Closed Formulas and Rating Schemes for Derivatives

PHD THESIS

to obtain the title of
PhD of Economics
(Dr. rer. pol.)

of Frankfurt School of Finance & Management

Specialty: Quantitative Finance

defended by

Carlos Manuel Antunes Veiga

Thesis Advisor: Prof. Dr. Uwe Wystup

March 2010

Revised on August 2010
Defended on August 16th, 2010

Frankfurt School of Finance & Management
Bankakademie | HfB

Jury:

Prof. Dr. Tomas Björk
Stockholm School of Economics

Prof. Dr. Robert Tompkins
Frankfurt School of Finance & Management

Prof. Dr. Uwe Wystup
Frankfurt School of Finance & Management

Declaration of Authorship

I hereby certify that unless otherwise indicated in the text or references, or acknowledged, this thesis is entirely the product of my own scholarly work. Appropriate credit is given where I have used language, ideas, expressions, or writings of another from public or non-public sources. This thesis has not been submitted, either in whole or part, for a degree at this or any other university or institution.

Frankfurt am Main,

Signed: ______________________________

Date: ______________________________

"I find, that if I just sit down to think... The solution presents itself!"

Prof. Henry Walton Jones, Sr.

Abstract

In the first and second parts of the thesis we develop closed form pricing and hedging formulas for financial derivatives. The first part takes a simple form of the Black-Scholes model to address the issue of the impact of discrete dividend payments in financial derivatives valuation and hedging. It successfully resolves the problem of considering a known absolute jump in a geometric Brownian motion by making use of a Taylor series expansion and measure changes. The second part extends the Black-Scholes model to cover multiple currency zones and assets and uses it to develop a valuation framework that covers a significant class of exotic derivatives. This framework also enables the decoupling of the processes of payoff definition and coding of pricing functions by creating an implicit payoff language. On both parts examples are presented with analysis of accuracy and performance of the formulas. The third part develops a critique of the rating schemes of structured products addressing their ability to rank and select products by the criteria they announce based on properties we believe rating schemes should show. We then examine to what extent the ranking of structured products is possible and propose a route based on issuers' binding commitments.

To those who still tag along even after all these years.

Acknowledgements

I start by thanking my employer, Millennium bcp, that provided me the necessary support to undertake this PhD program. Without Millennium bcp, none of this would have been possible. I also wish to make record of the efforts my managing director Luís Feria put into making that support real. I specially wish to show my appreciation to him for the kindness of allowing me to get what I asked for and not what others thought was best for me.

I thank next Prof. Dr. Uwe Wystup, my professor, my PhD advisor and my friend. Uwe has helped me in almost every way, from overcoming the most basic challenges a foreigner faces when settling in a foreign land to the most difficult mathematical problems I met while researching the subjects that make up this thesis. I thank him for allowing me to pursue the research topics I chose and for sharing with me his own research projects.

I would also like to show my gratitude for the remaining members of the faculty Prof. Dr. Tom Heidorn, Prof. Dr. Wolfgang Schmidt, Prof. Dr. Robert Tompkins, and Prof. Dr. Ursula Walther for taking interest in my work and helping me improve it. I include here also Prof. Dr. Manuel L. Esquível, from Universidade Nova de Lisboa, with whom I have been sharing research work for last eight years.

I thank Ms. Brigitte Gebhart for going to great lengths to make sure I would feel welcomed and at home in Frankfurt and in the Frankfurt School of Finance & Management.

Special thanks go to the fine friends with whom I had the pleasure to share the challenges of a PhD as well as many others. I am happy to have made more friends than those I can mention here, but still I name these much Christoph Becker, Dietmar Frenz, Susanne Griebsch, Stefanie Kammer, Denis

Karlow, Jian Liang, Natalie Packham, Dimitri Reiswich, Christian Schmaltz, Peter Scholz, and Andreas Weber.

My last word of acknowledgment goes to my family, the foundations that allow me to stand up straight every day. I thank them for the unconditional support, belief, and patience. I thank in particular Isabel, my wife, for bearing with me when my will was fainting and my spirit close to wearing out. Miguel, our son and ultimate challenge, is the kindest kid in town allowing good nights of sleep and plenty of entertainment during the day.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

Derivatives were the means that enabled the modernization of the banking industry. Through them, risk is now managed more effectively, investment is more customized to investors needs, and trading costs are lower. They also propelled a deeper understanding of the financial markets and risk as the industry and the research community took on the challenges they embody. The quest has not been free of failure and disappointment though, as recent history stands as a sharp reminder of the consequences of overconfidence in models and underestimation of risk.

Here we chose to tackle three problems that arise with the use of derivatives. The first two problems address valuation and hedging of derivatives, while the third focuses on the issue of comparing alternative investments through derivatives.

The first problem we address is the problem of correctly assessing the impact of the existence of dividend payments by the stocks that underlie some derivative contracts. These dividends are typically paid a few times a year and have an impact on the price of the stock itself and, in turn, on the derivative's price and risk profile. So far, the correct assessment of this impact was only possible through numerical methods but we succeeded in finding a closed formula for the most common derivatives, European calls and puts. However, the technique used to accomplish it may also be applied successfully to other option types. The valuation formula is composed by three terms: (i) the Black–Scholes (BS) price, (ii) the delta of the BS price[1] with a modified stock price argument, and (iii) the gamma[2] of the BS price also with a modified stock price argument. Thus, the price of such a derivative is obtained by the sum of these three readily accessible terms. The derivation below covers options with an arbitrary number of dividend payments and provides formulas not only for the price but also for the partial derivatives, necessary to carry out the hedging strategy, all with the same simple structure.

The second problem we address is the unification of valuation formulas of exotic options. The search for formulas is mainly motivated by the fact that a closed formula is more precise, requires less computational effort, and is easier to implement than a numerical evaluation method. However valuation formulas for exotic options are typically developed with one specific payoff feature in mind and, therefore, apply to a very narrow set of instruments. The unification of these formulas allows for significant improvements of enabling the mix of the features of different option types in one single instrument, as is common practice in the structured products business, and the decoupling of the processes of implementation of valuation formulas and the option payoff definition. It also saves the work of going through the extensive bibliography on this issue to collect the desired formulas. Together, these improvements give industry participants the ability to combine features to develop new exotic option payoffs, within a specified class, without having to undertake a development project to implement a new valuation formula in their management systems. In addition to the valuation formula we also develop formulas for the partial derivatives with respect to all model parameters and state variables, necessary to carry out the hedging strategy as in the case above.

Finally, the third problem focuses on the problems of comparing and ordering derivatives, in particular, structured products - securities issued by financial institutions that contain derivatives. These problems arise due to the large number and diversity of structured products existing in the market, reaching several hundred thousand products and tens of different issuers. The problem boils down to answering these two questions: (i) how to order products that have different characteristics, and (ii) how to order products

[1] first derivative of the price function with respect to the price of the underlying asset.

[2] second derivative of the price function with respect to the price of the underlying asset.

that have homogeneous characteristics. A strong motivation to address this issue was the industry trend of attempting to apply rating schemes to solve the comparison and order problem. Besides analyzing the existing rating schemes, we develop a general analysis able to cover future schemes that may appear. We find that existing rating schemes are not effective in solving problem (i), producing nothing but arbitrary orderings. Furthermore, our findings lead us to believe that no rating scheme would be able to solve the problem at hand. On the much simpler case (ii), ordering is still unfeasible since choosing the cheaper of a set of homogeneous structured products is not necessarily the best choice. We conclude that further information is needed to produce an order, in particular binding commitments from issuers with respect to bid prices over the course of the products' life. Accordingly, we propose a framework for such commitments and provide examples considering the most common products.

Each of the three problems we considered in this thesis is developed in an autonomous chapter, Chapter 2, Chapter 3 and Chapter 4 respectively.

Chapter 2

Closed Formulas with Discrete Dividends

2.1 Introduction

We present a closed pricing formula for European options under the Black–Scholes model as well as formulas for its partial derivatives. The formulas are developed making use of Taylor series expansions and a proposition that relates expectations of partial derivatives with partial derivatives themselves. The closed formulas are attained assuming the dividends are paid in any state of the world. The results are readily extensible to time dependent volatility models. For completeness, we reproduce the numerical results in Vellekoop and Nieuwenhuis, covering calls and puts, together with results on their partial derivatives. The closed formulas presented here allow a fast calculation of prices or implied volatilities when compared with other valuation procedures that rely on numerical methods. We also discuss implementation issues targeting calculation speed. Parts of this chapter were used in [46] and [47].

2.2 Motivation

The motivation to return to this issue is the fact that whenever a new product, model or valuation procedure is developed, the problem that arises with discrete dividends is dismissed or overlooked by applying the usual approximation that transforms the discrete dividend into a continuous stream of dividend payments proportional to the stock price. After all that has been said about the way to handle discrete dividends, there are still strong reasons to justify such an approach.

We here recall the reasons that underlie the use of this method by the majority of market participants and pricing tools currently available. We choose the word *method* instead of *model*, although one could look for what model would justify such calculations and find the Escrowed Model, as it is known in the literature. We do not follow this reasoning because we consider that such a model would be unacceptable since it admits arbitrage. The reason being that such a model would imply two different diffusion price processes for the same underlying stock under the same measure if two options were considered with different maturities and spanning over a different number of dividend payments. We thus refuse the model interpretation and consider the procedure that replaces the discrete dividend into a continuous stream of dividend payments as an approximation to the price of an option, under a model that remains arbitrage free when several options coexist. The models in Section 2.3 belong to that class.

The drivers behind the huge popularity of this method are mostly due to the (i) tractability of the valuation formulas, (ii) applicability to any given model for the underlying stock, and (iii) the preserved continuity of the option price when crossing each dividend date.

However, the method has some significant drawbacks. First and foremost, no proof has ever been present that this method would yield the correct result under an acceptable model in the sense above. In fact, for the natural extensions to the Black-Scholes model described in Section 2.3, the error grows larger as the dividend date is farther away from the valuation date. This is exactly the opposite behavior of what one would expect from an approximation – a larger period of time between the valuation date and the dividend date implies that the option valuation functions are smoother, and thus easier to approximate. The other side of the inaccurate pricing coin is the fact that this method does not provide a hedging strategy that will guarantee the replication of the option payoff at maturity. To sum up, no numerical procedure based on this method returns (or converges to) the true value of the option, in any of the acceptable models we are aware of. It still seems like the advantages outweigh the drawbacks since it is the most widely used method.

An example may help to demonstrate this. Consider a stochastic volatility model with jumps. Now consider the valuation problem of an American style option under this model. The complexity of this task is such that a rigorous treatment of discrete dividends, i.e., a modification of the underlying diffusion to account for that fact, would render the model intractable.

2.3 Description of the Problem

In the presence of discrete dividend payments, diffusion models like the Black–Scholes (BS) model, are no longer an acceptable description of the stock price dynamics. The risks that occur in this context are mainly the potential losses arising from incorrect valuation and ineffective hedging strategy. We address both of these issues. Björk [5] (1998) has one of the clearest descriptions of the problem of option pricing in the presence of discrete dividends.

The most natural extension to a diffusion model to allow for the existence of discrete dividends is to consider the same diffusion, for example, under the risk neutral measure Q,

$$\mathrm{d}S_t = S_t \left(r\mathrm{d}t + \sigma \mathrm{d}W_t \right), \tag{2.1}$$

where S is the stock price, r is the constant interest rate, σ is the volatility and W is a standard Brownian motion, and add a negative jump with size equal to the dividend amount. On the dividend payment date we then have $S_{t_D} = S_{t_D^-} - D$, with $S_{t_D^-}$ the stock price at the moment immediately before the dividend-payment moment t_D, D the dividend amount, and S_{t_D} the stock price at the moment immediately after the dividend payment moment. Here and throughout the text we assume that the stock price S_t is an ex-dividend price. This gives rise to the new model diffusion

$$\mathrm{d}S_t = S_t \left(r\mathrm{d}t + \sigma \mathrm{d}W_t \right) - D\mathrm{d}N_t, \tag{2.2}$$

where $N_t = \mathbb{I}_{\{t \geq t_D\}}$ is the the Heaviside function. Like the Brownian motion W_t, the Heaviside function does not have a time derivative, so $\mathrm{d}N_t$ should be taken in the same way as $\mathrm{d}W_t$, that is, a short hand notation for the integral $\int_0^t N_s \mathrm{d}s$.

There are though some common objections to this formulation. A first caveat may be the assumption that the stock price will fall by the amount of the dividend size. This objection is mainly driven by the effects taxes have on the behavior of financial agents and thus market prices. Here we will not consider this objection and, thus, assume Model (2.2) to be valid. A second objection may be that the dividend payment date and amount are not precisely known until a few months before their payment. We also believe this to be the case but a more realistic model in this respect would significantly grow in complexity. Our goal is rather to devise a simple variation that can be applied to a wide class of models, that does not worsen the tractability of the model and produces accurate results.

Finally, the model admits negative prices for the stock price S. This is in fact true and can easily be seen if one takes the stock price $S_{t_D^-}$ to be smaller than D at time t_D^-. A simple solution to this problem is to add an extra condition in (2.2) where the dividend is paid only if $S_{t_D^-} > D$, i.e.,

$$\mathrm{d}S_t = S_t \left(r\mathrm{d}t + \sigma \mathrm{d}W_t \right) - D\mathbb{I}_{\{S_{t_D^-} > D\}} \mathrm{d}N_t. \tag{2.3}$$

However, in most practical applications, this is of no great importance since the vast majority of the companies that pay dividends, have dividend amounts that equal a small fraction of the stock price, i.e., less than 10% of it, rendering the probability assigned to negative prices very small. For this reason we may drop this condition whenever it would add significant complexity.

In the remainder of this section we review the existing literature on the subject and the reasons that underlie the use of the method most popular among practitioners. We then turn to develop the formulas in Section 2.5, and in Section 2.7 we reproduce the numerical results in Vellekoop and Nieuwenhuis [50] together with put prices and partial derivatives.

2.4 Literature Review

The fact that discrete dividends posed a real problem in option pricing was recognized and addressed very early and gave rise to immediate significant contributions. The most prominent example is Merton [33] (1973), who analyzed the effect of discrete dividends in American calls and states that the only reason for early exercise is the existence of dividends. Remarkably though, at odds with the way the pricing problem of European and American options was solved, the American option case benefited from most attention and developments. This is justified in part because the problem of pricing an American call option in the presence of one discrete dividend ended up being much simpler than the European counterpart. Even simpler than the problem of pricing an American call on an underlying that pays a continuous yield because, in such a case, the exercise choice may happen at any moment. It is so because such a problem may be translated into a compound option problem, in which the holder of the option is entitled to choose, at the date just before the dividend payment, between (i) the option's intrinsic value, and (ii) an American call option in the presence of no dividends, i.e., an option as valuable as an European call. Based on Geske [18] (1979) solving the compound option problem, Roll [38] (1977) develops the following formula for an American call with one known dividend payment under the Black–Scholes model

$$\begin{aligned} C(S_t, t) = {} & S'_t\, N(d_+) + e^{-r(t_D - t)}(D - K)\, N(d_-) + \\ & S'_t\, N_2\left(g_+, -d_+; -\sqrt{\frac{t_D - t}{T - t}}\right) - e^{-r(T-t)} K\, N_2\left(g_-, -d_-; -\sqrt{\frac{t_D - t}{T - t}}\right), \end{aligned} \tag{2.4}$$

where

$$d_\pm = \frac{\log\left(\frac{S'_t}{S^*_{t_D}}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(t_D - t)}{\sigma\sqrt{t_D - t}}, \quad g_\pm = \frac{\log\left(\frac{S'_t}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}},$$

S_t the current stock price, r the risk free rate, σ the volatility of the stock returns, K the strike price, and D the dividend amount. The current date is t, the dividend payment date is t_D and the option's maturity date is T. N and N_2 are the cumulative distribution function of the univariate and of the bivariate normal distribution, respectively. Finally, $S'_t = S_t - e^{-r(t_D - t)}D$ and $S^*_{t_D}$ the S_{t_D} that divides the exercise/no-exercise regions. In the Black–Scholes model $S^*_{t_D}$ is constant and can be calculated using a simple numerical procedure.

A reader with a trained eye for pricing formulas would immediately recognize that the first line of Formula (2.4) is nothing more than an European call with maturity t_D that pays off $S'_{t_D} + (D - K) = S_{t_D} - K$ if $S_{t_D} > S^*_{t_D}$ and zero otherwise. Thus, the first line of the formula covers the cases in which the option is exercised prior to the dividend payment. The second line covers the other possible cases. By evaluating the bivariate normal on $-d_\pm$, it covers the complementary cases of the first line. The second dimension of the distribution is evaluated at $g_\pm$ to produce the probability of the option being exercised at maturity T (given that it was not exercised at t_D) under two martingale measures, one that considers the bank account as the *numéraire* and the other that considers it to be S_t.

However, for European options the problem cannot be circumvent in this fashion and one needs to handle the Model (2.3) with the deterministic absolute jump. The absolute nature of the jump is the really problematic feature and disrupts both closed formulas and numerical procedures alike. In the development of a formula we face the problem of handling a term of the form $\log(S_t - D)$ which cannot be factorized in anything more elementary that would enable the separation of S_t and D. In the numerical scheme alternative, like the binomial tree for example, an absolute jump renders the tree of the stock price S_t non-recombining and, therefore, implies and dramatic increase of the nodes to be calculated. The presence of multiple dividends adds to the problem by compounding on the problems of one dividend.

For these reasons, the problem was circumvent by replacing the Model (2.3) by a model that would yield tractable results, though not without a cost. The solution was to make the necessary modifications to be able to apply the *Black-76* formula, by Black [3] (1976), that prices options on futures prices and reads as

$$C(F_t, t) = e^{-r(T-t)} \left(F_t N(d_+) + K N(d_-)\right), \tag{2.5}$$
$$d_\pm = \frac{\log(\frac{F_t}{K}) \pm \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{T-t}}.$$

with F_t the current future's price that matures at time T and the remaining symbols as above. The model that underlies this formula assumes that the diffusion, under the risk neutral measure Q, of the future's price is $\mathrm{d}F_t = F_t \sigma \mathrm{d}W_t$. While this diffusion is compatible with the Black–Scholes model in (2.1) it is not compatible with (2.3). This can be easily seen if one takes the arbitrage free relationship between the stock price S_t and the future's price F_t. Assuming only constant interest rates and that the dividend is always paid, we have

$$F_t = e^{r(T-t)} S_t - e^{r(T-t_D)} D\, \mathbb{I}_{\{t<t_D\}} \Leftrightarrow S_t = e^{-r(T-t)} F_t + e^{-r(t_D-t)} D\, \mathbb{I}_{\{t<t_D\}}. \tag{2.6}$$

A simple application of the Itô formula yields

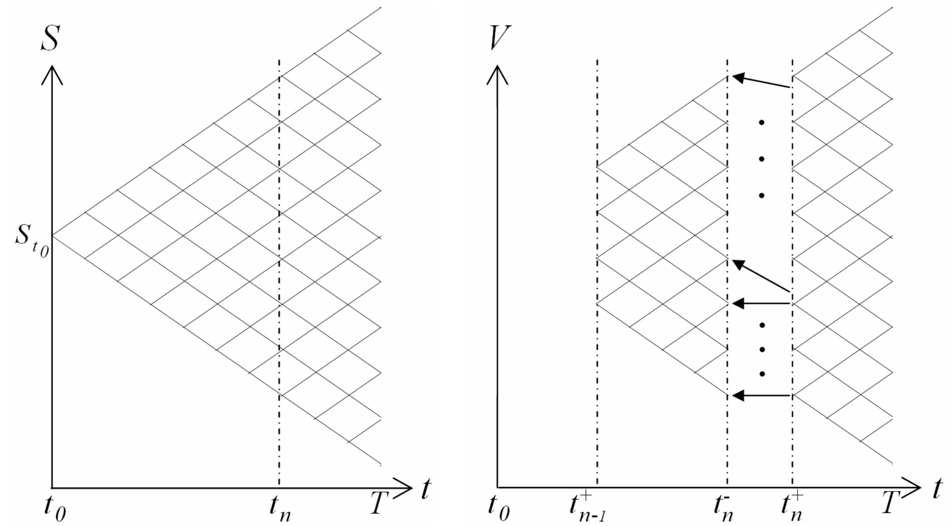
$$\mathrm{d}S_t = S_t \left(r\mathrm{d}t + \sigma \mathrm{d}W_t\right) - e^{-r(t_D-t)} D \mathbb{I}_{\{t<t_D\}} \sigma \mathrm{d}W_t - D\mathrm{d}N_t. \tag{2.7}$$

which is markedly different from (2.2), having an extra term linear in the Brownian motion W_t, $-e^{-r(t_D-t)} D \mathbb{I}_{\{t<t_D\}} \sigma \mathrm{d}W_t$, compromising thus the geometric nature of the dynamics of S_t.

Even nowadays, this approach is still the most commonly used solution. It is described in John Hull [27] (1989) and in many other sources. It is equivalent to subtracting from the current asset price S_t the net present value of all dividends to occur during the life of the option. The weakness in this model is that the model depends on the maturity of the option and, therefore, options with different maturities are priced with different models, producing incoherent results and giving rise to arbitrage opportunities within the model. That is obvious to see if one considers options with a maturity $T < t_D$, in which case all the terms that depend on D in (2.7) would disappear and we would get the geometric Brownian motion back.

Because Model (2.7) produces the greatest errors, if we assume Model (2.2) to be true, when the dividends are closer to the option's maturity, Musiela and Rutkowski [34] (1997) propose a model that adds the future value at maturity of all dividends paid during the lifetime of the option to the strike price. This approach is complementary to Model (2.7) as it produces the greatest errors when dividends are close to valuation date. Aiming at a compromise between these two last methods, Bos and Vandermark [8] (2002) devise a method that divides the dividends in "near" and "far" and subtracts the "near" dividends from the stock price and adds the "far" dividends to the strike price. Bos *et al.* [7] (2003) devise a method that adjusts the volatility parameter to correct the subtraction method stated above. Haug *et al.* [24] (2003) reviews all these models and reports that they produce systematic errors, when we assume Model (2.3) to be true, depending on the relative date of the dividend payment and moneyness of the option. They conclude reiterating that Model (2.3) is the most appropriate and propose a numerical quadrature scheme to solve the valuation problem. It is also well known that if, instead of absolute discrete dividends, considering proportional discrete dividends of the type $\delta\, S_{t_D^-}$, with $0 < \delta < 1$, yields a closed pricing formula for European options. This result is mentioned in Wilmott *et al.* [55] (1995), Björk [5] (1998), and Shreve [40] (2004). Korn and Rogers [29] (2005) model the discrete dividends as a stochastic process directly, deducing stock prices from it as well as option prices.

Finally, one of the most recent works in this subject deserves a special remark. Vellekoop and Nieuwenhuis [50] (2006) managed to resolve the numerical problems that arise with Model (2.3). In particular they propose a valuation approach that preserves the crucial recombining property of binomial trees. Instead of building a tree for the stock price, considering the absolute jumps that arise from discrete dividends, they build a stock price tree ignoring any dividend payments. The impact of the discrete dividend is then applied during the backward induction procedure, assigning to the option value nodes just before the dividend payment at a given stock price S, the value of the option just after the dividend at that stock price S minus the dividend amount D. Figures 2.1(a) and 2.1(b) illustrate the procedure.



(a) Stock price binomial tree. (b) Option valuation procedure.

Figure 2.1: Modified binomial tree numerical procedure.

2.5 Pricing Formula

The derivation of the closed formula assumes a Black–Scholes model as in (2.1) with constant interest rate r and constant volatility σ. However, the following can be easily modified to allow for time-dependent volatility. Furthermore, our arguments consider and are only valid for European style options.

We assume a problem with n dividends D_i, with $i = 1, \cdots, n$, having their respective payment dates on t_i ordered in this manner $t_0 < t_1 < \cdots < t_n < T$, and having t_0 and T the valuation date and the option's maturity date respectively.

We take a vanilla call option as our working example with payoff $X = \max[S_T - K, 0]$. We start by focusing on the time point just after the last dividend payment, which we will refer to as t_n. We choose this point in time because it is the earliest moment on which we can make a conjecture with respect to the price of the option, i.e., from this point on, we know how to price and hedge a claim, for there are no dividends left until the option matures. The price of our call would thus be a function $C(S_{t_n}, t_n)$[1] of the stock price and time, the celebrated Black–Scholes formula that we state here

[1] We omit the model parameters and the option specific quantities, like maturity or strike, from the function C to preserve clarity.

for completeness' sake

$$\begin{aligned} C(S_t,t) =& S_t N(d_+) + K e^{-r(T-t)} N(d_-), \\ d_\pm =& \frac{\log(\frac{S_t}{K}) + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}. \end{aligned} \tag{2.8}$$

As usual, K and T are the strike price and the maturity date respectively. Since it is going to be used extensively throughout the text, we take here the opportunity to also present the general formula for the i^{th} derivative with respect to the first variable, S_t, of Formula (2.8) developed by Carr [11]

$$\begin{aligned} \partial_1^i C\left(S_t,t\right) = S_t^{-i} \sum_{j=1}^{i} \mathcal{S}_1\left(i,j\right)\delta^j, \\ \delta^j = S_t N\left(d_+\right) + K e^{-r(T-t)} \frac{N'\left(d_-\right)}{\sigma\sqrt{T-t}} \sum_{h=0}^{j-2} \frac{H_h\left(d_-\right)}{\left(-\sigma\sqrt{T-t}\right)^h}, \end{aligned} \tag{2.9}$$

where $N'\left(x\right)$ denotes the probability density function of the standard normal distribution, $\mathcal{S}_1\left(i,j\right)$ the Stirling number of the first kind, and $H_i\left(d\right)$ are Hermite polynomials.

The problem we face now is how to move one step back in time to $t < t_n$. For that we take Assumption (2.3) in Section 2.3. This assumption implied that the call price at time t_n, i.e., just after the dividend payment moment, is given by

$$C\left(S_{t_d^-} - D, t_d\right) \mathbb{I}_A + C\left(S_{t_d^-}, t_d\right) \mathbb{I}_{\bar{A}} \tag{2.10}$$

with $A = \left\{\omega : S_{t_n^-} > D_n\right\}$.

A straightforward application of option pricing theory allows us to formally write the value of the option at t_{n-1}, i.e., just after the second to last dividend payment moment with only one dividend until maturity date T, as

$$e^{-r(t_n - t_{n-1})} \mathbb{E}_{t_{n-1}}^Q \left[C\left(S_{t_n^-}, t_n\right) + \left(C\left(S_{t_n^-} - D_n, t_n\right) - C\left(S_{t_n^-}, t_n\right)\right) \cdot \mathbb{I}_A\right]. \tag{2.11}$$

We then rewrite the problem by replacing the difference

$$C\left(S_{t_n^-} - D_n, t_n\right) - C\left(S_{t_n^-}, t_n\right)$$

by its corresponding Taylor series expansion. This is valid since the expansion of the Black–Scholes formula for calls is convergent for shifts D of size smaller than $S_{t_n^-}$. The call function is not even defined for $D > S_{t_n^-}$. We then get

$$e^{-r(t_d-t)}\mathbb{E}_t^Q\left[C\left(S_{t_d^-}, t_d\right) + \sum_{i=1}^{\infty}\frac{(-D)^i}{i!}\partial_1^i C\left(S_{t_d^-}, t_d\right)\mathbb{I}_A\right]. \tag{2.12}$$

This approach did lead to a closed formula for the case of problems with only one dividend payment (see [46]). However, the cost of this rigor was a highly complex formula that cannot be generalized to fit multiple dividends problems. For this reason we here take a different route.

Taking advantage of the fact that in practice $Q(A) \approx 1$, we proceed by removing the indicator function, changing thus our Assumption to (2.2), and accepting the the fact that our model allows negative prices whenever $D > S_{t_n^-}$. This simplification entails an additional problem as the Taylor series expansion diverges in such cases. Since our model assigns a small but positive probability to such an event, the expectation that the call price calculation involves also diverges, yielding and indefinite price.

To address this issue, we truncate the Taylor series formula assuming that

$$C\left(S_{t_n^-} - D_n, t_n\right) \approx \sum_{i=0}^{\eta_n}\frac{(-D_n)^i}{i!}\partial_1^i C\left(S_{t_n^-}, t_n\right) \tag{2.13}$$

with η_n high enough, approximates $C\left(S_{t_n^-} - D_n, t_n\right)$ reasonably well, for all $S_{t_n^-}$. We thus trade the error of this approximation for the tractability that it enables. We do so because we believe that in almost all realistic scenarios the error is not significant. In fact, our findings in Section 2.7 based on this assumption do provide very good results with scenarios even more demanding than realistic market conditions.

Hence, we rewrite Expression (2.12) as

$$e^{-r(t_n-t_{n-1})}\mathbb{E}^Q_{t_{n-1}}\left[\sum_{i=0}^{\eta_n}\frac{(-D_n)^i}{i!}\partial_1^i C\left(S_{t_n^-},t_n\right)\right]. \tag{2.14}$$

Since we have a finite series as integrand function, we can safely interchange the integral with the summation, yielding

$$e^{-r(t_n-t_{n-1})}\sum_{i=0}^{\eta_n}\frac{(-D_n)^i}{i!}\mathbb{E}^Q_{t_{n-1}}\left[\partial_1^i C\left(S_{t_n^-},t_n\right)\right]. \tag{2.15}$$

Finally, to turn Expression (2.15) above into an explicit formula we use Proposition 2.5.1 below, which proof relies on the following result by Carr [11]

Theorem 2.5.1. *The value, delta, gamma, and higher-order derivatives of path-independent claims in the Black-Scholes model are given by*

$$\partial_1^i C\left(S_t,t\right) = e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t)}\mathbb{E}^{S^i}_t\left[\partial_1^i f\left(S_T\right)\right] \quad (i=0,1,\ldots), \tag{2.16}$$

where the operator $\mathbb{E}^{S^i}$ indicates that the expectation is calculated from the diffusion

$$dS_t = S_t\left((r+i\sigma^2)dt+\sigma dW_{i,t}\right), \tag{2.17}$$

$W_{i,t}$ is a standard Brownian motion under the measure S^i, and $\partial_1^i f\left(S_T\right)$ is the i^{th} derivative of the payoff function f with respect to S_T.

For example, the delta of a call, $\partial_1^1 C\left(S_{t_n},t_n\right)$, is the value, at time t_n, of a derivative with a payoff function $\mathbb{I}_{\{S_T>K\}}$ evaluated under the measure S^1,

$$\partial_1^1 C\left(S_t,t\right) = \mathbb{E}^{S^1}_t\left[\mathbb{I}_{\{S_T>K\}}\right].$$

Proposition 2.5.1. *Let $C\left(S_t,t\right)$ be the value of a European call option under the Black-Scholes model, then*

$$\mathbb{E}^Q_t\left[\partial_1^i C\left(S_{t_k},t_k\right)\right] = e^{-\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(t_k-t)}\partial_1^i C\left(S_t e^{-i\sigma^2(t_k-t)},t\right). \tag{2.18}$$

Proof. By applying Theorem 2.5.1 to the left side of (2.18) we get

$$\mathbb{E}_t^Q\left[\partial_1^i C\left(S_{t_k},t_k\right)\right] = \mathbb{E}_t^Q\left[e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t_k)}\mathbb{E}_{t_k}^{S^i}\left[\partial_1^i f\left(S_T\right)\right]\right] \tag{2.19}$$

$$= e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t_k)}\mathbb{E}_t^Q\left[\mathbb{E}_{t_k}^{S^i}\left[\partial_1^i f\left(S_T\right)\right]\right]. \tag{2.20}$$

The diffusions with respect to which the expectations $\mathbb{E}_t^Q$ and $\mathbb{E}_t^{S^i}$ are taken, respectively equations (2.1) and (2.17), differ only in the drift term since $W_{i,t}$ and W_t are Brownian motions under their respective measures. Thus, to change the measure from S^i to Q, one can compensate the different drift by changing the starting value condition such that the solutions of the diffusions are equal.

Let, $S_{t,Q}$ and S_{t,S^i} be the initial conditions for diffusions (2.1) and (2.17) respectively. Then,

$$S_{t_n} = S_{t,Q}e^{\left(r-\frac{\sigma^2}{2}\right)(t_n-t)+\sigma W_{t_n}} \quad \text{under } Q\text{, and} \tag{2.21}$$

$$S_{t_n} = S_{t,S^i}e^{\left(r+i\sigma^2-\frac{\sigma^2}{2}\right)(t_n-t)+\sigma W_{i,t_n}} \quad \text{under } S^i. \tag{2.22}$$

If we now set $S_{t,S^i} = S_{t,Q}e^{-i\sigma^2(t_n-t)}$ then, under S^i,

$$S_{t_n} = S_{t,Q}e^{\left(r-\frac{\sigma^2}{2}\right)(t_n-t)+\sigma W_{i,t_n}}. \tag{2.23}$$

This solution is now equivalent, in a weak sense, to (2.21), since the distribution of S_{t_n} is, in both cases, a log-normal distribution with parameters $\mu_{LN} = \left(r-\frac{\sigma^2}{2}\right)(t_n-t)$ and $\sigma_{LN} = \sigma\sqrt{t_n-t}$.

The equality of distributions is enough to state the following equality of expectations,

$$\mathbb{E}_t^Q\left[X(S_{t_n})\right] = \mathbb{E}_t^{S^i}\left[X(S_{t_n})\right]\Big|_{S_{t,S^i}=S_{t,Q}e^{-i\sigma^2(t_n-t)}} \tag{2.24}$$

with X a measurable function.

Figure 2.2 illustrates the relationship between two expectations of the same random variables under the measures Q and S^i.

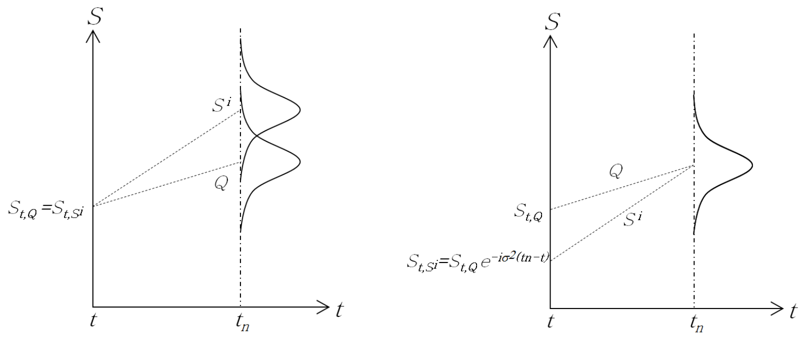


Figure 2.2: Equating expectations under different measures.

Recalling Expression (2.20), we have

$$\begin{aligned}
\mathbb{E}_t^Q\left[\partial_1^i C\left(S_{t_k}, t_k\right)\right] &= e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t_k)}\mathbb{E}_t^Q\left[\mathbb{E}_{t_k}^{S^i}\left[f^{(i)}\left(S_T\right)\right]\right] \\
&= e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t_k)}\ \mathbb{E}_t^{S^i}\left[\mathbb{E}_{t_k}^{S^i}\left[f^{(i)}\left(S_T\right)\right]\right]\Big|_{S_{t,S^i}=S_{t,Q}e^{-i\sigma^2(t_k-t)}} \\
&= e^{\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(T-t_k)}\ \mathbb{E}_t^{S^i}\left[f^{(i)}\left(S_T\right)\right]\Big|_{S_{t,S^i}=S_{t,Q}e^{-i\sigma^2(t_k-t)}} \\
&= e^{-\left(r+\frac{1}{2}i\sigma^2\right)(i-1)(t_k-t)}\partial_1^i C\left(S_t e^{-i\sigma^2(t_k-t)}, t\right),
\end{aligned}$$

where we have used Theorem 2.5.1 again in the last step. □

Hence, to get a closed formula for a call option maturing at T with one discrete dividend payment at time t_n of amount D_n, we explicitly rewrite Expression (2.15) and we denominate it as $C_n\left(S_{t_{n-1}}, t_{n-1}\right)$,

$$\sum_{i=0}^{\eta_n}\frac{(-D_n)^i}{i!}e^{-\left(r+\frac{1}{2}(i-1)\sigma^2\right)i(t_n-t_{n-1})}\partial_1^i C\left(S_{t_{n-1}}e^{-i\sigma^2(t_n-t_{n-1})}, t_{n-1}\right). \quad (2.25)$$

In what follows, we will require a more condensed notation, so we introduce the abbreviations below and suppress the time variable from all C functions.

$$f_{t_n}^h = \exp\left\{-(r+\frac{1}{2}(h-1)\sigma^2)h(t_n-t_{n-1})\right\} \quad (2.26)$$

$$g_{t_n}^j = \exp\left\{-j\sigma^2(t_n-t_{n-1})\right\} \quad (2.27)$$

Now, Formula (2.25) becomes

$$C_n\left(S_{t_{n-1}}\right)=\sum_{i=0}^{\eta_n}\frac{(-D_n)^i}{i!}f_{t_n}^i\partial_1^iC\left(g_{t_n}^iS_{t_{n-1}}\right). \tag{2.28}$$

We can now resume our movement backwards in the time axis using the same program that led us here, namely: apply (2.2) to move over the dividend date t_{n-1}; apply Approximation (2.13) now for C_n yielding[2]

$$\begin{aligned}C_n\left(S_{t_{n-1}^-}-D_{n-1}\right)&\approx\sum_{j=0}^{\eta_{n-1}}\frac{(-D_{n-1})^j}{j!}\partial_1^jC_n\left(S_{t_{n-1}^-}\right)\\&=\sum_{j=0}^{\eta_{n-1}}\frac{(-D_{n-1})^j}{j!}\frac{\partial^j}{\partial^jS_{t_{n-1}^-}}\left\{\sum_{i=0}^{\eta_n}\frac{(-D_n)^i}{i!}f_{t_n}^i\partial_1^iC\left(g_{t_n}^iS_{t_{n-1}^-}\right)\right\}\\&=\sum_{j=0}^{\eta_{n-1}}\sum_{i=0}^{\eta_n}\frac{(-D_{n-1})^j}{j!}\frac{(-D_n)^i}{i!}f_{t_n}^ig_{t_n}^{ij}\partial_1^{i+j}C\left(g_{t_n}^iS_{t_{n-1}^-}\right);\end{aligned}$$

take the discounted expectation under the measure Q with respect to the σ-algebra $\mathcal{F}_{t_{n-2}}$; and apply Proposition 2.5.1 to solve the expectation and get

$$C_{n-1}\left(S_{t_{n-2}}\right)=\sum_{j=0}^{\eta_{n-1}}\sum_{i=0}^{\eta_n}\frac{(-D_{n-1})^j}{j!}\frac{(-D_n)^i}{i!}f_{t_{n-1}}^{i+j}f_{t_n}^ig_{t_n}^{ij}\partial_1^{i+j}C\left(g_{t_{n-1}}^{i+j}g_{t_n}^iS_{t_{n-2}}\right). \tag{2.29}$$

Running this program for all n dividends, returns the formula for an arbitrary number of dividend payments

$$C_1\left(S_{t_0}\right)=\sum_{i_1=0}^{\eta_1}\cdots\sum_{i_n=0}^{\eta_n}\prod_{j=1}^{n}\frac{(-D_j)^{i_j}}{i_j!}f_{t_j}^{I_j}\prod_{k=j+1}^{n}\left(g_{t_k}^{I_k}\right)^{i_j}\partial_1^{I_1}C\left(G_IS_{t_0}\right), \tag{2.30}$$

with $I_l=\sum_{m=l}^n i_m$ and $G_I=\prod_{h=1}^n g_{t_h}^{I_h}$.

Before we conclude this section, we remark that even though we developed our analysis focused on a European call, it remains valid for other types of options. In fact, the above analysis is valid for all options that satisfy all con-

[2] Please note that $S_{t_{n-1}^-}$ is known at time t_{n-1} and thus $\partial^j/\partial^jS_{t_{n-1}^-}\{\cdots\}$ below is only a derivative with respect to an argument of the function and not a derivative with respect to a stochastic variable.

ditions it involved, namely: European style options which are priced by only taking expectations under the risk-neutral measure; Approximation (2.13); $\partial_1^n C$ is continuous for all $n \in \mathbb{N}$ to apply Leibniz integral rule. The existence of a closed formula for an arbitrary derivative of the option price, e.g. Formula (2.9), greatly accelerates the calculation process. However, the analysis remains valid if the derivatives are replaced by numerical approximations. This alternative may be useful for problems solved by finite difference methods that return a vector of option prices for different stock prices, thus enabling the calculation of numerical derivatives for all the necessary stock price levels.

Therefore, a European put is an example of another option type covered in this analysis and for which a closed formula for an arbitrary derivative is also available in [11]. In Section 2.7 we also consider European puts and observe that their prices are coherent with the respective call prices.

2.6 The Greeks

A closed formula for the derivative[3] of the option price of arbitrary order is a straightforward application of the chain rule. Thus, for the d^{th} derivative of the call price with one discrete dividend payment we have

$$\partial_1^d C_1\left(S_{t_0}\right) = \sum_{i_1=0}^{\eta_1} \cdots \sum_{i_n=0}^{\eta_n} \prod_{j=1}^{n} \frac{\left(-D_j\right)^{i_j}}{i_j!} f_{t_j}^{I_j} \prod_{k=j+1}^{n} \left(g_{t_k}^{I_k}\right)^{i_j} \left(G_I\right)^d \partial_1^{d+I_1} C\left(G_I S_{t_0}\right). \tag{2.31}$$

The derivatives with respect to other variables, namely σ and r, require similar derivations that we skip here since they constitute simple calculus exercises. There is one exception worth mentioning though: the theta, i.e. the derivative with respect to valuation time t. The theta can be calculated by making use of the Black–Scholes partial differential equation, yielding

$$\partial_2^1 C_1\left(S_t, t\right) = rC_1\left(S_t, t\right) - rS_t \partial_1^1 C_1\left(S_t, t\right) - \frac{1}{2} S_t^2 \sigma^2 \partial_1^2 C_1\left(S_t, t\right). \tag{2.32}$$

[3]The derivatives of the option price are usually called *Greeks* because Greek alphabet letters are commonly used to denote them.

2.7 Results

For ease of reference we reproduce the results stated in Vellekoop and Nieuwenhuis [50] for European call options with seven discrete dividend payments. The model parameters are set at $S_0 = 100$, $\sigma = 25\%$ and $r = 6\%$. Furthermore, the stock will pay one dividend per year, with each dividend one year after the previous, of amount 6, 6.5, 7, 7.5, 8, 8 and 8 for the first seven years respectively. We consider three different scenarios of dividend stream payments referenced by the payment date of the first dividend t_1, set at 0.1, 0.5 and 0.9. With respect to the call and put option specifications, we consider three different options, all with seven years maturity, with strikes of 70, 100 and 130. The calculations reported in Table 2.1 were performed taking a second order approximation for each of the dividend payments, i.e., $\eta_1, ..., \eta_7 = 2$. This approximation order proved to be very effective in this case, producing price differences of 0.01 in the worst cases when compared to the results reported in Vellekoop and Nieuwenhuis [50].

The table displays the price, the first and second derivatives with respect to S, the derivatives with respect to σ, t and r, i.e., delta, gamma, vega, theta and rho respectively. Each row of the table is a set of quantities that relate to one single option. These were calculated together in one single program run that took between 2.3 and 2.6 hundreds of a second[4]. We note here that a *naïve* and straightforward implementation of Formula (2.30) would be very inefficient due to a large number of repetitive calculations contained in it as well as in the Black–Scholes general derivative Formula (2.9).

The key to the performance of our implementation is the caching of all quantities that are needed more than once. We start by breaking up the implementation problem in two routines: one to calculate the Black-Scholes prices and respective derivatives (Formulas (2.8) and (2.9)) and the other routine to calculate the price of the option with dividends and its derivatives (Formulas (2.30) and (2.31)).

The first routine is implemented as an object that is initialized with all the parameters K, t_0, T, σ, r, $S_t = 1$, and η_{max}, which is the maximum derivative order that will be required during the entire calculation. The stock price is

[4] using a C++ ".xll" added to MS Excel03 running on a Intel Core2 4400@2GHz.

set to 1 because it is the only argument that will differ from call to call. The initialization calculates several quantities that will be used repeatedly and stores them in memory together with look up tables for factorials and the numbers $\mathcal{S}_1$, in particular

$$\frac{1}{\sigma\sqrt{T-t}}, \quad \frac{Ke^{-r(T-t)}}{\sigma\sqrt{T-t}}, \quad \frac{1}{(-\sigma\sqrt{T-t})^i}, \quad d_\pm.$$

This object exposes a method that returns all derivatives of the Black-Scholes formula, from arbitrary order a to b, for a given stock price S_t. This method starts by correcting $d_\pm$ by adding $\log S_t/(\sigma\sqrt{T-t})$, computing $N'(d_-)$ and $N(d_+)$. It moves on to calculate all δ^j's in (2.9). It starts by filling a vector indexed by j with the summation term[5] followed by the calculation of the δ^j vector. With all δ^j's in place, each derivative is just a sum of the numbers $\mathcal{S}_1$ multiplied by δ^j divided by a power of S_t.

The second routine also stores several quantities that are used repeatedly. It also takes into account that multiplications are more time consuming that summations. Thus, instead of storing all $f^h_{t_n}$ and $g^j_{t_n}$, the routine stores their logarithms. Accordingly, all products of these terms in (2.30) and (2.31) are implemented as sums of their logarithms and their final sum is taken through the exponential function. This routine also pre-calculates and stores all $(-D_j)^{i_j}/i_j!$ that involve powers, divisions and factorials that are particularly time consuming. The routine then iterates over all possible combinations of i_1, ..., i_n. At each combination it calculates the aggregate factor multiplying C and its argument $G_I S_{t_0}$. At this point it is important to note that the argument of C is the same for the price Formula (2.30) and for derivatives Formula (2.31), allowing the call of the first routine to request all derivatives between order I_1 and $d + I_1$[6]. The factor multiplying C in the derivatives formula is also very similar to the factor multiplying C in the price formula, differing only by $(G_I)^d$. It is thus quite efficient to calculate the price and all derivatives of interest in the same program run as their calculations largely

[5] Please note that each element of the vector is just the previous plus an extra term composed by the Hermite polynomial (that should be computed by the well known recursive relation) multiplied by a quantity already available in memory.

[6] and even derivatives of I_1 with respect to r and to σ if the first routine is ready to return them.

overlap.

In any of the cases under scrutiny, to calculate the price, the number of evaluations of the Black–Scholes pricing formula or any of its derivatives amounts to 2187 or 3^7. In general, the number of evaluations amounts to $\prod_{i=1}^{n}(\eta_i + 1)$.

Table 2.1: European calls and puts, $\sigma = 25\%$, $r = 6\%$, $S_0 = 100$, $T = 7$

Option		Price	Delta	Gamma	Vega	Theta	Rho
$t_1 = 0.1$							
K=70	Call	24.8862	70.6821	69.2653	68.9332	-4.9123	216.9129
	Put	13.0212	-29.3179			0.3758	-234.1280
K=100	Call	17.4394	56.0090	77.3505	80.7711	-4.7314	191.5356
	Put	25.2859	-43.9910			1.7394	-397.4851
K=130	Call	12.4114	43.8271	75.9637	81.9970	-4.2588	160.8653
	Put	39.9693	-56.1729			3.3947	-566.1352
$t_1 = 0.5$							
K=70	Call	26.0752	71.1645	66.2195	70.8947	-4.7747	225.5784
	Put	13.2109	-28.8355			0.4534	-238.8582
K=100	Call	18.4890	56.9270	74.3512	83.3331	-4.6298	200.6573
	Put	25.3362	-43.0730			1.7811	-401.7592
K=130	Call	13.2968	44.9643	73.6551	85.2207	-4.2018	169.9771
	Put	39.8554	-55.0357			3.3917	-570.4191
$t_1 = 0.9$							
K=70	Call	27.2117	71.6629	63.4400	72.6905	-4.6496	233.7131
	Put	13.3718	-28.3371			0.5200	-243.4113
K=100	Call	19.4905	57.8120	71.6694	85.6678	-4.5390	209.1948
	Put	25.3620	-42.1880			1.8133	-405.9094
K=130	Call	14.1419	46.0412	71.6077	88.1568	-4.1517	178.5016
	Put	39.7248	-53.9588			3.3833	-574.5825

From the analysis of the table we see that call and put prices are coherent with put-call parity. In every pair we get the relationship $Call - Put = S - e^{-r(T-t)}K - \sum_{i=1}^{n} e^{-r(T-t_i)}D_i$ and complementary deltas.

Perhaps a more interesting analysis is the comparison of these results with the results produced by *Modified stock price* and *Modified strike price*. Table 2.2 shows the same scenarios as above for $t_1 = 0.1$ and only calls using these two methods referred to as *MS* and *MK* respectively.

It should be noted that all values these methods produce differ rather strongly from the closed formula approximation and from each other. The theta appears to be very problematic since the closed formula value does not stand between both methods' values as is the case for the other quantities.

Table 2.2: European calls, $\sigma = 25\%$, $r = 6\%$, $S_0 = 100$, $T = 7$

Option		Price	Delta	Gamma	Vega	Theta	Rho
$t_1 = 0.1$							
K=70	MS	20.1576	75.1016	82.8568	48.5396	-4.1634	260.0109
	MK	30.7358	69.9048	52.6414	92.1224	-3.9952	200.4516
K=100	MS	12.3709	55.5057	103.2509	60.4870	-3.6682	209.8567
	MK	23.1768	58.5707	58.9171	103.1049	-3.9648	193.3094
K=130	MS	7.7555	39.8123	100.8274	59.0672	-2.9782	158.3466
	MK	17.5976	48.5136	60.2725	105.4769	-3.7385	176.2017

This fact seriously undermines the validity of methods that rely on the averaging of these two alternatives. The behavior of the gamma and vega are also worth noting. These two quantities, that are crucial for the effectiveness of the hedging strategy, show an almost erratic behavior with one method returning almost twice as much as the other for gamma and vice-versa for vega. These are in fact the quantities where the relative differences between the methods and the closed formula are greater.

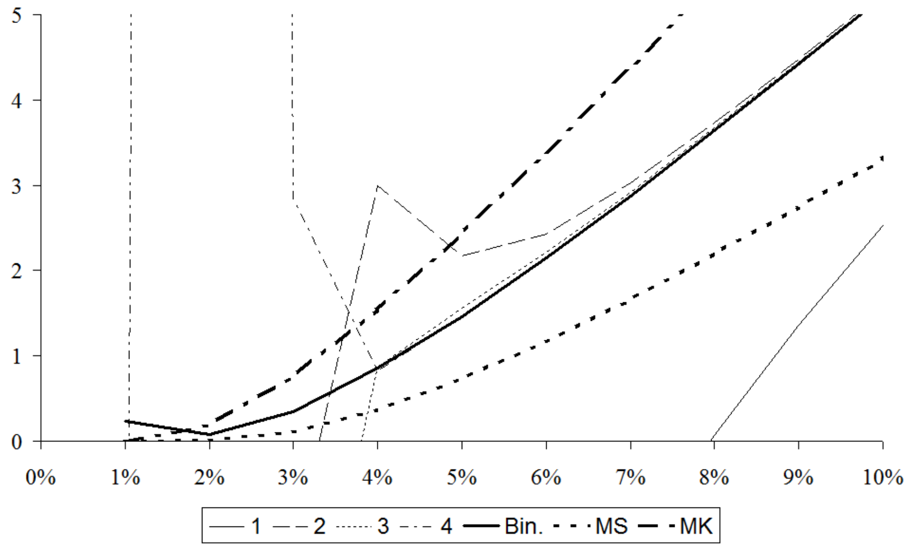


Figure 2.3: Formula performance in a low volatility environment.

Finally, one should expect that cases different from the ones here presented may require different approximation orders to achieve convergence of the closed formula. On the one hand, problems with larger individual dividends, with dividends very close to maturity or in the presence of very low volatilities should require a higher approximation order. The issue at stake is how smooth the function being approximated is - the smoother the function the lower the required approximation order. On the other hand, the higher the volatility the greater the probability of having negative stock prices and thus divergence on the Taylor series approximation. To inspect these problems we choose one of the options above, namely the call with $K = 100$ and $t_1 = 0.1$ and observe how the formula performs on different levels of volatility.

Figure 2.3 shows, for volatilities from 1% up to 10%, the value calculated by a binomial tree as in [50], the values for the methods *MS* and *MK* above, and the closed formula with all η_is equal to 1, 2, 3 and 4. We can see that the formula with all η_is equal to 2 is effective for volatilities 8% and above, while greater η_is provide good results from 4% on. Below 4% the function is not smooth enough to be approximated and the results of the formula are far from the binomial tree value.

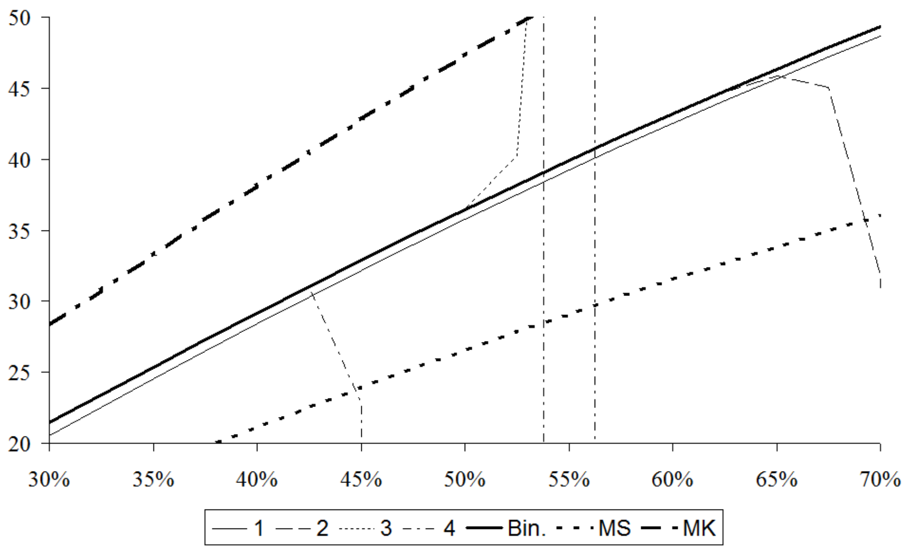


Figure 2.4: Formula performance in a high volatility environment.

Figure 2.4 shows, for volatilities from 30% up to 70%, the same functions. As anticipated, high volatilities will eventually lead the formula to diverge. What we see from the graph, though, is that, for η_is higher than 2, no significant extra precision is obtained but the stability of the formula at lower volatilities is compromised. In this example, with η_is equal to 2, the approximation starts to diverge only after 60% volatility. If η_is equal to 3 or 4 are taken, the approximation diverges at roughly 50% and 40% respectively. It is also interesting to note that the case with all η_is equal to 1 runs consistently close to the binomial tree value while never being exact. This quantity may be used for control purposes as may the upper and lower bounds given by the methods *MK* and *MS* respectively.

2.8 Summary

Departing from the well-known behavior of the option price at the dividend payment date, we approximate it by Taylor series expansion and successfully manipulate it to arrive at a closed form pricing formula. The formula relies on an approximation and a proposition that relates expectations of partial derivatives with partial derivatives themselves. Besides the pricing formula, we also present formulas for its derivatives with respect to the stock price and with respect to other model parameters.

We present applications of the formulas and successfully reproduce the results for calls reported by Vellekoop and Nieuwenhuis [50] and calculate the corresponding put options together with the usual hedging quantities. We compare them with the most commonly used pricing methods that rely on modification of the stock price and of the strike price. We observe severe differences both on the pricing and on the hedging quantities.

Our results show that for a setup that is more demanding than usual market conditions, a second order approximation is fast and sufficient to attain precise results. We also inspect the performance of the formula in extreme scenarios of very high and low volatilities. On the one hand, low volatilities require higher approximation orders to attain precise results. On the other hand, a second order approximation seems to be the most appropriate for very high volatilities since it shows signs of breaking down and diverging at higher volatilities compared to higher order approximations without any significant loss of precision.

Future research should focus on developing rules to control the effectiveness of the formula in extreme scenarios of very high or very low volatility environments. The extension of these results to other models than the Black–

Scholes one and the extension of this approach to multi-asset options are also research topics worth pursuing.

Chapter 3

Unifying Exotic Option Closed Formulas

3.1 Introduction

This chapter focuses on unifying exotic option closed formulas by generalizing a large class of existing formulas and by setting a framework that allows for further generalizations. The formula presented covers options from the plain vanilla to most, if not all, mountain range exotic options and is developed in a multi-asset, multi-currency Black-Scholes model with time dependent parameters. The general formula not only covers existing cases but also enables the combination of diverse features from different types of exotic options. It also creates implicitly a language to describe payoffs that can be used in industrial applications to decouple the functions of payoff definition from pricing functions. Examples of several exotic options are presented, benchmarking the closed formulas' performance against Monte Carlo simulations. Results show a consistent over performance of the closed formula, in some cases reducing calculation time by double digit factors. Parts of this chapter were used in [49].

3.2 Motivation

The pricing of exotic options, defined in most references as every option type apart from the European and American vanilla options, is performed either by using a closed formula or by relying on a numerical method to evaluate the integral the pricing function involves. Whenever available, a closed formula is more precise and requires less computational effort. However, pricing and respective partial derivatives formulas are scattered over many papers and books making it difficult to take full advantage of the existing knowledge. These are the reasons behind our search for general closed formulas that unify exotic option pricing.

The closed formulas for pricing exotic options have mainly been developed to price options whose payoffs exhibit one, and only one, very specific feature, and they assume an elementary market setup. However, the industry requirements go well beyond these simplifications. Exotic options underlying assets spread across several currency zones, and exotic options payoff profiles include features from multiple exotic option types.

This need to account for multiple features in a computationally simple process calls for a unification of the existing closed exotic option pricing formulas. Thus, instead of proceeding to develop formulas for specific option types, we propose a general approach that is able to accommodate several of the features seen in most exotics. Hence, we produce a formula for a general payoff, covering thus all exotic options whose features are included in it. The market setting underlying the formula is also able to accommodate very diverse market setups, covering as many currency zones as needed.

The general formula allows the development of payoff languages. Payoff languages are extremely useful in industrial pricing applications as they enable the decoupling the payoff definition process from the pricing routines. Thus, as long as the payoff only uses the features covered by this general formula, the development of a new payoff profile does not necessitate the development of a new pricing routine but only the parameterization of its payoff. This means that industry agents can freely combine the desired features, while using the same pricing routines.

Finally, we chose a Black-Scholes type model because it is the model type in which pricing and partial derivatives closed formulas can be found for more exotic option types.

3.3 Literature Review

Literature on exotic options is vast and dates back to the late 1970s. It is not our intention to give a complete chronology of the works related to this field but just to refer some landmark contributions for each of the main threads of research. Compilations of exotic options descriptions and pricing formulas may be found in Nelken [35] (1995), Zhang [58] (1997), Haug (1998), and Hakala and Wystup [22] (2002). This chapter also makes use of early works by Harrison and Kreps [20] (1979) and Harrison and Pliska [21] (1981) on the theory of continuous trading.

According to our exotic option definition above, there are three threads of research in exotics, the first of which deals with options on multiple underlyings. The distinctive characteristic of these options is their high sensitivity to correlations. The landmark closed formulas were Margrabe [31] (1978) for exchange options, Stulz [43] (1982) for maximum/minimum of two assets, and Johnson [28] (1987) for maximum/minimum of several assets. One other thread deals with path-dependent options, namely lookback and barrier, which we only consider in their discrete version. The main contributions on this thread are Rubinstein and Reiner (1991) for barrier options and Goldman *et al.* [19] (1979) and Conze and Viswanathan [12] (1991) for lookbacks. Further developments on barrier options were due to work by Heynen and Kat [25] (1994), Carr [10] (1995) and Wystup [57] (2003). For a remarkable description of the barrier option problem see Björk [5] (1998), whose general approach covers a wide class of payoffs. The last thread deals with Asian option and basket options. Their distinctive characteristic is the need to handle sums of geometric Brownian motions. Initial contributions for simpler geometric average problems are from Vorst [51] (1992), and a major development for arithmetic average problems is due to Večeř [44] (2001). This chapter extends and develops previous work by the author on this subject.

3.4 Formula Development

3.4.1 Model Description

The model on which we develop a closed formula can be classified as a multivariate Black–Scholes model. It is a multi-asset model in which all assets are tradable including for example stocks, currencies, precious metals and indexes composed by these.

We choose a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ carrying a n-dimensional standard Brownian motion $\bar{\mathbf{W}}(t)$ under the measure P that generates the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We also have a fixed time horizon T, thus having $\mathcal{F} = \mathcal{F}_T$. We also consider a n-dimensional correlated Brownian motion $\mathbf{W}(t)$ constructed in the usual way. That is, $\mathbf{W}(t) = \Lambda(t)\bar{\mathbf{W}}(t)$ with $\Lambda(t)$ a deterministic matrix process such that, for all t: the rows i of $\Lambda(t)$ have Euclidean norm 1 ($\|\Lambda_i(t)\| = 1$), $\rho(t) = \Lambda(t)\Lambda(t)'$ is a nonsingular positive semi-definite matrix, and $\int_0^T \Lambda_{ij}^2(t)\mathrm{d}t < \infty$. The deterministic correlation matrix process of $\mathbf{W}(t)$ is thus $\rho(t)$. We state without proof that $\mathcal{F}_t = \mathcal{F}_t^{\mathbf{W}}$, as the invertibility of $\rho(t)$ allows the calculation of $\bar{\mathbf{W}}(t)$ given $\mathbf{W}(t)$ and vice-versa.

We assume the existence of $n+1$ assets A_i, and the respective bank accounts B_i where asset A_i may be deposited, with $i = 1, \ldots, n+1$. Each of the accounts yields a return, in units of the same asset, at a deterministic continuosly compounded rate of $r_i(t)$. We assume $\int_0^T |r_i(s)|\mathrm{d}s < \infty$ and $B_i(0) = 1$. Such a rate may be interpreted as an interest rate of a currency or as a repo rate[1] of a stock. Although it is also common also to use this rate to represent dividend payments for individual stocks, we advise against it since dividend payments are typically not payed continuously and are not proportional to the asset price, see [47] for details. Each bank account thus follows the dynamics

$$\mathrm{d}B_i(t) = B_i(t)r_i(t)\mathrm{d}t. \tag{3.1}$$

We furthermore assume the existence of one, and only one, price process for each asset A_i allowing its expression in units of another asset A_j, thus totaling n processes. This structure is usually referred to as a tree structure.

[1] rate paid on a repurchase agreement or stock lending contract.

Though here, the definition of the root (asset) of the tree is not critical, any asset can play that role. What is critical is to have one path, and only one path, to express the price of one asset in terms of any other. Such a structure excludes triangular relationships as for example EUR/USD, USD/JPY and EUR/JPY foreign exchange pairs. We exclude these relationships because they impose restrictions on the volatilities and correlations between the assets, see [22] for details.

Hence, we assume the existence of strictly positive price processes $S_{ij}(t)$, that is the price of one unit of A_i expressed in units of A_j, with the dynamics following the stochastic differential equation (SDE)

$$\mathrm{d}S_{ij}(t) = S_{ij}(t)\left(\mu_{ij}(t)\mathrm{d}t + \sigma_{ij}(t)\mathrm{d}W_{ij}(t)\right), \tag{3.2}$$

where $W_{ij}(t)$ is a component of the n-dimensional correlated Brownian motion $\mathbf{W}(t)$, $\mu_{ij}(t)$ and $\sigma_{ij}(t)$ are deterministic functions of time such that $\int_0^T |\mu_{ij}(t)|\mathrm{d}t < \infty$, $\int_0^T \sigma_{ij}^2(t)\mathrm{d}t < \infty$ and $\sigma_{ij}(t) > 0$ for $0 \leq t \leq T$. Let also $\rho_{ij,kl}(t)$ the correlation between $W_{ij}(t)$ and $W_{kl}(t)$, another component of $\mathbf{W}(t)$, and $\varsigma_{ij,kl}(t)$ their respective covariance $\rho_{ij,kl}(t)\sigma_{ij}(t)\sigma_{kl}(t)$.

Although other setups are also plausible, we choose this one for three reasons: it is general enough to accommodate most exotic options we have encountered, the formulas generated are still manageable, and the volatilities and correlations can be freely specified. Figure 3.1 illustrates a model setup that would underlie the valuation of a typical structured product that depends on several equity indexes spread across the world.

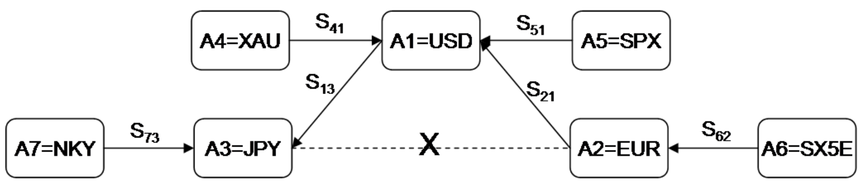


Figure 3.1: Example of market setup. The abbreviations refer to the following: USD to United States dollars, EUR to the euro currency, JPY to the Japanese yen, XAU to the gold ounce, SPX to the S&P500 index, SX5E to the DJ Eurostoxx 50 index, and NKY to the Nikkei index.

It shows a market with seven assets and six prices. It includes the currencies of the three main monetary zones and the most popular indexes of each. The currency pairs $S_{21}(t)$ and $S_{13}(t)$ are the most liquid and are defined according to market standards, EUR/USD and USD/JPY respectively. The prices of the baskets of stocks that compose each of the equity indexes A_5, A_6 and A_7 are naturally expressed in terms of their respective currencies.

We now look for a martingale measure in this market as that would imply, by the so called First Fundamental Theorem (see Björk [5] or Shreve [40] for details), that the model is arbitrage free. As *numéraire* we choose a simple portfolio, composed by one unit of the bank account $B_k(t)$.

To characterize the dynamics of the prices $S_{ij}(t)$ under the measure Q_k, we need to derive the equation that relates the Brownian motions under both measures. As all admissible portfolios are martingales under Q_k when expressed in terms of units of $B_k(t)$, we shall consider first the static portfolios composed only by one unit of $B_i(t)$. The number of units of $B_k(t)$ of such a portfolio is given by

$$1 \cdot R_{ik}(t) = \prod_{h=1}^{n} \frac{B_{i_{h-1}}(t) \left(S_{(i_h,i_{h-1})}(t)\right)^{\lambda_{i_h}}}{B_{i_h}(t)}, \tag{3.3}$$

with n denoting the number of price conversions needed to express $A_i(t)$ in units of $A_k(t)$, i_h is the index of all the assets in the conversion chain (with $i_0 = i$ and $i_n = k$), $S_{(i_h,i_{h-1})}(t)$ the prices between two of these assets (where we use the notation (i, j) instead of ij or ji because we do not know the direction of each price conversion), and λ_{i_h} accounts for the direction of each of the prices, which may be natural ($\lambda_{i_h} = 1$) or inverse ($\lambda_{i_h} = -1$). A price expressed in the natural direction, with respect to the price path from A_i to A_k, is one that multiplies the previous quantity to yield the next. Conversely, a price expressed in the inverse direction is one that divides.

We do not really need to include all terms $B_{i_h}(t)$ and $B_{i_{h-1}}(t)$ in (3.3) since they form a telescopic product yielding $B_i(t)/B_k(t)$. We do so because it simplifies the derivation below, allowing the analysis to be applicable to all possible $B_i(t)$ at the same time.

We now apply the Itô formula to (3.3) and get

$$\mathrm{d}R_{ik}(t) = R_{ik}(t) \sum_{h=1}^{n} d_h(t)\mathrm{d}t + \lambda_{i_h} \sigma_{(i_h,i_{h-1})}(t) \mathrm{d}W_{(i_h,i_{h-1})}(t) \tag{3.4}$$

with

$$d_h(t) = r_{i_{h-1}}(t) - r_{i_h}(t) + \lambda_{i_h} \mu_{(i_h,i_{h-1})}(t) - \frac{\lambda_{i_h} - 1}{2} \sigma^2_{(i_h,i_{h-1})}(t) + \sum_{m=1}^{h-1} \lambda_{i_h} \lambda_{i_m} \varsigma_{(i_h,i_{h-1}),(i_m,i_{m-1})}(t).$$

Analyzing (3.4), it becomes clear that we can turn $R_{ik}(t)$ into a martingale under Q_k if we transform each Brownian motion $W_{(i_h,i_{h-1})}(t)$ in a way that it cancels the drift term $d_h(t)$.

Therefore, we get

$$\mathrm{d}W_{(i_h,i_{h-1})}(t) = \mathrm{d}W_{(i_h,i_{h-1}),k}(t) - \frac{d_h(t)}{\lambda_{i_h} \sigma_{(i_h,i_{h-1})}(t)} \mathrm{d}t, \tag{3.5}$$

with $W_{(i_h,i_{h-1}),k}(t)$ a Brownian motion under the measure Q_k.

We have thus found a martingale measure and therefore our model is arbitrage free. Fortunately, in this search we have found not only a martingale measure but a martingale measure that is unique. That can be easily seen analyzing Equation (3.4). If we consider a $B_i(t)$ that is only one price conversion away from $B_k(t)$, the summation in (3.4) would have only one term. Choosing a $B_i(t)$ that requires two price conversions would add an extra term to the summation and not modify the previous term. Therefore, this transformation not only made the portfolio with one unit of $B_i(t)$ a martingale but it also made every portfolio of one unit of any bank account $B_{i_h}(t)$, a martingale as well. As we have selected an arbitrary branch of the tree structure of assets, such a transformation is valid for all, and therefore it turns all portfolios with only one unit of any bank account into martingales. The uniqueness of the martingale measure implies, by the Second Fundamental Theorem (see Björk [5] or Shreve [40] for details), that the market is complete, i.e., that every contingent claim can be replicated. We have thus got the necessary results to enable the use of arbitrage pricing theory to price contingent claims within this model, which will be done in Section 3.4.3 below.

We now need to apply the transformation to the price processes (3.2) to get the dynamics of $S_{ij}(t)$ under the measure Q_k. For clarity, we consider the cases of natural and inverse prices separately.

For a price that is in the natural direction when one considers the conversion path between $B_i(t)$ and $B_k(t)$ we have $\lambda_{i_h} = 1$ and get

$$\begin{aligned}
\mathrm{d}S_{i_{h-1}i_h}(t) &= S_{i_{h-1}i_h}(t)\left(\left(\mu_{i_{h-1}i_h}(t) - d_h(t)\right)\mathrm{d}t + \sigma_{i_{h-1}i_h}(t)\mathrm{d}W_{i_{h-1}i_h,k}(t)\right) \\
&= S_{i_{h-1}i_h}(t)\left(\left(r_{i_h}(t) - r_{i_{h-1}}(t) - \sum_{m=1}^{h-1}\lambda_{i_m}\varsigma_{i_{h-1}i_h,(i_m,i_{m-1})}(t)\right)\mathrm{d}t\right. \\
&\qquad\left. + \sigma_{i_{h-1}i_h}(t)\mathrm{d}W_{i_{h-1}i_h,k}(t)\right).
\end{aligned} \tag{3.6}$$

For a price that is in the inverse direction when one considers the conversion path between $B_i(t)$ and $B_k(t)$ we have $\lambda_{i_h} = -1$ and get

$$\begin{aligned}
\mathrm{d}S_{i_hi_{h-1}}(t) &= S_{i_hi_{h-1}}(t)\left(\left(\mu_{i_hi_{h-1}}(t) + d_h(t)\right)\mathrm{d}t + \sigma_{i_hi_{h-1}}(t)\mathrm{d}W_{i_hi_{h-1},k}(t)\right) \\
&= S_{i_hi_{h-1}}(t)\left(\left(r_{i_{h-1}}(t) - r_{i_h}(t) - \sum_{m=1}^{h}\lambda_{i_m}\varsigma_{i_hi_{h-1},(i_m,i_{m-1})}(t)\right)\mathrm{d}t\right. \\
&\qquad\left. + \sigma_{i_hi_{h-1}}(t)\mathrm{d}W_{i_hi_{h-1},k}(t)\right),
\end{aligned} \tag{3.7}$$

where we included the term $\sigma^2_{(i_h,i_{h-1})}(t)$ as an extra term in the summation.

Fortunately, both dynamics (3.6) and (3.7) are identical with only h and $h-1$ exchanged. One may therefore write them as

$$\mathrm{d}S_{ij}(t) = S_{ij}(t)\left(\left(r_j(t) - r_i(t) - \sum_{l=1}^{n}\lambda_{j_l}\varsigma_{ij,i_l}(t)\right)\mathrm{d}t + \sigma_{ij}(t)\mathrm{d}W_{ij,k}(t)\right), \tag{3.8}$$

with n denoting the number of price conversions between A_j and A_k, λ_{j_l} the direction of each of the conversions, and i_l the indexes of the price of each conversion.

In the example above, if the EUR bank account is chosen as *numéraire*, the dynamics of the index NKY are

$$\mathrm{d}S_{73}(t) = S_{73}(t)\left((r_3(t) - r_7(t) - (-\varsigma_{73,13}(t) - \varsigma_{73,21}(t)))\,\mathrm{d}t \right.$$
$$\left. + \sigma_{73}(t)\mathrm{d}W_{73,2}(t)\right). \quad (3.9)$$

Now that we have all dynamics of all prices S under one arbitrary martingale measure Q_k, the relevant information concerning the location of A_i in the tree structure is condensed in the summation $\sum_{l=1}^{n} \lambda_{j_l} \varsigma_{ij,i_l}(t)$. Therefore, we can replace the subscripts of S, σ, ρ and ς that tracks both assets linked by one price. Thus, from now on, we replace the subscript ij with a reindexed subscript i that indexes all prices in the model. Furthermore, we will also assign the symbol $d_i(t)$ to the drift term function in (3.8) to write the equation in a more economic form as

$$\mathrm{d}S_i(t) = S_i(t)\left(d_i(t)\mathrm{d}t + \sigma_i(t)\mathrm{d}W_{ik}(t)\right), \quad (3.10)$$

where we also removed the comma on the diffusion term because, from now on, we shall only need one letter to refer to an asset.

3.4.2 Abstract Assets

To allow the closed formula to cover as much option types as possible, we now look for reference values, other than the prices $S_i(t)$, to determine the options payoffs. The reference values we allow are functions of the prices $S_i(t)$ that preserve a set of properties necessary to develop the pricing and hedging formulas. This lead us to the define the following concept

Definition 3.4.1. *An abstract asset $\mathbb{A}_i$ exists whenever we have processes $\mathbb{S}_i(t)$ and $\mathbb{B}_i(t)$, such that*

$$\mathrm{d}\mathbb{S}_i(t) = \mathbb{S}_i(t)\left(d_{\mathbb{S}_i}(t)\mathrm{d}t + \sum_{j=1}^{m} \theta_{\mathbb{S}_i}^{j}(t)\mathrm{d}W_{jk}(t)\right), \quad (3.11)$$

$$\mathrm{d}\mathbb{B}_i(t) = \mathbb{B}_i(t) r_{\mathbb{S}_i}(t)\mathrm{d}t, \quad (3.12)$$

$$\frac{\mathbb{B}_i(t)\mathbb{S}_i(t)}{B_k(t)} \text{ is a martingale under } Q_k, \quad (3.13)$$

with $r_{\mathbb{S}_i}(t)$*,* $d_{\mathbb{S}_i}(t)$ *and* $\theta^j_{\mathbb{S}_i}(t)$ *deterministic functions of time such that* $\int_0^T |r_{\mathbb{S}_i}(t)|\mathrm{d}t < \infty$*,* $\int_0^T |d_{\mathbb{S}_i}(t)|\mathrm{d}t < \infty$ *and* $\int_0^T \left(\theta^j_{\mathbb{S}_i}(t)\right)^2 \mathrm{d}t < \infty$*.*

The price of $\mathbb{A}_i$ *in terms of* A_k *is* $\mathbb{S}_i(t)$*, and its bank account expressed in units of* $\mathbb{A}_i$ *evolves as* $\mathbb{B}_i(t)$*.*

To verify that abstract assets exist, we just need to find a process $\mathbb{S}_i(t)$ that follows (3.11), since setting

$$r_{\mathbb{S}_i}(t) = r_k(t) - d_{\mathbb{S}_i}(t) \tag{3.14}$$

satisfies both (3.12) and (3.13). A trivial example is $\mathbb{S}_i(t) = S_1(t)$.

Definition 3.4.2. *A martingale measure* $Q_{\mathbb{S}_i}$ *for the abstract asset* $\mathbb{A}_i$ *is such that*

$$\frac{X(t_0)}{\mathbb{S}_i(t_0)\mathbb{B}_i(t_0)} = E^{Q_{\mathbb{S}_i}}_{t_0}\left[\frac{X(T)}{\mathbb{S}_i(T)\mathbb{B}_i(T)}\right], \quad T > t_0 \tag{3.15}$$

for any adapted process $X(t)$ *such that* $\frac{X(t)}{B_k(t)}$ *is a* $Q_k -$ *martingale, i.e.,*

$$\frac{X(t_0)}{B_k(t_0)} = E^{Q_k}_{t_0}\left[\frac{X(T)}{B_k(T)}\right]. \tag{3.16}$$

Theorem 3.4.1. *There is one, and only one, martingale measure* $Q_{\mathbb{S}_i}$ *for each abstract asset* $\mathbb{A}_i$*.*

Proof. To prove the theorem we will search for a unique transformation of the sort of (3.5) that makes the process $Z(t) = \frac{X(t)}{\mathbb{S}_i(t)\mathbb{B}_i(t)}$ a martingale. To that end, we use the Martingale Representation Theorem (see Corolary 5.48 of Hunt and Kennedy [26] for details) to find the differential of $M(t) = X(t)/B_k(t)$. In turn, this requires that $\mathcal{F}^{\mathbf{W}}_t = \mathcal{F}^{\mathbf{R}}_t$ with $\mathbf{R}$ the vector of processes in (3.3), the martingales of Q_k. The equality $\mathcal{F}^{\mathbf{W}}_t = \mathcal{F}^{\mathbf{W_k}}_t$ is straight forward as it is clear how to recover $\mathbf{W}(t)$ given $\mathbf{W_k}(t)$ and vice-versa.

The assertion $\mathcal{F}^{\mathbf{R}}_t \subset \mathcal{F}^{\mathbf{W}_k}_t$ is trivial since all $R_{ik}(t)$ are $\mathcal{F}^{\mathbf{W}_k}_t$-measurable by construction. To prove $\mathcal{F}^{\mathbf{W}_k}_t \subset \mathcal{F}^{\mathbf{R}}_t$ we adapt a similar proof from Harrison and Kreps [20].

Let A_i be an asset that is only one price conversion away from A_k. That is, $(S_i(t))^{\lambda_i}$ is the number of A_k one unit of A_i is worth. By (3.4) and (3.5), we have under Q_k

$$R_{ik}(t) = R_{ik}(0) + \int_0^t R_{ik}(s)\lambda_i\sigma_i(s)\mathrm{d}W_{ik}(s). \tag{3.17}$$

Define $V_i(t) = R_{ik}(t) - R_{ik}(0) = \int_0^t R_{ik}(s)\lambda_i\sigma_i(s)\mathrm{d}W_{ik}(s)$ and

$$W_N^i(t) = \sum_{m=0}^{2^N-1} \frac{V_i(t_{n+1}) - V_i(t_n)}{R_{ik}(t_n)\lambda_i\sigma_i(t_n)} \tag{3.18}$$

for integer N and $t_n = nt/2^N$. Thus, $W_N^i(t)$ is $\mathcal{F}_t^{\mathbf{R}}$-measurable and $W_N^i(t) \to W_{ik}(t)$ a.s. as $N \to \infty$. Thus $W_{ik}(t)$ is $\mathcal{F}_t^{\mathbf{R}}$-measurable. Hence, we have shown that all Brownian motions that drive all prices $S_i(t)$ are $\mathcal{F}_t^{\mathbf{R}}$-measurable.

Let now A_j be an asset that is two price conversions away from A_k. Then we have

$$R_{jk}(t) = R_{jk}(0) + \int_0^t R_{jk}(s)\lambda_i\sigma_i(s)\mathrm{d}W_{ik}(s) + \int_0^t R_{jk}(s)\lambda_j\sigma_j(s)\mathrm{d}W_{jk}(s). \tag{3.19}$$

We now define $V_j(t) = R_{jk}(t) - R_{jk}(0) - \int_0^t R_{jk}(s)\lambda_i\sigma_i(s)\mathrm{d}W_{ik}(s)$, which is $\mathcal{F}_t^{\mathbf{R}}$-measurable, and proceed as above to show that all $W_{jk}(t)$ are $\mathcal{F}_t^{\mathbf{R}}$-measurable.

In this constructive fashion, we show that all components of $W(t)$ are $\mathcal{F}_t^{\mathbf{R}}$-measurable and therefore we also have $\mathcal{F}_t^{\mathbf{W}_k} \subset \mathcal{F}_t^{\mathbf{R}}$.

We may now apply the Martingale Representation Theorem that enables us to write the differential of $M(t)$ as

$$\mathrm{d}M(t) = \sum_{g=1}^{n} H_g(t)\mathrm{d}R_{gk}(t) \tag{3.20}$$

with $M(0) = X(0)/B_k(0)$ and all $H_g(t)$ are $\mathcal{F}_t^{\mathbf{R}}$-adapted processes.

Since $Z(t) = \frac{M(t)B_k(t)}{\mathbb{S}_i(t)\mathbb{B}_i(t)}$, making use of the Itô formula (and some patience) we are able to write its differential as

$$\mathrm{d}Z(t) = \frac{B_k(t)}{\mathbb{S}_i(t)\mathbb{B}_i(t)} \left(\sum_{g=1}^{n} H_g(t) R_{gk}(t) \sum_{h=1}^{n_g} \lambda_{g_h} \sigma_{(g_h,g_{h-1})}(t) \right. \tag{3.21a}$$

$$\left. \left(\mathrm{d}W_{(g_h,g_{h-1})k}(t) - \sum_{j=1}^{m} \theta^j_{\mathbb{S}_i}(t) \rho_{(g_h,g_{h-1})j}(t) \mathrm{d}t \right) \right) \tag{3.21b}$$

$$+Z(t) \left(\left(r_k(t) - r_{\mathbb{S}_i}(t) - d_{\mathbb{S}_i}(t) + \sigma^2_{\mathbb{S}_i}(t) \right) \mathrm{d}t - \sum_{j=1}^{m} \theta^j_{\mathbb{S}_i}(t) \mathrm{d}W_{jk}(t) \right). \tag{3.21c}$$

with $\sigma^2_{\mathbb{S}_i}(t) = \sum_{j=1}^{m} \sum_{l=1}^{m} \theta^j_{\mathbb{S}_i}(t) \theta^l_{\mathbb{S}_i}(t) \rho_{jl}(t)$.

To cancel the drift of $\mathrm{d}Z(t)$ we start be looking for a transformation that would cancel the drift terms in (3.21b). The transformation is thus

$$\mathrm{d}W_{(g_h,g_{h-1})k}(t) = \mathrm{d}W_{(g_h,g_{h-1})\mathbb{S}_i}(t) + \sum_{j=1}^{m} \theta^j_{\mathbb{S}_i}(t) \rho_{(g_h,g_{h-1})j}(t) \mathrm{d}t. \tag{3.22}$$

As it turns out, this is all we need. The drift in (3.21c) also cancels out using transformation (3.22) and the relationship (3.14). □

Hence, we have not only shown that there is one, and only one, measure but we were also able to describe it by transformation (3.22).

We are now able to write the dynamics of any abstract asset price $\mathbb{S}_j(t)$ under $Q_{\mathbb{S}_i}$ as

$$\mathrm{d}\mathbb{S}_j(t) = \mathbb{S}_j(t) \left(\left(d_{\mathbb{S}_j}(t) + \zeta_{\mathbb{S}_j\mathbb{S}_i}(t) \right) \mathrm{d}t + \sum_{j=1}^{m} \theta^j_{\mathbb{S}_j}(t) \mathrm{d}W_{j\mathbb{S}_i}(t) \right), \tag{3.23}$$

with $\zeta_{\mathbb{S}_j\mathbb{S}_i}(t) = \sum_{g=1}^{m_j} \sum_{l=1}^{m_i} \theta^g_{\mathbb{S}_j}(t) \theta^l_{\mathbb{S}_i}(t) \rho_{gl}(t)$.

Taking advantage of the fact that abstract assets prices also include the stock prices $S_j(t)$ themselves, henceforth we shall only consider abstract assets prices $\mathbb{S}_j(t)$ in our analysis.

Furthermore, in the sequel we shall only be taking expectations of the processes $\mathbb{S}_j(t)$ and functions thereof. For these purposes the knowledge of the distribution of all $\mathbb{S}_j(t)$ for all $0 < t \leq T$ is sufficient. Thus, to ease the notation and the analysis below, we consider the processes

$$\mathrm{d}\mathbb{S}_j(t) = \mathbb{S}_j(t)\left(\left(d_{\mathbb{S}_j}(t) + \zeta_{\mathbb{S}_j\mathbb{S}_i}(t)\right)\mathrm{d}t + \sigma_{\mathbb{S}_j}(t)\mathrm{d}W^*_{\mathbb{S}_j\mathbb{S}_i}(t)\right), \tag{3.24}$$

with $W^*_{\mathbb{S}_j\mathbb{S}_i}(t)$ $Q_{\mathbb{S}_i}$-Brownian motions and, for any two abstract asset prices $\mathbb{S}_j(t)$ and $\mathbb{S}_g(t)$, the quadratic variation

$$\mathrm{d}\left[W^*_{\mathbb{S}_j\mathbb{S}_i}, W^*_{\mathbb{S}_g\mathbb{S}_i}\right](t) = \frac{\zeta_{\mathbb{S}_j\mathbb{S}_g}(t)}{\sigma_{\mathbb{S}_j}(t)\sigma_{\mathbb{S}_g}(t)}\mathrm{d}t \tag{3.25}$$

if $\sigma_{\mathbb{S}_j}(t) \neq 0$ and $\sigma_{\mathbb{S}_j}(t) \neq 0$ and zero otherwise.

Standard stochastic calculus allows us to write the distribution, under $Q_{\mathbb{S}_i}$, of the vector of the logarithms of abstract assets prices $\mathbb{S}(t)$, which follows under both models (3.23) and (3.24)

$$\log(\mathbb{S}(t)) \sim \mathcal{N}\left(\log(\mathbb{S}(0)) + \int_0^t d_{\mathbb{S}}(s) + \zeta_{\mathbb{S}\mathbb{S}_i}(s) - \frac{\sigma^2_{\mathbb{S}}(s)}{2}\mathrm{d}s; \int_0^t \zeta(s)\mathrm{d}s\right). \tag{3.26}$$

with $\mathcal{N}$ the multivariate normal distribution, $d_{\mathbb{S}}(t)$ a vector with elements $d_{\mathbb{S}_j}(t)$, $\zeta_{\mathbb{S}\mathbb{S}_i}(t)$ a vector with elements $\zeta_{\mathbb{S}_j\mathbb{S}_i}(t)$, $\sigma^2_{\mathbb{S}}(t)$ a vector with elements $\sigma^2_{\mathbb{S}_j}(t)$, and $\zeta(t)$ a covariance matrix with entries $\zeta_{\mathbb{S}_j\mathbb{S}_g}(t)$.

We now proceed by specifying the type of abstract assets we will consider in closed formula derivation that follows.

Proposition 3.4.1. *The process*

$$\mathbb{S}_i(t) = \prod_{j=1}^{m}\left(S_{i_j,t_j}(t)\right)^{\alpha_j}, \tag{3.27}$$

where $S_{i_j,t_j}(t)$ is the process $S_{i_j}(t)$ frozen at time t_j, i.e., $S_{i_j}(t \wedge t_j)$ and i_j is an index of an asset, yields an abstract asset $\mathbb{A}_i$.

Proof. By the Itô formula, the differential of $\mathbb{S}_i(t)$ under Q_k is

$$\mathrm{d}\mathbb{S}_i(t) = \mathbb{S}_i(t)\left(d_{\mathbb{S}_i}(t)\mathrm{d}t + \sum_{j=1}^{m}\theta^j_{\mathbb{S}_j}(t)\mathrm{d}W_{jk}(t)\right), \tag{3.28}$$

with

$$d_{\mathbb{S}_i}(t) = \left(\sum_{j=1}^{m} \alpha_j \left(d_{i_j}(t) - \frac{\sigma_{i_j}^2(t)}{2} + \sum_{h=1}^{m} \frac{\alpha_h \varsigma_{i_j i_h}(t)}{2} \mathbb{I}_{\{t<t_h\}}\right)\right) \mathbb{I}_{\{t<t_j\}}, \quad (3.29)$$

$$\theta_{\mathbb{S}_j}^j(t) = \alpha_j \sigma_{i_j}(t) \mathbb{I}_{\{t<t_j\}}. \quad (3.30)$$

both deterministic and obeying the respective integral conditions. □

Before we conclude this subsection, we would like to make a remark on how these abstract assets fit the arbitrage theory framework. As in Björk [5], Chapter 24, arbitrage theory requires that the *numéraire* of a given model definition must be a *traded* asset. Clearly these abstract assets are not traded *per se*, and they cannot be replicated by any self-financing portfolio. However, we were able to find the $Q_{\mathbb{S}_i}$ measure in which all the linear combinations of abstract assets prices, i.e., "portfolios of abstract assets" expressed in units of B_k, are martingales when translated into units of $\mathbb{B}_i$, i.e., when multiplied $\frac{B_k(t)}{\mathbb{B}_i(t)\mathbb{S}_i(t)}$.

3.4.3 General Contract

Now we need a definition of a general contract, or claim, that should include as many features and existing contracts as possible. Hence, we propose the following payoff definition expressed in terms of asset A_k

$$\Phi_k = \sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(T_i) \mathbb{I}_{C_i}, \quad (3.31)$$

with $c_i \in \mathbb{R}$, $\mathbb{S}_{I_i,t_i}(T_i)$ as the price of an abstract asset $\mathbb{A}_{I_i}$ expressed in terms of A_k, observed at time t_i, to be settled at time T_i, and $\mathbb{I}_{C_i}$ the indicator function of the set C_i that will be defined in Section 3.4.4. For the payoff to be adapted, we need $t_i \leq T_i \leq T$. We note that $\mathbb{S}_{I_i,t_i}(t)$ may be an abstract asset price of the type of Proposition 3.4.1 with all price processes $S_{i_j}(t)$ frozen at time t_i the latest. Thus, $d_{\mathbb{S}_{I_i,t_i}}(t)$, $\sigma_{\mathbb{S}_{I_i,t_i}}(t)$ and $\zeta_{\mathbb{S}_{I_i,t_i}\mathbb{S}_i}(t)$ are zero for $t > t_i$. The return rate $r_{\mathbb{S}_{I_i,t_i}}(t) = r_k(t)$ for $t > t_i$.

As our model is complete and arbitrage free, the price of contract (3.31) is, as usual, the discounted expected payoff under the martingale measure Q_k, thus

$$V(t_0) = \sum_{i=1}^{n} c_i B_k(t_0) E_{t_0}^{Q_k} \left[\frac{\mathbb{S}_{I_i,t_i}(T_i)\mathbb{I}_{C_i}}{B_k(T_i)} \right], \tag{3.32}$$

where $E_{t_0}^{Q_k}$ is the conditional expectation, under the measure Q_k, conditioned on the σ-algebra $\mathcal{F}_{t_0}$. We also used the fact that the conditional expectation is a linear operator to interchange it with the summation.

For each term of the summation we may write, with $V(t_0) = \sum_{i=1}^{n} v_i(t_0)$,

$$\frac{v_i(t_0)}{B_k(t_0)} = E_{t_0}^{Q_k} \left[c_i \frac{\mathbb{S}_{I_i,t_i}(T_i)\mathbb{I}_{C_i}}{B_k(T_i)} \right], \tag{3.33}$$

which is a martingale by definition of Q_k.

We now translate the price and the payoff expressed in units of B_k in units of the process $\mathbb{B}_{I_i,t_i}(T_i)$. These new quantities are martingales under the measure $Q_{\mathbb{S}_{I_i,t_i}}$, and therefore

$$\frac{v_i(t_0)}{B_k(t_0)} \frac{B_k(t_0)}{\mathbb{S}_{I_i,t_i}(t_0)\mathbb{B}_{I_i,t_i}(t_0)} = E_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}} \left[c_i \frac{\mathbb{S}_{I_i,t_i}(T_i)\mathbb{I}_{C_i}}{B_k(T_i)} \frac{B_k(T_i)}{\mathbb{S}_{I_i,t_i}(T_i)\mathbb{B}_{I_i,t_i}(T_i)} \right]. \tag{3.34}$$

This step can be view as a change of *numéraire* from B_k to $\mathbb{B}_{I_i,t_i}$ as in Geman *et al.* [16]. However, despite the similarity, this is not a standard change of *numéraire* because neither $\mathbb{S}_{I_i,t_i}(T_i)$ is the price of a traded asset nor is $\mathbb{B}_{I_i,t_i}(T_i)$ a portfolio of tradable assets. Similar measures may be found in Carr [11] and Björk and Landén [6][2].

Canceling terms and rearranging we get

$$V(t_0) = \sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(t_0)\mathbb{B}_{I_i,t_i}(t_0) E_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}} \left[\frac{\mathbb{I}_{C_i}}{\mathbb{B}_{I_i,t_i}(T_i)} \right]. \tag{3.35}$$

Additionally, we know that $\mathbb{B}_i$ is a deterministic processes under the meas-

[2]The author wishes to thank Prof. Tomas Björk for his advice on this issue.

ure $Q_{\mathbb{S}_{I_i,t_i}}$ and can thus be taken out of the expectation, yielding

$$\begin{aligned} V(t_0) &= \sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(t_0) \frac{\mathbb{B}_{I_i,t_i}(t_0)}{\mathbb{B}_{I_i,t_i}(T_i)} P_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}}(C_i) \\ &= \sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(t_0) \exp\left\{ -\int_{t_0}^{T_i} r_{\mathbb{S}_{I_i,t_i}}(u) \mathrm{d}u \right\} P_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}}(C_i), \end{aligned} \tag{3.36}$$

where $P_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}}(C_i)$ is the probability of the set C_i, under the risk neutral measure $Q_{\mathbb{S}_{I_i,t_i}}$, and considering the prices at time t_0.

However, in general the expression on the right hand side of (3.36) does not lead to a closed formula and may require numerical integration. Hence, we need to impose some restrictions on the shape of the set C_i to make sure the probability terms $P_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}}(C_i)$ can be evaluated using a closed form expression. More specifically, we will constrain the set C_i in a way that guarantees that $P_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}}(C_i)$ can be evaluated by a sum of multivariate normal cumulative distribution functions.

3.4.4 Set Definition and its Probability

Proposition 3.4.2. *Let the set C_i be of the form*

$$\bigcap_{l=1}^{m_i} \left\{ \frac{\mathbb{S}_{I_{l_u},t_{l_u}}(T_i)}{\mathbb{S}_{I_{l_d},t_{l_d}}(T_i)} < h_l \right\}, \tag{3.37}$$

with I_{l_u}, I_{l_d} denoting the indexes of abstract assets, $h_l \geq 0$ and t_{l_u}, $t_{l_d} \leq T_i$.

Then $P_t^{Q_{\mathbb{S}_{I_i,t_i}}}(C_i)$ is of the form

$$\mathcal{N}_{m_i}^{\mathbb{S}_{I_i,t_i}}(v; \phi, \Sigma), \tag{3.38}$$

with $\mathcal{N}_{m_i}$ denoting the m_i-dimensional multivariate normal cumulative distribution function, with covariance matrix Σ and mean vector ϕ, evaluated at vector v.

Proof. By (3.27) and recalling that all $\mathbb{S}_i(t)$ are positive by definition, we have

$$\bigcap_{l=1}^{m_i} \left\{ \frac{\mathbb{S}_{I_{l_u},t_{l_u}}(T_i)}{\mathbb{S}_{I_{l_d},t_{l_d}}(T_i)} < h_l \right\} = \tag{3.39}$$

$$\bigcap_{l=1}^{m} \left\{ \log\left(\mathbb{S}_{I_{l_u},t_{l_u}}(T_i)\right) - \log\left(\mathbb{S}_{I_{l_d},t_{l_d}}(T_i)\right) < \log\left(h_l\right) \right\}. \tag{3.40}$$

By (3.26) we know that the logs all abstract assets prices are jointly normally distributed. Being a linear combination of two jointly normal random variables, the random variable $X_l = \log\left(\mathbb{S}_{I_{l_u},t_{l_u}}(T_i)\right) - \log\left(\mathbb{S}_{I_{l_d},t_{l_d}}(T_i)\right)$ is also normally distributed. Furthermore, all X_l, with $l = 1, \ldots, m_i$ are jointly normally distributed. We now calculate their means and the respective covariance matrix using (3.26).

The vector of means $\phi = [\phi_1, \ldots, \phi_{m_i}]^T$ is given by

$$\begin{aligned} \phi_l = {} & \log\left(\mathbb{S}_{I_{l_u},t_{l_u}}(t_0)\right) + \int_{t_0}^{T_i} d_{\mathbb{S}_{I_{l_u},t_{l_u}}}(s) + \zeta_{\mathbb{S}_{I_i,t_i}\mathbb{S}_{I_{l_u},t_{l_u}}}(s) - \frac{\sigma^2_{\mathbb{S}_{I_{l_u},t_{l_u}}}(s)}{2} \mathrm{d}s \\ & - \left(\log\left(\mathbb{S}_{I_{l_d},t_{l_d}}(t_0)\right) + \int_{t_0}^{T_i} d_{\mathbb{S}_{I_{l_d},t_{l_d}}}(s) + \zeta_{\mathbb{S}_{I_i,t_i}\mathbb{S}_{I_{l_d},t_{l_d}}}(s) - \frac{\sigma^2_{\mathbb{S}_{I_{l_d},t_{l_d}}}(s)}{2} \mathrm{d}s \right). \end{aligned} \tag{3.41}$$

Let us denote Σ_{ef}, the elements of Σ, with $e, f = 1, \ldots, m_i$. The covariance between two of the random variables X_e, X_f is, by definition,

$$\Sigma_{ef} = E_{t_0}^{Q_{\mathbb{S}_{I_i,t_i}}} \left[\left(X_e - \phi_e\right)\left(X_f - \phi_f\right) \right], \tag{3.42}$$

which yields, after some simple algebra,

$$\begin{aligned} \int_{t_0}^{T_i} & \zeta_{\mathbb{S}^e_{I_{l_u},t_{l_u}} \mathbb{S}^f_{I_{l_u},t_{l_u}}}(s) - \zeta_{\mathbb{S}^e_{I_{l_u},t_{l_u}} \mathbb{S}^f_{I_{l_d},t_{l_d}}}(s) \\ & - \zeta_{\mathbb{S}^e_{I_{l_d},t_{l_d}} \mathbb{S}^f_{I_{l_u},t_{l_u}}}(s) + \zeta_{\mathbb{S}^e_{I_{l_d},t_{l_d}} \mathbb{S}^f_{I_{l_d},t_{l_d}}}(s) \mathrm{d}s, \end{aligned} \tag{3.43}$$

where $\mathbb{S}^e_{\cdot}$ and $\mathbb{S}^f_{\cdot}$ are the assets in conditions $l = e$ and $l = f$ respectively. It is worth noting that the covariance is the same whatever the measure under which the expectation is taken. This follows the well known fact that changes of martingale measures only modify the location of the distribution and not its shape.

The variance of X_e is obtained with $f = e$. The elements of the vector v are $v_l = \log(h_l)$, with $l = 1, \ldots, m_i$. □

3.4.5 Pricing Formula

Proposition 3.4.2 together with Equation (3.36) allow us to write the following:

Theorem 3.4.2. *The arbitrage free price of the claim with payoff* Φ_k *as in* (3.31) *and the sets* C_i *as in* (3.37) *can be calculated using the formula*

$$V(t_0) = \sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(t_0) \exp\left\{ - \int_{t_0}^{T_i} r_{\mathbb{S}_{I_i,t_i}}(u)\mathrm{d}u \right\} \mathcal{N}_{m_i}^{\mathbb{S}_{I_i,t_i}}(v; \phi, \Sigma). \tag{3.44}$$

Finally, we may also consider the complement of sets C_i in Proposition 3.4.2, as we have $P_t^{Q_{\mathbb{S}_{I_i,t_i}}}\left(\overline{C_i}\right) = 1 - \mathcal{N}_{m_i}^{\mathbb{S}_{I_i,t_i}}(v; \phi, \Sigma)$, by the properties of cumulative distribution functions.

3.5 The Greeks

Developing a closed pricing formula has immediate benefits when it comes to pricing the claims and also opens the possibility of allowing the calculation of the quantities relevant for hedging strategies and risk management, i.e., the Greeks[3] or partial derivatives, also by closed formulas. This assumes special importance as the numeric methods to calculate the price typically show significant degradation when used to evaluate partial derivatives.

The approach we take to calculate the relevant partial derivatives relies on the works of Carr [11] and of Reiß and Wystup [37]. The first paper shows how to calculate spatial derivatives, i.e., derivatives with respect to the asset prices, by deriving the payoff function instead of the pricing formula. The second enables us to write the derivatives with respect to the other parameters in the model as functions of the spatial derivatives, in particular with respect to correlation parameters.

[3]The derivatives of the option price are usually called *Greeks* because Greek alphabet letters are commonly used to denote them.

We start by writing the partial differential equation (PDE) implicit in the pricing Formula (3.32) by using the Feynman-Kac theorem

$$V_t + \frac{1}{2}\sum_{i,j=1}^{n} \zeta_{\mathbb{S}_{I_i}\mathbb{S}_{I_j}} \mathbb{S}_{I_i}\mathbb{S}_{I_j} V_{\mathbb{S}_{I_i}\mathbb{S}_{I_j}} + \sum_{i=1}^{n} d_{\mathbb{S}_{I_i}} \mathbb{S}_{I_i} V_{\mathbb{S}_{I_i}} = r_k V, \tag{3.45}$$

where we removed the parameters of all the functions and processes to promote clarity and also the freeze time subscript making $\mathbb{S}_{I_i,t_i}(t) = \mathbb{S}_{I_i}$. n is the number of abstract assets in the model and the subscripts of V denote partial derivatives. See Björk [5] for details.

If we derive PDE (3.45) with respect to any $\mathbb{S}_{I_i}$ we get a PDE for the derivative function, the quantity needed for delta-hedging the claim. Consecutive derivations yield PDEs for all higher order spatial derivatives.

We now need to write the PDE (3.45) derivative with respect to an arbitrary sequence of variables. Hence, we write it as

$$V_{tD_p} + \frac{1}{2}\sum_{i,j=1}^{n} \zeta_{\mathbb{S}_{I_i}\mathbb{S}_{I_j}} \mathbb{S}_{I_i}\mathbb{S}_{I_j} V_{\mathbb{S}_{I_i}\mathbb{S}_{I_j}D_p} + \sum_{i=1}^{n} a_i(D_p) \mathbb{S}_{I_i} V_{\mathbb{S}_{I_i}D_p} = b(D_p)\, V_{D_p}, \tag{3.46}$$

with D_p denoting the sequence of derivations, formally $D_p = \prod_{h=1}^{p} \mathbb{S}_{I_h}$, and I_h an index of an abstract asset in the model. Additionally, a_i and b are functions of time t defined by

$$a_i(D_p) = d_{\mathbb{S}_{I_i}} + \sum_{h=1}^{p} \zeta_{\mathbb{S}_{I_i}\mathbb{S}_{I_h}}, \quad i = 1, \ldots, n, \tag{3.47}$$

$$b(D_p) = r_k - \sum_{h=1}^{p} d_{\mathbb{S}_{I_h}} - \sum_{f=1}^{p}\sum_{g=f+1}^{p} \zeta_{\mathbb{S}_{I_f}\mathbb{S}_{I_g}}. \tag{3.48}$$

We list a_i and b in Table 3.1 for the first and second order derivatives.

It is worth noting that a_i and b of the first order derivatives recover the trend of all $\mathbb{S}_{I_i}$s under the measure $Q_{\mathbb{S}_{I_1}}$ in Equation (3.24) and the deposit rate $r_{\mathbb{S}_{I_1}}$ in Equation (3.14), respectively. The volatilities and covariances are trivially recovered since they do not change. Hence, the measure under which we should take the expectation of the first derivative of the payoff, with

Table 3.1: Process trends and discount rate implicit in the PDE of the first and second derivative of the price function.

	$p=0$	$p=1$	$p=2$
D_p	1	$\mathbb{S}_{I_1}$	$\mathbb{S}_{I_1}\mathbb{S}_{I_2}$
$a_i(D_p)$	$d_{\mathbb{S}_{I_i}}$	$d_{\mathbb{S}_{I_i}} + \zeta_{\mathbb{S}_{I_i}\mathbb{S}_{I_1}}$	$d_{\mathbb{S}_{I_i}} + \zeta_{\mathbb{S}_{I_i}\mathbb{S}_{I_1}} + \zeta_{\mathbb{S}_{I_i}\mathbb{S}_{I_2}}$
$b(D_p)$	r_k	$r_k - d_{\mathbb{S}_{I_1}}$	$r_k - d_{\mathbb{S}_{I_1}} - d_{\mathbb{S}_{I_2}} - \zeta_{\mathbb{S}_{I_1}\mathbb{S}_{I_2}}$

respect to $\mathbb{S}_{I_1}$, is the measure where $\mathbb{S}_{I_1}$ itself is the reference abstract asset, i.e., $Q_{\mathbb{S}_{I_1}}$.

To apply the Feynman-Kac theorem, now in the reverse direction as above, to the PDE (3.46), all we need is to calculate the respective boundary condition. We do so on a term by term basis of contract Function (3.31) and get

$$\frac{\partial^p \Phi_k}{\partial D_p} = \sum_{i=1}^{n} c_i \frac{\partial^p \left(\mathbb{S}_{I_i,t_i}(T_i)\mathbb{I}_{C_i}\right)}{\partial D_p}. \tag{3.49}$$

We can now proceed to write an expression for the spatial derivatives.

Theorem 3.5.1. *Spatial derivatives of the pricing function of the* $V(t(0))$ *are given by the expression*

$$\frac{\partial^p V(t_0)}{\partial D_p(t_0)} = \sum_{i=1}^{n} c_i \exp\left\{-\int_{t_0}^{T_i} r_{D_p}(u)\mathrm{d}u\right\} E_{t_0}^{Q_{D_p}}\left[\frac{\partial^p \mathbb{S}_{I_i,t_i}(T_i)\mathbb{I}_{C_i}}{\partial D_p}\right]. \tag{3.50}$$

with $r_{D_p}(t) = b(D_p)(t)$, *and*

$$\mathrm{d}\mathbb{S}_{I_i,t_i}(t) = \mathbb{S}_{I_i,t_i}(t)\left(a_i(D_p)(t)\mathrm{d}t + \sigma_{\mathbb{S}_{I_i,t_i}}(t)\mathrm{d}W_{\mathbb{S}_{I_i,t_i}D_p}(t)\right). \tag{3.51}$$

Despite its condensed look, this formula harbors some amount of complexity. To clarify, and for completeness, we write the evaluation formulas for the first order derivatives. We start by writing explicitly the first derivative of the contract function. To consider only sets C_i that yield closed form solutions, we use the definition in Proposition 3.4.2 as a product of indicator functions. Thus,

$$\mathbb{I}_{C_i} = \prod_{l=1}^{m_i} \mathbb{I}_{\left\{\frac{\mathbb{S}_{I_{l_u},t_{l_u}}(T_i)}{\mathbb{S}_{I_{l_d},t_{l_d}}(T_i)} < h_l\right\}}. \tag{3.52}$$

To derive the contract function, all we need is to apply the product rule and to recall that

$$\frac{\partial \mathbb{I}_{\left\{\frac{a}{b}<c\right\}}}{\partial b} = \delta(a-bc), \quad \text{and} \quad \frac{\partial \mathbb{I}_{\left\{\frac{a}{b}<c\right\}}}{\partial a} = -\delta(a-bc), \tag{3.53}$$

with $\delta(x)$ the Dirac delta function[4].

We find

$$\begin{aligned}
\frac{\partial \Phi_k}{\partial \mathbb{S}_{I_x,t_x}(T_i)} &= \sum_{i=1}^{n} c_i \Bigg(\mathbb{I}_{\{(I_x,t_x)=(I_i,t_i)\}} \mathbb{I}_{C_i} \\
&+ \mathbb{S}_{I_i,t_i}(T_i) \Bigg(\sum_{j=1}^{m_i} \mathbb{I}_{\left\{(I_x,t_x)=(I_{j_d},t_{j_d})\right\}} \delta\left(\mathbb{S}_{I_{j_u},t_{j_u}}(T_i) - h_j \mathbb{S}_{I_{j_d},t_{j_d}}(T_i)\right) \\
&\qquad \prod_{l\neq j}^{m_i} \mathbb{I}_{\left\{\frac{\mathbb{S}_{I_{l_u},t_{l_u}}(T_i)}{\mathbb{S}_{I_{l_d},t_{l_d}}(T_i)}<h_l\right\}} \\
&\quad - \sum_{j=1}^{m_i} \mathbb{I}_{\{(I_x,t_x)=(I_{j_u},t_{j_u})\}} \delta\left(\mathbb{S}_{I_{j_u},t_{j_u}}(T_i) - h_j \mathbb{S}_{I_{j_d},t_{j_d}}(T_i)\right) \\
&\qquad \prod_{l\neq j}^{m_i} \mathbb{I}_{\left\{\frac{\mathbb{S}_{I_{l_u},t_{l_u}}(T_i)}{\mathbb{S}_{I_{l_d},t_{l_d}}(T_i)}<h_l\right\}} \Bigg)\Bigg).
\end{aligned}$$

Taking advantage of the fact that $Q_{D_p} = Q_{\mathbb{S}_{I_x,t_x}}$, the first order derivative formula turns out to be

[4]The Dirac delta function is characterized by the two properties

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x) \mathrm{d}x = 1.$$

$$\frac{\partial V(t_0)}{\partial \mathbb{S}_{I_x,t_x}(t_0)} = \exp\left\{-\int_{t_0}^{T_i} r_{\mathbb{S}_{I_x,t_x}}(u)\mathrm{d}u\right\} \cdot \tag{3.54a}$$

$$\left(\sum_{i=1}^{n} \mathbb{I}_{\{(I_x,t_x)=(I_i,t_i)\}} c_i \mathcal{N}_{m_i}^{\mathbb{S}_{I_x,t_x}}(v;\phi,\Sigma) \right. \tag{3.54b}$$

$$+\sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(t_0) \sum_{j=1}^{m_i} \mathbb{I}_{\left\{(I_x,t_x)=(I_{j_d},t_{j_d})\right\}} \mathcal{N}_{m_i}^{\mathbb{S}_{I_x,t_x}}\left(v;\phi,\Sigma \,\middle|\, \frac{\mathbb{S}_{I_{j_u},t_{j_u}}(T_i)}{\mathbb{S}_{I_{j_d},t_{j_d}}(T_i)} = h_j\right) \tag{3.54c}$$

$$\left. -\sum_{i=1}^{n} c_i \mathbb{S}_{I_i,t_i}(t_0) \sum_{j=1}^{m_i} \mathbb{I}_{\{(I_x,t_x)=(I_{j_u},t_{j_u})\}} \mathcal{N}_{m_i}^{\mathbb{S}_{I_x,t_x}}\left(v;\phi,\Sigma \,\middle|\, \frac{\mathbb{S}_{I_{j_u},t_{j_u}}(T_i)}{\mathbb{S}_{I_{j_d},t_{j_d}}(T_i)} = h_j\right)\right). \tag{3.54d}$$

For performance reasons, it is important to observe that the probabilities $\mathcal{N}_{m_i}^{\mathbb{S}_{I_x,t_x}}(v;\phi,\Sigma)$ in (3.54b) are also calculated in the context of the pricing function.

In order to recover the derivatives with respect to real asset prices S_l, all we need is to apply the chain rule. Thus,

$$\frac{\partial V(t_0)}{\partial S_l(t_0)} = \sum_{x=1}^{n} \frac{\partial V(t_0)}{\partial \mathbb{S}_{I_x,t_x}(t_0)} \frac{\partial \mathbb{S}_{I_x,t_x}(t_0)}{\partial S_l(t_0)}. \tag{3.55}$$

The first factor in the summation is the one we derived above; the second factor is a simple derivative that either yields zero, if $S_l(t_0)$ no longer affects $\mathbb{S}_{I_x,t_x}(t_0)$, or yields $\mathbb{S}_{I_x,t_x}(t_0)\frac{\alpha_l}{S_l(t_0)}$ otherwise, with α_l as in the Definition (3.27).

Finally, we can use the result from Reiß and Wystup [37] to calculate the derivatives with respect to the other model parameters. As these relationships are valid for contracts with a single maturity date, we write the contract value as $V(t_0) = \sum_{i=1}^{n} v_i(t_0)$, with $v_i(t_0)$ the value of the i^{th} term of the summation in the contract payoff Φ_k in (3.31).

In a model with constant volatilities and correlations, a derivative with respect to the correlation between two asset prices is then given by

$$\frac{\partial v_i(t_0)}{\partial \rho_{jl}} = \sigma_j \sigma_l S_j(t_0) S_l(t_0) \frac{\partial^2 v_i(t_0)}{\partial S_j(t_0) \partial S_l(t_0)} (t^* - t_0), \tag{3.56}$$

with t^* the maximum t, with $t_0 \leq t \leq T_i$, such that both $S_j(t)$ and $S_l(t)$ still influence the payoff function.

The derivative with respect to each volatility is given by

$$\sigma_j \frac{\partial v_i(t_0)}{\partial \sigma_j} = \sum_{l=1}^{n} \rho_{jl} \sigma_j \sigma_l S_j(t_0) S_l(t_0) \frac{\partial^2 v_i(t_0)}{\partial S_j(t_0) \partial S_l(t_0)} (t_l^* - t_0), \tag{3.57}$$

again with t_l^* the maximum t, with $t_0 \leq t \leq T_i$, such that both $S_j(t)$ and $S_l(t)$ still influence the payoff function.

The derivative with respect to the bank account rates is given by

$$\frac{\partial v_i(t_0)}{\partial r_j} = -\sum_{l=1}^{f} S_{jl}(t_0) \frac{\partial v_i(t_0)}{\partial S_{jl}(t_0)} (t_l^* - t_0) + \sum_{l=1}^{g} S_{lj}(t_0) \frac{\partial v_i(t_0)}{\partial S_{lj}(t_0)} (t_l^* - t_0), \tag{3.58}$$

where we reverted to the initial notation in which asset prices are identified by a two index subscript. As before, t_l^* is the maximum t, with $t_0 \leq t \leq T_i$, such that $S_{jl}(t)$ or $S_{lj}(t)$ still influence the payoff function. For the rate of the bank account of the asset in which the option price is expressed, in our case r_k, the extra term $-v_i(t_0)(T_i - t_0)$ must be added.

3.6 Applications

We believe that these results constitute relevant contributions to several practical problems. First of all, it offers a multi-currency, multi-asset model description fit for implementation. The model itself is of the Black–Scholes type with time dependent parameters. The general description of the contract payoff allows for implementations where each instrument is defined through a payoff language. Such a payoff language enables addition of new instruments without additional development of the application. The payoff profiles that are covered by the general form of the contract, in (3.31) and (3.37), are the following: European style vanilla options, exchange options, digital options, forward start and cliquet options, options on the n^{th}-best/worst, options on the discretely observed maximum/minimum, most types of mountain range options, discrete barriers and lookbacks, power options and combinations of these. It allows the use of the following prices as underlying assets: stocks prices denominated on domestic currency, foreign currency (quanto), and foreign currency translated to domestic, as well as geometric averages of stocks prices to produce geometric Asian options or geometric basket options. These last two types are not as common in the industry as their arithmetic counterparts, but their prices are still very useful as control variates, which are very effective in reducing the variance of Monte Carlo simulations of the arithmetic versions. To illustrate the breadth of instruments covered by the contract definition above, we provide below a series of examples.

3.6.1 Performance

As the pricing formula for the contract requires several evaluations of multivariate normal probabilities, it is crucial to weight its computational cost against that of the alternative methods. To calculate the multivariate normal, we used the method developed by Genz [17]. The alternative, as far as we know, is only a Monte Carlo simulation that may, or may not, include variance reduction techniques. However, due to the fact that the convergence of Monte Carlo simulations depends strongly on the payoff profile of the contract, it is impossible to run a performance comparison valid for the contract general form (3.37). Therefore, we shall provide only case based performance analyses in each example of Section 3.6.2. For a performance comparison focused only on the calculation of the multivariate normal probabilities, we refer to Genz [17]. The Genz method also relies on a Monte Carlo simulation but does so in the context of a chain of unidimensional integrals. For this reason, the closed formula prices of the examples below will also show an error term.

In most cases, we have encountered that the closed formula outperforms the Monte Carlo simulation, though to different degrees depending on several factors. The addition of asset prices to the payoff implies an increase in the number of dimensions of both procedures, although it generally weighs heavier on Monte Carlo simulation. The presence of several time points at which stock prices are observed to compose the claim payoff greatly increases the dimensionality of the Monte Carlo simulation, degrading thus its performance.

Several time points also have an impact on the closed formula alternative, as they give rise to highly correlated random variables. The complexity of the payoff may require the evaluation of a large number of summands in (3.44), thus worsening the performance of the closed formula while not necessarily changing the Monte Carlo's performance.

Finally, the integrals of the parameter functions r, σ and ς typically have closed form solutions, as their definition is usually done as piece-wise linear functions or functions that have indefinite integrals. Therefore, its calculation has a residual impact on the overall computation time.

3.6.2 Examples

Our market setup for the cases included in this section is described as follows. The *numéraire* asset is chosen to be the asset in which the options pay off. It is the same for all options, and it yields risk free returns at the rate of 5%. We then have 5 currencies with risk free yields, from the first to the last, of 1%, 2%,...,5% respectively. The price of each currency is expressed in terms of the *numéraire* asset (in the natural direction) and they have volatilities, from the first to the last, of 11%, 12%, ..., 15% respectively. The correlation between the currencies' prices is 20% for all combinations. There are also five equity indexes that yield risk free returns, from the first to the last, of 2%, 4%, ...,10% respectively. The price of each index is expressed (in the natural direction) in terms of the currency with the same cardinal as the index. All indexes start with a price of 100. The volatilities of each index, from the first to the last, are 22%, 24%, ..., 30% respectively. The correlation between any two indexes is 60%. The correlation between any combination of index and currency is 10%.

We consider four options: a cliquet on the first index, a best of five indexes, a discrete lookback on the first index, and a Himalaya on the first three indexes. All options have a maturity of one year, $T = 1$, $t_0 = 0$.

The cliquet option has five periods of equal length. Hence, it can be viewed as a portfolio of a vanilla at-the-money (spot) option plus four forward start at-the-money (spot) options. As vanillas and forward start options involve only one condition, the cliquet option is evaluated instantly. In fact, in this case, the general Formula (3.44) reduces to the known closed formula for cliquets.

$$\Phi = \sum_{n=1}^{5} \Phi_n \left(\frac{n}{5}T\right), \quad \Phi_n \left(\frac{n}{5}T\right) = \left(S_1 \left(\frac{n}{5}T\right) - S_1 \left(\frac{n-1}{5}T\right)\right)^+ .$$

The best of five pays off the difference, if positive, between the maximum of the five index values at maturity and 100.

$$\Phi(T) = \left(\max\left(S_1(T), \ldots, S_5(T)\right) - 100\right)^+ .$$

The discrete lookback pays off the difference, if positive, of the highest stored value of the first index and 100. The index values are stored 12 times during the year at evenly spaced times, starting at 1/12.

$$\Phi(T) = \left(\max\left(S_1\left(\frac{1}{12}T\right), S_1\left(\frac{2}{12}T\right) \ldots, S_1\left(\frac{12}{12}T\right)\right) - 100\right)^+$$

At the end of each period of 1/3 units of time, the Himalaya option pays off the best return of the three first indexes over that period times 100, but only if the best return is positive. The indexes that pay out are not considered for any of the subsequent periods.

$$\Phi = \sum_{n=1}^{3} \Phi_n \left(\frac{n}{3}T\right),$$

$$\Phi_n \left(\frac{n}{3}T\right) = 100 \max\left(0, \eta_{n,1}\frac{S_1\left(\frac{n}{3}T\right)}{S_1\left(\frac{n-1}{3}T\right)}, \eta_{n,2}\frac{S_2\left(\frac{n}{3}T\right)}{S_2\left(\frac{n-1}{3}T\right)}, \eta_{n,3}\frac{S_3\left(\frac{n}{3}T\right)}{S_3\left(\frac{n-1}{3}T\right)}\right),$$

where $\eta_{n,i}$ equals 0 if the asset i has determined the payout of one of the payments at any time $t < \frac{n}{3}T$, and 1 otherwise.

The parameterization of these payoff functions, including the set definition for each of the terms in the payoff summation, is given in the appendix.

To assess the performance of the closed formula, we benchmark the results against a Monte Carlo experiment. The results are shown in Table 3.2. The

Table shows a price estimate and a 99% confidence error bound expressed in percentage of the price estimate. The pricing routines were allowed to run for 10 seconds and for five minutes.

Table 3.2: Formula performance against a Monte Carlo simulation for a selection of exotic options.

	Cliquet		Best of 5		Lookback		Himalaya	
calculation time = 10"								
MC	18.27	0.53%	19.16	0.43%	13.50	1.03%	174.46	0.57%
CF	18.33	—	19.16	0.29%	13.47	1.24%	173.90	0.05%
calculation time = 5'								
MC	18.33	0.10%	19.16	0.08%	13.50	0.19%	173.97	0.10%
CF	18.33	—	19.15	0.05%	13.51	0.23%	173.93	0.01%

The results show that the closed formula is superior in all cases but the lookback. The cliquet case just shows that the general formula is able to produce the already known formulas, namely for vanilla options, exchange options, forward starts, digitals and others of European style that constitute unidimensional problems. The best-of-5 is an example with low correlation between random variables, in this case between different stocks, and only one time horizon, the maturity date. The closed formula increases the precision by a factor of 1.45(=0.427%/0.294%). Hence, considering the rate of convergence of the Monte Carlo, the closed formula is 2.11(=1.45^2) times faster. In the Himalaya case, the performance is even more extreme with the precision increasing by a factor of 10.52(=0.571%/0.543%) or, equivalently, 111(=10.52^2) times faster. The Himalaya is a case in which the closed formula performs particularly well. Even though it requires the evaluation of 63 cumulative probability functions, they are of low dimensionality, 6.9 on average, while the Monte Carlo engine needs to account for a 9 dimensional problem (3 stocks observed at 3 time horizons). In the Lookback case, the dimensionality was 12 for both methods and required the evaluation of 13 cumulative probability functions.

The Lookback result came as a surprise as the closed formula performed worse than in the Monte Carlo simulation. To figure out what was causing the poor performance, we applied two variations to the initial problem. We first diminished the number of observation points to 4 to test if the dimensionality constituted a problem. Then we enlarged the time between two observations from 1 month to 3 months. The results for 5 minute simulations are listed in Table 3.3.

Table 3.3: Formula performance against a Monte Carlo simulation for a discrete lookback option.

	Observations							
	12				4			
	MC		CF		MC		CF	
$\Delta t = 1/12$	13.50	0.19%	13.51	0.23%	6.85	0.09%	6.85	0.10%
$\Delta t = 3/12$	20.99	0.20%	20.99	0.14%	11.33	0.10%	11.34	0.06%

These results lead us to conclude that the closed formula does not provide better performance when the time between observations is small and starts to perform better the larger the time between observations. Small intervals between observations give rise to highly correlated random variables, the asset prices at each observation moment. Such cases are known to carry convergence problems for numerical procedures, and thus it is not surprising that the multivariate normal numerical procedure performance shows degradation. What is surprising though is that it shows worst results than the Monte Carlo simulation, which also suffers from the same effect as it is also a numerical procedure.

3.7 Summary

The results above produce a closed formula that generalizes a large class of multivariate European style options, ranging from the plain vanilla to mountain range options. It does so in a generalized Black-Scholes model, with time dependent parameters, able to cope with an arbitrary number of currency zones. It introduces the concept of abstract assets as an intermediate random variable that allows the formula to cover variations like geometric averages, baskets, asset prices expressed in foreign currencies, and forward start features. In fact, abstract assets are a useful generalization of the asset concept and should be considered as a replacement of plain assets in Monte Carlo engines.

The closed formula performs better than the alternative Monte Carlo simulations in most cases, improving performance by more than 100 times in the most extreme. However, for problems with highly correlated random variables the performance was worse than Monte Carlo's. The examples show that even when the closed formula requires the evaluation of a large number of cumulative probability functions, it still outperforms Monte Carlo.

As a byproduct of the definition of the closed formula, a language for option payoff definition arises. This language acquires extreme importance in industrial systems as it enables the decoupling of the payoff definition function from the pricing function. Hence, the pricing function is able to price any option as long as its payoff is expressible in terms of the language.

Future research should focus on including other features of options for which there are closed formulas, namely continuous barrier and lookback features. The problem with including barrier options in the general formula above is that it requires the knowledge of the joint distribution of a Brownian motion with time dependent drift and its running maximum. The results on Brownian motion with constant drift are applicable neither to models with time dependent parameters nor to abstract asset's dynamics.

Chapter 4

Ratings of Structured Products and Issuers' Commitments

4.1 Introduction

This chapter analyzes the evolution of the structured products market focusing on the tools available for private investors, on which they rely for the selection process. The selection process is extremely difficult because there is a myriad of products, because of the dynamic nature of the market and market participants' actions, and because of the complexity of many of the products. We consider the existing types of tools, in particular the rating schemes that have been proposed by industry participants to provide guidance to the investor. We propose a set of properties that a rating scheme should show and check whether the existing schemes carry these properties. Our findings suggest that the existing rating schemes do not carry the desired properties. Furthermore, for the purpose of solving a highly indefinite selection process, an effective rating scheme may not exist. In light of this, we propose the introduction of a new quantity, the *floor*, that has a legal and financial meaning, on which issuers can also compete in addition to price and spread. Its acceptance and use would also yield standardization towards investors' interests by excluding some pricing practices and severely limiting others. Even though very little research has been produced in this area, we believe this to be a topic of high importance in establishing guidelines for healthy industry development and regulation that upholds investors' interests. Parts of this chapter were used in [48].

4.2 Motivation

Around the world there is a growing number of securities and contracts issued and written by financial institutions. Their purpose is to offer a customized risk/return profile that suits investors' preferences. These so called structured products[1] are linked to diverse underlying assets and are used by private and institutional investors alike. They cover short, medium and long term products from low risk to high risk and leverage.

In Germany and elsewhere this market shows significant activity with the number of issuers surpassing ten in the most liquid underlyings. For example, in June 2008, the most active German exchange for structured products, Börse Stuttgart's Euwax, reported 33 active issuers and more than 300,000 structured products listed. Other countries, specially in Europe and Australia, have also developed structured product markets with several issuers, thousands of products, and whose liquidity is close to 5% of the country's stock market. In the year 2008 the structured products' exchange traded volume, on the European exchanges members of FESE[2], amounted to €213 billion while equities volume amounted to €3,885 billion. In addition to the exchange traded volume, one should also consider the over the counter transactions of listed and unlisted structured products. These are surely a significant percentage of the total structured products trading, but for which, unfortunately, there are no aggregated statistics.

The key difference between structured products and the standardized derivative contracts, i.e. exchange traded options and futures, is the fact that they are issued as securities. This means that a structured product issue has a definite number of "shares" and is bound to the dynamics of securities trading. These dynamics differ strongly from those of standardized derivative

[1] We shall use the terms *products* and *structured products* interchangeably.
[2] Federation of European Securities Exchanges

contracts specially when selling is concerned. Simply stated, a security can only be sold if it is held (either by previous purchase or borrow), while taking up a selling position in standardized derivative contracts is not hindered by that constraint.

The importance of this difference is clear in light of arbitrage theory. It states that for a claim price (security or contract) to be coherent with the price(s) of its underlying asset(s) (that again may be securities or contracts), it is necessary that an agent be able to sell the claim short, if it is overpriced with respect to its underlying, and to buy it, if it is underpriced. However, in the case of structured products, borrowing is impossible[3] and, consequently, so is short selling. Thus, there is no market force driving the price of an overpriced security towards its arbitrage theory fair price. That is, the price it would have if short selling were possible.

Hence, the consequence of the impossibility of short selling is that the claim may be overpriced but may never be underpriced. However, arbitrage theory only states the overpricing *can* occur, not that it *does* occur. Though, it should come as no surprise that banks require a reward for going through the costs of issuing and maintaining these products and that profit is their true *raison d'être*.

There is some research corroborating this fact by Stoimenov and Wilkens [41] (2005) and [53] (2007), and Wilkens *et al.* [52] (2003) that detail the dynamics of the overpricing over the life cycle of a product. Figure 4.1 shows schematically the various identified behaviors. These authors identify an overpricing margin decay over the life of the product. The first example depicted in the figure would imply that the cost to the investor due to the overpricing decay would be proportional to the length of the investment period. The second example of a pronounced decay after issuance penalizes more the investors who buy products as they are issued than those who do so later on. The last two graphs show an overpricing margin decay driven by the amount of the issue sold to investors at each moment. The figure shows an example in

[3] Borrowing is impossible for several reasons, the most important of which are the unwillingness of the issuers to lend the securities, the dispersion of holders of such products, and the nonexistence of a securities lending market for these securities. Exchange traded funds (ETFs), though being securities, are different from structured products for they have built in, in the fund's by-laws, the borrowing possibility for their market makers.

which the issuer cuts sharply the overpricing margin when most of the issue has been sold, thus locking in the margin drop on the outstanding amount. Although new investors would benefit from the lower overpricing afterwards, there will certainly be a small number of them, either because there are not many more securities of the issue to sell or because the issuer may shift its marketing efforts to other products. If, later still, investors end up selling the securities back, the issuer may then increase the margin and revamp the product's marketing. Although this would benefit the residual investors that still hold the securities, it is a cost that the issuer can assess and, therefore, weigh against the benefit of recycling the product for further sale with a higher overpricing margin.

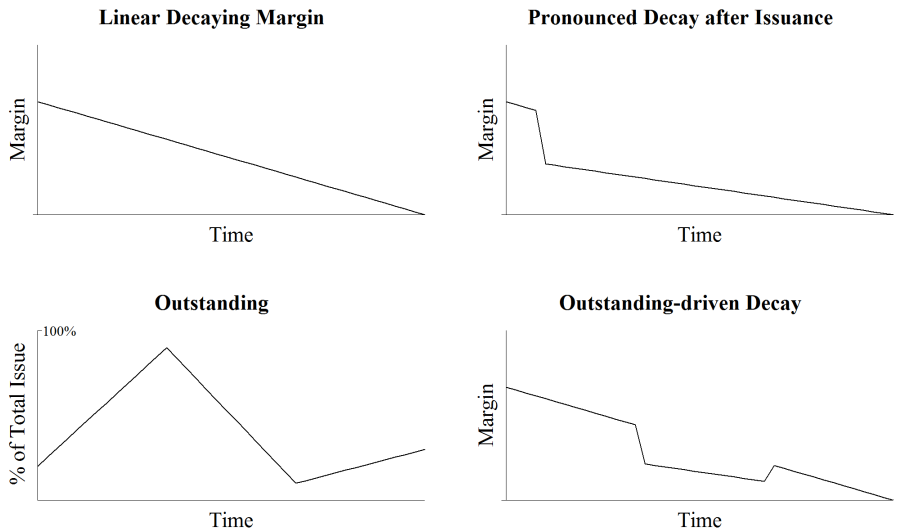


Figure 4.1: Reported patterns in overpricing margin decays.

One may rightfully ask also why does an investor even consider buying securities that are possibly overpriced. There are certainly several reasons for doing so but here we just state one: many investors do not have the size or will to invest in non-biased securities in a way that would replicate the structured product's payoff. Thus, the trade-offs are ones of price versus size or price versus convenience, which are also present in any other market, financial or not. What is not similar to other markets is the inability of an investor, due to lack of information, to choose the best trade-off available. This is the core subject of this section.

So far, the efforts to produce the lacking information have been devoted to the development of rating schemes that classify and order products according to a scale. We devote Section 4.3 to assess whether such ratings do produce relevant information, to enable the choice of the best trade-off, and conclude that they do not. In Section 4.4 we develop a formal analysis of the lack of information problem, propose a solution, and show that it produces relevant information. In Section 4.5 we, give our view of the development of the structured products market in connection with the lack of information problem.

4.3 Rating Schemes

Before analyzing existing rating schemes, we shall first state what we believe are the properties a rating should have in order to be effective in providing relevant information for the selection problem. Hence a rating should be:

- focused - the rating should measure only one well defined target feature;
- easily perceived - should allow for immediate perception of level and order between products;
- informative - produce additional information to set already available;
- impartial - consider only attributes specific to the product itself;
- current - the rating should be updated to reflect changes of the input data;
- robust - the rating should not be hindered by unusual or complex payoff profiles. It should be applicable to whatever product or contract.

These principles are probably easier to agree upon than to fulfill. Even the well-known and established ratings that classify the credit worthiness of issuers like the Moody's, Standard & Poor's or Fitch's credit ratings do not fulfill all the principles above. Common critics are that (i) ratings react slowly to changes in the environment, (ii) rating agencies choose the timing of rating reviews to be cautious about the political impact on the subject country or company, (iii) the fact that rated subjects pay for the rating service and that sole fact may bias the judgment (similar to an auditor's problem), (iv) rating agencies make significant subjective evaluations, and (v) that rating procedures are not robust enough to be standard across all industries and are not easily applicable to complex structures. This last issue has even been severely highlighted in the course of the current financial crisis. These critiques put into question all principles above except the first three; credit ratings are focused solely on measuring the ability to meet future payments, are easily perceived, and seem to add new information. Nevertheless, they are regarded by industry participants and regulators extremely useful classifications.

Given the success of credit rating schemes, several institutions started to apply the same concept to distill the large quantity of information present in the structured products market. Examples of these schemes are the ones from Institut für ZertifikateAnalyse[4] (IZA), Scope Group[5] and European Derivatives Group[6] (EDG), as are issuers' classification schemes. We shall analyze these rating schemes in general as our analysis is focused on the foundations and concepts that underlie these schemes. By keeping the analysis general, we believe that it remains valid not only for existing schemes but also for future ones that address the same problem. For illustration purposes we do take the mentioned schemes as examples to highlight the problems and implications that arise in connection with principles above.

We shall proceed by taking each principle above individually and examine what sort of procedures it rules out.

Focused excludes:

- a target feature created and defined within the rating process itself.

Examples of such target features are the *quality* of a product or its *appropriateness* to a given investor profile. These concepts are defined within the rating process, they do not mean anything outside of it. Examples of proper target features are the ability to meet future payments or the overall cost of a structured product. These concepts exist *a priori*. When costs, credit

[4]www.iza.de
[5]www.scope.de
[6]www.derivatives-group.com

rating, investor risk preferences, etc. are aggregated or composed into a single measure, the result is an arbitrary and meaningless concept that cannot be attached to anything outside the scope of the rating process. Furthermore, if such a concept were to be taken as reference, it would, at best, reflect the preferences of a theoretical investor that, for being so individually specific, no other investor could relate to. Any investor, other than the theoretical, with different preferences with respect to any of the attributes, would not rate the products in the same order or scale as the rating would.

Easily Perceived excludes:

- multidimensional rating assessments;
- use of the same symbols to order distinct groups of products.

A multidimensional rating assessment fails to fulfill its very purpose since it does not map the set of products to an ordered scale. For example, a two dimensional rating, e.g. a measure of cost and another of expected return, can be sorted in an infinite number of ways by linearly combining the two measures. Thus, the ordering is left unresolved and hence the investor still lacks a clear basis for a decision. The ease of perceptiveness also excludes the use of the same symbols on several subsets of products that are not comparable with each other. Although the definition of subsets would simplify the rating process, the reuse of the same symbols would yield and implicit comparison that is not intended by the rating itself.

Informative excludes:

- redundant measurements of target features.

A rating that orders by issue date or maturity date also does not add any information to the existing set. A uniform classification of all products also would not carry any information, as it would not order the products.

Impartial excludes:

- the inclusion of investor preferences;
- the use of valuation models;
- estimated parameters;

- arbitrary or subjective assessments.

The key to understand the impartiality concept can be found in measurement theory. The problem is that the inclusion of measurements of attributes that are not specific to the rated products will change the ordering and evaluation of the products. This inclusion shall never be consensual as it biases the rating towards some of the products. On the other hand, the inclusion of attributes that are specific to the product cannot be argued against for it is the product itself that is being rated. Investor preferences are evidently product non-specific. A valuation model implicitly biases the evaluations towards some products, just consider a barrier and a vanilla option in light of a model that assumes the existence of jumps and one that does not. One can very easily construct an example with two products where the two models yield different orders. The same is true for the inclusion of estimates. Estimates are sample and estimation method dependent and, furthermore, for the calibration of models to market prices, there may be several parameter sets that would calibrate the model. The same is true for arbitrary and subjective assessments that, if changed, would also change the ordering of the products. These assessments include the choice inherent to any aggregation or composition of measurements of different target features.

Current excludes:

- rating revision not linked to input data variability.

An immediate and dramatic consequence of this principle is that the rating should be reassessed every time the input data is refreshed. Thus, if the rating depends on live information, e.g. product price, stock prices or option prices, the rating must also be updated live. Given the nature of the structured products market, where prices are typically overpriced, the price of the structured product is a necessary input to assess the costs embedded in it. The other necessary inputs are the prices of the underlying asset and of related derivatives needed to calculate the theoretical price. Thus, including costs in the rating assessment implies that the rating should be updated as frequently as the product price updates and as often the underlying asset prices updates. For most exchange traded structured products this makes it infeasible to include the cost estimates as an input for the rating process. The same may be said with respect to estimated data, i.e., the rating should be updated as often as the sample that underlies the estimation develops.

Robust excludes:

- rating processes valid only for a specific product type or class;
- any specific model.

The robustness requires that the rating process is a general approach valid for any product that exists or may exist. Different rating processes for different product types raise the problem of comparison across types. Furthermore, the inclusion of future products in the requirement comes from the fact that if they are not included by construction, new products may be created specifically targeted to take advantage of the limitations of the rating process. By the same token, no model may be able to properly evaluate and describe the risk of all types of products. It is, in fact, quite well known that typical models of a given asset class do not preform well when applied to other asset classes.

Still on the robustness principle, one may argue that it is too demanding and should not be considered. Even the well accepted and established credit ratings do not fulfill this principle, so why should the structured products ratings do. We believe it should be upheld for the sole reason that the structured products market has seen a remarkable dynamic in its short history in terms of creation of new types of products. There is also no evidence that this trend is abating.

To complete our analysis of rating schemes for structured products, we check what is left after the exclusions implied by the principles. Although no existing rating scheme belongs to the class of ratings schemes satisfying the principles above, that class may be non-empty. We are unable to produce a formal proof of the existence or the non-existence of rating schemes that fulfill the principles. We can report though that we have not been able to find one. Given the market characteristics, we believe that a rating scheme should consider the overall costs of the products, but that implies the existence of a model to calculate the theoretical price, and that, in turn, is not allowed by the principles. It would also use live data that would imply a live rating. We also fail to see how two products with the same overall costs, but with different payoff profiles, can be ordered in a non-subjective way.

With respect to the rating schemes mentioned above, no single one satisfies the principles above. All three, IZA, Scope and EDG produce scores which do not measure any objective feature of the product. They all rely extensively on aggregation of measurements of both objective and subjective features, e.g., cost, risk, concept of the product, and information produced to describe the product. They consider cost and risk, which in turn require the choice of a model with parameters that need to be estimated and calibrated, and whose assessments are highly ephemeral and not consensual among market participants. Risk is measured by the value at risk only, even though risk may be assessed in multiple ways, yielding each of these its own order. Moreover, computing the value at risk requires a model assumption and possibly parameter assumptions on the distribution of the underlying, which is highly subjective. Though they all exhibit rating reassessment periods that are longer than two weeks. EDG and IZA even consider investor preferences as part of the rating process, as if an investor would know how to describe his or her risk profile in these terms, or check if it would match any of the predefined ones.

Another perspective of the problem is to ask what is the harm in choosing a rating scheme that does not fulfill the principles. Such a choice would foster standardization and all products would still be rated on an equal basis. Even though all principles stand relevant in such a case, the impartiality principle assumes increased importance. If a rating that does not fulfill it is taken as a standard, or even enforced by regulation, that would yield only a standardization towards the (subjective and arbitrary) rating definition and not towards investors' interests. Furthermore, even though investors still need to solve the selection problem on their own, as existing ratings are not effective in ordering products in a meaningful way, they bear their costs. Either payed directly to an agency or embedded in the price of the product (in which the issuer reflects all its costs including the rating related ones), investors end up paying for rating schemes.

Therefore, we believe that a rating scheme is not the answer to bring standardization and informed investor choice to the structured products market. We believe instead that it can be achieved by introducing more tangible information, of the sort of bid-ask spreads and prices.

4.4 Floor

The proposal we describe in this section builds on the work of Stoimenov and Wilkens [41] and [53], and Wilkens *et al.* [52] that describe the dynamics of the price of a structured product over its life cycle. This dynamics exhibit overpricing at issuance, overpricing decaying over the life of the product, significant overpricing drops after issuance, and order flow driven price behavior. The authors rely on the concept of theoretical value and super-hedging boundaries to establish a price reference. This price is then compared with market prices to determine the overpricing and its dynamics.

To formalize these observations, without loss of generality, we assume that the issuer determines its bid and ask prices according to the functions

$$Ask(t) \quad = f^A(t) + Markup^A(t), \tag{4.1}$$

$$Bid(t) \quad = f^B(t) + Markup^B(t), \tag{4.2}$$

where $f^{A,B}(t)$ is the issuer's estimate of the product's theoretical values, using the relevant spread sides for each variable, and $Markup^{A,B}(t)$ are arbitrary functions. The markup functions may depend on any factor, including the total quantity sold of the product up to time t.

The price policies described above generate profit for the issuers that can be decomposed in two parcels: interest and capital gains, denoted by P_i and P_{cg} respectively.

The interest is earned on the sale price markup $Markup^A(t_0)$ only, for we assume $f^A(t_0)$ was spent to purchase the issuer's hedge. If we assume a bank account yielding an overnight rate $r(i)$, the profit accumulated up to time t is just

$$P_i(t) = \left(\prod_{i=[t_0]}^{[t]-1} (1 + r(i)) - 1 \right) \times Markup^A(t_0), \tag{4.3}$$

where t_0 is the trade time and i running from the day of t_0, $[t_0]$ to the day before t, $[t] - 1$.

It is important to note that this parcel of the profit cannot be controlled by the issuer after the initial transaction. On the investor's perspective, the loss, corresponding to the issuer's profit P_i, is included in his or her overall carry cost of holding the product. That cost is, to a large extent, predictable and/or bounded.

For the capital gains we need to write first the capital gains or losses on the whole structured product transaction, that is

$$Ask(t_0) - Bid(t) = f^A(t_0) - f^B(t) + Markup^A(t_0) - Markup^B(t). \quad (4.4)$$

We now assume that $f^A(t_0) - f^B(t)$ is covered by the issuer's hedge. Therefore, the issuer's capital gain attributable to the pricing policy is just

$$P_{cg}(t) = Markup^A(t_0) - Markup^B(t). \quad (4.5)$$

Unlike P_i, P_{cg} does depend on the issuer's pricing policy. The issuer is free to change $Markup^B(t)$ at any point; even set it at negative values[7]. On the investor's perspective, an decrease of $Markup^B(t)$ constitutes a loss. Such a loss is unpredictable in its size and moment.

We now claim that investors are better off if the $Markup^B(t)$ is known in advance, that is, before the investor purchases the product. Better off for the sole reason that investors would have enough information to weigh the total costs of the product against the benefits it brings them. Without the knowledge of $Markup^B(t)$ there is a loose end in the costs side until the product's maturity is reached, time when, by definition, $Markup^B(t)$ is zero.

Accordingly, we proceed with our analysis assuming, from this point on, that the issuer has committed to use the function $Markup^B(t)$ and that it stated on the product's term sheet.

However, there is still one open problem. This analysis has assumed that the issuer's estimates of the product's theoretical value, $f^{A,B}(t)$, are not subject to arbitrary revisions. If they are, the commitment is hollow because $f^{A,B}(t)$ may include not only the issuer's estimate of theoretical value but also hide part of the $Markup^{A,B}(t)$. If that is allowed to happen, we are back to the initial situation, where there is not enough information to determine in advance the issuer's total profit. However, it is not reasonable to ask the issuer to disclose $f^{A,B}(t)$ for it may include trade secrets, be extremely complex

[7] This is equivalent to setting a bid price at a discount to the reference price. Stoimenov and Wilkens [41] provide evidence of this practice.

and unusable by other parties.

Therefore, we need to replace $f^{A,B}(t)$, chosen by the issuer, by new function $h^{A,B}(t)$, independent of the issuer's views, such that Expression (4.5) remains valid. In turn, this means $h^A(t_0) - h^B(t)$ is covered by the issuer's hedge.

This is easily accomplished if there is a static hedge for the structured product. Then $h^{A,B}(t)$ are just the prices of that static hedge portfolio, and $h^A(t_0) - h^B(t)$ is just the result from setting up and unwinding the hedge portfolio.

If there is a static super hedge, and $h^{A,B}(t)$ is the price of the super hedge portfolio, Expression (4.5) is still valid for all t before maturity. At maturity time T, $P_{cg}(T) \geq Markup^A(t_0) - Markup^B(T)$ because the payoff of the super hedge portfolio may be greater than the payoff of the product. However, if the investor sells the structured product before maturity, the additional loss is not incurred.

Thus, if there is a static hedge or a super hedge for the structured product, there are functions $h^{A,B}(t)$ to replace $f^{A,B}(t)$ that are independent from the issuer's assessments. Functions $h^{A,B}(t)$ may even track a dynamic hedge (or super hedge) self-financing portfolio that the issuer is able to trade.

For products that cannot be statically super-hedged there may or may not be functions $h^{A,B}(t)$ to replace $f^{A,B}(t)$. However, if a product that can be decomposed as a portfolio, with its elements only taking positive values, the bid price may track only those elements that can be statically hedged (or super hedged). In such a case, the bid and ask Functions (4.1) and (4.2) would be revised as

$$Ask(t) = h^A(t) + g^A(t) + Markup^A(t), \tag{4.6}$$

$$Bid(t) = h^B(t) + Markup^B(t), \tag{4.7}$$

with $h^{A,B}(t)$ the prices of the static hedge portfolio and $g^A(t) \geq 0$ the issuer's estimate of the price of the elements that are not statically hedgeable.

Hence, be $h^B(t)$ an hedge, super-hedge or sub-hedge, its determination is independent from the issuer's will. Furthermore, as $Markup^B(t)$ is defined before issuance, $Bid(t)$ does not depend on the issuer's will at any point in *time during the life of the product.*

For example, consider a capital guaranteed product composed by a zero coupon bond and an exotic option. Furthermore, assume the issuer considers the Reuters' or Bloomberg's zero coupon bond price estimate as a reference price for $h^{A,B}(t)$. Thus, the product would trade at least at the zero coupon bond price, which is still better than no lower boundary at all. We say at least at zero coupon bond price because, in some situations, the bid will significantly underprice the structured product. The issuer will then, most likely, bid the structured product above the bid commitment to prevent the bid-ask spread from getting too large and to show a more competitive price.

This example shows that issuers may have reasons to bid their structured products above their commitments. It is even likely that issuers do this on a consistent basis on all products, at least by a small amount. The reason being to avoid unintended breaches of the bid price commitments and diminish potential conflicts. This observation is what motivated us to name the issuer's commitment as *floor* and not bid price commitment. From now on we will refer to it only as floor.

The cases we considered so far are cases where it is simple to find a floor and where the floor does not charge the issuer with extra risks. However, the issuer is free to choose the floor, even floors that carry extra risks with them. For the cases we considered above, the term sheet of a structured product should include at least these additional clauses:

- Floor in the Secondary Market: applicable.
- Floor Guarantor: legal name of entity.
- Floor Type: sub-hedge, exact, super-hedge.
- Floor Reference Price: instrument identification and price location.
- Initial Floor Markup: X currency units.
- Floor Markup Daily Decay: Y currency units.

We remark that these rules, on the one hand, exclude some pricing policies reported in Stoimenov and Wilkens [41] and, on the other hand, make some others predictable. Arbitrary pricing policies are excluded as they cannot be described by any function. This is a major difference as the issuer is no longer free to charge investors that hold their structured products in an non-disclosed-in-advance way. Pricing policies that depend on transaction volume or total outstanding quantity would have to be described in advance in a function. Furthermore, its relevant quantities would have to be made public and refreshed at a rate set by the markup definition. This is probably enough to deter issuers from including such rules in the markup definition. Markup functions may still have a non-linear decay, as the reported large decays after issuance. However, as this information is known in advance, investors may postpone the purchase of the product until that period has passed. The floor still leaves room for regular and predictable pricing strategies that are essential for the issuer to be able to profit from its products.

We also make note that, the floor is a new value that should be disseminated through the information network. Just like it is done with the usual set of prices that include the bid and ask prices, the last traded price, the daily maximum and minimum, and the previous sessions' close price.

To conclude this section we cover the most common types of structured products and provide examples of static or dynamic (super) hedges.

We start with a simple example of a very common structured product. The product is called index-tracker and pays off the value of an equity price index on the maturity date. The choice of the index itself as the floor reference is problematic because an equity price index is not a valid static portfolio, for it suffers from cash withdrawals by the amount of the dividends its shares pay. Therefore, if the issuer would choose this index as the Floor Reference Price, the following three clauses should be reviewed to

- Floor Type: super-hedge.
- Initial Floor Markup: implicit in Floor Reference Price.
- Floor Markup Daily Decay: implicit in Floor Reference Price.

Figure 4.2 shows payoffs for vanilla warrants, discount certificates, bonus certificates and turbo warrants. The dashed lines represent the several possible values the product may pay off, depending on a barrier monitoring.

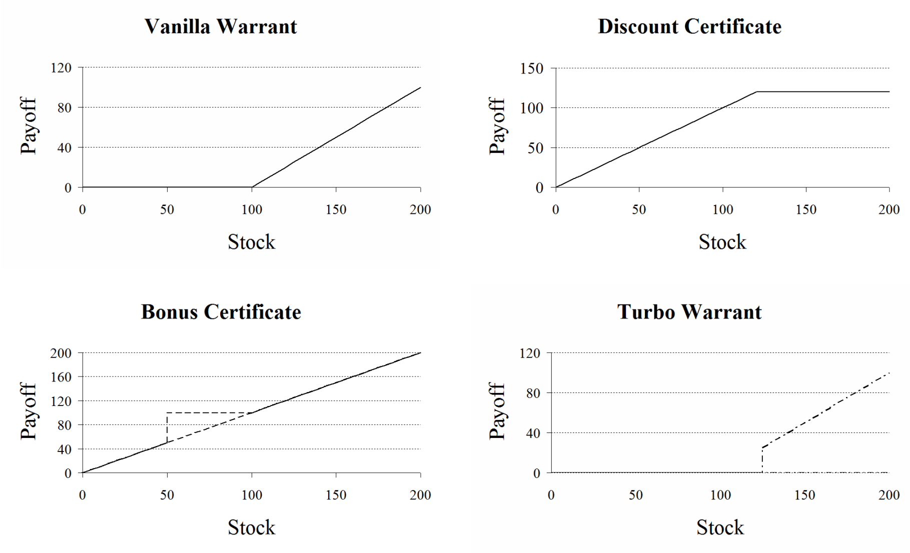


Figure 4.2: Payoff profiles of common structured products.

The proposals of hedge portfolios that follow, assume the existence of exchange traded options and futures on the same underlying asset and with the same maturity as the product. They also assume the availability of risk-free cash deposits for those maturities. A vanilla warrant has as static hedge an exchange traded option on the same underlying with the same strike and maturity. For call warrants, an exchange traded option with a lower strike constitutes a super hedge. The static hedge of a discount certificate is composed by: a short position on an exchange traded call, a long position on a future and a deposit of the total unused cash. A super hedge is obtained with a higher strike call. The price of bonus a certificate is always higher than the value of a portfolio with a future plus a deposit that pays out the contracted future price. A turbo warrant is a barrier option with the barrier located on the in-the-money side of the strike. There is no static hedge for it using the instruments we assumed. This is a typical case where an issuer may choose to assume a floor that introduces additional risks. Consider a turbo warrant call on a stock that does not pay dividends or on a total return index. Assume also a zero interest rate. A possible floor would be the intrinsic value of the turbo warrant, that is, just the difference of underlying price and strike. It is as if the turbo warrant were of American style, exercisable at any moment.

To hedge this new liability, the issuer has to buy one unit of the underlying. If its price never touches the barrier the hedge works. If the price does touch the barrier, the issuer needs to sell the unit of underlying at the barrier level to maximize his result. At least, the issuer needs to sell the hedge above the strike to prevent a loss. However, this may not be possible because stock prices and indexes sometimes evolve in a discontinuous fashion. This is thus the extra risk this floor involves: the risk of not being able to unwind the hedge above the strike price. This is an example where stating a floor would generate a more valuable product and also justify charging a higher price for it.

4.5 Summary

The goal of this analysis is to enable investors to be able to identify the best trade-off available in the structured products market. The trade-off is one of price versus benefit brought to the investor.

To do so we analyze existing tools that claim to contribute to this identification. In particular, we survey the effectiveness of rating schemes to this purpose. We conclude ratings are not effective for they are in essence arbitrary in their definition and, therefore, are only able to produce arbitrary orderings. In order to fairly analyze the schemes, we first present a list of principles we believe every effective rating scheme should have. The rating schemes are then analyzed in light of these principles. We conclude that none of the considered schemes satisfy them. We also report to have failed to develop a rating scheme that would fulfill those principles.

We proceed with a simplification of the problem by removing the benefit to the investor from the analysis. We do so because the investor's benefit is an individual assessment that does not lend itself to modeling. We are then left with the price and realize that, even in price, there is currently no way to have a precise assessment.

Fortunately price is a more tangible concept that allows for modeling and formal description. We offer a framework to study the concept and then develop a proposal that enables a clear assessment of the price side of the trade-off. The proposal is a floor guarantee until the product's maturity. The floor is a quantity that can be freely defined by the issuer but is legally binding. It is the lowest price the issuer can bid at each moment for the structured product. In a way, it constitutes the time continuous counterpart of the discrete time commitments stated in the term sheet. The floor excludes some of the pricing policies referenced in the literature and seriously limits others. However, it still leaves room for regular and predictable pricing strategies that are essential for the issuer to be able to profit from its products. We complete the proposal with examples of application to the most common types of structured products.

Having a guaranteed floor, or the lack of it, quoted by an issuer for a complex product is probably the best rating the investor can get, as the inability or unwillingness to define a floor is itself an indication of the complex balance of risks and rewards underlying the product.

As to the question of whether the floor would make products more expensive, the answer is: not necessarily. On the one hand, any floor is always better than none at all (even if it is theoretically redundant) and thus should imply an extra cost. On the other hand, as explained above, it is impossible to know how expensive existing structured products really are due to their lack of floors. Therefore, the floor introduces a qualitative change that makes the products that bear it incomparable to the ones that do not. Only with a floor is a proper assessment of the cost of a product possible.

Actual rating schemes, as we showed, are not primarily scientific constructions but instead, the result of accumulation of credibility over many years, even centuries, the most elegant scientific formulation would ever replace.

As to the acceptance of these proposals, we do not expect market leader issuers to take them up promptly, as standardization may harm their margins and market share. As in any market, it is more likely that smaller players, that want to grow their market share, use these proposals as a mean to develop products that are objectively superior to the products of their competitors.

Appendix A

Formula Parameterization

The payoff parameterization of the options considered in the examples section above follows expression (3.31) and uses that same notation. The C_i set definition follows expression (3.37) and also uses its notation. For each term in the payoff (3.31), it is still required to select if the set C_i or its complement, $\overline{C_i}$, determines the payment.

The index of the *numéraire* asset is $I_. = 0$. As the examples do not include payments of currency prices, we index the underlying equity indexes with $I_. = 1, \ldots, 5$.

Cliquet Option

This option payoff is composed of 10 terms. The terms follow a structure that can be summarized by iterating $t = 0.2, 0.4, 0.6, 0.8, 1$. In this case the sets C_i have only one condition for all terms, i.e., $m_i = 1$.

Term $i = 1, 3, 5, 7, 9$ *(strike payment for each* t*)*

c_i	I_i	t_i	T_i	Set Complement Flag
-1	1	$t - 0.2$	t	*false*

Set C_i

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
1	$t - 0.2$	1	t	1

Term $i = 2, 4, 6, 8, 10$ *(index price reception for each* t*)*

c_i	I_i	t_i	T_i	Set Complement Flag
1	1	t	t	*false*

Set C_i

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
1	$t-0.2$	1	t	1

Best of 5

This option payoff is composed of 6 terms, five for the reception of each of the five possible maximum index prices at maturity, and one payment of the exercise price 100.

The terms for each of the five index payments follow the following rule. Let $a_i = 1, \dots, 5$ and let $b_{i,1}, \dots, b_{i,4}$ be the elements of the set $\{1, 2, 3, 4, 5\} \setminus a_i$.

The terms for reception of each of the five index prices is parameterized by

Term $i = 1, \dots, 5$ *(index price reception)*

c_i	I_i	t_i	T_i	Set Complement Flag
1	a_i	1	1	*false*

Set C_i

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
0	0	a_i	1	1/100
$b_{i,1}$	1	a_i	1	1
$b_{i,2}$	1	a_i	1	1
$b_{i,3}$	1	a_i	1	1
$b_{i,4}$	1	a_i	1	1

The term of the strike payment is parameterized as

Term i=6 (strike payment)

c_i	I_i	t_i	T_i	Set Complement Flag
-100	0	1	1	*true*

Set C_6

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
a_1	1	0	0	100
a_2	1	0	0	100
a_3	1	0	0	100
a_4	1	0	0	100
a_5	1	0	0	100

Discrete Lookback

This option payoff is composed of 13 terms, 12 for the reception of each of the 12 possible maximum values of the index prices during the life of the option, and one payment of the exercise price 100.

The terms for each of the 12 index payments follow the following rule. Let $u_i = 1/12,\ 2/12, \ldots, 12/12$ and let $v_{i,1}, \ldots, v_{i,11}$ be the elements of the set $\{1/12, 2/12, \ldots, 12/12\} \setminus u_i$.

The terms for reception of each of the 12 possible maximum index prices is parameterized by

Term $i = 1, \ldots, 12$ *(index price reception)*

c_i	I_i	t_i	T_i	Set Complement Flag
1	1	u_i	1	*false*

Set C_i

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
0	0	1	u_i	1/100
1	$v_{i,1}$	1	u_i	1
1	$v_{i,2}$	1	u_i	1
		$\vdots$		
1	$v_{i,11}$	1	u_i	1

The term of the strike payment is parameterized as

Term i=13 (strike payment)

c_i	I_i	t_i	T_i	Set Complement Flag
-100	0	0	1	*true*

Set C_{13}

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
1	u_1	0	0	100
1	u_2	0	0	100
		$\vdots$		
1	u_{12}	0	0	100

Himalaya

This option has payments at three distinct times. For each of the periods that end at these payment dates, only the period return matters and not the accumulated return since inception. We represent each of these returns with an abstract asset price of the form

$$\mathbb{S}_j = S_{j,\frac{n}{3}T} / S_{j,\frac{n-1}{3}T},$$

the ratio of two versions of the same price process frozen at different moments in time. The numerator version, frozen at the end and the denominator at the beginning of the reference return period. As we have 3 assets and 3 return periods, we have 9 abstract assets for all combinations of both.

We shall index the abstract assets that represent the first period return on the first 3 assets by $a_{1,i} = 1, 2, 3$. The returns of the second period returns are indexed as $a_{2,i} = 4, 5, 6$. Finally, the returns of the third period are indexed by $a_{3,i} = 7, 8, 9$, with $i = 1, 2, 3$. We also have $b_{j,i,1}, b_{j,i,2}$, the elements of the set $\{a_{j,1}, a_{j,2}, a_{j,3}\} \setminus a_{j,i}$.

For the payments at the end of the first period, at $t = T/3$ we have 3 terms in the payoff function.

Term $i = 1, 2, 3$ *(index price return reception,* $t = T/3$*)*

c_i	I_i	t_i	T_i	Set Complement Flag
100	$a_{1,i}$	1	1	*false*

Set C_i

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{1,i}$	0	$a_{1,i}$	1	1
$b_{1,i,1}$	1	$a_{1,i}$	1	1
$b_{1,i,2}$	1	$a_{1,i}$	1	1

For the payments at the end of the first period, at $t = 2T/3$ we have 12 terms in the payoff function. For each of the 3 possible return payments there are 4 terms, all with the same asset payment and the 4 sets C_i below.

Term i*, with* $i = 1, 2, 3$ *(index price return reception,* $t = 2T/3$*)*

c_i	I_i	t_i	T_i	Set Complement Flag
100	$a_{2,i}$	2	2	*false*

Sets C_i*, with* g_1, g_2 *the elements of all the possible permutations of the elements of the set* $\{1, 2\}$

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{2,i}$	1	$a_{2,i}$	2	1
$b_{2,i,1}$	2	$a_{2,i}$	2	1
$b_{2,i,2}$	2	$a_{2,i}$	2	1
$a_{1,i}$	1	$b_{1,i,1}$	1	1
$a_{1,i}$	1	$b_{1,i,2}$	1	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{2,i}$	1	$a_{2,i}$	2	1
$b_{2,i,1}$	2	$a_{2,i}$	2	1
$b_{2,i,2}$	2	$a_{2,i}$	2	1
b_{1,i,g_1}	1	$a_{1,i}$	1	1
$a_{1,i}$	1	b_{1,i,g_2}	1	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{2,i}$	1	$a_{2,i}$	2	1
$b_{2,i,1}$	2	$a_{2,i}$	2	1
$b_{2,i,2}$	2	$a_{2,i}$	2	1
$b_{1,i,1}$	1	$a_{1,i}$	1	1
$b_{1,i,2}$	1	$a_{1,i}$	1	1
$a_{1,i}$	1	$a_{1,i}$	0	1

For the payments at the end of the first period, at $t = 3T/3$ we have 48 terms in the payoff function. For each of the 3 possible return payments there are 16 terms, all with the same asset payment and the 16 sets C_i below.

Term i, with $i = 1, 2, 3$ (index price return reception, $t = 3T/3$)

c_i	I_i	t_i	T_i	Set Complement Flag
100	$a_{3,i}$	3	3	*false*

Sets C_i, with g_1, g_2 the elements of all the possible permutations of the elements of the set $\{1, 2\}$, and h_1, h_2 also the elements of a similar permutation.

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$a_{1,i}$	1	$b_{1,i,1}$	1	1
$a_{1,i}$	1	$b_{1,i,2}$	1	1
$a_{2,i}$	2	$b_{2,i,1}$	2	1
$a_{2,i}$	2	$b_{2,i,2}$	2	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$a_{1,i}$	1	$b_{1,i,1}$	1	1
$a_{1,i}$	1	$b_{1,i,2}$	1	1
$a_{2,i}$	2	b_{2,i,g_1}	2	1
b_{2,i,g_2}	2	$a_{2,i}$	2	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$a_{1,i}$	1	b_{1,i,h_1}	1	1
b_{1,i,h_2}	1	$a_{1,i}$	1	1
$a_{2,i}$	2	$b_{2,i,1}$	2	1
$a_{2,i}$	2	$b_{2,i,2}$	2	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$a_{1,i}$	1	b_{1,i,h_1}	1	1
b_{1,i,h_2}	1	$a_{1,i}$	1	1
$a_{2,i}$	2	b_{2,i,g_1}	2	1
b_{2,i,g_2}	2	$a_{2,i}$	2	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$a_{1,i}$	1	$b_{1,i,1}$	1	1
$a_{1,i}$	1	$b_{1,i,2}$	1	1
$b_{2,i,1}$	2	$a_{2,i}$	2	1
$b_{2,i,2}$	2	$a_{2,i}$	2	1
$a_{2,i}$	2	$a_{2,i}$	1	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$a_{1,i}$	1	b_{1,i,g_1}	1	1
b_{1,i,g_2}	1	$a_{1,i}$	1	1
$b_{2,i,1}$	2	$a_{2,i}$	2	1
$b_{2,i,2}$	2	$a_{2,i}$	2	1
$a_{2,i}$	2	$a_{2,i}$	1	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$b_{1,i,1}$	1	$a_{1,i}$	1	1
$b_{1,i,2}$	1	$a_{1,i}$	1	1
$a_{1,i}$	1	$a_{1,i}$	0	1
$a_{2,i}$	2	$b_{2,i,1}$	2	1
$a_{2,i}$	2	$b_{2,i,2}$	2	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$b_{1,i,1}$	1	$a_{1,i}$	1	1
$b_{1,i,2}$	1	$a_{1,i}$	1	1
$a_{1,i}$	1	$a_{1,i}$	0	1
$a_{2,i}$	2	b_{2,i,g_1}	2	1
b_{2,i,g_2}	2	$a_{2,i}$	2	1

I_{l_u}	t_{l_u}	I_{l_d}	t_{l_d}	h_l
$a_{3,i}$	1	$a_{3,i}$	2	1
$b_{3,i,1}$	2	$a_{3,i}$	2	1
$b_{3,i,2}$	2	$a_{3,i}$	2	1
$b_{1,i,1}$	1	$a_{1,i}$	1	1
$b_{1,i,2}$	1	$a_{1,i}$	1	1
$a_{1,i}$	1	$a_{1,i}$	0	1
$b_{2,i,1}$	2	$a_{2,i}$	2	1
$b_{2,i,2}$	2	$a_{2,i}$	2	1
$a_{2,i}$	2	$a_{2,i}$	1	1

Bibliography

[1] Ahlert, M., Gubernatis, G. and Kliemt, H. (2001). Kidney Allocation in Eurotransplant. *Analyse and Kritik* **23**, 156–172

[2] Barone-Adesi, G. and Whaley, R. (1986). The Valuation of American Call Options and the Expected Ex-Dividend Stock Price Decline. *Journal of Financial Economics*, **17**(1), September, 91–111.

[3] Black, F. (1976). The pricing of commodity contracts. *Journal of Financial Economics*, **3**, 167–179.

[4] Berger, E. and Klein, D. (1998). Valuing Options on Dividend-Payment Stocks. *Bloomberg Analytics*, July, 116–120.

[5] Björk, T. (1998).*Arbitrage Theory in Continuous Time*. New York, Oxford University Press.

[6] Björk, T. and Landén, C. (2002). On the Term Structure of Futures and Forward Prices, *Mathematical Finance - Bachelier Congress 2000*, 111–150.

[7] Bos, R., Gairat A. and Shepeleva, A. (2003). Dealing with discrete dividends. *Risk Magazine*, January, 109–112.

[8] Bos, M. and Vandermark, S. (2002). Finessing fixed dividends. *Risk Magazine*, September, 157–158.

[9] Cabral, L. (2000). *Introduction to Industrial Organization*. MIT Press, USA.

[10] Carr, P. (1995). Two extensions to barrier options valuation. *Applied Mathematical Finance*, **2**(3), 173–209.

[11] Carr, P. (2001) Deriving Derivatives of Derivative Securities. *Journal of Computational Finance*, Winter 2000/2001, **4**(2), 5–29.

[12] Conze A. and Viswanathan (1991). Path dependent options: the case of lookback options. *Journal of Finance*, **46**(5), 1893–1907.

[13] Dudley, R. (2002). Real Analysis and Probability. UK, Cambridge University Press.

[14] Estrella, A. (1995). Taylor, Black and Scholes: series approximations and risk management pitfalls. *Research Paper* 9501, Federal Reserve Bank of New York.

[15] Frishling, V. (2002). A discrete question. *Risk Magazine*, January, 115–116.

[16] Geman, H., El Karoui, N. and Rochet, J.C. (1995). Changes of Numéraire, Changes of Probability Measure and Option Pricing *Journal of Applied Probability*, **32**, 443–458.

[17] Genz, A. (1992). Numerical Computation of Multivariate Normal Probabilities *Journal of Computational and Graphical Statistics*, **1**(2), 141–149.

[18] Geske, R. (1979). A Note on an Analytical Valuation Formula for Unprotected American Call Options on Stocks With Known Dividends. *Journal of Financial Economics*, **7**, 375–380.

[19] Goldman, M., Soson, H. and Gatto, M. (1979). Path dependent options: buy at the low, sell at the high. *Journal of Finance*, **34**(5), 1111–1127.

[20] Harrison, J. and Kreps, D. (1979). Martingales and Arbitrage in Multiperiod Securities Markets. *Journal of Economic Theory*, **20**, 381–408.

[21] Harrison, J. and Pliska, S. (1981). Martingales and Stochastic Integrals in the Theory of Continuous Trading. *Stochastic Processes and their Applications*, **11**, 215–260.

[22] Hakala, J. and Wystup, U. (2002).*Foreign Exchange Risk*. London, Risk Books.

[23] Haug, E. (1998). *The complete guide to option pricing formulas.* New York, McGraw-Hill Professional.

[24] Haug, E., Haug, J. and Lewis, A. (2003). Back to Basics: a New Approach to the Discrete Dividend Problem. *Wilmott Magazine*, September, 37–47.

[25] Heynen, R. and Kat, H. (1994). Partial barrier options. *Journal of Financial Engineering*, **3**(3), 253–274.

[26] Hunt, P., and Kennedy, J. (2004). *Financial Derivatives in Theory and Practice, revised edition.* England, John Wiley & Sons Ltd.

[27] Hull, J. (1989). *Options Futures and Other Derivatives.* New Jersey, Prentice Hall.

[28] Johnson, H. (1987). Options on the minimum and maximum of several assets. *Journal of Financial and Quantitative Analysis*, **22**, 277–283.

[29] Korn, R., and Rogers, L. (2005). Stocks paying discrete dividends: modelling and option pricing. *The Journal of Derivatives*, **13**(2), 44–48.

[30] Luce, R., Krantz, D., Suppes, P., and Tversky, A. (1990). *Foundations of Measurement, Vol. III.* Academic Press, San Diego.

[31] Margrabe, W. (1978). The value of an option to exchange one asset for another. *Journal of Finance*, **33**, 177–186.

[32] Matos, J., Dilão, R., and Ferreira, B. (2002). The Exact Value for European Options on a Stock Paying a Discrete Dividend. *http://arxiv.org/abs/math/0609212v1.*

[33] Merton, R. (1973). Theory of Rational Option Pricing. *Bell Journal of Economics and Management Science*, Spring, **4**(1), 141–183.

[34] Musiela, M. and Rutkowski, M.(1997), *Martingale Methods in Financial Modeling.* Berlin, Springer.

[35] Nelken, I. (1995). *The Handbook of Exotic Options: Instruments, Analysis, and Applications.* New York, McGraw-Hill Professional.

[36] Rubinstein, M. and Reiner, E. (1997). Breaking Down Barriers. *Risk Magazine*, **4**(9), 28–35.

[37] Reiß, O. and Wystup, U. (2001). Efficient Computation of Option Price Sensitivities Using Homogeneity and Other Tricks. *The Journal of Derivatives*, **9**(2), Winter.

[38] Roll, R. (1977). An Analytical Valuation Formula for Unprotected American Call Options on Stocks With Known Dividends. *Journal of Financial Economics*, **5**, 251–58.

[39] Shaw, W. (1998). *Modelling Financial Derivatives with Mathematica.* Cambridge, Cambridge University Press.

[40] Shreve, S. (2004). *Stochastic Calculus for Finance II, Continuous–Time Models.* New York, Springer.

[41] Stoimenov, P., Wilkens, S. (2005). Are structured products 'fairly' priced? An analysis of the German market for equity-linked instruments. *Journal of Banking & Finance*, **29**, 2971–2993.

[42] Stojanovic, S. (2003). *Computational Financial Mathematics using Mathematica.* Boston, Birkhäuser.

[43] Stulz, R. (1982). Options on the minimum and maximum of two risky assets. *Journal of Financial Economics*, **10**, 161–185.

[44] Večeř, J. (2001). A new PDE approach for pricing arithmetic average Asian options. *Journal of Computational Finance*, **4**(4), 105–113.

[45] Veiga, C. (2004). Expanding Further the Universe of Exotic Options Closed Pricing Formulas. In: *Stochastic Finance*, Proceedings of a 2004 conference in Lisbon, Portugal, edited by A. Shiryaev et al., published in 2006 by Springer.

[46] Veiga, C. and Wystup, W. (2007). Closed Formula for Options with Discrete Dividends and its Derivatives. *CPQF Working Papers*, Frankfurt School of Finance & Management, Nr. 16.

[47] Veiga, C. and Wystup, U. (2009). Closed Formula for Options with Discrete Dividends and its Derivatives. *Applied Mathematical Finance*, **16**(6), 517–531.

[48] Veiga, C. and Wystup, U. (2010). Issuers' commitments would add more value than any rating scheme could ever do. In: *Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen*. Chiarella, C. and Novikov, A. (Eds.) , New York, Springer.

[49] Veiga, C., Wystup, W. and Esquível, M. (2010). Unifying Exotic Option Closed Formulas. *CPQF Working Papers*, Frankfurt School of Finance & Management, Nr. 23.

[50] Vellekoop, M. and Nieuwenhuis, J. (2006). Efficient Pricing of Derivatives on Assets with Discrete Dividends. *Applied Mathematical Finance*, September, **13**(3), 265–284.

[51] Vorst, T. (1992). Prices and hedge ratios of average exchange rate options. *International Review of Financial Analysis*, **1**, 179–193.

[52] Wilkens, S., Erner, C., Röder, K. (2003). The Pricing of Structured Products in Germany. *The Journal of Derivatives*, Fall, 55–69

[53] Wilkens, S., Stoimenov, P. (2007). The pricing of leverage products: An empirical investigation of the German market for 'long' and 'short' stock index certificates. *Journal of Banking & Finance* **31**, 735–750.

[54] Wilmott, P., Dewynne, J. and Howison, S. (1993). *Option Pricing: Mathematical Models and Computation* . Oxford, Oxford Financial Press.

[55] Wilmott, P., Howison, S. and Dewynne, J. (1995). *The Mathematics of Financial Derivatives. A Student Introduction*. Cambridge, USA, Cambridge Press.

[56] Wilmott, P. (1998). *Derivatives: The Theory and Practice of Financial Engineering*. New York, John Wiley & Son Ltd.

[57] Wystup, U. (2003). The market price of one-touch options in foreign exchange markets. *Derivatives Week*, **12**(13), 8–9.

[58] Zhang, P. (1997). *Exotic Options*. Singapore, World Scientific.