

Contents

1	Introduction	1
1.1	Organization of the Text	3
2	Preliminary Definitions and Results	5
2.1	Fundamental Notions and Definitions	6
2.1.1	Critical Elements, Non-wandering Points, Stable and Unstable Sets	6
2.1.2	Limit Sets, Transitivity, Attractors and Repellers	6
2.1.3	Hyperbolic Critical Elements	10
2.1.4	Topological Equivalence, Structural Stability	10
2.2	Low Dimensional Flow Versus Chaotic Behavior	11
2.2.1	One-Dimensional Flows	11
2.2.2	Two-Dimensional Flows	12
2.2.3	Three Dimensional Chaotic Attractors	14
2.3	Hyperbolic Flows	16
2.3.1	Hyperbolic Sets and Singularities	18
2.3.2	Examples of Hyperbolic Sets and Axiom A Flows	18
2.4	Expansiveness and Sensitive Dependence on Initial Conditions	21
2.4.1	Chaotic Systems	22
2.4.2	Expansive Systems	24
2.5	Basic Tools	27
2.5.1	The Tubular Flow Theorem	27
2.5.2	Transverse Sections and the Poincaré Return Map	28
2.5.3	The Hartman-Grobman Theorem on Local Linearization	28
2.5.4	The (Strong) Inclination Lemma (or λ -Lemma)	29
2.5.5	Homoclinic Classes, Transitivity and Denseness of Periodic Orbits	30
2.5.6	The Closing Lemma	31
2.5.7	The Connecting Lemma	31
2.5.8	The Ergodic Closing Lemma	33
2.5.9	A Perturbation Lemma for Flows	34

2.5.10	Generic Vector Fields and Lyapunov Stability	35
2.6	The Linear Poincaré Flow	37
2.6.1	Hyperbolic Splitting for the Linear Poincaré Flow	37
2.6.2	Dominated Splitting for the Linear Poincaré Flow	39
2.6.3	Incompressible Flows, Hyperbolicity and Dominated Splitting	43
2.7	Ergodic Theory	44
2.7.1	Physical or SRB Measures	45
2.7.2	Gibbs Measures Versus SRB Measures	47
2.8	Stability Conjectures	53
3	Singular Cycles and Robust Singular Attractors	55
3.1	Singular Horseshoe	56
3.1.1	A Singular Horseshoe Map	56
3.1.2	A Singular Cycle with a Singular Horseshoe First Return Map	59
3.1.3	The Singular Horseshoe Is a Partially Hyperbolic Set with Volume Expanding Central Direction	65
3.2	Bifurcations of Saddle-Connections	68
3.2.1	Saddle-Connection with Real Eigenvalues	68
3.2.2	Inclination Flip and Orbit Flip	69
3.2.3	Saddle-Focus Connection and Shil'nikov Bifurcations	71
3.3	Lorenz Attractor and Geometric Models	73
3.3.1	Properties of the Lorenz System of Equations	74
3.3.2	The Geometric Model	77
3.3.3	The Geometric Lorenz Attractor Is a Partially Hyperbolic Set with Volume Expanding Central Direction	83
3.3.4	Existence and Robustness of Invariant Stable Foliation	84
3.3.5	Robustness of the Geometric Lorenz Attractors	93
3.3.6	The Geometric Lorenz Attractor Is a Homoclinic Class	96
4	Robustness on the Whole Ambient Space	99
4.1	No Equilibria Surrounded by Regular Orbits with Dominated Splitting	100
4.2	Homogeneous Flows and Dominated Splitting	103
4.2.1	Dominated Splitting over the Periodic Orbits	103
4.2.2	Dominated Splitting over Regular Orbits from the Periodic Ones	105
4.2.3	Bounded Angles on the Splitting over Hyperbolic Periodic Orbits	106
4.2.4	Dominated Splitting for the Linear Poincaré Flow Along Regular Orbits	109
4.3	Uniform Hyperbolicity for the Linear Poincaré Flow	113
4.3.1	Subadditive Functions of the Orbits of a Flow and Exponential Growth	114

4.3.2	Uniform Hyperbolicity for the Linear Poincaré Flow on the Whole Manifold	120
5	Robust Transitivity and Singular-Hyperbolicity	123
5.1	Definitions and Statement of Results	124
5.1.1	Equilibria of Robust Attractors Are Lorenz-Like	126
5.1.2	Robust Attractors Are Singular-Hyperbolic	127
5.1.3	Brief Sketch of the Proofs	128
5.2	Higher Dimensional Analogues	129
5.2.1	Singular-Attractor with Arbitrary Number of Expanding Directions	129
5.2.2	The Notion of Sectionally Expanding Sets	130
5.2.3	Homogeneous Flows and Sectionally Expanding Attractors	130
5.3	Attractors and Isolated Sets for C^1 Flows	130
5.3.1	Proof of Sufficient Conditions to Obtain Attractors	132
5.3.2	Robust Singular Transitivity Implies Attractors or Repellers	135
5.4	Attractors and Singular-Hyperbolicity	142
5.4.1	Uniformly Dominated Splitting over the Periodic Orbits	144
5.4.2	Dominated Splitting over a Robust Attractor	146
5.4.3	Robust Attractors Are Singular-Hyperbolic	147
5.4.4	Flow-Boxes Near Equilibria	150
5.4.5	Uniformly Bounded Angle Between Stable and Center-Unstable Directions on Periodic Orbits	151
6	Singular-Hyperbolicity and Robustness	163
6.1	Cross-Sections and Poincaré Maps	168
6.1.1	Stable Foliations on Cross-Sections	169
6.1.2	Hyperbolicity of Poincaré Maps	171
6.1.3	Adapted Cross-Sections	175
6.1.4	Global Poincaré Return Map	180
6.1.5	The One-Dimensional Piecewise Expanding Map	184
6.1.6	Denseness of Periodic Orbits and the One-Dimensional Map	184
6.1.7	Crossing Strips and the One-Dimensional Map	187
6.2	Homoclinic Class	188
6.3	Sufficient Conditions for Robustness	189
6.3.1	Denseness of Periodic Orbits and Transitivity with a Unique Singularity	190
6.3.2	Unstable Manifolds of Periodic Orbits Inside Singular-Hyperbolic Attractors	198
7	Expansiveness and Physical Measure	203
7.1	Statements of the Results and Overview of the Arguments	203
7.1.1	Robust Sensitiveness	204
7.1.2	Existence and Uniqueness of a Physical Measure	206
7.2	Expansiveness	208
7.2.1	Proof of Expansiveness	208

7.2.2	Infinitely Many Coupled Returns	211
7.2.3	Semi-global Poincaré Map	212
7.2.4	A Tube-Like Domain Without Singularities	213
7.2.5	Every Orbit Leaves the Tube	215
7.2.6	The Poincaré Map Is Well-Defined on Σ_j	216
7.2.7	Expansiveness of the Poincaré Map	218
7.2.8	Singular-Hyperbolicity and Chaotic Behavior	218
7.3	Non-uniform Hyperbolicity	220
7.3.1	The Starting Point	220
7.3.2	The Hölder Property of the Projection	221
7.3.3	Integrability of the Global Return Time	223
7.3.4	Suspending Invariant Measures	225
7.3.5	Physical Measure for the Global Poincaré Map	228
7.3.6	Suspension Flow from the Poincaré Map	229
7.3.7	Physical Measures for the Suspension	234
7.3.8	Physical Measure for the Flow	234
7.3.9	Hyperbolicity of the Physical Measure	235
7.3.10	Absolutely Continuous Disintegration of the Physical Measure	236
7.3.11	Constructing the Disintegration	239
7.3.12	The Support Covers the Whole Attractor	247
8	Singular-Hyperbolicity and Volume	249
8.1	Dominated Decomposition and Zero Volume	249
8.1.1	Dominated Splitting and Regularity	250
8.1.2	Uniform Hyperbolicity	256
8.2	Singular-Hyperbolicity and Zero Volume	257
8.2.1	Partial Hyperbolicity and Zero Volume on C^{1+} Flows	258
8.2.2	Positive Volume Versus Transitive Anosov Flows	262
8.2.3	Zero-Volume for C^1 Generic Singular-Hyperbolic Attractors	265
8.2.4	Extension to Sectionally Expanding Attractors in Higher Dimensions	266
9	Global Dynamics of Generic 3-Flows	269
9.1	Spectral Decomposition	272
9.2	A Dichotomy for C^1 Generic 3-Flows	276
9.2.1	Some Consequences of the Generic Dichotomy	276
9.2.2	Generic 3-Flows, Lyapunov Stability and Singular- Hyperbolicity	278
9.3	C^1 Generic Incompressible Flows	283
9.3.1	Conservative Tubular Flow Theorem	284
9.3.2	Realizable Linear Flows	286
9.3.3	Blending Oseledets Directions Along an Orbit Segment	295
9.3.4	Lowering the Norm: Local Procedure	297
9.3.5	Lowering the Norm: Global Procedure	301
9.3.6	Proof of the Dichotomy with Singularities (Theorem 9.4)	305

10 Related Results and Recent Developments	309
10.1 More on Singular-Hyperbolicity	309
10.1.1 Topological Dynamics	309
10.1.2 Attractors that Resemble the Lorenz Attractor	311
10.1.3 Unfolding of Singular Cycles	312
10.1.4 Contracting Lorenz-Like Attractors	312
10.1.5 Unfolding of Singular Cycles	314
10.2 Dimension Theory, Ergodic and Statistical Properties	314
10.2.1 Large Deviations for the Lorenz Flow	315
10.2.2 Central Limit Theorem for the Lorenz Flow	316
10.2.3 Decay of Correlations	317
10.2.4 Decay of Correlations for the Return Map and Quantitative Recurrence on the Geometric Lorenz Flow	318
10.2.5 Non-mixing Flows and Slow Decay of Correlations	319
10.2.6 Decay of Correlations for Flows	320
10.2.7 Thermodynamical Formalism	321
10.3 Generic Conservative Flows in Dimension 3	322
Appendix A Lyapunov Stability on Generic Vector Fields	325
Appendix B A Perturbation Lemma for Flows	331
Appendix C Robustness of Dominated Decomposition	337
References	343
Index	355