \to W$ by $\delta(gB, hB) = w$ if and only if $g^{-1}h \in BwB$ for all $g, h \in G$. Then $(G/B, \delta)$ is a thick building of type (W, S) and G is strongly transitive on G/B.*

We want see how a Tits system can arise in the context of semi-simple Lie groups. Let K be a compact connected Lie group, G its complexification and t be a maximal torus of K. We consider the complexification $T = t_{\mathbb{C}}$ of t and regard the standard Borel subgroup B of G. Let $N = N_G(T)$ be the normaliser in G of T and S be the set of simple reflections in $W = N/T$ with respect to the basis Δ of the root system $R(L(G), L(T))$ determined by B. Then by [19] or [57] we obtain that (B, N) is a Tits system of G.

We want recall some facts for a Tits system (B, N) of group G. Proofs can be found in [57].

Lemma A.7.5 *The set S of simple reflections of the Weyl group $W = N/T$ contains precisely those $w \in W$ for which $B \cup BwB$ is a group.*

Theorem A.7.6 *If $I \subseteq S$. Then $P_I = BW_IB$ with $W_I = \langle I \rangle$ is a subgroup of G. Furthermore for $w, w' \in W$ we get $BwB = Bw'B$ if and only if $w = w'$.*

Theorem A.7.7
- *The only subgroups of G containing B are those of the form $P_I = BW_IB$ with $I \subset S$.*
- *If P_I is conjugate to P_J then $P_I = P_J$.*
- *$N_G(P_I) = P_I$*
- *If $W_I \subseteq W_J$ then $I \subseteq J$.*
- *If $P_I \subseteq P_J$ then $I \subseteq J$.*

Let (G, B, N, S) be a Tits system, where G is the complexification of a compact connected Lie group K, B a Borel group including a maximal torus T, which is the complexification of a maximal torus t of K, $N = N_G(T)$, W the Weyl group and S the set of simple reflections with respect to the basis Δ of the root system $\Phi = R(L(G), L(T)$ determined by B.

The parabolic subgroups of G containing B (not one of its conjugate) are called **standard parabolic subgroups** of G relative to B. Let $\mathfrak{g} = L(G)$ be the lie algebra of G, then the lie algebra of P_I is $\mathfrak{p}_I = \mathfrak{t}_{\mathbb{C}} \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha$, where $\mathfrak{t}$ is a Cartan subalgebra of $L(K)$ and $\mathfrak{t}_{\mathbb{C}}$ the complexification of $\mathfrak{t}$ and Λ is some subset of the root system Φ containing Φ^+.

Theorem A.7.8
- *Each parabolic subgroup of G is conjugate to one and only one subgroup P_I, where $I \subseteq \Delta$.*
- *The roots of P_I relative to T are those in Φ^+ along with those roots in Φ^- which are $\mathbb{Z}$-linear combinations of I.*

Recall, that for any integer $g \geq 2$ or for $g = \infty$, a generalized m-gon is a connected rank 2 geometry $\mathcal{G}$ over $I = \{p, l\}$ with of diameter m and grith $2m$, in which each element of type p is incident to at least two different elements of type l. Thus the incidence graph $\Gamma_{\mathcal{G}}$ of $\mathcal{G}$ is a connected bipartite graph with diameter m and grith $2m$ in which each vertex has at least two neighbors.

Proposition A.7.9 *A rank two building of type $\circ\overset{m}{\text{———}}\circ$ is a generalized m-gon, and vice versa.*

Proof: This is proposition 3.2 in [76]. ∎

Thus the shortest cycle in a rank two building of type $\circ\overset{2}{\text{———}}\circ$ is of length four and the shortest cycle in a rank two building of type $\circ\overset{3}{\text{———}}\circ$ is of length six.

A.8 Phan theory

In 1977 Kok-Wee Phan [74] gave a method for identifying a group G as a quotient of the finite unitary group $\mathrm{SU}_{n+1}(\mathbb{F}_{q^2})$ by finding a generating configuration of subgroups

$$\mathrm{SU}_3(\mathbb{F}_{q^2}) \quad \text{and} \quad \mathrm{SU}_2(\mathbb{F}_{q^2}) \times \mathrm{SU}_2(\mathbb{F}_{q^2})$$

in G. Suppose $n \geq 2$ and suppose q is a prime power. Consider $G = \mathrm{SU}_{n+1}(\mathbb{F}_{q^2})$ acting as matrices on a unitary $(n+1)$-dimensional vector space over $\mathbb{F}_{q^2}$ with respect to an orthonormal basis and let $U_i \cong \mathrm{SU}_2(\mathbb{F}_{q^2})$, $i = 1, 2, \dots, n$, be the subgroups of G, represented as matrix groups with respect to the chosen orthonormal basis, corresponding to the (2×2)-blocks along the main diagonal. Let T_i be the diagonal subgroup in U_i, which is a maximal torus of U_i of size $q + 1$. When $q \neq 2$ the following hold for $1 \leq i, j \leq n$:

(P1) if $|i - j| > 1$, then $[x, y] = 1$ for all $x \in U_i$ and $y \in U_j$;

(P2) if $|i - j| = 1$, then $\langle U_i, U_j \rangle$ is isomorphic to $\mathrm{SU}_3(\mathbb{F}_{q^2})$; moreover $[x, y] = 1$ for all $x \in T_i$ and $y \in T_j$; and

(P3) the subgroups U_i, $1 \leq i \leq n$, generate G.

Suppose G is an arbitrary group containing a system of subgroups $U_i \cong \mathrm{SU}_2(\mathbb{F}_{q^2})$, and suppose a maximal torus T_i of size $q + 1$ is chosen in each U_i. If the conditions (P1)–(P3) above hold for G, we will say that G contains a *Phan system of type A_n over $\mathbb{F}_{q^2}$*.

In [74] Kok-Wee Phan proved the following result:

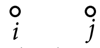
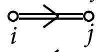
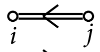
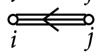
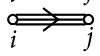
Phan's Theorem:
Let $q \geq 5$ and let $n \geq 3$. If G contains a Phan system of type A_n over $\mathbb{F}_{q^2}$, then G is isomorphic to a central quotient of $\mathrm{SU}_{n+1}(\mathbb{F}_{q^2})$.

In [75] Phan proved similar results for finite groups corresponding to all simply laced Dynkin diagrams. Phan's theorems were used for the identification of simple groups. Thus Phan's theorem are important for the revision of the classification of the finite simple groups [29], [30], [31], [32], [33], [34]. Also the question was raised whether one could generalise and unify Phan's results. The program described in [5] led to new proofs of some of Phan's old results, see [6], [41], and to new unexpected Phan-type theorems, see [36], [37]. The ideas for the new proofs can be found in the theory of flag-transitive diagram geometries and the area of amalgams of groups. In fact, Phan's theorem is a characterisation of the geometry $\mathcal{N} = \mathcal{N}(n, \mathbb{F}_{q^2})$ of all proper non-degenerate subspaces in the unitary vector space U_n over $\mathbb{F}_{q^2}$. The connection between Phan's theorems and diagram geometries was first observed by M. Aschbacher in [3].

Here we give the exact results, which are proved in [6], [36], [42] and [44].

For $n \geq 2$, let Δ be a Dynkin diagram of rank n and $I = \{1, \ldots, n\}$ be the set of labels of Δ. A group H admits a **weak Phan system of type Δ over the complex numbers**, if H is generated by subgroups U_i, $i \in I$, that are central quotients of simply connected compact semisimple Lie groups of rank one, i.e., $U_i \cong \mathrm{SU}_2(\mathbb{C})$ or $U_i \cong \mathrm{SO}_3(\mathbb{R}) \cong \mathrm{PSU}_3(\mathbb{C})$ for all $1 \leq i \leq n$, and if the groups U_i are embedded as rank one groups with respect to a fundamental system of roots of the groups $U_{ij} = \langle U_i, U_j \rangle$, which have following isomorphism types:

$\langle U_i, U_j \rangle \cong$

- $(U_i \times U_j)/Z$ in case  where Z is a central subgroup of $U_i \times U_j$
- $\mathrm{SU}_3(\mathbb{C})$ or $\mathrm{PSU}_3(\mathbb{C})$ in case 
- $\mathrm{U}_2(\mathbb{H})$ or $\mathrm{SO}_5(\mathbb{R})$ in case  or 
- $G_{2,-14}$ in case  or  .

Theorem A.8.1 (Main Theorem of [42]) *Let Δ be a Dynkin diagram and let G be a group admitting a weak Phan system of type Δ over $\mathbb{C}$. Then G is a central quotient of the simply connected compact semisimple Lie group whose complexification is the simply connected complex semisimple Lie group of type Δ. In particular, for irreducible Dynkin diagrams, the group G is a central quotient of*

- $\mathrm{SU}_{n+1}(\mathbb{C})$, *if* $\Delta = A_n$,
- $\mathrm{Spin}_{2n+1}(\mathbb{R})$, *if* $\Delta = B_n$,
- $\mathrm{U}_n(\mathbb{H})$, *if* $\Delta = C_n$,
- $\mathrm{Spin}_{2n}(\mathbb{R})$, *if* $\Delta = D_n$,
- $E_{6,-78}$, *if* $\Delta = E_6$,
- $E_{7,-133}$, *if* $\Delta = E_7$,
- $E_{8,-248}$, *if* $\Delta = E_8$,
- $F_{4,-52}$, *if* $\Delta = F_4$.

For a finite field $\mathbb{F}_{q^2}$, we have the following definition and result. We will say that subgroups U_1 and U_2 of $\mathrm{SU}_3(\mathbb{F}_{q^3})$ form a standard pair whenever each U_i is the stabilizer in $\mathrm{SU}_3(\mathbb{F}_{q^2})$ of a non-singular vector v_i (v_i is then unique up to a scalar factor) and, furthermore, v_1 and v_2 are perpendicular. By Witt's theorem, standard pairs are exactly the conjugates of the pair formed by the two subgroups $\mathrm{SU}_2(\mathbb{F}_{q^2})$ arising from the 2×2 blocks on the main diagonal. Standard pairs in $\mathrm{PSU}_3(\mathbb{F}_{q^2})$ will be defined as the images under the natural homomorphism of the standard pairs from $\mathrm{SU}_3(\mathbb{F}_{q^2})$. For $n \geq 2$, let Δ be a Dynkin diagram of rank n and $I = \{1, \dots, n\}$ be the set of labels of Δ. A group H admits a weak **Phan system of type Δ over the** $\mathbb{F}_{q^2}$, if H is generated by subgroups U_i, $i \in I$, such that $U_i \cong \mathrm{SU}_2(\mathbb{F}_{q^2})$ for all $1 \leq i \leq n$, and if the groups U_i are embedded as rank one groups with respect to a fundamental system of roots of the groups $U_{ij} = \langle U_i, U_j \rangle$, which have following isomorphism types:

$$\langle U_i, U_j \rangle \cong \begin{cases} (U_i \times U_j)/Z & \text{in case } \underset{i}{\circ} \quad \underset{j}{\circ} \quad \text{where} \\ & Z \text{ is a central subgroup of } U_i \times U_j \\ \mathrm{SU}_3(\mathbb{F}_{q^2}) \text{ or } \mathrm{PSU}_3(\mathbb{F}_{q^2}) & \text{in case } \underset{i}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{j}{\circ} \\ \mathrm{Sp}_4(\mathbb{F}_q)/Z & \text{in case } \underset{i}{\circ}\!\Rightarrow\!\underset{j}{\circ} \quad \text{or} \quad \underset{i}{\circ}\!\Leftarrow\!\underset{j}{\circ} \end{cases} .$$

Theorem A.8.2 (theorem of [44]) *Let $n \geq 3$, q be some prime power, Δ be a spherical irreducible Dynkin diagram of rank at least three and G be a group admitting a Phan system of type Δ over $\mathbb{F}_{q^2}$. Then G is a central quotient of*

- $\mathrm{SU}_{n+1}(\mathbb{F}_{q^2})$, *if* $\Delta = A_n$, $q \geq 4$,
- $\mathrm{Spin}_{2n+1}(\mathbb{F}_q)$, *if* $\Delta = B_n$, $q \geq 4$,
- $\mathrm{Sp}_{2n}(\mathbb{F}_q)$, *if* $\Delta = C_n$, $q \geq 3$,
- $\mathrm{Spin}^{\pm}_{2n}(\mathbb{F}_q)$, *if* $\Delta = D_n$, $q \geq 4$,
- ${}^2E_6(\mathbb{F}_{q^2})$, *if* $\Delta = E_6$, $q \geq 4$,
- $E_7(\mathbb{F}_q)$, *if* $\Delta = E_7$, $q \geq 4$,
- $E_8(\mathbb{F}_q)$, *if* $\Delta = E_8$, $q \geq 4$,
- $F_4(\mathbb{F}_q)$, *if* $\Delta = F_4$, $q \geq 11$.

A.9 Root systems of type A_n and E_6

In this part we will explicitly describe the root system of type A_n and of E_6. A good reference is [10] or [55].

root system of type A_n for $n \geq 1$

- V is the hyperplane of $\mathbb{R}^{n+1}$ equipped with the standard scalar product $(\cdot, \cdot)$, consisting of the vectors whose coordinates add up to 0 with respect to the standard basis $\alpha : e_1, \dots, e_{n+1}$.

- The root system $\Phi_{A_n} = \Phi$ is the set of all vectors of length $\sqrt{2}$ in the intersection of $V \cap \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{n+1}$. So Φ consists of the $n \cdot (n+1)$ different vectors
$$e_i - e_j \text{ for } 1 \leq i, j \leq n+1,\ i \neq j.$$

- A basis Δ for Φ is
$$\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_n = e_n - e_{n+1}.$$

  Then the root of maximal height with respect to Δ is
$$\tilde{\alpha} = e_1 - e_{n+1} = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

root system of type E_6

- We start with vector space $\mathbb{R}^8$ equipped with the standard scalar product $(\cdot,\cdot)$. We consider the set $L = \{\sum_{i=1}^n \lambda_i e_i \mid \lambda_i \in \mathbb{Z} \text{ and } \sum_{i=1}^8 \lambda_i \text{ is even}\}$. We take all vectors $v \in L + \mathbb{Z}(\frac{1}{2}\sum_{i=1}^8 e_i)$ such that $(v,v) = 2$. The six vectors
$$\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8),$$
$$\alpha_2 = e_1 + e_2,$$
$$\alpha_i = e_{i-1} - e_{i-2} \text{ for } 3 \leq i \leq 6$$
  are elements of $L + \mathbb{Z}(\frac{1}{2}\sum_{i=1}^8 e_i)$. We define $V = \langle \alpha_1, \ldots, \alpha_6 \rangle$.

- The root system $\Phi_{E_6} = \Phi$ is the set of all 72 vectors of length $\sqrt{2}$ lying in V. So Φ consists of the different vectors:
$$\pm e_i - \pm e_j \text{ for } 1 \leq i, j \leq 5,\ i \neq j,$$
$$\pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^5 \pm e_i),$$
  where the number of minus signs in the sum is odd.

- A basis Δ for Φ
$$\alpha_1, \ldots, \alpha_6.$$

  Then the root of maximal height with respect to Δ is
$$\tilde{\alpha} = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6.$$

APPENDIX TWO

Enumeration in finite unitary spaces

In this appendix for convenience of the reader we collect a number of known results that will be used extensively throughout the thesis. Let U be a finite dimensional vector space over the finite field $\mathbb{F}_{q^2}$. The finite field $\mathbb{F}_{q^2}$ has an automorphism of order two $\sigma : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ with $a \mapsto \overline{a} = \sigma(a) = a^q$. By $\mathbb{F}_0 = \{a \in \mathbb{F}_{q^2} \mid a = \overline{a}\}$ we denote the fixed field of order q of $\mathbb{F}_{q^2}$ under the automorphism σ. We start with the following lemma which is proved in [86] or [98].

Lemma B.1.1 *For any non-zero scalar λ of $\mathbb{F}_0$ the equation $x \cdot \overline{x} = \lambda$ has exactly $q+1$ solutions in $\mathbb{F}_{q^2}^\times$ and the equation $x + \overline{x} = \mu$ has precisely q solutions in $\mathbb{F}_{q^2}$ for any $\mu \in \mathbb{F}_0$.*

Notation: With $\mathbb{F}^{\sigma,1}$ we will denote the set of scalars solving the equation $\lambda + \overline{\lambda} = 0$, so $\mathbb{F}^{\sigma,1} = \{\lambda \in \mathbb{F}_{q^2} \mid \lambda + \overline{\lambda} = 0\}$ and $\mathbb{F}^{\sigma,1\times} = \mathbb{F}^{\sigma,1} \backslash \{0\}$.

Next we fix a non-degenerate sesquilinear form $(\cdot,\cdot)$ on the n-dimensional vector space U. The **Gram matrix** $G_\alpha = ((v_i, v_j))_{1\leq i,j\leq n}$ has full rank with respect to any basis $\alpha : v_1, \ldots, v_n$ of U. A vector v of U is said to be **isotropic** (degenerate, singular) resp. **non-isotropic** (non-degenerate, regular) if $(v, v) = 0$ resp. $(v, v) \neq 0$. If the dimension of U is at least two then the unitary vector space U contains isotropic and non-isotropic vectors, which is proved in [86] or [98]. In the first part of this section we want to classify all subspaces up to dimension six in an n-dimensional unitary vector space U_n and simultaneously we will also fix some notation.

Let W be an m-dimensional vector subspace of U_n such that $m \leq n$ then W has **rank** $p \leq m$, in symbols $\mathrm{rk}(W) = p$ if the rank of the Gram matrix G_β^W with respect to some basis β of W is p. We call the subspace W **non-degenerate** (**regular**) if $\mathrm{rk}(W) = \dim(W)$. Otherwise if $\mathrm{rk}(W) < \dim(W)$ then W is a **degenerate** (**singular**) subspace of U. We say W is a **totally singular subspace** of the vector space U_n if $\mathrm{rk}(W) = 0$. Furthermore a regular subspace of dimension two is also called a **hyperbolic line**.

An n-dimensional vector space U_n over $\mathbb{F}_{q^2}$ is isomorphic to $\mathbb{F}_{q^2}^n$, so U_n contains $(q^2)^n$ vectors. Consequently the number of k-dimensional subspaces, $k \leq n$, in U_n is described by the **Gaussian coefficent**

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = \frac{\prod_{i=n-k+1}^{n}(q^{2i} - 1)}{\prod_{i=1}^{k}(q^{2i} - 1)}. \tag{B.1}$$

Next let W be some k-dimensional subspace of U_n then exactly

$$\begin{bmatrix} n-k \\ m-k \end{bmatrix}_{q^2} \tag{B.2}$$

different m-dimensional subspaces of the n-dimensional vector space U_n contains the subspace W.

For quick reference we refer to tables B.1 and B.2 on page 262.

Our next goal is to count and to arrange all k-dimensional subspaces of U_n with respect to their order of singularity. Since U_n is a non-degenerate unitary vector space for each subset M of U_n we denote the orthogonal subspace of M by

$$M^\pi = \{u \in U_n \mid (u, m) = 0 \text{ for all } m \in M\}.$$

The radical of a subspace W of U_n, denoted by $\mathrm{rad}(W)$, contains all elements w of W such that $(w, v) = 0$ for all elements $v \in W$. Certainly in a non-degenerate unitary vector space U a subspace W is totally singular if and only if $W \subseteq W^\pi$ and W is non-degenerate if and only if $W \cap W^\pi = \{0\}$. Moreover the radical of a degenerate subspace W of U is not trivial and we get the identity $\mathrm{rad}(W) = W \cap W^\pi$ for every subspace W of a non-degenerate unitary vector space U. Two different subspaces W and V of the unitary vector space U are orthogonal to each other if and only if $W \subseteq V^\pi$ and equivalent $V \subseteq W^\pi$.

From lemma 5.19 of [98] we get that an n-dimensional non-degenerate unitary vector space U_n contains

$$q^{r(n+r-2m)} \frac{\prod_{i=n+r-2m+1}^{n}(q^i - (-1)^i)}{\prod_{i=1}^{r}(q^i - (-1)^i) \prod_{i=1}^{m-r}(q^{2i} - 1)} \tag{B.3}$$

different subspaces of dimension m and rank r for $2r \leq 2m \leq n+r$. Furthermore in a $(n+l)$-dimensional unitary vector space U_{n+l} of rank n are

$$\sum_{k=\max\{0,\frac{2m-n-r+1}{2}\}}^{\min\{l,m-r\}} q^{r(n+r-2m+2k)+2(m-k)(l-k)} \cdot \frac{\prod_{i=n+r-2m+2k+1}^{n}(q^i-(-1)^i)\prod_{i=l-k+1}^{l}(q^{2i}-1)}{\prod_{i=1}^{r}(q^i-(-1)^i)\prod_{i=1}^{m-r-k}(q^{2i}-1)\prod_{i=1}^{k}(q^{2i}-1)} \tag{B.4}$$

different m-dimensional subspaces of rank r for $\max\{0,\frac{2m-n-r}{2}\} \leq \min l, m-r$. Again for quick reference regarding these formulas, we list the possibilities for all m-dimensional subspace with the rank of a n-dimensional non-degenerate unitary vector space U_n, $m \leq n$, $1 \leq n \leq 6$ as well as of a $n+l$-dimensional rank n unitary vector space for $1 \leq n+l \leq 6$ in table B.3 on page 264.

Before we start to determine the number of certain subspaces in some unitary vector space, we make a simple observation.

Lemma B.1.2 *Let W be an m-dimensional subspace of rank d of the unitary vector space U_n. An $(m-d)$-dimensional subspace H of the subspace W is regular if and only if $H \cap \mathrm{rad}(W) = \{0\}$.*

Proof: Since one implication is certainly true we fix an $(m-d)$-dimensional subspace H with $H \cap \mathrm{rad}(W) = \{0\}$. Suppose $\dim(\mathrm{rad}(H)) \geq 1$ then $\mathrm{rad}(W) \subsetneq \langle \mathrm{rad}(H), \mathrm{rad}(W)\rangle = \mathrm{rad}(\langle H, \mathrm{rad}(W)\rangle) = \mathrm{rad}(W)$, contradiction. ∎

Next let W be an m-dimensional vector space of rank r. It is natural to ask for the number of all k-dimensional subspaces of rank l of the vector space U_n containing the given subspace W. We determine this number for some special cases, which we need later.

Observation B.1.3 Let P be a plane of rank one, then any two-dimensional subspace different from the radical of P has rank one.

Next we consider a plane P of rank two.

Lemma B.1.4 *Let P be a singular plane of rank two, so the radical of P is a one-dimensional subspace x, a point. Every line not through the point x of P is a hyperbolic line. Also every point y different from x is incident to q^2 different hyperbolic lines.*

Proof: Let l be a two-dimensional subspace of P not incident to the point x, so $x \notin l$. Suppose l is a singular subspace of P then we choose z to be a point in $\mathrm{rad}(l)$. The three-dimensional space $\langle l, x\rangle$ is the plane P. However the radical of $\langle l, x\rangle$ contains the two-dimensional space $\langle z, x\rangle$, contradiction. In fact l is a hyperbolic line.

For the second claim let y be a point of P distinct from x and denote the set of lines containing the point y with L. Suppose all $q^2 + 1$ elements of L are singular lines then any element of L contains the radical x. Thus any line $l \in L$ contains the point y and the point x implying that $l = \langle y, x \rangle$, contradiction. Hence the point y is incident to exactly one singular line $s = \langle y, x \rangle$ and all other lines different from s containing y are hyperbolic lines, which proves the statement. ■

Now we look at the non-degenerate planes P of the unitary polar space U. Recall that a non-degenerate plane P of U only contains lines of rank two and rank one by the formula B.3 of page 255 and table B.3.

Lemma B.1.5 *In a non-degenerate plane, any regular point is incident to $q^2 - q$ different hyperbolic lines and to $q+1$ different two-dimensional rank one subspaces. Any singular point of a regular plane is contained in q^2 hyperbolic lines and in one singular line of this regular plane.*

Proof: Let s be a singular point in P. The orthogonal space s^π of s in P is a two-dimensional singular subspace of P containing the point s itself as radical, which show that the points s lies at least on one singular line in P. Suppose we can find two different singular lines l and m through the singular point s in P. We denote the radical of the line l by x_l resp. the radical of the line m by x_m. A line of rank one contains exactly one singular point by the formula B.4 of page 255 and table B.3 and we obtain that $x_l = x_m = s$. The space H spanned by the two different lines l and m in the plane P has dimension three, thus $H = P$. However the space H has the radical $\mathrm{rad}(H) = s$, contradiction. In fact a singular point of a regular plane is incident to one singular line and to q^2 hyperbolic lines.

Now we consider the case that z is a regular point of P. The orthogonal space z^π of the point z is a hyperbolic line in P, which contains $q + 1$ singular points and $q^2 - q$ regular points by the formula B.3 of page 255 and table B.3. We pick one of these points, call this point p, and obtain the two-dimensional subspace $h = \langle z, p \rangle$ in P. The Gram matrix of the line h with respect to the basis $\beta : z_v, p_v$ where z_v is a vector of z as well as p_v is a vector of the point p, is of the form $\begin{pmatrix} (z_v, z_v) & 0 \\ 0 & (p_v, p_v) \end{pmatrix}$ implying that h is a hyperbolic line if p is a regular point and h has rank one if p is a singular point of z^π. By construction the intersection of line h and z^π is the point p, thus each regular point z of the regular plane P is incident to $q^2 - q$ different hyperbolic lines and to $q + 1$ different two-dimensional rank one subspaces. ■

The last statement implies that any two different singular points of a non-degenerate three-dimensional space P lie on different singular lines. Indeed let z and s be two singular points in P and suppose there is a singular line l containing both points, z and s. It follows that $\mathrm{rad}(l) = s = z$, contradiction.

With the same methods we determine the number of all k-dimensional subspaces of rank l of a four-dimensional spaces H containing a fixed point, line or plane of H in the unitary vector space U over $\mathbb{F}_{q^2}$.

Lemma B.1.6 *Let H be a space of rank four. Then every hyperbolic line h of H is incident to $q^2 - q$ non-degenerate planes and to $q+1$ planes of rank one. Furthermore a regular point is incident to $q^4 - q^3 + q^2$ hyperbolic lines and a singular point of H is on q^4 different hyperbolic lines.*

Proof: To prove this statement, let h be a hyperbolic line and h^π its two-dimensional non-degenerate orthogonal space in H. Due to the formula B.3 of page 255 and table B.3 the hyperbolic line h^π has $q^2 - q$ regular points p_i, $1 \leq i \leq q^2 - q$, and $q+1$ singular points s_j, $1 \leq i \leq q+1$. The planes $\langle h, p_i \rangle$ are non-degenerate since $\operatorname{rad}(\langle h, p_i \rangle) \subseteq h^\pi \cap p_i^\pi = \{0\}$ and the spaces $\langle h, s_j \rangle$ are planes of rank one using that the Gram matrix of the subspace $\langle h, s_j \rangle$ has rank two.

Next let p be some point of H. Suppose p is regular then the polar space p^π of p is a non-degenerate plane, which contains $q^4 - q^3 + q^2$ regular points r_i and $q^3 + 1$ singular points s_j by the formula B.4 on page 255. Since $\langle p, r_i \rangle$ is a hyperbolic line in H for each point r_i and $\langle p, s_j \rangle$ is a rank one line in H for any point s_j, the statement follows. On the other hand if p is a singular point, then of course p^π is a rank two plane with radical p. Every line running through the point p in the plane p^π is singular, thus p is incident to at least $q^2 + 1$ singular lines in H. Let l be some two-dimensional subspace in H with $p \in l$ and $l \notin p^\pi$. Hence $l = \langle p, r \rangle$ for some point $r \in l \backslash p^\pi$ and thus the points p and r are not orthogonal to each other implying that l is a hyperbolic line in H. Counting the number of lines through p which are not contained in the plane p^π, we conclude that p is incident to q^4 different hyperbolic lines. ■

Lemma B.1.7 *Let H be a space of rank three. Then every hyperbolic line l of H lies on q^2 non-degenerate planes and on one plane of rank one. Every non-radical point p of H is incident either to q^4 hyperbolic and to $q^2 + 1$ singular lines, if p is singular, or to $q^4 - q^3$ hyperbolic and to $q^3 + q^2 + 1$ singular lines under the condition that p is regular.*

Proof: Let l be a hyperbolic line in H. The orthogonal space l^π of the line l is a line of rank one containing the radical of H, since the space H has rank three. Clearly l is contained in exactly one singular plane $P = \langle l, \operatorname{rad}(H) \rangle$ and in q^2 non-degenerated planes $\langle l, y \rangle$ where y is a regular point of l^π.

For the second statement let p be a non-radical point of the 3-space H. If p is a regular point then p^π is a rank two plane, which contains $q^4 - q^3$ different regular points and $q^3 + q^2 + 1$ distinct singular point. In fact the regular point p is incident to $q^4 - q^3$ different hyperbolic lines and to $q^3 + q^2 + 1$ distinct singular lines in H.

Alternatively if p is a non-radical singular point then p^π is rank one plane and thus every line in p^π through the point p is singular. On the other hand every line incident to the point p in H, which is not contained in the subspace p^π is a hyperbolic line. Therefore the singular point p lies on q^4 hyperbolic lines and on $q^2 + 1$ singular lines in H. ∎

Observation B.1.8 In a four-dimensional rank two space H every hyperbolic line h of H is incident to $q^2 + 1$ planes of rank one.

Finally we consider a five-dimensional non-degenerate unitary space W and determine the number of regular planes as well as singular planes of W, which contains a fixed hyperbolic line h of W.

Lemma B.1.9 *Let W be a non-degenerate five-dimensional space. Then every hyperbolic line h of W lies on $q^4 - q^3 + q^2$ non-degenerate planes and is incident to $q^3 + 1$ different singular planes.*

Proof: Let h be a hyperbolic line in W and h^π its three-dimensional non-degenerate orthogonal space, a regular plane in W. Using the formula B.3 from page 255 and the tabular B.3 we obtain that h^π contains $q^4-q^3+q^2$ regular points p_i, $1 \le i \le q^4-q^3+q^2$ and $q^3 + 1$ singular points s_j, $1 \le i \le q + 1$. The planes $\langle h, p_i \rangle$ are regular and the planes $\langle h, s_j \rangle$ are of rank one. ∎

Later we will need an easy formula to determine the number of hyperbolic lines in an n-dimensional non-degenerate unitary vector space U_n. We find this formula by using the formula B.3 on page 255 for $m = r = 2$ and some $n \ge 2$.

Observation B.1.10 The number of different hyperbolic lines in an n-dimensional regular unitary space U_n is

$$\frac{q^{2m-4} \cdot (q^m - 1)(q^{m-1} + 1)}{(q+1)(q^2-1)}$$

if n is odd and

$$\frac{q^{2m-4} \cdot (q^m + 1)(q^{m-1} - 1)}{(q+1)(q^2-1)}$$

if n is even.

From table B.3 we get that a regular plane has $q^4-q^3+q^2$ hyperbolic lines. A singular plane of rank two contains q^4 hyperbolic lines, while in a plane of rank less than two are no hyperbolic lines. Thus by counting the number of hyperbolic lines in a given plane we can determine if the plane is regular or not. Moreover in a regular plane we can also determine if a given point is singular or not by counting the number of hyperbolic lines through this given point by lemma B.1.5. The just described method of classifying a given point p of a unitary vector space U_n is only usable if every point p of U_n is incident to both some regular plane and some singular plane of rank two.

Lemma B.1.11 *Let p be a point in the n-dimensional non-degenerate unitary vector space U_n for $n \geq 4$. Then the point p is incident to a non-degenerate plane P of U_n and also p lies in some rank two plane S with a radical $y \neq p$.*

Proof: Let p be some point of the unitary vector space U_n. We consider the two different cases that either p is a singular point or that p is a regular point of U_n.

Suppose p is a singular point of U_n then by the regularity of the space U_n we choose a point z in U_n such that $(p,z) \neq 0$, which implies that $\langle p,z\rangle$ is a hyperbolic line h of U_n. Moreover the orthogonal space h^π of h is a $(n-2)$-dimensional regular subspace containing a singular point y and a regular point w. It follows that $\langle h,w\rangle = \langle p,z,w\rangle$ is a non-degenerate plane and $\langle h,y\rangle = \langle p,z,w\rangle$ is a rank two plane with the radical point y of U_n.

Alternatively if p is a regular point in U_n, then p^π is $(n-1)$-dimensional regular space containing a hyperbolic line l and a rank one line h by the fact that $n-1 \geq 3$. Certainly $\langle p,l\rangle$ is a non-degenerate plane of U_n and $\langle p,h\rangle$ is a plane of rank two with a radical y different from p. ■

In the last part of this section we collect some properties of points on a hyperbolic lines l and the vectors which generate a given point on l. For this purpose we recall the definition of the fixed field $\mathbb{F}_0 = \{x \in \mathbb{F}_{q^2} \mid \bar{x} = x\}$ of $\mathbb{F}_{q^2}$ and the definition of $\mathbb{F}^{\sigma,1} = \{\lambda \in \mathbb{F}_{q^2} \mid \lambda + \overline{\lambda} = 0\}$.

Lemma B.1.12 *Let $l = \langle a,b\rangle$ be a hyperbolic line spanned by two different singular points of the unitary vector space U_n and let a_v a non-zero vector of a,so $\langle a_v\rangle = a$ and b_v be a non zero vector of b, so $\langle b_v\rangle = b$. Then every singular point s of l is spanned by a vector $s_v = a_v + \mu(a_v,b_b)b_v$ with $\mu \in \mathbb{F}^{\sigma,1}$.*

Proof: Let $\mu \in \mathbb{F}^{\sigma,1}$ then the point $\langle a_v + \mu(a_v,b_v)b_v\rangle = p_{a,b,\mu}$ spanned by the vector $a_v + \mu(a_v,b_v)b_v$ is clearly incident to the line $\langle a,b\rangle$. Because

$$\begin{aligned}
& (a_v + \mu(a_v,b_v)b_v, a_v + \mu(a_v,b_v)b_v) \\
= \; & (a_v,a_v) + \mu(a_v,b_v)(b_v,a_v) + \overline{\mu}\overline{(a_v,b_v)(a_v,b_v)} + \mu(a_v,b_v)\overline{\mu}\overline{(a_v,b_v)}(b_v,b_v) \\
= \; & \mu(a_v,b_v)(b_v,a_v) + \overline{\mu}\overline{(a_v,b_v)}(a_v,b_v) \\
= \; & (a_v,b_v)(b_v,a_v)(\mu + \overline{\mu}) \\
= \; & 0
\end{aligned}$$

we see that $p_{a,b,\mu}$ is a singular point for every $\mu \in \mathbb{F}^{\sigma,1}$.

Conversely let s be a singular point of l and s_v be a non-zero vector of s then $s = \langle s_v \rangle$ and $(s_v, s_v) = 0$. Moreover $s_v = a_v + \delta(a_v, b_v)b_v$ for some $\delta \in \mathbb{F}_{q^2}$ and we get that

$$\begin{aligned} 0 = \ & (s_v, s_v) = (a_v + \delta(a_v, b_v)b_v, a_v + \delta(a_v, b_v)b_v) \\ = \ & (a_v, a_v) + \delta(a_v, b_v)(b_v, a_v) + \overline{\delta(a_v, b_v)}(a_v, b_v) + \delta(a_v, b_v)\overline{\delta(a_v, b_v)}(b_v, b_v) \\ = \ & \delta(a_v, b_v)(b_v, a_v) + \overline{\delta(a_v, b_v)}(a_v, b_v) \\ = \ & (a_v, b_v)(b_v, a_v)(\delta + \overline{\delta}), \end{aligned}$$

which implies that $\delta \in \mathbb{F}^{\sigma,1}$ as $(a_v, b_v) \neq 0$. ■

Lemma B.1.13 *Let $l = \langle a, b \rangle$ be a hyperbolic line of the unitary vector space U_n spanned by two different singular points a and b. Furthermore let a_v an non-zero vector of a, so $\langle a_v \rangle = a$, and b_v be an non-zero vector of b, so$\langle b_v \rangle = b$, such that $(a_v, b_v) = 1$.*

For three non-identity scalars $r, s, t \in \mathbb{F}^{\sigma,1}$ either the vector $a_v - tb_v - rta_v$ spans the point b or there are scalars $u, v, w \in \mathbb{F}^{\sigma,1\times}$ such that

$$\begin{aligned} & x - t(x, b_v)b_v - r(x, a_v)a_v + rt(x, b_v)a_v - w(x, b_v)b_v + rw(x, a_v)b_v - wrt(x, b_v)b_v \\ = \ & x - u(x, a_v)a_v - s(x, b_v)b_v + su(x, a_v)b_v - v(x, a_v)a_v + vs(x, b_v)a_v - vsu(x, a_v)a_v \end{aligned}$$

for all vectors $x \in U_n$.

Proof: Suppose $a_v - tb_v + rta_v$ spans the point b then $a_v - tb_v + rta_v = \mu b_v$ for some $\mu \in \mathbb{F}_{q^2}$, which implies that $(1 + rt)a_v - (t - \mu)b_v = 0$ and $1 + rt = 0 = t - \mu$ using that a_v and b_v are two linearly independent vectors, in particular $rt = -1$.

In the other case we start with the fact that $rt \neq -1$ and choose $u = \frac{r(s-t)}{s(1+tr)}$, $v = \frac{tr}{s}$ and $w = \frac{s-t}{1+tr}$. Then

$$\begin{aligned} & x - t(x, b_v)b_v - r(x, a_v)a_v + rt(x, b_v)a_v - w(x, b_v)b_v + rw(x, a_v)b_v - wrt(x, b_v)b_v \\ = \ & x - t(x, b_v)b_v - r(x, a_v)a_v + rt(x, b_v)a_v - \tfrac{s-t}{1+tr}(x, b_v)b_v + r\tfrac{s-t}{1+tr}(x, a_v)b_v - \\ & \tfrac{s-t}{1+tr}rt(x, b_v)b_v \\ = \ & x - r(x, a_v)a_v + rt(x, b_v)a_v + \tfrac{r(s-t)}{1+tr}(x, a_v)b_v + (x, b_v)b_v\left(-t - \tfrac{(s-t)}{(1+tr)}(1 + tr)\right) \\ = \ & x - r(x, a_v)a_v + rt(x, b_v)a_v + \tfrac{r(s-t)}{1+tr}(x, a_v)b_v - s(x, b_v)b_v \end{aligned}$$

for all $x \in U_n$.

Furthermore

$$
\begin{aligned}
& x-u(x,a_v)a_v-s(x,b_v)b_v+su(x,a_v)b_v-v(x,a_v)a_v+vs(x,b_v)a_v-vsu(x,a_v)a_v \\
&= x-\tfrac{r(s-t)}{s(1+tr)}(x,a_v)a_v-s(x,b_v)b_v+s\tfrac{r(s-t)}{s(1+tr)}(x,a_v)b_v-\tfrac{tr}{s}(x,a_v)a_v+\tfrac{tr}{s}s(x,b_v)a_v- \\
&\quad \tfrac{tr}{s}s\tfrac{r(s-t)}{s(1+tr)}(x,a_v)a_v \\
&= x-s(x,b_v)b_v+\tfrac{r(s-t)}{s(1+tr)}(x,a_v)b_v+tr(x,b_v)a_v+(x,a_v)a_v\left(\tfrac{-r(s-t)}{s(1+tr)}(1+tr)-\tfrac{tr}{s}\right) \\
&= x-s(x,b_v)b_v+\tfrac{r(s-t)}{s(1+tr)}(x,a_v)b_v+tr(x,b_v)a_v-r(x,a_v)a_v
\end{aligned}
$$

for all vectors x of the unitary vector space U_n. ■

	number of k-dimensional subspace in U_n					
n	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
1	1					
2	$q^2 + 1$	1				
3	$q^4 + q^2 + 1$	$q^4 + q^2 + 1$	1			
4	$q^6 + q^4 + q^2 + 1$	$q^8 + q^6 + 2q^4 + q^2 + 1$	$q^6 + q^4 + q^2 + 1$	1		
5	$q^8 + q^6 + q^4 + q^2 + 1$	$q^{12} + q^{10} + 2q^8 + 2q^6 + 2q^4 + q^2 + 1$	$q^{12} + q^{10} + 2q^8 + 2q^6 + 2q^4 + q^2 + 1$	$q^8 + q^6 + 2q^4 + q^2 + 1$	1	
6	$q^{10}+q^8+q^6+q^4+q^2+1$	$q^{16}+q^{14}+2q^{12}+2q^{10}+3q^8+2q^6+2q^4+q^2+1$	$q^{18}+q^{16}+2q^{14}+3q^{12}+3q^{10}+3q^8+3q^6+2q^4+q^2+1$	$q^{16}+q^{14}+2q^{12}+2q^{10}+3q^8+2q^6+2q^4+q^2+1$	$q^{10}+q^8+q^6+q^4+q^2+1$	1

Table B.1: number of k-dimensional subspace in U_n

		number of m-dimensional subspaces of U_n containing W	
n	$\dim(W) = k$	$m = 2$	$m = 3$
3	1	$q^2 + 1$	
4	1	$q^4 + q^2 + 1$	$q^4 + q^2 + 1$
4	2	1	$q^2 + 1$

		number of m-dimensional subspaces of U_n containing W			
n	$\dim(W) = k$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
5	1	$q^6 + q^4 + q^2 + 1$	$q^8 + q^6 + 2q^4 + q^2 + 1$	$q^6 + q^4 + q^2 + 1$	
5	2	1	$q^4 + q^2 + 1$	$q^4 + q^2 + 1$	
5	3		1	$q^2 + 1$	
6	1	$q^8+q^6+q^4+q^2+1$	$q^{12} + q^{10} + 2q^8 + 2q^6 + 2q^4 + q^2 + 1$	$q^{12} + q^{10} + 2q^8 + 2q^6 + 2q^4 + q^2 + 1$	$q^8+q^6+q^4+q^2+1$
6	2	1	$q^6 + q^4 + q^2 + 1$	$q^8 + q^6 + 2q^4 + q^2 + 1$	$q^6 + q^4 + q^2 + 1$
6	3		1	$q^4 + q^2 + 1$	$q^4 + q^2 + 1$
6	4			1	$q^2 + 1$

Table B.2: number of m-dimensional subspaces of U_n containing W

		number of m-dimensional subspace of rank r in U_{n+l}				
		one-dimensional subspace		two-dimensional subspace		
$n+l$	n	$r=1$	$r=0$	$r=2$	$r=1$	$r=0$
2	2	q^2-q	$q+1$			
	1	q^2	1			
	0		q^2+1			
3	3	$q^4-q^3+q^2$	q^3+1	$q^4-q^3+q^2$	q^3-1	
	2	q^4-q^3	q^3+q^2+1	q^4	q^2-1	$q+1$
	1	q^4	q^2+1		q^4+q^2	1
	0	q^4+q^2+1				q^4+q^2+1
4	4	$q^6-q^5+a^4-q^3$	$q^5+q^3+q^2+1$	$q^8-q^7+2q^6-q^5+q^4$	$q^7-q^6+q^5-q^3+q^2-q$	q^4+q^3+q+1
	3	$q^6-q^5+q^4$	q^5+q^2+1	$q^8-q^7+q^6$	$q^7+2q^4-q^3+q^2$	q^3+1
	2	q^6-q^5	$q^5+q^4+q^2+1$	q^8	$q^6-q^5+q^4-q^3$	$q^5+q^4+q^3+q^2+1$
5	5	$q^8-q^7+q^6-q^5+q^4$	$q^7+q^5+q^2+1$	$q^{12}-q^{11}+2(q^{10}-q^9+q^8)-q^7+q^6$	$q^{11}-q^{10}+2q^9-q^8)+q^7+q^6-q^5+2q^4-q^3+q^2$	$q^8+q^5+q^3+1$
	4	$q^8-q^7+q^6-q^5$	$q^7+q^5+q^4+q^2+1$	$q^{12}-q^{11}+2q^{10}-q^9+q^8$	$q^{11}-q^{10}+q^9-q^7+2q^6-2q^5+q^4-q^3$	$q^8+q^7+2q^5++q^4+q^3+q^2+1$
6	6	$q^{10}-q^9+q^8-q^7+q^6-q^5$	$q^9+q^7+q^5+q^4+q^2+1$	$q^{16}-q^{15}+2q^{14}-2q^{13}+3q^{12}-2q^{11}+2q^{10}-q^9+q^8$	$q^{15}-q^{14}+2q^{13}-2q^{12}+2q^{11}-q^{10}+q^8-2q^7+2q^6-2q^5+q^4-q^3$	$q^{12}+q^{10}+q^9+q^8+2q^7+2q^5+q^4+q^3+q^2+1$

		number of m-dimensional subspace of rank r in U_{n+l}						
		three-dimensional subspace				four-dimensional subspace		
$n+l$	n	$r=3$	$r=2$	$r=1$	$r=0$	$r=4$	$r=3$	$r=2$
4	4	$q^6-q^5+q^4-q^3$	$q^5+q^3+q^2+1$					
	3	q^6	$q^4-q^3+q^2$	q^3+1				
	2		q^6+q^4	q^2-q	$q+1$			
5	5	$q^{12}-q^{11}+2q^{10}-2q^9+2q^8-q^7+q^6$	$q^{11}-q^{10}+2q^9-q^8)+q^7+q^6-q^5+2q^4-q^3+q^2$	$q^8+q^5+q^3+1$		$q^8-q^7+q^6-q^5+q^4$	$q^7+q^5+q^2+1$	
	4	$q^{12}-q^{11}+2q^{10}-q^9$	$q^{11}+q^9+2q^8-q^7+3q^6-q^5+q^4$	$q^7-q^6+q^5-q^3+q^2-q$	q^4+q^3+q+1	q^8	$q^6-q^5+q^4-q^3$	$q^5+q^3+q^2+1$

Table B.3: number of m-dimensional subspace of rank r in U_{n+l}

APPENDIX THREE

Graph isomorphisms

As mentioned at the end of section 4.4 we state explicitly for each vertex $\mathbf{z}_{mn}^{ij}$ with indices $i \in \{1,2\}$, $j,m,n \in \{1,\ldots,6\}, |\{j,m,n\}| = 3$ of proposition 4.4.18 the desired isomorphism $\gamma_{\mathbf{z}_{mn}^{ij}}$ between $\Sigma_{\mathbf{z}_{mn}^{ij}}$ and $\Sigma_{\mathbf{x}}$ given in proposition 4.4.28.

We consider case 2 of section 4.4, thus $\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i4}$ for $i \in \{1,2\}$. Let Σ be the induced subgraph of Γ on the 32 vertices

$$\mathcal{V}(\Sigma) = \{\mathbf{x}, \mathbf{y}^{ij}, \mathbf{z}_{cd}^{kl} \mid 1 \le i < j \le 6,\, k \in \{1,2\}, 1 \le c < d \le 6,\, l \in \{1,\ldots,6\} \backslash \{c,d\}\}$$

with the relation list:

$$\mathbf{z}_{12}^{i3} = \mathbf{z}_{56}^{i4} = \mathbf{z}_{45}^{i6} = \mathbf{z}_{46}^{i5} = \mathbf{z}_{13}^{i2} = \mathbf{z}_{23}^{i1} \text{ for } i \in \{1,2\}$$

$$\mathbf{z}_{12}^{i4} = \mathbf{z}_{56}^{i3} = \mathbf{z}_{35}^{i6} = \mathbf{z}_{36}^{i5} = \mathbf{z}_{14}^{i2} = \mathbf{z}_{24}^{i1} \text{ for } i \in \{1,2\}$$

$$\mathbf{z}_{12}^{i5} = \mathbf{z}_{46}^{i3} = \mathbf{z}_{36}^{i4} = \mathbf{z}_{34}^{i6} = \mathbf{z}_{15}^{i2} = \mathbf{z}_{25}^{i1} \text{ for } i \in \{1,2\}$$

$$\mathbf{z}_{12}^{i6} = \mathbf{z}_{45}^{i3} = \mathbf{z}_{35}^{i4} = \mathbf{z}_{34}^{i5} = \mathbf{z}_{16}^{i2} = \mathbf{z}_{26}^{i1} \text{ for } i \in \{1,2\}$$

$$\mathbf{z}_{56}^{i2} = \mathbf{z}_{13}^{i4} = \mathbf{z}_{14}^{i3} = \mathbf{z}_{34}^{j1} = \mathbf{z}_{25}^{j6} = \mathbf{z}_{26}^{j5} \text{ for } \{i,j\} = \{1,2\}$$

$$\mathbf{z}_{45}^{i2} = \mathbf{z}_{13}^{i6} = \mathbf{z}_{16}^{i3} = \mathbf{z}_{36}^{j1} = \mathbf{z}_{24}^{j5} = \mathbf{z}_{25}^{j4} \text{ for } \{i,j\} = \{1,2\}$$

$$\mathbf{z}_{46}^{i2} = \mathbf{z}_{13}^{i5} = \mathbf{z}_{15}^{i3} = \mathbf{z}_{35}^{j1} = \mathbf{z}_{24}^{j6} = \mathbf{z}_{26}^{j4} \text{ for } \{i,j\} = \{1,2\}$$

$$\mathbf{z}_{56}^{j1} = \mathbf{z}_{23}^{j4} = \mathbf{z}_{24}^{j3} = \mathbf{z}_{34}^{i2} = \mathbf{z}_{15}^{i6} = \mathbf{z}_{16}^{i5} \text{ for } \{i,j\} = \{1,2\}$$

$$\mathbf{z}^{j1}_{45} = \mathbf{z}^{j6}_{23} = \mathbf{z}^{j3}_{26} = \mathbf{z}^{i2}_{36} = \mathbf{z}^{i5}_{14} = \mathbf{z}^{i4}_{15} \text{ for } \{i,j\} = \{1,2\}$$

$$\mathbf{z}^{j1}_{46} = \mathbf{z}^{j5}_{23} = \mathbf{z}^{j3}_{25} = \mathbf{z}^{i2}_{35} = \mathbf{z}^{i6}_{14} = \mathbf{z}^{i4}_{16} \text{ for } \{i,j\} = \{1,2\}$$

For each vertex $\mathbf{z}^{kl}_{cd} \in \{\mathbf{z}^{i3}_{12}, \mathbf{z}^{i4}_{12}, \mathbf{z}^{i5}_{12}, \mathbf{z}^{i6}_{12}, \mathbf{z}^{mn}_{56}, \mathbf{z}^{mn}_{45}, \mathbf{z}^{mn}_{46} \mid i,m,n \in \{1,2\}\}$ we determine in detail the vertex set $\mathcal{V}(\Sigma_{\mathbf{z}^{kl}_{cd}})$ and the isomorphism $\gamma_{\mathbf{z}^{kl}_{cd}}$.

- $\mathbf{z}^{i3}_{12} = \mathbf{z}^{i2}_{13} = \mathbf{z}^{i1}_{23}$ for $i \in \{1,2\}$

$$\mathcal{V}(\Sigma_{\mathbf{z}^{i3}_{12}}) = \{\mathbf{y}^{12}, \mathbf{y}^{13}, \mathbf{y}^{23}, \mathbf{y}^{45}, \mathbf{y}^{46}, \mathbf{y}^{56}, \mathbf{z}^{j4}_{12}, \mathbf{z}^{j5}_{12}, \mathbf{z}^{j6}_{12}, \mathbf{z}^{j4}_{13}, \mathbf{z}^{j5}_{13}, \mathbf{z}^{j6}_{13}, \mathbf{z}^{j4}_{23}, \mathbf{z}^{j5}_{23}, \mathbf{z}^{j6}_{23} \mid j \in \{1,2\} \setminus \{i\}\}$$

$$\begin{array}{llll}
\mathbf{z}^{j4}_{12} \mapsto \mathbf{y}^{34} & \mathbf{z}^{j4}_{13} \mapsto \mathbf{y}^{24} & \mathbf{z}^{j4}_{23} \mapsto \mathbf{y}^{14} & \in \{\mathbf{y}^{56}\}^{\perp} \\
\mathbf{z}^{j5}_{12} \mapsto \mathbf{y}^{35} & \mathbf{z}^{j5}_{13} \mapsto \mathbf{y}^{25} & \mathbf{z}^{j5}_{23} \mapsto \mathbf{y}^{15} & \in \{\mathbf{y}^{46}\}^{\perp} \\
\mathbf{z}^{j6}_{12} \mapsto \mathbf{y}^{36} & \mathbf{z}^{j6}_{13} \mapsto \mathbf{y}^{26} & \mathbf{z}^{j6}_{23} \mapsto \mathbf{y}^{16} & \in \{\mathbf{y}^{45}\}^{\perp} \\
\in \{\mathbf{y}^{12}\}^{\perp} & \in \{\mathbf{y}^{13}\}^{\perp} & \in \{\mathbf{y}^{23}\}^{\perp} &
\end{array}$$

- $\mathbf{z}^{i4}_{12} = \mathbf{z}^{i2}_{14} = \mathbf{z}^{i1}_{24}$ for $i \in \{1,2\}$

$$\mathcal{V}(\Sigma_{\mathbf{z}^{i4}_{12}}) = \{\mathbf{y}^{12}, \mathbf{y}^{14}, \mathbf{y}^{24}, \mathbf{y}^{35}, \mathbf{y}^{36}, \mathbf{y}^{56}, \mathbf{z}^{j3}_{12}, \mathbf{z}^{j5}_{12}, \mathbf{z}^{j6}_{12}, \mathbf{z}^{j3}_{14}, \mathbf{z}^{j5}_{14}, \mathbf{z}^{j6}_{14}, \mathbf{z}^{j3}_{24}, \mathbf{z}^{j5}_{24}, \mathbf{z}^{j6}_{24} \mid j \in \{1,2\} \setminus \{i\}\}$$

$$\begin{array}{llll}
\mathbf{z}^{j3}_{12} \mapsto \mathbf{y}^{43} & \mathbf{z}^{j3}_{14} \mapsto \mathbf{y}^{23} & \mathbf{z}^{j3}_{24} \mapsto \mathbf{y}^{13} & \in \{\mathbf{y}^{56}\}^{\perp} \\
\mathbf{z}^{j5}_{12} \mapsto \mathbf{y}^{45} & \mathbf{z}^{j5}_{14} \mapsto \mathbf{y}^{25} & \mathbf{z}^{j5}_{24} \mapsto \mathbf{y}^{15} & \in \{\mathbf{y}^{36}\}^{\perp} \\
\mathbf{z}^{j6}_{12} \mapsto \mathbf{y}^{46} & \mathbf{z}^{j6}_{14} \mapsto \mathbf{y}^{26} & \mathbf{z}^{j6}_{24} \mapsto \mathbf{y}^{16} & \in \{\mathbf{y}^{35}\}^{\perp} \\
\in \{\mathbf{y}^{12}\}^{\perp} & \in \{\mathbf{y}^{14}\}^{\perp} & \in \{\mathbf{y}^{24}\}^{\perp} &
\end{array}$$

- $\mathbf{z}^{i5}_{12} = \mathbf{z}^{i2}_{15} = \mathbf{z}^{i1}_{25}$ for $i \in \{1,2\}$

$$\mathcal{V}(\Sigma_{\mathbf{z}^{i5}_{12}}) = \{\mathbf{y}^{12}, \mathbf{y}^{15}, \mathbf{y}^{25}, \mathbf{y}^{34}, \mathbf{y}^{36}, \mathbf{y}^{46}, \mathbf{z}^{j3}_{12}, \mathbf{z}^{j4}_{12}, \mathbf{z}^{j6}_{12}, \mathbf{z}^{j3}_{15}, \mathbf{z}^{j4}_{15}, \mathbf{z}^{j6}_{15}, \mathbf{z}^{j3}_{25}, \mathbf{z}^{j4}_{25}, \mathbf{z}^{j6}_{25} \mid j \in \{1,2\} \setminus \{i\}\}$$

$$\begin{array}{llll}
\mathbf{z}^{j3}_{12} \mapsto \mathbf{y}^{35} & \mathbf{z}^{j3}_{15} \mapsto \mathbf{y}^{23} & \mathbf{z}^{j3}_{25} \mapsto \mathbf{y}^{13} & \in \{\mathbf{y}^{46}\}^{\perp} \\
\mathbf{z}^{j4}_{12} \mapsto \mathbf{y}^{45} & \mathbf{z}^{j4}_{15} \mapsto \mathbf{y}^{24} & \mathbf{z}^{j4}_{25} \mapsto \mathbf{y}^{14} & \in \{\mathbf{y}^{36}\}^{\perp} \\
\mathbf{z}^{j6}_{12} \mapsto \mathbf{y}^{56} & \mathbf{z}^{j6}_{15} \mapsto \mathbf{y}^{26} & \mathbf{z}^{j6}_{25} \mapsto \mathbf{y}^{16} & \in \{\mathbf{y}^{34}\}^{\perp} \\
\in \{\mathbf{y}^{12}\}^{\perp} & \in \{\mathbf{y}^{15}\}^{\perp} & \in \{\mathbf{y}^{25}\}^{\perp} &
\end{array}$$

- $\mathbf{z}^{i6}_{12} = \mathbf{z}^{i2}_{16} = \mathbf{z}^{i1}_{26}$ for $i \in \{1, 2\}$

$$\mathcal{V}(\Sigma_{\mathbf{z}^{i6}_{12}}) = \{\mathbf{y}^{12}, \mathbf{y}^{16}, \mathbf{y}^{26}, \mathbf{y}^{34}, \mathbf{y}^{35}, \mathbf{y}^{45}, \mathbf{z}^{j3}_{12}, \mathbf{z}^{j4}_{12}, \mathbf{z}^{j5}_{12}, \mathbf{z}^{j3}_{16}, \mathbf{z}^{j4}_{16}, \mathbf{z}^{j5}_{16}, \mathbf{z}^{j3}_{26}, \mathbf{z}^{j4}_{26}, \mathbf{z}^{j5}_{26} \mid j \in \{1, 2\} \setminus \{i\}\}$$

$$\begin{array}{llll}
\mathbf{z}^{j3}_{12} \mapsto \mathbf{y}^{36} & \mathbf{z}^{j3}_{16} \mapsto \mathbf{y}^{23} & \mathbf{z}^{j3}_{26} \mapsto \mathbf{y}^{13} & \in \{\mathbf{y}^{45}\}^{\perp} \\
\mathbf{z}^{j4}_{12} \mapsto \mathbf{y}^{46} & \mathbf{z}^{j4}_{16} \mapsto \mathbf{y}^{24} & \mathbf{z}^{j4}_{26} \mapsto \mathbf{y}^{14} & \in \{\mathbf{y}^{35}\}^{\perp} \\
\mathbf{z}^{j5}_{12} \mapsto \mathbf{y}^{56} & \mathbf{z}^{j5}_{16} \mapsto \mathbf{y}^{25} & \mathbf{z}^{j5}_{26} \mapsto \mathbf{y}^{15} & \in \{\mathbf{y}^{34}\}^{\perp} \\
\in \{\mathbf{y}^{12}\}^{\perp} & \in \{\mathbf{y}^{16}\}^{\perp} & \in \{\mathbf{y}^{26}\}^{\perp} &
\end{array}$$

- $\mathbf{z}^{i1}_{56} = \mathbf{z}^{j6}_{15} = \mathbf{z}^{j5}_{16}$ for $\{i, j\} = \{1, 2\}$

$$\mathcal{V}(\Sigma_{\mathbf{z}^{i1}_{56}}) = \{\mathbf{y}^{23}, \mathbf{y}^{24}, \mathbf{y}^{34}, \mathbf{y}^{15}, \mathbf{y}^{16}, \mathbf{y}^{56}, \mathbf{z}^{j2}_{56}, \mathbf{z}^{j3}_{56}, \mathbf{z}^{j4}_{56}, \mathbf{z}^{i2}_{15}, \mathbf{z}^{i3}_{15}, \mathbf{z}^{i4}_{15}, \mathbf{z}^{i2}_{16}, \mathbf{z}^{13}_{16}, \mathbf{z}^{14}_{16} \mid \{i, j\} = \{1, 2\}\}$$

$$\begin{array}{llll}
\mathbf{z}^{j2}_{56} \mapsto \mathbf{y}^{12} & \mathbf{z}^{i2}_{15} \mapsto \mathbf{y}^{26} & \mathbf{z}^{i2}_{16} \mapsto \mathbf{y}^{25} & \in \{\mathbf{y}^{34}\}^{\perp} \\
\mathbf{z}^{j3}_{56} \mapsto \mathbf{y}^{13} & \mathbf{z}^{i3}_{15} \mapsto \mathbf{y}^{36} & \mathbf{z}^{i3}_{16} \mapsto \mathbf{y}^{35} & \in \{\mathbf{y}^{24}\}^{\perp} \\
\mathbf{z}^{j4}_{56} \mapsto \mathbf{y}^{14} & \mathbf{z}^{i4}_{15} \mapsto \mathbf{y}^{46} & \mathbf{z}^{i4}_{16} \mapsto \mathbf{y}^{45} & \in \{\mathbf{y}^{23}\}^{\perp} \\
\in \{\mathbf{y}^{56}\}^{\perp} & \in \{\mathbf{y}^{15}\}^{\perp} & \in \{\mathbf{y}^{16}\}^{\perp} &
\end{array}$$

- $\mathbf{z}^{i2}_{56} = \mathbf{z}^{j6}_{25} = \mathbf{z}^{j5}_{26}$ for $\{i, j\} = \{1, 2\}$

$$\mathcal{V}(\Sigma_{\mathbf{z}^{i2}_{56}}) = \{\mathbf{y}^{13}, \mathbf{y}^{14}, \mathbf{y}^{34}, \mathbf{y}^{25}, \mathbf{y}^{26}, \mathbf{y}^{56}, \mathbf{z}^{j1}_{56}, \mathbf{z}^{j3}_{56}, \mathbf{z}^{j4}_{56}, \mathbf{z}^{i1}_{25}, \mathbf{z}^{i3}_{25}, \mathbf{z}^{i4}_{25}, \mathbf{z}^{i1}_{26}, \mathbf{z}^{13}_{26}, \mathbf{z}^{14}_{26} \mid \{i, j\} = \{1, 2\}\}$$

$$\begin{array}{llll}
\mathbf{z}^{j1}_{56} \mapsto \mathbf{y}^{12} & \mathbf{z}^{i1}_{25} \mapsto \mathbf{y}^{16} & \mathbf{z}^{i1}_{26} \mapsto \mathbf{y}^{15} & \in \{\mathbf{y}^{34}\}^{\perp} \\
\mathbf{z}^{j3}_{56} \mapsto \mathbf{y}^{23} & \mathbf{z}^{i3}_{25} \mapsto \mathbf{y}^{36} & \mathbf{z}^{i3}_{26} \mapsto \mathbf{y}^{35} & \in \{\mathbf{y}^{14}\}^{\perp} \\
\mathbf{z}^{j4}_{56} \mapsto \mathbf{y}^{24} & \mathbf{z}^{i4}_{25} \mapsto \mathbf{y}^{46} & \mathbf{z}^{i4}_{26} \mapsto \mathbf{y}^{45} & \in \{\mathbf{y}^{13}\}^{\perp} \\
\in \{\mathbf{y}^{56}\}^{\perp} & \in \{\mathbf{y}^{25}\}^{\perp} & \in \{\mathbf{y}^{26}\}^{\perp} &
\end{array}$$

- $\mathbf{z}^{i1}_{46} = \mathbf{z}^{j6}_{14} = \mathbf{z}^{j4}_{16}$ for $\{i, j\} = \{1, 2\}$

$$\mathcal{V}(\Sigma_{\mathbf{z}^{i1}_{46}}) = \{\mathbf{y}^{23}, \mathbf{y}^{25}, \mathbf{y}^{35}, \mathbf{y}^{14}, \mathbf{y}^{16}, \mathbf{y}^{46}, \mathbf{z}^{j2}_{46}, \mathbf{z}^{j3}_{46}, \mathbf{z}^{j5}_{46}, \mathbf{z}^{i2}_{14}, \mathbf{z}^{i3}_{14}, \mathbf{z}^{i5}_{14}, \mathbf{z}^{i2}_{16}, \mathbf{z}^{13}_{16}, \mathbf{z}^{15}_{16} \mid \{i, j\} = \{1, 2\}\}$$

$$\begin{array}{llll}
\mathbf{z}^{j2}_{46} \mapsto \mathbf{y}^{12} & \mathbf{z}^{i2}_{14} \mapsto \mathbf{y}^{26} & \mathbf{z}^{i2}_{16} \mapsto \mathbf{y}^{24} & \in \{\mathbf{y}^{35}\}^{\perp} \\
\mathbf{z}^{j3}_{46} \mapsto \mathbf{y}^{13} & \mathbf{z}^{i3}_{14} \mapsto \mathbf{y}^{36} & \mathbf{z}^{i3}_{16} \mapsto \mathbf{y}^{34} & \in \{\mathbf{y}^{25}\}^{\perp} \\
\mathbf{z}^{j5}_{46} \mapsto \mathbf{y}^{15} & \mathbf{z}^{i5}_{14} \mapsto \mathbf{y}^{56} & \mathbf{z}^{i5}_{16} \mapsto \mathbf{y}^{45} & \in \{\mathbf{y}^{23}\}^{\perp} \\
\in \{\mathbf{y}^{46}\}^{\perp} & \in \{\mathbf{y}^{14}\}^{\perp} & \in \{\mathbf{y}^{16}\}^{\perp} &
\end{array}$$

- $\mathbf{z}_{46}^{i2} = \mathbf{z}_{24}^{j6} = \mathbf{z}_{26}^{j4}$ for $\{i, j\} = \{1, 2\}$

$$\mathcal{V}(\Sigma_{\mathbf{z}_{46}^{i2}}) = \{\mathbf{y}^{13}, \mathbf{y}^{15}, \mathbf{y}^{35}, \mathbf{y}^{24}, \mathbf{y}^{26}, \mathbf{y}^{46}, \mathbf{z}_{46}^{j1}, \mathbf{z}_{46}^{j3}, \mathbf{z}_{46}^{j5}, \mathbf{z}_{24}^{i1}, \mathbf{z}_{24}^{i3}, \mathbf{z}_{24}^{i5}, \mathbf{z}_{26}^{i1}, \mathbf{z}_{26}^{13}, \mathbf{z}_{26}^{15} \,|\, \{i,j\} = \{1,2\}\}$$

$$\begin{array}{llll}
\mathbf{z}_{46}^{j1} \mapsto \mathbf{y}^{12} & \mathbf{z}_{24}^{i1} \mapsto \mathbf{y}^{16} & \mathbf{z}_{26}^{i1} \mapsto \mathbf{y}^{14} & \in \{\mathbf{y}^{35}\}^{\perp} \\
\mathbf{z}_{46}^{j3} \mapsto \mathbf{y}^{23} & \mathbf{z}_{24}^{i3} \mapsto \mathbf{y}^{36} & \mathbf{z}_{26}^{i3} \mapsto \mathbf{y}^{34} & \in \{\mathbf{y}^{15}\}^{\perp} \\
\mathbf{z}_{46}^{j5} \mapsto \mathbf{y}^{25} & \mathbf{z}_{24}^{i5} \mapsto \mathbf{y}^{56} & \mathbf{z}_{26}^{i5} \mapsto \mathbf{y}^{45} & \in \{\mathbf{y}^{13}\}^{\perp} \\
\in \{\mathbf{y}^{46}\}^{\perp} & \in \{\mathbf{y}^{24}\}^{\perp} & \in \{\mathbf{y}^{26}\}^{\perp} &
\end{array}$$

- $\mathbf{z}_{45}^{i1} = \mathbf{z}_{14}^{j5} = \mathbf{z}_{15}^{j4}$ for $\{i, j\} = \{1, 2\}$

$$\mathcal{V}(\Sigma_{\mathbf{z}_{45}^{i1}}) = \{\mathbf{y}^{23}, \mathbf{y}^{26}, \mathbf{y}^{36}, \mathbf{y}^{14}, \mathbf{y}^{15}, \mathbf{y}^{45}, \mathbf{z}_{45}^{j2}, \mathbf{z}_{45}^{j3}, \mathbf{z}_{45}^{j6}, \mathbf{z}_{14}^{i2}, \mathbf{z}_{14}^{i3}, \mathbf{z}_{14}^{i6}, \mathbf{z}_{15}^{i2}, \mathbf{z}_{15}^{13}, \mathbf{z}_{15}^{16} \,|\, \{i, j\} = \{1, 2\}\}$$

$$\begin{array}{llll}
\mathbf{z}_{45}^{j2} \mapsto \mathbf{y}^{12} & \mathbf{z}_{14}^{i2} \mapsto \mathbf{y}^{25} & \mathbf{z}_{15}^{i2} \mapsto \mathbf{y}^{24} & \in \{\mathbf{y}^{36}\}^{\perp} \\
\mathbf{z}_{45}^{j3} \mapsto \mathbf{y}^{13} & \mathbf{z}_{14}^{i3} \mapsto \mathbf{y}^{35} & \mathbf{z}_{15}^{i3} \mapsto \mathbf{y}^{34} & \in \{\mathbf{y}^{26}\}^{\perp} \\
\mathbf{z}_{45}^{j6} \mapsto \mathbf{y}^{16} & \mathbf{z}_{14}^{i6} \mapsto \mathbf{y}^{56} & \mathbf{z}_{15}^{i6} \mapsto \mathbf{y}^{46} & \in \{\mathbf{y}^{23}\}^{\perp} \\
\in \{\mathbf{y}^{45}\}^{\perp} & \in \{\mathbf{y}^{14}\}^{\perp} & \in \{\mathbf{y}^{15}\}^{\perp} &
\end{array}$$

- $\mathbf{z}_{45}^{i2} = \mathbf{z}_{24}^{j5} = \mathbf{z}_{25}^{j4}$ for $\{i, j\} = \{1, 2\}$

$$\mathcal{V}(\Sigma_{\mathbf{z}_{45}^{i2}}) = \{\mathbf{y}^{13}, \mathbf{y}^{16}, \mathbf{y}^{36}, \mathbf{y}^{24}, \mathbf{y}^{25}, \mathbf{y}^{45}, \mathbf{z}_{45}^{j1}, \mathbf{z}_{45}^{j3}, \mathbf{z}_{45}^{j6}, \mathbf{z}_{24}^{i1}, \mathbf{z}_{24}^{i3}, \mathbf{z}_{24}^{i6}, \mathbf{z}_{25}^{i1}, \mathbf{z}_{25}^{13}, \mathbf{z}_{25}^{16} \,|\, \{i, j\} = \{1, 2\}\}$$

$$\begin{array}{llll}
\mathbf{z}_{45}^{j1} \mapsto \mathbf{y}^{12} & \mathbf{z}_{24}^{i1} \mapsto \mathbf{y}^{15} & \mathbf{z}_{25}^{i1} \mapsto \mathbf{y}^{14} & \in \{\mathbf{y}^{36}\}^{\perp} \\
\mathbf{z}_{45}^{j3} \mapsto \mathbf{y}^{23} & \mathbf{z}_{24}^{i3} \mapsto \mathbf{y}^{35} & \mathbf{z}_{25}^{i3} \mapsto \mathbf{y}^{34} & \in \{\mathbf{y}^{16}\}^{\perp} \\
\mathbf{z}_{45}^{j6} \mapsto \mathbf{y}^{26} & \mathbf{z}_{24}^{i6} \mapsto \mathbf{y}^{56} & \mathbf{z}_{25}^{i6} \mapsto \mathbf{y}^{46} & \in \{\mathbf{y}^{13}\}^{\perp} \\
\in \{\mathbf{y}^{45}\}^{\perp} & \in \{\mathbf{y}^{24}\}^{\perp} & \in \{\mathbf{y}^{25}\}^{\perp} &
\end{array}$$

APPENDIX FOUR

Some open problems

- **local recognition of all connected graphs Γ which are locally the hyperbolic line graph $\mathbf{G}(U_6)$**

  See definition 1.1.10, to recall the definition of $\mathbf{G}(U_6)$.

  • Determine all connected locally $\mathbf{G}(U_6)$ graphs Γ.

  The local structure should determine the isomorphism type. Let $\widehat{\Gamma}$ be the 2-simply connected cover of Γ (as a 2-dimensional simplicial complex). Looking back to chapter 4, a central problem is to define a subgroup $G_{\widehat{\Gamma}}$ of Aut $(\widehat{\Gamma})$ such that for each vertex $\mathbf{x} \in \widehat{\Gamma}$ the group $G_{\widehat{\Gamma}}$ contains a subgroup $\mathrm{SU}_2(\mathbb{F}_{q^2})_{\mathbf{x}} \cong \mathrm{SU}_2(\mathbb{F}_{q^2}) \cong \mathrm{SL}_2(\mathbb{F}_q)$ acting naturally on the two-dimensional regular subspace $x_{\mathbf{y}}$ for some vertex $\mathbf{y} \in \widehat{\Gamma}$ and fixing $\mathbf{x}^\perp$ elementwise or, stronger, fixing the vector space $U(\mathcal{G}_{\mathbf{x}})$ of the unitary projective space $\mathcal{G}_{\mathbf{x}}$ elementwise.

- **local recognition of all connected graphs Γ which are locally the line-hyperline graph $\mathbf{L}(\mathbb{P}(V_6))$**

  Definition Let $n \in \mathbb{N}$ and V be an n-dimensional vector space over a division ring $\mathbb{F}$. We consider the projective space $\mathbb{P}(V_n(\mathbb{F})) = \mathbb{P}_{n-1}(\mathbb{F})$. The line-hyperline graph $\mathbf{L}(P_{n-1}(\mathbb{F})) = \mathbf{L}_n(\mathbb{F})$ of $\mathbb{P}_{n-1}(\mathbb{F})$ is the graph whose vertices are the non-intersecting line-hyperline pairs of $\mathbb{P}_{n-1}(\mathbb{F})$ and in which one vertex (a, A) is adjacent to another vertex (b, B), in symbols $(a, A) \perp (b, B)$, if and only if $a \subseteq B$ and $b \subseteq A$.

  It has been shown in [37] that the graph $\mathbf{L}_{n+2}(\mathbb{F})$ is locally $\mathbf{L}_n(\mathbb{F})$ (cf. Proposition 2.2 of [37]) and that, for $n \geq 7$, a connected locally $\mathbf{L}_n(\mathbb{F})$ graph is isomorphic to $\mathbf{L}_{n+2}(\mathbb{F})$, with the possible exception of the case $(\mathbb{F}, n) = (\mathbb{F}_2, 7)$ (cf. Theorem 1 of [37]).

• Classify all connected locally $\mathbf{L}_6(\mathbb{F})$ graphs Γ.

We denote again with $\widehat{\Gamma}$ be the 2-simply connected cover of Γ. Also in this case one problem to solve is to define a subgroup $\mathrm{SL}_2(\mathbb{F})_{\mathbf{x}} \cong \mathrm{SL}_2(\mathbb{F})$ of $\mathrm{Aut}(\widehat{\Gamma})$, which acts in a natural sense on the line $x_{\mathbf{y}}$ for some $\mathbf{y} \in \widehat{\Gamma}$ and fixing $\mathbf{x}^\perp$ elementwise. Then a transvection subgroup A of $\mathrm{SL}_2(\mathbb{F})_{\mathbf{x}}$ acts as such on the projective space structure of $\mathbf{y}^\perp$. The idea is now to define the subgroup $G_{\widehat{\Gamma}} = \langle \mathrm{SL}_2(\mathbb{F})_{\mathbf{x}} \mid \mathbf{x} \in \widehat{\Gamma} \rangle$ of $\mathrm{Aut}(\widehat{\Gamma})$ and show that this group is generated by a class of abstract root subgroups in the sense of definition 1.1 of chapter II of [91] and subsequently conclude from Timmesfeld's theorem 5 of [90] (see also §9 of chapter III of [91]) that the group $G/Z(G)$ is isomorphic to $\mathrm{PSL}_{n+2}(\mathbb{F})$ or to the adjoint version of $E_6(\mathbb{F})$.

- **local recognition of all connected graphs Γ which are locally the hyperbolic line graph $\mathbf{W}(W_6)$**

  Definition Let $W_n = W$ denote an n-dimensional vector space over $\mathbb{F}_q$ endowed with a non-degenerate symplectic form $(\cdot, \cdot)$. For a subspace $U \subseteq W$ the orthogonal space of U is $U^\pi = \{x \in W : (x, w) = 0 \text{ for all } u \in U\}$. The **hyperbolic line graph $\mathbf{W}(W_n)$** is the graph on the hyperbolic lines, i.e., the non-degenerate two-dimensional subspaces, of W_n, where hyperbolic lines l and m are adjacent (in symbols $l \perp m$) if and only if l is perpendicular to m with respect to the symplectic form.

  It has been proven in [39], that for $n \geq 8$ a connected locally $\mathbf{W}(W_n)$ graph Γ is isomorphic to $\mathbf{W}(W_{n+2})$.

  • Find all connected locally $\mathbf{W}(W_6)$ graphs Γ.

  Here one has not only to deal with the problem of defining a suitable automorphism subgroup $G_{\widehat{\Gamma}}$ of $\mathrm{Aut}(\widehat{\Gamma})$, where $\widehat{\Gamma}$ is the 2-simply connected cover of Γ. Also one has to identify all possibilities for induced subgraphs Σ of $\widehat{\Gamma}$, which are locally the reflection graph $\mathbf{W}(F_4)$ of the root system $\Phi_{\mathbb{F}_4}$. That implies directly that one now has to consider roots of different lengths.

- **local recognition of all connected graphs Γ which are locally the line graph $\mathbf{S}(V_n(\mathbb{R}))$**

  Here we take the n-dimensional vector space V over $\mathbb{R}$ equipped with the scalar product $(\cdot, \cdot)$. The line graph $\mathbf{S}(V_n(\mathbb{R}))$ is the graph on the two-dimensional subspaces of V, where two different lines l and m are adjacent if and only if $l \subseteq m^\pi$ or equivalent if $m \subseteq l^\pi$.

  • Characterise all connected locally $\mathbf{S}(V_n(\mathbb{R}))$ graphs Γ for $n \geq 6$.

Especially for the case $n = 6$, it might be possible to define the group $G_{\widehat{\Gamma}}$ as has been done in section 4.3. But since there are no results and structure theory for groups G admitting a (weak) Phan system of type Δ over $\mathbb{R}$, the technique to identify the group $G_{\widehat{\Gamma}}$ used in this work is not yet available.

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