

Contents

Part I Cantor and Dedekind

1	Cantor's CBT Proof for Sets of the Power of (II)	3
1.1	The Generation Principles	3
1.2	Proof of CBT for Sets of the Power of (II)	5
1.3	The Limitation Principle	9
1.4	The Union Theorem	11
1.5	The Principles of Arithmetic	13
2	Generalizing Cantor's CBT Proof	15
2.1	The Scale of Number-Classes	16
2.2	The Induction Step	18
2.3	The Declaration of Infinite Numbers	23
3	CBT in Cantor's 1878 <i>Beitrag</i>	27
3.1	Equivalence Classes	27
3.2	The Order Relation Between Powers	28
3.3	A Direct Allusion	29
3.4	CBT for the Continuum	30
3.5	Generalized Proofs	31
3.6	The Different Powers of 1878 <i>Beitrag</i>	33
3.6.1	The Comparison Program	34
3.6.2	The Set of Real Functions	34
3.6.3	The Set of Denumerable Numbers (II)	34
3.6.4	Dating the Infinity Symbols	35
3.6.5	The Berlin Circumstances	37
4	The Theory of Inconsistent Sets	39
4.1	Inconsistent Sets Contain an Image of Ω	41
4.2	Views in the Literature	43

4.3	The Origin of the Inconsistent Sets Theory	44
4.4	The End of the Limitation Principle	46
5	Comparability in Cantor's Writings	49
5.1	The Definition of Order Between Cardinal Numbers	51
5.2	The Comparability Theorem for Cardinal Numbers	52
5.3	The Corollaries to the Comparability Theorem	53
5.4	The Comparability of Sets	53
6	The Scheme of Complete Disjunction	57
6.1	The Scheme and Schoenflies	57
6.2	The Scheme and Dedekind	59
6.3	The Scheme and Borel	60
6.4	Schröder's Scheme	62
6.5	The Origin of the Scheme of Complete Disjunction	64
7	Ruptures in the Cantor-Dedekind Correspondence	67
7.1	Views in the Literature	67
7.2	The <i>Aufgabe</i> Complex	68
7.3	The 1874 Rupture	69
7.4	The 1899 Rupture	72
7.5	By the Way	75
8	The Inconsistency of Dedekind's Infinite Set	77
8.1	Dedekind's Infinite Set	77
8.2	Bernstein's Recollections	79
8.3	Cantor's Criticism	82
8.4	Dedekind's Concerns	84
9	Dedekind's Proof of CBT	87
9.1	Summary of the Theory of Chains	90
9.2	Dedekind's Proofs	92
9.2.1	The First Proof	92
9.2.2	The Second Proof	93
9.2.3	The Third Proof	94
9.2.4	Comparing the Proofs	94
9.3	The Origin of Dedekind's Proof	96
9.4	Descriptors for Dedekind's Proof	98
9.5	Comparison to Cantor's Proof	100
 Part II The Early Proofs		
10	Schröder's Proof of CBT	105
10.1	Schröder's Proof	106
10.2	Criticism of Schröder's Proof	111
10.3	Comparison with Cantor and Dedekind	116

11 Bernstein, Borel and CBT 117

11.1 Borel’s Proof 119

11.2 Bernstein’s Original Proof 121

11.3 Comparison with Earlier Proofs 123

12 Schoenflies’ 1900 Proof of CBT 125

12.1 Cantor’s 1899 Proof 125

12.2 Schoenflies’ Proof 126

12.3 Comparisons 127

13 Zermelo’s 1901 Proof of CBT 129

13.1 The Proof 130

13.2 The Reemergence Argument 134

13.3 Convex-Concave 135

14 Bernstein’s Division Theorem 139

14.1 The Proof’s Plan 140

14.2 The Proof 142

14.3 Generalizations of the Theorem 148

14.4 The Inequality-BDT 152

Part III Under the Logician Sky

15 Russell’s 1902 Proof of CBT 155

15.1 The Core Arguments 156

15.2 The Definition of \aleph_0 160

16 The Role of CBT in Russell’s Paradox 165

16.1 Russell’s Proof of Cantor’s Theorem 166

16.2 Derivation of Russell’s Paradox 167

16.3 The Crossly-Bunn Reconstruction 169

16.4 Corroborating Lakatos 169

17 Jourdain’s 1904 Generalization of *Grundlagen* 171

17.1 The Ordinals and the Alephs 172

17.2 The Power of the Continuum 174

17.3 Inconsistent Aggregates 175

17.4 The Corollaries 178

17.5 Jourdain’s Rendering of Zermelo’s 1901 CBT Proof 179

17.6 The Sum and Union Theorems 181

17.7 Comparison with Cantor’s Theory 184

18 Harward 1905 on Jourdain 1904 185

18.1 Proof of CBT 186

18.2 Harward’s Unlimited Classes and Other Basic Notions 187

18.3 Harward’s Partitioning Theorem 189

18.4 Constructing the Number-Classes 189

18.5 The Union Theorem 190

19	Poincaré and CBT	195
19.1	The First Proof by Complete Induction	197
19.2	The Second Proof by Complete Induction	199
19.3	The Russell-Like Argument	201
19.4	On Impredicativity and Poincaré's Influence on Russell	203
19.5	Criticism of Zermelo's Proof	204
19.6	Criticism of Cantor's Proof	206
19.7	CBT from the Well-Ordering Theorem	207
20	Peano's Proof of CBT	209
20.1	Peano's Inductive Proof	210
20.2	Addressing Poincaré's Challenge	213
20.3	A Model for Arithmetic	214
21	J. König's Strings Gestalt	217
21.1	J. König's Ideology	218
21.2	J. König's CBT Proof	220
21.3	More Comments on the Proof	221
21.4	Bernstein's 1906 Proof	223
21.5	Comparison with Earlier Proofs	223
22	From Kings to Graphs	227
22.1	D. König's Proof that $m = m + m$	227
22.2	D. König's 1914 Proof that $vm = vn \rightarrow m = n$	229
22.3	Into the Land of Graphs	231
22.4	Factoring Finite Graphs	234
22.5	Factoring Denumerable Graphs	236
23	Jourdain's Improvements Round	239
23.1	The 1907 CBT Proof	240
23.2	The 1908 Proof of the Union Theorem	242
24	Zermelo's 1908 Proof of CBT	245
24.1	CBT and Its Proof	246
24.2	The Main Notions of Zermelo's Set Theory	248
24.2.1	Sets and Elements	249
24.2.2	Subsets, Parts and Components; Transitivity of \subseteq	250
24.2.3	Equality	251
24.2.4	Definiteness	252
24.2.5	Equivalence and Related Notions	253
24.2.6	Intersection	255
24.3	Comparison with the Proof in Poincaré 1906b	255
24.4	Final Words	256
25	Korselt's Proof of CBT	259
25.1	Theorem and Proof	260
25.2	An Application	261

26	Proofs of CBT in Principia Mathematica	265
26.1	The First Two Formulations	267
26.2	The Impredicative Proof	268
26.3	Without the Reducibility Axiom	269
26.4	The Inductive Proof	271
26.5	The Drawings	274
26.6	The Cardinal Version	276
26.7	Comparisons with Earlier Proofs	279
27	The Origin of Hausdorff Paradox in BDT	283
27.1	The Metaphor	284
27.2	Proof of Hausdorff's Paradox	286
27.3	Analytic Geometry Background	289
 Part IV At the Polish School		
28	Sierpiński's Proofs of BDT	293
28.1	Sierpiński's First Proof	293
28.2	Sierpiński's Second Proof	298
28.3	$2m \leq 2n \rightarrow m \leq n$	299
29	Banach's Proof of CBT	303
29.1	Proof of the Partitioning Theorem and Consequences	304
29.2	Aspects of the Proof	306
30	Kuratowski's Proof of BDT	309
30.1	The Theorem and Proof	310
30.2	Examples and Generalizations	313
30.3	D. König on Kuratowski's Paper	314
31	Early Fixed-Point CBT Proofs: Whittaker; Tarski-Knaster	317
31.1	Whittaker's Partitioning Theorem and CBT	318
31.2	The Tarski-Knaster Fixed-Point Theorem	321
32	CBT and BDT for Order-Types	323
32.1	Comparability Theorems	323
32.2	Division Theorems	325
33	Sikorski's Proof of CBT for Boolean Algebras	329
33.1	Theorems and Proofs	330
33.2	Sikorski's Open Problems	332
34	Tarski's Proofs of BDT and the Inequality-BDT	335
34.1	Tarski's 1949b Proof	336
34.2	The Proof of the Simplified Theorem	340
34.3	The Doyle-Conway Proof	342

35 Tarski’s Fixed-Point Theorem and CBT 343

35.1 Lattices and Boolean Algebras 344

35.2 A Proof of CBT 344

35.3 The Mean-Value Theorem 346

35.4 The Partitioning Theorem for Boolean Algebras 349

35.5 The Fixed-Point Theorem for Lattices 350

35.6 The Relation \preceq 351

35.7 Classifying CBT Proofs by Their Fixed-Points 353

36 Reichbach’s Proof of CBT 357

Part V Other Ends and Beginnings

37 Hellmann’s Proof of CBT 363

37.1 Theorems and Proofs 363

37.2 Discussion of the Proofs 364

37.3 Teaching Concerns 365

38 CBT and Intuitionism 367

38.1 Brouwer’s Counterexample 368

38.2 Myhill’s CBT for 1–1 Recursive Functions 370

38.3 Van Dalen’s Counterexamples to CBT for Fans 372

38.3.1 Basic Notions 373

38.3.2 The First Counterexample 374

38.3.3 The Main Counterexample 375

38.4 Troelstra’s CBT for Lawlike Sequences 381

39 CBT in Category Theory 387

39.1 Terminology 388

39.2 The Cantor-Bernstein Category and Relatives 389

39.3 Every Brandt Category is a Banach Category 391

39.4 A Non-Brandt Category is Not a CBC 393

39.5 Another Approach to CBT in Category Theory 397

39.6 On Possible Origins of the Commutative Diagram 398

Conclusion 401

Bibliography 403

Names Index 415

Subject Index 421