DEGENERATE DIFFUSION

Behaviour at the boundary and kernel estimates

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Abstract: We study evolution equations of the form:

$$\frac{\partial u}{\partial t}(t,x) = m(x)(\triangle u)(t,x) \qquad t \in \mathbb{R}_+,\, x \in \Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ and the function $m : \Omega \to (0,\infty)$ is assumed to be measurable. Dirichlet boundary conditions are posed. We investigate under which conditions on m and $\partial\Omega$ the operator $m\triangle$ generates a strongly continuous semigroup on $C_0(\Omega)$. In the second part of the thesis we obtain various estimates on the kernel of the semigroup generated by $m\triangle$ on weighted L^p-spaces.

Contents

Notation

$C_c(\Omega)$	continuous functions with compact support in Ω
$\partial\Omega$	the boundary of an open set $\Omega \subset \mathbb{R}^N$
$\overline{\Omega}$	the closure of Ω in $\mathbb{R}^N$
$\omega \subset\subset \Omega$	ω is an open subset of $\mathbb{R}^N$ such that $\overline{\omega} \subset \Omega$
$\hat{N}$	$\max\{N, 2\}$
$\mathcal{D}(\Omega) = C_c^\infty(\Omega)$	test functions (C^∞ functions with compact support)
$\mathcal{D}(\Omega)_+$	$\{v \in \mathcal{D}(\Omega) : v \geq 0\}$
$\mathcal{D}(\Omega)'$	the space of all distributions (continuous functionals on $\mathcal{D}(\Omega)$)
$D_j = \frac{\partial}{\partial x_j}$	(weak) derivative with respect to the j^{th} coordinate
$H^1(\Omega)$	$\{u \in L^2(\Omega) : D_j u \in L^2(\Omega),\ j = 1, ..., d\}$-the first Sobolev space
$H_0^1(\Omega)$	the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$
$L_{loc}^p(\Omega)$	$\{u : \Omega \to \mathbb{R}$ measurable: $\int_\omega \|u(x)\|^p \, dx < \infty$ whenever $\omega \subset\subset \Omega\}$
$H_{loc}^1(\Omega)$	$\{u \in L_{loc}^2(\Omega) : D_j u \in L_{loc}^2(\Omega)$ for $j = 1, ..., d\}$.
$C_0(\Omega)$	$\{u \in C(\overline{\Omega}) : u_{\|\partial\Omega} = 0\}$
χ_A	characteristic function of a set A
$a \wedge b$	$\min\{a, b\}$
$a \vee b$	$\max\{a, b\}$
Σ_θ	$\{z \in \mathbb{C} : z \neq 0, \|\arg z\| < \theta\}$
Σ_θ^ζ	$\{z \in \mathbb{C} : z \neq 0, \theta < \arg z < \zeta\}$
e^{At}	the semigroup generated by an (unbounded) operator A
$R(\lambda, A)$	$:= (\lambda - A)^{-1}$, the resolvent of A

Introduction

In this work, the main object of investigation is the equation

$$\frac{\partial u}{\partial t}(t,x) = m(x)(\triangle u)(t,x) \qquad t \in \mathbb{R}_+,\ x \in \Omega, \tag{1}$$

where Ω is a bounded domain (open connected set) in $\mathbb{R}^N$ and Dirichlet boundary conditions are posed. The function $m : \Omega \to (0,\infty)$ is assumed to be measurable and the Laplace operator is understood to operate on functions of $x \in \mathbb{R}^N$. It is the desire to understand deeper the interplay of growth (or decay) properties of m and properties of the kernels of semigroups generated by $m\triangle$ on various function spaces that motivates our work in this thesis.

There are two main themes that are elaborated. Firstly, it is the question whether the operator $m\triangle$ (after a proper definition) generates a strongly continuous semigroup on $C_0(\Omega)$, the space of continuous functions vanishing at the boundary of Ω. More precisely, we study conditions on Ω and m that guarantee the existence of such a semigroup. There are good reasons for studying the operator on the space $C_0(\Omega)$. One reason is that one obtains a Feller semigroup in this way with the corresponding relations to stochastic processes (see [27], [30], [32] and [66] for the role of $C_0(\Omega)$ in the theory of Markov processes). Another reason concerns possible applications to non-linear problems and dynamical systems. For semilinear problems the space $C_0(\Omega)$ is much better suited than $L^p(\Omega)$-spaces since composition with a locally Lipschitz continuous function is locally Lipschitz continuous on $C_0(\Omega)$ but never on $L^p(\Omega)$ unless the function is already globally Lipschitz continuous, see the treatise of Cazenave-Haraux [18], for example. Studying arbitrary measurable functions m seems to be useful for possible applications to quasilinear equations.

The second main theme of the work are kernel estimates for the semigroup on weighted L^p-spaces (with the weight $\frac{1}{m}$). Here we first give a condition on the function m so that a bounded kernel for the semigroup (on the weighted L^2) exists. After demonstrating by an example that this condition is optimal we proceed to refine the estimates for bounded kernels. This refinement is of twofold nature - firstly we prove estimates where a Gaussian factor $e^{-c\frac{|x-y|^2}{t}}$ is incorporated and secondly we obtain upper and lower kernel estimates depending on the behaviour of the first eigenfunction of the operator.

A few words of explanation concerning the title of this work should be said before we describe the content of the chapters. We are motivated by the fact that the evolution equation

$$u_t(t,x) = (\triangle_x u)(t,x) \qquad (+f(t,x)) \tag{2}$$

is used in various models to describe diffusion. If one changes the operator on the right-hand side of (2) by a multiplicative factor $m(x)$ one obtains the equation

$$u_t(t,x) = m(x)(\triangle_x u)(t,x). \tag{3}$$

From the probabilistic point of view this perturbation results in the change of time in the underlying Markov process (see [44], [45] and the references there). The change is governed by the behaviour of m and m may, in general, blow up or vanish fast at the boundary of Ω. So much to the word degenerate.
The operator on the right-hand side of (3) is a particular kind of a general second order elliptic operator in non-divergence form having merely a principal part

$$a_{ij}(x)\frac{\partial u}{\partial x_i \partial x_j}. \tag{4}$$

Here, the coefficients a_{ij} take a special form $a_{ii} = m(x)$ for $i = 1, ..., N$ and $a_{ij} = 0$ for $i \neq j$. The word isotropically refers to the fact that the coefficients do not depend on a particular direction.
Note that the operator $m\triangle$ is elliptic (in the terminology of [34]) since we assume that $m(x) > 0$, but we do not assume strict ellipticity (this would require $m(x) > \varepsilon$ for some $\varepsilon > 0$).

We comment on the content of the chapters.

We start in Chapter 1 by introducing all relevant notions and theorems which are needed in the main body of the work. We try to keep the text as self-contained as possible and thus prove some of the results even in this introductory chapter. In particular, since the theory of submarkovian semigroups provides a most natural setting for the operators analysed in this thesis we treat in detail the Beurling-Deny criteria for the generation of such semigroups.
Similarly, we devote a whole section to the notion of a regularised distance function and prove in detail its basic properties. These are of decisive importance at various places in the work; firstly when developing a local theory for points of weak diffusion in Section 3.4. It reappears later confirming optimality of the ultracontractivity result in Theorem 4.4.1 and once more when proving intrinsic ultracontractive estimates in Chapter 6. On the other hand, interpolation theorems, Sobolev embeddings and the spectral theorem are given without proof here. There exists abundant high-quality literature concerning all these topics. It is listed in Notes and Comments to this chapter.

Our own work starts in Chapter 2. Here we introduce general forms acting on weighted L^p-spaces and prove basic properties of the sesquilinear forms in order to be able to obtain an associated operator - the generator of a strongly continuous semigroup. Furthermore, since we are interested in obtaining a submarkovian semigroup, we study conditions which have to be posed on coefficients of the form in order to make the associated operator L^∞-contractive.

The next four chapters constitute the core of the thesis.
In Chapter 3 we give a precise meaning to the operator $m\triangle$ and develop both global and local theory of regular points for this operator. We stress here that the results of this chapter depend strongly on the fact that the operator in question is isotropic. Our results show in particular that for m larger than a positive constant (even $\frac{1}{m} \in L^q(\Omega)$, $q > \frac{\max\{N,2\}}{2}$ suffices) the regular points of $m\triangle$ are the same as for $\triangle$. For general (non-isotropic) elliptic operators in non-divergence form this is no longer true in both directions. In fact Miller [49] showed that there may be regular points for the Laplacian which are non-regular for a particular elliptic operator and vice versa.
The theory of this chapter culminates in the Theorem 3.5.1 where we develop a local theory for the generation on $C_0(\Omega)$. In order to obtain a strongly continuous semigroup generated by $m\triangle$ on $C_0(\Omega)$ we require each point in $\partial\Omega$ to fulfil (at least) one of the conditions: the point should be regular for the Laplace equation or the diffusion should be weak enough in the neighbourhood of the point (see Sections 3.3 and 3.4 for precise definitions).

In Chapter 4 the existence of a bounded kernel for the semigroup on the weighted spaces is investigated. We prove the abstract Dunford-Pettis theorem and reformulate the question in terms of the ultracontractivity of the semigroup. We continue by a positive result for the operator $m\triangle$ with a perturbing function m s.t. $\frac{1}{m} \in L^q(\Omega, dx)$ for some $q > \frac{\hat{N}}{2}$. We show the optimality of the result by considering the operator $\sigma^2\triangle$, where σ is a regularised distance function. The semigroup associated to this operator is not ultracontractive on $L^2(\Omega, \frac{dx}{\sigma(x)^2})$. We finish the chapter by listing various consequences of ultracontractivity. In particular, we obtain a representation of the kernel in terms of a series containing denumerably many eigenfunctions of the operator.

In Chapter 5 we refine the ultracontractive estimates of the previous chapter and incorporate a Gaussian factor. This can be used to prove the existence of a holomorphic extension for the semigroup on the weighted L^1-space. However, the strong continuity of the semigroup at zero is not guaranteed. Further possible consequences include the investigation of degenerate operators on unbounded domains. Although certainly interesting and useful, these topics lie beyond the scope of the present thesis and must be left for the future work.

Chapter 6 refines the ultracontractive estimates of Chapter 4 in a different way. Namely, since we work with Dirichlet boundary conditions throughout this thesis one expects the kernel of the semigroup to vanish at the boundary. One may also ask how fast the convergence is once it takes place. For a particular class of operators we show that the behaviour of the kernel at the boundary is controlled by the first eigenfunction. It is also interesting to note that upper kernel estimates of this form automatically imply corresponding lower ones.
Here we also show the following result interesting on its own. If the perturbing function m is strictly positive on a bounded Dirichlet regular domain Ω and if $\frac{1}{m} \in L^q(\Omega, dx)$ for some $q > \frac{\hat{N}}{2}$ then the first eigenfunction of the operator $m\triangle$ (on $L^p(\Omega, \frac{dx}{m(x)})$) is also strictly positive at all points of Ω.

Chapter 7 concludes the thesis. Here we prove L^p-maximal regularity for realisations of the operator $m\triangle$ on weighted L^p-spaces. We deduce our result from a more general theorem guaranteeing L^p-maximal regularity for generators of positive contractions on L^p-spaces. This result on its own is based on an estimate of ergodic type for such generators.

At the end of the introduction we would like to express our acknowledgement. I would like to thank Wolfgang Arendt, without whom the whole project might have never started. His expertise was admirable as were his acute remarks and suggestions to the preliminary versions of the work. For all this and much more, I express my sincere gratitude. At this point I would also like to thank Jaroslav Milota for leading me to the realm of higher mathematics.

Chapter 1

Preliminaries

In this introductory chapter we collect the most important results and methods which will be needed in the sequel.

1.1 Sesquilinear forms

We introduce basic terminology concerning form techniques. Note that throughout this work we could do only with quadratic forms in most of the text, the only place where we need sesquilinear forms in general is in Chapter 5 when working with twisted forms.
Let H be a Hilbert space, denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$ and let $a(\cdot,\cdot)$ be a sesquilinear form i.e. a mapping defined on a linear subspace $\mathcal{D}(a)$ of H satisfying

$$\begin{aligned} a(\cdot,\cdot) :\, \mathcal{D}(a) \times \mathcal{D}(a) &\longrightarrow \mathbb{K} \\ a(\alpha u + v, w) &= \alpha a(u,w) + a(v,w) \\ a(u, \alpha v + w) &= \overline{\alpha} a(u,v) + a(u,w) \end{aligned}$$

for all $\alpha \in \mathbb{K}$ and $u, v, w \in \mathcal{D}(a)$. We define the *adjoint* form of a to be the sesquilinear form a^*:

$$a^*(u,v) := \overline{a(v,u)} \quad \text{with the domain} \quad \mathcal{D}(a^*) := \mathcal{D}(a).$$

We call *a symmetric* if $a^* = a$.
We say that a is *accretive* if $Re\; a(u,u) \geq 0$ for all $u \in \mathcal{D}(a)$. If a is accretive we may define a scalar product on $\mathcal{D}(a)$ in the following way:

$$\langle u, v\rangle_a := \frac{1}{2}[a(u,v) + a^*(u,v)] + \langle u, v\rangle \qquad \forall\, u, v \in \mathcal{D}(a).$$

Equipped with this scalar product $D(a)$ becomes a pre-Hilbert space. The expression $\|\cdot\|_a := \sqrt{Re\; a(u,u) + \|u\|^2}$ then defines a norm on $\mathcal{D}(a)$. We say that

- a is *densely defined* if $\mathcal{D}(a)$ is dense in H.
- a is *continuous* if it is accretive and there exists $M \geq 0$ such that

$$|a(u,v)| \leq M\, \|u\|_a \, \|v\|_a \qquad \forall\, u, v \in \mathcal{D}(a). \tag{1.1}$$

- a is *closed* if a is accretive and $(\mathcal{D}(a), \|\cdot\|_a)$ is a complete space.

We say that a linear subspace $\mathcal{D} \subset \mathcal{D}(a)$ is a *core* of a if $\mathcal{D}$ is dense in $(\mathcal{D}(a), \|\cdot\|_a)$. We note that instead of the norm $\|\cdot\|_a$ we could also use an equivalent norm $\|u\|_{a,w} := \sqrt{Re\ a(u,u) + w\,\|u\|^2}$ for any $w > 0$. Analogously for the scalar product $\langle \cdot, \cdot \rangle_a$.
Any sesquilinear form a may be written as a sum of two symmetric forms. This is easily accomplished as follows. Define the forms a_1 and a_2 by

$$\begin{aligned} a_1 &:= \frac{1}{2}(a + a^*), \\ a_2 &:= \frac{1}{2i}(a - a^*) \end{aligned}$$

with $\mathcal{D}(a_1) := \mathcal{D}(a) =: \mathcal{D}(a_2)$. Then we have

$$a = a_1 + ia_2$$

and also

$$a_1(u,u) = Re\ a(u,u) \qquad \text{for any } u \in \mathcal{D}(a).$$

The form a_1 is called the *real* part of a. Similarly, a_2 is called the *imaginary* part of a. One may check continuity of a given sesquilinear form by verifying the (sectoriality) assumption of the next proposition. The result is known as Schwarz's inequality.

Proposition 1.1.1 *Let a be a sesqulinear form on a Hilbert space H. If there exists a constant c such that*

$$|Im\ a(u,u)| \leq c\, Re\ a(u,u), \qquad \forall\, u \in \mathcal{D}(a),$$

then the inequality

$$|a(u,v)| \leq (c+1)\sqrt{Re\ a(u,u)}\sqrt{Re\ a(v,v)} \tag{1.2}$$

is valid for all $u, v \in \mathcal{D}(a)$.

Proof. Choose $u, v \in \mathcal{D}(a)$ arbitrarily. Without loss of generality we assume that $a(u,v) \in \mathbb{R}$ since we may replace u by $e^{i\theta}u$, $\theta \in (0, 2\pi)$ without affecting the inequality (1.2). We have

$$a(u,v) = \frac{1}{4}\left(a(u+v, u+v) - a(u-v, u-v)\right)$$

and thus by the assumption and the parallelogram identity (for the form a_1) we obtain

$$\begin{aligned} |a(u,v)| &\leq \frac{c+1}{4}\left(Re\ a(u+v, u+v) + Re\ a(u-v, u-v)\right) \\ &= \frac{c+1}{2}\left(Re\ a(u,u) + Re\ a(v,v)\right). \end{aligned}$$

Using the last inequality for αu and $\frac{1}{\alpha}v$ with an arbitrary $\alpha > 0$ we have

$$|a(u,v)| \leq \frac{c+1}{2}\left(\alpha^2 Re\ a(u,u) + \frac{1}{\alpha^2} Re\ a(v,v)\right).$$

In case $Re\ a(u,u) \neq 0$ we put $\alpha^2 := \left(\frac{Re\ a(v,v)}{Re\ a(u,u)}\right)^{\frac{1}{2}}$ and get the result. If $Re\ a(u,u) = 0$ we let α tend to ∞ and obtain $a(u,v) = 0$. Thus (1.2) holds in both cases. □

1.2 Associated operator, fundamentals of the semigroup theory

We assume that a is a densely defined, accretive, continuous and closed sesquilinear form on a Hilbert space H. We define the operator A associated with a in the following way:

$$\mathcal{D}(A) := \{u \in \mathcal{D}(a) : \text{ there exists } v \in H \text{ such that } a(u,\phi) = \langle v, \phi\rangle, \ \forall\, \phi \in \mathcal{D}(a)\}$$

$$Au := -v.$$

We have the following result.

Proposition 1.2.1 *The operator A is densely defined. For each $\lambda > 0$, the operator $\lambda - A$ is invertible (as a mapping from $\mathcal{D}(A)$ to H) and its inverse $R(\lambda, A)$ is a bounded operator on H. We have for all $\lambda > 0$ and $f \in H$*

$$\|\lambda R(\lambda, A) f\| \leq \|f\|.$$

Remark 1.2.2 *We shall frequently use the following terminology. If we write*

$$u \leq c(-A)u \qquad u \in \mathcal{D}(A) \tag{1.3}$$

in the quadratic form sense (for some constant $c \in \mathbb{R}$) we always mean that

$$\langle u, u\rangle \leq c \cdot a(u,u)$$

should hold for such a u.

Proof of Proposition 1.2.1. For a fixed $\lambda > 0$ we have already defined the norm

$$\|u\|_\lambda := \sqrt{Re\ a(u,u) + \lambda \|u\|^2} \qquad \forall\, u \in \mathcal{D}(a).$$

Obviously, the norm $\|\cdot\|_\lambda$ is equivalent to the norm $\|\cdot\|_a$. Thus $V := (\mathcal{D}(a), \|\cdot\|_\lambda)$ is a Hilbert space. The continuity of the form a implies the boundedness of the form $\lambda + a^*$ on $V \times V$. Also the coercivity of $\lambda + a^*$ on V is easily seen. Let now $f \in H$ and set

$$\phi(v) := \langle v, f\rangle, \qquad v \in V.$$

The mapping ϕ defines a linear continous functional on V and hence by the lemma of Lax-Milgram (Lemma A.1.3 in Appendix) there exists a unique $u \in V$ such that

$$\phi(v) = a^*(v,u) + \lambda\langle v, u\rangle = \overline{a(u,v)} + \lambda\langle v,u\rangle \qquad \forall\, v \in V.$$

This and the definition of A imply that $u \in \mathcal{D}(A)$ and $(\lambda - A)u = f$. Thus the range of $\lambda - A$ equals H. The injectivity of the mapping $\lambda - A$ follows easily from the accretivity assumption. By the closed graph theorem we see that $\lambda - A$ is invertible. Choose now $f \in H$ and find $u \in \mathcal{D}(A)$ fulfilling $(\lambda - A)u = f$. Because of

$$Re\ \langle -Au, u\rangle = Re\ a(u,u) \geq 0, \tag{1.4}$$

we have

$$Re\ \langle f, u\rangle \geq \lambda \|u\|^2.$$

We conclude that

$$\|\lambda R(\lambda, A) f\| = \lambda \|u\| \geq \|f\|.$$

It remains to check that A is densely defined. In order to prove this, let $v \in H$ fulfil

$$\langle v, u\rangle = 0 \text{ for all } u \in \mathcal{D}(A).$$

From the invertibility of $I - A$ we find $\psi \in \mathcal{D}(A)$ with $v = (I - A)\psi$. We obtain

$$0 = \langle v, \psi \rangle = \langle (I - A)\psi, \psi \rangle = \|\psi\|^2 - \langle A\psi, \psi \rangle.$$

Because of (1.4) we see that $\psi = 0$ and therefore $v = 0$. □

The operation of taking adjoints is preserved when constructing the associated operator.

Proposition 1.2.3 *The operator associated with a^* is A^*. If a is symmetric then A is self-adjoint.*

Proof. The second part of the assertion is an obvious consequence of the first one. To prove the first assertion, use $\tilde{A}$ to denote the operator associated with a^* and pick $u \in \mathcal{D}(\tilde{A})$ arbitrarily. We have for all $\phi \in \mathcal{D}(A) \subset \mathcal{D}(a^*) = \mathcal{D}(a)$,

$$\langle -\tilde{A}u, \phi \rangle = a^*(u, \phi) = \overline{a(\phi, u)} = \overline{\langle -A\phi, u \rangle}.$$

This shows that $u \in \mathcal{D}(A^*)$ and $A^*u = \tilde{A}u$.
Choose now $v \in \mathcal{D}(A^*)$, we show that $v \in \mathcal{D}(\tilde{A})$. Proposition 1.2.1 yields the existence of $\psi \in \mathcal{D}(\tilde{A})$ fulfilling $(I - A^*)v = (I - \tilde{A})\psi$. Hence by what we have already proved

$$(I - A^*)v = (I - A^*)\psi$$

and

$$\langle v - \psi, (I - A)u \rangle = \langle (I - A^*)(v - \psi), u \rangle = 0$$

for all $u \in \mathcal{D}(A)$. The invertibility of $I - A$ implies now $v = \psi \in \mathcal{D}(\tilde{A})$. □

We define *a semigroup* on a Banach space X to be a family $\{T(t)\}_{t \geq 0}$ of bounded linear operators on X which satisfy

(i) $T(0) = 1$

(ii) $T(s)T(t) = T(s + t) \quad 0 \leq s, t < \infty$.

We say that a semigroup is *strongly continuous* if in addition the following condition holds.

(iii) $\lim_{t \to 0} T(t)u = u$ for any $u \in X$.

Using the semigroup property it is easy to check that the last condition - strong continuity at $t_0 = 0$ is equivalent to the strong continuity at any $t_0 \in \mathbb{R}_+$ i.e.

(iii') $\lim_{t \to 0} T(t_0 + t)u = T(t_0)u$ for any $u \in X$ and any $t_0 \in \mathbb{R}_+$.

The *generator* of a strongly continuous semigroup is the operator A defined by

$$\mathcal{D}(A) := \left\{ u \in X : \lim_{t \to 0} \frac{T(t)u - u}{t} \text{ exists} \right\}$$

$$Au := \lim_{t \to 0} \frac{T(t)u - u}{t} \quad \text{for all } u \in \mathcal{D}(A).$$

Lemma 1.2.4 *The subspace $\mathcal{D}(A)$ is dense in X and invariant under $T(t)$, i.e.*

$$T(t)\mathcal{D}(A) \subset \mathcal{D}(A) \quad \forall\, t \geq 0.$$

The following basic commutativity property holds true:

$$T(t)Au = AT(t)u, \quad \forall\, u \in \mathcal{D}(A), \quad \forall\, t \geq 0.$$

Proof. For $u \in X$ we set $u(t) := \int_0^t T(t)u\,dt$. We obtain

$$\begin{aligned}\lim_{h\to 0}\frac{T(h)u(t)-u(t)}{h} &= \lim_{h\to 0}\frac{\int_h^{t+h}T(s)u\,ds-\int_0^t T(s)u\,ds}{h}\\ &= \lim_{h\to 0}\frac{\int_t^{t+h}T(s)u\,ds-\int_0^h T(s)u\,ds}{h} = T(t)u-u.\end{aligned}$$

We conclude that $u(t) \in \mathcal{D}(A)$ and

$$Au(t) = T(t)u - u.$$

The density of $\mathcal{D}(A)$ in X follows, since $\frac{u(t)}{t} \to u$ in norm as $t \to 0$.
For $u \in \mathcal{D}(A)$ and $t \geq 0$ we have

$$\lim_{h\to 0}\frac{T(h)-I}{h}T(t)u = \lim_{h\to 0}T(t)\frac{(T(h)-I)f}{h} = T(t)Au.$$

Therefore $T(t)u \in \mathcal{D}(A)$ and $T(t)Au = AT(t)u$. □

Lemma 1.2.5 *For any $u \in \mathcal{D}(A)$ one has*

$$T(t)u - u = \int_0^t T(s)Au\,ds.$$

Proof. For $u \in \mathcal{D}(A)$ and $\phi \in X^*$ we define $h(t) : \mathbb{R}_+ \to \mathbb{C}$ by

$$h(t) := \left(T(t)u - u - \int_0^t T(s)Au\,ds, \phi\right).$$

We have (denoting by D^+ the right hand derivative)

$$D^+h(t) = (AT(t)u - T(t)Au, \phi) = 0.$$

We conclude that $h \equiv 0$ since h is continuous and $h(0) = 0$.
Since ϕ in X^* was arbitrary, the Hahn-Banach theorem concludes the proof. □

We now come to the relation of semigroups and Cauchy problems.

Lemma 1.2.6 *For any $u \in \mathcal{D}(A)$ the function $u(t) := T(t)u$ is continuously differentiable on $\mathbb{R}_+$ and*

$$u'(t) = Au(t).$$

Proof. We recall that we already proved the right-differentiability in Lemma 1.2.4. We compute the left derivative for $t > 0$ using the result of the last Lemma,

$$D^-T(t)u = \lim_{h\to 0}\frac{T(t)u - T(t-h)u}{h} = \lim_{h\to 0}\frac{\int_{t-h}^t T(s)Au\,ds}{h} = T(t)Au = AT(t)u.$$

□

Henceforth, armed with the result of the last lemma, we use the notation e^{At} for the semigroup $T(t)$ generated by A. We state the classical generation theorem due to Hille and Yosida.

Theorem 1.2.7 *Let A be a closed densely defined linear operator on a Banach space X and let $\omega \in \mathbb{R}$. The operator A is the generator of a strongly continuous semigroup $T(t)$ satisfying $\|T(t)\| \leq ce^{\omega t}$ for all $t \geq 0$ if and only if all $\lambda > \omega$ lie in the resolvent set of A and we have the estimate*

$$(\lambda - \omega)^m \|R(\lambda, A)^m\| \leq c \qquad \forall\, \lambda > \omega, \qquad m = 1, 2, \ldots\,. \tag{1.5}$$

Proof. See e.g. [22]. □

Combining Proposition 1.2.1 and the Hille-Yosida theorem 1.2.7 we see that the operator A associated to a densely defined, accretive, continuous and closed sesquilinear form on a Hilbert space H is the generator of a contractive C_0-semigroup on H. This semigroup is actually always holomorphic as we state in the next section.

1.3 Holomorphic semigroups

In this section we introduce semigroups which may be defined for a complex parameter z in a sector of the complex plane. Although not every C_0-semigroup can be continued holomorphically outside $\mathbb{R}_+$, there is a substantial class of semigroups (i.e. semigroups associated to forms) which may be extended in such a way. The reward for the additional assumption of holomorphy are nice regularity properties of such semigroups.

Definition 1.3.1 *Let $\theta \in (0, \frac{\pi}{2}]$, $z \in \Sigma_\theta$ and X be a Banach space. We say that a family of bounded operators $T(z) \in \mathcal{L}(X)$ forms a* bounded holomorphic semigroup[1] *if the following conditions are satisfied.*

(i) $T(z)$ is a holomorphic function of $z \in \Sigma_\theta$.

(ii) $T(z_1)T(z_2) = T(z_1 + z_2)$ for any $z_1, z_2 \in \Sigma_\theta$.

(iii) For any $\theta_1 < \theta$ and any $u \in X$ we have

$$\lim_{z \to 0,\, z \in \Sigma_{\theta_1}} T(z)u = u.$$

(iv) For any $\theta_1 < \theta$ there exists M_{θ_1} such that

$$\|T(z)\| \leq M_{\theta_1} \qquad \forall\, z \in \Sigma_{\theta_1}.$$

Any angle θ satisfying the conditions $(i) - (iv)$ will be called an angle of analyticity *of $T(z)$.*

The generator of $T(z)$ is then defined as the generator of the (restricted) semigroup $T(z)|_{z\in\mathbb{R}}$.

We also say that a closed linear operator A generates a *holomorphic* semigroup (not necessarily bounded) if there is $w \in \mathbb{R}$ such that $A - w$ generates a bounded holomorphic semigroup.

We have the following regularity result.

Proposition 1.3.2 *Let $T(z)$ be a bounded analytic semigroup on a Banach space X and denote by A its generator. Then for all $u \in X$ and $z \in \Sigma_\theta$ we have*

$$T(z)u \in \mathcal{D}(A).$$

Proof. We assume first that $z := t > 0$. Then we have

$$\lim_{s\to 0} \frac{1}{s}[T(s)T(t)u - T(t)u] = \lim_{s\to 0} \frac{1}{s}[T(t+s)u - T(t)u] = T'(t)u$$

[1] Equivalently we use the term: *bounded analytic semigroup*.

since $T(t)$ is differentiable. This shows that

$$T(t)u \in \mathcal{D}(A) \qquad \text{and} \qquad AT(t)u = T'(t)u.$$

This is the assertion of the proposition in the case $z \in \mathbb{R}$.
For a general complex z we use the result above together with the fact that for any $\theta_1 < \theta$ the operator $e^{i\theta_1}A$ is the generator of the semigroup $S(s)$ defined by $S(s) := T(e^{i\theta_1}s)$. The last fact is easy to check, see e.g. [22], Theorem 2.33. □

We quote the following important characterization which identifies generators of analytic semigroups with sectorial operators.

Theorem 1.3.3 *Let X be a Banach space. If $T(z)$ is a bounded analytic semigroup on X with the generator A then*

$$\sigma(A) \subset \left\{ z : \, |\arg z| \geq \theta + \frac{\pi}{2} \right\}$$

where θ is an angle of analyticity of $T(z)$. One has the estimate

$$|\lambda R(\lambda, A)| \leq M_{\theta_1} \qquad \forall\, \lambda \in \Sigma_{\theta_1 + \frac{\pi}{2}}$$

for any $\theta_1 < \theta$.
On the other hand, if A is a closed densely defined operator on X with $\sigma(A) \subset \left\{ z : \, |\arg z| \geq \frac{\pi}{2} + \theta \right\}$ for some $\theta \in (0, \frac{\pi}{2}]$ and fulfilling the estimate

$$|\lambda R(\lambda, A)| \leq M_{\theta_1} \qquad \forall\, \lambda \in \Sigma_{\theta_1 + \frac{\pi}{2}}$$

for any $\theta_1 < \theta$, then A is the generator of a bounded analytic semigroup.

Finally we state the theorem showing that holomorphic semigroups appear naturally; namely any semigroup associated to a sesquilinear form on a Hilbert space H is automatically holomorphic.

Theorem 1.3.4 *Let a be a densely defined accretive closed and continuous sesquilinear form on a Hilbert space H. Denote by A the associated operator. Then A generates a holomorphic semigroup on the sector Σ_θ where $\theta = \frac{\pi}{2} - \arctan M$ and M is the constant from the continuity assumption (1.1).*

1.4 Beurling-Deny conditions

In this section we assume that η is a $\sigma-$finite Borel measure on an open set $\Omega \subset \mathbb{R}^N$ and consider the Hilbert space $L^2(\Omega, \eta)$. Our aim is to give a characterisation of positive $L^\infty-$contractive semigroups in terms of the associated sesquilinear form. We start by giving criteria for a given closed convex set of H to be left invariant by a semigroup acting on H. We state these criteria in an abstract setting of a general Hilbert space. The precise assumptions follow.
Let H be a Hilbert space and a be a densely defined, accretive, continuous and closed sesquilinear form on H giving rise to an operator A and a semigroup e^{At}. Let C be a non-empty closed convex subset of H and P be the orthogonal projection on C, i.e. P is the mapping $P : H \to C$ satisfying

$$\|u - Pu\| = \min v \in C \, \|u - v\| \, .$$

The projection P may also be characterised in the following way. The element $w \in H$ satisfies $Pu = w$ for some $u \in H$ if and only if

$$w \in C \qquad \text{and} \qquad Re\, \langle u - w, v - w \rangle \leq 0 \text{ for all } v \in C.$$

The next lemma asserts that the invariance under the semigroup is equivalent to the invariance under the resolvent.

Lemma 1.4.1 *The following statements are equivalent:*

(i) $e^{At}C \subset C \qquad \forall\, t \geq 0.$

(ii) $\lambda R(\lambda, A)C \subset C \qquad \forall\, \lambda > 0.$

Proof. See [52], Proposition 2.1. □

Lemma 1.4.2 *Under the assumptions above the following statements are equivalent:*

(i) $e^{At}C \subset C \quad \forall\, t \geq 0.$

(ii) $P(\mathcal{D}(a)) \subset \mathcal{D}(a)$ *and* $Re\ a(Pu, u - Pu) \geq 0 \quad \forall\, u \in \mathcal{D}(a).$

(iii) $P(\mathcal{D}(a)) \subset \mathcal{D}(a)$ *and* $Re\ a(u, u - Pu) \geq 0 \quad \forall\, u \in \mathcal{D}(a).$

(iv) There exists a core $\mathcal{D}$ *of* a *such that* $P(\mathcal{D}) \subset \mathcal{D}(a)$ *and* $Re\ a(Pu, u - Pu) \geq 0$ *for all* $u \in \mathcal{D}$.

Proof. See [52], Theorem 2.2. □

For most of the applications in this work the following symmetric version of the last theorem suffices.

Theorem 1.4.3 *Let* a *be a densely defined, accretive, symmetric and closed sesquilinear form on a Hilbert space* H*. The following statements are equivalent.*

(i) $e^{At}C \subset C$ *for all* $t \geq 0$.

(ii) $P\mathcal{D}(a) \subset \mathcal{D}(a)$ *and* $a(Pu, Pu) \leq a(u, u)$ *for any* $u \in \mathcal{D}(a)$.

(iii) There exists a core $\mathcal{D}$ *of* a *such that* $P\mathcal{D} \subset \mathcal{D}(a)$ *and* $a(Pu, Pu) \leq a(u, u)$ *for all* $u \in \mathcal{D}$.

Proof. $(ii) \Rightarrow (i)$ By Cauchy-Schwarz,

$$Re\ a(u, u - Pu) = a(u, u) - Re\ a(u, Pu) \geq a(u, u) - \sqrt{a(u, u)}\sqrt{a(Pu, Pu)} \geq 0.$$

The assertion (i) follows from the last theorem.

$(i) \Rightarrow (ii)$ By the last theorem we may write for any $u \in C$,

$$\begin{aligned} a(Pu, Pu) &= Re\ a(Pu, Pu - u) + Re\ a(Pu, u) \\ &\leq Re\ a(Pu, u) \quad \leq \quad \sqrt{a(Pu, Pu)}\sqrt{a(u, u)}. \end{aligned}$$

This proves (ii).

$(iii) \Rightarrow (ii)$ Let $u_n \in \mathcal{D}$ converge to $u \in \mathcal{D}(a)$ in $\|\cdot\|_a$. For any $n \in \mathbb{N}$ we have

$$a(Pu_n, Pu_n) \leq a(u_n, u_n)$$

and therefore $(Pu_n)_{n \in \mathbb{N}}$ is a bounded sequence in $(\mathcal{D}(a), \|\cdot\|_a)$. Using the continuity of P and lemma A.1.1 it follows that $Pu \in \mathcal{D}(a)$ and

$$a(Pu, Pu) \leq \liminf_{n \to \infty} a(Pu_n, Pu_n) \leq \liminf_{n \to \infty} a(u_n, u_n) = a(u, u). \qquad \square$$

We apply the foregoing abstract theorems to obtain the Beurling-Deny criteria for positivity and $L^\infty-$contractivity. First we need some definitions.
Henceforth we consider the (complex) Hilbert space $H := L^2(\Omega, \eta)$ where η is a $\sigma-$finite Borel measure on an open set $\Omega \subset \mathbb{R}^N$. We set $H_{\mathbb{R}} := L^2(\Omega, \eta, \mathbb{R})$ and

$$H_+ := \{u \in H : u(x) \geq 0 \text{ for } \eta - \text{almost all } x \in \Omega\}.$$

Definition 1.4.4 *We say that a strongly continuous semigroup e^{At} on H is* real *if $e^{At}H_{\mathbb{R}} \subset H_{\mathbb{R}}$ for all $t \geq 0$. A C_0-semigroup e^{At} is called* positive *if $e^{At}H_+ \subset H_+$ for all $t \geq 0$. It is called $L^\infty-$*contractive *if*

$$\left\|e^{At}u\right\|_{L^\infty(\Omega,\eta)} \leq \|u\|_{L^\infty(\Omega,\eta)}$$

for any $u \in L^2(\Omega,\eta) \cap L^\infty(\Omega,\eta)$ and any $t \geq 0$.
If e^{At} is both positive and L^∞-contractive it is called submarkovian. *We call a C_0-semigroup* symmetric *if e^{At} is a self-adjoint operator for all $t > 0$.*

The function $\operatorname{sign} u$ is defined by

$$\operatorname{sign} u := \begin{cases} \frac{u(x)}{|u(x)|} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

The inequalities and equalities on H should always be understood in the η-almost everywhere sense although this is often suppressed in the notation.

Remark 1.4.5 *Any positive semigroup is in particular real.*

Proposition 1.4.6 *Let a be a densely defined, accretive, continuous and closed form on $H := L^2(\Omega,\eta)$. The following statements are equivalent.*

(i) The semigroup e^{At} is real.

(ii) $u \in \mathcal{D}(a)$ implies $Re\ u \in \mathcal{D}(a)$ and $a(Re\ u, Im\ u) \in \mathbb{R}$.

(iii) $u \in \mathcal{D}(a)$ implies $\overline{u} \in \mathcal{D}(a)$ and $a(u,v) \in \mathbb{R}$ for any $u, v \in \mathcal{D}(a) \cap H_R$.

(iv) There exists a core $\mathcal{D}$ of a such that $Re\ u \in \mathcal{D}(a)$ and $a(Re\ u, Im\ u) \in \mathbb{R}$ for all $u \in \mathcal{D}$.

Proof. We apply Theorem 1.4.2 for the closed convex set $H_{\mathbb{R}}$. The projection on this set is given by $Pu = Re\ u$. We obtain that (i) is equivalent to

$$u \in \mathcal{D}(a) \Longrightarrow Re\ u \in \mathcal{D}(a) \text{ and } Re\ a(Re\ u, iIm\ u) \geq 0.$$

Applying this inequality to $(-Re\ u + iIm\ u)$ yields $a(Re\ u, Im\ u) \in \mathbb{R}$. □

Theorem 1.4.7 *Let a be a densely defined, accretive, continuous and closed form on $H := L^2(\Omega,\eta)$. The following statements are equivalent.*

(i) The semigroup e^{At} is positive.

(ii) $u \in \mathcal{D}(a)$ implies $(Re\ u)^+ \in \mathcal{D}(a)$, $a(Re\ u, Im\ u) \in \mathbb{R}$ and $a((Re\ u)^+, (Re\ u)^-) \leq 0$.

(iii) The semigroup is real and $u^+ \in \mathcal{D}(a)$, $a(u^+, u^-) \leq 0$ for any $u \in \mathcal{D}(a) \cap H_{\mathbb{R}}$.

(iv) There exists a core $\mathcal{D}$ of a such that $(Re\ u)^+ \in \mathcal{D}(a)$, $a(Re\ u, Im\ u) \in \mathbb{R}$ and $a((Re\ u)^+, (Re\ u)^-) \leq 0$ for all $u \in \mathcal{D}$.

Proof. Theorem 1.4.2 yields the equivalence of (i), (ii) and (iv). The fact that (iii) is implied by any of the other conditions is obvious. Assume now that (iii) holds. Then by Proposition 1.4.6 for all $u \in \mathcal{D}(a)$ we have $Re\ u \in \mathcal{D}(a) \cap H_{\mathbb{R}}$ and $a(Re\ u, Im\ u) \in \mathbb{R}$. The conclusion (ii) follows on applying (iii) to $Re\ u$. □

Theorem 1.4.8 *Let a be a densely defined, accretive, continuous and closed form on $H := L^2(\Omega,\eta)$. The following statements are equivalent.*

(i) The semigroup e^{At} *is* L^∞*−contractive.*

(ii) $u \in \mathcal{D}(a)$ *implies* $(1 \wedge |u|)\operatorname{sign} u \in \mathcal{D}(a)$ *and* $Re\ a(u, (|u|-1)^+ \operatorname{sign} u) \geq 0$.

(iii) $u \in \mathcal{D}(a)$ *implies* $(1 \wedge |u|)\operatorname{sign} u \in \mathcal{D}(a)$ *and* $Re\ a((1 \wedge |u|)\operatorname{sign} u, (|u|-1)^+ \operatorname{sign} u) \geq 0$.

(iv) There exists a core $\mathcal{D}$ *of* a *such that* $(1 \wedge |u|)\operatorname{sign} u \in \mathcal{D}(a)$ *and* $Re\ a((1 \wedge |u|)\operatorname{sign} u, (|u|-1)^+ \operatorname{sign} u) \geq 0$ *for all* $u \in \mathcal{D}$.

Proof. We define a closed convex set in H by

$$C := \left\{u \in L^2(\Omega, \eta),\ |u| \leq 1\right\}.$$

The projection on C is given by $Pu = (1 \wedge |u|)\operatorname{sign} u$. We also have

$$u = (1 \wedge |u|)\operatorname{sign} u + (|u|-1)^+ \operatorname{sign} u.$$

The theorem now follows directly from Theorem 1.4.2. □

The assumption of accretivity in the last theorem is superfluous for the equivalence of (i) and (iii).

Theorem 1.4.9 *Let* a *be a densely defined form on* $H := L^2(\Omega, \eta)$ *such that for some constant* $\omega \in \mathbb{R}$ *the form*

$$(a + \omega) := a(u, v) + \omega\langle u, v\rangle \qquad u, v \in \mathcal{D}(a)$$

is accretive, continuous and closed. Then the following statements are equivalent.

(i) The semigroup e^{At} *is* L^∞*−contractive.*

(ii) $u \in \mathcal{D}(a)$ *implies* $(1 \wedge |u|)\operatorname{sign} u \in \mathcal{D}(a)$ *and* $Re\ a((1 \wedge |u|)\operatorname{sign} u, (|u|-1)^+ \operatorname{sign} u) \geq 0$.

Proof. We may assume $\omega > 0$, otherwise there is nothing new to prove.
$(i) \Rightarrow (ii)$ Assumption (i) implies that also the semigroup $e^{A-\omega}$ is L^∞−contractive. Thus

$$(1 \wedge |u|)\operatorname{sign} u \in \mathcal{D}(a) \qquad \text{for all } u \in \mathcal{D}(a).$$

The fact that $|e^{At}((1 \wedge |u|)\operatorname{sign} u)| \leq 1$ and Lemma A.1.2 from Appendix imply

$$\begin{aligned}
&Re\ a((1 \wedge |u|)\operatorname{sign} u, (|u|-1)^+ \operatorname{sign} u) = \\
&= \lim_{t\to 0} \frac{1}{t} Re \int_\Omega \left[(1 \wedge |u|)\operatorname{sign} u - e^{At}(1 \wedge |u|)\operatorname{sign} u\right](|u|-1)^+ \operatorname{sign} \overline{u}\, d\eta = \\
&= \lim_{t\to 0} \frac{1}{t} Re \int_\Omega (|u|-1)^+ \left[1 - \operatorname{sign}(\overline{u}) e^{At}((1 \wedge |u|)\operatorname{sign} u)\right] d\eta \geq 0.
\end{aligned}$$

$(ii) \Rightarrow (i)$ Choose $u \in L^2(\Omega, \eta)$ with $|u| \leq 1$ and define

$$\phi(t) = \frac{1}{2} \int_\Omega \left[(|e^{At}u| - 1)^+\right]^2 d\eta.$$

For any $t > 0$ we have (by the holomorphy of e^{At}, Lemma 1.2.6 and Lemma 2.1.7 in the next section)

$$\frac{d}{dt}|e^{At}u| = Re\ (\operatorname{sign}(\overline{e^{At}u}) A e^{At}u).$$

Thus for any $t > 0$,

$$\begin{aligned}\phi'(t) &= Re \int_\Omega Ae^{At}u(|e^{At}u| - 1)^+ \operatorname{sign}(\overline{e^{At}u})\, d\eta \\ &= -Re\, a(e^{At}u, (|e^{At}u| - 1)^+ \operatorname{sign} e^{At}u\end{aligned}$$

Here we used (ii) and also that $e^{At}u \in \mathcal{D}(a)$ for any $t > 0$. Since

$$e^{At}u = (1 \wedge |e^{At}u|) \operatorname{sign}(e^{At}u) + (|e^{At}u)| - 1)^+ \operatorname{sign}(e^{At}u)$$

we obtain (using (ii) again)

$$\phi'(t) \leq -Re\, a((|e^{At}u)| - 1)^+ \operatorname{sign}(e^{At}u), (|e^{At}u)| - 1)^+ \operatorname{sign}(e^{At}u)).$$

The accretivity of $a + \omega$ implies now

$$\phi'(t) \leq 2\omega\phi'(t) \qquad \forall\, t > 0.$$

But $\lim_{t\to 0} \phi(t) = 0$ and thus $\phi(t) = 0$ for any $t > 0$. This means that $|e^{At}u| \leq 1$ and since $u \in H$ was arbitrary subject only to $|u| \leq 1$ the conclusion (i) follows. □

We may simplify the criteria given in the foregoing theorems if the considered sesquilinear form is symmetric. We start with the positivity.

Theorem 1.4.10 *Let a be a densely defined, accretive, symmetric and closed form on $H := L^2(\Omega, \eta)$. Then the following statements are equivalent.*

(i) *The semigroup e^{At} is positive.*

(ii) *$u \in \mathcal{D}(a)$ implies $(Re\ u)^+ \in \mathcal{D}(a)$ and $a((Re\ u)^+, (Re\ u)^+) \leq a(u, u)$.*

(iii) *The semigroup is real and for any $u \in \mathcal{D}(a) \cap H_\mathbb{R}$ we have $|u| \in \mathcal{D}(a)$ and $a(|u|, |u|) \leq a(u, u)$.*

(iv) *There exists a core $\mathcal{D}$ of a such that $(Re\ u)^+ \in \mathcal{D}(a)$ and $a((Re\ u)^+, (Re\ u)^+) \leq a(u, u)$ for all $u \in \mathcal{D}$.*

Proof. The equivalence of (i), (ii) and (iv) follows directly from Theorem 1.4.3 on setting $C := H^+$ and $Pu = (Re\ u)^+$.
For any $u \in \mathcal{D}(a) \cap H_\mathbb{R}$ we use the decompositions $u = u^+ - u^-$ and $|u| = u^+ + u^-$. Then $u^+ \in \mathcal{D}(a)$ for all $u \in \mathcal{D}(a) \cap H_\mathbb{R}$ if and only if $|u| \in \mathcal{D}(a)$ for all $u \in \mathcal{D}(a) \cap H_\mathbb{R}$. The equivalence of (i) and (iii) follows now from Theorem 1.4.7 and the fact that $a(u^+, u^-) \leq 0$ is equivalent to $a(|u|, |u|) \leq a(u, u)$. □

Next we formulate the characterisation of L^∞-contractivity for symmetric forms.

Theorem 1.4.11 *Let a be a densely defined, accretive, symmetric and closed form on $H := L^2(\Omega, \eta)$. The following statements are equivalent.*

(i) *The semigroup e^{At} is $L^\infty-$contractive.*

(ii) *$u \in \mathcal{D}(a)$ implies $(1 \wedge |u|) \operatorname{sign} u \in \mathcal{D}(a)$ and $a((1 \wedge |u|) \operatorname{sign} u, (1 \wedge |u|) \operatorname{sign} u) \leq a(u, u)$.*

(iv) *There exists a core $\mathcal{D}$ of a such that $(1 \wedge |u|) \operatorname{sign} u \in \mathcal{D}(a)$ and $a((1 \wedge |u|) \operatorname{sign} u, (1 \wedge |u|) \operatorname{sign} u) \leq a(u, u)$ for all $u \in \mathcal{D}$.*

Proof. This follows directly from Theorem 1.4.2 on choosing C and P as in the proof of Theorem 1.4.8. □

Next we give a characterisation of submarkovian semigroups in terms of the behaviour of the sesquilinear form under normal contractions. We say that a function $c : \mathbb{C} \to \mathbb{C}$ is a *normal contraction* if $p(0) = 0$ and

$$|p(x) - p(y)| \leq |x - y| \qquad \text{for all } x, y \in \mathbb{C}.$$

For any $u \in H = L^2(\Omega, \eta)$ we define[2] $p(u)$ by $p(u)(x) := p(u(x))$. Then we have

Theorem 1.4.12 *Suppose that a is densely defined, accretive, closed and symmetric form on $H = L^2(\Omega, \eta)$. Then the following statements are equivalent.*

(i) The semigroup e^{At} is submarkovian.

(ii) For every normal contraction p and any $u \in \mathcal{D}(a)$ we have $p(\mathcal{D}(a)) \subset \mathcal{D}(a)$ and $a(p(u), p(u)) \leq a(u, u)$.

Proof. *(ii)* ⇒ *(i)* This follows from Theorems 1.4.10 and 1.4.11 by using (ii) for $p(z) := (Re\ z)^+$ and $p(u) := (1 \wedge |z|)\,\mathrm{sign}\, z$ consecutively.

(i) ⇒ *(ii)* By Lemma A.1.2 in the Appendix it suffices to prove

$$\langle p(u) - e^{At} p(u), p(u)\rangle \leq \langle u - e^{At} u, u\rangle \qquad \forall\, t > 0, \quad \forall\, u \in H.$$

First we verify this inequality for simple functions of the form $u = \sum_{n=1}^{n} \alpha_i \chi_{A_i}$ where $A_i \subset \Omega$ are measurable and disjoint with $\eta(A_i) < \infty$ for all i. By sesquilinearity this reduces to proving

$$\sum_{i,j=1}^{n} \langle \chi_{A_i} - e^{At}\chi_{A_i}, \chi_{A_j}\rangle p(\alpha_i)\overline{p(\alpha_j)} \leq \sum_{i,j=1}^{n} \langle \chi_{A_i} - e^{At}\chi_{A_i}, \chi_{A_j}\rangle \alpha_i \overline{\alpha_j}. \tag{1.6}$$

Define

$$b_{ij} := \langle \chi_{A_i} - e^{At}\chi_{A_i}, \chi_{A_j}\rangle, \quad \lambda_i := \langle \chi_{A_i}, \chi_{A_i}\rangle, \quad a_{ij} := \langle e^{At}\chi_{A_i}, \chi_{A_j}\rangle.$$

Using the self-adjointness of e^{At} we have

$$\sum_{i,j=1}^{n} b_{ij} p(\alpha_i)\overline{p(\alpha_j)} = \sum_{i<j} a_{ij} |p(\alpha_i) - p(\alpha_j)|^2 + \sum_{j} \left[\lambda_j - \sum_{i} a_{ij}\right] |p(\alpha_j)|^2. \tag{1.7}$$

Since we assume that the semigroup e^{At} is submarkovian we have $a_{ij} \geq 0$ and $\sum_{i=1}^{n} a_{ij} \leq \lambda_j$. Now the inequality (1.6) follows from (1.7) and the definition of a normal contraction. □

One of the motivating reasons to study whether a given symmetric[3] semigroup on $L^2(\Omega, \eta)$ is submarkovian is the possibility of extrapolating such semigroups to all $L^p(\Omega, \eta)$, $1 \leq p \leq \infty$.

Theorem 1.4.13 *Let e^{At} be a submarkovian symmetric semigroup on $L^2(\Omega, \eta)$. Then e^{At} may be extended from $L^1(\Omega, \eta) \cap L^\infty(\Omega, \eta)$ to a positive contraction semigroup $e^{A_p t}$ on all $L^p(\Omega, \eta)$, $1 \leq p \leq \infty$. The semigroups are consistent i.e. for all $f \in L^p(\Omega, \eta) \cap L^q(\Omega, \eta)$, $1 \leq p$, $q \leq \infty$ we have*

$$e^{A_p t} f = e^{A_q t} f.$$

[2] with a slight abuse of notation

[3] Without the assumption of symmetry one is still able to extrapolate between $L^2(\Omega, \eta)$ and $L^\infty(\Omega, \eta)$.

For $1 \le p < \infty$ the semigroup $e^{A_p t}$ is strongly continuous and we also have

$$(e^{A_p t})^* = e^{A_q t}$$

where $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 1.4.14 *We should comment on how the notation $e^{A_\infty t}$ is to be interpreted. Since the semigroup $T_\infty(t)$ on $L^\infty(\Omega, \eta)$ is often not strongly continuous we have to explain how A_∞ is defined. One possible way is to define it by the relation*

$$(\lambda - A_\infty)^{-1} = R(\lambda, A_1)^* \qquad \forall \lambda > 0.$$

Note though that usually $\mathcal{D}(A_\infty)$ is not dense in $L^\infty(\Omega, \eta)$.

Proof of Theorem 1.4.13. We have for any $u \in L^1(\Omega, \eta) \cap L^\infty(\Omega, \eta) \subset L^2(\Omega, \eta) \cap L^\infty(\Omega, \eta)$ and any $t \ge 0$

$$\left\| e^{At} u \right\|_{L^\infty(\Omega,\eta)} \le \|u\|_{L^\infty(\Omega,\eta)}.$$

Hence for any $u, v \in L^1(\Omega, \eta) \cap L^\infty(\Omega, \eta)$,

$$|\langle e^{At} u, v\rangle| = |\langle u, e^{At} v\rangle| \le \|u\|_{L^1(\Omega,\eta)} \left\| e^{At} v \right\|_{L^\infty(\Omega,\eta)} \le \|f\|_{L^1(\Omega,\eta)} \|u\|_{L^\infty(\Omega,\eta)}.$$

A proper choice of v yields

$$\left\| e^{At} u \right\|_{L^1(\Omega,\eta)} \le \|u\|_{L^1(\Omega,\eta)}.$$

Interpolating between $L^1(\Omega, \eta)$ and $L^\infty(\Omega, \eta)$ by using the Riesz-Thorin theorem (Theorem 1.6.8) we obtain a contraction $T_p(t)$ on every $L^p(\Omega, \eta)$, $1 \le p \le \infty$. By density arguments the semigroup property extends to all elements of $L^p(\Omega, \eta)$, $1 \le p \le \infty$ and thus we write $e^{A_p t} := T_p(t)$. We now show that $T_1(t)$ is strongly continuous on $L^1(\Omega, \eta)$. We start by showing the strong continuity for functions of the form $u := \chi_\omega$ where $\omega \subset \Omega$ is a measurable subset with $\eta(\omega) < \infty$. The general case will then follow from the boundedness of $T_1(t)$ on $L^1(\Omega, \eta)$. We write by using Hölder's inequality

$$\left| \int_\omega |e^{At} \chi_\omega| \, d\eta - \int_\omega \chi_\omega \, d\eta \right| \le \sqrt{\eta(\omega)} \left\| e^{At} \chi_\omega - \chi_\omega \right\|_{L^2(\Omega,\eta)}.$$

By the strong continuity of e^{At} on $L^2(\Omega, \eta)$ the last inequality yields

$$\lim_{t \to \infty} \left[\int_\omega |e^{At} \chi_\omega| \, d\eta - \int_\omega \chi_\omega \, d\eta \right] = 0.$$

On denoting by ω^c the complement of ω in Ω we also have

$$\int_{\omega^c} |e^{At} \chi_\omega| \, d\eta + \int_\omega e^{At} \chi_\omega \, d\eta = \left\| e^{At} \chi_\omega \right\|_{L^1(\Omega,\eta)} \le \|\chi_\omega\|_{L^1(\Omega,\eta)}$$

and hence

$$\lim_{t \to 0} \int_{\omega^c} |e^{At} \chi_\omega| \, d\eta = 0.$$

We may now estimate

$$\begin{aligned} \left\| e^{At} \chi_\omega - \chi_\omega \right\|_{L^1(\Omega,\eta)} &= \int_\omega |e^{At} \chi_\omega - \chi_\omega| \, d\eta + \int_{\omega^c} |e^{At} \chi_\omega| \, d\eta \\ &\le \sqrt{\eta(\omega)} \left\| e^{At} \chi_\omega - \chi_\omega \right\|_{L^2(\Omega,\eta)} + \int_{\omega^c} |e^{At} \chi_\omega| \, d\eta. \end{aligned}$$

By the strong continuity of e^{At} on $L^2(\Omega,\eta)$ we obtain that

$$\lim_{t\to 0}\left\|e^{At}\chi_\omega-\chi_\omega\right\|_{L^1(\Omega,\eta)}=0. \tag{1.8}$$

An easy density argument using the linearity of $e^{A_1 t}$ and the density of simple functions in $L^1(\Omega,\eta)$ concludes the proof. □

In the following theorem we show that for $1<p<\infty$ the extrapolated semigroups are actually holomorphic on $L^p(\Omega,d\eta)$ and give an angle of holomorphy. This angle may be further refined as can be seen from the Remark 1.4.17 below.

Theorem 1.4.15 *Let e^{At} be a symmetric submarkovian semigroup on $L^2(\Omega,d\eta)$ and let $1<p<\infty$. Then the extrapolated semigroup on $L^p(\Omega,d\eta)$ (given by Theorem 1.4.13) is bounded holomorphic with angle*

$$\theta_p\geq\frac{\pi}{2}\left(1-\left|\frac{2}{p}-1\right|\right). \tag{1.9}$$

Proof. We will make use of the spectral theorem (Theorem 1.8.1). Pick $f\in L^1(\Omega,d\eta)\cap L^\infty(\Omega,d\eta)$ and $g\in L^2(\Omega,d\eta)\cap L^\infty(\Omega,d\eta)$ arbitrarily and fix $r>0$ and $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$. On the strip $\{z: 0\leq Re\ z\leq 1\}$ consider the function $s(z):=re^{i\theta z}$ and define the operator $S(z)$ by

$$\langle S(z)f,g\rangle=\langle e^{As(z)}f,g\rangle. \tag{1.10}$$

This is made possible by Theorem 1.8.1. By the same theorem we see that the left-hand side of (1.10) is bounded on the strip. Also

$$|\langle S(z)f,g\rangle|\leq\begin{cases}\|f\|_{L^1(\Omega,d\eta)}\|g\|_{L^\infty(\Omega,dx)} & \text{when } Re\ z=0\\ \|f\|_{L^2(\Omega,d\eta)}\|g\|_{L^2(\Omega,dx)} & \text{when } Re\ z=1.\end{cases}$$

The Stein interpolation theorem (Theorem 1.6.7) yields now the bound

$$\|S(t)f\|_{p(t)}\leq\|f\|_{p(t)}$$

where $0<t<1$ and $p(t)$ is defined by

$$\frac{1}{p(t)}=1-t+\frac{t}{2}.$$

This is equivalent to the bound

$$\left\|e^{Aw}f\right\|_{L^p(\Omega,d\eta)}\leq\|f\|_{L^p(\Omega,d\eta)}$$

for complex w with $|\arg w|\leq\frac{\pi}{2}(2-\frac{2}{p})$. This is the statement of the theorem in case $1<p<2$.
The estimate (1.9) in the case $2<p<\infty$ follows from the case $1<p<2$ by duality. □

Remark 1.4.16 *The result (and the method) of the Theorem 1.4.15 holds true in a more general setting. If e^{At} is a bounded holomorphic semigroup on $L^2(\Omega,d\eta)$ and a bounded C_0-semigroup on some $L^{p_0}(\Omega,d\eta)$ with $p_0>2$, then the semigroup extrapolates to all $L^p(\Omega,d\eta)$, $2<p<p_0$ and is bounded holomorphic there.*

Remark 1.4.17 *In the setting of the Theorem 1.4.15, a stronger result may be proved. Namely the sector of holomorphy may be improved to Σ_{θ_0} with $\theta_0:=\frac{\pi}{2}-\arctan\frac{|p-2|}{2\sqrt{p-1}}$. This improvement is optimal as witnessed by the Ornstein-Uhlenbeck operator. This operator also serves as a counterexample to a possible generalisation of the extrapolation of holomorphy to $p=1$. The semigroup generated by the Ornstein-Uhlenbeck operator is not holomorphic on the weighted space $L^1(\mathbb{R}^N,e^{-\frac{x^2}{2}}dx)$.*

1.5 Irreducibility

As in the last section we assume that η is a $\sigma-$finite Borel measure on an open set $\Omega \subset \mathbb{R}^N$ and consider the Hilbert space $L^2(\Omega, \eta)$. The main goal of this section is to apply Theorem 1.4.2 to give conditions for irreducibility in terms of a form. First, though, we give the definition of irreducibility and an equivalent characterisation.

Definition 1.5.1 *We say that a strongly continuous semigroup e^{At} on $H := L^2(\Omega, \eta)$ is* irreducible *if the following condition holds true: if for some fixed $\omega \subset \Omega$ one has $e^{At}L^2(\omega, \eta) \subset L^2(\omega, \eta)$ for all $t > 0$ then either $\eta(\omega) = 0$ or $\eta(\omega^c) = 0$.*

Here we use the notation $\omega^c := \Omega \backslash \omega$. We also remind the reader that all the equalities and inequalities between $L^2(\Omega, \eta)$ functions are understood η almost everywhere.

Theorem 1.5.2 *Let a be a densely defined, accretive, continuous and closed form on $H := L^2(\Omega, \eta)$ such that the associated semigroup e^{At} is positive. Then the following statements are equivalent.*

(i) For any $0 \neq u \in H^+$ and any $t > 0$ we have $e^{At}u(x) > 0$ for η-almost all $x \in \Omega$.

(ii) The semigroup e^{At} is irreducible.

Proof. *(i)* $\Rightarrow$ *(ii)* Suppose that $\omega \subset \Omega$ is invariant under e^{At} for all $t > 0$. Then $e^{At}u = 0$ on ω^c for any $t > 0$ and any $u \in L^2(\omega, \eta)$. In case $\eta(\omega) > 0$ there exists a nontrivial $0 \neq u \in L^2(\omega, \eta)$ and the irreducibility implies now that $\eta(\omega^c) = 0$.
(ii) $\Rightarrow$ *(i)* Pick $0 \neq u \in H^+$, fix $s > 0$ and define

$$\omega := \left\{x \in \Omega :\, (e^{As}u)(x) = 0\right\}$$

Assume for contradiction that $\eta(\omega) > 0$. For any $\phi \in L^2(\omega, \eta)$ we have $\langle e^{As}u, \phi\rangle = 0$. The strong continuity of e^{At} yields the existence of a sequence $t_n \in (0, s)$ such that

$$\left\|e^{At_n}u - u\right\| \leq 2^{-n}.$$

Define $u_n := e^{At_n}u$ and $v_n := u - \sum_{k \geq n}(u - u_k)^+$. Then

$$v_n \leq v - (v - v_m)^+ = \inf\left\{u, u_m\right\} \leq u_m$$

for all $m \geq n$. We then have for all $0 \leq \phi \in L^2(\omega, \eta)$ and all $m \geq n$,

$$0 \leq \langle e^{A(s-t_m)}v_n^+, \phi\rangle \leq \langle e^{A(s-t_m)}u_m, \phi\rangle = \langle e^{As}u, \phi\rangle = 0.$$

It follows that $\langle e^{A(s-t_m)}v_n^+, \phi\rangle$ for all $m \geq n$. Since the semigroup e^{At} is holomorphic on H and s was arbitrary it follows that

$$\langle e^{At}v_n^+, \phi\rangle = 0 \qquad \text{for all } t \geq 0.$$

Taking the limit $n \to \infty$ we have for any $\phi \in L^2(\omega, \eta)$,

$$\langle e^{At}u, \phi\rangle = 0 \qquad \text{for all } t \geq 0. \tag{1.11}$$

Setting $w := e^{As}u$ the last equality yields

$$\langle e^{At}w, \phi\rangle = 0 \qquad \text{for all } t \geq 0. \tag{1.12}$$

Choose now any $0 \leq v \in L^2(\omega^c, \eta)$ and $\phi \in L^2(\omega, \eta)$. We decompose v as

$$v = (v - nw)^+ + \inf\left\{v, nw\right\}.$$

We have by (1.12),

$$|\langle e^{At}(\inf\{v, nw\}), \phi\rangle| \leq n\langle e^{At}w, |\phi|\rangle = 0.$$

Thus

$$\langle e^{At}v, \phi\rangle = \langle e^{At}(v - nw)^+, \phi\rangle. \tag{1.13}$$

Since $g \in L^2(\omega^c, \eta)$ and $\omega = \{w = 0\}$ we have $(v - nw)^+ \to 0$ in $L^2(\Omega, \eta)$ as $n \to \infty$. Hence by (1.13),

$$\langle e^{At}v, \phi\rangle = 0 \qquad \forall\, t \geq 0.$$

The last equation remains true for all $v \in L^2(\omega^c, \eta)$ and $\phi \in L^2(\omega, \eta)$. This shows that $e^{At}L^2(\omega^c, \eta) \subset L^2(\omega^c, \eta)$ for any $t \geq 0$. Since we assume that $\eta(\omega) > 0$ it follows from the assumption (ii) that $\eta(\omega^c) = 0$. By (1.11) for $t = 0$ we see that $u \equiv 0$, which is a contradiction. □

We apply Theorem 1.4.2 to obtain a characterisation of irreducibility in terms of the sesquilinear form associated to e^{At}.

Theorem 1.5.3 *Let a be a densely defined, accretive, continuous and closed form on $H := L^2(\Omega, \eta)$ such that the associated semigroup e^{At} is positive. Then the following statements are equivalent.*

(i) The semigroup e^{At} is irreducible.

(ii) If $\omega \subset \Omega$ fulfills $\chi_\omega u \in \mathcal{D}(a)$ and Re $a(\chi_\omega u, \chi_{\omega^c} u) \geq 0$ for all $u \in \mathcal{D}(a)$, then either $\eta(\omega) = 0$ or $\eta(\omega^c) = 0$.

(iii) If $\omega \subset \Omega$ is such that $\chi_\omega u \in \mathcal{D}(a)$ and Re $a(\chi_\omega u, \chi_{\omega^c} u) \geq 0$ for all u in a core $\mathcal{D}$ of a, then either $\eta(\omega) = 0$ or $\eta(\omega^c) = 0$.

Proof. We set $C := L^2(\omega, \eta)$. Then C is a closed convex subset of H and the projection on this subset is given by $Pu = \chi_\omega u$. The conclusion follows from Theorems 1.5.2 and 1.4.2. □

One of the important consequences of irreducibility is that it guarantees positivity of the first eigenfunction (if it exists). This will follow from Theorem 1.5.5 (see Section 6.2).

Until the end of this section we shall assume that we are given a consistent family of positive contractive semigroups e^{At} on $L^p(\Omega, \eta)$, $1 \leq p \leq \infty$, where η is a σ-finite Borel measure on $\Omega \subset \mathbb{R}^N$. We also assume e^{At} is strongly continuous when $1 \leq p < \infty$ and that e^{At} is *irreducible* on $L^2(\Omega, \eta)$. In our cases of interest, the irreducibility will follow from the assumption that Ω be connected.

We need a preparatory lemma. For the definition of a *lattice* and a *sublattice* see Appendix or [68].

Lemma 1.5.4 *The set $S := \{f \in \mathcal{D}(A_p),\ A_p f = 0\}$ is a closed linear sublattice of $L^p(\Omega, \eta)$ for any p, $1 \leq p < \infty$.*

Proof. We check first that $A_p f = 0$ if and only if $e^{A_p t} f = f$ for all $t \geq 0$. This follows easily from the definition of the generator of a semigroup and the fact that

$$\frac{d}{dt} e^{A_p t} f = A_p e^{A_p t} f = e^{A_p t} A_p f = 0$$

for $f \in S$. Therefore we obtain the equality $S = \bigcap_{t \geq 0} S_t$, where

$$S_t := \left\{ f \in L^p(\Omega, \eta) : e^{A_p t} f = f \right\}.$$

We show that S_t is a closed linear sublattice of $L^p(\Omega,\eta)$. For $f \in S_t$ we have

$$0 \le |f| = |e^{A_p t} f| \le e^{A_p t}|f|.$$

Assuming that $|f| \ne e^{A_p t}|f|$, we would have $\||f|\|_p < \left\|e^{A_p t}|f|\right\|_p$ which contradicts the contractivity of $e^{A_p t}$. □

Theorem 1.5.5 *For the set S defined in the foregoing lemma we always have*

$$\text{Dim } S \le 1. \tag{1.14}$$

If Dim $S = 1$ then there exists an $f \in S$ such that $f > 0$ a.e.

Proof. Assume that Dim $S > 0$ otherwise there is nothing to prove. We have proved above that S is a sublattice and hence there exists $0 \ne f \in S$ such that $f \ge 0$. For any $g \in L^p(\Omega,\eta)$ such that $|g| \le \alpha f$ for some $\alpha \ge 0$ we have

$$|e^{A_p t} g| \le e^{A_p t}|g| \le e^{A_p t}(\alpha f) = \alpha f.$$

Therefore on denoting

$$T := \{g \in L^p(\Omega,\eta) : \ |g| \le \alpha f \text{ for some } \alpha \ge 0\}$$

we have $e^{A_p t} T \subset T$ for all $t \ge 0$. It is also easy to verify that

$$\overline{T} = \{g \in L^p(\Omega,\eta), \ \text{ supp } g \subset \text{supp } f\}.$$

We have thus proved that $e^{A_p t}$ leaves $L^p(\text{supp } f, \eta)$ invariant. By irreducibility, it follows that supp $f = \Omega$, i.e. $f > 0$ a.e.
Now, if $f \in S$, then also f^+, $f^- \in S$ and by the argument above f^+ or f^- vanishes. If f and g are two positive elements of S, then for any $\lambda \in \mathbb{R}$, $f + \lambda g$ is either positive or negative. There must occur a sign change as λ increases and thus we find λ_0 so that $f + \lambda_0 g \equiv 0$. It follows that Dim $S = 1$. □

1.6 Interpolation theorems

In this section we formulate some of the fundamental interpolation theorems. The full beauty of the fruit they bear for us will be revealed only later.
We start by setting down the assumptions. Although many of the concepts introduced below can be investigated in greater generality[4] we tailor the assumptions to our needs. Thus we shall assume the following.
We suppose that (Ω, Σ, η) is a σ-finite measure space[5]. Consider an operator T defined on a linear space D of measurable functions on (Ω, Σ, η) with values in the space of all measurable functions on (Ω, Σ, η). We assume that D contains all finite linear combinations of characteristic functions of sets of finite measure. We also assume that if $f \in D$ then also all truncations[6] of f lie in D.

Definition 1.6.1 *Let $1 \le p, q \le \infty$. We say that an operator T satisfying the conditions above is of* type (p, q) *if there is a constant c such that*

$$\|Tf\|_{L^q(\Omega,\eta)} \le c\,\|f\|_{L^p(\Omega,\eta)} \tag{1.15}$$

for all $f \in D \cap L^p(\Omega,\eta)$. The infimum of all such constants c is then defined to be the (p, q)-norm *of T.*

[4]In particular, the most natural extension would be to consider the values of T as functions on a different measure space. We shall not need this in this work.

[5]In applications we have in mind, Ω will be a subset of $\mathbb{R}^N$ equipped with the Borel $\sigma-$algebra. In that case we shall suppress Σ in the notation.

[6]By a *truncation* of f we understand any of the functions depending on $r_1, r_2 \ge 0$ and defined by: $h(x) := f(x)$ if $r_1 < |f(x)| \le r_2$, and $h(x) := 0$ otherwise.

Operators of type $(p,\, q)$ always satisfy a weaker property defined in terms of the distribution function of Tf.

Definition 1.6.2 *For a measurable function g on (Ω, Σ, η) and $s > 0$ we define the* level set

$$F_s := F_{s,g} := \{x \in \Omega : \ |g(x)| > s\}$$

and the distribution function *of g by*

$$\lambda(s) := \lambda_g(s) := \eta(F_s) = \eta(x : \ |g(x)| > s).$$

Lemma 1.6.3 *Let $1 \le p \le \infty$ and $1 \le q < \infty$. An operator of type $(p,\, q)$ satisfies the inequality*

$$\lambda_{Tf}(s) \le \left(\frac{c\, \|f\|_p}{s} \right)^q .$$

where c is any constant fulfilling (1.15).

Proof. We denote by F_s the level set of Tf and by $\lambda(s)$ the distribution function of Tf. We have

$$s^q \lambda(s) = s^q \int_{F_s} d\eta \le \int_{F_s} |Tf|^q \, d\eta \le \|Tf\|_q^q \le (c\, \|f\|_p)^q.$$

□

Definition 1.6.4 *Let $1 \le p \le \infty$ and $1 \le q < \infty$. We say that T (satisfying the conditions before Definition 1.6.1) is of* weak type $(p,\, q)$ *if there is a constant c such that*

$$\lambda_{Tf} \le \left(\frac{c\, \|f\|_p}{s} \right)^q$$

for all $f \in D \cap L^p(\Omega, \eta)$. For $1 \le p \le \infty$ we say that T is of weak type (p, ∞) *if T is of type (p, ∞).*

Since the interpolation theorem we are about to state can be formulated for a more general class than linear operators (and this generalisation is indeed useful) we need one more definition.

Definition 1.6.5 *We say that T (satisfying the conditions given before Definition 1.6.1) is* subadditive *if*

$$|[T(f_1 + f_2)](x)| \le |(Tf_1)(x)| + |(Tf_2)(x)|$$

for almost all $x \in \Omega$ and all $f_1, f_2 \in D$.

We state the Marcinkiewicz interpolation theorem in a form most suitable for our purposes.

Theorem 1.6.6 *Let T be a subadditive operator of weak type (p_j, q_j), $j = 0, 1$ where $1 \le p_j, q_j \le \infty$, $p_0 \le p_1$, $q_0 \ne q_1$. Then T is also of type (p_t, q_t) where $0 < t < 1$ is arbitrary and p_t, q_t satisfy the relations:*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \text{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

The proof can be found e.g. in [StWe] where the Marcinkiewicz interpolation theorem is formulated in its most natural setting of general Lorentz spaces.

In the following we investigate the possibility of interpolating operators depending analytically on a complex parameter z. We assume that we are given a family of linear operators indexed by the complex numbers belonging to the strip $S := \{z \in \mathbb{C} : 0 \le Re\, z \le 1\}$. Each operator $T(z)$ is defined on the simple functions in $L^1(\Omega, \eta)$ and takes values in the space of all measurable functions on Ω. We suppose that $(T(z)f)g$ is integrable whenever f and g are simple functions in $L^1(\Omega, \eta)$ and that the mapping

$$z \to \int_\Omega (T(z)f)g\, d\eta$$

is analytic in the interior of S and continuous on S. Finally we assume that there is a constant $c_1 < \pi$ such that

$$e^{-c_1|y|} \log \left| \int_\Omega (T(z)f)g\, d\eta \right|, \qquad z = x + iy,$$

is uniformly bounded from above in S.
We are in a position to state the Stein interpolation theorem.

Theorem 1.6.7 *Let $1 \le p_j, q_j \le \infty$, $j = 0,1$ and assume that $T(z)$, $z \in S$, is a family of linear operator satisfying the assumptions above. Suppose further that there are real functions $M_j(y)$, $j = 0,1$ satisfying*

$$\sup_{y \in \mathbb{R}} e^{-c_2|y|} \log M_j(y) < \infty$$

for some $c_2 < \pi$ and such that

$$\|T(iy)f\|_{q_0} \le M_0(y)\, \|f\|_{p_0} \quad \text{and} \quad \|T(1+iy)f\|_{q_1} \le M_1(y)\, \|f\|_{p_1}$$

for all simple functions $f \in L^1(\Omega, \eta)$.
Then for each t, $0 \le t \le 1$, there is a constant M_t such that

$$\|T(t)f\|_{q_t} \le M_t\, \|f\|_{p_t}$$

for all simple functions $f \in L^1(\Omega, \eta)$. Here p_t and q_t are given by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

For the proof we refer the reader to the book [StWe].

In the connection with the Stein interpolation theorem we now state the Riesz-Thorin interpolation theorem. Although conceptually simpler, the Riesz-Thorin interpolation theorem suffices to handle a variety of (important) situations. In order to state the theorem we come back to the assumptions from the beginning of the section. Thus we assume (implicitly in the definition of the type) that T is defined at least on all finite linear combinations of characteristic functions of sets of positive measure and that the domain D of T contains all truncations of its members.

Theorem 1.6.8 *Assume that a linear operator T is of type (p_i, q_i) with its (p_i, q_i)-norm equal to M_i, $i = 0,1$. Then T is of type (p_t, q_t) for each $0 \le t \le 1$ and its (p_t, q_t)-norm is smaller or equal to $M_0^{(1-t)} M_1^t$. Here*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \text{and} \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

We finish this section with a result dealing with interpolation of compact operators.

Theorem 1.6.9 *Let $1 \leq p_0 < p < p_1 \leq \infty$ and let A be a linear operator defined on $L^{p_0}(\Omega,\eta) \cap L^{p_1}(\Omega,\eta)$ with values in $L^{p_0}(\Omega,\eta) \cap L^{p_1}(\Omega,\eta)$. Assume that A can be extended to a bounded operator on $L^{p_1}(\Omega,\eta)$ and to a compact operator on $L^{p_0}(\Omega,\eta)$. Then A can be extended to a compact operator on $L^{p}(\Omega,\eta)$. The spectrum of A is the same for all $p_0 \leq p < p_1$ and the spectral projections corresponding to non-zero eigenvalues are independent of p.*

1.7 Sobolev embeddings

We state here the Sobolev embedding theorems which are needed in this work. Since Sobolev embeddings form a topic of many a textbook, our presentation will be deliberately very brief. Throughout this thesis we assume the familiarity of the reader with weak derivatives.

Let $\Omega \subset \mathbb{R}^N$ be open and bounded. We define the Sobolev space $W^{1,p}(\Omega)$ for $1 \leq p \leq \infty$ as

$$W^{1,p}(\Omega) := \{f \in L^p(\Omega, dx) : \nabla f \in L^p(\Omega, dx)\}$$

where ∇f is understood in the weak sense. On $W^{1,p}(\Omega)$ we are given the norm

$$\|f\|_{W^{1,p}(\Omega)} := \left\{ \|f\|^p_{L^p(\Omega,dx)} + \|\nabla f\|^p_{L^p(\Omega,dx)} \right\}^{\frac{1}{p}}$$

and $W^{1,p(\Omega)}$ equipped with this norm becomes a Banach space.
The space $W_0^{1,p}(\Omega)$ is then defined as the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

Remark 1.7.1 *We also use the notation $H^1(\Omega) := W^{1,2}(\Omega)$ and $H_0^1(\Omega) := W_0^{1,2}(\Omega)$.*

We have the following embeddings.

Theorem 1.7.2 *Let $1 \leq p < N$ and*

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{N}.$$

Then $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ and there is a constant c such that

$$\|f\|_{L^q(\Omega,dx)} \leq c \|\nabla f\|_{L^p(\Omega,dx)}$$

for all $f \in W_0^{1,p}(\Omega)$. In addition, the embedding is compact in case

$$\frac{1}{q} > \frac{1}{p} - \frac{1}{N}.$$

Theorem 1.7.3 *Let $N < p \leq \infty$. Then $W_0^{1,p}(\Omega) \subset C_0(\overline{\Omega})$ and there is a constant c such that*

$$\|f\|_\infty \leq c \|\nabla f\|_{L^p(\Omega,dx)} \cdot \operatorname{diam}(\Omega)^{1-\frac{N}{p}}$$

for any $f \in W_0^{1,p}(\Omega)$. Furthermore, the embedding is compact.

Theorem 1.7.4 *Let $N = p$. Then there exist constants $c_1 > 0$ and c_2 (depending only on N and not on Ω) such that for any $u \in W_0^{1,p}(\Omega)$ we have*

$$\int_\Omega \exp\left[\left(c_1 \frac{|u|}{\|\nabla u\|_p}\right)^{\frac{N}{N-1}}\right] dx \leq c_2 |\Omega|.$$

In particular, $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for any $1 \leq q < \infty$.

1.8 Spectral theorem

We state here the spectral theorem in the form we shall use in this work. There are at least three common versions of the spectral theorem, each having its advantage. From the perspective of a mathematician, the following version seems to be the most illuminating. At the same time, this is the version we use in this thesis.

Theorem 1.8.1 *Let A be a self-adjoint operator on a Hilbert space H. Then there exists a measure space $(\Omega, d\eta)$ such that the space H can be unitarily identified with $L^2(\Omega, d\eta)$ and a real measurable function g on Ω such that*

$$\mathcal{D}(A) = \left\{ f : \int_\Omega [1 + g(y)]|f(y)|^2 \, dy < \infty \right\}$$

and

$$(Af)(y) = g(y)f(y)$$

for all $y \in \Omega$ and $f \in \mathcal{D}(A)$.

Remark 1.8.2 *The space Ω may be chosen of the form $\Omega := \sigma(A) \times \mathbb{N}$ where $\sigma(A)$ denotes the spectrum of A on H.*

As an application of the spectral theorem we state the following theorem which will be of much importance for our work in further sections.

Theorem 1.8.3 *Let H be a Hilbert space and A a self-adjoint operator on H such that the spectrum of A is contained in $[0, \infty)$. Let λ be a negative real number. The following statements are equivalent.*

(i) *The resolvent $R(\lambda, A)$ is compact.*

(ii) *The spectrum of A consists of countably many eigenvalues $\lambda_n \geq 0$ with finite multiplicity. The eigenvalues λ_n converge to $+\infty$ as $n \to \infty$. The corresponding eigenvectors may be chosen in the way that they form a complete orthonormal set in H.*

1.9 Regularized distance function

At various places throughout this thesis we shall use powers of the distance function, both as a technical tool and as a good source of examples in order to demonstrate the scope of our results. Let $\Omega \subsetneq \mathbb{R}^N$ be an open set and consider the distance function

$$d(x) := \operatorname{dist}(x, \Omega^c) = \inf_{y \in \Omega^c} |x - y|.$$

This function is Lipschitz (evidently) but, in general, not smoother as evidenced by an example of the interval $\Omega := (0, 1) \subset \mathbb{R}$. However for various purposes (e.g. to guarantee that it belongs to a domain of some operator) it is desirable to have a smooth function at hand. Thus we would like to find a smooth proxy for the distance function and it is this task which we set out to accomplish in this section.

We start describing the so called Whitney decomposition of an open set $\Omega \subsetneq \mathbb{R}^N$. Our approach will be based on exhausting Ω by closed cubes, alternatively we could work with balls. We denote by F the complement of Ω, i.e. $F := \mathbb{R}^N \backslash \Omega$.

Definition 1.9.1 *We say that two (closed) cubes*[7] *are* disjoint *if their interiors are disjoint. Also we say that two disjoint cubes* touch *if their boundaries have a point in common.*

[7] Whenever we write cubes, we mean closed cubes. Thus a cube C is defined uniquely by its centre $x = (x_1, \dots, x_N)$ and its side length $l > 0$ by $C := [x_1 - \frac{l}{2}, x_1 + \frac{l}{2}] \times \cdots \times [x_N - \frac{l}{2}, x_N + \frac{l}{2}]$.

Theorem 1.9.2 *There exists a family of mutually disjoint cubes $\mathcal{Q} = \{Q_1, Q_2, \dots\}$ satisfying*

(i) $\bigcup_{k=1}^{\infty} Q_k = \Omega$

(ii) $\operatorname{diam} Q_k \leq \operatorname{dist}(Q_k, F) \leq 4 \operatorname{diam} Q_k$

(iii) if $Q_i \in \mathcal{Q}$ and $Q_j \in \mathcal{Q}$ touch then

$$\frac{1}{4} \operatorname{diam} Q_i \leq \operatorname{diam} Q_j \leq 4 \operatorname{diam} Q_i$$

(iv) for any $Q \in \mathcal{Q}$ there are at most $C := 3 \cdot 12^N$ cubes in $\mathcal{Q}$ which touch Q.

Proof. Set

$$M_0 := \{Q : \exists x = (x_1, \dots, x_N),\, x_i \in \mathbb{Z}, \forall i \text{ s.t. } Q = [x_1, x_1 + 1] \times \cdots \times [x_N, x_N + 1]\}$$

In other words, M_0 is the collection of all cubes of unit length with integral coordinates of the vertices. The mesh M_0 will be a basic building block defining a collection of meshes $\{M_k\}_{k=-\infty}^{\infty}$, where $M_k := 2^{-k} M_0$. Obviously the mesh M_k contains cubes with side length 2^{-k} i.e. of diameter $\sqrt{N} 2^{-k}$. By bisecting the sides of a cube in M_k we obtain 2^N cubes belonging to M_{k+1}.
We decompose Ω to layers, i.e. we write $\Omega = \bigcup_{k=-\infty}^{\infty} \Omega_k$ with

$$\Omega_k := \left\{x : c2^{-k} \leq \operatorname{dist}(x, F) \leq c2^{-k+1}\right\}$$

where $c := 2\sqrt{N}$. The reason for this choice of c will be seen below (see (1.16) and (1.17)).
We now define a preliminary choice of the collection $\mathcal{Q}_0$. After eliminating redundant elements this will result in defining $\mathcal{Q}$. We set

$$\mathcal{Q}_0 := \bigcup_{k=-\infty}^{\infty} \{Q \in M_k : Q \cap \Omega_k \neq \emptyset\}.$$

Obviously, we have $\bigcup_{Q \in \mathcal{Q}_0} Q = \Omega$. We now check (ii). Pick $Q \in \mathcal{Q}_0$. Then $Q \in M_k$ for some k and hence $\operatorname{diam} Q = \sqrt{N} 2^{-k}$. Since $Q \cap \Omega_k \neq \emptyset$ there exists $x \in Q \cap \Omega_k$ and thus

$$\operatorname{dist}(Q, F) \leq \operatorname{dist}(x, F) \leq c2^{-k+1} = 2\sqrt{N} 2^{-k+1} = 4 \operatorname{diam} Q \tag{1.16}$$

and also

$$\operatorname{dist}(Q, F) \geq \operatorname{dist}(x, F) - \operatorname{diam} Q \geq 2\sqrt{N} 2^{-k} - \sqrt{N} 2^{-k} = \sqrt{N} 2^{-k} = \operatorname{diam} Q, \tag{1.17}$$

i.e. (ii) holds.
Thus we are left with the task of making the collection $\mathcal{Q}_0$ disjoint while still keeping enough elements to compose Ω. For any cube $Q \in \mathcal{Q}_0$ consider now its *maximal* cube $\widetilde{Q} \in \mathcal{Q}_0$. This is defined as the largest cube in $\mathcal{Q}_0$ containing Q. It is unique as the following reasoning shows. The inequality (ii) yields

$$\operatorname{diam} \widehat{Q} \leq 4 \operatorname{diam} Q \qquad \text{for any } \widehat{Q} \in \mathcal{Q}_0,\ Q \subset \widehat{Q}.$$

Also for any two cubes $\widehat{Q}_1, \widehat{Q}_2 \in \mathcal{Q}_0$ containing Q we have that[8] either $\widehat{Q}_1 \subset \widehat{Q}_2$ or $\widehat{Q}_2 \subset \widehat{Q}_1$. Thus given any $Q \in \mathcal{Q}_0$ there exists a unique maximal cube $\widetilde{Q} \in \mathcal{Q}_0$

[8] This follows from the general simple fact that if $Q_a \in M_k$, $Q_b \in M_l$ for $k \leq l$ and Q_a, Q_b are not disjoint then $Q_b \subset Q_a$.

containing Q. These maximal cubes are disjoint by the same argument as above. Thus setting

$$\mathcal{Q} := \left\{ \widetilde{Q} \in \mathcal{Q}_0,\ \widetilde{Q} \text{ is maximal to some } Q \in \mathcal{Q}_0 \right\}$$

and indexing

$$\mathcal{Q} = \{Q_1, Q_2, \dots\}$$

we see that the elements of $\mathcal{Q}$ are mutually disjoint and the properties (i) and (ii) are satisfied. In order to prove (iii) we note that as a consequence of (ii) and the assumption that Q_i and Q_j touch we may estimate

$$\operatorname{dist}(Q_j, F) \leq \operatorname{dist}(Q_i, F) + \operatorname{diam} Q_i \leq 5 \operatorname{diam} Q_i.$$

However, in general, we have $\operatorname{diam} Q_j = 2^k \operatorname{diam} Q_i$ for some $k \in \mathbb{Z}$ and thus

$$\operatorname{diam} Q_j \leq 4 \operatorname{diam} Q_i.$$

The full conclusion of (iii) is now obtained by symmetry.
If $Q \in \mathcal{Q}$ belongs to M_k then there are $3^N - 1$ cubes in M_k touching Q. Also any cube in this M_k can contain at most $(4^N + 2^N + 1)$ cubes of $\mathcal{Q}$ with diameter larger or equal than $\frac{1}{4} \operatorname{diam} Q$. These observations combined with (iii) prove (iv). □

Pick now any cube $Q_k \in \mathcal{Q}$ and denote by $x^{(k)}$ its centre and by l_k its side length. Evidently, $\operatorname{diam} Q_k = \sqrt{N} l_k$. Fix now $0 < \varepsilon < \frac{1}{4}$ once and for all and set

$$\widehat{Q}_k := (1 + \varepsilon) \left[Q_k - x^{(k)} \right] + x^{(k)}.$$

Thus $\widehat{Q}_k$ has the same centre as Q_k and is dilated by a factor $(1 + \varepsilon)$. We have

$$\Omega = \bigcup_{k=1}^{\infty} \widehat{Q}_k$$

and although the cubes $\widehat{Q}_k$, $k = 1, 2, \dots$ are not disjoint in general, we still have

Proposition 1.9.3 *Any $x \in \Omega$ is contained in at most $C = 3 \cdot 12^N$ of the cubes $\widehat{Q}_k$, $k = 1, 2, \dots$.*

Proof. Since any point $x \in \Omega$ is contained in some $Q \in \mathcal{Q}$ the assertion will follow from the Theorem 1.9.2 part (iv) once we show that for any $k \in \mathbb{N}$, $\widehat{Q}_k$ intersects Q only if Q_k touches Q.
In fact, fix $k \in \mathbb{N}$ and consider the union of Q_k with all the cubes in $\mathcal{Q}$ that touch Q_k. The diameters of all these cubes are larger or equal than $\frac{1}{4} \operatorname{diam} Q_k$ and hence the union must contain $\widehat{Q}_k$. Thus Q intersects $\widehat{Q}_k$ only if Q touches Q_k. □

Remark 1.9.4 *The proof of the Proposition 1.9.3 shows actually more, namely that every point $x \in \Omega$ has a neighbourhood which intersects at most $3 \cdot 12^N$ of the cubes $\widehat{Q}_k$, $k \in \mathbb{N}$.*

With each cube Q_k we now associate a smooth function in the following way. Let Q_0 be the unit cube centered at the origin of $\mathbb{R}^N$ and fix a C^∞-function $\varphi : \mathbb{R}^N \to [0, 1]$ satisfying

$$\varphi(x) = 1 \quad \text{for } x \in Q_0 \quad \text{and} \quad \varphi(x) = 0 \quad \text{for } x \notin (1 + \varepsilon) Q_0.$$

Denote by φ_k the adjustment of φ to Q_k, i.e.

$$\varphi_k(x) := \varphi \left(\frac{x - x^{(k)}}{l_k} \right).$$

We have that

$$\varphi_k(x) = 1 \quad \text{for } x \in Q_k \quad \text{and} \quad \varphi_k(x) = 0 \quad \text{for } x \notin \widehat{Q}_k.$$

Also we may easily find constants a_α fulfilling

$$\left| \left(\frac{\partial}{\partial x} \right)^{\alpha} \varphi_k(x) \right| \leq a_\alpha (\operatorname{diam} Q_k)^{-|\alpha|} \tag{1.18}$$

for any multiindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$. Here $|\alpha| := \sum_{i=1}^{N} \alpha_i$.
We may now construct a regularized distance function. For any open set $\Omega \subsetneq \mathbb{R}^N$ we denote by $d(x)$ the distance to the boundary of Ω, i.e. we set

$$d(x) := \operatorname{dist}(x, \Omega^{\mathrm{c}}) = \inf_{y \in \Omega^{\mathrm{c}}} |x - y|.$$

Theorem 1.9.5 *Let $\Omega \subsetneq \mathbb{R}^N$ be open. Then there exist a C^∞−function σ and positive constants c, $c_\alpha \in \mathbb{R}$ fulfilling*

(a) $c^{-1} d(x) \leq \sigma(x) \leq c d(x)$

(b) $\left| \frac{\partial}{\partial x^\alpha} \sigma(x) \right| \leq c_\alpha d(x)^{1-|\alpha|}$.

Proof. We set

$$\sigma(x) := \sum_{k=1}^{\infty} \operatorname{diam} Q_k \cdot \varphi_k(x).$$

We need the following two observations.

$$x \in Q_k \Rightarrow d(x) \leq 5 \operatorname{diam} Q_k. \tag{1.19}$$

$$x \in \widehat{Q}_k \Rightarrow d(x) \geq \frac{3}{4} \operatorname{diam} Q_k. \tag{1.20}$$

The assertion (1.19) follows since by the Theorem 1.9.2 (ii) we may write for $x \in Q_k$,

$$\operatorname{dist}(x, F) \leq \operatorname{dist}(Q_k, F) + \operatorname{diam} Q_k \leq 5 \operatorname{diam} Q_k.$$

Also by the Theorem 1.9.2 (ii) we have for $x \in \widehat{Q}_k$,

$$d(x) \geq \operatorname{dist}(Q_k, F) - \frac{1}{4} \operatorname{diam} Q_k \geq \frac{3}{4} \operatorname{diam} Q_k$$

and (1.20) follows. We also have

$$x \in \widehat{Q}_k \Rightarrow d(x) \leq 6 \operatorname{diam} Q_k, \tag{1.21}$$

since for $x \in \widehat{Q}_k$ there exists $\overline{x} \in Q_k$ with $\operatorname{dist}(x, \overline{x}) \leq \frac{1}{4} \operatorname{diam} Q_k$ and hence

$$d(x) \leq \operatorname{diam} Q_k + d(\overline{x}) \leq 6 \operatorname{diam} Q_k.$$

Using the observations (1.19), (1.20) and (1.21) we have for any $x \in \Omega$, $x \in Q_k$,

$$\sigma(x) \geq \operatorname{diam} Q_k \varphi_k(x) = \operatorname{diam} Q_k \geq \frac{1}{5} d(x).$$

Since x lies in at most $3 \cdot 12^N$ of the $\widehat{Q}_k$ we also have

$$\sigma(x) \leq \sum_{x \in \widehat{Q}_k} \operatorname{diam} Q_k \leq 3 \cdot 12^N \cdot \frac{4}{3} d(x).$$

Thus (a) has been proved. In order to prove (b) we invoke (1.18) and (1.21), resp. (1.20) in case $|\alpha| \le 1$. We estimate for $x \in \widehat{Q}_k$,

$$\left|\frac{\partial}{\partial x^\alpha}\sigma(x)\right| \le \sum_{k=1}^{\infty} \operatorname{diam} Q_k \cdot \left|\frac{\partial}{\partial x^\alpha}\varphi_k(x)\right| \le 3 \cdot 12^N \cdot a_\alpha (\operatorname{diam} Q_k)^{1-|\alpha|} \le c_\alpha \cdot d(x)^{1-|\alpha|}.$$

□

Remark 1.9.6 *It is easily seen that the exponent $1 - |\alpha|$ in the estimate (b) of Theorem 1.9.5 is optimal, in general. This is already clear in $\mathbb{R}$ if we take $\Omega := \bigcup_{j=-\infty}^{\infty}(2^{-j}, 2^{-j+1})$.*

1.10 Notes and comments

Section 1.1
The theory of sesquilinear forms can be found in many books, we mention Davies [22], Ouhabaz [52], Fukushima [33].
Sections 1.2, 1.3
The theory of one-parameter linear semigroups is well exposed in various books. See Davies [22], Engel-Nagel [31], Pazy [55].
Section 1.4 The criteria of Beurling and Deny are very well-known. For the proofs see [33],[23]. Our presentation based on abstract statements about invariance of closed convex sets follows [52].
Section 1.5
The material of this section is also classical. We follow [22] and [52].
Section 1.6
We describe here only interpolation theorems which are needed in this work. Many generalizations to general Banach spaces are possible. See the monograph of Bergh and Löfström [14]. The Marcinkiewicz interpolation theorem in its basic form for the diagonal goes back to Marcinkiewicz [47], but the version we use here is due to Zygmund [69]. For the proof of the Riesz-Thorin theorem see Thorin [65] or Zygmund and Tamarkin [64]. Interpolation of analytic families is due to Stein [60]. Our exposition follows closely the book of Stein and Weiss [63].
Section 1.7
The literature concerning Sobolev spaces is large. We mention e.g. the books of Gilbarg-Trudinger [34] or Adams [1].
Section 1.8
There are various approaches to the spectral theorem. The version we state here is from [22]. For a detailed presentation and a proof see also [25].
Section 1.9
The described decomposition is based on ideas of Whitney [67], however, the precise form of Theorem 1.9.2 is taken from Stein [61]. Theorem 1.9.5 is in Calderón and Zygmund [16]. Our presentation follows [63].

Chapter 2

Elliptic operators on weighted spaces

This chapter consists of two sections. In the first one we introduce general sesquilinear forms acting on weighted L^p-spaces. In the second section we study conditions for L^∞-contractivity of the semigroups which arise in the first section. The material of this chapter is partly new, see also Notes and Comments at the end of the chapter.

2.1 Forms on a weighted space

In this section we introduce a general form acting on the weighted space $L^2(\Omega, \frac{dx}{m(x)})$, where $\Omega \subset \mathbb{R}^N$ is open and bounded and $m : \Omega \to (0, \infty)$ is a bounded measurable function such that $\frac{1}{m} \in L^1_{\text{loc}}(\Omega, dx)$ The space $L^2(\Omega, \frac{dx}{m(x)})$ is a Hilbert space where the scalar product is given by

$$\langle u|v\rangle = \int_\Omega u(x)\overline{v(x)}\,\frac{dx}{m(x)}.$$

Let $a : \Omega \to \mathbb{C}^{N\times N}$, $b : \Omega \to \mathbb{C}^N$, $c : \Omega \to \mathbb{C}^N$ and $a_0 : \Omega \to \mathbb{C}$ be bounded measurable functions. We also assume the following ellipticity condition on the matrix a: there exists a constant c_E such that

$$Re \sum_{k,l=1}^{N} a_{kl}(x)\xi_k\overline{\xi_l} \geq c_E|\xi|^2 \tag{2.1}$$

for all $\xi \in \mathbb{C}^N$ and almost all $x \in \Omega$.
Then we define a sesquilinear form a by

$$\begin{aligned} a(u,v) : = &\int_\Omega \nabla u(x)\cdot[\mathbf{a}(x)\overline{\nabla v(x)}]\,dx + \int_\Omega b(x)\cdot\nabla u(x)\overline{v(x)}\,dx \\ &+ \int_\Omega u(x)c(x)\cdot\overline{\nabla v(x)}\,dx + \int a_0(x)u(x)\overline{v(x)}\,dx \end{aligned} \tag{2.2}$$

with the domain $\mathcal{D}(a) = H^1_0(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$. Here the dot product of two N-dimensional vectors w and z is given by $w \cdot z := \sum_{j=1}^N w_j z_j$.

Since throughout the work we assume the function m to be bounded from above we have

Lemma 2.1.1 $L^2(\Omega, \frac{dx}{m(x)})$ *is continuously embedded in* $L^2(\Omega, dx)$.

Proof. This follows immediately from the assumption of boundedness (from above) of m. $\square$

We need show that the domain of a is dense in the Hilbert space $L^2(\Omega, \frac{dx}{m(x)})$. We shall do this by the means of the following lemma. We show that the test functions (i.e. infinitely smooth functions with compact support in Ω) denoted by $\mathcal{D}(\Omega)$ are dense in $\mathcal{D}(a)$. Since they are also dense in $L^2(\Omega, \frac{dx}{m(x)})$ the density of $\mathcal{D}(a)$ in $L^2(\Omega, \frac{dx}{m(x)})$ will follow. We let

$$\mathcal{D}(\Omega)_+ := \{v \in \mathcal{D}(\Omega) : \, v \geq 0\}$$

and

$$\mathcal{D}(a)_+ := \{u \in \mathcal{D}(a) : \, u \geq 0 \text{ a.e.}\}\,.$$

Proposition 2.1.2 $\mathcal{D}(\Omega)$ *is dense in* $\mathcal{D}(a)$ *and* $\mathcal{D}(\Omega)_+$ *is dense in* $D(a)_+$.

Proof. We prove the second assertion. The first assertion then follows since $\mathcal{D}(a) = \mathcal{D}(a)_+ - \mathcal{D}(a)_+$.
a) Let $u \in \mathcal{D}(a)_+$. There exists a sequence $\varphi_n \in \mathcal{D}(\Omega)$ s.t. $\varphi_n \to u$ in $H^1(\Omega)$. Let $u_n := (\varphi_n \wedge u) \vee 0$. Then $0 \leq u_n \leq u$ and $u_n \to u$ in $H^1(\Omega)$. Moreover $u_n \to u$ a.e. (for a subsequence which we denote also by u_n). Hence $u_n \to u$ in $L^2(\Omega, \frac{dx}{m(x)})$ by the dominated convergence theorem. We have shown that $\mathcal{D}(a)_+ \cap L_c^\infty(\Omega)$ is dense in $\mathcal{D}(a)_+$, where

$$L_c^\infty(\Omega) := \{u \in L^\infty(\Omega) : \text{supp } u \subset \Omega \text{ is compact}\}$$

b) Let $u \in \mathcal{D}(a)_+ \cap L_c^\infty(\Omega), u_n := \rho_n * u$, where ρ_n is a mollifier. Then $u_n \in \mathcal{D}(\Omega)$, supp $u_n \subset K \subset\subset \Omega$ (for $n \geq n_0$) and $\|u_n\|_\infty \leq c$ (for $n \geq n_0$), $u_n \to u$ in $H^1(\Omega)$ and $u_n \to u$ a. e. after choosing a subsequence. Hence $u_n \to u$ in $L^2(\Omega, \frac{dx}{m(x)})$. $\square$

In general, we have defined the domain of a to be $\mathcal{D}(a) = H_0^1(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$. However, if we pose more restrictive conditions on m we have the following. Here (and everywhere else throughout the work) $\hat{N}$ is defined by $\hat{N} := \max\{2, N\}$.

Lemma 2.1.3 *Assume that* $\frac{1}{m} \in L^q(\Omega, dx)$ *for some* $q \geq \hat{N}/2$ *in case* $N \neq 2$, *if* $N = 2$ *assume that* $\frac{1}{m} \in L^q(\Omega, dx)$ *for some* $q > 1$. *Then* $H_0^1(\Omega)$ *is dense and continuously embedded in* $L^2(\Omega, \frac{dx}{m(x)})$. *Consequently,* $\mathcal{D}(a) = H_0^1(\Omega)$.

Proof. We have

$$\begin{aligned}\int_\Omega \frac{|u|^2}{m(x)}\, dx &\leq \left(\int_\Omega \frac{1}{m(x)^q}\, dx\right)^{\frac{1}{q}} \left(\int_\Omega |u|^{2q'}\, dx\right)^{\frac{1}{q'}} \\ &\leq c_N \left\|\frac{1}{m}\right\|_{L^q(\Omega)} \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}} = c_N \left\|\frac{1}{m}\right\|_{L^q(\Omega)} \|u\|_{H_0^1(\Omega)}\end{aligned}$$

where we used the Hölder inequality and a Sobolev embedding i.e. Theorem 1.7.2, 1.7.3 or 1.7.4 depending on the dimension N. $\square$

In order to prove continuity of the form a we shall make use of the following (sectoriality) estimate.

Lemma 2.1.4 *There exists a constant* w *which is of the form*

$$w := c_{\boldsymbol{a},N,m}\left(\||b| + |c|\|_\infty^2 + \|a_0\|_\infty\right) \tag{2.3}$$

such that

$$|Im\ a(u,u)| \le c_{\boldsymbol{a}} Re\ a(u,u) + w\, \|u\|^2_{L^2(\Omega, \frac{dx}{m(x)})}\, .$$

Here $c_{\boldsymbol{a}}$ is a constant depending on the matrix $\boldsymbol{a}$, it may be chosen as $c_{\boldsymbol{a}} := 3$ in case $\boldsymbol{a}$ equals the identity matrix.

Proof. We estimate

$$\begin{aligned}
Re\ a(u,u) = Re\ &\int_\Omega \nabla u(x)\cdot \mathbf{a}(x)\overline{\nabla u(x)}\, dx + \int_\Omega Re\ (b(x)+c(x)) Re\ (\overline{u}(x)\nabla u(x))\, dx \\
&- \int_\Omega Im\ (b(x)-c(x)) Im\ (\overline{u}(x)\nabla u(x))\, dx + \int_\Omega a_0(x)|u(x)|^2\, dx \\
&\ge c_E \int_\Omega |\nabla u(x)|^2\, dx - \|Re\ a_0\|_\infty \int_\Omega |u(x)|^2\, dx \\
&- \sqrt{N}(\|Re\ (b+c)\|_\infty + \|Im\ (c-b)\|_\infty) \int_\Omega |\overline{u}(x)\nabla u(x)|\, dx \\
&\ge c_E \int_\Omega |\nabla u(x)|^2\, dx - \frac{1}{2} c_E \int_\Omega |\nabla u(x)|^2\, dx \\
&- \left(\frac{N}{c_E}(\|Re\ (b+c)\|^2_\infty + \|Im\ (c-b)\|^2_\infty) + \|Re\ a_0\|_\infty\right) \int_\Omega |u(x)|^2\, dx
\end{aligned}$$

where we used the inequality $|2vw| \le \frac{1}{c_E}v^2 + c_E w^2$ for $v, w \in \mathbb{R}$. We may also write

$$\begin{aligned}
Im\ a(u,u) = Im\ &\int_\Omega \nabla u(x)\cdot \mathbf{a}(x)\overline{\nabla u(x)}\, dx + \int_\Omega Re\ (b(x)-c(x)) Im\ (\overline{u}(x)\nabla u(x))\, dx \\
&+ \int_\Omega Im\ (b(x)+c(x)) Re\ (\overline{u}(x)\nabla u(x))\, dx + \int_\Omega a_0(x)|u(x)|^2\, dx
\end{aligned}$$

and estimate

$$\begin{aligned}
|Im\ a(u,u)| &\le \|a\|_\infty \int_\Omega |\nabla u(x)|^2\, dx + \|Im\ a_0\|_\infty \int_\Omega |u(x)|^2\, dx \\
&+ \sqrt{N}(\|Re\ (b-c)\|_\infty + \|Im\ (b+c)\|_\infty) \int_\Omega |\overline{u}(x)\nabla u(x)|\, dx \\
&\le (\|a\|_\infty + \frac{1}{2}) \int_\Omega |\nabla u(x)|^2\, dx \\
&+ (N\, \|Re\ (b-c)\|^2_\infty + \|Im\ (b+c)\|^2_\infty) + \|Im\ a_0\|_\infty) \int_\Omega |u(x)|^2\, dx
\end{aligned}$$

Combining the estimates for $Re\ a(u,u)$ and $Im\ a(u,u)$ we obtain

$$|Im\ a(u,u)| \le c_{\mathbf{a}} Re\ a(u,u) + c_{\mathbf{a},N}\left(\||b| + |c|\|^2_\infty + \|a_0\|_\infty\right) \int_\Omega |u(x)|^2\, dx.$$

The last estimate and Lemma 2.1.1 conclude the proof. □

Remark 2.1.5 *We point out here the fact that the proof of Lemma 2.1.4 follows the usual proof valid in the non-weighted setting, what is important here is only the validity of the embedding $L^2(\Omega, \frac{dx}{m(x)}) \hookrightarrow L^2(\Omega, dx)$ guaranteed by the assumption of boundedness from above of the function m.*

We see from the results above that if we define in $L^2(\Omega, \frac{dx}{m(x)})$ a form

$$a_w(u,v) := a(u,v) + w\langle u, v\rangle_{L^2(\Omega, \frac{dx}{m(x)})} \text{ with } \mathcal{D}(a) := H^1_0(\Omega) \tag{2.4}$$

then a_w is a densely defined, accretive, closed and continuous sesquilinear form in $L^2(\Omega, \frac{dx}{m(x)})$ (the continuity is seen by applying Proposition 1.1.1). Therefore, by the theory of Section 1.2, there exists an associated operator[1] A_w which is a generator of contractive C_0-semigroup on $L^2(\Omega, \frac{dx}{m(x)})$. We will see in the next chapter that in the case $\mathbf{a} \equiv I$ and $b \equiv c \equiv a_0 \equiv 0$ (in which case we may take $w = 0$) the operator A_0 equals $m\triangle$ (with the Dirichlet boundary condition). We shall use the general theory of this section (i.e. nonzero b, c and a_0) when proving pseudo-Gaussian estimates in Chapter 5 via twisted form.

For the sake of reference we summarise what we have done in the following theorem.

Theorem 2.1.6 *Let Ω be a bounded open subset of $\mathbb{R}^N$ and $m : \Omega \to (0, \infty)$ be a bounded measurable function such that $\frac{1}{m} \in L^q_{\text{loc}}(\Omega, dx)$ for some $q > \frac{\hat{N}}{2}$. Let*

$$\boldsymbol{a} : \Omega \to \mathbb{C}^{N\times N},\ b : \Omega \to \mathbb{C}^N,\ c : \Omega \to \mathbb{C}^N \quad \text{and} \quad a_0 : \Omega \to \mathbb{C}$$

be bounded measurable functions such that (2.1) holds. Define $a(\cdot, \cdot)$ by (2.2). Then there exists $w \in \mathbb{R}_+$ such that $a_w(\cdot, \cdot)$ defined by (2.4) is a densely defined, accretive, closed and continuous sesquilinear form in $L^2(\Omega, \frac{dx}{m(x)})$. The associated operator A_w is the generator of a contractive C_0-semigroup on $L^2(\Omega, \frac{dx}{m(x)})$. The constant w may be chosen to be of the form

$$w := c_{\boldsymbol{a},N,m} \left(\||b| + |c|\|^2_\infty + \|a_0\|_\infty \right).$$

We now investigate positivity, L^∞-contractivity and irreducibility of the semigroup $e^{A_w t}$. First, though, we need the following two lemmata.

Lemma 2.1.7 *For any $u \in H^1_0(\Omega)$ we have $(Re\ u)^+$, $|u| \in H^1_0(\Omega)$ and*

$$\nabla (Re\ u)^+ = \nabla (Re\ u) \chi_{\{Re\ u > 0\}} \tag{2.5}$$

$$\nabla |u| = Re\ (\text{sign}(\overline{u}) \nabla u). \tag{2.6}$$

We also have for any $u \in H^1(\Omega)$,

$$\chi_{\{|u|>0\}} \nabla \operatorname{sign} u = \chi_{\{|u|>0\}} \operatorname{sign} u \, \frac{i Im\ (\nabla u \operatorname{sign} \overline{u})}{|u|}. \tag{2.7}$$

Proof. Consider the complex function

$$g_\varepsilon(z) := \sqrt{|z|^2 + \varepsilon^2} - \varepsilon.$$

By an easy computation we see that g_ε is continuously differentiable with partial derivatives[2] $\frac{\partial}{\partial t} g_\varepsilon$ and $\frac{\partial}{\partial s} g_\varepsilon$ bounded on $\mathbb{C}$. Thus we conclude from Theorem 1.4.12 that $g_\varepsilon(u) \in H^1(\Omega)$ for any $u \in H^1(\Omega)$. We have

$$\begin{aligned} \nabla g_\varepsilon(u) &= \frac{\partial}{\partial t} g_\varepsilon(Re\ u, Im\ u) \nabla (Re\ u) + \frac{\partial}{\partial s} g_\varepsilon (Re\ u, Im\ u) \nabla (Im\ u) \\ &= \frac{1}{\sqrt{|u|^2 + \varepsilon^2}} (Re\ u \nabla (Re\ u) + Im\ u \nabla (Im\ u)) \\ &= Re \left(\nabla u \frac{\overline{u}}{\sqrt{|u|^2 + \varepsilon^2}} \right). \end{aligned}$$

[1] We shall denote by $A := A_0$ the operator associated to the (unshifted) form $a(\cdot, \cdot)$.

[2] Here we use the notation $t := Re\ z$ and $s := Im\ z$

Pick $\varphi \in C_c^\infty(\Omega)$. Then for any $\varepsilon > 0$,

$$\int_\Omega g_\varepsilon(u)(x)\nabla\varphi(x)\,dx = -\int_\Omega Re\,\left(\nabla u\frac{\overline{u}}{\sqrt{|u|^2+\varepsilon^2}}\right)(x)\varphi(x)\,dx.$$

Sending $\varepsilon \to 0$ we obtain

$$\int_\Omega |u|(x)\nabla\varphi(x)\,dx = -\int_\Omega Re\,\left(\operatorname{sign}(\overline{u})\nabla u\right)(x)\varphi(x)\,dx.$$

Thus $|u| \in H^1(\Omega)$ and $\nabla|u| = Re\,(\operatorname{sign}(\overline{u})\nabla u)$. Using what we have just proved for $Re\,u$ and also the fact that $(Re\,u)^+ = \frac{1}{2}(|Re\,u| + Re\,u)$ we obtain

$$(Re\,u)^+ \in H^1(\Omega) \qquad \text{and} \qquad \nabla(Re\,u)^+ = \nabla(Re\,u)\chi_{\{Re\,u>0\}}.$$

In particular, we have proved that $u^+ \in H^1(\Omega)$ for all real-valued $u \in C_c^\infty(\Omega)$. The conclusion of the first part of the lemma follows now from Proposition A.3.1 and the fact that any function in $H^1(\Omega)$ with compact support in Ω is in $H_0^1(\Omega)$. This last fact can be easily seen by aproximation with a mollifier.
In order to prove the second part we have (arguing as above)

$$\nabla\left(\frac{u}{\sqrt{|u|^2+\varepsilon}}\right) = \frac{\nabla u}{\sqrt{|u|^2+\varepsilon}} - \frac{u|u|Re\,(\nabla u\operatorname{sign}\overline{u})}{(|u|^2+\varepsilon)^{\frac{3}{2}}}$$

and hence also (by considering the sets $\{|u| > k\}$ for any $k > 0$ and reasoning as in the first part of the proof)

$$\begin{aligned}\chi_{\{|u|>0\}}\nabla\operatorname{sign}u &= \chi_{\{|u|>0\}}\left[\frac{\nabla u}{|u|} - \frac{\operatorname{sign}u Re\,(\nabla u\operatorname{sign}\overline{u})}{|u|}\right]\\ &= \chi_{\{|u|>0\}}\operatorname{sign}u\,\frac{iIm\,(\nabla u\operatorname{sign}\overline{u})}{|u|}.\end{aligned}$$

□

Lemma 2.1.8 *For any $u \in H_0^1(\Omega)$ both $(|u|-1)^+\operatorname{sign}u$ and $(1\wedge|u|)\operatorname{sign}u$ belong to $H_0^1(\Omega)$. We have*

$$\nabla((|u|-1)^+\operatorname{sign}u) = \chi_{\{|u|>1\}}\left[\nabla u - i\frac{\operatorname{sign}u\cdot Im\,(\nabla u\operatorname{sign}\overline{u})}{|u|}\right] \tag{2.8}$$

$$= \chi_{\{|u|>1\}}\operatorname{sign}u\left[\nabla|u| + i(|u|-1)\frac{Im\,(\nabla u\operatorname{sign}\overline{u})}{|u|}\right] \tag{2.9}$$

and

$$\nabla\big((1\wedge|u|)\operatorname{sign}u\big) = i\frac{Im\,(\nabla u\operatorname{sign}\overline{u})}{|u|}\operatorname{sign}u\chi_{\{|u|>1\}} + \nabla u\chi_{\{|u|\le 1\}}. \tag{2.10}$$

Proof. We define

$$g_\varepsilon(t) := \begin{cases}\sqrt{(t-1)^2+\varepsilon^2}-\varepsilon & \text{if } t > 1,\\ 0 & \text{if } t \le 1.\end{cases}$$

Since g_ε has bounded derivative on $\mathbb{R}$, we have $g_\varepsilon(u) \in H^1(\Omega)$ for all real-valued $u \in H^1(\Omega)$ by Theorem 1.4.12. By Lemma 2.1.7 we thus obtain $g_\varepsilon(|u|) \in H^1(\Omega)$ for all $u \in H^1(\Omega)$. We have

$$\nabla g_\varepsilon(|u|) = \frac{(t-1)}{\sqrt{(t-1)^2+\varepsilon^2}}\nabla|u|$$

and hence (by an argument as in the proof of Lemma 2.1.7) we see that $(|u|-1)^+ \in H^1(\Omega)$ with

$$\nabla(|u|-1)^+ = \chi_{\{|u|>1\}}\nabla|u| = \chi_{\{|u|>1\}} Re\,(\nabla u \,\mathrm{sign}\,\overline{u}).$$

We also see that $\frac{u}{\sqrt{|u|^2+\varepsilon}}(|u|-1)^+ \in H^1(\Omega)$ for any $\varepsilon > 0$ and

$$\begin{aligned}\nabla\Big(\frac{u}{\sqrt{|u|^2+\varepsilon}}(|u|-1)^+\Big) &= \chi_{\{|u|>1\}}\frac{\nabla u}{\sqrt{|u|^2+\varepsilon}}(|u|-1)^+ \\ &\quad - \chi_{\{|u|>1\}}\frac{u(|u|-1)^+|u| Re\,(\nabla u\,\mathrm{sign}\,\overline{u})}{(|u|^2+\varepsilon)^{\frac{3}{2}}} \\ &\quad + \chi_{\{|u|>1\}}\frac{u Re\,(\nabla u\,\mathrm{sign}\,\overline{u})}{\sqrt{|u|^2+\varepsilon}}.\end{aligned}$$

Sending $\varepsilon \to 0$ it follows[3] that $(|u|-1)^+ \,\mathrm{sign}\, u \in H^1(\Omega)$ and we have (2.8). We write

$$\begin{aligned}&\chi_{\{|u|>1\}}\,\mathrm{sign}\,u\left(\nabla|u| + i(|u|-1)\frac{Im\,(\nabla u\,\mathrm{sign}\,\overline{u})}{|u|}\right) \\ &= \chi_{\{|u|>1\}}\,\mathrm{sign}\,u\left(Re\,(\nabla u\,\mathrm{sign}\,\overline{u}) + iIm\,(\nabla u\,\mathrm{sign}\,\overline{u}) - i\frac{Im\,(\nabla u\,\mathrm{sign}\,\overline{u})}{|u|}\right) \\ &= \chi_{\{|u|>1\}}\left(\nabla u - i\frac{\mathrm{sign}\,u Im\,(\nabla u\,\mathrm{sign}\,\overline{u})}{|u|}\right)\end{aligned}$$

and obtain (2.9). The expression (2.10) for $u \in H^1(\Omega)$ follows from the decomposition

$$u = (1 \wedge |u|)\,\mathrm{sign}\,u + (|u|-1)^+\,\mathrm{sign}\,u$$

and the fact that

$$Re\,(\nabla u\,\mathrm{sign}\,\overline{u})\,\mathrm{sign}\,u + iIm\,(\nabla u\,\mathrm{sign}\,\overline{u})\,\mathrm{sign}\,u = \nabla u. \tag{2.11}$$

It remains to check that $(1 \wedge |u|)\,\mathrm{sign}\,u \in H_0^1(\Omega)$ for any $u \in H_0^1(\Omega)$. However, this can be easily seen for $u \in C_c^\infty(\Omega)$ (since such functions are in $H^1(\Omega)$ and have compact support in Ω) and the assertion for a general $u \in H_0^1(\Omega)$ follows by approximation. □

We now prove positivity of $e^{A_w t}$.

Proposition 2.1.9 *The semigroup $e^{A_w t}$ is positive (and hence in particular real).*

Proof. This follows from Theorem 1.4.7. We observe that $u \in H_0^1(\Omega)$ implies $(Re\ u)^+ \in H_0^1(\Omega)$ by Lemma 2.1.7. We also have

$$a_w((Re\ u)^+, (Re\ u)^-) = 0$$

since $\nabla(Re\ u)^+ = \nabla(Re\ u)\chi_{\{Re\ u>0\}}$. □

In the following we investigate irreducibility of $e^{A_w t}$.

Theorem 2.1.10 *If the open set Ω is connected then the semigroup $e^{A_w t}$ is irreducible.*

[3]once again by an argument analogous to the one in Lemma 2.1.7

Proof: We check the condition (ii) from Lemma 1.5.3. Assume in the contrary that there exists a measurable set ω_1 with $\lambda(\omega_1) > 0$ and $\lambda(\omega_2) > 0$ (where $\omega_2 := \Omega\backslash\omega_1$) and $\chi_{\omega_1}u \in H_0^1(\Omega)$ for any $u \in H_0^1(\Omega)$. Choose $u \in C_c^\infty(\Omega)$. Since $\chi_{\{v=0\}}\nabla v = 0$ for all $v \in H^1(\Omega)$ we have $\nabla(\chi_{\omega_1}u) = \chi_{\omega_1}\nabla u$. Therefore $\chi_{\omega_1}u \in W_0^{1,p}(\tilde{\Omega})$ for all $1 \le p \le \infty$, where $\tilde{\Omega}$ is an open subset of Ω, $\partial\tilde{\Omega}$ is C^∞ and *supp* $u \in \tilde{\Omega}$. A Sobolev embedding (Theorem 1.7.3) implies that $\chi_{\omega_1}u$ may be represented by a continuous function v on $\tilde{\Omega}$.

We now claim that there exists $x_0 \in \Omega$ s.t. for every $\delta > 0$

$$\lambda(B(x_0,\delta)\cap\omega_1) > 0 \text{ and } \lambda(B(x_0,\delta)\cap\omega_2) > 0. \tag{2.12}$$

If not, then for any $x \in \Omega$ we find δ such that either $\lambda(B(x_0,\delta\cap\omega_1) = 0$ or $\lambda(B(x_0,\delta\cap\omega_2) = 0$. Define Ω_1 (resp. Ω_2) as the union of all balls $B(x,\delta)$ where x and δ fulfil $\lambda(B(x,\delta)\cap\omega_1) = 0$ (resp. $\lambda(B(x,\delta)\cap\omega_2) = 0$). Then Ω_1, Ω_2 are disjoint open subsets of Ω such that $\Omega \subset \Omega_1\cup\Omega_2$. Moreover, if $\Omega \subset \Omega_i$ (i=1,2), then $\lambda(\omega_i) = 0$ which is not possible by our assumption. Hence $\Omega \not\subseteq \Omega_1$, $\Omega \not\subseteq \Omega_2$ and $\Omega = \Omega_1 \cup \Omega_2$ which contradicts the connectivity of Ω. Therefore (2.12) holds true and we may choose $\delta > 0$ small enough so that $B(x_0, 2\delta) \subset \Omega$. Consider $u \in C_c^\infty(\Omega)$ s.t. $u\chi_{B(x_0,\delta)} \equiv 1$. For a.e. $x \in B(x_0,\delta)\cap\omega_1$ and a.e. $y \in B(x_0,\delta)\cap\omega_2$ we have

$$1 = |\chi_{\omega_1}u(x) - \chi_{\omega_1}u(y)| = |v(x) - v(y)|$$

and this is a contradiction to the continuity of v. Hence $e^{A_w t}$ must be irreducible.□

We devote the next section to studying conditions guaranteeing L^∞-contractivity of $e^{A_w t}$.

2.2 Criteria for L^∞-contractivity of elliptic operators

In this section we continue to work in the setting of Theorem 2.1.6. Thus we assume that the functions a_{jk}, b_k, c_k, a_0, $j,k = 1,\dots,N$ are complex-valued, bounded and that (2.1) holds. We set ourselves the task of understanding under which conditions on a_{jk}, b, c, a_0 the semigroup $e^{A_w t}$ is contractive on $L^\infty(\Omega, \frac{dx}{m(x)})$.

The theory of this section is made possible by two facts. Firstly, the spaces $L^\infty(\Omega, dx)$ and $L^\infty(\Omega, \frac{dx}{m(x)})$ coincide since we assume throughout the work that m is strictly positive on Ω. Secondly, the criteria for L^∞-contractivity of semigroups arising from sesquilinear forms are given purely in terms of the forms (cf. Theorem 1.4.11).

We start with a preliminary characterisation which is actually valid for a much broader class of boundary conditions. Afterwards we refine the result for our setting of the Dirichlet boundary conditions. We use the notation φ to denote the function $\varphi(u) := \chi_{\{u\neq 0\}}\frac{Im\ (\nabla u\,\text{sign}\,\overline{u})}{|u|}$. Its very close relation to the derivative of $\text{sign}\,u$ can be seen from Lemma 2.1.7. We also use the componentwise notation $\varphi_k(u) := \chi_{\{u\neq 0\}}\frac{Im\ (D_k u\,\text{sign}\,\overline{u})}{|u|}$, where $D_k := \frac{\partial}{\partial x_k}$. We omit the dependence of φ on u when no confusion arises and write simply $\varphi := \varphi(u)$ and $\varphi_k := \varphi_k(u)$.

Theorem 2.2.1 *The semigroup e^{At} is L^∞-contractive if and only if for any $u \in H_0^1(\Omega)$ such that $\varphi_k(u)\varphi_j(u)|u| \in L^1(\Omega)$ and $\varphi_k D_j|u| \in L^1(\Omega)$ (where $j,k =$*

$1,\dots,N$) we have

$$\int_\Omega \left(\sum_{j,k=1}^N Re\,(a_{kj})\varphi_k\varphi_j|u| - \sum_{j,k=1}^N Im\,(a_{kj})\varphi_k D_j|u| + \sum_{j=1}^N Im\,(c_j-b_j)\varphi_j|u| \right) dx + \int_\Omega \left(\sum_{j=1}^N Re\,(c_j) D_j|u| + Re\,(a_0)|u| \right) dx \geq 0. \tag{2.13}$$

Proof. Using the expressions from Lemmas 2.1.7 and 2.1.8 we have for any $u \in H_0^1(\Omega)$,

$$\begin{aligned} & Re\, a\big((|u|\wedge 1)\,\mathrm{sign}\,u, (|u|-1)^+\,\mathrm{sign}\,u\big) \\ &= \int_\Omega \left(\sum_{j,k=1}^N Re\,(a_{kj})\varphi_k\varphi_j(|u|-1)^+ - \sum_{j,k=1}^N Im\,(a_{kj})\varphi_k D_j|u|\chi_{\{|u|>1\}} \right) dx \qquad (2.14) \\ &+ \int_\Omega \left(\sum_{j=1}^N Im\,(c_j-b_j)\varphi_j(|u|-1)^+ + \sum_{j=1}^N Re\,(c_j)D_j|u|\chi_{\{|u|>1\}} \right) dx \\ &+ \int_\Omega Re\,(a_0)(|u|-1)^+\,dx. \end{aligned}$$

Assume now that the condition (2.13) of the theorem is satisfied. For any $u \in H_0^1(\Omega)$ we set $v := (|u|-1)^+\,\mathrm{sign}\,u$. We recall from Lemma 2.1.8 that $v \in H_0^1(\Omega)$. We have also that $|v| = (|u|-1)^+$ and on plugging the expression (2.9) for ∇v into the definition of $\varphi(v)$ we obtain also $\varphi(v) := \varphi(u)\chi_{\{|u|>1\}}$. Hence $\varphi_k(v)\varphi_j(v)|v|$ and $\varphi_k(v)D_j|v|$ belong to $L^1(\Omega)$. Applying the inequality (2.13) for v we see that

$$Re\, a\big((|u|\wedge 1)\,\mathrm{sign}\,u, (|u|-1)^+\,\mathrm{sign}\,u\big) \geq 0$$

and e^{At} is L^∞-contractive by Theorem 1.4.9.
In the opposite direction, assume that e^{At} is L^∞-contractive and let $u \in H_0^1(\Omega)$ fulfil $\varphi_k(u)\varphi_j(u)|u| \in L^1(\Omega)$ and $\varphi_k D_j|u| \in L^1(\Omega)$ (where $j,k = 1,\dots,N$). We apply Theorem 1.4.9 to $\frac{u}{k}$ for any $k>0$ and obtain (2.14) with $(|u|-k)^+$ instead of $(|u|-1)^+$ and $\chi_{\{|u|>k\}}$ instead of $\chi_{\{|u|>1\}}$. Because of the assumptions on u we may take limit as $k \to 0$ and use Lebegue's dominated convergence to conclude that (2.13) holds true. □

As we have already mentioned the last theorem remains valid for a larger class of boundary conditions (after a proper realisation of the operator A). In the following characterisation we shall see that for the Dirichlet boundary conditions a much more precise statement can be made.

Theorem 2.2.2 *The semigroup e^{At} is L^∞-contractive if and only if the following conditions hold:*

(i) $Im\,(a_{kj}+a_{jk}) = 0$ *for all* $j,k = 1,\dots,N$,

(ii) $g_0 = Re\,(a_0) - \sum_{j=1}^N D_j(Re\,c_j)$ *is a positive Radon measure on* Ω,

(iii) $g_k = \sum_{j=1}^N D_j(Im\,a_{kj}) \in L^1_{\mathrm{loc}}(\Omega), \qquad k = 1,\dots N,$

(iv) $\sum_{k,j=1}^N Re\,(a_{kj})\xi_k\xi_j + \sum_{j=1}^N (Im\,(c_j-b_j)+g_j)\xi_j + g_{0,\mathrm{reg}} \geq 0$ *a.e. on* Ω *for any* $\xi \in R^N$.

We use the notation $g_{0,\mathrm{reg}}$ to denote the regular (absolutely continuous) part of the measure g_0.

Proof. Assume first that e^{At} is L^∞-contractive. We apply (2.13) to $u = re^{i\psi}$ where $r \in C_c^\infty(\Omega, \mathbb{R}^+)$ and $\psi \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$. Since $\varphi(u) = \nabla\psi\chi_{\{|u|>0\}}$ (this can be seen easily from (2.7)) the inequality (2.13) yields for any $\psi \in C^\infty(\mathbb{R}^N, \mathbb{R})$,

$$\sum_{k,j=1}^N Re\ (a_{kj}) D_k\psi D_j\psi + \sum_{j=1}^N Im\ (c_j - b_j) D_j\psi + g_0 \qquad (2.15)$$
$$+ \sum_{j=1}^N g_j D_j\psi + \sum_{k,j=1}^N Im\ (a_{kj}) D_j D_k\psi \geq 0$$

in $(C_c^\infty(\Omega))'$.
Choosing $\psi \equiv 0$ in (2.15) we obtain (ii).
We see from (2.15) that for any $\psi \in C^\infty(\mathbb{R}^N, \mathbb{R})$ the expression $g_0 + \sum_{j=1}^N g_j D_j\psi$ defines a Radon measure on Ω with a nonnegative singular part. If we set $\psi(x) = \lambda x_k,\ \lambda \in \mathbb{R},\ k = 1, ..., N$, we conclude that g_k has a trivial singular part, i.e. (iii) holds.
Now choose x_0 to be a common Lebesgue point for all $a_{kj}, b_j, c_j, g_j, g_{0,\mathrm{reg}},\ k, j = 1, ..., N$ and set $\psi(x) := \frac{\lambda}{2}\left((x - x_0)\cdot\xi\right)^2$ for $\xi \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$. Applying (2.15) for this choice of ψ we have then

$$\lambda^2 \sum_{k,j=1}^N Re\ (a_{kj})\xi_k\xi_j \left((x - x_0)\cdot\xi\right)^2 + \lambda \sum_{j=1}^N \left(Im\ (c_j - b_j) + g_j\right)\left((x - x_0)\cdot\xi\right)\xi_j$$
$$+ g_{0,\mathrm{reg}} + \lambda \sum_{j,k=1}^N Im\ (a_{jk})\xi_j\xi_k \geq 0$$

for a.e. $x \in \Omega$. In particular, for $x = x_0$ we obtain

$$g_{0,\mathrm{reg}}(x_0) + \lambda \sum_{j,k=1}^N Im\ (a_{jk}(x_0)\xi_k\xi_j \geq 0, \qquad \lambda \in \mathbb{R}.$$

Here, $\lambda \in \mathbb{R}$ is arbitrary and hence we have $\sum_{j,k=1}^N Im\ (a_{jk})\xi_j\xi_k = 0$ a.e. on Ω for any $\xi \in \mathbb{R}^N$. Thus we proved (i).
The statement (iv) follows immediately on applying (2.15) to $\psi(x) := \xi \cdot x$.
Suppose now that the condition $(i) - (iv)$ hold true. Using the density of C_c^∞ in $H_0^1(\Omega)$, Theorem 1.4.9 and the expression (2.14) we need check that for any $u \in C_c^\infty$, we have (writing $u = re^{i\psi}$),

$$\int_\Omega \left(\sum_{j,k=1}^N Re\ (a_{kj})\varphi_k\varphi_j(|u| - 1)^+ - \sum_{j,k=1}^N Im\ (a_{kj})\varphi_k D_j(|u| - 1)^+ \right) dx \qquad (2.16)$$
$$+ \int_\Omega \left(\sum_{j=1}^N Im\ (c_j - b_j)\varphi_j(|u| - 1)^+ + \sum_{j=1}^N Re\ (c_j) D_j(|u| - 1)^+ \right) dx$$
$$+ \int_\Omega Re\ (a_0)(|u| - 1)^+\, dx \qquad \geq 0.$$

The function $(|u|-1)^+ \in W^{1,\infty}(\Omega)$ has compact support in Ω and $\sum_{j=1}^N D_j(Re\ c_j) = Re\ (a_0) - g_0$ is a Radon measure on Ω by assumption. Hence

$$\int_\Omega \sum_{j=1}^N Re\ (c_j) D_j(|u| - 1)^+ = \int_\Omega (g_0 - Re\ a_0)(|u| - 1)^+.$$

We also have that φ (which equals $\nabla\psi\chi_{\{|u|>0\}}$) belongs to $C^\infty(\{|u|>0\})$ and $D_j\varphi_k = D_k\varphi_j$. This follows from the fact that on the open set $\{|u|>0\}$ we have (see (2.7))

$$\varphi_k = -i\,\mathrm{sign}\,\overline{u}D_k(\mathrm{sign}\,u)$$

and hence also

$$D_j\varphi_k = -i(-\varphi_j\varphi_k + (\mathrm{sign}\,\overline{u})D_kD_j(\mathrm{sign}\,u)) \tag{2.17}$$

since

$$\nabla\,\mathrm{sign}\,\overline{u} = \mathrm{sign}\,\overline{u}\frac{iIm\ (\mathrm{sign}\,u\nabla\overline{u})}{|u|} = -\,\mathrm{sign}\,\overline{u}\frac{iIm\ (\mathrm{sign}\,\overline{u}\nabla u)}{|u|}.$$

on the set $\{|u|>0\}$. The expression (2.17) is symmetric with respect to k and j. Therefore (applying the assumptions (i) and (iii)),

$$\sum_{j,k=1}^N D_jIm\ (a_{jk}\varphi_k) = \sum_{k=1}^N\left(\sum_{j=1}^N D_jIm\ (a_{jk})\right)\varphi_k + \sum_{j,k=1}^N Im\ (a_{jk})D_j\varphi_k = \sum_{k=1}^N g_k\varphi_k$$

in $(C_c^\infty(\{|u|>0\}))'$. Since $(|u|-1)^+$ has compact support in $\{|u|>0\}$, we have

$$-\int_\Omega\sum_{j,k=1}^N (Im\ (a_{jk})\varphi_kD_j(|u|-1)^+ = \int_\Omega\sum_{k=1}^N g_k\varphi_k(|u|-1)^+.$$

Hence the left-hand side in (2.16) reads

$$\int_\Omega\left(\sum_{j,k=1}^N Re\ (a_{jk})\varphi_k\varphi_j + \sum_{j=1}^N(Im\ (c_j-b_j)+g_j)\varphi_j + g_{0,\mathrm{reg}}\right)(|u|-1)^+\,dx \tag{2.18}$$
$$+\int_\Omega(|u|-1)^+\,d(g_{0,\mathrm{sing}}),$$

where $g_{0,\mathrm{sing}} := g_0 - g_{0,\mathrm{reg}}$. Applying assumptions (ii) and (iv) we see that (2.18) is nonnegative. □

Corollary 2.2.3 *Let* $Re\ (c_j)\in W^{1,\infty}(\Omega)$ *and* $Im\ a_{jk} = -Im\ a_{kj}\in W^{1,\infty}(\Omega)$ *for* $j,k=1,...,N$. *Let* $w\in\mathbb{R}$ *be such that*

$$w + g_0 - \frac{1}{4}\sum_{j=1}^N|Im\ (c_j-b_j)+g_j|^2 \geq 0$$

on Ω. *Then the semigroup* $e^{(A-w)t}$ *is* L^∞*-contractive.*

Proof. The conditions of Theorem 2.2.2 are satisfied if we consider a_0+w instead of a_0. We just use an easy inequality

$$\sum_{j=1}^N(Im\ (c_j-b_j)+g_j)\,\xi_j \geq -\frac{1}{4}\sum_{j=1}^N|Im\ (c_j-b_j)+g_j|^2 - \sum_{j=1}^N\xi_j^2.$$

□

Corollary 2.2.4 *Let* $a_{jk}=\delta_{jk}$ *for* $j,k=1,...,N$ *and let* b, c *and* a_0 *be real. Suppose also that* $c_j\in W^{1,\infty}(\Omega)$, $j=1,\dots,N$ *and let* $w\in\mathbb{R}$ *be a constant such that*

$$w \geq \sum_{j=1}^N D_jc_j - a_0$$

on Ω. *Then the semigroup* $e^{(A-w)t}$ *is* L^∞*-contractive.*

2.3 Notes and comments

Section 2.1
The generalisations to the weighted spaces are due to the author (Lemmas 2.1.3, 2.1.1 and 2.1.4). Other results in this section are well-known (see e.g. [52]).
Section 2.2
Although the theory in this section seems to be *de facto* new due to the fact that we work on weighted spaces, it is basically a careful repetition of arguments valid in the non-weighted setting [52]. This strategy works as explained at the beginning of the section.

Chapter 3

The operator $m\triangle$ - Introduction and generation on $C_0(\Omega)$

In this chapter we introduce the operator $m\triangle$ where $m : \Omega \to (0, \infty)$ is a measurable function. This multiplicative perturbation of the Laplace operator serves as the most important application for the theory developed in this thesis. The aim is to pose as little restrictions on the regularity (and the 'ellipticity') of m as possible, but to be still able to develop a meaningful theory under such weak assumptions. In particular, two sorts of questions will be addressed in this work; firstly, generation results for the operator $m\triangle$ considered on $C_0(\Omega)$ and secondly, estimates for the kernel of the semigroup generated by $m\triangle$ on weighted L^p-spaces (where the weight is given by $\frac{1}{m(x)}$).

3.1 The operator $m\triangle$ - introduction and definition

The goal of this chapter is to investigate in deep the problem of obtaining the semigroup on $C_0(\Omega) := \{u \in C(\overline{\Omega}) : u_{|\partial\Omega} = 0\}$ for the operator $m\triangle$. We will see that a positive answer (indeed, a characterisation) can be given. In a natural way we will be led to conditions of two kinds (both of them applied locally): either the boundary is regular or the perturbing function vanishes at the boundary fast enough. Combining these two we shall then prove our main result in Theorem 3.5.1.

In this section we give precise meaning to the operator $m\triangle$, state and prove corresponding generation results for L^p-spaces. This will be our starting point for the deeper analysis of $m\triangle$ on the space $C_0(\Omega)$ which shall then be the content of the next section.

We will need Sobolev spaces $H^1(\Omega)$, $H_0^1(\Omega)$ and the local analogue $H^1_{\text{loc}}(\Omega)$ as defined in the Notation. In general, we have the inclusion $H^1(\Omega) \cap C_0(\Omega) \subset H_0^1(\Omega)$. However,

$$H_0^1(\Omega) \cap C(\overline{\Omega}) \subset C_0(\Omega) \quad \text{if and only if } \Omega \text{ is regular in capacity}$$

(see [15]). The spaces $H_0^1(\Omega)$ and $H^1(\Omega)$ are sublattices of $L^2(\Omega)$. More precisely

$$u \in H^1(\Omega) \quad \text{implies } D_j u^+ \in H^1(\Omega) \text{ and } D_j u^+ = \chi_{\{u>0\}} D_j u \quad j = 1, ..., d,$$

where by χ_A we denote the characteristic function of a set A. If $u \in H_0^1(\Omega)$, then also $u^+ \in H_0^1(\Omega)$. All this can be seen from Lemma 2.1.7.

If $u \in L^1_{loc}(\Omega)$, then the Laplacian $\triangle u$ is a distribution. By

$$-\triangle u \leq 0 \quad \text{in } \mathcal{D}(\Omega)'$$

we mean that

$$-\langle \triangle u, v\rangle \leq 0 \qquad \text{whenever } 0 \leq v \in \mathcal{D}(\Omega).$$

If $u \in H^1_{loc}(\Omega)$, this is equivalent to

$$\int_\Omega \nabla u(x) \nabla v(x)\, dx \leq 0 \quad \text{for } 0 \leq v \in \mathcal{D}(\Omega) \tag{3.1}$$

and if $u \in H^1(\Omega)$, both inequalities remain true for all $0 \leq u \in H_0^1(\Omega)$. In fact, the cone $\mathcal{D}(\Omega)_+$ of all positive test functions is dense in $H_0^1(\Omega)_+ := \left\{u \in H_0^1(\Omega) : \ u \geq 0\right\}$.

We frequently use the following maximum principle.

Lemma 3.1.1 *Let $u \in H^1(\Omega)$ such that*

$$-\triangle u \leq 0.$$

If $u^+ \in H_0^1(\Omega)$, then $u \leq 0$.

Proof. Taking $v = u^+$ in (3.1) we obtain $\int_\Omega |\nabla u(x)^+|^2\, dx \leq 0$. By Poincaré's inequality, this implies that $u^+ = 0$. □

The strategy now is as follows. Firstly we obtain a semigroup $e^{m\triangle t}$ on $L^2(\Omega, \frac{dx}{m(x)})$, which will then be extrapolated to all $L^p(\Omega, \frac{dx}{m(x)})$, $1 \leq p \leq \infty$. In this way we obtain a semigroup on $L^\infty(\Omega, \frac{dx}{m(x)}) = L^\infty(\Omega)$ and may pose the question whether there is a corresponding semigroup on the smaller space $C_0(\Omega) \subset L^\infty(\Omega)$. Although this does not hold true in general, it will be the case under some further (optimal) conditions.

Let $m : \Omega \to (0, \infty)$ be measurable such that $\frac{1}{m} \in L^1_{loc}(\Omega)$. We recall the Hilbert space $L^2(\Omega, \frac{dx}{m(x)})$ (already used in Section 2.1) with the scalar product

$$\langle u|v\rangle = \int_\Omega u(x)v(x)\, \frac{dx}{m(x)}.$$

On $L^2(\Omega, \frac{dx}{m(x)})$ we define the operator $m\triangle_2$ by

$$\begin{aligned} \mathcal{D}(m\triangle_2) \quad &:= \quad \left\{u \in H_0^1(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)}) : \exists f \in L^2(\Omega, \frac{dx}{m(x)}) \text{ s.t. } \triangle u = \frac{f}{m}\right\} \\ (m\triangle_2)u \quad &:= \quad f \end{aligned}$$

Note that $\frac{f}{m} \in L^1_{loc}(\Omega)$ since for $\omega \subset\subset \Omega$

$$\int_\omega \frac{|f(x)|}{m(x)}\, dx \leq (\int_\omega |f(x)|^2 \frac{dx}{m(x)})^{\frac{1}{2}} (\int_\omega \frac{dx}{m(x)})^{\frac{1}{2}}.$$

Thus the identity $\triangle u = \frac{f}{m}$ is well-defined in $\mathcal{D}(\Omega)'$. The expression $m\triangle_2$ is purely symbolic and has to be understood in the sense of the above definition. In fact, in general $\triangle u$ is merely in $\mathcal{D}(\Omega)'$ and $m\triangle u$ cannot be defined as a distribution.
We will prove the following theorem. Recall the definition of a submarkovian semigroup given in Section 1.4.

Theorem 3.1.2 *The operator $m\triangle_2$ is self-adjoint and generates a positive, contractive C_0-semigroup T_2 on $L^2(\Omega, \frac{dx}{m(x)})$. Moreover, the semigroup is submarkovian.*

Proof. Set $\mathcal{D}(a) := H_0^1(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$ and let $a : \mathcal{D}(a) \times \mathcal{D}(a) \to \mathbb{R}$ be given by

$$a(u, v) = \int_\Omega \nabla u(x) \nabla v(x)\, dx.$$

Then (as seen in Section 2.1) a is densely defined, accretive, closed and continuous bilinear form. Thus we may apply Theorem 2.1.6 and denote by A the operator associated with a . Then A is self-adjoint and A generates a contractive semigroup T_2 on $L^2(\Omega, \frac{dx}{m(x)})$. We show that $m\triangle_2 = A$. In fact, for $u, f \in L^2(\Omega, \frac{dx}{m(x)})$ we have by definition,

$$\begin{aligned} u \in \mathcal{D}(A) \text{ and } Au = f \qquad & \text{if and only if} \\ a(u, v) = -\int_\Omega f(x)v(x)\frac{dx}{m(x)} \qquad & \text{for all } v \in \mathcal{D}(a). \end{aligned}$$

Taking $v \in \mathcal{D}(\Omega)$, this implies that $\triangle u = \frac{f}{m}$. Hence $u \in \mathcal{D}(m\triangle_2)$ and $(m\triangle_2)u = f$. Conversely, if $u \in \mathcal{D}(m\triangle_2)$ and $(m\triangle_2)u = f$, then $\triangle u = \frac{f}{m}$ in $\mathcal{D}(\Omega)'$. Since $u \in H_0^1(\Omega)$, this implies that

$$\int_\Omega \nabla u(x) \nabla v(x)\, dx = -\langle \triangle u, v\rangle = -\int_\Omega f(x)v(x)\frac{dx}{m(x)}$$

for all $v \in \mathcal{D}(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}(a)$ it follows that $u \in \mathcal{D}(A)$ and $Au = f$. It follows from Lemmas 2.1.7 and 2.1.8 and the Beurling-Deny criteria of Theorems 1.4.10 and 1.4.11 that the semigroup is submarkovian. □

Because of the results of the last Theorem (symmetry and submarkovianity) we may use Theorem 1.4.13 to find a consistent family T_p, $1 \le p \le \infty$, of semigroups on $L^p(\Omega, \frac{dx}{m(x)})$, such that T_2 is the given semigroup generated by $m\triangle_2$. Here T_p is a positive, contractive C_0-semigroup for $1 \le p < \infty$ and $T_\infty(t) = T_1'(t)$ for all $t \ge 0$. We denote the generator of T_p by $m\triangle_p$. Thus $m\triangle_\infty = (m\triangle_1)'$.

We note that the consistency of semigroups implies the consistency of the resolvents. This follows from the representation of the resolvents as the Laplace transform of the semigroups. In particular

$$R(\lambda, m\triangle_\infty)f = R(\lambda, m\triangle_2)f \tag{3.2}$$

for all $\lambda > 0$, $f \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$. We also note that

$$R(\lambda, m\triangle_\infty) \ge 0 \qquad \text{for all } \lambda > 0.$$

We will frequently use the following local regularity of the Laplacian.
Let $\frac{\hat{N}}{2} < p \le \infty$. Then

$$u \in L^1_{loc}(\Omega),\ \triangle u \in L^p_{loc}(\Omega) \quad \text{implies } u \in C(\Omega). \tag{3.3}$$

See Appendix Theorem A.4.1. To avoid confusion in the case $N = 1$ we shall tacitly assume $p \ge 1$ throughout even if not explicitly stated.

If $m \equiv 1$, then the operator $\triangle_p = m\triangle_p$ is just the Dirichlet Laplacian on $L^p(\Omega)$. We need the following properties of this operator.

Proposition 3.1.3 *The operator $\triangle_p$ is invertible. Moreover, for $\frac{\hat{N}}{2} < p \le \infty$ the following holds:*

(a) $\mathcal{D}(\triangle_p) = \{u \in H_0^1(\Omega) : \triangle u \in L^p(\Omega)\}$ *and* $\triangle_p u = \triangle u$ *in* $\mathcal{D}(\Omega)'$ *for all* $u \in \mathcal{D}(\triangle_p)$

(b) $\mathcal{D}(\triangle_p) \subset C^b(\Omega) := \{u : \Omega \to \mathbb{R} :$ *u is bounded and continuous* $\}$

We give the proof in the Appendix, see Proposition A.4.2.

Now we can add the following local regularity of the Laplacian.

Proposition 3.1.4 *Let $\frac{\hat{N}}{2} < p \le \infty$ and Ω be an open set in $\mathbb{R}^N$. Then*

$$u \in L^1_{loc}(\Omega),\ \triangle u \in L^p_{loc}(\Omega) \text{ implies } u \in H^1_{loc}(\Omega). \tag{3.4}$$

Proof. Let $u \in L^1_{loc}(\Omega)$ such that $\triangle u \in L^p_{loc}(\Omega)$. Let $\omega \subset\subset \Omega$ be arbitrary and $f = \triangle u_{|\omega} \in L^p(\omega)$. Consider the operator $\triangle_p$ on $L^p(\omega)$. Then $w := \triangle_p^{-1} f \in H_0^1(\omega)$ by Proposition 3.1.3. Since $\triangle w = f = \triangle u$ in $\mathcal{D}(\Omega)'$, the function $u - w$ is harmonic and hence in $C^\infty(\omega)$. Thus $u \in H^1(\omega)$. □

In the following we consider again a function $m : \Omega \to (0, \infty)$ satisfying $\frac{1}{m} \in L^1_{loc}(\Omega)$. We first show how $m\triangle_\infty$ operates on functions.

Proposition 3.1.5 **(a)** *Let $u \in \mathcal{D}(m\triangle_\infty)$, $f = (m\triangle_\infty)u$. Then*

$$\triangle u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

(b) *If $\frac{1}{m} \in L^p_{loc}(\Omega)$ for some $p > \frac{\hat{N}}{2}$, then*

$$\mathcal{D}(m\triangle_\infty) \subset C^b(\Omega) \cap H^1_{loc}(\Omega).$$

(c) *If $m \in L^\infty_{loc}(\Omega)$, then $\mathcal{D}(\Omega) \subset \mathcal{D}(m\triangle_\infty)$ and $(m\triangle_\infty)u = m \cdot \triangle u$ for $u \in \mathcal{D}(\Omega)$.*

Proof. *(a)* Let $\lambda > 0$. Define $g := \lambda u - f \in L^\infty(\Omega)$. Then $u = R(\lambda, m\triangle_\infty)g$. If $g \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$, then the claim follows from the fact that $R(\lambda, m\triangle_\infty)g = R(\lambda, m\triangle_2)g$. In the general case there exist $g_k \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$ such that $g_k \to g$ for $\sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}))$. Let $u_k = R(\lambda, m\triangle_\infty)g_k$. Then

$$-\triangle u_k = \frac{g_k - \lambda u_k}{m}$$

Now we use that $R(\lambda, m\triangle_\infty) = R(\lambda, m\triangle_1)'$ is continuous for the *weak*∗-topology $\sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}))$. Hence $u_k \to u$ for $\sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}))$. Since $\mathcal{D}(\Omega) \subset L^1(\Omega, \frac{dx}{m(x)})$ we conclude that $u_k \to u$ in $\mathcal{D}(\Omega)'$. Hence $\triangle u_k \to \triangle u$ in $\mathcal{D}(\Omega)'$. Since $g_k - \lambda u_k \to g - \lambda u$ for $\sigma(L^\infty(\Omega), L^1(\Omega, \frac{dx}{m(x)}))$, it follows that $\frac{g_k - \lambda u_k}{m} \to \frac{g - \lambda u}{m}$ in $\mathcal{D}(\Omega)'$. Thus

$$-\triangle u = \frac{g - \lambda u}{m} = -\frac{f}{m}.$$

The proof of *(a)* is complete.
(b) This follows now from (3.3) and (3.4).
(c) Assume that $m \in L^\infty_{loc}(\Omega)$. Let $u \in \mathcal{D}(\Omega)$, $f = m \cdot \triangle u$. Then $u \in H_0^1(\Omega)$, $f \in L^2(\Omega, \frac{dx}{m(x)})$ and $\triangle u = \frac{f}{m}$. Thus $u \in \mathcal{D}(m\triangle_2)$ and $(m\triangle_2)u = f$. Let $\lambda > 0$ and set $g := \lambda u - f$. Then $g \in L^\infty(\Omega) \cap L^2(\Omega, \frac{dx}{m(x)})$ and $R(\lambda, m\triangle_\infty)g = R(\lambda, m\triangle_2)g = u$.

Thus $u \in \mathcal{D}(m\triangle_\infty)$ and $\lambda u - (m\triangle_\infty)u = g = \lambda u - f$, i.e., $(m\triangle_\infty)u = f$. □
In Proposition 3.1.5, the boundary condition is not incorporated. But if $\frac{1}{m} \in L^1(\Omega)$, then $L^\infty(\Omega) \subset L^2(\Omega, \frac{dx}{m(x)})$ and the operator $m\triangle_\infty$ is just the part of $m\triangle_2$ in $L^\infty(\Omega)$. Thus, if $\frac{1}{m} \in L^1(\Omega)$, then

$$\mathcal{D}(m\triangle_\infty) = \left\{u \in H_0^1(\Omega) \cap L^\infty(\Omega) : \exists f \in L^\infty(\Omega) \text{ s.t. } \triangle u = \frac{f}{m}\right\} \quad (3.5)$$
$$(m\triangle_\infty)u = f.$$

If $\frac{1}{m} \in L^p(\Omega)$ for some $\infty \geq p > \frac{\hat{N}}{2}$, we can even assert more.

Proposition 3.1.6 *Assume that $\frac{1}{m} \in L^p(\Omega)$ where $\frac{\hat{N}}{2} < p \leq \infty$. Then $m\triangle_\infty$ is invertible.*

Proof. Let $f \in L^\infty(\Omega)$. Then $\frac{f}{m} \in L^p(\Omega)$. Thus by Proposition 3.1.3 there exists $u \in H_0^1(\Omega)$ such that $\triangle u = \frac{f}{m}$. This shows that $m\triangle_\infty$ is surjective. If $u \in \mathcal{D}(m\triangle_\infty)$, $(m\triangle_\infty)u = 0$, then by (3.5) we have $u \in H_0^1(\Omega)$ and $\triangle u = 0$. This implies that $u = 0$. Thus $(m\triangle_\infty)$ is injective. Since the operator is closed, the proof is finished.

□

3.2 The operator $m\triangle_0$ on $C_0(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let $m : \Omega \to (0, \infty)$ be a measurable function such that $m \in L^\infty_{loc}(\Omega)$ and $\frac{1}{m} \in L^p_{loc}$ where $p > \frac{\hat{N}}{2}$. We want to define a maximal realization of $m\triangle$ in $C_0(\Omega)$. If $u \in C_0(\Omega)$ then $\triangle u \in \mathcal{D}(\Omega)'$, but $m\triangle u$ may not be defined as a distribution. Thus the following definition is natural.

Definition 3.2.1 *We define the operator $m\triangle_0$ on $C_0(\Omega)$ by*

$$\begin{aligned} \mathcal{D}(m\triangle_0) &:= \left\{u \in C_0(\Omega) : \exists f \in C_0(\Omega) \textit{ s.t. } \triangle u = \frac{f}{m}\right\} \\ (m\triangle_0)u &:= f \end{aligned} \quad (3.6)$$

Since $\frac{f}{m} \in L^1_{loc} \subset \mathcal{D}(\Omega)'$ this definition makes sense. The notation $(m\triangle_0)$ is purely symbolic. But if $u \in C_0(\Omega) \cap C^2(\Omega)$ such that $m \cdot \triangle u \in C_0(\Omega)$, then $u \in \mathcal{D}(m\triangle_0)$ and $(m\triangle_0)u = m \cdot \triangle u$.

Proposition 3.2.2 *The operator $m\triangle_0$ is closed and dissipative. Moreover, if $R(\lambda_0, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$ for some $\lambda_0 > 0$, then $m\triangle_0$ generates a C_0-semigroup of positive contractions on $C_0(\Omega)$. In that case*

$$(0, \infty) \subset \rho(m\triangle_0)$$
$$R(\lambda, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega) \quad \textit{for all } \lambda > 0 \qquad \textit{and}$$
$$R(\lambda, m\triangle_0) = R(\lambda, m\triangle_\infty)_{|C_0(\Omega)}.$$

Note that in general, $\mathcal{D}(\Omega) \nsubseteq \mathcal{D}(m\triangle_0)$ since we do not assume that m is continuous. Thus in Proposition 3.2.2 density of the domain (which is necessary for the generation property) needs a separate argument.
Since $m\triangle_0$ is dissipative, it follows in particular that no proper restriction of $m\triangle_0$ may generate a C_0-semigroup on $C_0(\Omega)$.
We first prove dissipativity.

Lemma 3.2.3 *Let $\lambda > 0,\ u = \mathcal{D}(m\triangle_0),\ f = \lambda u - (m\triangle_0)u$. Let $c > 0$ be such that*

$$f(x) \le c \quad \text{for all } x \in \Omega.$$

Then $\lambda u(x) \le c$ for all $x \in \Omega$.

Proof. By the definition of the operator we have

$$\lambda \frac{u}{m} - \triangle u = \frac{f}{m} \le \frac{c}{m}.$$

Since by (3.4) $u \in H^1_{loc}(\Omega)$, this implies that for $0 \le v \in \mathcal{D}(\Omega)$

$$\int_\Omega \frac{(\lambda u(x) - c)}{m(x)} v(x)\, dx + \int_\Omega \nabla u(x) \nabla v(x)\, dx \le 0. \tag{3.7}$$

Since $u \in C_0(\Omega)$, $(\lambda u - c)^+$ has compact support. Let $\omega \subset\subset \Omega$ such that supp $(\lambda u - c)^+ \subset \omega$. Then $(\lambda u - c)^+ \in H^1_0(\omega)$ and $(\lambda u - c) \in H^1(\omega)$. Now (3.7) implies that

$$\int_\omega \frac{(\lambda u(x) - c)}{m(x)} v(x)\, dx + \frac{1}{\lambda} \int_\omega \nabla(\lambda u(x) - c) \nabla v(x)\, dx \le 0$$

for all $0 \le v \in H^1_0(\omega)$. Taking in particular, $v := (\lambda u - c)^+$ we see that

$$\int_\omega \frac{(\lambda u(x) - c)^{+2}}{m(x)}\, dx + \frac{1}{\lambda} \int_\omega |\nabla(\lambda u(x) - c)^+|^2\, dx \le 0$$

This implies that $(\lambda u - c)^+ = 0$, i.e., $\lambda u \le c$. □

Applying Lemma 3.2.3 to $\pm u$, we see that

$$\|\lambda u\|_{L^\infty(\Omega)} \le \|\lambda u - (m\triangle_0)u\|_\infty$$

for all $u \in \mathcal{D}(m\triangle_0)$, i.e., $m\triangle_0$ is dissipative. But in fact, Lemma 3.2.3 shows that the operator $m\triangle_0$ is *dispersive.* We refer to ([8], [50],Chapter II) for this notion.

Proof of Proposition 3.2.2. The dissipativity has been proved above and the closedness is easy to see. Let now $R(\lambda, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$ for some $\lambda > 0$. We show that $\lambda \in \rho(m\triangle_0)$ and $R(\lambda, m\triangle_0) = R(\lambda, m\triangle_\infty)_{|C_0(\Omega)}$. Let $f \in C_0(\Omega)$ and consider $u = R(\lambda, m\triangle_\infty)f \in C_0(\Omega)$. Then (by Proposition 3.1.5)

$$\lambda \frac{u}{m} - \triangle u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

It follows that $u \in \mathcal{D}(m\triangle_0)$ and $(\lambda u - (m\triangle_0)u) = f$. We have shown that $\lambda - (m\triangle_0)$ is surjective. Since the injectivity of $(\lambda - m\triangle_0)$ follows from the dissipativity of $m\triangle_0$, the closed graph theorem implies now that $\lambda \in \rho(m\triangle_0)$. The calculation above shows also that $R(\lambda, m\triangle_0)f = u = R(\lambda, m\triangle_\infty)f$.
By the resolvent identity (see ([2], Proposition 3.II.2)) we have for $0 \le f \in C_0(\Omega)$ and $\lambda > \lambda_0$

$$0 \le R(\lambda, m\triangle_\infty)f \le R(\lambda_0, m\triangle_\infty)f \in C_0(\Omega).$$

Since by Proposition 3.1.3 the function $R(\lambda, m\triangle_\infty)f$ is continuous, it follows from the domination property above that $R(\lambda, m\triangle_\infty)f \in C_0(\Omega)$. Thus $C_0(\Omega)$ is invariant for all $\lambda \ge \lambda_0$. Hence $[\lambda_0, \infty) \subset \rho(m\triangle_0)$.
Next we show that $\mathcal{D}(m\triangle_0)$ is dense in $C_0(\Omega)$. Since $m \in L^\infty_{loc}(\Omega)$, we have $\mathcal{D}(\Omega) \subset \mathcal{D}(m\triangle_\infty)$ by Proposition 3.1.5. Hence $C_0(\Omega) \subset \overline{\mathcal{D}(m\triangle_\infty)}$. Thus, for $f \in C_0(\Omega)$ one has

$$\lim_{\lambda\to\infty} \lambda R(\lambda, m\triangle_0)f = \lim_{\lambda\to\infty} \lambda R(\lambda, m\triangle_\infty)f = f.$$

Since $\lambda R(\lambda, m\triangle_0)f \in \mathcal{D}(m\triangle_0)$, density of the domain is proved. Now the Lumer-Phillips theorem implies that $m\triangle_0$ generates a contractive C_0-semigroup. Since the resolvent of $m\triangle_0$ is positive, this semigroup is positive. It follows also that $(0,\infty) \subset \rho(m\triangle_0)$. □

We will now consider two cases which imply the invariance given in Proposition 3.2.2 namely that Ω is Dirichlet regular or that the diffusion coefficient $m(x)$ tends to 0 fast enough as x approaches the boundary. We start discussing Dirichlet regularity.

3.3 Regular points

Let $\Omega \subset \mathbb{R}^N$ be open, bounded set. Consider the Dirichlet problem.

$$\begin{aligned} h \in C(\overline{\Omega}) \cap C^2(\Omega) \\ \triangle h = 0 \text{ in } \Omega \\ h_{|\partial\Omega} = \varphi \end{aligned} \tag{3.8}$$

where $\varphi \in C(\partial\Omega)$ is given.

Definition 3.3.1 *We say that Ω is called Dirichlet regular, if for each $\varphi \in C(\partial\Omega)$ a (necessarily unique) solution of (3.8) exists.*

If Ω has Lipschitz boundary then Ω is Dirichlet regular. Much weaker geometric properties of the boundary suffice, though. In dimension $N = 1$ each bounded open subset Ω of $\mathbb{R}$ is Dirichlet regular. If $N = 2$ then each simply connected bounded open set is Dirichlet regular. This is no longer true in $\mathbb{R}^3$. The Lebesgue cusp gives an example of a simply connected domain with continuous boundary, which is not Dirichlet regular (see [10] for more information).

Let $\frac{\hat{N}}{2} < p \leq \infty$. Let $m : \Omega \to (0,\infty)$ be measurable such that $m \in L^\infty_{loc}(\Omega)$ and $\frac{1}{m} \in L^p_{loc}(\Omega)$. We shall prove the following theorem.

Theorem 3.3.2 *If Ω is Dirichlet regular, then $m\triangle_0$ generates a positive contractive C_0-semigroup on $C_0(\Omega)$.*

Thus in the case of a Dirichlet regular set, no condition on $m(x)$ as x approaches the boundary is needed. We merely impose a (very weak) regularity condition on m in the interior of Ω.
It will be useful to prove an individual version of Theorem 3.3.2 first. For this we have to recall the notion of regular points. Consider the problem (3.8) again.

Definition 3.3.3 *A function $u \in C(\overline{\Omega})$ is called a subsolution if*

$$-\triangle u \leq 0 \text{ in } \mathcal{D}(\Omega)' \quad \text{and} \quad \limsup_{x\to z,\, x\in\Omega} u(x) \leq \varphi(z) \quad \text{for all } z \in \partial\Omega.$$

A function $u \in C(\overline{\Omega})$ is called a supersolution if

$$-\triangle u \geq 0 \text{ in } \mathcal{D}(\Omega)' \quad \text{and} \quad \liminf_{x\to z,\, x\in\Omega} u(x) \geq \varphi(z) \quad \text{for all } z \in \partial\Omega.$$

One has the following well-known result.

Theorem 3.3.4 ***(Perron)***
Let $\varphi \in C(\partial\Omega)$. Then for all $x \in \Omega$

$$h_\varphi(x) := \sup\{u(x) : \quad u \text{ is a subsolution}\}$$

exists. Moreover,

$$h_\varphi(x) = \inf\{v(x) : \quad v \textit{ is a supersolution}\}.$$

The function h_φ is harmonic and

$$\inf_{\partial\Omega} \varphi \le h_\varphi(x) \le \sup_{\partial\Omega} \varphi$$

for all $x \in \Omega$. If (3.8) has a solution h, then $h_\varphi = h$.

Definition 3.3.5 *The function h_φ is called the Perron solution of (3.8). A point $z \in \partial\Omega$ is called regular if*

$$\lim_{x\to z,\, x\in\Omega} h_\varphi(x) = \varphi(z)$$

for all $\varphi \in C(\partial\Omega)$.

Thus Ω is Dirichlet regular if and only if each point $z \in \partial\Omega$ is regular. It is possible to characterize regular points by the existence of a barrier or by a capacity condition (Wiener's theorem). We refer to [40].

Now we can formulate the local version of Theorem 3.3.2 which we want to prove.

Theorem 3.3.6 *Let Ω be bounded and open. Let $z \in \partial\Omega$ be a regular point. Let $\lambda > 0,\ f \in C_0(\Omega),\ u = R(\lambda, m\triangle_\infty)f$. Then*

$$\lim_{x\to z,\, x\in\Omega} u(x) = 0.$$

Thus, if Ω is Dirichlet regular, then $C_0(\Omega)$ is invariant under $R(\lambda, m\triangle_\infty)$ and Theorem 3.3.2 follows from Proposition 3.2.2.
For the proof of Theorem 3.3.6 we use the following variational characterization of the Perron solution (see [10]). We use the notation $H^{-1}(\Omega) := H_0^1(\Omega)'$.

Theorem 3.3.7 *Let $\Phi \in C(\overline{\Omega})$ be such that $\triangle\Phi \in H^{-1}(\Omega)$. Let $\varphi = \Phi_{|\partial\Omega}$. Let u be the unique solution of*

$$\begin{aligned} u &\in H_0^1(\Omega) \\ -\triangle u &= \triangle\Phi. \end{aligned}$$

Then $h_\varphi = \Phi + u$.

Corollary 3.3.8 *Let $v \in C_0(\Omega)$ such that $\triangle v \in H^{-1}(\Omega)$. Then $v \in H_0^1(\Omega)$.*

Proof. We set $\Phi := v$ in Theorem 3.3.7. Then $\varphi = 0$ and $h_\varphi = 0$. Hence $u + v = 0$, i.e. $v = -u \in H_0^1(\Omega)$. □

For our purposes the following consequence of Theorem 3.3.7 is important. Recall that by Proposition 3.1.3 for all $f \in L^p(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ such that

$$-\triangle u = f \quad \text{in } \mathcal{D}(\Omega)'.$$

In fact, $u = R(0, \triangle_p)f$ where $\triangle_p$ denotes the Dirichlet Laplacian on $L^p(\Omega)$. Moreover, one has $u \in C^b(\Omega)$.

Corollary 3.3.9 *Let $f \in L^p(\Omega)$, $u = R(0, \triangle_p)f$. Then*

$$\lim_{x\to z,\, x\in\Omega} u(x) = 0$$

for each regular point $z \in \partial\Omega$. Thus, if Ω is Dirichlet regular, then $u \in C_0(\Omega)$.

Proof. It follows from the Sobolev embedding theorem that $L^p(\Omega) \subset H^{-1}(\Omega)$. Let $f \in L^p(\Omega)$. Let $\Phi = E * f$, where E is the Newtonian potential. Then (by [21], II.3, Proposition 6) $\Phi \in C(\mathbb{R}^N)$ and in $\mathcal{D}(\Omega)'$ we have

$$\triangle\Phi = f \in L^p(\Omega) \subset H^{-1}(\Omega).$$

Let $u = R(0, \triangle_p)f$. Then it follows from Theorem 3.3.7 that $h_\varphi = \Phi + u$. Thus

$$\lim_{x\to z,\, x\in\Omega} h_\varphi(x) = \varphi(z) \qquad \text{if } z \in \partial\Omega \text{ is regular.}$$

Consequently[1], $\lim_{x\to z} u(x) = 0$. □

Remark. a) In [3] a more special case of Corollary 3.3.9 is proved with the help of H^1-barriers (proof of Theorem 3.8 in [3]).
b) Special cases of Theorem 3.3.7 were obtained before by Hildebrandt [37] and Simader [59].

Proof of Theorem 3.3.6. (a) Let $\lambda > 0$, $0 \le f \in C_c(\Omega)$, $u = R(\lambda, m\triangle_\infty)f$. Then $u \in H_0^1(\Omega)$ and

$$\lambda\frac{u}{m} - \triangle u = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

Moreover $0 \le u \in C^b(\Omega)$. Observe that $0 \le \frac{f}{m} \in L^p(\Omega)$. Let $w = R(0, \triangle_p)\frac{f}{m}$. Then we know that $0 \le w \in H_0^1(\Omega) \cap C^b(\Omega)$ and, by Corollary 3.3.9, $\lim_{x\to z} w(x) = 0$ for all regular points $z \in \partial\Omega$. By definition

$$-\triangle w = \frac{f}{m} \quad \text{in } \mathcal{D}(\Omega)'.$$

Thus $-\triangle(u-w) \le 0$ in $\mathcal{D}(\Omega)'$. Since $u-w \in H^1(\Omega)$ and $(u-w)^+ \in H_0^1(\Omega)$, it follows from the maximum principle (Lemma 3.1.1) that $u \le w$. Hence $\lim_{x\to z} u(x) = 0$ for each regular point $z \in \partial\Omega$.
(b) Let $z \in \partial\Omega$ be a regular point. Then by *(a)*

$$\lim_{x\to z, x\in\Omega} (R(\lambda, m\triangle_\infty)f)(x) = 0$$

for each $0 \le f \in C_c(\Omega)$, hence also for each $f \in C_c(\Omega)$. Since $C_c(\Omega)$ is dense in $C_0(\Omega)$, this remains true for all $f \in C_0(\Omega)$. □

Next we show a converse of Theorem 3.3.2. If the diffusion coefficient m is not weak enough at the boundary, then Dirichlet regularity is necessary for $m\triangle_0$ to generate a C_0-semigroup. More precisely, the following holds. Recall that $\frac{\hat{N}}{2} < p \le \infty$.

Theorem 3.3.10 *Assume that $\frac{1}{m} \in L^p(\Omega)$. Then $m\triangle_0$ generates a C_0-semigroup if and only if Ω is Dirichlet regular.*

For the proof we need the following.

[1] We will sometimes use the notation $\lim_{x\to z} f(x) := \lim_{x\to z,\, x\in\Omega} f(x)$ for $f : \Omega \to \mathbb{R}$

Proposition 3.3.11 *Let $u \in C_0(\Omega)$ be such that $-\triangle u = f \in L^p(\Omega)$ for some $p > \frac{\hat{N}}{2}$. Then $u \in H_0^1(\Omega)$, hence $u = R(0, \triangle_p)f$.*

Proof. This follows from Corollary 3.3.8 since $L^p(\Omega) \subset H^{-1}(\Omega)$. □

Proof of Theorem 3.3.10. Assume that $m\triangle_0$ generates a C_0-semigroup. Since $\frac{1}{m} \in L^p(\Omega)$, we know from Proposition 3.1.6 that $[0, \infty) \subset \rho(m\triangle_\infty)$ and $R(\lambda, m\triangle_\infty) \geq 0$ for all $\lambda \geq 0$. We now claim that $R(\lambda, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$ and $R(\lambda, m\triangle_0) = R(\lambda, m\triangle_\infty)_{|C_0(\Omega)}$ for any $\lambda > 0$. Let $f \in C_0(\Omega)$, $u = R(\lambda, m\triangle_0)f$. Then

$$-\triangle u = \frac{f}{m} - \lambda\frac{u}{m} \in L^p(\Omega).$$

Since $u \in C_0(\Omega)$, it follows from Proposition 3.3.11 that $u \in H_0^1(\Omega)$. Since $\frac{1}{m} \in L^p(\Omega)$ we have $L^\infty(\Omega) \subset L^2(\Omega, \frac{dx}{m(x)})$. Thus by (3.5) we have $u \in \mathcal{D}(m\triangle_\infty)$ and $\lambda u - (m\triangle_\infty)u = f$. Hence $u = R(\lambda, m\triangle_\infty)f$. This proves the claim.
Since $0 \in \rho(m\triangle_\infty)$, the claim implies that

$$\limsup_{\lambda\to 0} \|R(\lambda, m\triangle_0)\|_{\mathcal{L}(C_0(\Omega))} < \infty,$$

hence $0 \in \rho(m\triangle_0)$ and $R(0, m\triangle_0) \geq 0$.
Let $0 \leq f \in C_0(\Omega)$, $f(x) > 0$ for all $x \in \Omega$, $u = R(0, m\triangle_0)f$. Then $u \in C_0(\Omega)$ and $-\triangle u = \frac{f}{m}$ in $\mathcal{D}(\Omega)'$. Hence $R(0, \triangle_p)\frac{f}{m} = u \in C_0(\Omega)$ by Proposition 3.3.11. We deduce that $R(0, \triangle_p)g \in C_0(\Omega)$ for all $g \in L^p(\Omega)$ such that $|g| \leq \frac{f}{m}$ for some $0 \leq f \in C_0(\Omega)$. The space of all such functions g is dense in $L^p(\Omega)$. Thus $R(0, \triangle_p)L^p(\Omega) \subset C_0(\Omega)$. Now it follows from Theorem A.4.3 (Appendix) that Ω is Dirichlet regular. □

3.4 Points of weak diffusion

Let $\Omega \subset \mathbb{R}^N$ be open and bounded and let $m : \Omega \to (0, \infty)$ be a bounded measurable function such that $\frac{1}{m} \in L^p_{loc}(\Omega)$ for some $\frac{\hat{N}}{2} < p \leq \infty$. Instead of regularity we may assume that m is small in a neighbourhood of a boundary point. We say that $z \in \partial\Omega$ is a *point of weak diffusion* (for the operator $m\triangle$) if there exist $r > 0$, $c > 0$ such that

$$m(x) \leq c \cdot dist(x, \partial\Omega)^2 \tag{3.9}$$

for all $x \in \Omega \cap B(z, r)$. If $z \in \partial\Omega$ is a point of weak diffusion, then we show that

$$\lim_{x\to z,\ x\in\Omega} (R(\lambda, m\triangle_\infty)f)(x) = 0 \tag{3.10}$$

for all $f \in C_0(\Omega)$. We will also show that the condition (3.9) is optimal in the sense that

$$m(x) \leq c \cdot dist(x, \partial\Omega)^\alpha$$

for some $0 < \alpha < 2$ does not suffice to enforce (3.10).
We use a regularized distance function. For a construction of such a function see the Section 1.9. From the Theorem 1.9.5 we obtain directly the following corollary.

Corollary 3.4.1 *There exist a constant $c_\sigma > 0$ and a function $\sigma : \Omega \to (0, +\infty)$, which is of class $C^\infty(\Omega)$ and fulfils:*

$$\begin{array}{rcl} c_\sigma^{-1} d(x) \leq & \sigma(x) & \leq c_\sigma d(x) \\ |\nabla\sigma|^2 & \leq & c_\sigma \\ |\sigma\triangle\sigma| & \leq & c_\sigma \end{array}$$

for all $x \in \Omega$, where $d(x) := \inf\left\{\|x - y\|,\ y \in \mathbb{R}^d\backslash\Omega\right\}$.

Proof. This follows immediately from the Theorem 1.9.5. □

Since $\sigma \in C_0(\Omega)$ it follows in particular that $\sigma \in H_0^1(\Omega)$. At first we now consider the case $m(x) := \sigma(x)^2$.

Proposition 3.4.2 *The operator $\sigma^2\triangle_0$ generates a strongly continuous semigroup of positive contractions on $C_0(\Omega)$.*

Proof. Let $\lambda \geq c_\sigma + 1$ where c_σ is a constant from Lemma 3.4.1. Set $u = R(\lambda, \sigma^2\triangle_\infty)\sigma$. Since $\sigma \in L^2(\Omega, \frac{dx}{\sigma(x)^2})$ it follows from (3.2) that $0 \leq u \in H_0^1(\Omega) \cap L^2(\Omega, \frac{dx}{\sigma(x)^2})$ and

$$\lambda\frac{u}{\sigma^2} - \triangle u = \frac{\sigma}{\sigma^2} \quad \text{in } \mathcal{D}(\Omega)'.$$

Since $\sigma\triangle\sigma \leq c_\sigma$, it follows that $\sigma \leq \lambda\sigma - c_\sigma\sigma \leq \lambda\sigma - \sigma^2\triangle\sigma$. Thus

$$\lambda\frac{u}{\sigma^2} - \triangle u = \frac{1}{\sigma^2}\sigma \leq \lambda\frac{\sigma}{\sigma^2} - \triangle\sigma \quad \text{in } \mathcal{D}(\Omega)'.$$

Hence

$$\lambda\frac{(u-\sigma)}{\sigma^2} - \triangle(u-\sigma) \leq 0 \quad \text{in } \mathcal{D}(\Omega)'.$$

Since $u - \sigma \in H^1(\Omega)$ and $(u-\sigma)^+ \leq u \in H_0^1(\Omega)$, it follows that $(u-\sigma)^+ \in H_0^1(\Omega)$. Now the maximum principle (Lemma 3.1.1) implies that $(u-\sigma)^+ \leq 0$, i.e., $u \leq \sigma$. We have shown that

$$R(\lambda, \sigma^2\triangle_\infty)\sigma \leq \sigma \qquad (\lambda \geq \lambda_0 := 1 + c_\sigma). \tag{3.11}$$

Thus, for $f \in C_0(\Omega)$ such that $|f| \leq c\sigma$ one has

$$|R(\lambda, \sigma^2\triangle_\infty)f| \leq cR(\lambda, \sigma^2\triangle_\infty)\sigma \leq c\sigma.$$

Consequently, $R(\lambda, \sigma^2\triangle_\infty)f \in C_0(\Omega)$ for $\lambda \geq \lambda_0$. Since functions satisfying $|f| \leq c\sigma$ for some $c \geq 0$ are dense in $C_0(\Omega)$ we deduce that $R(\lambda, \sigma^2\triangle_\infty)C_0(\Omega) \subset C_0(\Omega)$ for $\lambda \geq \lambda_0$. Now the claim follows from Proposition 3.2.2. □

We comment that the result of Proposition 3.4.2 may be alternatively deduced from ([24], Theorem 5.4). However, our argument given here is quite different from [24].

We need a local extension of the resolvents of $\sigma^2\triangle$. Recall that $\frac{\hat{N}}{2} < p \leq \infty$.

Lemma 3.4.3 *Let $\omega \subset\subset \Omega$, $\lambda > 0$. There exists an operator*

$$Q(\lambda, \omega) \in \mathcal{L}(L^p(\omega), C_0(\Omega))$$

such that

$$Q(\lambda, \omega)f = R(\lambda, \sigma^2\triangle_0)f \quad \text{for all } f \in L^p(\omega) \cap C_0(\Omega).$$

For $f \in L^p(\omega)$ the function $u = Q(\lambda, \omega)f$ is the unique solution of

$$\begin{aligned} & u \in C_0(\Omega) \\ & \lambda\frac{u}{\sigma^2} - \triangle u = \frac{f}{\sigma^2} \text{ in } \mathcal{D}(\Omega)'. \end{aligned} \tag{3.12}$$

Moreover, $u \in H_0^1(\Omega)$.

Here we consider $L^p(\omega)$ as a subspace of $L^p(\Omega)$ extending functions by 0 outside ω. Similarly, we consider $C_c(\omega) \subset C_0(\omega) \subset C_0(\Omega)$.

Proof. *(a)* Let $0 \le f \in C_c(\omega)$. There exists $\delta > 0$ such that $\sigma^2 \ge \delta$ on ω. Let $u = R(\lambda, \sigma^2\triangle_0)f = R(\lambda, \sigma^2\triangle_2)f$. Then $0 \le u \in H_0^1(\Omega)$ and

$$\lambda\frac{u}{\sigma^2} - \triangle u = \frac{f}{\sigma^2} \le \frac{1}{\delta}f.$$

Let $w := \frac{1}{\delta}R(0, \triangle_p)f$, where $\triangle_p$ denotes the Dirichlet Laplacian on $L^p(\Omega)$. Then $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and

$$-\triangle w = \frac{1}{\delta}f \quad \text{ in } \mathcal{D}(\Omega)'.$$

Moreover $\|w\|_{L^\infty(\Omega)} \le c_1\|f\|_{L^p(\omega)}$ where $c_1 = \frac{1}{\delta}\|R(0,\triangle_p)\|_{\mathcal{L}(L^p(\Omega),L^\infty(\Omega))}$ (see Proposition 3.1.3(b)). We show that $u \le w$. In fact, we have

$$\begin{aligned}-\triangle u &\le \lambda\frac{u}{\sigma^2} - \triangle u \le \frac{1}{\delta}f \quad \text{ and}\\ -\triangle w &= \frac{1}{\delta}f,\end{aligned}$$

hence $-\triangle(u - w) \le 0$ in $\mathcal{D}(\Omega)'$. Consequently, by the maximum principle (Lemma 3.1.1), $u \le w$. Thus

$$\|u\|_{L^\infty(\Omega)} \le \|w\|_{L^\infty(\Omega)} \le c_1\|f\|_{L^p(\omega)}.$$

We have shown that

$$\|R(\lambda, \sigma^2\triangle_0)f\|_{L^\infty(\Omega)} \le c_1\|f\|_{L^p(\omega)} \tag{3.13}$$

for $0 \le f \in C_c(\omega)$. Since for arbitrary $f \in C_c(\omega)$,

$$|R(\lambda, \sigma^2\triangle_0)f| \le R(\lambda, \sigma^2\triangle_0)|f|,$$

the estimate (3.13) remains true for all $f \in C_c(\omega)$. By the density of $C_c(\omega)$ in $L^p(\omega)$ the first claim is proved.
(b) In order to prove the second, let $f \in L^p(\omega)$, $u = Q(\lambda, \omega)f$. Let $f_k \in C_c(\omega)$ be such that $f_k \to f$ in $L^p(\omega)$. Then $u_k := Q(\lambda, \omega)f_k \to u$ in $C_0(\Omega)$. We have $u_k \in H_0^1(\Omega) \cap C_0(\Omega)$ and

$$\lambda\frac{u_k}{\sigma^2} - \triangle u_k = \frac{f_k}{\sigma^2} \quad \text{ in } \mathcal{D}(\Omega)'. \tag{3.14}$$

Passing to the limit as $k \to \infty$ shows that (3.12) holds.
It remains to show that $u \in H_0^1(\Omega)$. Multiplying (3.14) by u_k and integrating yields

$$\begin{aligned}&\lambda\int_\Omega \frac{u_k(x)^2}{\sigma(x)^2}\,dx + \int_\Omega |\nabla u_k(x)|^2\,dx = \int_\Omega \frac{f_k(x)u_k(x)}{\sigma(x)^2}\,dx \le\\ &\le \|u_k\|_{L^\infty(\Omega)}\frac{1}{\delta^2}\cdot|\Omega|^{\frac{1}{p'}}\|f_k\|_{L^p(\Omega)}.\end{aligned}$$

This shows that $(u_k)_{k\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Thus, passing to a subsequence we may assume that $u_k \rightharpoonup w \in H_0^1(\Omega)$. Since $u_k \to u$ in $C_0(\Omega)$, it follows that $u = w \in H_0^1(\Omega)$. □

Now we consider a more general function m satisfying the hypothesis formulated in the beginning of this section. We prove regularity of $m\triangle_\infty$ at points of weak diffusion.

Theorem 3.4.4 *Let $z \in \partial\Omega$ be a point of weak diffusion (in the sense of (3.9)). Let $f \in C_0(\Omega)$, $\lambda > 0$, $u = R(\lambda, m\triangle_\infty)f$. Then*

$$\lim_{x \to z,\, x \in \Omega} u(x) = 0.$$

Proof. Let $r_1 > 0$ be a large radius such that $\overline{\Omega} + \overline{B}(0,r) \subset B(0,r_1)$. Consider the open set

$$\tilde{\Omega} := (\Omega \cap B(z,r)) \cup (B(0,r_1) \backslash \overline{B}(z, \frac{r}{2})).$$

Then $\Omega \subset \tilde{\Omega}$ and $\overline{B}(z,\frac{r}{2}) \cap \partial\Omega \subset \partial\tilde{\Omega}$. In particular, $z \in \partial\tilde{\Omega}$. Consider a regularized distance $\tilde{\sigma}$ with respect to $\tilde{\Omega}$. Then there exists a constant $c > 0$ such that

$$m(x) \le c\tilde{\sigma}(x)^2 \qquad \text{for all } x \in \Omega. \tag{3.15}$$

In fact, for $x \in B(z,r) \cap \Omega$ this follows from (3.9). But for $x \in \Omega \backslash B(z, \frac{3}{4}r)$, one has $dist(x, \partial\tilde{\Omega}) \ge \frac{r}{4}$. Since m is bounded, it follows that

$$m(x) \le c_2 (\frac{r}{4})^2 \le c_2 dist(x, \partial\tilde{\Omega})^2$$

for all $x \in \Omega \backslash B(z, \frac{3}{4}r)$. This shows that (3.15) is valid for a suitable constant $c > 0$. Now let $\lambda > 0$. Let $0 \le f \in C_c(\Omega)$, $u = R(\lambda, m\triangle_\infty)f$. Then $u \in C^b(\Omega) \cap H_0^1(\Omega)$ and

$$\lambda \frac{u}{m} - \triangle u = \frac{f}{m} \qquad \text{in } \mathcal{D}(\Omega)'.$$

Let $\rho := \frac{m}{\tilde{\sigma}^2}$. Then $0 < \rho \le c$ on Ω and

$$\frac{1}{c} \le \frac{1}{\rho} = \frac{\tilde{\sigma}^2}{m} \in L^p_{loc}(\Omega).$$

Hence

$$\frac{\lambda}{c} \frac{u}{\tilde{\sigma}^2} \le \frac{\lambda}{\rho} \frac{u}{\tilde{\sigma}^2} = \frac{\lambda u}{m}.$$

Thus

$$\frac{\lambda}{c} \frac{u}{\tilde{\sigma}^2} - \triangle u \le \frac{f}{m} = \frac{1}{\tilde{\sigma}^2} \frac{f}{\rho}.$$

Let $\omega \subset\subset \Omega$ be such that $supp\ f \subset \omega$. Consider the operator $Q(\lambda, \omega) \in \mathcal{L}(L^p(\omega), C_0(\tilde{\Omega}))$ of Lemma 3.4.3 defined with respect to $\tilde{\sigma}$. Let $w = Q(\frac{\lambda}{c}, \omega)\frac{f}{\rho}$. Note that w is well defined, since $\frac{f}{\rho} \in L^p(\omega)$. Then $0 \le w \in C_0(\tilde{\Omega}) \cap H_0^1(\tilde{\Omega})$ and by (3.12),

$$\frac{\lambda}{c} \frac{w}{\tilde{\sigma}^2} - \triangle w = \frac{1}{\tilde{\sigma}^2} \frac{f}{\rho} \qquad \text{in } \mathcal{D}(\tilde{\Omega})'$$

and hence also in $\mathcal{D}(\Omega)'$. Thus

$$\frac{\lambda}{c} \frac{(u-w)}{\tilde{\sigma}^2} - \triangle(u-w) \le 0 \qquad \text{in } \mathcal{D}(\Omega)'.$$

Recall that $u \in H_0^1(\Omega) \cap C^b(\Omega)$. Thus $(u-w) \in H^1(\Omega)$. Hence

$$\frac{\lambda}{c} \int_\Omega \frac{(u(x) - w(x))}{\tilde{\sigma}(x)^2} v(x)\, dx + \int_\Omega \nabla(u(x) - w(x)) \nabla v(x)\, dx \le 0 \tag{3.16}$$

for all $0 \le v \in \mathcal{D}(\Omega)$. Since $(u-w)^+ \in H^1(\Omega)$ and $(u-w)^+ \le u \in H_0^1(\Omega)$, it follows that $(u-w)^+ \in H_0^1(\Omega)$.
Since $u = R(\lambda, m\triangle_\infty)f = R(\lambda, m\triangle_2)f$, it follows that

$$u \in L^2(\Omega, \frac{dx}{m(x)}) \subset L^2(\Omega, \frac{dx}{\tilde{\sigma}(x)^2})$$

because of (3.15). It follows (since also $w \in L^2(\Omega, \frac{dx}{\tilde{\sigma}(x)^2})$ that

$$v_1 := (u-w)^+ \in V := L^2(\Omega, \frac{dx}{\widetilde{\sigma}(x)^2}) \cap H_0^1(\Omega).$$

Since $\mathcal{D}(\Omega)_+$ is dense in V_+ by Proposition 2.1.2, (3.16) remains true for $v := v_1$. This means that

$$\frac{\lambda}{c}\int_\Omega \frac{(u(x)-w(x))^{+2}}{\widetilde{\sigma}(x)^2}\,dx + \int_\Omega |\nabla(u(x)-w(x))^+|^2\,dx \le 0.$$

This implies that $(u-w)^+ = 0$. Hence $0 \le u \le w$.
Since

$$\lim_{x\to z, x\in\tilde{\Omega}} w(x) = 0,$$

it follows that

$$\lim_{x\to z, x\in\Omega} u(x) = 0.$$

We have proved the theorem for the case when $0 \le f \in C_c(\Omega)$. Hence it is also true for arbitrary $f \in C_c(\Omega)$. Since $R(\lambda, m\triangle_\infty) \in \mathcal{L}(L^\infty(\Omega))$, and $C_c(\Omega)$ is dense in $C_0(\Omega)$ it follows that

$$\lim_{x\to z, x\in\Omega} (R(\lambda, m\triangle_\infty)f)(x) = 0$$

for all $f \in C_0(\Omega)$. □

Corollary 3.4.5 *Assume that each $z \in \partial\Omega$ is a point of weak diffusion (in the sense of (3.9)). Then $m\triangle_0$ generates a positive, contractive C_0-semigroup on $C_0(\Omega)$.*

3.5 Generation theorem for e^{mAt} on $C_0(\Omega)$

We may now formulate the following general generation theorem. Let $\Omega \subset \mathbb{R}^N$ be bounded, open and $\frac{\hat{N}}{2} < p \le \infty$. Let $m : \Omega \to (0,\infty)$ be bounded and such that $\frac{1}{m} \in L^p_{loc}(\Omega)$.

Theorem 3.5.1 *Assume that for each point $z \in \partial\Omega$ one of the following conditions is satisfied:*

(a) *z is a regular point or*

(b) *z is a point of weak diffusion (in the sense of (3.9)).*

Then $m\triangle_0$ generates a positive, contractive C_0-semigroup on $C_0(\Omega)$.

Proof. Theorem 3.3.6 and Theorem 3.4.4 show that $C_0(\Omega)$ is invariant. Thus the claim follows from Proposition 3.2.2. □

Finally, we show that the condition (3.9) of being a point of weak diffusion is optimal.

Let $N = 2$ and $\Omega = \{x \in \mathbb{R}^2 : 0 < |x| < 2\}$. Then $\partial\Omega = \mathbb{T} \cup \{0\}$ where $\mathbb{T} = \{x \in \mathbb{R}^2 : |x| = 2\}$. The points in $\mathbb{T}$ are regular but 0 is not regular.
Consider the function d given by $d(x) = |x|$, $x \in \Omega$. Thus $d(x) = dist(x, \partial\Omega)$ for $0 < |x| < \frac{1}{2}$. Then $\frac{1}{d} \in L^q(\Omega)$ if and only if $q < 2$. Now let $0 < \beta < 2$. Then $\frac{1}{d^\beta} \in L^p(\Omega)$ for some $p > 1 = \frac{\hat{N}}{2}$. Since Ω is not Dirichlet regular, it follows from

Theorem 3.3.10 that $d^{\beta}\triangle_0$ is not a generator.
On the other hand, if $\beta \geq 2$, then for $m = d^{\beta}$, the point 0 is of weak diffusion. Since the other boundary points are regular, it follows from Theorem 3.5.1 that $d^{\beta}\triangle_0$ generates a C_0-semigroup on $C_0(\Omega)$.

An interesting open set in $\mathbb{R}^3$ with continuous boundary and exactly one singular point is the Lebesgue cusp (see e.g. [11] for a detailed investigation).

3.6 Notes and comments

The results of this section arose in collaboration with W. Arendt and are contained in the joint-paper [9].

Chapter 4

Kernel estimates I - Ultracontractivity

4.1 Dunford-Pettis Theorem

Throughout this section we assume that $\Omega \subset \mathbb{R}^N$ is an open set and η is a $\sigma-$finite Borel measure on Ω. Our aim is to prove the Dunford-Pettis criterion for the kernels. Given $k \in L^\infty(\Omega \times \Omega, d\eta \times d\eta)$ the mapping

$$(T_k u)(x) := \int_\Omega k(x,y)u(y)d\eta(y)$$

defines a bounded operator $T_k \in \mathcal{L}(L^1(\Omega, d\eta), L^\infty(\Omega, d\eta))$ and

$$\|T_k\|_{1\to\infty} \leq \|k\|_{L^\infty(\Omega\times\Omega, d\eta\times d\eta)}$$

The thrust of the Dunford-Pettis theorem is that all elements of $\mathcal{L}(L^1(\Omega, d\eta), L^\infty(\Omega, d\eta))$ are associated to a kernel as above.

Theorem 4.1.1 *The mapping $k \to T_k$ is an isometric isomorphism from $L^\infty(\Omega \times \Omega, d\eta \times d\eta)$ onto $\mathcal{L}(L^1(\Omega, d\eta), L^\infty(\Omega, d\eta))$. Furthermore*

$$T_k \geq 0 \text{ if and only if } k \geq 0.$$

Proof. First we define $u \otimes v \in L^1(\Omega \times \Omega, d\eta \times d\eta)$ by

$$(u \otimes v)(x,y) := u(x)v(y).$$

We obviously have $\|u \otimes v\|_{L^1(\Omega\times\Omega, d\eta\times d\eta)} = \|u\|_{L^1(\Omega, d\eta)} \cdot \|v\|_{L^1(\Omega, d\eta)}$. The space

$$G := \left\{ \sum_{n=1}^{n} c_i \chi_{E_i} \otimes \chi_{F_i} \,:\, n \in \mathbb{N},\ c_i \in \mathbb{R},\ \eta(E_i) < \infty,\ \eta(F_i) < \infty \right\}$$

is dense in $L^1(\Omega \times \Omega, d\eta \times d\eta)$ by the construction of the product measure. Pick now $T \in \mathcal{L}(L^1(\Omega, d\eta), L^\infty(\Omega, d\eta))$ arbitrarily and define $\phi : G \to \mathbb{R}$ by

$$\phi(w) = \sum_{i=1}^{m} c_i \int_\Omega (T\chi_{E_i})(y)\chi_{F_i}(y)\, d\eta(y)$$

for $w = \sum_{i=1}^m c_i \chi_{E_i} \chi_{F_i}$. It can be checked immediately that ϕ is well-defined (does not depend on the particular representation of w). It is also obvious that ϕ is linear. Now we prove that

$$|\phi(w)| \leq \|T\|_{\mathcal{L}(L^1(\Omega, d\eta), L^\infty(\Omega, d\eta))} \cdot \|w\|_{L^1(\Omega\times\Omega, d\eta\times d\eta)} \,.$$

We may assume that $E_i \times F_i$ is disjoint from $E_j \times F_j$ if $i \neq j$. We have

$$\|w\|_{L^1(\Omega\times\Omega,d\eta\times d\eta)} = \sum_{i=1}^m |c_i|\eta(E_i)\eta(F_i).$$

We estimate

$$\begin{aligned}
|\phi(w)| &\leq \sum_{i=1}^m |c_i| \, \|T\chi_{E_i}\|_{L^\infty(\Omega,d\eta)} \, \|\chi_{F_i}\|_{L^1(\Omega,d\eta)} \\
&\leq \sum_{i=1}^m |c_i| \, \|T\|_{\mathcal{L}(L^1(\Omega,d\eta),L^\infty(\Omega,d\eta))} \, \|\chi_{E_i}\|_{L^1(\Omega,d\eta)} \, \|\chi_{F_i}\|_{L^1(\Omega,d\eta)} \\
&= \|T\|_{\mathcal{L}(L^1(\Omega,d\eta),L^\infty(\Omega,d\eta))} \, \|w\|_{L^1(\Omega,d\eta)}.
\end{aligned}$$

We use the fact that $(L^1(\Omega\times\Omega, d\eta\times d\eta))' = L^\infty(\Omega\times\Omega, d\eta\times d\eta)$. Hence we find a function $k \in L^\infty(\Omega\times\Omega, d\eta\times d\eta)$ such that

$$\|k\|_{L^\infty(\Omega\times\Omega,d\eta\times d\eta)} \leq \|T\|_{\mathcal{L}(L^1(\Omega,d\eta),L^\infty(\Omega,d\eta))}$$

and

$$\phi(w) = \int_\Omega\int_\Omega k(x,y)w(x,y)\,d\eta(y)\,d\eta(x) \qquad w \in G.$$

Thus for simple functions $u, v \in L^1(\Omega, d\eta)$ we have

$$\begin{aligned}
\int_\Omega (Tu)(x)v(x)\,d\eta(x) &= \phi(u\otimes v) = \int_\Omega\int_\Omega k(x,y)u(y)\,d\eta(y)\,v(x)\,d\eta(x) \\
&= \int_\Omega (T_k u)(x)v(x)\,d\eta(x).
\end{aligned}$$

We have shown that $Tu = T_k u$ for any simple $u \in L^1(\Omega, d\eta)$ and therefore $T = T_k$. Thus we have proved the first part of the theorem.
In order to check the last statement, we recall that the functions

$$w = \sum_{i=1}^n u_i \otimes v_i \qquad u_i, v_i \in L^1(\Omega, d\eta)_+$$

are dense in $L^1(\Omega\times\Omega, d\eta\times d\eta)_+$. Hence

$$T_k \geq 0 \qquad \text{if and only if} \qquad \int_{\Omega\times\Omega} w(x,y)k(x,y)\,d\eta(x)\,d\eta(y) \geq 0$$

for all $w \in L^1(\Omega\times\Omega, d\eta\times d\eta)_+$. This happens if and only if $k \geq 0$ $\eta-$almost everywhere. □

4.2 Kernel representation

Throughout this section we assume that $\Omega \subset \mathbb{R}^N$ is an open set and η is a $\sigma-$finite Borel measure on Ω. We assume that we are given a symmetric submarkovian semigroup e^{At} on $L^2(\Omega, \eta)$. We know (see the Theorem 1.4.13) that such a semigroup can be extrapolated to all $L^p(\Omega, \eta)$, $1 \leq p \leq \infty$.

Definition 4.2.1 *We say that e^{At} is ultracontractive if e^{At} is bounded from $L^2(\Omega, d\eta)$ to $L^\infty(\Omega, d\eta)$ for all $t > 0$. We use the notation $c_t := \left\|e^{At}\right\|_{2\to\infty}$.*

Remark 4.2.2 *The bound c_t is a nonincreasing function of t. This is easy to see using the assumed submarkovian property of e^{At}.*

By the Dunford-Pettis theorem of the last section (Theorem 4.1.1) and an easy duality argument (using the self-adjointness of A) we see that the ultracontractivity is equivalent to the fact that the semigroup e^{At} may be represented by a bounded kernel $k(t,x,y)$ so that we have for any $u \in L^2(\Omega,\eta)$,

$$(e^{At}u)(x) = \int_\Omega k(t,x,y)u(y)\,d\eta(y) \qquad \text{for a.a. } x \in \Omega.$$

We have the following spectral consequences of the ultracontractivity.

Theorem 4.2.3 *Assume that $\eta(\Omega) < \infty$ and let e^{At} be ultracontractive. Then e^{At} is compact on all $L^p(\Omega,\eta), 1 \le p \le \infty$ for any $t > 0$. The spectrum $\sigma(A_p)$ is purely discrete and independent of p, $1 \le p \le \infty$. If we denote by E_n, $n = 0,1,\dots$ the eigenvalues of A in decreasing order (possibly repeated if the multiplicity is larger than one) and by φ_n the corresponding normalised[1] eigenfunctions we have*

$$k(t,x,y) = \sum_{n=0}^{\infty} e^{E_n t}\varphi_n(x)\varphi_n(y) \qquad \text{for a.a. } x,y \in \Omega. \tag{4.1}$$

The convergence in (4.1) is uniform on $[t,\infty) \times \Omega \times \Omega$ for any $t > 0$. All the functions φ_n, $n = 0,1,\dots$ are in $L^\infty(\Omega)$ and we have the estimate

$$\|\varphi_n\|_\infty \le c_n := \inf_{t>0}(c_t e^{-E_n t}).$$

If φ_n are continuous for all $n = 0,1,\dots$ then the expression (4.1) is valid for all $x,y \in \Omega$ and the kernel $k(t,x,y)$ is a continuous function on $(0,\infty) \times \Omega \times \Omega$. In that case we also have the following formula for the trace

$$\operatorname{Tr} e^{At} = \int_\Omega k(t,x,x)\,d\eta(x). \tag{4.2}$$

Proof. We see easily that e^{At} is Hilbert-Schmidt for any $t > 0$. In fact, the square of the Hilbert-Schmidt norm is equal to

$$\int_{\Omega\times\Omega} k(t,x,y)^2\,d\eta(x)d\eta(y) \le c_t\eta(\Omega)^2 < \infty.$$

Hence we see that the operator e^{At} is compact on $L^2(\Omega,\eta)$ for any $t > 0$. We also see that the trace of e^{At} is finite since for any orthonormal basis $\{\psi_n\}_{n\in\mathbb{N}}$ in $L^2(\Omega,\eta)$ we have by the semigroup property and the assumption of symmetry (denoting by $\langle\cdot,\cdot\rangle$ the scalar product in $L^2(\Omega,\eta)$)

$$\text{Tr } e^{At} = \sum_{n=0}^{\infty}\langle e^{At}\psi_n,\psi_n\rangle = \sum_{n=0}^{\infty}\langle e^{A\frac{t}{2}}\psi_n, e^{A\frac{t}{2}}\psi_n\rangle$$

and the last expression is finite, since e^{At} is a Hilbert-Schmidt operator for any $t > 0$. We will prove now that $e^{A_1 t}$ is compact on $L^1(\Omega,\eta)$. This follows from the identity

$$e^{A_1 t} = Ie^{A_2\frac{t}{2}}e^{A_1\frac{t}{2}}$$

on noting that $e^{A_1\frac{t}{2}}$ is bounded from $L^1(\Omega,\eta)$ to $L^2(\Omega,\eta)$ by the ultracontractivity and duality, $e^{A_2\frac{t}{2}}$ is compact on $L^2(\Omega,\eta)$ and I =identity is bounded from $L^2(\Omega,\eta)$

[1] $\|\varphi_n\|_{L^2(\Omega,d\eta)} = 1$.

to $L^1(\Omega,\eta)$ since we assume that $\eta(\Omega) < \infty$. The first two assertions of the theorem follow now from Theorem 1.6.9 and duality. By the representation

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} e^{At}\, dt$$

(valid for any λ with $Re\,\lambda > 0$ by the contractivity of e^{At}) we see that also the resolvent of A is compact for any $\lambda > 0$. Thus by Theorem 1.8.3 (used for $-A$) we conclude that the spectrum of A is discrete, consists of countably many eigenvalues and that the corresponding eigenfunctions φ_n, $n = 0, 1, \dots$ form a complete orthonormal system on $L^2(\Omega,\eta)$. We show that φ_n are bounded functions for all $n = 0, 1, \dots$. We have by ultracontractivity

$$\|\varphi_n\|_\infty = \left\| e^{-E_n \frac{t}{3}} e^{A\frac{t}{3}} \varphi_n \right\|_\infty \le c_{t/3} e^{-E_n \frac{t}{3}} \|\varphi_n\|_{L^2(\Omega,\eta)} = c_{t/3} e^{-E_n\frac{t}{3}} \tag{4.3}$$

valid for any $t > 0$ and thus

$$\|\varphi_n\|_\infty \le c_n := \inf_{t>0} (c_{\frac{t}{3}} e^{-E_n \frac{t}{3}}) < \infty.$$

Therefore (using also that the trace of e^{At} is finite for any $t > 0$) we have

$$\sum_{n=0}^\infty e^{E_n t} |\varphi_n(x)\varphi_n(y)| \le c_{t/3}^2 \sum_{n=0}^\infty e^{E_n \frac{t}{3}} = c_{t/3}^2 \operatorname{Tr}(e^{A\frac{t}{3}}) < \infty \tag{4.4}$$

and the convergence is uniform on $[t,\infty) \times \Omega \times \Omega$ by the Weierstrass M-test and the monotonicity of c_t.
For any $u \in L^2(\Omega,\eta)$ we have the Fourier expansion

$$u = \sum_{n=0}^\infty \langle u, \varphi_n\rangle_{L^2(\Omega,\eta)} \varphi_n$$

where the series converges in $L^2(\Omega,\eta)$. Since e^{At} is a bounded operator on $L^2(\Omega,\eta)$ for any $t > 0$ we also have

$$e^{At} u = \sum_{n=0}^\infty \langle u, \varphi_n\rangle_{L^2(\Omega,\eta)} e^{At}\varphi_n = \sum_{n=0}^\infty e^{\lambda_n t} \langle u, \varphi_n\rangle_{L^2(\Omega,\eta)} \varphi_n.$$

Thus for any $u \in L^\infty(\Omega)$ and a.a. $x \in \Omega$ we may write

$$\begin{aligned}\int_\Omega k(t,x,y) u(y)\, d\eta(y) = (e^{At}u)(x) &= \sum_{n=0}^\infty e^{\lambda_n t} \varphi_n(x) \int_\Omega \varphi_n(y) u(y)\, d\eta(y) \\ &= \int_\Omega \sum_{n=0}^\infty e^{\lambda_n t} \varphi_n(x)\varphi_n(y) u(y)\, d\eta(y)\end{aligned}$$

where interchanging the sum and the integral is allowed by the uniform convergence of the series in 4.4 and the boundedness of u. Since u was arbitrary subject only to the condition $u \in L^\infty(\Omega)$ we obtain

$$k(t,x,y) = \sum_{n=0}^\infty e^{E_n t} \varphi_n(x)\varphi_n(y) \qquad \text{for a.a. } x, y \in \Omega. \tag{4.5}$$

If the φ_n are continuous for all $n = 0, 1, \dots$ then the expression (4.5) is valid for all $x, y \in \Omega$ and the kernel $k(t,x,y)$ is a continuous function on $(0,\infty) \times \Omega \times \Omega$. This follows from (4.4) and the well-known fact that if a sequence of continuous functions converges uniformly then the limiting function has to be continuous, too. In that case we have

$$\operatorname{Tr} e^{At} = \sum_{n=0}^{\infty} \langle e^{At}\varphi_n, \varphi_n\rangle = \int_\Omega \sum_{n=0}^{\infty} e^{E_n t}\varphi_n(x)\varphi_n(x)\, d\eta(x) = \int_\Omega k(t,x,x)\, d\eta(x)$$

where, once again, the interchange of summation and integration is justified since the sum converges uniformly on Ω. □

4.3 Characterisations of Ultracontractivity

We assume that e^{At} is the symmetric submarkovian semigroup associated with a densely defined, accretive, continuous and closed sesquilinear form a on $L^2(\Omega, \eta)$, where $\Omega \subset \mathbb{R}^N$ and η is a $\sigma-$finite Borel measure. We use the abbreviation $\|\cdot\|_p :$ $= \|\cdot\|_{L^p(\Omega,\eta)}$ and also use the notation $\|\cdot\|_{2\to\infty}$ to denote the norm of an operator in $\mathcal{L}(L^2(\Omega, d\eta), L^\infty(\Omega, d\eta))$.
Our goal is to prove the equivalence of ultracontractive estimates with so called logarithmic Sobolev inequalities. We start by the following lemma which already gives a hint as to why it is the logarithmic function that appears in 'logarithmic' Sobolev inequalities.

Lemma 4.3.1 *Let $0 \geq u \in \mathcal{D}(A) \cap L^1(\Omega, \eta) \cap L^\infty(\Omega, \eta)$ and put $u(s) := e^{As}u$. Let*

$$p : [0, t) \to [2, \infty)$$

be a continuously differentiable function. Then we have

$$\frac{d}{ds}\, \|u(s)\|_{p(s)}^{p(s)} = p(s)\langle u'(s), u(s)^{p(s)-1}\rangle + p'(s) \int_\Omega u(s)^{p(s)} \log u(s)\, d\eta.$$

Proof. On a formal level, this is an easy computation. See [23], Lemma 2.2.1. for a rigorous justification. □

We now show how ultracontractivity implies logarithmic Sobolev inequalities.

Theorem 4.3.2 *Suppose that e^{At} is ultracontractive with*

$$\left\|e^{At}\right\|_{2\to\infty} \leq e^{U(t)} \quad \forall\, t > 0$$

where U is a monotonically decreasing continuous function. Then for any $0 \leq u \in \mathcal{D}(a) \cap L^1(\Omega, \eta) \cap L^\infty(\Omega, \eta)$ we have $u^2 \log u \in L^1(\Omega, \eta)$ and

$$\int_\Omega u^2 \log u\, d\eta \leq \varepsilon a(u, u) + U(\varepsilon)\, \|u\|_2^2 + \|u\|_2^2 \log \|u\|_2 \quad \forall\, \varepsilon > 0. \tag{4.6}$$

Proof. We prove the assertion under an additional assumption

$$0 \leq u \in \mathcal{D}(A) \cap L^1(\Omega, \eta) \cap L^\infty(\Omega, \eta) =: M \qquad \text{and } \|u\|_2 = 1. \tag{4.7}$$

The general case follows by approximation as in [23]. Let us for a moment assume also that $\|u\|_2 = 1$. So for a normalised u fulfilling (4.7) we define $u(s) := e^{As}u$, fix $\varepsilon > 0$ and set $p(s) := \frac{2\varepsilon}{\varepsilon - s}$ for $0 \leq s < \varepsilon$. Then by the ultracontractivity assumption and the Stein interpolation theorem (Theorem 1.6.7) we have

$$\left\|e^{As}\right\|_{2\to p(s)} \leq e^{\frac{U(\varepsilon)s}{\varepsilon}}$$

for all $0 \le s < \varepsilon$. Hence

$$\|u(s)\|_{p(s)}^{p(s)} \le e^{\frac{U(\varepsilon)p(s)s}{\varepsilon}}$$

and

$$\frac{d}{ds}\,\|u(s)\|_{p(s)}^{p(s)}\,\|_{s=0} \le 2\frac{U(\varepsilon)}{\varepsilon}$$

By Lemma 4.3.1 we thus have

$$\int_\Omega u^2 \log u d\eta \le \varepsilon\langle -Au, u\rangle + U(\varepsilon) \tag{4.8}$$

which is the statement of the Theorem since we have assumed that $\|u\|_2 = 1$. For a general $0 \le u \in \mathcal{D}(A) \cap L^1(\Omega,\eta) \cap L^\infty(\Omega,\eta)$ we apply (4.8) to $\frac{u}{\|u\|_2}$ and (4.6) follows. □

We use the abbreviation $e^{At}(L^1 \cap L^\infty)_+$ to denote the set

$$\left\{v \in L^1(\Omega,\eta) \cap L^\infty(\Omega,\eta) : \text{there ex. } 0 \le u \in L^1(\Omega,\eta) \cap L^\infty(\Omega,\eta) \text{ s.t. } e^{At}u = v\right\}.$$

Remark 4.3.3 *The logarithmic Sobolev inequality (4.6) valid for $0 \le u \in \mathcal{D}(a) \cap L^1(\Omega,\eta) \cap L^\infty(\Omega,\eta)$ implies automatically its variant for the L^p-norm instead of the L^2-norm, namely:*

$$\int_\Omega u^p \log u\, d\eta \le \varepsilon(p)\langle -Au, u^{p-1}\rangle + \frac{2U(\varepsilon)}{p}\,\|u\|_p^p + \|u\|_p^p \log \|u\|_p \tag{4.9}$$

valid for all $2 < p < \infty$ and $u \in \bigcup_{t>0} e^{At}(L^1 \cap L^\infty)_+$. The proof is not difficult, one uses $u^{\frac{p}{2}}$ in (4.6) and the inequality

$$a(u^{\frac{p}{2}}, u^{\frac{p}{2}}) \le \frac{p^2}{4(p-1)}\langle -Au, u^{p-1}\rangle$$

(see [23]). Note that (4.9) is the assumption (4.10) of the next theorem if we set $g(p) := \frac{2U(\varepsilon)}{p}$.

In the opposite direction we shall show that logarithmic Sobolev inequalities imply ultracontractivity.

Theorem 4.3.4 *Assume that for two continuous functions $\varepsilon(p) > 0$ and $g(p)$ defined on $(2,\infty)$ the following inequality holds true*

$$\int_\Omega u^p \log u\, d\eta \le \varepsilon(p)\langle -Au, u^{p-1}\rangle + g(p)\,\|u\|_p^p + \|u\|_p^p \log \|u\|_p \tag{4.10}$$

for all $2 < p < \infty$ and $u \in \bigcup_{t>0} e^{At}(L^1 \cap L^\infty)_+$. If the quantities defined by

$$t := \int_2^\infty \frac{\varepsilon(p)}{p}\, dp, \qquad U := \int_2^\infty \frac{g(p)}{p}\, dp$$

are finite, then e^{At} is well defined as a mapping from $L^2(\Omega,\eta)$ to $L^\infty(\Omega,\eta)$ and

$$\left\|e^{At}\right\|_{2\to\infty} \le e^U.$$

Proof. We will make use of the Lemma 4.3.1 with the function $p(s)$ defined as the solution of the ordinary differential equation

$$p'(s) = \frac{p(s)}{\varepsilon(p(s))}, \qquad p(0) = 2. \tag{4.11}$$

The right-hand side $\frac{p}{\varepsilon(p)}$ is by assumption a continuous function in p and thus the existence of a local solution to (4.11) follows from Peano's theorem. The assumed positivity of $\varepsilon(p)$ and the initial condition $p(0) = 2$ guarantee that any solution to (4.11) is increasing. Fix now one maximal solution $\widetilde{p}$ to (4.11) defined on an interval $[0, r_1)$ with values in $[2, r_2)$. Since $\widetilde{p}$ is increasing its inverse function $z := \widetilde{p}^{-1}$ exists, it is increasing, we have $z : [2, r_2) \to [0, r_1)$ and the derivative of z is given by

$$z'(w) = \frac{1}{p'(z(w))} = \frac{\varepsilon(p(z(w)))}{p(z(w))} = \frac{\varepsilon(w)}{w}.$$

Thus z is given by

$$z(w) = \int_2^w \frac{\varepsilon(v)}{v}\,dv$$

and from the assumption that t is finite we conclude that z is well-defined on the interval $[2, \infty)$ and takes values in $[0, t)$. Since z is the inverse of $\widetilde{p}$ we have proved that there is a unique solution to (4.11) and we have

$$\lim_{s \to t} p(s) = +\infty.$$

Having defined $p(s)$, we can define another auxiliary function $f(s)$ defined as the solution of

$$f'(s) = \frac{g(p(s))}{\varepsilon(p(s))}, \qquad f(0) = 0. \tag{4.12}$$

Here f is given by

$$f(s) = \int_0^s \frac{g(p(w))}{\varepsilon(p(w))}\,dw = \int_2^{p(s)} \frac{g(x)}{x}\,dx \tag{4.13}$$

where we used the substitution $p(w) = x$ and the fact that $p(w)$ satisfies (4.11). We see from (4.13) and the assumption that U is finite that $f(s) \to U$ as $s \to t$.
For $u_0 \in \bigcup_{t>0} e^{At}(L^1 \cap L^\infty)_+$ we have (as before, we set $u(s) := e^{As}u_0$ for $0 < s < t$)

$$\begin{aligned}
&\frac{d}{ds}\log\left(e^{-f(s)}\|u(s)\|_{p(s)}\right) = \frac{d}{ds}\left(-f(s) + \frac{1}{p(s)}\log\|u(s)\|_{p(s)}^{p(s)}\right)\\
&= -\frac{g(p(s))}{\varepsilon(p(s))} - \frac{1}{p^2(s)}\frac{p(s)}{\varepsilon(p(s))}\log\|u(s)\|_{p(s)}^{p(s)}\log\|u(s)\|_{p(s)}^{p(s)}\\
&\quad + \frac{1}{p(s)}\frac{1}{\|u(s)\|_{p(s)}^{p(s)}}\left[p\langle Au(s), u(s)^{p(s)-1}\rangle + \frac{p(s)}{\varepsilon(p(s))}\int_\Omega u(s)^{p(s)}\log u(s)\,d\eta\right]\\
&= \frac{1}{\varepsilon(p)\|u\|_p^p}\left[\int_\Omega u^p \log u\,d\eta + \varepsilon\langle Au, u^{p-1}\rangle - g(p)\|u\|_p^p - \|u\|_p^p\log\|u\|_p^p\right]\\
&\le 0.
\end{aligned}$$

We conclude that

$$e^{-f(s)}\|u(s)\|_p(s) \le e^{-f(0)}\|u_0\|_{p(0)} = \|u_0\|_2 \qquad \text{for all } 0 \le s < t$$

and then (on recalling the definition of $u(s)$ and using the submarkovianity of e^{As})

$$\left\|e^{At}u_0\right\|_{p(s)} \le \left\|e^{As}u_0\right\|_{p(s)} \le e^{f(s)}\|u_0\|_2 \qquad \text{for all } 0 \le s < t. \tag{4.14}$$

This yields

$$\left\|e^{At}u_0\right\|_\infty \le e^U\|u_0\|_2 \tag{4.15}$$

in the limit as $s \to t$.
Given now $0 \le u \in L^2(\Omega, \eta)$ there exists a sequence $u_n \in \bigcup_{t>0} e^{At}(L^1 \cap L^\infty)_+$ s.t. $\|u_n - u\|_2 \to 0$. Then also $\left\|e^{At}u_n - e^{At}u\right\|_2 \to 0$ and since by (4.15) we have

$$\left\|e^{At}u_n\right\|_\infty \le e^U \|u_n\|_2$$

we obtain

$$\left\|e^{At}u\right\|_\infty \le e^U \|u\|_2 .$$

Finally, given $u \in L^2(\Omega, \eta)$ we have by the positivity of e^{At},

$$|e^{At}u| \le e^{At}|u|.$$

Hence

$$\left\|e^{At}u\right\|_\infty \le \left\|e^{At}|u|\right\|_\infty \le e^U \||u|\|_2 = e^U \|u\|_2 .$$

□

Corollary 4.3.5 *Suppose that $h(\varepsilon)$ is a monotonically decreasing continuous function such that*

$$\int_\Omega u^2 \log u \, d\eta \le \varepsilon a(u,u) + h(\varepsilon) \|u\|_2^2 + \|u\|_2^2 \log \|u\|_2 \tag{4.16}$$

for all $0 \le u \in \mathcal{D}(a) \cap L^1(\Omega, \eta) \cap L^\infty(\Omega, \eta)$ and $\varepsilon > 0$. Assume that

$$U(t) := \frac{1}{t} \int_0^t h(\varepsilon) \, d\varepsilon < \infty \quad \forall\, t > 0.$$

Then e^{At} is ultracontractive with

$$\left\|e^{At}\right\|_{2\to\infty} \le e^{U(t)} \quad 0 < t < \infty.$$

Proof. Fix $t > 0$ and set

$$\varepsilon(p) := \frac{2t}{p} \qquad \text{and} \qquad g(p) := \frac{2h(\varepsilon(p))}{p}.$$

Using the Remark 4.3.3 and Theorem 4.3.4 we

$$U(t) = \int_2^\infty \frac{2h(\varepsilon(p))}{p^2} \, dp = \int_2^\infty 2h(\frac{2t}{p}) \frac{1}{p^2} \, dp = \frac{1}{t} \int_0^t h(\varepsilon) \, d\varepsilon.$$

□

Example 4.3.6 *The existence of constants c and $\mu \ge 0$ satisfying*

$$\left\|e^{At}\right\|_{2\to\infty} \le ct^{-\frac{\mu}{4}} \quad \forall\, t > 0$$

is equivalent to the existence of c and $h(\varepsilon)$ satisfying (4.16) and

$$h(\varepsilon) \le c - \frac{\mu}{4} \log \varepsilon \quad \forall\, \varepsilon > 0.$$

Proof. This follows immediately from Theorem 4.3.2 and Corollary 4.3.5. □

We give now a characterization of ultracontractivity by means of a Sobolev inequality. This result is due to Varopoulos.

Theorem 4.3.7 *If $\mu > 2$ then a bound of the form*

$$\|e^{At}u\|_\infty \le ct^{-\frac{\mu}{4}}\|u\|_2 \tag{4.17}$$

for all $t > 0$ and all $f \in L^2(\Omega, d\eta)$, is equivalent to a bound of the form

$$\|u\|^2_{L^{\frac{2\mu}{\mu-2}}(\Omega,d\eta)} \le c_2 a(u,u) \tag{4.18}$$

for all $u \in \mathcal{D}(a)$.

Proof. Since A is self-adjoint (4.17) implies

$$\left\|e^{At}u\right\|_\infty \le ct^{-\frac{\mu}{2}}\left\|u\right\|_1.$$

The Riesz-Thorin theorem (Theorem 1.6.8) implies

$$\left\|e^{At}u\right\|_\infty \le ct^{-\frac{\mu}{2q}}\left\|u\right\|_q$$

for all $t > 0$, $f \in L^q(\Omega, \eta)$ and $1 < q < \mu$. We now use the decomposition

$$(-A)^{-\frac{1}{2}}u = g + h$$

where

$$\begin{aligned} g &= \Gamma(\frac{1}{2})^{-1}\int_0^T t^{-\frac{1}{2}}e^{At}u\,dt \\ h &= \Gamma(\frac{1}{2})^{-1}\int_T^\infty t^{-\frac{1}{2}}e^{At}u\,dt. \end{aligned}$$

We have

$$\left\|h\right\|_\infty \le \Gamma(\frac{1}{2})^{-1}\int_T^\infty ct^{-\frac{1}{2}-\frac{\mu}{2q}}\left\|u\right\|_q\,dt = c\left\|u\right\|_q T^{\frac{1}{2}-\frac{\mu}{2q}}.$$

For given $\lambda > 0$ we now define T by

$$\frac{\lambda}{2} = c\left\|u\right\|_q T^{\frac{1}{2}-\frac{\mu}{2q}}.$$

Then (using also that e^{At} is a contraction on $L^q(\Omega, \eta)$)

$$\begin{aligned} \left|\left\{x : |(-A)^{-\frac{1}{2}}u(x)| \ge \lambda\right\}\right| &\le \left|\left\{x : |g(x)| \ge \frac{\lambda}{2}\right\}\right| \\ &\le \left(\frac{\lambda}{2}\right)^{-q}\|g\|_q^q \le \left(\frac{\lambda}{2}\right)^{-q}\left[\Gamma\left(\frac{1}{2}\right)^{-1}2T^{\frac{1}{2}}\left\|u\right\|_q\right]^q. \end{aligned}$$

We deduce for $\frac{1}{r} = \frac{1}{q} - \frac{1}{\mu}$ the following weak type estimate

$$|\left\{x : |(-A)^{-\frac{1}{2}}u(x)| \ge \lambda\right\}| \le c\lambda^{-q}\left(\frac{\lambda}{\left\|f\right\|_q}\right)^{\frac{q}{1-\frac{\mu}{q}}}\left\|u\right\|_q^q = c\lambda^{-r}\left\|u\right\|_q^r.$$

We have proved that $(-A)^{-\frac{1}{2}}$ is of weak type (q, r) for all $1 < q < \mu$. The Marcinkiewicz interpolation theroem yields the boundedness of $(-A)^{-\frac{1}{2}}$ from $L^2(\Omega, \eta)$ to $L^p(\Omega, \eta)$ where $\frac{1}{p} = \frac{1}{2} - \frac{1}{\mu}$. This shows the first implication.

Assume now that (4.18) is valid and define p by $\frac{1}{p} = \frac{1}{2} - \frac{1}{\mu}$. For fixed $u \ge 0$ normalized by $\left\|u\right\|_2 = 1$ define the measure θ by

$$d\theta(x) = f(x)^2 d\eta(x).$$

The concavity of the logarithm and Jensen's inequality (note that $\theta(\Omega)=1$) implies

$$\begin{aligned}\int_\Omega u^2\log u\,d\eta &= \frac{1}{p-2}\int_\Omega \log(u^{p-2})\,d\theta \le \frac{1}{p-2}\log\int_\Omega u^{p-2}\,d\theta\\ &= \frac{1}{p-2}\log\|u\|_p^p \le \frac{p}{2(p-2)}\log\|u\|_p^2\\ &\le \frac{\mu}{4}(-\log\varepsilon+\varepsilon\|u\|_p^2) \le \frac{\mu}{4}(-\log\varepsilon + c\varepsilon a(u,u))\end{aligned}$$

We may apply Example 4.3.6 to conclude the proof. □

4.4 Applications

We are now ready to prove ultracontractivity for $e^{m\triangle t}$. For the clarity of exposition we formulate the result separately for dimensions $N=1$ and $N=2$.

Theorem 4.4.1 *a) Let Ω be a bounded open set in $\mathbb{R}^N$, $N\ge 3$ and $m:\Omega\to(0,\infty)$ such that $\frac{1}{m}\in L^q(\Omega)$ for some $q>\frac{N}{2}$. Then the semigroup $e^{m\triangle t}$ is ultracontractive i.e. there exists $c>0$ s.t.*

$$\|e^{m\triangle t}f\|_\infty \le c\cdot t^{-\frac{N(q-1)}{2(2q-N)}}\|f\|_{L^2(\Omega,\frac{dx}{m(x)})} \tag{4.19}$$

for all $f\in L^2(\Omega,\frac{dx}{m(x)})$ and all $t>0$.
b) Let Ω be a bounded open set in $\mathbb{R}^N$, $N=1$ or $N=2$ and let $m:\Omega\to(0,\infty)$ be such that $\frac{1}{m}\in L^q(\Omega)$ for some $q>1$. Then the semigroup $e^{m\triangle t}$ is ultracontractive i.e. for each $\alpha>\frac{1}{2}$ there exists a constant c_α such that

$$\|e^{m\triangle t}f\|_\infty \le c_\alpha\cdot t^{-\alpha}\|f\|_{L^2(\Omega,\frac{dx}{m(x)})} \tag{4.20}$$

for all $f\in L^2(\Omega,\frac{dx}{m(x)})$ and all $t>0$.

Proof of Theorem 4.4.1: Define $\widetilde N := N$ if $N\ge 3$, otherwise let $\widetilde N>2$ be arbitrary subject to the condition $\widetilde N<2q$. We check (4.18). Take $u\in\mathcal{D}(a)=H_0^1(\Omega)$, set $\mu:=\frac{2\widetilde N(q-1)}{2q-\widetilde N}$ and estimate

$$\begin{aligned}\|u\|^2_{L^{\frac{2\mu}{\mu-2}}(\Omega,\frac{dx}{m(x)})} &= \left[\int_\Omega |u(x)|^{\frac{2\mu}{\mu-2}}\frac{dx}{m(x)}\right]^{\frac{\mu-2}{\mu}}\\ &\le \left[\left(\int_\Omega\frac{dx}{m(x)^q}\right)^{\frac1q}\left(\int_\Omega |u(x)|^{\frac{2\mu q'}{\mu-2}}\,dx\right)^{\frac{1}{q'}}\right]^{\frac{\mu-2}{\mu}}\\ &= c(m,\widetilde N,q)\cdot\left(\int_\Omega |u(x)|^{\frac{2\widetilde N}{\widetilde N-2}}\,dx\right)^{\frac{1}{q'}\frac{\mu-2}{\mu}\frac{\widetilde N-2}{2\widetilde N}\frac{2\widetilde N}{\widetilde N-2}}\\ &= c(m,\widetilde N,q)\left(\int_\Omega |u(x)|^{\frac{2\widetilde N}{\widetilde N-2}}\,dx\right)^{2\frac{(\widetilde N-2)}{2\widetilde N}}\\ &\le \tilde c(m,\widetilde N,q)\int_\Omega |\nabla u(x)|^2\,dx = c\cdot a(u,u)\end{aligned}$$

where Hölder's inequality, the equality $\frac{\mu-2}{\mu}\frac{q-1}{q}\frac{\widetilde N}{\widetilde N-2}=\frac{2q(\widetilde N-2)}{2\widetilde N(q-1)}\frac{q-1}{q}\frac{\widetilde N}{\widetilde N-2}=1$ and a Sobolev inequality (i.e. Theorem 1.7.2, 1.7.3 or 1.7.4 depending on the dimension

N) were used. We conclude the proof by applying Theorem 4.3.7. □

We will show (in Theorem 4.4.3) that the result of Theorem 4.4.1 is rather sharp (given no regularity assumption on $\partial\Omega$). First though, we prove that there is no ultracontractivity (in case $N \geq 3$) for the operator $\sigma^2\triangle$ where σ is a regularized distance to the boundary of Ω (see Theorem 1.9.5) and Ω is a bounded open set in $\mathbb{R}^N$.

We use the method of [54] (where smoothness of the underlying domain is assumed) and provide a complete proof in the general case of arbitrary bounded open sets.

Let $b \in \mathbb{R}$ and Ω be a bounded open set in $\mathbb{R}^N$. Since all the considered sesquilinear forms will be symmetric it suffices to define them for pairs of the form (u, u). The unique extension to general pairs (u, v) is then given by using the polarisation identity. Define a on $H_0^1(\Omega) \cap L^2(\Omega, \sigma^{-b}(x)dx)$ by

$$a(u, u) := \int_\Omega |\nabla u(x)|^2 \, dx.$$

and let A be the associated generator on $L^2(\Omega, \sigma^{-b}(x)dx)$. Here σ is a regularized distance function, i.e. a function fulfilling the conclusions of Theorem 1.9.5.
Let ω be an open subset of Ω with piecewise smooth boundary and ρ a smooth function on $\overline{\omega}$. Let A_0 be the self-adjoint operator on $L^2(\omega, \rho^{-b}(x)\, dx)$ associated with the form a_0 defined on $H_0^1(\omega)$ by

$$a_0(u, u) := \int_\omega |\nabla u(x)|^2 \, dx.$$

Let $U : L^2(\omega, dx) \to L^2(\omega, \rho^{-b}(x)dx)$ be defined by $Uf := \rho^{\frac{b}{2}} f$ for $f \in L^2(\omega, dx)$. Let A_1 be the self-adjoint operator on $L^2(\omega, dx)$ associated with the form defined for $u \in H_0^1(\omega)$ by

$$\begin{aligned} a_1(u, u) := a_0(Uu, Uu) &= \int_\omega |\nabla(\rho^{\frac{b}{2}} u)|^2 \, dx \\ &= \int_\omega \rho^b |\nabla u|^2 \, dx + \int_\omega V|u|^2 \, dx \end{aligned}$$

where

$$V := \frac{b}{2}\rho^{b-1}\triangle\rho - \frac{b(b-1)}{2}|\nabla\rho|^2\rho^{b-2} + \frac{b^2}{4}\rho^{b-2}|\nabla\rho|^2.$$

Hence for all $u \in C_c^\infty(\omega)$

$$A_1 u = \sum_{i=1}^N \frac{\partial}{\partial x_i}(\rho^b \frac{\partial u}{\partial x_i}) + Vu =: Ju + Vu.$$

Theorem 4.4.2 *If $N \geq 3$ and $b \geq 2$ then e^{At} is not ultracontractive.*

Proof: Let $0 < \varepsilon_0 < 1$ and set $\partial\Omega(\varepsilon_0) := \{x \in \Omega : \ d(x) < \epsilon_0\}$. Here d is the distance to the boundary of Ω. Let $x_0 \in \partial\Omega(\varepsilon_0)$ and define

$$\omega := \prod_{i=1}^N \left(x_{0,i} - \frac{1}{2\sqrt{N}} d(x_0),\, x_{0,i} + \frac{1}{2\sqrt{N}} d(x_0) \right) \subset B(x_0, \frac{d(x_0)}{2}),$$

where $x_{0,i}$ are the coordinates of x_0. Then

$$0 < \frac{c^{-1}d(x_0)}{2} \leq \sigma(x) \leq \frac{3cd(x_0)}{2} \quad \text{for any } x \in \omega. \tag{4.21}$$

Let ρ be the restriction of σ to ω.
Now suppose that e^{At} be ultracontractive and fix $t > 0$. Then by domination

$$0 \le K_{A_0}(t,x,y) \le K_A(t,x,y) \le \delta_1(t) < \infty \text{ where } x, y \in \omega,\ t > 0.$$

Since the function σ^{-b} is bounded on ω we may use Theorem 4.4.5 below and obtain the continuity of the kernel $K_{A_0}(t,x,y)$. Thus by Mercer's formula (4.2) we have

$$tr\ (e^{A_0 t}) = \int_\omega K_{A_0}(t,x,x)\sigma^{-b}(x)\,dx \le \delta_2(t) d^{-b+N}(x_0). \tag{4.22}$$

On the other hand, A_0 is unitarily equivalent to A_1 in $L^2(\omega, dx)$, where V is a bounded function on ω (to see the boundedness of V apply Theorem 1.9.5). Denote by L and L_0 the Dirichlet Laplacians in $L^2(\omega, dx)$ and $L^2((0,1)^N, dx)$ respectively and by $\lambda_n^{\cdot}$ eigenvalues of the operators corresponding to the superscript. We have

$$\begin{aligned}
\mathrm{Tr}(e^{A_0 t}) &= \mathrm{Tr}(e^{A_1 t}) \\
&\ge e^{-\|V\|_\infty t} \cdot \mathrm{Tr}(e^{Jt}) = e^{-\|V\|_\infty t} \cdot \sum_{n=0}^{\infty} e^{\lambda_n^J t} && (4.23)\\
&\ge e^{-\|V\|_\infty t} \cdot \sum_{n=0}^{\infty} e^{\delta_3 d^b(x_0)\lambda_n^L t} && (4.24)\\
&= e^{-\|V\|_\infty t} \cdot \sum_{n=0}^{\infty} e^{\delta_4 d^{b-2}(x_0)\lambda_n^{L_0} t} && (4.25)\\
&= \delta_5(t) e^{-\|V\|_\infty t} \cdot d^{-\frac{N}{2}b+N}(x_0). && (4.26)
\end{aligned}$$

The inequality (4.23) follows from the Golden-Thompson inequality (cf.[56], p.333) which yields

$$\mathrm{Tr}(e^{Jt}) \le \mathrm{Tr}(e^{-\frac{V}{2}t} e^{A_1 t} e^{-\frac{V}{2}t}).$$

The kernel of the operator on the right-hand side is given by

$$e^{-\frac{V(x)}{2}t} k_{A_1}(t,x,y) e^{-\frac{V(y)}{2}t}$$

and hence

$$\mathrm{Tr}(e^{-\frac{V}{2}t} e^{A_1 t} e^{-\frac{V}{2}t}) = \int_\omega k_{A_1}(t,x,x) e^{-V(x)t}\,dx \le \mathrm{Tr}(e^{A_1 t}) e^{\|V\|_\infty t}.$$

The inequality (4.24) follows from (4.21) and the minimax formula which allows computing the eigenvalues by means of the corresponding sesquilinear forms. In detail, we have (cf. e.g. [23], p.6)

$$\lambda_n^L = \inf_V \sup \left\{ a_L(\psi,\psi) :\ \psi \in V,\ \|\psi\|_{L^2(\omega,dx)} = 1 \right\}$$

where the infimum is taken through all subspaces $V \subset \mathcal{D}(L) = H_0^1(\omega)$ with $\dim V = n+1$. Also a_L is given by

$$a_L(u,v) = \int_\omega \nabla u(x) \overline{\nabla v(x)}\,dx \qquad u, v \in H_0^1(\omega).$$

An analogous formula holds for λ_n^J, $n \in \mathbb{N}$. Note that the operator J in the space $L^2(\omega, dx)$ is associated to the sesquilinear form given by

$$a_J(u,v) = \int_\omega \sigma^b(x) \nabla u(x) \overline{\nabla v(x)}\,dx \qquad u, v \in H_0^1(\omega).$$

The equality (4.25) follows from comparison of the eigenvalues for the Dirichlet Laplacian on dilated cubes. The last equality (4.26) is the well-known evaluation of the trace of the Dirichlet Laplacian on $(0,1)^N$ (see Theorem A.4.4 in Appendix). Combining the last estimate with (4.22) and letting x_0 tend to $\partial\Omega$ we arrive at a contradiction. Therefore e^{At} is not ultracontractive. □

In order to show that the ultracontractivity result 4.4.1 is, in general (i.e. for rough domains, without any regularity condition on the boundary), optimal, we give the following example.
Consider the unit ball in $\mathbb{R}^N$, $N \geq 3$, without its center, i.e. set

$$\Omega := B(0,1)\backslash\{0\},$$

and define a function m to be equal to $|x|^b$, $b > 0$. Note that the function m is smooth on Ω and equals the distance to the boundary in a neighbourhood of the point 0. By an easy computation we also have

$$\frac{1}{m} \in L^q(\Omega) \Longleftrightarrow qb < N.$$

Thus if $b < 2$ there exists a $q > \frac{N}{2}$ such that $\frac{1}{m} \in L^q(\Omega)$. Consequently, the assumptions of Theorem 4.4.1 are satisfied and the Theorem yields the ultracontractivity for the semigroup $e^{m\triangle t}$ on $L^2(\Omega, \frac{dx}{m(x)})$.
On the other hand, if $b \geq 2$, since the function m is equal to the distance to the boundary in a neighbourhood of 0, the method of Theorem 4.4.2 is applicable (just replace $d(x)$ in the definition of $\partial\Omega(\varepsilon_0)$ by $|x|$ and consider $x_0 \to 0$). Thus $e^{m\triangle t}$ is not ultracontractive on $L^2(\Omega, \frac{dx}{m(x)})$.
We have thus shown the following (taking $b = 2$ in the definition of m).

Theorem 4.4.3 *Assume $N \geq 3$. There exists an open set $\Omega \subset \mathbb{R}^N$ and a function $m : \Omega \to (0,\infty)$ such that $\frac{1}{m} \in L^q(\Omega)$ for all $q \in [1, \frac{N}{2})$ but the semigroup $e^{m\triangle t}$ is not ultracontractive on $L^2(\Omega, \frac{dx}{m(x)})$.*

As a first consequence of the ultracontractivity we obtain compactness of the resolvent.

Corollary 4.4.4 *Under the assumption $\frac{1}{m} \in L^q(\Omega)$ for some $q > \frac{\hat{N}}{2}$ the resolvent operator $R(\lambda, m\triangle)$ is compact on $L^2(\Omega, \frac{dx}{m(x)})$ for all $\lambda \geq 0$.*

Proof. We use the formula

$$R(\lambda, m\triangle) = \int_0^\infty e^{-\lambda t} e^{m\triangle t}\, dt \tag{4.27}$$

together with the fact that for any $t > 0$ the semigroup $e^{m\triangle t}$ is compact on $L^2(\Omega, \frac{dx}{m(x)})$ since it is given by an L^2-kernel (we use the finiteness of the measure $\frac{dx}{m(x)}$). The last fact also implies that the integrand in (4.27) is norm continuous and thus the result follows by approximation of (4.27) with finite sums (cf. [50], p.41). □

The assumption $\frac{1}{m} \in L^q(\Omega)$ for some $q > \frac{\hat{N}}{2}$ together with the boundedness of Ω imply that

$$\eta(\Omega) := \int_\Omega \frac{dx}{m(x)} < \infty.$$

Thus we may conclude from the Theorem 4.2.3 that in this case the spectrum of the operator $m\triangle$ is independent of p, $1 \leq p \leq \infty$. The spectrum is purely discrete and

consists of eigenvalues. We write the eigenvalues in decreasing order and denote them by $\{E_n\}_{n=0}^{\infty}$. We denote the corresponding eigenfunctions by $\{\varphi_n\}_{n=0}^{\infty}$ and normalize them by $\|\varphi_n\|_{L^2(\Omega,\frac{dx}{m(x)})} = 1$.
The eigenfunctions φ_n, $n = 0, 1, \dots$ belong to $\mathcal{D}(m\triangle_\infty)$ and they fulfil the equation

$$\triangle\varphi_n = E_n \frac{\varphi_n}{m} \qquad \text{in } \mathcal{D}(\Omega)' \tag{4.28}$$

(see Proposition 3.1.5). If we now, in addition, assume that Ω is Dirichlet regular then since the right-hand side of (4.28) belongs to $L^q(\Omega)$ for some $q > \frac{\hat{N}}{2}$ and $\varphi_n \in \mathcal{D}(m\triangle_2) \subset H_0^1(\Omega)$ we have from Corollary 3.3.9 that $\varphi_n \in C_0(\Omega)$ for all $n = 0, 1, \dots$.

We then have the following representation for the kernel.

Theorem 4.4.5 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded Dirichlet regular set and $m : \Omega \to (0, \infty)$ be a measurable function such that $\frac{1}{m} \in L^q(\Omega)$ for some $q > \frac{\hat{N}}{2}$. Then the spectrum of the operator $m\triangle$ is independent of $1 \le p \le \infty$, it is purely discrete and consists only of eigenvalues. The corresponding (normalised) eigenfunctions φ_n are continuous on $\overline{\Omega}$ for all $n \in \mathbb{N}_0$. We have the representation for the kernel $k(t, x, y)$ of $e^{m\triangle t}$ on $L^2(\Omega, \frac{dx}{m(x)})$:*

$$k_t(x, y) := k(t, x, y) = \sum_{n=0}^{\infty} e^{E_n t}\varphi_n(x)\varphi_n(y), \qquad t > 0, \tag{4.29}$$

where the series converges uniformly on $[T, \infty) \times \Omega \times \Omega$ for any $T > 0$. Thus

$$0 \le k_t(x, y) \in C_0(\Omega) \times C_0(\Omega).$$

If we define $q_t : \Omega \to \mathbb{R}^+$ for $t > 0$ by

$$q_t(x) := \sqrt{k(t, x, x)} \tag{4.30}$$

then q_t is a continuous function in $L^2(\Omega, \frac{dx}{m(x)})$ and for all $n \in \mathbb{N}_0$, $t > 0$, $x \in \Omega$, we have the estimate

$$|\varphi_n(x)| \le e^{-\frac{E_n}{2}t} q_t(x). \tag{4.31}$$

The trace of the operator $e^{m\triangle t}$ is equal to $\|q_t\|^2_{L^2(\Omega,\frac{dx}{m(x)})}$ and is finite for all $t > 0$.

Proof. The first part of the theorem follows directly from the Theorem 4.2.3. The fact that $\varphi_n \in C_0(\Omega)$ has been proved above and since the convergence in (4.29) is uniform by Theorem 4.2.3, we also obtain

$$k_t(x, y) \in C_0(\Omega) \times C_0(\Omega).$$

Since the kernel is continuous, the function q_t is well-defined by (4.30). Using the semigroup property we have

$$k(2t, x, y) = \int_\Omega k(t, x, w)k(t, w, y)\,\frac{dw}{m(w)} \tag{4.32}$$

Combining the fact that the kernel is non-negative (since the semigroup is positive), continuous and symmetric we obtain from (4.32) by the Cauchy-Schwarz inequality

$$0 \le k(t, x, y) \le q_t(x)q_t(y). \tag{4.33}$$

Using Mercer's formula (4.2) we also obtain easily that

$$\|q_t\|_{L^2(\Omega,\frac{dx}{m(x)})} = Tr\ e^{m\triangle t} < \infty$$

since the kernel $k(t,x,y)$ is in $L^\infty(\Omega)$ and the measure $\frac{dx}{m(x)}$ is finite. We now estimate

$$\begin{aligned} |\varphi_n(x)| &= |e^{-E_n t} \int_\Omega k(t,x,y)\varphi_n(y)\,\frac{dy}{m(y)}| \\ &\le e^{-E_n t} \int_\Omega q_t(x)q_t(y)|\varphi_n(y)|\,\frac{dy}{m(y)} \le e^{-E_n t} q_t(x)\,\|q_t\|_{L^2(\Omega,\frac{dx}{m(x)})}\,. \end{aligned} \tag{4.34}$$

We also see that

$$e^{E_n t}|\varphi_n(x)|^2 \le \sum_{m=0}^{\infty} e^{E_m t}|\varphi_m(x)|^2 = k(t,x,x) = q_t(x)^2.$$

□

At this place we may add another characterisation of the Dirichlet regularity (complementing Theorem 3.3.10). We formulate this in the following theorem.

Theorem 4.4.6 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and $m : \Omega \to (0,\infty)$ be a locally bounded measurable function such that $\frac{1}{m} \in L^q(\Omega)$ for some $q > \frac{\hat{N}}{2}$. Then Ω is Dirichlet regular if and only if*

$$k_t(x,y) \in C_0(\Omega) \times C_0(\Omega)$$

where $k_t(x,y)$ is the kernel of the semigroup $e^{m\triangle t}$ on $L^2(\Omega, \frac{dx}{m(x)})$.

Proof. One implication is contained in Theorem 4.4.5 above. Thus we assume that

$$k_t(x,y) \in C_0(\Omega) \times C_0(\Omega).$$

Since

$$(e^{m\triangle t}u)(x) = \int_\Omega k_t(x,y)u(y)\,\frac{dy}{m(y)}$$

for a.a. $x \in \Omega$, we easily see that

$$e^{m\triangle_\infty t}C_0(\Omega) = e^{m\triangle t}C_0(\Omega) \subset C_0(\Omega) \tag{4.35}$$

by continuity of parameter integrals (a majorising function of the form $c\frac{1}{m(y)}$ is easily found by the assumptions). We now show that $C_0(\Omega)$ lies within the subspace of $L^\infty(\Omega)$ where the semigroup $e^{m\triangle_\infty t}$ is strongly continuous with respect to $\|\cdot\|_\infty$. By $\langle\cdot,\cdot\rangle$ we denote the scalar product on $L^2(\Omega, \frac{dx}{m(x)})$. We have for $u, v \in \mathcal{D}(\Omega)$

$$\langle u, e^{m\triangle t}v\rangle = \langle u, v\rangle + \int_0^t \langle u, e^{m\triangle s}m\triangle v\rangle\,ds$$

where the interchange of integrals will be justified by the estimate below. We have

$$|\langle u, e^{m\triangle_\infty t}v - v\rangle| \le t\,\|u\|_{L^1(\Omega,\frac{dx}{m(x)})}\,\|m\triangle_\infty v\|_\infty$$

by the contractivity of $e^{m\triangle_\infty t}$. Note that $\|m\triangle_\infty v\|_\infty$ is finite by Theorem 3.1.5 c) and the assumption $v \in \mathcal{D}(\Omega)$. Since the test functions are dense in $L^1(\Omega, \frac{dx}{m(x)})$ we obtain

$$\left\|e^{m\triangle t}v - v\right\|_\infty \le t\,\|m\triangle_\infty v\|_\infty\,.$$

It follows that

$$\left\|e^{m\triangle_\infty t}v - v\right\|_\infty \to 0$$

as $t \to 0$. For a general $v \in C_0(\Omega)$ and arbitrary $\varepsilon > 0$ we find $\tilde{v} \in (\Omega)$ such that $\|v - \tilde{v}\|_\infty < \varepsilon$ and t_0 such that $\left\|e^{m\triangle_\infty t}\tilde{v} - \tilde{v}\right\|_\infty < \varepsilon$ for all $t < t_0$. Then for all $t < t_0$,

$$\left\|e^{m\triangle_\infty t}v - v\right\|_\infty \leq \left\|e^{m\triangle_\infty t}v - e^{m\triangle_\infty t}\tilde{v}\right\|_\infty + \left\|e^{m\triangle_\infty t}\tilde{v} - \tilde{v}\right\|_\infty + \|v - \tilde{v}\|_\infty \leq 3\varepsilon$$

and thus

$$\left\|e^{m\triangle_\infty t}v - v\right\|_\infty \to 0$$

as $t \to 0$. By the semigroup property we conclude that $e^{m\triangle_\infty t}v$ is a continuous function with respect to $\|\cdot\|_\infty$ for any $v \in C_0(\Omega)$. We write

$$R(\lambda, m\triangle_\infty)v = \int_0^\infty e^{-\lambda t} e^{m\triangle_\infty t} v\, dt$$

and the integral may be approximated by the Riemannian sums in $\|\cdot\|_\infty$. Thus (4.35) implies

$$R(\lambda, m\triangle_\infty)C_0(\Omega) \subset C_0(\Omega).$$

By Theorem 3.2.2 we see that $m\triangle_0$ generates a C_0-semigroup on $C_0(\Omega)$ and by Theorem 3.3.10 we conclude that Ω must be Dirichlet regular. □

At the end of this section we give yet another characterizations of ultracontractivity (and in general, supercontractivity[2]).
Note that in the assumptions of the next proposition the hypothesis $tr\ e^{At} < \infty$ subsists which we proved in our cases of interest by proving ultracontractivity *a priori*. So it may seem that we just list consequences of ultracontractivity. However, when studying intrinsic ultracontractivity, we may use the theorem independently of the validity of intrinsic ultracontractivity, since then the trace hypothesis is fulfilled due to the preservation of trace under unitary transformations.

Proposition 4.4.7 *Let A be the generator of a symmetric submarkovian semigroup e^{At} on $L^p(\Omega, \eta)$, $1 \leq p \leq \infty$ where η is a σ-finite Borel measure. Assume that the trace of e^{At} is finite. The following statements are equivalent:*

(i) e^{At} *is bounded from* $L^2(\Omega, \eta)$ *to* $L^p(\Omega, \eta)$ *for all* $t > 0$.

(ii) For all $t > 0$ *there exists* $c_t < \infty$ *such that*

$$\|\varphi_n\|_{L^p(\Omega,\eta)} \leq c_t e^{-E_n t} \qquad \forall\, n \in \mathbb{N}_0.$$

(iii) $\|q_t\|_{L^p(\Omega,\eta)} < \infty \quad \forall\, t > 0$.

(iv) $\|k(t, \cdot, \cdot)\|_{p,2} := \left(\int \left(\int_\Omega |k(t,x,y)|^2\, d\eta(y)\right)^{\frac{p}{2}}\, d\eta(x)\right)^{\frac{1}{p}} < \infty \quad \forall\, t > 0$.

The outer integral in the last statement is to be understood as $\sup_{x\in\Omega}(\cdot(x))$ *in case* $p = \infty$.

Proof. *(i)* ⇒ *(ii)*

$$\|\varphi_n\|_{L^p(\Omega,\eta)} = \left\|e^{-E_n t}e^{At}\varphi_n\right\|_{L^p(\Omega,\eta)} \leq e^{-E_n t}\left\|e^{At}\right\|_{2\to p}.$$

[2]Here we say that e^{At} is supercontractive if for *some* $p \geq 2$ the semigroup e^{At} is bounded from $L^2(\Omega, d\eta)$ to $L^p(\Omega, d\eta)$.

$(ii) \Rightarrow (iii)$

$$\begin{aligned}\|q_t\|^p_{L^p(\Omega,\eta)} &= \left\|q_t^2\right\|^{\frac{p}{2}}_{L^2(\Omega,\eta)} \le \left(\sum_{n=0}^{\infty} e^{E_n t} \left\||\varphi_n|^2\right\|_{L^{\frac{p}{2}}(\Omega,\eta)}\right)^{\frac{p}{2}} = \left(\sum_{n=0}^{\infty} e^{E_n t} \|\varphi_n\|^2_{L^p(\Omega,\eta)}\right)^{\frac{p}{2}} \\ &\le \left(\sum_{n=0}^{\infty} e^{E_n \frac{t}{3}} c^2_{\frac{t}{3}}\right)^{\frac{p}{2}} = c^p_{\frac{t}{3}} \left(|\, tr\ e^{A\frac{t}{3}}|\right)^{\frac{p}{2}} < \infty\end{aligned}$$

$(iii) \Rightarrow (iv)$ By (4.33) we have

$$\|k(t,\cdot,\cdot)\|_{p,2} \le \|q_t\|_{L^p(\Omega,\eta)} \|q_t\|_{L^2(\Omega,\eta)} = \|q_t\|_{L^p(\Omega,\eta)} \left(tr\ |e^{At}|\right)^{\frac{1}{2}} < \infty.$$

$(iv) \Rightarrow (i)$ On setting $h_t(x) := \left(\int_\Omega |k(t,x,y)|^2 d\eta(y)\right)^{\frac{1}{2}}$ we have for any $f \in L^2(\Omega,\eta)$,

$$|(e^{At}f)(x)| = |\int_\Omega k(t,x,y) f(y)\, d\eta(y)| \le h_t(x) \cdot \|f\|_{L^2(\Omega,\eta)}.$$

Hence

$$\left\|e^{At}f\right\|_{L^p(\Omega,\eta)} \le \|h_t\|_{L^p(\Omega,\eta)} \|f\|_{L^2(\Omega,\eta)} = \|k(t,\cdot,\cdot)\|_{p,2} \|f\|_{L^2(\Omega,\eta)}.$$

□

4.5 Notes and comments

Section 4.1
The Dunford-Pettis Theorem is well-known. It is ascribed to Kantorovich and Vulikh. We follow [4], where also more references to the history of the theorem are given.
Section 4.2
This material is standard. See e.g. [23]. The importance of the notion of hypercontractivity (see the footnote by Proposition 4.4.7 for the definition) in quantum theory was discovered by Nelson in 1966, see [51]. The notion of ultracontractivity was introduced later. See the notes in [26] and [23] for an overview of the history.
Section 4.3
Ideas behind the logarithmic Sobolev inequalities are due to Gross [35]. We use a modified approach as in [26].
Section 4.4
Theorems 4.4.1-4.4.6 are due to the author, although Theorem 4.4.2 for smooth domains has been known (cf. [54]). Proposition 4.4.7 is adapted from [26].

Chapter 5

Pseudo-Gaussian estimates for $e^{m\triangle t}$

In this chapter we refine the estimates from Theorem 4.4.1 to incorporate a Gaussian factor depending on the space variables. Thus we assume that Ω is a bounded open set in $\mathbb{R}^N$ and $m : \Omega \to (0, \infty)$ such that $\frac{1}{m} \in L^q(\Omega)$ for some $q > \frac{\hat{N}}{2}$. Under this assumptions we have proved in Theorem 4.4.1 the estimate

$$\|e^{m\triangle t} f\|_\infty \le c \cdot t^{-\frac{N(q-1)}{2q-N}} \|f\|_2$$

for all $f \in L^2(\Omega, \frac{dx}{m(x)})$.

5.1 The twisted form

Our goal is to obtain a Gaussian estimate for $e^{m\triangle t}$. In order to do so, we use Davies' method. Fix $x_0 \in \Omega,\ a \in \mathbb{R}^N$ with $\|a\| = 1$ and define

$$\phi(x) = \phi_{x_0,a}(x) := \langle x - x_0, a\rangle$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^N$.
Introduce the form:

$$a_\lambda(u, v) := a(e^{\lambda\phi} u, e^{-\lambda\phi} v) \text{ for } u, v \in H_0^1(\Omega).$$

The form a_λ satisfies the assumptions of Section 2.1 with

$$\begin{aligned} b(x) :&= -\lambda \nabla\phi(x) \\ c(x) :&= \lambda\nabla\phi(x) \\ a_0(x) :&= -\lambda^2 \langle \nabla\phi(x), \nabla\phi(x)\rangle. \end{aligned}$$

We also note that the adjoint form to a_λ looks the same as a_λ only the roles of b and c interchange.
The constant w in Lemma 2.1.4 is of the form $w_\lambda = \tilde{c} \cdot \lambda^2$, where $\tilde{c}$ depends only on $\|m\|_\infty$. Therefore, if we denote by A_λ the operator associated with a_λ, we have

$$\left\|e^{A_\lambda t - w_\lambda t}\right\|_2 \le 1.$$

Now we investigate under which assumption we may obtain a contractive semigroup on $L^\infty(\Omega, \frac{dx}{m(x)})$. Here we use the results of Section 2.2. In particular, from Corollary 2.2.4 we see that upon choosing

$$w_{\infty,\lambda} := \lambda^2 |\nabla\phi(x)|^2 + \lambda\triangle\phi(x) = \lambda^2 \|a\|^2 + 0 = \lambda^2$$

we have

$$\left\|e^{A_\lambda t - w_{\lambda,\infty} t}\right\|_\infty \le 1.$$

By interpolation we get

$$\left\|e^{A_\lambda t - \max\,(w_\lambda, w_{\lambda,\infty})t}\right\|_p \le 1 \text{ for any } p \in [2,\infty]\,.$$

Since an analogous estimate holds also for the adjoint form a_λ^*, we obtain finally by duality

$$\left\|e^{A_\lambda t - w_\lambda t}\right\|_p \le 1 \tag{5.1}$$

for all $1 \le p \le \infty$ with $w_\lambda := \tilde{c}\lambda^2$ where $\tilde{c}$ is a constant larger than one and otherwise depending only on $\|m\|_\infty$.

5.2 Ultracontractivity for the twisted form

We start with the estimate from Theorem 4.4.1 which states that

$$\|f\|^2_{L^{\frac{2\mu}{\mu-2}}(\Omega,\frac{dx}{m(x)})} \le c_1 \int_\Omega |\nabla u(x)|^2\,dx \tag{5.2}$$

where $\mu := \frac{2\hat{N}(q-1)}{2q-\hat{N}}$ and $\hat{N} := N$ in case $N \ge 3$ and $\hat{N}$ is arbitrary subject to the condition $2 < \hat{N} < 2q$ in case $N = 1$ or $N = 2$.
We now prove a hypercontractive estimate for the twisted semigroup.

Theorem 5.2.1 *For the semigroup A_λ and for w_λ defined in (5.1) we have*

$$\left\|e^{A_\lambda t - wt}\right\|_{L^2\to L^\kappa} \le c_2 t^{-\frac{1}{2}} \text{ for all } t > 0$$

where $\kappa := \frac{2\mu}{\mu-2}$ and $L^2 \to L^\kappa$ denotes the norm of a linear operator from $L^2(\Omega,\frac{dx}{m(x)})$ to $L^\kappa(\Omega,\frac{dx}{m(x)})$.

Proof. Combining Lemma 2.1.4 with the estimate (5.2) we have

$$\|u\|^2_{L^\kappa(\Omega,\frac{dx}{m(x)})} \le c_2(\mathrm{Re}\ a(u,u) + w_\lambda\langle u,u\rangle_m)$$

where $\langle\cdot,\cdot\rangle_m$ denotes the scalar product on $L^2(\Omega,\frac{dx}{m(x)})$.
Define $S(t) := e^{A_\lambda t - w_\lambda t}$. Then $\|S(t)\|_{L^\kappa\to L^\kappa} \le 1$ and for $f \in L^2(\Omega,\frac{dx}{m(x)}) \cap L^\kappa(\Omega,\frac{dx}{m(x)})$ we have

$$\begin{aligned}
c_2^{-1} t\,\|S(t)f\|^2_{L^\kappa(\Omega,\frac{dx}{m(x)})} &= c_2^{-1}\int_0^t \|S(t)f\|^2_{L^\kappa}\,ds = c_2^{-1}\int_0^t \|S(s)S(t-s)f\|^2_{L^\kappa}\,ds\\
&\le c_2^{-1}\int_0^t \|S(s)f\|^2_{L^\kappa}\,ds \le \int_0^t \mathrm{Re}\ a(S(s)f,S(s)f) + w_\lambda\langle S(s)f,S(s)f\rangle_m\,ds\\
&= \int_0^t -\frac{d}{ds}\|S(s)f\|^2_{L^2(\Omega,\frac{dx}{m(x)})}\,ds = \|f\|_2^2 - \|S(s)f\|_2^2 \le \|f\|^2_{L^2(\Omega,\frac{dx}{m(x)})}\,.
\end{aligned}$$

Since such functions f are dense in $L^2(\Omega,\frac{dx}{m(x)})$ the conclusion of the theorem is proved. □

Now we prove ultracontractivity for the twisted semigroup.

Theorem 5.2.2 *There exists a constant c_3 independent of λ such that*

$$\left\|e^{A_\lambda t - w_\lambda t}\right\|_{L^2\to L^\infty} \le c_3 \cdot t^{-\frac{\mu}{4}} \text{ for all } t > 0.$$

Proof. By the previous theorem and (5.1) we have:

$$\begin{aligned}\|S(t)\|_{L^2(\Omega,\frac{dx}{m(x)})\to L^\kappa(\Omega,\frac{dx}{m(x)})} &\leq c_2 t^{-\frac{1}{2}}\\ \|S(t)\|_{L^r(\Omega,\frac{dx}{m(x)})} &\leq 1 \text{ for all } 1\leq r\leq\infty.\end{aligned}$$

For $\theta\in(0,1)$ define p_θ and q_θ by

$$\begin{aligned}\frac{1}{p_\theta} &= \frac{\theta}{2}+\frac{1-\theta}{r}\\ \frac{1}{q_\theta} &= \frac{\theta}{\kappa}+\frac{1-\theta}{r}\end{aligned}$$

By applying the Riesz-Thorin interpolation theorem we obtain

$$\|S(t)\|_{L^{p_\theta}\to L^{q_\theta}}\leq c_2^\theta\cdot t^{-\frac{\theta}{2}}.$$

Set $\theta=\frac{1}{p}$ and $r=2(p-1)$. Then $p_\theta=p$ $q_\theta=p\frac{\mu}{\mu-1}$ and

$$\|S(t)\|_{p\to p\frac{\mu}{\mu-1}}\leq c_2^{\frac{1}{p}}\cdot t^{-\frac{1}{2p}}.$$

Define $R=\frac{\mu}{\mu-1}$, $t_k=\frac{\mu+1}{2\mu}(2R)^{-k}$, $p_k=2R^k$, where k is a nonnegative integer. Then

$$\sum_{k=0}^{\infty}t_k=1 \text{ and } \sum_{k=0}^{\infty}\frac{1}{p_k}=\frac{\mu}{2}.$$

Therefore

$$\|S(t)\|_{L^2\to L^\infty}\leq\prod_{k=0}^{\infty}\|S(tt_k)\|_{L^{p_k}\to L^{p_{k+1}}}\leq\prod_{k=0}^{\infty}c_2^{\frac{1}{p_k}}\cdot t^{-\frac{1}{2p_k}}\cdot t_k^{-\frac{1}{2p_k}}=C_\mu\cdot t^{-\frac{\mu}{4}}.$$

□

Since an analogous theorem may be proved for A_λ^* (the corresponding form is of the same type; only the coefficients b and c interchange) we obtain by duality

Theorem 5.2.3 *There exists a constant c_4 independent of λ such that*

$$\left\|e^{A_\lambda t-w_\lambda t}\right\|_{L^1\to L^\infty}\leq c_4\cdot t^{-\frac{\mu}{2}} \textit{ for all } t>0.$$

This is equivalent to the following estimate for the kernel $K_\lambda(t,x,y)$ of the semigroup $e^{A_\lambda t}$:

$$|K_\lambda(t,x,y)|\leq c_4\cdot t^{-\frac{\mu}{2}}\cdot e^{\tilde{c}\lambda^2} \textit{ for all } t>0.$$

The constant $\tilde{c}$ depends only on $\|m\|_\infty$.

5.3 Gaussian estimates for $m\triangle$

In order to prove Gaussian estimates we need to estimate the kernel of the operator $e^{-\lambda\phi}e^{m\triangle}e^{\lambda\phi}$ where $\phi=\phi_{x_0,a}$ is a smooth function defined in the beginning of Section 2. However, the associated form to this operator is the form a_λ investigated in the previous two sections. Therefore we have by Theorem 5.2.3

$$\left\|e^{-\lambda\phi}e^{m\triangle}e^{\lambda\phi}\right\|_{L^1(\Omega,\frac{dx}{m(x)})\to L^\infty(\Omega,\frac{dx}{m(x)})}\leq c_4\cdot t^{-\frac{\mu}{2}}\cdot e^{\tilde{c}\lambda^2 t} \text{ for all } t>0.$$

This implies an estimate for the kernel $K(t,x,y)$ of $e^{m\triangle}$:

$$0 \le K(t,x,y) \le C \cdot t^{-\frac{\mu}{2}} \cdot e^{\tilde{c}\lambda^2 t + \lambda(\phi(x)-\phi(y))}$$

for all $t > 0$. Setting $\lambda := \frac{\phi(x)-\phi(y)}{2\tilde{c}t}$ and then optimizing over a in the definition of ϕ we finally obtain the bound:

$$0 \le K(t,x,y) \le C \cdot t^{-\frac{\mu}{2}} e^{-\frac{|x-y|^2}{4\tilde{c}t}} \text{ for all } t > 0.$$

We summarize in the main theorem.

Theorem 5.3.1 *Let Ω be an open bounded subset of $\mathbb{R}^N$ and $m : \Omega \to (0,\infty)$ be a bounded measurable function such that $m^{-1} \in L^q(\Omega, dx)$ for some $q > N/2$. Then we have the following estimate for the kernel $K(t,x,y)$ of the operator $e^{m\triangle}t$:*

$$0 \le K(t,x,y) \le C_{N,q} \cdot t^{-\frac{\hat{N}(q-1)}{2q-\hat{N}}} e^{-\frac{|x-y|^2}{4\tilde{c}t}} \text{ for all } t > 0$$

where the constant $\tilde{c}$ depends only on $\|m\|_\infty$. The constant $\hat{N}$ is equal to the dimension N in case $N \ge 3$, otherwise ($N = 1, 2$) the constant $\hat{N}$ may be chosen arbitrarily subject to $2 < \hat{N} < 2q$.

5.4 Notes and comments

The main result in this chapter (Theorem 5.3.1) is due to the author. We use Davies' method (cf. [23], [52]), the important step in the proof is the determination of the dependence of the constant w_λ on λ in (5.1). This is made possible by the results of Sections 2.1 and 2.2 for weighted spaces.

Chapter 6

Intrinsic ultracontractivity

6.1 Intrinsic ultracontractivity - motivation

In Chapter 4 and Chapter 5 we obtained estimates for the kernel $K(t,x,y)$ of the semigroup $e^{m\triangle t}$ for $m : \Omega \to (0,\infty)$ such that $\frac{1}{m} \in L^q(\Omega)$ for some $q > \frac{\hat{N}}{2}$ (the condition on q is optimal in order to guarantee the existence of a bounded kernel as can be seen from Theorem 4.4.3). However, if the operator is considered on a bounded domain with Dirichlet boundary conditions, one would expect the kernel to tend to zero as $x \to \partial\Omega$ or $y \to \partial\Omega$. This is not reflected in the estimates of Theorem 4.4.1 neither in those of Theorem 5.2.3.
There are also other reasons to study such kind of kernel estimates (dubbed intrinsic ultracontractivity for the reasons which will become apparent later - see the Definition 6.3.2). Namely, one can prove Sobolev estimates up to the boundary. To explain this, we first prove the following lemma.

Lemma 6.1.1 *Let η be a Borel measure on Ω, where Ω is an open set in $\mathbb{R}^N$ and e^{At} be a symmetric submarkovian semigroup on $L^2(\Omega,\eta)$. Assume that $\mu > 0$ and $\lambda > 0$. If for all $u \in \mathcal{D}(A^\mu)$*

$$\|u\|_\infty \le c\,\|(\lambda - A)^\mu u\|_2$$

then we have

$$\left\|e^{At}u\right\|_\infty \le c\max(1,t^{-\mu})\,\|u\|_2 \quad \forall\, u \in L^2(\Omega,\eta), \forall\, t > 0. \tag{6.1}$$

In the other direction, assuming (6.1) we obtain

$$\|u\|_\infty \le c\,\|(\lambda - A)^\nu u\|_2 \quad \forall\, \nu > \mu, \forall\, u \in \mathcal{D}(A^\nu). \tag{6.2}$$

Proof. By the assumed submarkovian property of e^{At} it suffices to prove (6.1) for $0 < t < 1$. We have by the holomorphy of e^{At} that $e^{At}u \in \mathcal{D}(A^\mu)$ for any $u \in L^2(\Omega,\eta)$ and

$$\left\|t^\mu(\lambda - A)^\mu e^{At}u\right\|_2 \le c\,\|u\|_2\,.$$

Hence

$$\left\|e^{At}u\right\|_2 \le c\left\|(\lambda - A)^\mu e^{At}u\right\|_2 \le ct^\mu\,\|u\|_2\,.$$

In the opposite direction we use the representation

$$(\lambda - A)^{-\nu}u = \Gamma(\nu)^{-1}\int_0^\infty t^{\nu-1}e^{-(\lambda-A)t}u\,dt \qquad \forall\, u \in L^2(\Omega,\eta)$$

which by (6.1) converges in L^∞-norm for $\nu > \mu$. The estimate (6.1) yields now (6.2). □

Consider the Dirichlet Laplacian $A := \triangle$ on $L^2(\Omega, dx)$, where Ω is a bounded domain in $\mathbb{R}^N$ (this operator is constructed by the theory of Chapter 2 with $m \equiv 1$). The semigroup $e^{\triangle t}$ is ultracontractive with

$$\left\|e^{\triangle t}\right\|_{2\to\infty} \le ct^{-\frac{N}{4}}.$$

This is very well-known and can also be obtained as a limiting case ($q \to \infty$) of our Theorem 4.4.1. Thus by a slight modification of Lemma 6.1.1 for $\lambda = 0$ (using the fact that $0 \in \rho(\triangle)$ - see Proposition 3.1.6), we obtain

$$\|u\|_\infty \le c\,\|(-\triangle)^\nu u\|_{L^2(\Omega,dx} \qquad \forall\,\nu > \frac{N}{4} \quad \forall\,u \in \mathcal{D}(A^\nu). \tag{6.3}$$

If we pose an additional regularity condition on $\partial\Omega$ and assume that Ω is Dirichlet regular (an exterior cone condition as in Definition 6.5.1 implies Dirichlet regularity) then it is known that u is continuous and vanishes on $\partial\Omega$ (see e.g. [26], Cor. C.4.). It is interesting and natural to ask how fast the convergence $u(x) \to 0$ is as $x \to \partial\Omega$. It is well-known that if Ω is not smooth (for instance, if Ω possesses linear or worse cusps), the rate of convergence depends on the geometry of the boundary even for the Laplace operator (see e.g. Section 4.6 in [23]). However, one expects the convergence to be the slowest for the first eigenfunction φ_0 of the operator (if it exists). Thus in the line with (6.3) we ask whether the following estimate holds for some $\mu > 0$ and $u \in \mathcal{D}(\triangle^\mu)$:

$$u(x) \le c\varphi_0(x)\,\|(-\triangle)^\mu u\|_{L^2(\Omega,dx)}$$

or more generally, under the assumptions of Lemma 6.1.1, whether for $\lambda > 0$ an estimate

$$u(x) \le c\varphi_0(x)\,\|(\lambda - A)^\mu u\|_{L^2(\Omega,d\eta(x)} \tag{6.4}$$

holds true for some constant $c \ge 0$ and some (large) μ depending on the geometry of the domain.

Indeed, in certain cases, one can prove that the ground state controls the behaviour of all $u \in \mathcal{D}(A^\mu)$, $\mu > \alpha$ where $\alpha > \frac{N}{4}$ is a constant depending on the geometry of $\partial\Omega$. This is equivalent to the so called intrinsic ultracontractivity (see Lemma 6.1.1 and Definition 6.3.2) together with a polynomial time decrease in the ultracontractive estimate and has been studied for the Laplace operator in various papers, starting with the seminal paper of Davies and Simon [DS84]. See notes for further reference.

Here we shall prove such a result for the operator $\sigma^b\triangle$, where $0 \le b < 1$ and σ is a regularized distance function (see Theorem 6.6.3).

6.2 Positivity of the kernel

We now come back to our main example and consider Ω, a bounded Dirichlet regular domain of $\mathbb{R}^N$ (open connected set) and a function $m : \Omega \to (0, \infty)$ such that $\frac{1}{m} \in L^q(\Omega)$ for some $q > \frac{\hat{N}}{2}$. Under these assumptions we have seen in Corollary 4.4.4 that the resolvent of $m\triangle$ is compact. The spectrum of $m\triangle$ is independent of $1 \le p \le \infty$ and there exist a non-increasing sequence of eigenvalues E_n and a corresponding sequence of normalized eigenfunctions φ_n, $n \in \mathbb{N}_0$ (see Theorem

4.4.5). The eigenfunctions φ_n belong to $C_0(\Omega)$, in particular they are continuous (see the reasoning below Corollary 4.4.4). We may apply Theorem 1.5.5 to the operator $A := m\triangle + E_0$ on $L^2(\Omega, \frac{dx}{m(x)})$. The corresponding semigroup is still contractive due to the fact that the spectral bound and the growth bound coincide for holomorphic semigroups (see e.g. [2]). Hence we see that $\varphi_0 > 0$ a.e. on Ω. By Theorem 1.5.5 we also know that the multiplicity of the first eigenvalue is one and thus we have $E_1 < E_0$. We shall prove that actually $\varphi_0(x) > 0$ everywhere under the assumptions above. We define

$$\Omega_0 := \{x \in \Omega : \varphi_0(x) > 0\} \tag{6.5}$$

and we set out to prove that $\Omega_0 = \Omega$. We denote by $T_\Omega(t)$ the semigroup generated by $m\triangle$ on $C_0(\Omega)$. The semigroup exists since we assume that Ω is Dirichlet regular (see Theorem 3.3.10). We have the following description of the domain of the generator on $C_0(\Omega)$ (see (3.6))

$$\mathcal{D}((m\triangle_\Omega)_0) = \left\{u \in C_0(\Omega) : \exists f \in C_0(\Omega) \text{ s.t. } \triangle u = \frac{f}{m} \text{ in } \mathcal{D}(\Omega)'\right\}. \tag{6.6}$$

We show that $C_0(\Omega_0)$ remains invariant under the action of $T_\Omega(t)$. We start by showing the invariance for the following class of functions

$$S := \{u \in C_0(\Omega_0) : \exists c \geq 0 \text{ s.t. } |u(x)| \leq c\varphi_0(x) \quad \forall x \in \Omega\}.$$

Indeed, let $u \in S$ and pick $c \geq 0$ fulfilling $u(x) \leq c\varphi_0(x)$ for all $x \in \Omega$. We have by the positivity of $T_\Omega(t)$,

$$|T_\Omega(t)u| \leq cT_\Omega(t)\varphi_0 = ce^{\lambda_0 t}\varphi_0$$

and hence $T_\Omega(t)u \in C_0(\Omega_0)$ for any $t > 0$ and any $u \in S$. Since S is dense in $C_0(\Omega_0)$ we conclude that

$$T_\Omega(t)C_0(\Omega_0) \subset C_0(\Omega_0) \qquad \forall t > 0. \tag{6.7}$$

If we denote by A the generator of $T_\Omega(t)\mid_{C_0(\Omega_0)}$ then we have

$$\mathcal{D}(A) = \left\{u \in C_0(\Omega_0) : \exists f \in C_0(\Omega_0) \text{ s.t. } \triangle u = \frac{f}{m} \text{ in } \mathcal{D}(\Omega)'\right\} \tag{6.8}$$

since A is the part of $(m\triangle_\Omega)_0$ in $C_0(\Omega_0)$ and $\mathcal{D}((m\triangle_\Omega)_0)$ is given by (6.6). Recall that there is also the operator $(m\triangle_{\Omega_0})_0$ defined on $C_0(\Omega_0)$ with

$$\mathcal{D}((m\triangle_{\Omega_0})_0) = \left\{u \in C_0(\Omega_0) : \exists f \in C_0(\Omega_0) \text{ s.t. } \triangle u = \frac{f}{m} \text{ in } \mathcal{D}(\Omega_0)'\right\} \tag{6.9}$$

and we know by Theorem 3.2.2 that $(m\triangle_{\Omega_0})_0$ is dissipative. Since $\Omega_0 \subset \Omega$ we see from (6.8) and (6.9) that

$$\mathcal{D}(A) \subset \mathcal{D}((m\triangle_{\Omega_0})_0).$$

However, A (as the generator of a contractive C_0-semigroup) is m-dissipative. Therefore, A and $(m\triangle_{\Omega_0})_0$ coincide. As a consequence, $(m\triangle_{\Omega_0})_0$ is a generator and Ω_0 must be Dirichlet regular by Theorem 3.3.10. Since $C_0(\Omega_0)$ is dense in

$$L^2\Big(\Omega_0, \frac{dx}{m(x)}\Big) = L^2\Big(\Omega, \frac{dx}{m(x)}\Big)$$

we see that

$$e^{m\triangle_\Omega t} = e^{m\triangle_{\Omega_0} t} \qquad \forall t > 0,$$

where $e^{m\triangle_\Omega t}$ and $e^{m\triangle_{\Omega_0} t}$ denote the semigroups generated by $m\triangle$ on $L^2(\Omega, \frac{dx}{m(x)})$ and $L^2(\Omega_0, \frac{dx}{m(x)})$, respectively. But then the domains of the corresponding sesquilinear forms must coincide, too, and thus we find that

$$\mathcal{D}(a_\Omega) = H_0^1(\Omega) = H_0^1(\Omega_0) = \mathcal{D}(a_{\Omega_0}) \tag{6.10}$$

since $H_0^1(\Omega_0)$ and $H_0^1(\Omega)$ are both contained in $L^2(\Omega, \frac{dx}{m(x)})$ by the assumption $\frac{1}{m} \in L^q(\Omega)$ for some $q > \frac{\hat{N}}{2}$ (see Lemma 2.1.3). The equality (6.10) implies that

$$\mathrm{cap}(\Omega \backslash \Omega_0) = 0 \tag{6.11}$$

(see Corollary 3.9 in [5]). But both Ω and Ω_0 are Dirichlet regular and thus (6.11) implies that

$$\Omega = \Omega_0$$

(cf. Proposition 3.13 b) in [5]). We have thus proved the following

Theorem 6.2.1 *Let Ω be a bounded Dirichlet regular domain of $\mathbb{R}^N$ and $m : \Omega \to (0, \infty)$ be a function such that $\frac{1}{m} \in L^q(\Omega)$ for some $q > \frac{\hat{N}}{2}$. Denote by φ_0 the first eigenfunction of the operator $m\triangle$ on $L^p(\Omega, \frac{dx}{m(x)})$, $1 \le p \le \infty$. Then $\varphi_0 \in C_0(\Omega)$ and φ_0 is strictly positive everywhere on Ω, i.e.*

$$\varphi_0(x) > 0 \qquad \textit{for all } x \in \Omega.$$

We know already that $e^{m\triangle t}$ possesses a continuous kernel $k(t,x,y) \ge 0$ which is also bounded on Ω (see Theorem 4.4.5). Now we show that actually k is strictly positive.

Theorem 6.2.2 *Under the assumptions of Theorem 6.2.1 we have $k(t,x,y) > 0$ for any $x, y \in \Omega$ and any $t > 0$.*

Proof. We fix $x, y \in \Omega$, $t > 0$ and start by showing that if $k(t,x,y) > 0$ then $k(t+s,x,y) > 0$ for any $s > 0$. We have from (4.29) for any $s > 0$,

$$k(s,y,y) \ge e^{E_0 s} \varphi_0(y)^2 > 0. \tag{6.12}$$

Hence, given $s > 0$, by the continuity of k we find a neighbourhood U of y so that $k(t,x,z) > 0$ and $k(s,z,y) > 0$ for all $z \in U$. We see that

$$k(t+s,x,y) = \int_\Omega k(t,x,z) k(s,z,y) \frac{dz}{m(z)} \ge \int_U k(t,x,z) k(s,z,y) \frac{dz}{m(z)} > 0$$

for any $s > 0$. The kernel $k(t,x,y)$ is analytic in t (for fixed x and y in Ω it is given by a series of analytic functions converging uniformly on a half-space $Re\,\lambda > T$ for any $T > 0$ - cf. Theorem 4.4.5) and thus the observation above shows that for fixed $x, y \in \Omega$ either $k(t,x,y) = 0$ for all $t > 0$ or $k(t,x,y) > 0$ for all $t > 0$. We now show that $k(t,x,y)$ cannot be 0 for all $t > 0$. To this purpose recall that $E_1 < E_0$ and write

$$\begin{aligned}\lim_{s\to\infty} k(t+s,x,y) e^{-E_0 s} &= \sum_{n=0}^{\infty} \lim_{s\to\infty} e^{E_n(t+s)} e^{-E_0 s} \varphi_n(x)\varphi_n(y) \\ &= \lim_{s\to\infty} e^{E_0 t} \varphi_0(x)\varphi_0(y) + \sum_{n=1}^{\infty} 0 = e^{E_0 t}\varphi_0(x)\varphi_0(y) > 0.\end{aligned}$$

Note that interchanging the limit and the sum is allowed, since the series converges uniformly (Theorem 4.4.5). Hence $k(t,x,y) > 0$ for all $t > 0$ and all $x, y \in \Omega$. □

By a domination argument we are able to extend the result of the last Theorem to all bounded domains.

Theorem 6.2.3 *Let Ω be a bounded domain in $\mathbb{R}^N$ and $m : \Omega \to (0, \infty)$ be a function such that $\frac{1}{m} \in L^q(\Omega)$ for some $q > \frac{\hat{N}}{2}$. Then the kernel $k(t, x, y)$ of the semigroup $e^{m\triangle t}$ on $L^2(\Omega, \frac{dx}{m(x)})$ is strictly positive i.e. for all $t > 0$ and $x, y \in \Omega$ we have*

$$k(t, x, y) > 0.$$

Proof. Any bounded domain Ω may be approximated by smooth domains Ω_n so that

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n$$

and each $\Omega_n \subset \Omega_{n+1}$ has C^∞ boundary (in particular, it is Dirichlet regular). Denote by $k_n(t, x, y)$ the kernel of the semigroup $e^{m\triangle t}$ on $L^2(\Omega_n, \frac{dx}{m(x)})$. We will show that for any $n \in \mathbb{N}$ we have

$$k_n(t, x, y) \le k(t, x, y) \qquad \forall\, t > 0, \forall\, x, y \in \Omega_n \tag{6.13}$$

and since $k_n(t, x, y) > 0$ for any $n \in \mathbb{N}$ by Theorem 6.2.2 above and every $x, y \in \Omega$ are eventually contained in some Ω_n the result will follow. In order to prove (6.13) it suffices (by an easy argument or by the last statement of Theorem 4.1.1) to prove that

$$e^{m\triangle_{\Omega_n} t} u \le e^{m\triangle_\Omega t} u \tag{6.14}$$

for any $0 \le u \in L^2(\Omega_n, \frac{dx}{m(x)})$. The inequality (6.14) will in turn follow from the inequality

$$R(\lambda, m\triangle_{\Omega_n})u \le R(\lambda, m\triangle_\Omega)\, u, \qquad \lambda > 0,\ 0 \le u \in L^2(\Omega_n, \frac{dx}{m(x)}) \tag{6.15}$$

and the Euler formula

$$T(t)u = \lim_{n\to\infty} \Big(\frac{n}{t} R(\frac{n}{t}, A)\Big)^n u, \qquad u \in H$$

valid for any strongly continuous semigroup $T(t)$ on a Hilbert space H with the generator A. Thus we shall concentrate on proving (6.15). Let $\lambda > 0$, $0 \le u \in L^2(\Omega_n, \frac{dx}{m(x)})$ and set

$$v_n := R(\lambda, m\triangle_{\Omega_n})u \quad \text{and} \quad v := R(\lambda, m\triangle_\Omega)u.$$

Then $v_n \ge 0$, $v \ge 0$ and we have

$$\int_{\Omega_n} \lambda v_n(x) w(x) \frac{dx}{m(x)} + \int_{\Omega_n} \nabla v_n(x) \nabla w(x)\, dx = \int_{\Omega_n} u(x) w(x) \frac{dx}{m(x)} \tag{6.16}$$

for any $w \in H_0^1(\Omega_n)$ and similarly

$$\int_{\Omega} \lambda v_n(x) w(x) \frac{dx}{m(x)} + \int_{\Omega} \nabla v_n(x) \nabla w(x)\, dx = \int_{\Omega} u(x) w(x) \frac{dx}{m(x)} \tag{6.17}$$

for any $w \in H_0^1(\Omega)$. Hence (on trivially extending functions in $H_0^1(\Omega_n)$ to $H_0^1(\Omega)$ by zero) we obtain by subtracting (6.17) from (6.16),

$$\int_{\Omega_n} \lambda(v_n(x) - v(x)) w(x) \frac{dx}{m(x)} + \int_{\Omega_n} \nabla(v_n(x) - v(x)) \nabla w(x)\, dx = 0 \tag{6.18}$$

for any $w \in H_0^1(\Omega_n)$. In particular, choosing $w := (v_n - v)^+ \in H_0^1(\Omega_n)$ in (6.18) we have

$$\int_{\Omega_n} \lambda((v_n(x) - v(x))^+)^2 \frac{dx}{m(x)} + \int_{\Omega_n} |\nabla(v_n(x) - v(x))|^2\, dx = 0. \tag{6.19}$$

We conclude that

$$v_n(x) = (R(\lambda, m\triangle_{\Omega_n})u)(x) \le v(x) = (R(\lambda, m\triangle_\Omega)u)(x), \qquad \text{a.a. } x \in \Omega_n$$

which is (6.15). This concludes the proof. □

6.3 Intrinsic ultracontractivity

We will work under the following standing assumptions throughout the section. Let e^{At} be an irreducible symmetric submarkovian semigroup on $L^2(\Omega, d\eta)$, where η is a σ-finite Borel measure on an open domain Ω, $\Omega \subset \mathbb{R}^N$. Assume that e^{At} may be represented by a jointly continuous kernel $0 \le k(t,x,y)$, i.e. the following holds for any $u \in L^2(\Omega, d\eta)$,

$$(e^{At}u)(x) = \int_\Omega k(t,x,y)u(y)\, d\eta(y) \qquad \text{for a.a. } x \in \Omega.$$

Assume that the trace of e^{At} is finite for all $t > 0$, this may be equivalently expressed by the assumption that

$$\int_\Omega k(t,x,x)\, d\eta(x) < \infty \tag{6.20}$$

(see Remark 6.3.3 below). Assume further that the bottom of the spectrum of $(-A)$ is an eigenvalue $-E_0$. Then its multiplicity is 1 (cf. Theorem 1.5.5). Denote the corresponding eigenfunction by φ_0 and normalize it to $\|\varphi_0\|_2 = 1$. The function φ_0 is strictly positive almost everywhere on Ω (cf. Theorem 1.5.5 again). Define the unitary operator

$$\begin{aligned} U : L^2(\Omega, d\eta) &\;\to\; L^2(\Omega, \varphi_0(x)^2 d\eta(x)) \\ Uf &\;:=\; \varphi_0^{-1} f \end{aligned}$$

and an operator

$$\widetilde{A} := U(A - E_0)U^{-1}$$

on

$$\mathcal{D}(\widetilde{A}) := \left\{ g \in L^2(\Omega, \varphi_0^2 d\eta),\ U^{-1}g \in \mathcal{D}(A) \right\}.$$

Lemma 6.3.1 *Under the assumptions above $e^{\widetilde{A}t}$ is a symmetric submarkovian semigroup on $L^2(\Omega, \varphi_0^2 d\eta)$.*

Proof. The semigroup properties follow from the corresponding properties of e^{At}, the symmetry and positivity is also obvious. We check that L^∞-contractivity holds. We have

$$e^{\widetilde{A}t}1 = \varphi_0^{-1} e^{A-E_0} t \varphi_0 = 1 \qquad \forall\, t \ge 0$$

and hence for a measurable u with $-1 \le u \le 1$

$$-1 \le e^{\widetilde{A}t} u \le 1 \qquad \forall\, t \ge 0$$

by positivity. □

Definition 6.3.2 *Under the assumptions above we say that e^{At} is intrinsically ultracontractive if $e^{\widetilde{A}t}$ is ultracontractive.*

We obviously have the relations:

$$\widetilde{k}(t,x,y) = e^{-E_0 t}\frac{k(t,x,y)}{\varphi_0(x)\varphi_0(y)} \tag{6.21}$$

$$\widetilde{q}_t(x) = \frac{q_t(x)}{\varphi_0(x)} \tag{6.22}$$

and for the other eigenfunctions (if they exist)

$$\widetilde{\varphi}_n(x) = \frac{\varphi_n(x)}{\varphi_0(x)}. \tag{6.23}$$

Here we recall the notation $q_t(x) := \sqrt{k(t,x,x)}$, (see (4.30).

Remark 6.3.3 *We note that in the applications we have in mind the function φ_0 is continuous and strictly positive everywhere (cf. Theorem 6.2.1). For the fact that under the assumptions above the finiteness of the trace is equivalent to (6.20), see [57], p. 65.*

The following theorem shows a first application of intrinsic ultracontractivity.

Theorem 6.3.4 *If $e^{\widetilde{A}t}$ is ultracontractive with*

$$\left\| e^{\widetilde{A}t} \right\|_{2\to\infty} \le c_t \quad \forall\, t > 0,$$

then there exists a complete orthonormal set of eigenfunctions in $L^2(\Omega,\eta)$ satisfying

$$A\varphi_n = E_n\varphi_n$$

and for some constants c_n we have a bound

$$|\varphi_n(x)| \le c_n\varphi_0(x) \quad \forall\, x \in \Omega \;\; \forall\, n \in \mathbb{N}. \tag{6.24}$$

Proof. The ultracontractivity assumption yields

$$0 \le \widetilde{k}(t,x,y) = e^{-E_0 t}\frac{k(t,x,y)}{\varphi_0(x)\varphi_0(y)} \le c_t$$

(see the argument below Definition 4.2.1). If we define the measure μ by

$$d\mu(x) := \varphi_0(x)^2 d\eta(x)$$

i.e. as the absolutely continuous measure (with respect to the measure η) given by the density φ_0^2, then $\eta(\Omega) < \infty$ and the assumption means that $e^{\widetilde{A}t}$ is ultracontractive on $L^2(\Omega, d\mu)$. We may apply Theorem 4.2.3 and obtain a complete set of eigenfunctions $\widetilde{\varphi}_n$ fulfilling

$$\|\widetilde{\varphi}_n\|_\infty \le c_n \tag{6.25}$$

for some finite constants c_n. We put

$$\varphi_n := \widetilde{\varphi}_n\varphi_0$$

and the properties of φ_n follow from the analogous properties of $\widetilde{\varphi}_n$ since U is unitary. The estimate (6.24) follows directly from (6.25). □

6.4 Rosen's criterion for intrinsic ultracontractivity

In this section we establish an important tool for studying intrinsic ultracontractivity. However, we start by giving another characterisation of ultracontractivity.

Lemma 6.4.1 *Let η be a Borel measure on Ω, where Ω is a subset of $\mathbb{R}^N$ and e^{At} be a symmetric submarkovian semigroup on $L^2(\Omega,\eta)$. Assume that $\mu > 2$. Then a bound of the form*

$$\left\|e^{At}u\right\|_\infty \le ct^{-\frac{\mu}{4}} \left\|u\right\|_2 \quad \forall\, t>0, \forall\, u \in L^2(\Omega,\eta) \tag{6.26}$$

is equivalent to a bound

$$g \le c\left\|g\right\|_{\frac{\mu}{2}}(-A) \quad \forall\, g \in L^{\frac{\mu}{2}}(\Omega, d\eta), \tag{6.27}$$

where the last bound is to be understood in the quadratic form sense.

Proof. The estimate (6.26) is by Theorem 4.3.7 equivalent to the bound

$$\left\|u\right\|^2_{\frac{2\mu}{\mu-2}} \le ca(u,u) \qquad \forall\, u \in \mathcal{D}(a). \tag{6.28}$$

We prove the equivalence of (6.27) and (6.28).
Assuming that (6.28) holds, we have for $u \in \mathcal{D}(a)$

$$\langle gu, u\rangle \le \left\|g\right\|_{\frac{\mu}{2}} \left\|u^2\right\|_\nu = \left\|g\right\|_{\frac{\mu}{2}} \left\|u\right\|^2_{2\nu}$$

with $\frac{2}{\mu} + \frac{1}{\nu} = 1$. Hence

$$\langle gu, u\rangle \le \left\|g\right\|_{\frac{\mu}{2}} \left\|u\right\|_{\frac{2\mu}{\mu-2}} \le c\left\|g\right\|_{\frac{\mu}{2}} a(u,u).$$

On the other hand from (6.27) we obtain

$$\int_\Omega g|u|^2 d\eta \le c\left\|g\right\|_{\frac{\mu}{2}} a(u,u)$$

for all $u \in \mathcal{D}(a)$. Choosing $g := |u|^{\frac{4}{\mu-2}}$ we have

$$\int_\Omega |u|^{\frac{2\mu}{\mu-2}} d\eta \le c\left(\int_\Omega |u|^{\frac{2\mu}{\mu-2}} d\eta\right)^{\frac{2}{\mu}} a(u,u)$$

and (6.28) is proved. □

We keep the assumptions from the previous section. We are ready to prove the following criterion.

Lemma 6.4.2 ***(Rosen's criterion for intrinsic ultracontractivity)***
In the setting of the assumptions given before Definition 6.3.2 suppose, in addition, that the semigroup e^{At} is ultracontractive, i.e. the following bound holds

$$\left\|e^{At}u\right\|_\infty \le ct^{-\frac{\mu}{4}} \left\|u\right\|_2 \quad \forall\, t>0, \forall\, u \in L^2(\Omega,\eta).$$

Suppose further that there exist $c \in \mathbb{R}$ and $\alpha > 0$ such that the following inequality:

$$-\log\varphi_0 \le \varepsilon(-A) + c - \alpha\log\varepsilon \qquad \forall\, \varepsilon > 0 \tag{6.29}$$

holds true in the quadratic form sense. Then e^{At} is intrinsically ultracontractive, more precisely, there exists $\widetilde{c} > 0$ such that

$$\|e^{\widetilde{A}t}\|_{2\to\infty} \le \widetilde{c}t^{-(\frac{\mu}{4}+\alpha)} \qquad \forall\, t > 0.$$

Proof. We prove first that for all $0 \le u \in L^2(\Omega, \varphi_0^2 d\eta)$ of norm one we have in the quadratic form sense

$$\log u \le \varepsilon(-A) + k - \Big(\frac{\mu}{4} + \alpha\Big)\log\varepsilon, \quad \forall\, \varepsilon > 0 \tag{6.30}$$

where α and k are real constants. To do so we define for fixed $\beta > 0$ the characteristic function χ by

$$\chi := \begin{cases} 1 & \text{if } \beta u\varphi_0 \ge 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\log(\beta u\varphi_0) \le \chi\log(\beta u\varphi_0) \le b\,\|\chi\log(\beta u\varphi_0)\|_{\frac{\mu}{2}}\,(-A).$$

Also, since we have $(\log s)^{\frac{\mu}{2}} \le b_2 s^2$ for $s \ge 1$, we obtain

$$\|\chi\log(\beta u\varphi_0)\|_{\frac{\mu}{2}}^{\frac{\mu}{2}} \le b_2\int_\Omega (\chi\beta u\varphi_0)^2 d\eta \le b_2\beta^2.$$

We have shown that

$$\log(\beta u\varphi_0) \le b b_2^{\frac{2}{\mu}}\beta^{\frac{4}{\mu}}(-A).$$

We now set $\beta := \left(\frac{\varepsilon}{2a}\right)^{\frac{\mu}{4}} b^{-\frac{1}{2}}$ and obtain

$$\log u \le -\frac{\mu}{4}\log\varepsilon + \widetilde{k} - \log\varphi_0 + \frac{\varepsilon}{2}(-A).$$

for a constant $\widetilde{k}$. This combined with the estimate on φ_0 from the assumption proves the inequality (6.30).
Thus we have proved

$$\varepsilon(-A) - \log u + k - \Big(\frac{\mu}{4} + \alpha\Big)\log\varepsilon \ge 0, \quad \forall\, \varepsilon > 0 \tag{6.31}$$

in the quadratic form sense on $L^2(\Omega, \eta)$. However, A is unitarily equivalent to $\widetilde{A}$ and hence we have

$$\varepsilon(-\widetilde{A}) - \log u + k - \Big(\frac{\mu}{4} + \alpha\Big)\log\varepsilon \ge 0, \quad \forall\, \varepsilon > 0 \tag{6.32}$$

in the form sense on $L^2(\Omega, \varphi_0^2 d\eta)$. This reads with $h(\varepsilon) := k - \left(\frac{\mu}{4} + \alpha\right)\log\varepsilon$,

$$\int_\Omega u^2\log u\varphi_0^2\,d\eta \le \varepsilon\langle\widetilde{A}u, u\rangle_{L^2(\Omega,\varphi_0^2 d\eta)} + h(\varepsilon)$$

for all $0 \le u \in L^2(\Omega, \varphi_0^2 d\eta)$ with $\|u\|_{L^2(\Omega,\varphi_0^2 d\eta)} = 1$. It follows that for any $u \in L^2(\Omega, \varphi_0^2 d\eta)$ we have

$$\int_\Omega u^2\log u\varphi_0^2\,d\eta \le \varepsilon\langle\widetilde{A}u, u\rangle_{L^2(\Omega,\varphi_0^2 d\eta)} + h(\varepsilon)\,\|u\|_{L^2(\Omega,\varphi_0^2 d\eta)} + \|u\|^2_{L^2(\Omega,\varphi_0^2 d\eta)}\log\|u\|_2\,.$$

We conclude from Example 4.3.6 that $e^{\widetilde{A}t}$ is ultracontractive and

$$\left\|e^{\widetilde{A}t}\right\|_{2\to\infty} \le ct^{-(\frac{\mu}{4}+\alpha)}.$$

for some constant $c \ge 0$ and all $t > 0$. □

6.5 Intrinsic ultracontractivity for $e^{m\triangle t}$

Let $\Omega \subset R^N$ be an open bounded Dirichlet regular domain in $\mathbb{R}^N$. We have seen in Chapter 2 (Theorem 2.1.6) that the operator $\sigma^b\triangle,\ b<1$ generates a semigroup of positive contractions on $L^p(\Omega, \sigma^{-b}(x)dx),\ 1 \le p \le \infty$. Note that the measure $\sigma^{-b}(x)dx$ is finite. Since $e^{\sigma^b\triangle}$ is ultracontractive (Theorem 4.4.1), we know that the spectrum of $\sigma^b\triangle$ is purely discrete and independent of p. There exists the smallest eigenvalue E with multiplicity 1, upon normalization there is a unique eigenfunction $\varphi \in L^2(\Omega, \sigma^{-b}(x)dx)$ corresponding to E and φ is strictly positive everywhere on Ω. For these facts see Theorem 6.2.1. In order to prove intrinsic ultracontractivity of $e^{\sigma^b\triangle}$ we need pose additional conditions on the boundary of Ω.

Definition 6.5.1 *(a) We say that Ω satisfies a uniform external ball condition if there exist $\alpha, \beta > 0$ s.t. for any $y \in \partial\Omega$ and $0 < s \le \beta$ there exists a ball $B_{b,r}$ centered at b satisfying $|b-y| \le s$, with radius $r \ge 2\alpha s$ such that $B_{b,r} \cap \Omega = \emptyset$.*
(b) We say that Ω satisfies a uniform internal cone condition if there exist $\alpha > 0$, $r > 0$ such that for any $x \in \Omega$ we may find a cone $\widetilde{C} \subset \Omega$ s.t. $x \in \widetilde{C}$. Here $\widetilde{C}$ is a translated and rotated congruent copy of the cone C defined by

$$C := \left\{ (x_1, x_2, ..., x_N) :\ 0 \le x_1 \le r,\ \sqrt{x_2^2 + x_3^2 + ... + x_N^2} \le \sin\alpha \cdot x_1 \right\}$$

We say that the cone C is centered at $(r/2, 0, ..., 0)$ and denote by y_x the centre of the cone $\widetilde{C}$ corresponding to x. We use also the notation $C_y := \widetilde{C}$.

Remark 6.5.2 *The condition (a) is implied by a uniform external cone condition. Both (a) and (b) are fulfilled for Lipschitz boundaries.*

In order to obtain intrinsic ultracontractivity by applying Rosen's lemma, we need a lower bound on the first eigenfunction.

Lemma 6.5.3 ***(Lower estimate for the first eigenfunction)***
Let Ω be a bounded domain in $\mathbb{R}^N$ satisfying a uniform internal cone condition. Let ϕ be the (normalized) first eigenfunction for the operator $A := \sigma^b\triangle,\ b < 1$ on $L^2(\Omega, \sigma^{-b}(x)dx)$ with Dirichlet boundary condition. Then there exist $k_1, k_2 \ge 0$ such that

$$-\log\phi(x) \le k_1 + k_2 d(x)^{-1} \qquad \text{on } \Omega.$$

For the proof we need an elliptic Harnack's inequality in the following form (see [34], Theorem 8.20).

Theorem 6.5.4 ***(Harnack's inequality)***
Let G be a bounded domain in $\mathbb{R}^N$. Consider the operator L given formally by

$$L := -\sum_{i,j=1}^{N} \left[\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j}) \right] + V$$

where a_{ij} and V are measurable and fulfill:

$(i)\qquad 0 < \lambda < \{a_{ij}\}$ *(bound for the eigenvalues of a_{ij} independent of x)*

$(ii)\qquad \sum_{i,j=1}^{N} |a_{ij}|^2 \le \Lambda^2$

$(iii)\qquad \lambda^{-1}|V| \le \nu^2$

for some constants λ, Λ and ν. Suppose that $0 \le u \in H^1(G)$ is the solution of the equation

$$\int_G \left[\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial \psi(x)}{\partial x_j} + V(x)u(x)\psi(x) \right] dx = 0$$

for all $\psi \in C_c^\infty(G)$.
Then there exists $c_0 = c_0(N) \ge 1$ s.t. for all $B(y,R) \subset G$

$$\sup \left\{ u(x) : |x-y| \le \frac{R}{4} \right\} \le c_0^{\sqrt{(\frac{\Lambda}{\lambda} + \nu R)}} \inf \left\{ u(x) : |x-y| \le \frac{R}{4} \right\}.$$

Proof of Lemma 6.5.3:
Fix $x \in \Omega$ and consider the cone C_y centered at some $y = y_x \in \Omega$ s.t. $x \in C_y$ (guaranteed by the uniform internal cone condition). Since C_y is congruent with C for any $y = y_x$ there exists a universal constant c_Ω s.t. $d(y_x) \ge c_\Omega$ for any $x \in \Omega$. Set $x_0 := x$ and $r_0 := \frac{d(x)}{8}$. Define $\gamma := \min(\alpha, \frac{\pi}{2} - \alpha)$, where α is the angle from the uniform internal cone condition. Construct inductively balls $B_1, ..., B_n$ in the following way:

$r_i := r := r_0 \sin\gamma$

all x_i, $i = 1, 2, ...$ lie on the segment connecting x to y_x, $|x_{i+1} - x_i| = \dfrac{r}{4}$

and the value $|y - x_i|$ decreases as i increases.

Stop the construction when the ball B_n contains y_x for the first time, i.e.

$$y_x \in B_n \text{ and } y_x \notin B_i,\ i = 1, 2, ...n-1.$$

We have

$$\begin{aligned} n &\le \frac{4 \cdot \text{diam } C}{r} + 1 \qquad \text{i.e.} \\ n &\le c_1 d(x)^{-1} + 1. \end{aligned}$$

Note that all B_i, $i = 1, 2, ..., n$ lie in Ω. Note also that

$$d(z) \ge \frac{d(x)}{2} \qquad \text{for all } z \in \bigcup_{i=1}^{n} B_i. \tag{6.33}$$

Since ϕ is the eigenfuction of the operator $\sigma^b \triangle$ corresponding to the eigenvalue E we have for any open $G \subset \Omega$ and any $\psi \in C_c^\infty(G)$

$$\int_G \left(\nabla\phi(w) \nabla\psi(w) + E\sigma^{-b}(w)\phi(w)\psi(w) \right) dw = 0.$$

In order to apply Harnack's inequality we choose:

$$\begin{aligned} G &:= B_i := B(x_i, r_i) \quad i = 0, 1, ..., n \\ R &:= r_i \\ \lambda &:= \Lambda := 1 \\ V(z) &:= E\sigma^{-b}(z). \end{aligned}$$

By virtue of the estimate (6.33) and the properties of a regularized distance function we obtain a constant $c_2 = c_2(\Omega, b)$ independent of $x \in \Omega$ and $z \in \bigcup_{i=1}^n B_i$ such that

$$\sigma^{-b}(z) \le c_2 d(x)^{-2} \quad (z \in \bigcup_{i=1}^{n} B_i).$$

On setting $\nu := \sqrt{E \cdot c_2} \cdot d(x)^{-1}$ we have

$$V(z) \leq \nu^2 \qquad \text{for } z \in \ G = B(x_i, r_i).$$

and also

$$\nu R \leq \sqrt{E \cdot c_2}.$$

We may now apply Harnack's inequality successively (we use that $B(x_i, r_i/4)$ and $B(x_{i+1}, r_{i+1}/4)$ have always a point in common by construction) and obtain for $\widetilde{c} := c_0^{-\sqrt{(1+\sqrt{Ec_2})}}$

$$\begin{aligned}
\phi(x) &\geq \inf\left\{\phi(z):\ z \in B(x_0, \frac{r_0}{4})\right\} \geq \widetilde{c}\sup\left\{\phi(z):\ z \in B(x_0, \frac{r_0}{4})\right\} \geq \\
&\geq \widetilde{c}\inf\left\{\phi(z):\ z \in B(x_1, \frac{r_1}{4})\right\} \geq \widetilde{c}^2\sup\left\{\phi(z):\ z \in B(x_1, \frac{r_1}{4})\right\} \geq \\
&\geq \ \dots \geq \\
&\geq \widetilde{c}^n\inf\left\{\phi(z):\ z \in B(x_n, \frac{r_n}{4})\right\} \geq \widetilde{c}^{n+1}\sup\left\{\phi(z):\ z \in B(x_n, \frac{r_n}{4})\right\} \geq \\
&\geq \widetilde{c}^{(c_1 d(x)^{-1}+2)}\phi(y_x) \geq \widetilde{c}^{(c_1 d(x)^{-1}+2)}\inf\left\{\phi(z):\ z \in \Omega \text{ s.t. } d(z) \geq c_\Omega)\right\} = \\
&= c_3\widetilde{c}^{(c_1 d(x)^{-1}+2)}
\end{aligned}$$

where $c_3 > 0$ since ϕ is a positive continuous function in Ω and the set

$$\{z:\ z \in \Omega,\ d(z) \geq c_\Omega\}$$

is compact. Hence (since $\widetilde{c} < 1$) we have

$$\log\phi(x) \geq c_4 - c_5 d(x)^{-1}$$

for some $c_4 \leq 0$, $c_5 \geq 0$. This yields the statement of the lemma. □

Lemma 6.5.5 ***(Hardy's inequality)***
Let Ω be a bounded domain in $\mathbb{R}^N$ fulfilling a uniform external ball condition. Let $L := \triangle$ on $L^2(\Omega, dx)$ with Dirichlet boundary condition. Then there exist $k_3 > 0$ and $k_4 \geq 0$ such that

$$(-L) \geq k_3 d(x)^{-2} - k_4$$

in the quadratic form sense on $L^2(\Omega, dx)$.

See ([Dav][23], Theorem 1.5.4) for a proof.

Corollary 6.5.6 *Let Ω be a bounded domain in $\mathbb{R}^N$ fulfilling a uniform external ball condition. For the operator $A := \sigma^b\triangle$, $0 \leq b < 1$, there exist $k_3 > 0$, $k_5 \geq 0$ such that*

$$(-A) \geq k_3 d(x)^{b-2} - k_5$$

in the quadratic form sense on $L^2(\Omega, \sigma^{-b}(x)dx)$.

Proof: Hardy's inequality states that

$$\int_\Omega |\nabla u(x)|^2\, dx \geq k_3 \int_\Omega \frac{|u(x)|^2}{d(x)^2}\, dx - k_4 \int_\Omega |u(x)|^2\, dx \qquad (6.34)$$

for all $u \in \mathcal{D}(\triangle^{\frac{1}{2}} = H_0^1(\Omega)$. We have to prove that

$$\int_\Omega |\nabla u(x)|^2\, dx \geq k_3 \int_\Omega \frac{|u(x)|^2}{d(x)^2}\, dx - k_5 \int_\Omega \frac{|u(x)|^2}{d(x)^b}\, dx \qquad (6.35)$$

for all $u \in \mathcal{D}(a) = H_0^1(\Omega)$. However, since we assume that $b \geq 0$, the function $d(x)^b$ is bounded, and thus (6.35) follows from (6.34). □

Theorem 6.5.7 *Let Ω be a bounded domain in $\mathbb{R}^N$ satisfying a uniform external ball condition and a uniform internal cone condition. Let $0 \leq b < 1$ and let $A := \sigma^b \triangle$ with Dirichlet boundary condition be the generator of the positive symmetric submarkovian semigroup e^{At} on $L^2(\Omega, \sigma^{-b}(x)dx)$. Denote by ϕ the first eigenfunction of A. Then the conditions of Rosen's lemma are satisfied for A, i.e., there exist $c \in \mathbb{R}$, $\alpha > 0$ such that*

$$-\log \phi \leq \varepsilon(-A) + c - \alpha \log \varepsilon \quad \forall\, \varepsilon > 0 \tag{6.36}$$

in the quadratic form sense. Consequently, the semigroup $e^{\sigma^b \triangle t}$ is intrinsically ultracontractive, more precisely, there exist $c > 0$ and $\alpha > 0$ such that

$$0 \leq K(t, x, y) \leq ct^{-\alpha}\phi(x)\phi(y) \quad t > 0,\ x, y \in \Omega$$

where $K(t, x, y)$ is the kernel of $e^{\sigma^b \triangle t}$ on $L^2(\Omega, \sigma^{-b}(x)dx)$.

Proof: All the assumptions stated at the beginning of Section 6.3 are satisfied as can be seen from Theorem 4.4.5 and Theorem 6.2.1. Theorem 6.2.1 also guarantees that the first eigenfunction is actually strictly positive everywhere on Ω. Combining Lemma 6.5.3 and Corollary 6.5.6 we compute for x with $d(x) < 1$

$$\begin{aligned} -\log \phi(x) &\leq k_1 + k_2 d(x)^{-1} \leq k_1 + k_2 \cdot \varepsilon d(x)^{b-2} - k_2 \cdot \log \varepsilon \\ &\leq k_6 + k_7 \cdot \varepsilon(-A) - k_2 \log \varepsilon. \end{aligned}$$

The bound (6.36) follows on redefining ε.
An analogous estimate for x with $d(x) \geq 1$ is straightforward since ϕ is continuous and bounded below on $\{x \in \Omega :\ d(x) \geq 1\}$. Also A is strictly elliptic on $\{x \in \Omega :\ d(x) \geq 1\}$. The final estimate follows directly from Theorem 6.4.2 and the relation (6.21). □

6.6 Applications of intrinsic ultracontractivity

Here we list further consequences (and characterisations) of intrinsic ultracontractivity, in particular, for operators $\sigma^b \triangle$, $0 \leq b < 1$, we prove a Sobolev estimate up to the boundary of Ω of the kind discussed in the introduction, cf. (6.4). This is the content of Theorem 6.6.3 below.
We start by a general theorem in the setting of Section 6.3. Note in particular that an upper estimate on the kernel implies automatically a lower estimate.

Theorem 6.6.1 *We keep all the assumptions given above Definition 6.3.2 and assume, in addition, that the first eigenfunction satisfies $\varphi_0(x) > 0$ for* every *$x \in \Omega$. Then the following relations between the conditions listed below hold true:*

$$(1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) \Rightarrow (7).$$

If the constants c_t in (3) *can be chosen as $c_t := ct^{-\alpha}$ for some fixed $c \geq 0$ and $\alpha \geq 0$ then also* (3) $\Rightarrow$ (2) *with any $n > \alpha$.*

(1) *There exists $n \in \mathbb{N}$ and a constant $c > 0$ such that*

$$|u(x)| \leq c\varphi_0(x) \left\|(I + E_0 - A)^n u\right\|_{L^2(\Omega, d\eta(x))} \qquad \forall\, x \in \Omega,\ \forall\, u \in \mathcal{D}(A^n).$$

(2) *There exists $n \in \mathbb{N}$ and a constant $c > 0$ such that*

$$\|\widetilde{u}\|_\infty \leq c \left\|(I - \widetilde{A})^n \widetilde{u}\right\|_{L^2(\Omega, \varphi_0(x)^2 d\eta(x))}$$

for all $\widetilde{u} \in \mathcal{D}(\widetilde{A}^n)$.

(3) *For all $t>0$ there exists $c_t>0$ such that*

$$\left\| e^{\widetilde{A}t}\widetilde{u}\right\|_\infty \le c_t \left\|\widetilde{u}\right\|_{L^2(\Omega,\varphi_0(x)^2 d\eta(x))}$$

for all $\widetilde{u}\in L^2(\Omega,\varphi_0(x)^2 d\eta(x))$.

(4) *For all $t>0$ there exists $c_t>0$ such that*

$$q_t(x)\le c_t\varphi_0(x) \quad \forall\, x\in\Omega.$$

(5) *For all $t>0$ there exists $c_t>0$ such that*

$$k(t,x,y)\le c_t\varphi_0(x)\varphi_0(y) \quad \forall\, x,y\in\Omega.$$

(6) *For all $t>0$ there exists $c_t>0$ such that*

$$k(t,x,y)\ge c_t q_t(x) q_t(y) \quad \forall\, x,y\in\Omega.$$

(7) *For all $t>0$ there exists $c_t>0$ such that*

$$k(t,x,y)\ge c_t\varphi_0(x)\varphi_0(y) \quad \forall\, x,y\in\Omega.$$

Remark 6.6.2 *Of course, the constant 1 in front of the identity operator in the statements (1) and (2) has no special status, we could prove an analogous result for λI, $\lambda>0$ instead of I. This is clear from the proof below and Lemma 6.1.1.*

Proof. We start by showing (2) $\Rightarrow$ (1). Pick $u\in\mathcal{D}(A^n)$ arbitrarily and set

$$\widetilde{u} := Uu = \varphi_0^{-1}u.$$

Then evidently $\widetilde{u}\in\mathcal{D}(\widetilde{A}^n)$ and we have for any $x\in\Omega$

$$\begin{aligned}
\varphi_0(x)^{-1}u(x) \le \left\|\varphi_0^{-1}u\right\|_\infty = \left\|\widetilde{u}\right\|_\infty \le c\left\|(I-\widetilde{A})^n\widetilde{u}\right\|_{L^2(\Omega,\varphi_0(x)^2 d\eta(x))} \\
= \left\|[U(I+E_0-A)U^{-1}]^n\widetilde{u}\right\|_{L^2(\Omega,\varphi_0(x)^2 d\eta(x))} \\
= \left\|U(I+E_0-A)^nU^{-1}\widetilde{u}\right\|_{L^2(\Omega,\varphi_0(x)^2 d\eta(x))} \\
= \left\|(I+E_0-A)^n u\right\|_{L^2(\Omega,d\eta(x))}.
\end{aligned}$$

Thus (1) follows.
On the other hand, assume (1) and choose $\widetilde{u}\in\mathcal{D}(\widetilde{A}^n)$ arbitrarily. Set

$$u := U^{-1}\widetilde{u} = \varphi_0\widetilde{u},$$

then $u\in\mathcal{D}(A^n)$. Dividing the inequality in (1) by $\varphi(x)$ and then taking supremum over all $x\in\Omega$ we obtain

$$\left\|\varphi_0^{-1}u\right\|_\infty \le c\left\|(I+E_0-A)^n u\right\|_{L^2(\Omega,d\eta(x))}. \tag{6.37}$$

But the left-hand side of 6.37 equals $\left\|\widetilde{u}\right\|_\infty$ and the right-hand side of (6.37) equals

$$c\left\|(I-\widetilde{A})^n\widetilde{u}\right\|_{L^2(\Omega,\varphi_0(x)^2 d\eta(x))}$$

by the same computation as in the first part of the proof. This shows (1) $\Rightarrow$ (2).

The implication (2) $\Rightarrow$ (3) was proved in Lemma 6.1.1. Also the opposite implication (3) $\Rightarrow$ (2) is contained in Lemma 6.1.1 under the additional assumption that $c_t := ct^{-\alpha}$ for some $c, \alpha \geq 0$.

The equivalence (3) $\Leftrightarrow$ (4) follows from (6.22) and Lemma 4.4.7, part $(i) \Leftrightarrow (iii)$ used for $p = \infty$.

Using the semigroup property we have

$$k(2t, x, y) = \int_\Omega k(t, x, w) k(t, w, y)\, d\eta(w). \tag{6.38}$$

Combining the fact that the kernel is non-negative (since the semigroup is positive), continuous and symmetric we obtain from (6.38) by the Cauchy-Schwarz inequality

$$0 \leq k(t, x, y) \leq q_t(x) q_t(y). \tag{6.39}$$

The equivalence (4) $\Leftrightarrow$ (5) follows from the definition of q_t, relations (6.21), (6.22) and the estimate (6.39).

We now prove (5) $\Rightarrow$ (6).
Let $K \subset \Omega$ be compact and such that $\int_K \varphi(x)^2\, d\eta(x) \geq 1 - \frac{e^{E_0 t}}{2c_t}$. Then

$$\begin{aligned} e^{E_0 t}\varphi_0(x) &= \int_K k(t, x, y)\varphi_0(y)\, d\eta(y) \\ &\leq \int_{\Omega\setminus K} c_t \varphi_0(x)\varphi_0(y)^2\, d\eta(y) + \int_K k(t, x, y)\varphi_0(y)\, d\eta(y) \\ &\leq \frac{1}{2} e^{E_0 t}\varphi_0(x) + \int_K k(t, x, y)\varphi_0(y)\, d\eta(y), \end{aligned}$$

i.e.,

$$\int_K k(t, x, y)\varphi_0(y)\, d\eta(y) \geq \frac{1}{2} e^{E_0 t}\varphi_0(x) \quad \forall\, x \in \Omega.$$

Set $\gamma := \min_{z,w\in K} \frac{k(t,z,w)}{\varphi_0(z)\varphi_0(w)} > 0$. Then

$$\begin{aligned} k(3t, x, y) &= \int_\Omega \int_\Omega k(t, x, z) k(t, z, w) k(t, w, y)\, d\eta(z)\, d\eta(w) \\ &\geq \gamma \int_K \int_K k(t, x, z)\varphi_0(z) k(t, w, y)\varphi_0(w)\, d\eta(z)\, d\eta(w) \\ &\geq \gamma e^{2E_0 t}\varphi_0(x)\varphi_0(y). \end{aligned}$$

We finish by applying (4) $\Leftrightarrow$ (5). Note that we have also proved (5) $\Rightarrow$ (7) in this step.

Finally, we show (6) $\Rightarrow$ (4). We have for any $x \in \Omega$,

$$\begin{aligned} \varphi_0(x) &= e^{-E_0 t} \int_\Omega k(t, x, y)\varphi_0(y)\, d\eta(y) \\ &\leq e^{-E_0 t} \int_\Omega q_t(x) q_t(y)\varphi_0(y)\, d\eta(y) \\ &\leq e^{-E_0 t} q_t(x)\, \|q_t\|_{L^2(\Omega,\eta)} = e^{-E_0 t} q_t(x)(\mathrm{tr}\ e^{At}) = c e^{-E_0 t} q_t(x) \end{aligned}$$

by the Cauchy-Schwarz inequality and the assumed finiteness of the trace. Thus

$$\varphi_0(x) = e^{-E_0 t} \int_\Omega k(t,x,y)\varphi_0(y)\, d\eta(y) \geq e^{-E_0 t} \int_\Omega c_t q_t(x) q_t(y) \varphi_0(y)\, d\eta(y)$$
$$\geq \frac{c_t}{c} q_t(x) \int_\Omega \varphi_0(y)^2\, d\eta(y) = \widetilde{c}_t q_t(x).$$

□

In the last theorem of this chapter we combine the results of Theorem 6.6.1 with Theorem 6.5.7.

Theorem 6.6.3 *Let Ω be a bounded domain in $\mathbb{R}^N$ satisfying a uniform external ball condition and a uniform internal cone condition. Let $A := \sigma^b \triangle$ $(0 \leq b < 1)$ with the Dirichlet boundary condition be the generator of the positive symmetric submarkovian semigroup e^{At} on $L^2(\Omega, \sigma^{-b}(x)dx)$. Denote by ϕ the first eigenfunction of $\sigma^b\triangle$, by $E < 0$ the first (largest) eigenvalue and by $K(t,x,y)$ the kernel of e^{At} on $L^2(\Omega, \sigma^{-b}(x)dx)$. Then there exist $c_t, c, \hat{c}, \alpha > 0$ and $n \in \mathbb{N}$ such that the following estimates hold true*

$$0 < c_t \phi(x)\phi(y) \leq K(t,x,y) \leq c t^{-\alpha} e^{Et} \phi(x)\phi(y), \qquad t > 0, \quad \forall\, x, y \in \Omega$$

and

$$|u(x)| \leq \hat{c}\phi_0(x) \left\| (-\sigma^b \triangle)^n u \right\|_{L^2(\Omega, d\eta(x))} \qquad \forall\, x \in \Omega,\ \forall\, u \in \mathcal{D}((\sigma^b\triangle)^n).$$

Proof. This follows from Theorems 6.6.1 and 6.5.7. The last estimate follows from Remark 6.6.2 with $\lambda := -E > 0$. □

6.7 Notes and comments

Section 6.1
Lemma 6.1.1 is easy and known (cf. [23]). The notion of intrinsic ultracontractivity was introduced in the important paper [26], where also the first results concerning Dirichlet Laplacians and Schrödinger operators may be found. A precise geometric characterisation of domains in $\mathbb{R}^2$ allowing intrinsic ultracontractivity for the Dirichlet Laplacian is given in [13]. Results for symmetric stable processes are contained in [19] and [20], variants for relativistic symmetric stable processes are to be found in [41] and [39], all these papers use mostly probabilistic methods. The last class of processes have as their infinitesimal generators Schrödinger operators based on fractional Laplacians. See [39] for the precise definitions.
Section 6.2
The result of Theorem 6.2.1 appears to be new in this generality. It arose in collaboration with W. Arendt. Theorem 6.2.2 is taken from [26] and simplified by the author.
Section 6.3
The presentation here goes along the lines of [23].
Section 6.4
Our presentation follows [23], although (as the name suggests) the origins of Lemma 6.4.2 are to be found in [58].
Section 6.5
The main result Theorem 6.5.7 is due to the author. A similar approach is used in [54] for a certain class of degenerate operators on $\mathbb{R}^N$. The Harnack inequality is well-known, the version we use here is taken from [34]. The Hardy inequality is taken from [23].
Section 6.6
Theorem 6.6.1 is adapted from [26], Theorem 6.6.3 is due to the author.

Chapter 7

L^p-maximal regularity for $m\triangle$

7.1 Introduction

We start by explaining the notion of maximal regularity in a general setting. Let A be a closed densely defined operator on a Banach space X, $f : \mathbb{R}_+ \to X$ be[1] a locally integrable function and $x_0 \in X$. Consider the inhomogeneous Cauchy problem

$$u' = Au + f \tag{7.1}$$
$$u(0) = x_0. \tag{7.2}$$

Let $1 \leq p < \infty$. We say that A has L^p-maximal regularity if for any $f \in L^p(\mathbb{R}_+, X)$ there exists a unique $u \in W^{1,p}(\mathbb{R}_+, X) \cap L^p(\mathbb{R}_+, \mathcal{D}(A))$ satisfying (7.1) in the L^p-sense. Then the closed graph theorem implies that there exists a constant $C > 0$ independent of f such that

$$\|u\|_{L^p(\mathbb{R}_+,X)} + \|u'\|_{L^p(\mathbb{R}_+,X)} + \|Au\|_{L^p(\mathbb{R}_+,X)} \leq C \|f\|_{L^p(\mathbb{R}_+,X)} .$$

It is well-known that a necessary condition for an operator A to enjoy maximal regularity is the sectoriality of A with an angle larger or equal $\frac{\pi}{2}$ i.e.

$$\mathbb{C}_+ := \{z \in \mathbb{C},\, Re\ z > 0\} \subset \rho(A)$$

and there exists $M > 0$ such that

$$|R(\lambda, A)| \leq M(1 + |\lambda|)^{-1}$$

for all $\lambda \in \mathbb{C}_+$. This is equivalent to the condition that A generate a bounded analytic semigroup on X.
If X is a Hilbert space, then the necessary condition above is also sufficient. Unfortunately, for a general Banach space X this is far from being the case and it is an important problem to find sufficient conditions on A which would guarantee maximal regularity. In case X is a UMD-space[2] there is a celebrated result (due to L. Weis) characterising operators which admit maximal regularity. In order to state and explain the result we need the notion of randomized boundedness and it is this notion to which we turn next.

[1] Here $\mathbb{R}_+$ denotes the closed interval $[0, \infty)$.
[2] A Banach space X is called UMD if the (vector-valued) Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for $1 < p < \infty$.

7.2 R-boundedness

We introduce here the notion of R-boundedness. Our treatment is by no means exhaustive, for a thorough study the reader is referred to the literature. We want to stress here the connection between R-bounds and (quadratic or square function) estimates from the harmonic analysis. We shall prove such estimates for contractive semigroups on L^p-spaces by means of the weak maximal estimate proved in the next section.
We start with the definition of Rademacher functions.

Definition 7.2.1 *The functions on* $[0,1]$ *defined by*

$$r_n(t) := \operatorname{sign}\sin(2^n \pi t) \qquad n \in N$$

are called the Rademacher functions.

The Rademacher functions form an orthonormal sequence in $L^2[0,1]$ as is easily seen. They are also symmetric and independent. We state here basic inequalities concerning the Rademacher functions.

Lemma 7.2.2 (Khinchine's inequality)
For $1 \le p < \infty$ *there is a constant* c_p *such that for all* $a_n \in \mathbb{C}$ *and all independent, symmetric,* $\{-1,1\}$*-valued random variables* ε_j *on a probability space* (Ω, Σ, η) *we have*

$$c_p^{-1}\left(\sum_{n=1}^N |a_n|^2\right)^{\frac{1}{2}} \le \left\|\sum_{n=1}^N \varepsilon_n a_n\right\|_{L^p(\Omega,\eta)} \le c_p \left(\sum_{n=1}^N |a_n|^2\right)^{\frac{1}{2}}$$

for any $N \in \mathbb{N}$.

Lemma 7.2.3 (Kahane's inequality)
For any Banach space X *and any* $1 \le p < \infty$ *there is a constant* c_p *such that for all independent, symmetric,* $\{-1,1\}$*-valued random variables* ε_j *on a probability space* (Ω, Σ, η) *we have*

$$c_p^{-1}\left\|\sum_{n=1}^N \varepsilon_n x_n\right\|_{L^2(\Omega,\eta,X)} \le \left\|\sum_{n=1}^N \varepsilon_n x_n\right\|_{L^p(\Omega,\eta,X)} \le c_p \left\|\sum_{n=1}^N \varepsilon_n x_n\right\|_{L^2(\Omega,\eta,X)}$$

for any $N \in \mathbb{N}$ *and any* $x_n \in X$.

Lemma 7.2.4 (Kahane's contraction principle)
Let X *be a Banach space and* ε_j *be independent, symmetric,* $\{-1,1\}$*-valued random variables on a probability space* (Ω, Σ, η). *For any* $N \in \mathbb{N}$ *and any* $a_n \in \mathbb{C}$ *we have*

$$\left\|\sum_{n=1}^N \varepsilon_n a_n x_n\right\|_{L^p(\Omega,\eta,X)} \le 2 \left\|\sum_{n=1}^N \varepsilon_n x_n\right\|_{L^p(\Omega,\eta,X)} \qquad \forall\, x_n \in X.$$

In case $a_n \in \mathbb{R}$, $n = 1, ...N$ *the constant* 2 *in the result is redundant.*

We may now proceed to the crucial definition of R-boundedness.

Definition 7.2.5 *Let* X *and* Y *be Banach spaces. We say that a set* $\tau \in \mathcal{L}(X,Y)$ *is* R-bounded *if for one (or all)* $1 \le p < \infty$ *there is a constant* c_p *such that*

$$\left\|\sum_{n=1}^N r_n T_n x_n\right\|_{L^p([0,1],Y)} \le c_p \left\|\sum_{n=1}^N r_n x_n\right\|_{L^p([0,1],X)} \tag{7.3}$$

for any $N \in \mathbb{N}$ *and arbitrary* $T_1, ..., T_N \in \tau$ *and* $x_1, ..., x_N \in X$.

The definition is independent of p due to Kahane's inequality.
Henceforth we restrict ourselves to the setting of L^p-spaces (in the role of X) and show the equivalence of R-boundedness with square function estimates. We assume that $X = L^q(\Omega, \eta)$ where (Ω, η) is a σ-additive measure space (not necessarily probabilistic) and $1 \le q < \infty$. We identify each $x_1, ..., x_m \in X$ with an almost everywhere defined function $x_j(\cdot)$ on Ω.

Theorem 7.2.6 *A set $\tau \in \mathcal{L}(L^q(\Omega, \eta))$ is R-bounded if and only if there exists a constant c such that for any $m \in \mathbb{N}$ and arbitrary $T_1, ..., T_m \in \tau$ and $x_1, ..., x_m \in L^q(\Omega, \eta)$ we have*

$$\left\| \left(\sum_{n=1}^{m} |T_n x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\Omega,\eta)} \le c \left\| \left(\sum_{n=1}^{m} |x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\Omega,\eta)} . \tag{7.4}$$

Proof. We write using Kahane's inequality, Fubini's theorem and Khinchine's inequality

$$\begin{aligned} &\left\| \sum_{n=1}^{m} r_n x_n \right\|_{L^2([0,1],X)} \sim \left\| \sum_{n=1}^{m} r_n x_n \right\|_{L^q([0,1],L^q(\Omega,\eta))} \\ &= \left(\int_\Omega \int_0^1 | \sum_{n=1}^{m} r_n(s) x_n(y)|^q \, ds \, d\eta(y) \right)^{\frac{1}{q}} \\ &\sim \left(\int_\Omega \left(\sum_{n=1}^{m} |x_n(y)|^2 \right)^{\frac{q}{2}} d\eta(y) \right)^{\frac{1}{q}} = \left\| \left(\sum_{n=1}^{m} |x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\Omega,\eta)} . \end{aligned}$$

Since we may repeat the argument with $T_n x_n$ in the role of x_n, the claim follows.□

Corollary 7.2.7 *Let $X = L^q(\Omega, \eta)$ with $1 < q < \infty$. Then $\tau \subset \mathcal{L}(X)$ is R-bounded if and only if $\tau' := \{T' : T \in \tau\} \subset \mathcal{L}(X')$ is R-bounded.*

Proof. We introduce the space $L^q(\Omega, \eta, l^2_m)$ of sequences $(x_n)_{n=1}^m$ of functions in $L^q(\Omega, \eta)$ with the norm

$$\|(x_n)\|_{L^q(\Omega,\eta,l^2_m)} := \left\| \left(\sum_{n=1}^{m} |x_n|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\Omega,\eta)} .$$

Then Theorem 7.2.6 shows that τ is R-bounded if and only if there exists a constant c such that for any $m \in \mathbb{N}$ and all $T_1, ..., T_m$ the operator $\widetilde{T}(x_n) := (T_n x_n)_{n=1}^m$ is bounded on $L^q(\Omega, \eta, l^2_m)$ with $\|\widetilde{T}\| \le c$.
We have $L^q(\Omega, \eta, l^2_m)' = L^{q'}(\Omega, \eta, l^2_m)$ with[3] respect to the duality

$$\langle (x_n), (y_n) \rangle = \sum_{n=1}^{m} \int_\Omega x_n(w) y_n(w) \, d\eta(w)$$

for $(x_n) \subset L^q(\Omega, \eta, l^2_m)$ and $(y_n) \subset L^{q'}(\Omega, \eta, l^2_m)$. For any operators $T_1, ..., T_m \in \tau$ we define $\widetilde{T}(x_n) := (T_n x_n)_{n=1}^m$ on $L^q(\Omega, \eta, l^2_m)$ and $\widetilde{S}(y_n) := (T'_n y_n)_{n=1}^m$ on $L^{q'}(\Omega, \eta, l^2_m)$. Then $\widetilde{T}$ and $\widetilde{S}$ are dual to each other and the corollary is proved since

$$\|\widetilde{T}\| = \|\widetilde{T}'\| = \|\widetilde{S}\|.$$

□

We shall also need the following result.

[3]Of course, here q' denotes the conjugate index to q defined by $\frac{1}{q} + \frac{1}{q'} = 1$.

Proposition 7.2.8 *Let X be a Banach space and let $0 < \theta' \leq 2\pi$. Let $\lambda \mapsto Z(\lambda) \in \mathcal{L}(X)$ be analytic on $\Sigma_{\theta'}$. Suppose that the set $\{Z(\lambda) : \ \lambda \in \partial\Sigma_\theta, \ \lambda \neq 0\}$ is R-bounded for some $\theta < \theta'$. Then also*

(i) $\{Z(\lambda) : \ \lambda \in \Sigma_\theta\}$ *is R-bounded.*

(ii) $\{\lambda \frac{d}{d\lambda} Z(\lambda) : \ \lambda \in \Sigma_{\theta_1}\}$ *is R-bounded for every $\theta_1 < \theta$.*

After describing one possible approach to proving R-boundedness, we now turn to the main reason for introducing the notion of R-boundedness. Namely, we state the (aforementioned) deep theorem characterising maximal regularity in terms of R-boundedness.

Theorem 7.2.9 *Let A be the generator of a bounded analytic semigroup in a UMD-space X. Then the following statements are equivalent.*

(i) *A has maximal L^p-regularity for one (all) $1 < p < \infty$.*

(ii) $\{\lambda R(\lambda, A) : \lambda \in \Sigma_\theta\}$ *is R-bounded for some $\theta \geq \frac{\pi}{2}$.*

(iii) $\{T(z) : z \in \Sigma_\delta\}$ *is R-bounded for some $\delta > 0$.*

(iv) $\{T(t), tAT(t) : t > 0\}$ *is R-bounded.*

If $T(t)$ satisfies the equivalent conditions of the Theorem 7.2.9 we say that $T(t)$ is *R-analytic*.

7.3 Maximal ergodic estimate for contractive semigroups

Our aim in this section is to prove a maximal ergodic estimate for contractive semigroups on L^p-spaces. This will be subsequently applied in the next section to obtain maximal regularity for generators of such semigroups. Thus we wish to show

Theorem 7.3.1 *Let (Ω, η) be a (positive) measure space. Let $T(t)$ be a contractive semigroup on all $L^p(\Omega, d\eta)$, $1 \leq p \leq \infty$, which is strongly continuous for $1 \leq p < \infty$. Fix $1 \leq p < \infty$. Let f_j, $j = 1, ..., m$ be a finite collection of functions in $L^p(\Omega, d\eta)$. Then for $N(t)$ defined by*

$$N(t)f := \frac{1}{t} \int_0^t T(s) f \, ds \qquad f \in L^p(\Omega, d\eta)$$

we have

$$\left\| \sup_j \sup_{t>0} |N(t) f_j| \right\|_{L^p(\Omega, d\eta)} \leq 2 \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \left\| \sup_j |f_j| \right\|_{L^p(\Omega, d\eta)}. \tag{7.5}$$

for any $1 < p < \infty$.

We start with the result showing that $N(t)$ is well-defined.

Lemma 7.3.2 *Fix $1 \leq p < \infty$. For any $f \in L^p(\Omega, d\eta)$ the integral*

$$\int_0^t T(s) f \, ds$$

is well-defined as an element of $L^p(\Omega, d\eta)$ and the operator $N(t)$ is a bounded operator on $L^p(\Omega, d\eta)$ for any $t > 0$. Moreover, for any fixed $f \in L^p(\Omega, d\eta)$ there exists a null-set $E(f)$ such that for all $x \notin E(f)$ we have the pointwise representation

$$(N(t)f)(x) = \frac{1}{t}\int_0^t (T(s)f)(x)\,ds$$

and for fixed $x \notin E(f)$ the function $(N(t)f)(x)$ is continuous in t.

Proof. See [28], Section VIII.7. □

First, we prove a discrete version of the maximal ergodic estimate. We set $T := T(1)$.

Theorem 7.3.3 *Under the assumptions of Theorem 7.3.1, define*

$$D_T(n)f := \frac{1}{n}\sum_{j=0}^{n-1} T^j f \qquad f \in L^p(\Omega, d\eta). \tag{7.6}$$

Then

$$\left\|\sup_{n\in\mathbb{N}} D_T(n)f\right\|_{L^p(\Omega,d\eta)} \le 2\left(\frac{p}{p-1}\right)^{\frac{1}{p}} \|f\|_{L^p(\Omega,d\eta)}.$$

The proof of the Theorem 7.3.3 can be given in a stroke provided we have the following weak estimate. We denote $f^* := \sup_{n\in\mathbb{N}} D_T(n)f$ (the function f^* is well-defined for a.a. $x \in \Omega$ as will be seen).

Proposition 7.3.4 *Under the assumptions of Theorem 7.3.1 we have for any $f \in L^p(\Omega, d\eta)$ and any $\alpha > 0$,*

$$\alpha\eta\left(\{x:\ f^*(x) > 2\alpha\}\right) \le \int_{\{x:\,|f(x)|>\alpha\}} |f(x)|\,d\eta(x).$$

After stating the major ingredients of the proof of Theorem 7.3.1 we start treating these results one by one. We need two auxiliary lemmas due to Hopf and we present these first. We assume throughout the whole section that (Ω, η) is a measure space.

Lemma 7.3.5 *Let S be a positive contraction operator in $L^\infty(\Omega, \eta)$. Let $m \in \mathbb{N}$ and let χ_i be a decreasing sequence of characteristic functions with $\chi_i = 0$ for $i > m$. Then there exist functions $h_i \in L^\infty(\Omega)$ $i \in \mathbb{N}_0$ satisfying*

(i) $0 \le h_i \le \chi_{i+1}h_i$

(ii) $0 \le \chi_{i+2}(h_{i+1} - h_i)$

(iii) $\chi_{i+1} = \chi_{i+1}(h_i + Sh_{i+1} + S^2h_{i+2} + \dots)$.

Proof. Set $h_i \equiv 0$ for $i \ge m$ and for $0 \le i \le m$ define h_i by downward induction as:

$$h_i := \chi_{i+1}(1 - Sh_{i+1} - S^2h_{i+2} - \dots). \tag{7.7}$$

Then, obviously *(iii)* is fulfilled. We show now

$$h_i + Sh_{i+1} + S^2h_{i+2} + \dots \le 1. \tag{7.8}$$

This is evident for $i \ge m$, while for $i < m$ one uses downward induction. Since $(1 - \chi_{i+1})h_i \equiv 0$ we have by the induction hypothesis

$$\begin{aligned}(1 - \chi_{i+1})(h_i + Sh_{i+1} + \dots) &= (1 - \chi_{i+1})S(h_{i+1} + Sh_{i+2} + \dots)\\ &\le (1 - \chi_{i+1})S1 \le 1 - \chi_{i+1}.\end{aligned}$$

The last inequality, combined with (iii), proves (7.8).
Combining (7.7) and (7.8) we see that

$$\chi_{i+1} h_i = h_i = \chi_{i+1}(1 - Sh_{i+1} - S^2 h_{i+2} - \dots) \geq 0$$

which proves (i). In order to prove (ii) we check first that

$$(h_i + Sh_{i+1} + \dots) - (h_{i+1} - Sh_{i+2} - \dots) \geq 0 \qquad i \in \mathbb{N}_0. \tag{7.9}$$

This is seen by downward induction as follows. Assume that (7.9) holds for $i > j$. Then by (iii) and (7.8)

$$\begin{aligned} &\chi_{j+1}(h_j + Sh_{j+1} + \dots - h_{j+1} - Sh_{j+2} - \dots) \\ &\qquad = \chi_{j+1} - \chi_{j+1}(h_{j+1} + Sh_{j+2} + \dots) \geq 0. \end{aligned} \tag{7.10}$$

Since $(1 - \chi_{j+1})h_j = (1 - \chi_{j+1})h_{j+1} = 0$, we obtain using the induction hypothesis

$$\begin{aligned} &(1 - \chi_{j+1})(h_j + Sh_{j+1} + \dots - h_{j+1} - Sh_{j+2} - \dots) \\ &\qquad = (1 - \chi_{j+1}) S(h_{j+1} + Sh_{j+2} + \dots - h_{j+2} - Sh_{j+3} - \dots) \geq 0. \end{aligned} \tag{7.11}$$

Putting together (7.10) and (7.11) we have verified (7.9).
Since $\chi_{j+2}\chi_{j+1} = \chi_{j+2}\chi_j = \chi_{j+2}$ we have from (7.7) and (7.9),

$$\chi_{i+2}(h_{i+1} - h_i) = \chi_{i+2} S(h_{i+1} + Sh_{i+2} + \dots - h_{i+2} - Sh_{i+3} - \dots) \geq 0.$$

This shows (ii). □

Lemma 7.3.6 *Let P be a positive operator in $L^1(\Omega, d\eta)$. For any real $f \in L^1(\Omega, d\eta)$ and $m \in \mathbb{N}$ one has*

$$\int_G f(x)\, d\eta(x) \geq 0$$

where

$$G := \left\{ x : \sup_{1 \leq n \leq m} (D_P(n) f)(x) \geq 0 \right\}.$$

Here (as in (7.6)) we use the notation

$$D(n)f := D_P(n)f := \frac{1}{n} \sum_{j=0}^{n-1} P^j f \qquad f \in L^p(\Omega, d\eta). \tag{7.12}$$

Proof. Define $H_i := \emptyset$ if $i > m$ and for $1 \leq i \leq m$ by

$$H_i := \{ x : (D(i)f)(x) \geq 0 \text{ and } (D(j)f)(x) < 0 \text{ for all } j < i \}.$$

Let $G_i := \bigcup_{j \geq i} H_j$ so that $G_1 = G$. Then

$$\chi_{H_i} \sum_{j=0}^{i-1} P^j f \geq 0 \quad \text{ and } \quad \chi_{H_i} \sum_{j=0}^{k-1} P^j f \leq 0 \quad \text{for } k < i,$$

hence

$$\chi_{H_i} \sum_{j=k}^{i-1} P^j f \geq 0 \quad \text{for } k < i.$$

We obtain adding last inequalities together for a fixed $k \geq 0$,

$$\sum_{i > k} \sum_{j=k}^{i-1} \chi_{H_i} P^j f = \sum_{j \geq k} \chi_{G_{j+1}} P^j f \geq 0. \tag{7.13}$$

We now apply Lemma 7.3.5 to the adjoint operator $P^* \in \mathcal{L}(L^\infty(\Omega, d\eta))$. We obtain functions $h_i \in L^\infty(\Omega, d\eta)$, $i \in \mathbb{N}_0$ satisfying (i), (ii) of Lemma 7.3.5 and also (iii') $\chi_G = \chi_{G_1} = \chi_{G_1}(h_0 + P^* h_1 + P^{*2} h_2 + \dots)$.

We have from (7.13) and (ii) (we set $h_{-1} \equiv 0$),

$$\begin{aligned} 0 &\le \sum_{k\ge 0} \chi_{G_{k+1}}(h_k - h_{k-1}) \sum_{j\ge k} \chi_{G_{j+1}} P^j f = \sum_{k\ge 0}\sum_{j\ge k} \chi_{G_{j+1}}(h_k - h_{k-1}) P^j f \\ &= \sum_{j\ge 0}\sum_{j\ge k} (h_k - h_{k-1}) \chi_{G_{j+1}} P^j f = \sum_{j\ge 0} h_j \chi_{G_{j+1}} P^j f = \sum_{j\ge 0} h_j P^j f. \end{aligned}$$

Therefore

$$0 \le \int_\Omega (\sum_{j\ge 0} h_j P^j f)(x)\, d\eta(x) = \int_\Omega \sum_{j\ge 0} \left(P^{*j} h_j\right)(x) f(x)\, d\eta(x).$$

Since $P^{*j} h_i \ge 0$ and f is negative on $\Omega\backslash G \subset \Omega\backslash H_1$ we obtain

$$0 \le \int_G \sum_{j\ge 0} (P^{*j} h_j)(x) f(x)\, d\eta(x).$$

However, we see from (iii') that $\sum_{j\ge 0} P^* h_j(x) = 1$ for $x \in G$. Hence

$$0 \le \int_G f(x)\, d\eta(x).$$

□

Armed with the last result we may now proceed to the proof of Proposition 7.3.4 and Theorem 7.3.3.

Proof of Proposition 7.3.4. In a first step, we assume that $\eta(\Omega) < \infty$. Since

$$\Big\{x : \sup_{n\in\mathbb{N}} |\left(D_T(n) f\right)(x)| > \alpha\Big\} \subset \Big\{x : \sup_{n\in\mathbb{N}} \left(D_P(n)|f|\right)(x) > \alpha\Big\}$$

where $P := P(1)$ and $P(t)$ is the modulus semigroup of $T(t)$, we may also assume that T is positive and $f \ge 0$.
Define

$$g(x) := \begin{cases} f(x) & \text{if } f(x) > \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f - 2\alpha \le g - \alpha$ and

$$D(n) f - 2\alpha \le D(n)(f - 2\alpha) \le D(n)(g - \alpha).$$

Hence

$$\begin{aligned} \Big\{x : \sup_n \left(D(n) f\right)(x) > 2\alpha\Big\} &\subset \Big\{x : \sup_n D(n)(f - 2\alpha)(x) \ge 0\Big\} \\ &\subset \Big\{x : \sup_n D(n)(g - \alpha)(x) \ge 0\Big\}. \end{aligned}$$

By Lemma 7.3.6 we have

$$\begin{aligned} \alpha\eta\Big(x : \sup_n \left(D(n) f\right)(x) > 2\alpha\Big) &\le \alpha\eta\Big(x : \sup_n D(n)(g - \alpha)(x) \ge 0\Big) \\ &\le \int_{\{x:\sup_n D(n)(g-\alpha)(x)\ge 0\}} g(x)\, d\eta(x). \end{aligned}$$

However, setting

$$H := \{x : \sup_n D(n)(g-\alpha)(x) \geq 0\}$$

we see that g vanishes on the complement of the set H. In fact, if $x \in \Omega$ and $g(x) \neq 0$ then

$$0 < g(x) - \alpha = D(1)(g-\alpha)(x)$$

and thus $x \in H$.
Therefore

$$\alpha\eta\Big(x : \sup_n \left(D(n)f\right)(x) > 2\alpha\Big) \leq \int_\Omega g(x)\, d\eta(x) = \int_{\{x:\, f(x)>\alpha\}} f(x)\, d\eta(x).$$

This gives the assertion under the assumption of the finiteness of η.

Let (Ω, η) be an arbitrary $\sigma-$finite measure space and $\Omega_i \nearrow \Omega$ with $\eta(\Omega_i) < \infty$ for each $i \in \mathbb{N}$. This means that Ω_i is an increasing sequence with the union the whole Ω. Denote by χ_i the characteristic function of the set Ω_i and define

$$T_i := M_{\chi_i} T M_{\chi_i}$$

where M_g is the multiplication operator with the function g. From the first part of the proof we have for any fixed $m \in \mathbb{N}$,

$$\alpha\eta\Big(x : \sup_{1\leq n\leq m} \left(D_{T_i}(n)\right)(x) > \alpha\Big) \leq \int_{\Omega_i \cap \{x:\, |f(x)|>\alpha\}} f(x)\, d\eta(x). \tag{7.14}$$

Since $T_i \to T$ strongly in $L^p(\Omega, \eta)$ we have

$$\Big\| \sup_{1\leq n\leq m} D_{T_i}(n)f - \sup_{1\leq n\leq m} D_T(n)f \Big\|_{L^p(\Omega,\eta)} \to 0.$$

Also, since T_i increases with i and $f \geq 0$, the sequence

$$\sup_{1\leq n\leq m} D_{T_i}(n)(x)$$

is increasing with i for a.a. $x \in \Omega$. Hence

$$\Big\{x : \sup_{1\leq n\leq m} \left(D_{T_i}(n)f\right)(x) > \alpha\Big\} \nearrow \Big\{x : \sup_{1\leq n\leq m} \left(D_T(n)f\right)(x) > \alpha\Big\}.$$

Taking the limit in (7.14) we obtain

$$\alpha\eta\Big(x : \sup_{1\leq n\leq m} \left(D_T(n)f\right)(x) > 2\alpha\Big) \leq \int_{\{x:\, |f(x)|>\alpha\}} f(x)\, d\eta(x).$$

The conclusion follows on letting $m \to \infty$. $\square$

Proof of Theorem 7.3.3. We have

$$\begin{aligned}
\int_\Omega |f^*(x)|^p\, d\eta(x) &= p \int_\Omega \int_0^{|f^*(x)|} \alpha^{p-1}\, d\alpha\, d\eta(x) \\
&= p \int_0^\infty \alpha^{p-1} \eta\{x : |f^*(x)| > \alpha\}\, d\alpha \\
&\leq 2p \int_0^\infty \alpha^{p-2} \int_{\{x:\, |f(x)|>\alpha/2\}} |f(x)|\, d\eta(x)\, d\alpha \\
&= 2p \int_\Omega |f(x)| \int_0^{2|f(x)|} \alpha^{p-2}\, d\alpha\, d\eta(x) \\
&= 2^p \frac{p}{p-1} \int_\Omega |f(x)|^p\, d\eta(x)
\end{aligned}$$

where we used Fubini's theorem and Proposition 7.3.4. □

Finally we are ready to prove the continuous version of the maximal ergodic estimate.

Proof of Theorem 7.3.1. Since $T(t)$ is strongly continous and contractive, we have for each non-negative rational α and each $1 \le j \le m$,

$$N(\alpha)f_j = \lim_{n\to\infty} \frac{1}{\alpha n!} \sum_{i=0}^{\alpha n!-1} T\left(\frac{i}{n!}\right) f_j$$

in $L^p(\Omega,\eta)$. Since every convergent sequence in $L^p(\Omega,\eta)$ has a subsequence which converges pointwise a.e., using Theorem A.2.1 and the Cantor diagonalisation procedure we have (denoting the subsequence with n again)

$$\left(N(\alpha)f_j\right)(x) = \lim_{n\to\infty} \frac{1}{\alpha n!} \sum_{i=0}^{\alpha n!-1} T\left(\frac{i}{n!}\right) f_j(x), \qquad 1 \le j \le m,$$

for each rational $\alpha \ge 0$ and $x \in \Omega \backslash E$, where

$$E \supset \bigcup_{j=1}^{m} E(f_j)$$

and $E(f_j)$ are the null-sets given in Lemma 7.3.2. Define

$$f^*_{j,n}(x) := \sup_{k\in\mathbb{N}} \left|\frac{1}{k}\sum_{i=0}^{k-1}\left(T\left(\frac{i}{n!}\right) f_j\right)(x)\right| \qquad x \in E^{\mathrm{c}},\ 1 \le j \le m.$$

Then for $1 \le j \le m\, \alpha \ge 0$ rational, $x \in E^{\mathrm{c}}$ and $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$f^*_{j,n}(x) \ge \left|\left(N(\alpha)f_j\right)(x)\right| - \varepsilon$$

for all $n \ge n_0$. Hence

$$\liminf_{n\to\infty} f^*_{j,n}(x) \ge \left|\left(N(\alpha)f_j\right)(x)\right| \qquad \text{for } \alpha \ge 0 \text{ rational},$$

but since $\left(N(t)f_j\right)(x)$ is continuous in t for $x \in E^{\mathrm{c}}$ and $1 \le j \le m$ (by Lemma 7.3.2) we have

$$\liminf_{n\to\infty} f^*_{j,n}(x) \ge \sup_{t\in\mathbb{R}_+} \left|\left(N(t)f_j\right)(x)\right| \qquad x \in E^{\mathrm{c}},\ 1 \le j \le m.$$

Let $P(t)$ be the modulus semigroup associated to $T(t)$. Then

$$\left|\frac{1}{k}\sum_{i=0}^{k-1} T\Big(\frac{i}{n!}\Big) f_j\right| \le \frac{1}{k}\sum_{i=0}^{k-1} P\Big(\frac{i}{n!}\Big)|f_j| \le \frac{1}{k}\sum_{i=0}^{k-1} P\Big(\frac{i}{n!}\Big) \sup_j |f_j|$$

for any $k \in \mathbb{N}$ and $1 \le j \le m$. Therefore

$$\begin{aligned}\liminf_{n\to\infty} \sup_{k\in\mathbb{N}} \frac{1}{k}\sum_{i=0}^{k-1} P\Big(\frac{i}{n!}\Big) \sup_j |f_j| &\ge \liminf_{n\to\infty} \sup_{k\in\mathbb{N}} \left|\frac{1}{k}\sum_{i=0}^{k-1} T\Big(\frac{i}{n!}\Big) f_j\right| \\ &= \liminf_{n\to\infty} f^*_{j,n} \ge \sup_{t>0} |N(t)f_j|.\end{aligned}$$

Denoting $f := \sup_{1\le j\le m} |f_j|$ and $P_n(t) := P(\frac{t}{n!})$ we have thus proved

$$\liminf_{n\to\infty} \sup_{k\in\mathbb{N}} D_{P_n}(k)f \ge \sup_{1\le j\le m} \sup_{t>0} |N(t)f_j|.$$

Fatou's lemma and the discrete weak estimate in Theorem 7.3.3 yield now

$$\int_\Omega \Big(\sup_{j=1,\dots,m} \sup_{t>0} |N(t)f_j|(x)\Big)^p \, d\eta(x) \le \int_\Omega \Big(\liminf_{n\to\infty} \sup_{k\in\mathbb{N}} D_{P_n}(k)f(x)\Big)^p \, d\eta(x)$$
$$\le \liminf_{n\to\infty} \int_\Omega \Big(\sup_{k\in\mathbb{N}} D_{P_n}(k)f\Big)^p (x)\, d\eta(x) \le \frac{2^p p}{p-1} \int_\Omega f^p(x)\, d\eta(x).$$

□

7.4 Maximal regularity for contractive semigroups on L^p-spaces

Throughout this section we assume that (Ω,η) is a σ-finite measure space and $T(t)$ is a family of consistent semigroups on $L^p(\Omega,\eta)$ with $\|T(t)\|_{L^p(\Omega,\eta)} \le 1,\ 1\le p\le\infty$. We also assume that $T(t)$ is strongly continuous for $1\le p<\infty$. The aim of this section is to prove the following theorem.

Theorem 7.4.1 *Assume that $T(t)$ is analytic on $L^q(\Omega,\eta)$ for some $1<q<\infty$. Then $T(t)$ is R-analytic on $L^q(\Omega,\eta)$.*

We start with an interpolation result which we will need on the way. We settle down the necessary notation first. We denote

$$\Sigma_{\sigma_0}^{\sigma_1} := \{\lambda\in\mathbb{C}:\ \sigma_0 < \arg\lambda < \sigma_1\} \qquad \text{and}$$

$$l_p^m := \{x=(x_1,\dots,x_m)\in\mathbb{C}^m\} \text{ with the norm } \|x\|_{l_p^m} := \Big(\sum_{i=1}^m |x_i|^p\Big)^{\frac{1}{p}}.$$

Lemma 7.4.2 *Let $1\le p_0,p_1,q_0,q_1\le\infty$ and $m\in\mathbb{N}$. Let $Z(\lambda)$ be a family of operators*

$$Z(\lambda):\ L^{q_0}(\Omega,\eta,l_{p_0}^m)\cap L^{q_1}(\Omega,\eta,l_{p_1}^m) \to L^{q_0}(\Omega,\eta,l_{p_0}^m) + L^{q_1}(\Omega,\eta,l_{p_1}^m)$$

such that for all $h\in L^{q_0}(\Omega,\eta,l_{p_0}^m)\cap L^{q_1}(\Omega,\eta,l_{p_1}^m)$ the function

$$\lambda\in\Big(\overline{\Sigma_{\sigma_0}^{\sigma_1}}\Big) \to Z(\lambda)h \in L^{q_0}(\Omega,\eta,l_{p_0}^m) + L^{q_1}(\Omega,\eta,l_{p_1}^m)$$

is continuous and analytic in $\Sigma_{\sigma_0}^{\sigma_1}$. Suppose also that there are constants C_1, C_2 such that

$$\|Z(\lambda)\|_{\mathcal{L}(L^{q_j}(\Omega,\eta,l_{p_j}^m)} \le C_j \qquad \arg\lambda=\sigma_j,\ j=0,1.$$

Fix $\theta\in(0,1)$ and set

$$\frac{1}{q} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1},\ \frac{1}{p} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1},\ \sigma = (1-\theta)\sigma_0 + \theta\sigma_1.$$

Then

$$\|Z(\lambda)\|_{\mathcal{L}(L^q(\Omega,\eta,l_p^m))} \le C_0^{1-\theta}C_1^\theta \qquad \arg\lambda=\sigma.$$

Proof. Consider the strip $S := \{\lambda : 0 < Re\, \lambda < 1\}$ and define

$$Y(\lambda) = Z(e^{i(\sigma_0+(\sigma_1-\sigma_0)\lambda)}).$$

Then

$$\|Y(\lambda)\|_{\mathcal{L}(L^{q_j}(\Omega,\eta,l^m_{p_j})} \le C_j \text{ for } Re\, \lambda = j,\, j = 0, 1.$$

We have by complex interpolation

$$L^q(\Omega, \eta, l^m_p) = \left[L^{q_0}(\Omega, \eta, l^m_{p_0}), L^{q_1}(\Omega, \eta, l^m_{p_1})\right]_\theta$$

and hence for $h \in L^q(\Omega, \eta, l^m_p)$, $\|h\|_{L^q} \le 1$ there exists a function

$$G : \overline{S} \to L^{q_0}(\Omega, \eta, l^m_{p_0}) + L^{q_1}(\Omega, \eta, l^m_{p_1})$$

continuous in $\overline{S}$. analytic in S, such that $G(\theta) = h$ and

$$\|G(\lambda)\|_{(L^{q_j}(\Omega,\eta,l^m_{p_j})} \le 1 \qquad Re\, \lambda = j;\ j = 0, 1.$$

Fix now $\mu := re^{i\sigma}$ and define

$$W(\lambda) := Y\left(\lambda - i\frac{\log r}{\sigma_1 - \sigma_0}\right) G(\lambda).$$

Then we have

$$\|W(\lambda)\|_{(L^{q_j}(\Omega,\eta,l^m_{p_j})} \le C_j \qquad Re\, \lambda = j,\ j = 0, 1$$

and by the three-lines theorem we have

$$\|Z(\mu)h\|_{(L^{q_j}(\Omega,\eta,l^m_{p_j})} \le C_0^{1-\theta} C_1^{\theta}.$$

The last inequality holds for any $h \in L^q(\Omega, \eta, l^m_p)$, $\|h\|_{L^q} \le 1$ and any μ with $\arg \mu = \sigma$. □

Now we prove a crucial lemma connecting maximal estimates to R-boundedness.

Lemma 7.4.3 *Let $1 \le q < 2$ and $0 < \delta \le 2\pi$. Let $N(\lambda)$ be an analytic $\mathcal{L}(L^q(\Omega, \eta))$-valued function defined on $\overline{\Sigma}_\delta$, which is bounded and strongly continuous on $\overline{\Sigma}_\delta$ with*

$$\|N(\lambda)\|_{\mathcal{L}(L^q(\Omega,\eta))} \le C_q.$$

Suppose that the following maximal estimate holds true for $N(t)$, $t > 0$: there exists $C_\infty \ge 0$ such that for any $m \in \mathbb{N}$ and any $t_1, ..., t_m \in \mathbb{R}_+$ and $f_1, ..., f_m \in L^q(\Omega, \eta)$,

$$\left\| \sup_{j=1,\dots,m} |N(t_j) f_j| \right\|_{L^q(\Omega,\eta)} \le C_\infty \left\| \sup_{j=1,\dots,m} |f_j| \right\|_{L^q(\Omega,\eta)}. \tag{7.15}$$

Then $\{N(\lambda),\ \lambda \in \Sigma_{\frac{q}{2}\delta}\}$ is R-bounded and its R-bound does not exceed $C_q^{\frac{q}{2}} C_\infty^{1-\frac{q}{2}}$.

Proof. We show a square function estimate, namely that for all $m \in \mathbb{N}$ and arbitrary $\lambda_1, ..., \lambda_m$ with $|\arg \lambda_j| = \frac{q}{2}\delta$ and any $g_1, ..., g_m \in L^q(\Omega, \eta)$,

$$\left\| \left(\sum_{n=1}^m |N(\lambda_j) g_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\Omega,\eta)} \le C_q^{\frac{q}{2}} C_\infty^{1-\frac{q}{2}} \left\| \left(\sum_{n=1}^m |g_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q(\Omega,\eta)}. \tag{7.16}$$

The claim then follows from Proposition 7.2.8 and Theorem 7.2.6.
By the assumption and the Fubini theorem we have for all $m \in \mathbb{N}$, arbitrary $\lambda_1, ..., \lambda_m$ with $|\arg \lambda_j| = \frac{q}{2}\delta$ and any $g_1, ..., g_m \in L^q(\Omega, \eta)$,

$$\left\| \left(\sum_{n=1}^{m} |N(\lambda_j)g_j|^q \right)^{\frac{1}{q}} \right\|_{L^q(\Omega,eta)}^q = \sum_{n=1}^{m} \|N(\lambda_j)g_j\|_{L^q(\Omega,\eta)}^q \tag{7.17}$$
$$\leq C_q^q \sum_{n=1}^{m} \|g_j\|_{L^q(\Omega,\eta)}^q = C_q^q \left\| \left(\sum_{n=1}^{m} |g_j|^q \right)^{\frac{1}{q}} \right\|_{L^q(\Omega,\eta)}^q .$$

We fix $m \in \mathbb{N},\ t_1, ..., t_m \in \mathbb{R}_+$ and set

$$Z(\lambda)(g_1, ..., g_m) := (N(t_1\lambda)g_1, ..., N(t_m\lambda)g_m)) .$$

Then $Z(\lambda) \in \mathcal{L}(L^q(\Omega, \eta, l_q^m))$ for $\lambda \in \overline{\Sigma_\delta}$ and $Z(\lambda)$ is strongly continuous in $\overline{\Sigma_\delta}$ and analytic in Σ_δ. The estimates (7.15) and (7.17) yield:

$$\|Z(\lambda)\|_{\mathcal{L}(L^q(\Omega,\eta,l_q^m))} \leq C_q \qquad \text{for } |\arg \lambda| = \delta$$

and

$$\|Z(\lambda)\|_{\mathcal{L}(L^q(\Omega,\eta,l_\infty^m))} \leq C_\infty \qquad \text{for } \lambda \in \mathbb{R}_+.$$

Lemma 7.4.2 implies

$$\|Z(\lambda)\|_{\mathcal{L}(L^q(\Omega,\eta,l_2^m))} \leq C_q^{\frac{q}{2}} C_\infty^{1-\frac{q}{2}} \qquad \text{for } |\arg \lambda| = \frac{q}{2}\delta.$$

Since the last inequality does not depend on $m \in \mathbb{N}$ nor t_j, the bound (7.16) follows. □

We are ready to prove the main result.

Proof of Theorem 7.4.1. There is nothing to prove if $q = 2$, since on a Hilbert space a collection of uniformly bounded operators is R-bounded. Assume now $1 < q < 2$ and that $T(t)$ extends analytically to a bounded semigroup $T(\lambda)$, $\lambda \in \Sigma_{\delta_1}$ for some $\delta_1 > 0$. We set

$$N(\lambda) := \frac{1}{\lambda} \int_0^\lambda T(z)\, dz \qquad \lambda \in \Sigma_{\delta_1}.$$

Fix $0 < \delta < \delta_1$. The ergodic estimate (7.5) implies the estimate (7.15) in the assumptions of Lemma 7.4.3. Therefore there exists $\delta_2 > 0$ such that

$$\{N(\lambda),\, \lambda \in \Sigma_{\delta_2}\}$$

is R-bounded. Lemma 7.2.8 implies that for some δ_3 $0 < \delta_3 < \delta_2$ the set

$$\{\lambda \frac{d}{d\lambda} N(\lambda),\, \lambda \in \Sigma_{\delta_3}\}$$

is R-bounded. The conclusion of the Theorem (for $1 < q < 2$) follows now from the equality

$$T(\lambda) = N(\lambda) + \lambda \frac{d}{d\lambda} N(\lambda).$$

In case $2 < q < \infty$, the adjoint semigroup $T^*(\lambda)$ is R-bounded on $L^{q'}(\Omega, \eta)$ where $\frac{1}{q'} + \frac{1}{q} = 1$. The result follows on applying Corollary 7.2. □

As a corollary we state the corresponding result concerning $m\triangle$. Note that the operator enjoys maximal regularity despite the fact that m is in general only a measurable function without any regularity properties. Also no assumption on the boundary of Ω is needed.

Theorem 7.4.4 *Let Ω be a bounded domain in $\mathbb{R}^N$ and $m : \Omega \to (0, \infty)$ be a measurable function such that $\frac{1}{m} \in L^1_{\text{loc}}(\Omega)$. Then the operator $m\triangle$ with the Dirichlet boundary conditions has maximal regularity on $L^q(\Omega, \frac{1}{m(x)}dx)$, $1 < q < \infty$.*

Proof. This follows from Theorem 7.2.9 and Theorem 7.4.1. □

7.5 Notes and comments

Section 7.2
This material is already standard and may be found in various works, see e.g. [38], [42]. The breakthrough result 7.2.9 is due to L. Weis.
Section 7.3
The proof of the maximal ergodic estimate for contractive semigroups (Theorem 7.3.1) follows [28].
Section 7.4
Theorem 7.4.1 is taken from [42]. Our Theorem 7.4.4 follows then easily.

Appendix A

Appendix

A.1 Sesquilinear forms

Lemma A.1.1 *Let a be a densely defined, accretive, continuous and closed form on a Hilbert space H. Suppose that $u_n \in \mathcal{D}(a)$ converges to u in H and*

$$\|u_n\|_a \leq M$$

for some constant M. Then $u \in \mathcal{D}(a)$ and

$$Re\ a(u,u) \leq \liminf_{n\to\infty} Re\ a(u_n, u_n).$$

Proof. See [52], Lemma 1.36. □

Lemma A.1.2 *Let a be a densely defined, accretive, continuous and closed form on a Hilbert space H and let $u \in H$. Then $u \in \mathcal{D}(a)$ if and only if*

$$\sup_{t>0} \frac{Re\ \langle u - e^{At}u, u\rangle}{t} < \infty.$$

For any $u, v \in \mathcal{D}(a)$ one has

$$a(u,v) = \lim_{t\to 0} \frac{\langle u - e^{At}u, v\rangle}{t}$$

Proof. See [52], Lemma 1.52. □

Lemma A.1.3 *Let a be a sesquilinear form on a Hilbert space H with $\mathcal{D}(a) = H$. Assume that a is continuous and coercive i.e. there exist constants $c_1, c_2 > 0$ s.t.*

$$|a(u,v)| \leq c_1 \|u\|_H \|v\|_H \qquad \forall\, u, v \in H$$

and

$$Re\ a(u,u) \geq c_2 \|u\|_H^2 \qquad \forall\, u \in H.$$

Let ϕ be a continuous linear functional on H. Then there exists a unique $v \in H$ s.t.

$$\phi(u) = a(u,v) \qquad \forall\, u \in H.$$

Proof. See [52], Lemma 1.3. □

A.2 Measure theory

For an ease of reference we state here the following well-known result which is used in the proof of the continuous maximal ergodic estimate (Theorem 7.3.1).

Theorem A.2.1 *Let X be a (real or complex) Banach space. Let $(\Omega_1, \Sigma_1, \eta_1)$ and $(\Omega_2, \Sigma_2, \eta_2)$ be $\sigma-$finite measure spaces and denote by (Ω, Σ, η) their product. Fix $1 \leq p \leq \infty$ and let*

$$F : \Omega_1 \to L^p(\Omega_2, \eta_2, X)$$

be a η_1-measurable function. Then there exists a η-measurable function

$$f : \Omega \to X$$

such that $f(x, \cdot) = F(x)$ for η_1-a.a. $x \in \Omega_1$, $f(\cdot, y)$ is η_1-integrable for η_2-a.a. $y \in \Omega_2$ and

$$\int_{\Omega_1} f(x, y)\, d\eta_1(x) = \left(\int_{\Omega_1} F(x)\, d\eta_1(x) \right)(y)$$

for η_2-a.a. $y \in \Omega_2$.

Proof. See Theorem III.11.17 in [28]. □

A.3 Sobolev space $H^1(\Omega)$

Here we prove the following continuity result which is used in the proof of Lemma 2.1.7. We use the notation $u_n \rightharpoonup u$ to denote the weak convergence in $H^1(\Omega)$. The functions u_n and u are assumed to be real-valued.

Proposition A.3.1 *Let Ω be an open set in $\mathbb{R}^N$. The mapping $u \mapsto |u|$ is continuous on $H^1(\Omega)$.*

Proof. Let $u_n \in H^1(\Omega)$ be a sequence converging to u in $H^1(\Omega)$. We know from the first part of the proof of Lemma 2.1.7 that $|u_n|$, $|u| \in H^1(\Omega)$ for all $n \in \mathbb{N}$. We will show that $|u_n| \to |u|$ in $H^1(\Omega)$. It suffices to show that there exists a subsequence u_{n_k} such that $|u_{n_k}| \to |u|$ (for then each subsequence has a subsequence which converges to $|u|$). By reflexivity of $H^1(\Omega)$ we may assume that $|u_n| \rightharpoonup v$ in $H^1(\Omega)$ for some $v \in H^1(\Omega)$. However, since $|u_n| \to |u|$ in $L^2(\Omega)$ we have $v = |u|$. Using (2.6) we also have

$$\limsup_{n\to\infty} \| |u_n| \|_{H^1(\Omega)} = \limsup_{n\to\infty} \|u_n\|_{H^1(\Omega)} = \|u\|_{H^1(\Omega)} = \| |u| \|_{H^1(\Omega)} .$$

The proof is finished by applying the well-known result below. □

Lemma A.3.2 *Let x_n, $n \in \mathbb{N}$ and x be elements of a Hilbert space H such that $x_n \rightharpoonup x$ and $\limsup_{n\to\infty} \|x_n\| \leq \|x\|$. Then $x_n \to x$.*

Proof. We have

$$\begin{aligned} 0 \leq \limsup_{n\to\infty} \|x - x_n\|^2 &= \limsup_{n\to\infty} (\langle x - x_n, x\rangle - \langle x, x_n\rangle + \langle x_n, x_n\rangle) \\ &= - \|x\|^2 + \limsup_{n\to\infty} \|x_n\|^2 \leq 0. \end{aligned}$$

□

A.4 Laplace operator

In the next result, the Laplace operator is considered in the distributional sense.

Theorem A.4.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $\frac{\hat{N}}{2} < p \leq \infty$. Then*

$$u \in L^1_{loc}(\Omega),\ \triangle u \in L^p_{loc}(\Omega) \quad \textit{implies } u \in C(\Omega). \tag{A.1}$$

Proof. See ([21], II.3 Proposition 6). □

By $\triangle_p$ we will understand the Dirichlet Laplacian on $L^p(\Omega, dx)$. This operator is constructed by the theory of Section 2.1 with $m \equiv 1$. Similarly, we denote by $\triangle_0$ the Dirichlet Laplacian on $C_0(\Omega)$ as constructed in Section 3.2. We list some standard properties of $\triangle_p$ (resp. $\triangle_0$) that we need (and do not prove) in the main body of the work.

Proposition A.4.2 *The operator $\triangle_p$ is invertible. Moreover, for $\frac{\hat{N}}{2} < p \leq \infty$ the following holds:*

(a) $\mathcal{D}(\triangle_p) = \{u \in H^1_0(\Omega) : \triangle u \in L^p(\Omega)\}$ *and* $\triangle_p u = \triangle u$ *in* $\mathcal{D}(\Omega)'$ *for all* $u \in \mathcal{D}(\triangle_p)$

(b) $\mathcal{D}(\triangle_p) \subset C^b(\Omega) := \{u : \Omega \to \mathbb{R} : u \textit{ is bounded and continuous } \}$

Proof. The invertibility follows from ([23], Theorem 1.6.3), for example. Note that for $\frac{\hat{N}}{2} < p \leq \infty$

$$\|T_p(t)\|_{\mathcal{L}(L^p(\Omega), L^\infty(\Omega))} \leq c t^{-\frac{N}{2p}} e^{-\omega t} \quad (t \geq 0)$$

for some $c > 0$, $\omega > 0$ ([52] Lemma 6.5). Thus

$$R(0, \triangle_p) = \int_0^\infty T_p(t)\, dt \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega)).$$

Let $f \in L^p(\Omega)$, $u = R(0, \triangle_p)f$. Then $u \in L^\infty(\Omega)$. Moreover, $-\triangle u = f$ in $\mathcal{D}(\Omega)'$. In fact, let $f_k \to f$ in $L^p(\Omega)$ where $f_k \in L^2(\Omega) \cap L^p(\Omega)$. Then $u_k := R(0, \triangle_p)f_k \to u$ in $L^\infty(\Omega)$. Moreover, since $R(0, \triangle_p)f_k = R(0, \triangle_2)f_k$, one has $u_k \in H^1_0(\Omega)$ and $-\triangle u_k = f_k$ in $\mathcal{D}(\Omega)'$. Since $u_k \to u$ in $L^\infty(\Omega) \hookrightarrow \mathcal{D}(\Omega)'$, it follows that $\triangle u_k \to \triangle u$ in $\mathcal{D}(\Omega)'$. Thus $-\triangle u = f$. It follows from (3.3) that $u \in C(\Omega)$. Finally, by the definition of $\triangle_2$, one has

$$\int_\Omega |\nabla u_k(x)|^2\, dx = \int_\Omega f_k(x) u_k(x)\, dx \leq \|f_k\|_{L^p(\Omega)} \|u_k\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{p'}}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Thus $(u_k)_{k\in\mathbb{N}}$ is bounded in $H^1_0(\Omega)$. Taking a subsequence, we may assume that $u_k \rightharpoonup w \in H^1_0(\Omega)$. Since $u_k \to u \in L^\infty(\Omega)$, it follows that $u = w \in H^1_0(\Omega)$. Thus (b) and one inclusion in (a) are proved.
Let $u \in H^1_0(\Omega)$ such that $f := \triangle u \in L^p(\Omega)$. It remains to show that $u \in \mathcal{D}(\triangle_p)$ and $\triangle_p u = \triangle u$. Let $w = R(0, \triangle_p)f$. Then $w \in H^1_0(\Omega)$ and $-\triangle w = f$ by what has been proved above. Thus $u + w \in H^1_0(\Omega)$ and $\triangle(u + w) = 0$. By the maximum principle (Lemma 3.1.1) this implies $u + w = 0$. □

Theorem A.4.3 *Let Ω be a bounded and open set in $\mathbb{R}^N$. Then the following statements are equivalent:*

(i) Ω is Dirichlet regular.

(ii) $\rho(\triangle_0) \neq \emptyset$.

(iii) $\mathcal{D}(\triangle_p) \subset C_0(\Omega)$ *for all* $p \in (\frac{\pi}{2}, \infty]$.

(iv) *There exist* $p \in (\frac{\pi}{2}, \infty]$, $\lambda \in \rho(\triangle_p)$ *and* $f \in L^p(\Omega, dx)$ *such that* $f(x) > 0$ *a.e. and* $R(\lambda, \triangle_p)f \in C_0(\Omega)$.

Proof. See [3], Theorem 2.4. □

Theorem A.4.4 *Consider* $\Omega := (-1,1)^N \subset \mathbb{R}^N$. *The trace of the semigroup generated by the Dirichlet Laplace operator on* Ω *is given by*

$$\operatorname{Tr} e^{\triangle t} = ct^{-\frac{N}{2}}.$$

Proof. The eigenfunctions of the operator are given by

$$\varphi_n(x) = \prod_{i=1}^{N} \psi_{n_i}(x)$$

where

$$\psi_k(x) = \cos(k\frac{\pi}{2}x), \qquad k = 1, 3, 5, \ldots$$
$$\psi_k(x) = \sin(k\frac{\pi}{2}x), \qquad k = 2, 4, 6, \ldots.$$

and $n = (n_1, \ldots, n_N)$ runs through all elements of $\mathbb{N}^N$. Hence the eigenvalues are

$$E_n = c\sum_{i=1}^{N} n_i^2, \qquad n \in \mathbb{N}^N$$

with $c := -(\frac{\pi}{2})^2$. Thus the trace is computed by

$$\operatorname{Tr} e^{\triangle t} = \sum_{n_i \in \mathbb{N}^N} e^{E_n t} = \sum_{n_i \in \mathbb{N}^N} e^{c\sum_{i=1}^{N} n_i^2 t} = \sum_{n_i \in \mathbb{N}^N} \prod_{i=1}^{N} e^{cn_i^2 t} = (\sum_{n=1}^{\infty} e^{cn^2 t})^N = \tilde{c}t^{-\frac{N}{2}}$$

where we used the Cauchy method of summation (this is justified by the absolute convergence of the series). □

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