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Numerical Methods for Control of Second Order Hyperbolic Equations

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Contents

Abstract

This thesis is devoted to the numerical treatment of optimal control problems governed by second order hyperbolic partial differential equations. Adaptive finite element methods for optimal control problems of differential equations of this type are derived using the dual weighted residual method (DWR) and separating the influences of time, space, and control discretization. Moreover, semismooth Newton methods for optimal control problems of wave equations with control constraints and their convergence are analyzed for different types of control action. These two approaches are applied to optimal control problems governed by the dynamical Lamé system. The thesis ends with a discussion of numerical techniques to solve exact controllability problems for the wave equation.

Zusammenfassung

Diese Arbeit beschäftigt sich mit der numerischen Behandlung von Optimalsteuerungsproblemen für hyperbolische partielle Differentialgleichungen zweiter Ordnung. Adaptive Finite-Elemente-Verfahren für Optimalsteuerungsprobleme mit Differentialgleichungen dieser Art werden basierend auf der dual-gewichteten-Residuum-Methode (DWR) hergeleitet und dabei die Einflüsse der Zeit-, Orts- und Kontrolldiskretisierungsfehler separiert. Weiter werden semi-glatte Newtonverfahren für Optimalsteuerungsprobleme für Wellengleichungen mit Kontrollbeschränkungen und ihr Konvergenzverhalten für unterschiedliche Wahl der Kontrolle untersucht. Diese beiden Methoden werden auf Optimalsteuerungsprobleme mit dem dynamischen Lamé System angewandt. Die Arbeit endet mit einer Diskussion über numerische Methoden zum Lösen von Problemen der exakten Steuerbarkeit für die Wellengleichung.

1 Introduction

This thesis is devoted to numerical methods for control problems governed by second order hyperbolic partial differential equations. Thereby we consider optimal control as well as exact controllability problems. In optimal control of partial differential equations one is interested in minimizers of a cost functional depending on a control and a corresponding state. The relation between the control and the state is given by a partial differential equation; optionally, there are additional constraints on the state and control. In exact controllability one is interested in finding a control entering a partial differential equation, which drives the corresponding solution of the equation at a given time point to a final target exactly.

The main issues of this thesis are adaptive finite element methods for optimal control problems of second order hyperbolic equations, semi-smooth Newton methods for optimal control problems of wave equations with additional constraints on the controls, the application of these two methods to optimal control of the dynamical Lamé system, and numerical methods for exact controllability problems of the wave equation.

There exists a rich literature on optimal control of elliptic and parabolic partial differential equations; see, e.g., the monographs by Lions [87] and Tröltzsch [126] and for its numerical treatment Hinze et al. [58] and the references therein. For optimal control of second order hyperbolic equations we refer to Lions [87] and Lasiecka & Triggiani [84]. However, in contrast to optimal control of elliptic and parabolic equations there exist only few results on numerical methods for optimal control of hyperbolic equations of second order. There is the work by Gerdts, Greif & Pesch [46] on optimal boundary control of a string to rest in finite time. Domain decomposition in the context of optimal control of the wave equation is considered in Lagnese & Leugering [79, 80]. For state constrained optimal control problems of the wave equation see Gugat, Keimer & Leugering [51], Gugat [49] as well as Mordukhovich & Raymond [107, 105] in case of Dirichlet boundary control and [106] for Neumann boundary control. In Kowalewski, Lasiecka & Sokolowksi [71] sensitivity analysis for optimal control problems of hyperbolic equations is considered. In Kunisch & Wachsmuth [134] a time optimal control problem for the wave equation is analyzed.

For optimal control of first order hyperbolic equations, there exist also only few results; cf. Ulbrich [130], Gugat et al. [50], Ngnotchouye et al. [113], Castro, Palacios and Zuazua [30], even though control of first order equations is not a subject of this thesis. In the following we always write hyperbolic equations, instead of second order hyperbolic equations.

Optimal control of hyperbolic equations plays an important role in applications, e.g. in noise suppression problems, in medical applications as focusing of ultrasound waves and in problems in elastodynamics. Furthermore, interpretating the optimal control problem as a parameter estimation problem, it is closely related to questions arising in seismic problems as well as in noise emission problems. A discussion of these aspects in more detail is presented later.

In contrast to optimal control of wave equations there exist a lot of publications on exact controllability of the wave equation. For an introduction to this topic we refer to the monograph by Lions [89] and for an overview to the review article by Zuazua [141].

The new contributions of this thesis are the following:

- *Adaptive finite element methods using the dual weighted residual method (DWR; cf. Becker & Rannacher [13]) applied to optimal control problems governed by hyperbolic equations.* To the knowledge of the author these are the first results on the DWR method for optimal control of hyperbolic equations. They are published in Kröner [73]; see also Kröner [74]. There are several publications on the DWR method for optimal control of elliptic and parabolic equations showing that the method works very well for efficiently solving optimal control problems reducing the numerical effort. Here, we transfer techniques developed for optimal control of parabolic equations, cf. Meidner & Vexler [99], to optimal control of hyperbolic problems. An important aspect when analyzing wave like phenomena described by hyperbolic equations is the conservation of energy, which should also be taken into account on the discrete level. We analyze the question of conservation of energy on adaptively in time changing meshes.

- *Semi-smooth Newton methods for optimal control problems of wave equations with constraints on the control.* The results are published in Kröner, Kunisch & Vexler [76, 75]. Control constraints are a natural additional condition, since in physical applications the appearing quantities are mostly bounded. The incorporation of these constraints lead to non-smooth operator equations. For solving these equations we introduce the framework of semi-smooth Newton methods and analyze its behaviour of convergence using techniques based on Hintermüller, Ito and Kunisch [56]. Semi-smooth Newton methods, which can be equivalently formulated as primal-dual active set methods, have shown to work well in many situations. For monographs on these Newton methods we refer the reader to Ito & Kunisch [64] and Ulbrich [129] and for the application to optimal control of parabolic equations see Kunisch & Vexler [78]. Numerical examples confirm our theoretical results.

- *Adaptive finite element methods and semi-smooth Newton methods applied to optimal control of the dynamical Lamé system.* We consider optimal control problems with respect to the linearized Lamé-Navier system resulting in the elastic wave equation. This system can be considered as a model for seismic waves or acoustic waves traveling in solid materials. We apply the methods described above to this system. Although the Lamé system is subject of many publications, to the knowledge of the author the presented results are the first contribution on dual weighted residual methods for optimal control of the dynamical Lamé system and convergence analysis of semi-smooth Newton methods solving these control problems with additional constraints on the control.

Further, we finish this thesis with a discussion of numerical methods for exact controllability problems of the wave equation. We recall some main aspects from the literature and confirm them by a numerical example. It is well-known, see, e.g. Zuazua [141] that exact controllability problems can be formulated as observability problems and that the discretization of these problems leads to spurious high frequencies. We consider the relation between optimal control and exact controllability and consider two numerical approaches to solve the exact controllability problem. On the one hand we interpret it as an optimal Dirichlet boundary control problem and on the other hand as an optimization problem over the space of initial data. We conclude with a numerical example.

The thesis is organized as follows.

Chapter 2: Continuous problem
In this chapter we formulate an abstract optimal control problem and present some examples for optimal control problems. Further, we recall some results on optimality conditions and formulate existence and regularity results for linear hyperbolic equations as well as for the inhomogeneous Neumann and Dirichlet boundary problem for the wave equation and the inhomogeneous Dirichlet boundary problem for the strongly damped wave equation. Thereby, we recall results on the hidden regularity for the wave equation. Further, we formulate some basic properties of the wave equation, as the propagation along characteristics, conservation of energy and the relation to conservation laws.

Chapter 3: Adaptive finite element methods
In this chapter we consider optimal control problems of hyperbolic equations without control constraints and develop adaptive finite element methods to solve these problems reducing the computational costs. Therefore, we derive a posteriori error estimates separating the error arising from time, space and control discretization using the dual weighted residual method. We transfer techniques developed in Meidner & Vexler [99] for parabolic equations to hyperbolic equations. The problem is discretized by space-time finite elements. We discretize the problem first in time using a Petrov-Galerkin method, then in space by conforming finite elements and finally we discretize the control space. Numerical examples are presented. Furthermore, we analyze the behaviour of the energy of the homogeneous discrete wave equation on meshes changing dynamically in time and confirm the results by numerical examples.

Chapter 4: Semi-smooth Newton methods
In this chapter we consider optimal control problems governed by wave equations with additional constraints on the control. To solve these problems we consider semi-smooth Newton methods and analyze the convergence of these methods for different types of control action. We consider distributed control, Neumann boundary and Dirichlet boundary control. In case of distributed and Neumann boundary control we prove superlinear convergence, in case of Dirichlet boundary control however, the operator mapping the control to a trace of the adjoint state has no smoothing property which we need for superlinear convergence. This motivates to consider the strongly damped wave equation, which models the behaviour of waves in case of loss of energy. For the strongly damped wave equation we prove superlinear convergence. The problems are discretized by finite elements and to solve the optimization problems computationally we formulate the semi-smooth Newton method equivalently as a primal-dual active set method (PDAS). We present some numerical ex-

amples confirming the theoretical results.

Chapter 5: Application to the dynamical Lamé system
In this chapter we apply the techniques, adaptive finite elements and semi-smooth Newton methods, developed in the previous two chapters to the linearized dynamical Lamé system. The dynamical Lamé system is used to model acoustic waves in solid materials or seismic waves. We apply the adaptive finite element method from Chapter 3 to an optimal control problem with a time-dependent control. Then in the second part we consider semi-smooth Newton methods for optimal control problems of the dynamical Lamé system with control constraints and consider distributed, Neumann boundary and Dirichlet boundary control. Thereby, we transfer the proofs presented in Chapter 4 for the wave equation to the dynamical Lamé system. As in Chapter 4 we derive superlinear convergence in case of distributed and Neumann boundary control. For Dirichlet control the operator mapping the control to a trace of the adjoint state has no smoothing property, so we consider the strongly damped dynamical Lamé system and prove superlinear convergence in this situation. The theoretical results are confirmed by numerical examples.

Chapter 6: Controllability
In this chapter we consider exact boundary controllability problems for the wave equation. In this case a final target at a given time T has to be reached exactly. We analyze the relation to optimal control problems and recall the difficulties when solving exact controllability problems numerically. Finally, we consider two approaches to solve exact controllability problems for the wave equation numerically and present an example.

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2 Continuous problem

In this chapter we discuss some aspects concerning existence and regularity of solutions of optimal control problems governed by second order hyperbolic equations.

Optimal control problems governed by hyperbolic equations are considered in the monographs by Lions [87, 88], Lions & Magenes [91, 92] and Lasiecka & Triggiani [84]. In case of additional constraints on the state optimality conditions for optimal Neumann boundary control problems of the wave equation are derived in Mordukhovich & Raymond [105], and for optimal Dirichlet boundary control in Mordukhovich & Raymond [107, 106]. Optimal control of nonlinear wave equations are analyzed in Clason, Kaltenbacher & Veljovic [33] and Farahi, Rubio & Wilson [41].

For existence and regularity results for general linear hyperbolic equations we refer the reader to Lions & Magenes [92], for boundary value problems for the wave equation to Lasiecka & Triggiani [86] in case of the Neumann problem and to Lasiecka, Lions and Triggiani [85] in case of the Dirichlet problem. In this chapter we recall some main results from these publications.

Furthermore, we will consider the strongly damped wave equation which is used in models with loss of energy. We derive a regularity result, which is published in Kröner, Kunisch & Vexler [76]. There exists several publications on the strongly damped wave equation; cf. Chill & Srivastava [32], Avrin [4], Mugnolo [108], Pata & Squassina [114], Massatt [95], Larsson, Thomee & Wahlbin [81]. Further regularity results for some structurally damped problems can be found in Triggiani [125]. In Bucci [23] an existence and regularity result for an optimal Dirichlet boundary control problem for the strongly damped wave equation is analyzed considering controls in $H^1(L^2(\partial\Omega))$ in contrast to the results presented in this thesis, where the controls are in $L^2(L^2(\partial\Omega))$.

When considering wave equations, in particular when applying numerical methods to solve these equations, the main properties of the continuous equation should be taken into account as conservation of energy, transport of singularities along characteristics and the relation to conservation laws. We will discuss these aspects at the end of this chapter.

The chapter is organized as follows. After some preliminary remarks in Section 2.1, we formulate an abstract optimal control problem in Section 2.2, present some examples and formulate optimality conditions. In Section 2.3 we formulate existence and regularity results for several state equations; we consider linear hyperbolic equations and boundary value problems for the wave equation as well as for the strongly damped wave equation. In Section 2.4 we discuss specific properties of the wave equation.

2.1 Notation

Throughout this thesis (if not defined else wise), let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be a bounded domain with C^2-boundary $\partial\Omega$ (for $d \neq 1$) and $I = (0, T)$ a time interval for given $0 < T < \infty$.

We set

$$Q = I \times \Omega, \quad \Sigma = I \times \partial\Omega.$$

Further, we employ the usual definitions of Lebesgue $L^p(D)$ and Sobolev spaces $W^{k,p}(D)$ and $W_0^{k,p}(D)$, respectively, for sufficiently smooth $D = \Omega$, $D = \partial\Omega$ or $D = I$, $1 \le p \le \infty$ and a non-negative integer k; cf. Adams [1]. We set $H^k(D) = W^{k,2}(D)$, $H_0^k(D) = W_0^{k,2}(D)$, and $H^s(D) = [H^m(D), L^2(D)]_{1-\frac{s}{m}}$ for any integer $m \ge s \ge 0$, $s \in \mathbb{R}$, and the interpolation space $[\cdot,\cdot]$; cf. Lions & Magenes [91, pp. 10]. Further we use the usual notation for the space $H_0^s(D)$ and its dual space denoted by $H^{-s}(D)$, $s \ge 0$; cf. Lions & Magenes [91, pp. 55]. For any Banach space Z we define the usual Banach space valued Lebesgue spaces $L^p(I,Z)$, Sobolev spaces $H^s(I,Z)$ and Hölder spaces $C^k(\bar{I},Z)$, $1 \le p \le \infty$, $s \ge 0$, $s \in \mathbb{R}$, $k \in \mathbb{N}_0$; cf. [91]. To shorten notations we set

$$\begin{aligned} H^s(Z) &= H^s(I,Z), \quad C^k(Z) = C^k(\bar{I},Z), \\ L^p(Z) &= L^p(I,Z), \quad H^0(Z) = L^2(Z). \end{aligned}$$

For Banach spaces X, Z we denote by $\mathcal{L}(X,Z)$ the set of continuous, linear mappings from X to Z and we denote the norm of Z by $\|\cdot\|_Z$, in case of the space $L^2(\Omega)$ we just write $\|\cdot\|$, and for $\mathbb{R}$ we denote the absolute value by $|\cdot|$. Moreover, let $\langle\cdot,\cdot\rangle_{Z^*,Z}$ denote the canonical dual pairing between Z and its dual Z^* and for a Hilbert space H let $(\cdot,\cdot)_H$ be the inner product in H. Further, we define

$$(u,v)_J = \int_J (u(t), v(t))_H dt$$

for an open interval $J \subset I$ and $u, v \in L^2(H)$ and the inner products

$$(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega)}, \quad \langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{L^2(\partial\Omega)}, \quad \langle\cdot,\cdot\rangle_I = \langle\cdot,\cdot\rangle_{L^2(L^2(\partial\Omega))}.$$

Finally, we denote by $C > 0$ a generic constant.

For an overview on the notation see Chapter 7.

2.2 Abstract optimal control problem

For given Hilbert spaces U and X we introduce a cost functional

$$J \colon U \times X \to \mathbb{R} \tag{2.2.1}$$

and call U control space and X state space. Further, let

$$U_{\text{ad}} \subset U,$$

be a convex, closed, and non-empty set, which we call the set of admissible controls and

$$S \colon U \to X, \quad u \mapsto y = S(u) \tag{2.2.2}$$

a control-to-state operator mapping a control u to a corresponding state y. The relation (2.2.2) between the control and the state let be given by a hyperbolic partial differential equation, which we will specify in the next sections.

After these preparations we introduce the following general optimal control problem:

$$\begin{cases} \text{Minimize} \quad J(u,y), \quad u \in U_{\text{ad}},\ y \in X, \ \text{subject to (s.t.)} \\ y = S(u). \end{cases} \tag{P}$$

In this thesis we will consider control problems of type (P).

Remark 2.2.1. Throughout this thesis the cost functional (2.2.1) will mostly be given in the form

$$J(u,y) = J_A(y) + \frac{\alpha}{2} \|u\|_U^2 ,$$

as the sum of a functional $J_A\colon X \to \mathbb{R}$ and control costs with parameter $\alpha > 0$; often we choose $J_A(y) = \frac{1}{2} \|y - y_{\text{d}}\|_{L^2(Q)}^2$ with a desired state $y_{\text{d}} \in L^2(Q)$.

Before we recall some results on existence and uniqueness and formulate optimality conditions for (P), we present four examples. We start with the classical wave equation.

Example 2.2.2 (Classical wave equation)**.** The classical *wave equation* with homogeneous boundary conditions

$$\begin{cases} y_{tt} - \Delta y = u & \text{in } Q, \\ y(0) = y_0 & \text{in } \Omega, \\ y_t(0) = y_1 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma \end{cases} \tag{2.2.3}$$

with $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$, $u \in L^2(Q)$ can be seen as a model for small oscillations of a string ($d = 1$) and membrane ($d = 2$), respectively, which are fixed on the boundary. The displacement and velocity at time zero is given by the initial data y_0 and y_1. The function u is the control acting as a force on the time space cylinder Q. Let $y_{\text{d}} \in L^2(Q)$ be the desired state. Then we consider the following optimal control problem

$$\begin{cases} \text{Minimize} \quad J(u,y) = \frac{1}{2} \|y - y_{\text{d}}\|_{L^2(Q)}^2 + \frac{\alpha}{2} \|u\|_{L^2(Q)}^2 , \quad u \in U_{\text{ad}}, \quad y \in L^2(Q), \\ \text{s.t.} \\ \text{equation (2.2.3)} \end{cases}$$

with the set of admissible controls given by

$$U_{\text{ad}} = \{ u \in L^2(Q) : u_a \le u \le u_b \text{ a.e. in } Q \}$$

for some given lower and upper bounds $u_a, u_b \in L^2(Q)$ and a parameter $\alpha > 0$. The boundedness of the controls can be motivated by the fact that physical quantities are usually bounded.

Example 2.2.3 (Strongly damped wave equation)**.** Longitudinal vibrations in a homogeneous bar in which there are viscous effects are described by the *strongly damped wave* *equation*. This equation is not a hyperbolic equation but it can we be seen as a regularized hyperbolic equation. We introduce the following optimal Dirichlet boundary control problem

$$\begin{cases} \text{Minimize} \quad J(u,y) = \frac{1}{2}\|y - y_{\mathrm{d}}\|^2_{L^2(Q)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Sigma)}, \quad u \in L^2(\Sigma), \quad y \in L^2(Q), \\ \qquad \text{s.t.} \\ \qquad\qquad y_{tt} - \Delta y - \rho\Delta y_t = f \quad \text{in } Q, \\ \qquad\qquad y(0) = y_0 \quad \text{in } \Omega, \\ \qquad\qquad y_t(0) = y_1 \quad \text{in } \Omega, \\ \qquad\qquad y = u \quad \text{on } \Sigma \end{cases} \tag{2.2.4}$$

for $f \in L^2(Q)$, $\rho > 0$ and y_0, y_1, y_d, α as in Examples 2.2.2. The term $\rho\Delta u_t$ indicates that the stress is proportional not only to the strain, as with Hooke's law, but also to the strain rate as in a linearized Kelvin material, see Massatt [95], Fitzgibbon [43], cf. also Larsson, Thomee & Wahlbin [81]. Furthermore, the strongly damped wave equation can be considered as a regularization of the wave equation. For further details we refer to Section 2.3.2.

Example 2.2.4 (Elastic wave equation)**.** For modeling of elastic waves, which arise e.g. in seismic problems or are caused by acoustic waves traveling through solid material structures, the *elastic wave equation* is applied. It is also used for acoustic emission problems, see, e.g. Schechinger [122]. We introduce the following optimal control problem

$$\begin{cases} \text{Minimize} \quad J(u,y) = \frac{1}{2}\|y - y_{\mathrm{d}}\|^2_{L^2(L^2(\Omega)^3)} + \frac{\alpha}{2}\|u\|^2_{L^2(I,\mathbb{R}^3)}, \\ \qquad u \in L^2(I,\mathbb{R}^3), \quad y \in L^2(Q)^3, \quad \text{s.t.} \\ \qquad y_{tt} - (\lambda+\mu)\nabla\mathrm{div}y - \mu\Delta y = \sum_{i=1}^{3} u_i g_i \quad \text{in } Q, \\ \qquad y(0) = y_0 \quad \text{in } \Omega, \\ \qquad y_t(0) = y_1 \quad \text{in } \Omega, \\ \qquad y = 0 \quad \text{on } \Sigma \end{cases} \tag{2.2.5}$$

for given $g_i \in L^2(\Omega)^3$, $i = 1, 2$, $\lambda, \mu > 0$, $y_0 \in H^1_0(\Omega)^3$, $y_1 \in L^2(\Omega)^3$, dimension $d = 3$ and $\alpha > 0$. So when, for example, the functions g_i are given as characteristic functions of some subsets of Ω, the control u acts only on some parts of Ω depending on time.

The elastic wave equation can be derived from the Lamé Navier equations after some linearizations; see Hughes [59]. For further details we refer to Chapter 5.

Example 2.2.5 (Westervelt equation)**.** This more advanced example is taken from Clason, Kaltenbacher & Veljović [33]. We consider optimal control of highly focused ultrasound, where the pressure fluctuations of the ultrasound are modeled by the *Westervelt equation*. As the strongly damped wave equation in Example 2.2.3 this is not a hyperbolic equation but it can be seen as a nonlinear strongly damped wave equation. The control problem reads as follows

$$\begin{cases} \text{Minimize} \quad J(u,y) = \dfrac{1}{2}\displaystyle\int_{\Omega} |y(T) - y_{\mathrm{d}}|^2 dx + \dfrac{\alpha}{2}\int_0^T \int_{\Gamma_0} |u|^2 d\sigma dt, \quad u \in U_{\mathrm{ad}}, \quad y \in X, \\ \qquad \text{s.t.} \\ \qquad\qquad \begin{aligned} y_{tt} - c\Delta y - b\Delta y_t &= \frac{\beta}{c\rho}(y^2)_{tt} && \text{in } Q, \\ y(0) &= y_0 && \text{on } \Omega, \\ y_t(0) &= y_1 && \text{on } \Omega, \\ \partial_n y &= u && \text{on } I \times \Gamma_0, \\ y_t + c\partial_n y &= 0 && \text{on } I \times \Gamma_1, \end{aligned} \end{cases} \tag{2.2.6}$$

with the spaces

$$\begin{aligned} U &= \{\, u \in L^2(I \times \Gamma_0) \mid \|u\|_U < \infty \,\}, \quad \|u\|_U^2 = \|u\|^2_{H^1(H^{\frac{1}{2}}(\Gamma_0))} + \|u_{tt}\|^2_{L^2(H^{-\frac{1}{2}}(\Gamma_0))}, \\ U_{\mathrm{ad}} &= \{\, u \in U \mid \|u\|_U \le K \text{ and } u(0,\cdot) = \partial_n y_0 \text{ on } \Gamma_0 \,\}, \\ X &= \{\, y \in L^\infty(Q) \mid y, y_t \in H^1(Q) \,\}, \end{aligned}$$

$y_0 \in H^2(\Omega)$, $y_1 \in H^1(\Omega)$, $K > 0$ and $\partial\Omega = \Gamma_0 \cup \Gamma_1$. The function y describes the acoustic pressure fluctuation, u the normal acceleration of transducers on the part Γ_0 of the boundary $\partial\Omega$, c the speed of the sound, $b > 0$ the diffusivity of sound, $\rho > 0$ the mass density and $\beta > 1$ a parameter of nonlinearity. To avoid artificial reflection we assume the mixed boundary condition on Γ_1. Instead of the Westervelt equation also the *Kutznov equation* can be used to describe ultrasound propagation. For further results on the Westervelt equation we refer the reader to Kaltenbacher & Lasiecka [67].

Now, we return to the abstract optimal control problem (P) and formulate an existence result and optimality conditions.

Existence of a solution of the optimal control problem

To prove existence of a solution under certain conditions, which we specify in the sequel, we apply the reduced ansatz. Therefore we define the reduced cost functional by

$$j\colon U \to \mathbb{R}, \quad j(u) = J(u, S(u)) \tag{2.2.7}$$

and reformulate the optimal control problem (P) equivalently as

$$\text{Minimize} \quad j(u), \quad u \in U_{\mathrm{ad}}. \tag{P_{red}}$$

Existence of a solution of (P_{red}) follows under weak assumptions.

Proposition 2.2.6. *Let the reduced cost functional $j\colon U_{ad} \to \mathbb{R}$ be weakly lower semicontinuous, i.e.*

$$\liminf_{n\to\infty} j(u_n) \ge j(u) \quad \textit{for } u_n \rightharpoonup u \in U_{ad}$$

and let j be coercive over U_{ad}, i.e.

$$j(u) \geq \gamma \|u\|_U + c$$

for all $u \in U_{ad}$ and $\gamma > 0$, $c \in \mathbb{R}$. Then problem (P_{red}) has at least one solution.

For a proof we refer to Lions [87, pp. 8].

Remark 2.2.7. In the following Chapters we will derive that the control-to-state operators associated with the Examples 2.2.2 - 2.2.4 are linear and continuous. Hence, in these three examples the reduced cost functionals are continuous and convex and consequently weakly lower semicontinuous; cf. Dacorogna [35]. Thus, we obtain existence of a solution in all three examples by Proposition 2.2.6. To derive existence of a solution of the optimal control problem given in Example 2.2.5, we refer to Clason, Kaltenbacher & Veljovic [33]. This example is more involved, since the state equation is nonlinear.

Remark 2.2.8. If j is strictly convex, the solution is unique. However, in particular in case of a nonlinear state equation the solution may not be unique.

This motivates the notion of a local solution.

Definition 2.2.9 (Local solution)**.** A function $\bar{u} \in U_{\text{ad}}$ is called a local solution of the optimal control problem if

$$j(u) \geq j(\bar{u})$$

for all $u \in \{\, u \in U_{\text{ad}} \mid \|u - \bar{u}\|_U \leq \varepsilon \,\}$ and some $\epsilon > 0$.

Next, we formulate optimality conditions for the solution of (P_{red}).

Optimality conditions

The necessary optimality condition of first order for the control problem (P_{red}) is given in the following proposition.

Proposition 2.2.10 (Necessary optimal conditions of first order)**.** *Let the reduced cost functional j be directionally differentiable on U_{ad}. Then for a local optimal solution $\bar{u} \in U_{ad}$ there holds the necessary optimal condition of first order*

$$j'(\bar{u})(u - \bar{u}) \geq 0 \quad \forall u \in U_{ad}. \tag{2.2.8}$$

Without control constraints, i.e. $U_{ad} = U$, this is equivalent to

$$j'(\bar{u})(u) = 0 \quad \forall u \in U_{ad}.$$

This is a standard result; see, e.g., Tröltzsch [126]. For a proof of the following sufficient optimality condition of second order we also refer to [126].

Proposition 2.2.11 (Sufficient optimality condition of second order). *Let the reduced cost functional j be twice continuously Fréchet differentiable in a neighborhood of a point $\bar{u} \in U_{ad}$. Further, let $\bar{u}$ satisfy the necessary optimality condition of first order, i.e.*

$$j'(\bar{u})(u-\bar{u}) \geq 0 \quad \forall u \in U_{ad}$$

and exists a $\gamma > 0$ with

$$j''(\bar{u})(u,u) \geq \gamma \|u\|_U^2 \quad \forall u \in U.$$

Then there exists $\varepsilon > 0$ and $\sigma > 0$ such that there holds

$$j(u) \geq j(\bar{u}) + \sigma \|u-\bar{u}\|_U^2$$

for all $u \in U_{ad}$ with $\|u-\bar{u}\|_U \leq \varepsilon$. Hence, $\bar{u}$ is a local solution of the optimal control problem.

Remark 2.2.12. Sometimes it is not possible to prove a sufficient optimality condition of second order as formulated in Proposition 2.2.11. This is, for example the case, if the reduced cost functional is twice continuously differentiable only with respect to a subspace $\tilde{U} \subset U$ and coercivity is only given with respect to U. Then, the so-called *two norm-discrepancy* can be applied using norms of U and $\tilde{U}$; see Tröltzsch [126].

2.3 Existence and regularity for solutions of the state equation

This section is devoted to a discussion of existence and regularity results for solutions of linear hyperbolic equations and boundary value problems for the wave equation. These equations are candidates to define the control-to-state mapping (2.2.2). Regularity results for general linear hyperbolic equations can be found in Lions & Magenes [91], for the Neumann problem for the wave equation in Lasiecka & Triggiani [86] and for the Dirichlet problem for the wave equation in Lasiecka, Lions & Triggiani [85] and Lasiecka & Triggiani [84, pp. 954]. Further results for nonhomogeneous problems can be found in Lions & Magenes [92, pp. 103].

At first we consider a general linear hyperbolic equation, then the inhomogeneous Neumann and Dirichlet problem for the wave equation and finally, the Dirichlet problem for the inhomogeneous strongly damped wave equation.

2.3.1 Linear hyperbolic equations of second order

Let H and V be Hilbert spaces forming a Gelfand triple

$$V \subset H, \quad V \text{ dense in } H, \quad V \hookrightarrow H \text{ is continuous.}$$

We identify H with its dual space, and let V^* be the dual space of V. Then we identify H with a subspace of V^*, and we obtain

$$V \subset H \subset V^*. \tag{2.3.1}$$

If $f \in V^*$ and $v \in V$, their inner product is also denoted by $(f, v) = \langle f, v\rangle_{V^*,V}$, which is permissible by the identification (2.3.1), cf. Lions & Magenes [91]. Usually we choose

$$V = \{\, v \in H^1(\Omega)^n \mid v|_{\Gamma_D} = 0 \,\}\,, \quad H = L^2(\Omega)^n \tag{2.3.2}$$

with the Dirichlet part $\Gamma_D \subset \partial\Omega$ of the boundary or in case of homogeneous Neumann conditions

$$V = H^1(\Omega)^n, \quad H = L^2(\Omega)^n \tag{2.3.3}$$

for $n = 1, 2, 3$.

To recall some existence and regularity results for a general linear hyperbolic equation we introduce the following semilinear form.

For $t \in \bar{I}$, let

$$b(t, u, v)\colon V \times V \to \mathbb{R}$$

be a family of continuous bilinear forms, such that

$$\begin{cases} \bar{I} \ni t \mapsto b(t, u, v) \text{ is continuously differentiable in } \bar{I} \text{ for all } u, v \in V, \\ \text{there exists } \lambda, \alpha > 0, \text{ such that } b(t, v, v) + \lambda \left\|v\right\|_H^2 \geq \alpha \left\|v\right\|_V^2 \text{ for all } v \in V \text{ and } t \in [0, T]. \end{cases}$$

(2.3.4a)

(2.3.4b)

For fixed $t \in \bar{I}$ there holds the following identity

$$b(t, u, v) = \langle B(t)u, v\rangle_{V^*,V}, \quad B(t)u \in V^*,$$

which defines

$$B(t) \in \mathcal{L}(V, V^*).$$

We consider equations

$$y_{tt}(t) + B(t)y(t) = f(t), \tag{2.3.5}$$

with initial data

$$y(0) = y_0, \quad y_t(0) = y_1. \tag{2.3.6}$$

Theorem 2.3.1. *Assume that the properties* (2.3.4a) *and* (2.3.4b) *hold. Then for*

$$f \in L^2(H), \quad y_0 \in V, \quad y_1 \in H \tag{2.3.7}$$

there exists a unique function y satisfying (2.3.5) *and* (2.3.6) *with*

$$y \in L^2(V), \quad y_t \in L^2(H). \tag{2.3.8}$$

For a proof we refer to Lions & Magenes [91].

Remark 2.3.2. If (2.3.8) holds, then $B(t)y \in L^2(V^*)$, such that (2.3.5) implies

$$y_{tt} \in L^2(V^*).$$

Thus, we define the solution space for linear hyperbolic equation with homogeneous boundary conditions by

$$X = L^2(V) \cap H^1(H) \cap H^2(V^*). \tag{2.3.9}$$

Further, there holds the following theorem; cf. Lions & Magenes [91, pp. 275, 288].

Theorem 2.3.3. *Assume that the conditions* (2.3.4a) *and* (2.3.4b) *hold. Then after a possible modification on a set of measure zero, the solution y of* (2.3.5) *and* (2.3.6) *with data*

- *given in* (2.3.7) *satisfies*

  $$(y, y_t) \in C(V) \times C^1(H),$$

  and the mapping

  $$L^2(H) \times V \times H \to C(V) \times C^1(H), \quad (f, y_0, y_1) \mapsto (y, y_t)$$

  is continuous.

- *given by $(f, y_0, y_1) \in L^2(V^*) \times H \times V^*$ satisfies*

  $$(y, y_t) \in C(H) \times C^1(V^*),$$

  and the mapping

  $$L^2(V^*) \times H \times V^* \to C(H) \times C^1(V^*), \quad (f, y_0, y_1) \mapsto (y, y_t)$$

  is continuous.

Remark 2.3.4. The theorem implies the well-definedness of $y(0)$ and $y_t(0)$, so that (2.3.6) has a meaning.

The variational formulation of (2.3.5), (2.3.6) with data (2.3.7) reads as follows: Find $y \in X$, such that $y(0) = y_0 \in V$, $y_t(0) = y_1 \in H$ and

$$(y_{tt}(t), v)_H + b(t, y(t), v) = (f(t), v)_H \quad \forall v \in V \quad \text{a.e. in } I. \tag{2.3.10}$$

Remark 2.3.5. For a discussion of regularity results for nonlinear hyperbolic equations on domains with conical points we refer the reader to Witt [136].

Now, we proceed with wave equations.

2.3.2 Wave equations

Here, we replace operator B by the Laplacian $(-\Delta)\colon H_0^1(\Omega) \to H^{-1}(\Omega)$, and thus, equation (2.3.5) becomes the linear wave equation, which we consider in the following with inhomogeneous boundary conditions.

Neumann problem for the wave equation

The Neumann problem for the wave equation is given by

$$\begin{cases} y_{tt} - \Delta y = f & \text{in } Q, \\ y(0) = y_0 & \text{in } \Omega, \\ y_t(0) = y_1 & \text{in } \Omega, \\ \partial_n y = u & \text{on } \Sigma. \end{cases} \tag{2.3.11}$$

Applying the method of transposition we obtain existence and uniqueness of a solution of (2.3.11).

Theorem 2.3.6. *For every* $(f,\, y_0,\, y_1,\, u) \in L^1((H^1(\Omega))^*) \times L^2(\Omega) \times (H^1(\Omega))^* \times L^2(\Sigma)$ *there exists a unique very weak solution*

$$y \in C(L^2(\Omega)) \tag{2.3.12}$$

of (2.3.11)*, i.e.*

$$(y, g)_I = (f, \zeta)_I - (y_0, \zeta_t(0)) + (y_1, \zeta(0)) + \langle u, \zeta \rangle_I, \tag{2.3.13}$$

where $\zeta = \zeta_g$ *is the solution of*

$$\begin{cases} \zeta_{tt} - \Delta \zeta = g & \textit{in } Q, \\ \zeta(T) = 0 & \textit{in } \Omega, \\ \zeta_t(T) = 0 & \textit{in } \Omega, \\ \partial_n \zeta = 0 & \textit{on } \Sigma \end{cases}$$

for any $g \in L^1(L^2(\Omega))$*. The mapping*

$$L^1((H^1(\Omega))^*) \times L^2(\Omega) \times (H^1(\Omega))^* \times L^2(\Sigma) \to C(L^2(\Omega)), \quad (f, y_0, y_1, u) \mapsto y,$$

is continuous.

If we assume that $(f,\, y_0,\, y_1,\, u) \in L^1(L^2(\Omega)) \times H^1(\Omega) \times L^2(\Omega) \times L^2(\Sigma)$*, then there holds*

$$(y, y_t) \in C(H^{\frac{1}{2}}(\Omega)) \times C((H^{\frac{1}{2}}(\Omega))^*). \tag{2.3.14}$$

Proof of Theorem 2.3.6. The regularity result (2.3.14) has been proved in Lasiecka & Triggiani [86].

To verify the assertion (2.3.12) we recall a proof following classical arguments, see Lions [87]. From Theorem 2.3.3 we deduce that

$$(\zeta, \zeta_t) \in C(H^1(\Omega)) \times C(L^2(\Omega))$$

and hence the mapping

$$g \mapsto \mathcal{F} = (f, \zeta)_I - (y_0, \zeta_t(0)) + (y_1, \zeta(0)) + \langle u, \zeta \rangle_I$$

defines a continuous linear form on $L^1(L^2(\Omega))$. Therefore, there exists a solution

$$y \in L^\infty(L^2(\Omega))$$

fulfilling (2.3.13). Further, there exists a constant C independent of

$$(f,\, y_0,\, y_1,\, u) \in L^1((H^1(\Omega))^*) \times L^2(\Omega) \times (H^1(\Omega))^* \times L^2(\Sigma)$$

such that

$$\|y\|_{L^\infty(L^2(\Omega))} \le C\, \|(f,\, y_0,\, y_1,\, u)\|_{L^1((H^1(\Omega))^*)\times L^2(\Omega)\times(H^1(\Omega))^*\times L^2(\Sigma)}\,.$$

Uniqueness of the weak solution and continuous dependence on the data follows from this estimate. With the result (2.3.14) we have for sufficiently smooth data that $y \in C(L^2(\Omega))$, so by extension by continuity the proposed regularity (2.3.12) follows.

□

Further, we recall a regularity result from Lasiecka, Triggiani [83] for the inhomogeneous Neumann problem with slightly smoother Neumann data.

Proposition 2.3.7. *For $y_0 = y_1 = 0$ and $f = 0$ and $u \in L^2(H^{\frac{1}{2}}(\partial\Omega))$ there holds*

$$(y, y_t) \in C(H^1(\Omega)) \times C(L^2(\Omega))$$

for the solution of the Neumann problem (2.3.11)*.*

Dirichlet problem for the wave equation

To analyze the Dirichlet boundary problem we recall three regularity results in Theorem 2.3.8, 2.3.10, and 2.3.12. In particular, all three theorems provide some hidden regularity for the solution of the wave equation. We derive a regularity result for the normal derivative of the solution of the wave equation on the boundary, which cannot obtained directly by a trace theorem. The idea for the proof of the hidden regularity result goes back to Rellich [118]. The inhomogeneous Dirichlet problem for the wave equation reads as follows

$$\begin{cases} y_{tt} - \Delta y = f & \text{in } Q, \quad (2.3.15a)\\ y(0) = y_0 & \text{in } \Omega, \quad (2.3.15b)\\ y_t(0) = y_1 & \text{in } \Omega, \quad (2.3.15c)\\ y = u & \text{on } \Sigma. \quad (2.3.15d)\end{cases}$$

Theorem 2.3.8 (Higher regularity)**.** *For every $(f,\, y_0,\, y_1,\, u) \in L^1(H^1(\Omega)) \times H^2(\Omega) \times H^1(\Omega) \times H^2(\Sigma)$ with $f_t \in L^1(L^2(\Omega))$ satisfying the compatibility condition*

$$u(0) = y_0, \quad u_t(0) = y_1 \quad \text{on } \Sigma, \tag{2.3.16}$$

there exists a unique solution

$$(y, y_t, y_{tt}) \in C(H^2(\Omega)) \times C(H^1(\Omega)) \times C(L^2(\Omega))$$

of (2.3.15)*. Further, there holds for the normal derivative $\partial_n y \in H^1(\Sigma)$ and the mapping*

$$\begin{aligned}\big(H^{1,1}(L^2(\Omega)) \cap L^1(H^1(\Omega))\big) \times H^2(\Omega) \times H^1(\Omega) \times H^2(\Sigma)&\\ \longrightarrow C(H^2(\Omega)) \times C(H^1(\Omega)) \times H^1(\Sigma),&\\ (f, y_0, y_1, u) \mapsto (y, y_t, \partial_n y)&\end{aligned} \tag{2.3.17}$$

is continuous.

For a proof see Lasiecka, Lions & Triggiani [85].

Remark 2.3.9. The compatibility condition (2.3.16) is satisfied e.g. for $y_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $y_1 \in H_0^1(\Omega)$ and homogeneous Dirichlet boundary condition $u \equiv 0$.

The reader should notice that in the following theorem we only assume $f \in L^1(L^2(\Omega))$ in contrast to Theorem 2.3.3.

Theorem 2.3.10 (Hidden regularity)**.** *For every* $(f, y_0, y_1, u) \in L^1(L^2(\Omega)) \times H^1(\Omega) \times L^2(\Omega) \times H^1(\Sigma)$ *satisfying the compatibility condition*

$$u(0) = y_0 \quad \text{on } \Sigma, \tag{2.3.18}$$

there exists a unique solution $(y, y_t) \in C(H^1(\Omega)) \times C(L^2(\Omega))$ *of* (2.3.15)*. Further, there holds for the normal derivative*

$$\partial_n y \in L^2(\Sigma).$$

and the mapping

$$\begin{aligned} L^1(L^2(\Omega)) \times H^1(\Omega) \times L^2(\Omega) \times H^1(\Sigma) \to C(H^1(\Omega)) \times C(L^2(\Omega)) \times L^2(\Sigma), \\ (f, y_0, y_1, u) \mapsto (y, y_t, \partial_n y), \end{aligned} \tag{2.3.19}$$

is continuous.

For a proof see Lasiecka, Lions & Triggiani [85]; cf. also Lions [88, pp. 233].

Remark 2.3.11. The compatibility condition (2.3.18) is satisfied e.g. for $y_0 \in H_0^1(\Omega)$ and homogeneous Dirichlet boundary condition $u \equiv 0$.

Theorem 2.3.12. *For every* $(f, y_0, y_1, u) \in L^1(H^{-1}(\Omega)) \times L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Sigma)$ *there exists a unique very weak solution*

$$(y, y_t) \in C(L^2(\Omega)) \times C(H^{-1}(\Omega))$$

of (2.3.15)*, i.e.*

$$(y, g)_I = (f, \zeta)_I - (y_0, \zeta_t(0)) + (y_1, \zeta(0)) - \langle u, \partial_n \zeta \rangle_I \tag{2.3.20}$$

where $\zeta = \zeta_g$ *is the solution of*

$$\begin{cases} \zeta_{tt} - \Delta\zeta = g & \text{in } Q, \\ \zeta(T) = 0 & \text{in } \Omega, \\ \zeta_t(T) = 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \Sigma \end{cases} \tag{2.3.21}$$

for any $g \in L^1(L^2(\Omega))$*.*

Furthermore, there holds $\partial_n y \in H^{-1}(\Sigma)$ *(see [85, pp. 463] for a definition of the dual space) and the mapping*

$$\begin{aligned} L^1(H^{-1}(\Omega)) \times L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Sigma) \to C(L^2(\Omega)) \times C(H^{-1}(\Omega)) \times H^{-1}(\Sigma), \\ (f, y_0, y_1, u) \mapsto (y, y_t, \partial_n y), \end{aligned} \tag{2.3.22}$$

is continuous.

Remark 2.3.13. Under these very weak regularity assumptions on the data, no compatibility conditions are necessary.

Proof of Theorem 2.3.12. We recall the main idea from Lasiecka, Lions & Triggiani [85, Theorem 2.3], cf. also Lions [88, pp. 239]. We verify the assertion by transposition and interpolation. First, let $f = 0$. From Theorem 2.3.10 we obtain

$$\partial_n \zeta \in L^2(\Sigma), \quad \zeta \in C(H_0^1(\Omega)), \quad \zeta(0) \in H_0^1(\Omega), \quad \zeta_t(0) \in L^2(\Omega).$$

Hence, the mapping

$$g \mapsto -(y_0, \zeta_t(0)) + (y_1, \zeta(0)) - \langle u, \partial_n \zeta \rangle_I$$

is a continuous linear form. Thus, there exists

$$y \in L^\infty(L^2(\Omega))$$

satisfying the very weak formulation (2.3.20) and there holds $\Delta y \in L^\infty(H^{-2}(\Omega))$. From [85, pp. 157] we derive $y_{tt} \in L^\infty(H^{-2}(\Omega))$ and further by interpolation

$$y_t \in L^\infty(H^{-1}(\Omega)).$$

Following [85, pp. 158] we further obtain $(y, y_t) \in L^\infty(L^2(\Omega)) \times L^\infty(H^{-1}(\Omega))$ for $f \neq 0$ and $f \in L^1(H^{-1}(\Omega))$. For the estimate (2.3.22) we refer to [85, Remark 2.2] and for the step to $(y, y_t) \in C(L^2(\Omega)) \times C(H^{-1}(\Omega))$ to [85, pp. 153]. The regularity of the normal derivative follows by [85, Theorem 2.3]. □

Remark 2.3.14. This result is different to results for parabolic equations, where for a given boundary condition in $L^2(\Sigma)$ the solution at a given time $t \in \bar{I}$ may be not in $L^2(\Omega)$; cf. the example in Lions [87, pp. 202].

When considering the strongly damped wave equation which can be seen as a regularized Dirichlet problem, we obtain higher regularity of the solution of the homogeneous problem, as we see in the following.

Dirichlet problem for the strongly damped wave equation

The strongly damped wave equation, cf. Example 2.2.3, with a damping parameter ρ, $0 < \rho < \rho_0$, $\rho_0 \in \mathbb{R}^+$, and Dirichlet boundary condition is given by

$$\begin{cases} y_{tt} - \Delta y - \rho \Delta y_t = f & \text{in } Q, \\ y(0) = y_0 & \text{in } \Omega, \\ y_t(0) = y_1 & \text{in } \Omega, \\ y = u & \text{on } \Sigma \end{cases} \tag{2.3.23}$$

for $u \in L^2(\Sigma)$.

To prove a regularity result we first consider the damped wave equation with homogeneous Dirichlet data

$$\begin{cases} y_{tt} - \Delta y - \rho \Delta y_t = f & \text{in } Q, \\ y(0) = y_0 & \text{in } \Omega, \\ y_t(0) = y_1 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma. \end{cases} \tag{2.3.24}$$

The following theorem can be obtained.

Theorem 2.3.15. *For $f \in L^2(L^2(\Omega))$, $y_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, and $y_1 \in H_0^1(\Omega)$, there exists a unique weak solution of* (2.3.24)

$$y \in H^2(L^2(\Omega)) \cap C^1(H_0^1(\Omega)) \cap H^1(H^2(\Omega)) \tag{2.3.25}$$

given by

$$(y_{tt}(s), \phi) + (\nabla y(s), \nabla \phi) + \rho(\nabla y_t(s), \nabla \phi) = (f(s), \phi) \quad \forall \phi \in H_0^1(\Omega) \text{ a.e. in } I \tag{2.3.26}$$

with

$$y(0) = y_0, \quad y_t(0) = y_1.$$

Moreover, there holds the a priori estimate

$$\|y\|_{H^2(L^2(\Omega)) \cap C^1(H_0^1(\Omega)) \cap H^1(H^2(\Omega))} \le C \left(\|f\|_{L^2(L^2(\Omega))} + \|\nabla y_0\| + \|\Delta y_0\| + \|\nabla y_1\| \right), \tag{2.3.27}$$

where the constant $C = C(\rho)$ tends to infinity as ρ tends to zero.

Here, we present a direct proof, which is published in Kröner, Kunisch & Vexler [76]. Similar results can also be extracted from Chill & Srivastava [32].

To prove Theorem 2.3.15 we proceed as follows. We assume the existence of a solution with the desired regularity and prove a priori estimates by the following Lemmas 2.3.16–2.3.19. Then the existence of a solution

$$y \in H^2(L^2(\Omega)) \cap W^{1,\infty}(H_0^1(\Omega)) \cap H^1(H^2(\Omega)) \tag{2.3.28}$$

can be ensured using a Galerkin procedure, and by an additional consideration, presented below, we obtain the regularity in (2.3.25).

Lemma 2.3.16. *Let the conditions of Theorem 2.3.15 be fulfilled. Then the following estimate holds for almost every $t \in I$:*

$$\|y_t(t)\|^2 + \|\nabla y(t)\|^2 + \rho \int_0^t \|\nabla y_t(s)\|^2 ds \le C \left(\|\nabla y_0\|^2 + \|y_1\|^2 + \|f\|^2_{L^2(L^2(\Omega))} \right).$$

Proof. We set $\phi = y_t$ in (2.3.26) and obtain:

$$(y_{tt}(s), y_t(s)) + (\nabla y(s), \nabla y_t(s)) + \rho \|\nabla y_t(s)\|^2 = (f(s), y_t(s)).$$

Hence,

$$\frac{1}{2} \frac{d}{dt} \|y_t\|^2 + \frac{1}{2} \frac{d}{dt} \|\nabla y\|^2 + \rho \|\nabla y_t(s)\|^2 = (f(s), y_t(s)).$$

Integrating in time from 0 to t we find:

$$\|y_t(t)\|^2 + \|\nabla y(t)\|^2 + 2\rho \int_0^t \|\nabla y_t(s)\|^2 \, ds \\ \leq \|f\|^2_{L^2(L^2(\Omega))} + \|y_1\|^2 + \|\nabla y_0\|^2 + \int_0^t \|y_t(s)\|^2 \, ds.$$

Using Gronwall's lemma we obtain:

$$\|y_t(t)\|^2 \leq C \left(\|\nabla y_0\|^2 + \|y_1\|^2 + \|f\|^2_{L^2(L^2(\Omega))} \right).$$

This gives the desired result. □

Lemma 2.3.17. *Let the conditions of Theorem 2.3.15 be fulfilled. Then the following estimate holds for almost every $t \in I$:*

$$\int_0^t \|\Delta y(s)\|^2 ds + \rho \|\Delta y(t)\|^2 \leq \frac{C}{\rho} \left(\|\nabla y_0\|^2 + \|\Delta y_0\|^2 + \|y_1\|^2 + \|f\|^2_{L^2(L^2(\Omega))} \right).$$

Proof. We use $\phi = -\Delta y$ as a test function in (2.3.26) and obtain:

$$-(y_{tt}(s), \Delta y(s)) + \|\Delta y(s)\|^2 + \rho(\Delta y_t(s), \Delta y(s)) = -(f(s), \Delta y(s))$$

or equivalently

$$-(y_{tt}(s), \Delta y(s)) + \|\Delta y(s)\|^2 + \frac{\rho}{2} \frac{d}{dt} \|\Delta y(s)\|^2 = -(f(s), \Delta y(s)).$$

Integrating in time from 0 to t implies that:

$$-\int_0^t (y_{tt}(s), \Delta y(s)) \, ds + \int_0^t \|\Delta y(s)\|^2 \, ds + \frac{\rho}{2} \|\Delta y(t)\|^2 \\ \leq \frac{1}{2} \|f\|^2_{L^2(L^2(\Omega))} + \frac{1}{2} \int_0^t \|\Delta y(s)\|^2 \, ds + \frac{\rho}{2} \|\Delta y_0\|^2.$$

For the first term on the left-hand side we get for almost every $t \in I$

$$-\int_0^t (y_{tt}(s), \Delta y(s)) \, ds = \int_0^t (y_t(s), \Delta y_t(s)) \, ds - (y_t(t), \Delta y(t)) + (y_t(0), \Delta y(0)) \\ = -\int_0^t \|\nabla y_t(s)\|^2 \, ds - (y_t(t), \Delta y(t)) + (y_1, \Delta y_0).$$

Here, we have used the fact that $y_{tt} = y_t = 0$ on Σ and $y_1 = 0$ on $\partial\Omega$. This yields

$$\int_0^t \|\Delta y(s)\|^2 \, ds + \frac{\rho}{2} \|\Delta y(t)\|^2 \\ \leq \frac{1}{2} \|f\|^2_{L^2(L^2(\Omega))} + \frac{1}{2} \int_0^t \|\Delta y(s)\|^2 \, ds + \frac{\rho}{2} \|\Delta y_0\|^2 \\ + \int_0^t \|\nabla y_t(s)\|^2 \, ds + \frac{1}{\rho} \|y_t(t)\|^2 + \frac{\rho}{4} \|\Delta y(t)\|^2 + \frac{1}{2} \|y_1\|^2 + \frac{1}{2} \|\Delta y_0\|^2.$$

Absorbing terms we obtain:

$$\frac{1}{2}\int_0^t \|\varDelta y(s)\|^2\,ds + \frac{\rho}{4}\|\varDelta y(t)\|^2$$
$$\leq \frac{1}{2}\|f\|^2_{L^2(L^2(\Omega))} + \frac{\rho+1}{2}\|\varDelta y_0\|^2 + \int_0^t \|\nabla y_t(s)\|^2\,ds + \frac{1}{\rho}\|y_t(t)\|^2 + \frac{1}{2}\|y_1\|^2.$$

Using the result from the previous lemma we obtain the desired estimate. □

Lemma 2.3.18. *Let the conditions of Theorem 2.3.15 be fulfilled. Then the following estimate holds for almost every $t \in I$:*

$$\|\nabla y_t(t)\|^2 + \|\varDelta y(t)\|^2 + \rho\int_0^t \|\varDelta y_t(s)\|^2\,ds \leq \frac{1}{\rho}\|f\|^2_{L^2(L^2(\Omega))} + \|\nabla y_1\|^2 + \|\varDelta y_0\|^2.$$

Proof. We proceed as in the proofs of the previous lemmas and choose $\phi = -\varDelta y_t$. This yields

$$-(y_{tt}(s), \varDelta y_t(s)) + (\varDelta y(s), \varDelta y_t(s)) + \rho\|\varDelta y_t(s)\|^2 = -(f(s), \varDelta y_t(s)).$$

We integrate by parts in the first term and obtain for almost every s:

$$\frac{1}{2}\frac{d}{dt}\|\nabla y_t(s)\|^2 + \frac{1}{2}\frac{d}{dt}\|\varDelta y(s)\|^2 + \rho\|\varDelta y_t(s)\|^2 = -(f(s), \varDelta y_t(s)).$$

Integrating in time from 0 to t we obtain:

$$\frac{1}{2}\|\nabla y_t(t)\|^2 + \frac{1}{2}\|\varDelta y(t)\|^2 + \rho\int_0^t \|\varDelta y_t(s)\|^2\,ds$$
$$\leq \frac{1}{2\rho}\|f\|^2_{L^2(L^2(\Omega))} + \frac{\rho}{2}\int_0^t \|\varDelta y_t(s)\|^2\,ds + \frac{1}{2}\|\nabla y_1\|^2 + \frac{1}{2}\|\varDelta y_0\|^2.$$

This implies the desired estimate. □

Lemma 2.3.19. *Let the conditions of Theorem 2.3.15 be fulfilled. Then the following estimate holds:*

$$\int_0^t \|y_{tt}(s)\|^2 ds \leq \frac{C}{\rho}\left(\|f\|^2_{L^2(L^2(\Omega))} + \|\nabla y_0\|^2 + \|\varDelta y_0\|^2 + \|\nabla y_1\|^2\right).$$

Proof. We proceed as in the proof of Lemma 2.3.17 and choose $\phi = y_{tt}$. This yields:

$$\|y_{tt}(s)\|^2 - (\varDelta y(s), y_{tt}(s)) - \rho(\varDelta y_t, y_{tt}) = (f(s), y_{tt}(s)).$$

Hence,

$$\int_0^t \|y_{tt}(s)\|^2\,ds + \int_0^t (\varDelta y_t(s), y_t(s))\,ds - (\varDelta y(t), y_t(t)) + (\varDelta y(0), y_t(0))$$
$$= \int_0^t (f, y_{tt})ds + \rho\int_0^t (\varDelta y_t(s), y_{tt}(s))$$

and thus, we obtain

$$\begin{aligned}\int_0^t \|y_{tt}(s)\|^2\,ds \le \|f\|^2_{L^2(L^2(\Omega))} + \frac{1}{4}\int_0^t \|y_{tt}(s)\|^2 ds + \frac{\rho^2}{2}\int_0^t \|\Delta y_t(s)\|^2 ds \\ + \frac{1}{2}\int_0^t \|y_{tt}(s)\|^2 ds + \int_0^t \|\nabla y_t(s)\|^2 ds + \frac{1}{2}\|\nabla y(t)\|^2 \\ + \frac{1}{2}\|\nabla y_t\|^2 + \frac{1}{2}\|\Delta y_0\|^2 + \frac{1}{2}\|y_1\|^2.\end{aligned}$$

Absorbing terms and using Lemma 2.3.16 and Lemma 2.3.18 we obtain the desired estimate. □

Now, we are able to prove Theorem 2.3.15.

Proof of Theorem 2.3.15. In a first step we verify (2.3.28). Therefore, we employ Galerkin's method, cf. Evans [40, pp. 308], Lions [87, pp. 257]. We are taking $\{w_k\}_{k=1}^\infty$ to be the collection of eigenfunctions for $-\Delta$ on $H_0^1(\Omega)$. Thus we have

$$\{w_k\}_{k=1}^\infty \text{ is an orthogonal basis of } H_0^1(\Omega)$$

and

$$\{w_k\}_{k=1}^\infty \text{ is an orthonormal basis of } L^2(\Omega),$$

where we intend to select the coefficients $d_m^k(t)$ $(0 \le t \le T,\ k = 1,\dots,m,\ m \in \mathbb{N})$ to satisfy

$$\begin{aligned} d_m^k(0) &= (y_0, w_k) \quad (k = 1,\dots,m), \\ \partial_t d_m^k(0) &= (y_1, w_k) \quad (k = 1,\dots,m), \end{aligned}$$

and

$$(y_{tt}^m(s), w_k) + (\nabla y^m(s), \nabla w_k) + \rho(\nabla y_t^m(s), \nabla w_k) = (f(s), w_k) \text{ in } I \quad \text{for } k = 1,\dots,m \tag{2.3.29}$$

for

$$y^m(t,x) = \sum_{k=0}^m d_m^k(t) w_k(x).$$

The finite-dimensional system has a unique solution. This can be proven by formulating the ordinary differential equation as a first order system

$$\begin{aligned} v_m'(t) &= V_m(t, v_m(t)), \\ v_m(0) &= v_{0,m} \end{aligned}$$

with

$$v_m(t) = \begin{pmatrix} d_m(t) \\ d_m'(t) \end{pmatrix}, \quad v_{0,m} = \begin{pmatrix} d_{m,0}(0) \\ d_{m,1}(0) \end{pmatrix}$$

and the function $V_m\colon \bar I\times\mathbb{R}^{2m}\to\mathbb{R}^{2m}$, $V_m(t,x)=Lx+H(t)$ defined by

$$L=\begin{pmatrix}0 & \mathrm{id}\\ -M_m^{-1}A & -\rho M_m^{-1}A\end{pmatrix}\in\mathbb{R}^{2m\times 2m},$$

$$H(t)=\begin{pmatrix}0\\ M_m^{-1}F(t)\end{pmatrix}\in\mathbb{R}^{2m}$$

with the Gramian matrix $M=((w_i,w_j))_{i,j=1}^m$, the matrix $A=((\nabla w_i,\nabla w_j))_{i,j=1}^m$, force vector $F=((f(t),w_j)_{j=1}^m)^T$ and $d_{m,0}=(d_m^1(0),\dots,d_m^m(0))$, $d_{m,1}=(\partial_t d_m^1(0),\dots,\partial_t d_m^m(0))$. Applying Carathéodory's theorem existence of a solution follows.

Using the a priori estimate (2.3.27) for the solution and passing to the limit $m\to\infty$ we build the solution of the strongly damped wave equation (2.3.26) having the desired regularity (2.3.28).

To obtain the full regularity of (2.3.25) we proceed as in Kunisch & Vexler [78]. We have shown that

$$\|y_t\|_{L^\infty(H_0^1(\Omega))}\le C\left(\|f\|_{L^2(Q)}+\|\nabla y_0\|+\|\Delta y_0\|+\|\nabla y_1\|\right). \tag{2.3.30}$$

Using the embedding

$$y\in H^2(L^2(\Omega))\hookrightarrow C^1(L^2(\Omega)),$$

we also have $y_t\in C(L^2(\Omega))$ and thus

$$y_t(t)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\epsilon}^0 y_t(t+\tau)d\tau\quad\text{in } L^2(\Omega). \tag{2.3.31}$$

Define for $t\in\bar I$

$$g_\varepsilon=\frac{1}{\varepsilon}\int_{-\epsilon}^0 y_t(t+\tau)d\tau\in H_0^1(\Omega),$$

using $y_t\in L^\infty(H_0^1(\Omega))$, then we obtain with (2.3.30)

$$\|g_\varepsilon\|_{H_0^1(\Omega)}\le C\left(\|f\|_{L^2(Q)}+\|\nabla y_0\|+\|\Delta y_0\|+\|\nabla y_1\|\right).$$

Therefore, there is a subsequence converging weakly in $H_0^1(\Omega)$ against some $\bar g$ with

$$\|\bar g\|_{H_0^1(\Omega)}\le C\left(\|f\|_{L^2(Q)}+\|\nabla y_0\|+\|\Delta y_0\|+\|\nabla y_1\|\right).$$

Using (2.3.31) we obtain $y_t(t)=\bar g$ and hence, the regularity given in (2.3.25).

Uniqueness of the solution follows by the estimate (2.3.27). □

Now, we consider the strongly damped wave equation with inhomogeneous Dirichlet boundary conditions (2.3.23). To derive some regularity results we apply the method of transposition:

For given $v\in L^2(L^2(\Omega))$ let ζ be the solution of the adjoint equation

$$\begin{cases}\zeta_{tt}-\Delta\zeta+\rho\Delta\zeta_t=v & \text{in } Q,\\ \zeta(T)=0 & \text{in } \Omega,\\ \zeta_t(T)=0 & \text{in } \Omega,\\ \zeta=0 & \text{on } \Sigma.\end{cases} \tag{2.3.32}$$

Using the transformation $t \mapsto T - t$ this equation can be written in the form as (2.3.24). Therefore, we can apply Theorem 2.3.15 leading to $\zeta \in H^2(L^2(\Omega)) \cap C^1(H_0^1(\Omega)) \cap H^1(H^2(\Omega))$. If a smooth solution of (2.3.23) exists, then there holds (by testing with ζ and integrating in time):

$$\begin{aligned}(\zeta_{tt} - \Delta\zeta + \rho\Delta\zeta_t, y)_I + (y_0, \zeta_t(0)) - (y_1, \zeta(0)) + \langle y, \partial_n \zeta\rangle_I \\ - \rho\langle y, \partial_n \zeta_t\rangle_I + \rho(y_0, \Delta\zeta(0)) - \rho\langle y_0, \partial_n\zeta(0)\rangle = (f, \zeta)_I.\end{aligned}$$

This observation suggests the following definition: A function $y \in L^2(L^2(\Omega))$ is called a very weak solution of (2.3.23) if the following variational equation holds for all $v \in L^2(L^2(\Omega))$

$$\begin{aligned}(v, y)_I = -(y_0, \zeta_t(0)) + (y_1, \zeta(0)) - \langle u, \partial_n\zeta\rangle_I \\ + \rho\langle u, \partial_n\zeta_t\rangle_I - \rho(y_0, \Delta\zeta(0)) + \rho\langle y_0, \partial_n\zeta(0)\rangle + (f, \zeta)_I, \quad (2.3.33)\end{aligned}$$

where ζ is the solution to (2.3.32). This leads to the following theorem.

Theorem 2.3.20. *For $u \in L^2(\Sigma)$, $f \in L^1(H^{-2}(\Omega))$, $y_0 \in L^2(\Omega)$, and $y_1 \in H^{-1}(\Omega)$ equation* (2.3.23) *possesses a unique very weak solution defined by* (2.3.33) *and there holds the following estimate*

$$\|y\|_{L^2(L^2(\Omega))} \le C\left(\|u\|_{L^2(\Sigma)} + \|f\|_{L^1(H^{-2}(\Omega))} + \|y_0\| + \|y_1\|_{H^{-1}(\Omega)}\right),$$

where the constant $C = C(\rho)$ tends to infinity as ρ tends to zero.

Proof. The right hand side of (2.3.33) defines a linear functional $G(v)$ on $L^2(L^2(\Omega))$. This functional is bounded. In fact as a consequence of Theorem 2.3.15 we have

$$\begin{aligned}\|\zeta_t(0)\| + \|\zeta(0)\|_{H_0^1(\Omega)} + \|\Delta\zeta(0)\| + \|\partial_n\zeta(0)\|_{L^2(\partial\Omega)} \\ + \|\partial_n\zeta\|_{L^2(\Sigma)} + \|\partial_n\zeta_t\|_{L^2(\Sigma)} + \|\zeta\|_{L^\infty(H^2(\Omega))} \le C\|v\|_{L^2(L^2(\Omega))}.\end{aligned}$$

The representative of this functional in $L^2(L^2(\Omega))$ is y. This implies the desired result. □

2.4 Properties of the wave equation

In this section we look at some basic properties of the wave equation as the simplest representative of the class of second order hyperbolic partial differential equations. These main properties should be taken into account when considering numerical methods.

The homogeneous wave equation on the full space is given by

$$\begin{cases} y_{tt} - c^2\Delta y = f & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ y(0) = y_0 & \text{in } \mathbb{R}^d, \\ y_t(0) = y_1 & \text{in } \mathbb{R}^d \end{cases} \quad (2.4.1)$$

for initial data $y_0 \in C^2(\mathbb{R}, \mathbb{R}^d)$, $y_1 \in C^1(\mathbb{R}, \mathbb{R}^d)$, $f \in C^1(\mathbb{R} \times \mathbb{R}^d)$ and $d \in \mathbb{N}$.

In one dimension, i.e. ($d = 1$), the solution is given by *d'Alembert's formula*

$$y(t,x) = \frac{1}{2}\left(y_0(x+ct) + y_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} y_1(x)dx + \frac{1}{2c}\int_{\mathcal{C}(t,x)} f(s,y)dyds, \quad (2.4.2)$$

where

$$\mathcal{C}(t,x) = \{\, (y,s) \in \mathbb{R}^d \times \mathbb{R} \mid |y-x| \le ct - s, s \ge 0 \,\}$$

is the *cone of dependence*. The value $y(t,x)$ depends only on the data given in $\mathcal{C}(t,x)$; cf. Eriksson et al. [37].

From (2.4.2) we derive that information of the solution of the wave equation propagates with finite speed of propagation c.

Propagation along characteristics

Any singularities given in the initial data are transported into the time-space cylinder without any smoothing as we will see in the following. Starting with d'Alembert's formula we deduce that for $y_1 = 0$ and $f = 0$ the solution can not be more regular than the initial state y_0. Assume y_0 has a singularity in a point $\bar{x}$. Then the solution y has this singularity in all points $x + t = \bar{x}$ and $x - t = \bar{x}$, i.e. the singularity is transported along these lines. They are called characteristics; see Figure 2.1. This shows that in contrast to the heat

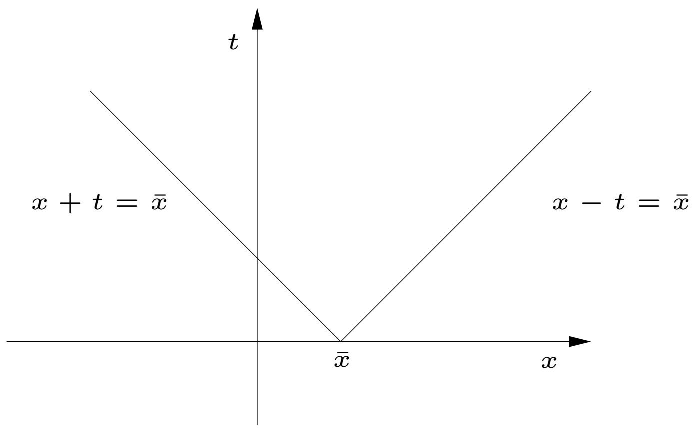


Figure 2.1: Characteristics

equation the wave equation has no smoothing effect with respect to the initial data.

Conservation of energy

The energy associated with the wave equation (2.3.15) for $f \equiv 0$ and $u \equiv 0$ is defined as follows:

Definition 2.4.1. Let $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$ and y be the corresponding solution of (2.3.15) with $f \equiv u \equiv 0$. Then the associated energy E is defined by

$$E(t) = \frac{1}{2}\left(\|y_t(t)\|^2 + \|\nabla y(t)\|^2\right).$$

With this definition we recall the following well-known result:

Proposition 2.4.2. *The energy of the homogeneous wave equation with zero Dirichlet data is constant in time and is determined by the initial data, i.e.*

$$E(t) = \frac{1}{2}\left(\|y_1\|^2 + \|\nabla y_0\|^2\right) = E(0) \quad \forall t \in \bar{I}.$$

Proof. Since $C_0^\infty(\Omega) \subset H_0^1(\Omega)$ and $C_0^\infty(\Omega) \subset L^2(\Omega)$ are dense, we approximate the initial data by smooth functions. Using the same notation for the solution of the wave equation with smooth data, we test the variational formulation by y_t and we obtain

$$\int_0^t \frac{d}{dt}\left(\frac{1}{2}\|y_t\|^2 + \frac{1}{2}\|\nabla y\|^2\right) dt = 0,$$

and consequently,

$$E(t) = \frac{1}{2}\left(\|y_t(t)\|^2 + \|\nabla y(t)\|\right)^2 = \frac{1}{2}\left(\|y_1\|^2 + \|\nabla y_0\|^2\right) = E(0) \quad \forall t \in \bar{I}.$$

With the a priori estimates from the previous section we derive that the result holds also for initial data $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$. □

This means that the energy remains in the system and is transported into the space-time cylinder. In case of the inhomogeneous wave equation the energy may grow or decline in dependence of the data. In contrast to parabolic equations we have no damping. In this thesis we will apply numerical methods which conserve these properties; cf. Chapter 3.

A disadvantage of conservative systems is the fact that they do not occur in nature, because there are always dissipative mechanisms leading to a reduction of the energy of the system. A widely accepted model reflecting the dissipative behaviour has the form

$$x'' + Bx' + Ax = 0, \tag{2.4.3}$$

where A and B are positive self-adjoint operators on a Hilbert space Z with domain $\mathcal{D}(A)$ and $\mathcal{D}(B)$, respectively, dense in Z and $(x, Ax)_Z \geq c\,\|x\|_Z$ for all $x \in Z$; cf. Chen & Russell [31] and Avrin [4]. If there exists a solution x of (2.4.3) twice continuously differentiable with $x(t) \in \mathcal{D}(A)$ and $x'(t) \in \mathcal{D}(B)$, then there holds for the associated energy

$$\frac{d}{dt}E(x(t), x'(t)) = \frac{d}{dt}\frac{1}{2}\left((x'(t), x'(t)) + (A^{\frac{1}{2}}x(t), A^{\frac{1}{2}}x(t))\right) = (x', x'' + Ax) = -(x', Bx') \leq 0.$$

That means, the energy declines with time t. This situation is given in the case of the strongly damped wave equation (2.3.24).

The wave equation as a conservation law

Finally, we recall that the one dimensional wave equation

$$y_{tt} - c^2 y_{xx} = 0$$

can be reformulated as a first order hyperbolic system. Therefore, we make the following formal consideration. We set $v = y_x$ and $w = y_t$ and obtain

$$v_t = (y_x)_t = (y_t)_x = w_x, \quad w_t - c^2 v_x = 0,$$

i.e.

$$\begin{pmatrix} v \\ w \end{pmatrix}_t + \begin{pmatrix} -w \\ -c^2 v \end{pmatrix}_x = 0.$$

Let

$$Y(t,x) = \begin{pmatrix} v(t,x) \\ w(t,x) \end{pmatrix}, \quad Y(0,x) = \begin{pmatrix} v(0,x) \\ w(0,x) \end{pmatrix},$$

with

$$v(0,x) = y_0(x), \quad w(0,x) = y_1(x).$$

Then, we have

$$Y_t + AY_x = 0, \quad A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}.$$

In this thesis we do not consider first order hyperbolic equations. For control of such equations we refer the reader to the references mentioned in the introduction of this thesis.

3 Adaptive finite element methods

In this chapter we derive a posteriori error estimates to solve optimal control problems governed by second order hyperbolic equations of the following type

$$\begin{cases} \text{Minimize} \quad J(u,y), \quad u \in U, \quad y \in X, \quad \text{s.t.} & \text{(3.0.1a)} \\ \begin{aligned} y_{tt} - A(u,y) &= f \quad \text{in } Q, \\ y(0) &= y_0 \quad \text{in } \Omega, \\ y_t(0) &= y_1 \quad \text{in } \Omega \end{aligned} & \text{(3.0.1b)} \end{cases}$$

with an operator A depending on the control u and the state y, a given force f, initial data y_0 and y_1 which may also depend on the control and a cost functional $J\colon U \times X \to \mathbb{R}$; cf. (2.2.1). We consider the case $U_{\text{ad}} = U$, i.e. we do not impose additional constraints on the controls and consequently, this fits in the general setting (P) with a control-to-state operator $S\colon U \to X$, $u \mapsto y = S(u)$ (cf. (2.2.2)) given by (3.0.1b).

This formulation in (3.0.1) incorporates optimal control as well as parameter identification problems.

The optimal control problem is discretized in time and space by space-time finite elements, see Section 3.2. Let (u, y) be the solution of the continuous problem from above and (u_σ, y_σ) the solution of the discretized control problem, where σ is a general discretization parameter including space, time, and control discretization. Then we want to estimate the error

$$J(u,y) - J(u_\sigma, y_\sigma)$$

in the cost functional. We separate the influences of time, space, and control discretization to obtain an efficient algorithm for estimating the error, i.e. we approximate the error in the following way

$$J(u,y) - J(u_\sigma, y_\sigma) \approx \eta_k + \eta_h + \eta_d,$$

where η_k describes the error given by time discretization, η_h by space discretization, and η_d by the discretization of the control.

Furthermore, the conservation of energy of the homogenous linear wave equation is analyzed with respect to meshes changing dynamically in time.

The results presented in this chapter are published in Kröner [73], see also Kröner [74].

Adaptive methods for solving hyperbolic equations of second order are developed in some publications; see, e.g., Rademacher [116], Bangerth & Rannacher [8, 9], Bangerth, Geiger & Rannacher [7], where the dual weighted residual method (DWR, cf. Becker, Kapp & Rannacher [11], Becker & Rannacher [13]) is applied. An adaptive Rothe's method is applied to the wave equation in Bornemann & Schemann [19]. In Adjerid [2] a posteriori error estimates for second-order hyperbolic equations are presented and their asymptotic correctness under mesh refinement is shown. In Bernardi & Sueli [18] a posteriori estimates are derived for the wave equation proving upper and lower bounds for temporal and spatial error indicators.

Adaptive methods for solving optimal control problems governed by elliptic and parabolic state equations are considered in many publications. For the case without control or state constraints; see, e.g., Meidner & Vexler [99], for the case with control constraints; see, e.g., Hintermüller & Hoppe [54], Vexler & Wollner [132], Hintermüller et al. [55], and with state constraints; see, e.g., Benedix [16], Benedix & Vexler [17], Wollner [137], Günther & Hinze [52] and Günther, Hinze & Tber [53].

The main contributions of this chapter are adaptive space-time finite element methods for solving optimal control problems governed by hyperbolic equations. We extend the techniques presented in Meidner & Vexler [99] and Schmich & Vexler [123]. In [123] adaptive finite element methods for parabolic equations are considered using the DWR method on dynamic meshes. In [99] adaptive finite element methods using the DWR technique are developed for optimal control problems governed by parabolic equations with respect to a quantity of interest. In contrast to these two publications, here we consider optimal control problems for hyperbolic equations. We formulate the state equation as a first order system in time and introduce a cG(r)cG(s) discretization for this system, which results for $r = s = 1$ in a Crank-Nicolson scheme when evaluating the right hand side by a trapedoizal rule. For the numerical solution of the control problem we derive a posteriori error estimates. Numerical examples for an optimal control problem with distributed control for the wave equation, a control problem with finite dimensional control and a nonlinear state equation and a control problem with distributed control and a nonlinear state equation are presented. Finally, we analyze the conservation of energy of the homogeneous discrete wave equation on meshes changing dynamically in time when applying a $cG(1)cG(1)$ method. To reflect the behaviour of the continuous equation the energy should be conserved on the discrete level. However, the energy of the discrete system remains only then constant if we allow refinement and coarsening in time but only refinement in space in every step from a time point t_m to t_{m+1} on a given discretization level; cf. also the results in Rademacher [116], Eriksson et al. [37], Bangerth, Geiger & Rannacher [7]. We present the difference of the energy in two neighboring time points and some numerical examples.

The chapter is organized as follows: In Section 3.1 we formulate the control problem in its functional analytic setting, in Section 3.2 we introduce the discretization of the problem, in Section 3.3 we present the optimization algorithm, in Section 3.4 we derive a posteriori error estimates and evaluate the weights of the estimator, in Section 3.5 we formulate the adaptive algorithm, in Section 3.6 we present numerical examples, in Section 3.7 we analyze the conservation of energy of the wave equation on dynamically in time changing meshes, and in Section 3.8 we give an outlook.

3.1 Optimal control problem

In this section we introduce the optimal control problem in its functional analytic setting, which fits in the setting given in (P).

We start by specifying the operator $S\colon U \to X$. Let $U \subset L^2(W)$ be the control space for a given Hilbert space W, X defined as in (2.3.9) and let

$$\bar{X} = L^2(H) \cap H^1(V^*), \quad Y = X \times \bar{X} \tag{3.1.1}$$

for V and H defined as in Section 2.3.1. Further, we introduce the semi-linear form

$$\tilde{a} \colon W \times V \times V \to \mathbb{R}$$

for a differential operator $A \colon W \times V \to V^*$ by

$$\tilde{a}(u, y)(\xi) = \langle A(u, y), \xi \rangle_{V^* \times V},$$

and define the form $a(\cdot, \cdot)(\cdot)$ on $U \times X \times X$ by

$$a(u, y)(\xi) = \int_0^T \tilde{a}(u(t), y(t))(\xi(t)) dt.$$

Moreover, let the initial data $y_0 \colon U \to V$ and $y_1 \colon U \to H$, and the force $f \in L^2(H)$ be given.

Then, we can introduce the state equation in a weak form in analogy to (2.3.10).

Definition 3.1.1. For $u \in U$ a function $\tilde{y} \in X$ is called a solution of the weak state equation if

$$\begin{aligned} (\tilde{y}_{tt}(t), \xi)_H + \tilde{a}(u(t), \tilde{y}(t))(\xi) &= (f(t), \xi)_H \quad \forall \xi \in V, \quad \text{a.e. in } [0, T], \\ \tilde{y}(0) &= y_0(u), \\ \tilde{y}_t(0) &= y_1(u). \end{aligned} \tag{3.1.2}$$

Remark 3.1.2. In the case of control of the initial data we choose U as the space of constant polynomials on $[0, T]$ with values in W being a subset of $L^2(W)$; cf. Meidner [97].

Remark 3.1.3. We do not formulate any further assumptions on $a(\cdot, \cdot)(\cdot)$, since the adaptive algorithm considered in the following sections does not depend on the specific structure of the semi-linear form.

We only assume that equation (3.1.2) admits a unique solution in X. According to Theorem 2.3.3 this is given if, e.g.,

$$a(u, \tilde{y})(\xi) = \int_0^T \bar{a}(\tilde{y}(t), \xi(t)) dt - \int_0^T (\tau(u)(t), \xi(t))_H dt \tag{3.1.3}$$

with $\bar{a} \colon V \times V \to \mathbb{R}$ satisfying (2.3.4a) and (2.3.4b) and $\tau \colon U \to L^2(H)$. Then, we even have

$$\tilde{y} \in C(V), \quad \tilde{y}_t \in C(H), \quad \tilde{y}_{tt} \in L^2(V^*),$$

such that $(f + \tau(u), y_0, y_1) \to (\tilde{y}, \tilde{y}_t)$ is continuous from $L^2(H) \times V \times H$ to $C(V) \times C(H)$. Thus, the initial conditions are well-defined.

The weak formulation (3.1.2) can be written equivalently as a first order system in time:

Lemma 3.1.4. *For $u \in U$ the state equation* (3.1.2) *admits a unique solution if and only if the following system admits a unique solution $y = (y^1, y^2) \in Y$:*

$$\begin{aligned} (y_t^2, \xi^1)_I + a(u, y^1)(\xi^1) + (y^2(0) - y_1(u), \xi^1(0))_H &= (f, \xi^1)_I \quad \forall \xi^1 \in X, \\ (y_t^1, \xi^2)_I - (y^2, \xi^2)_I - (y_0(u) - y^1(0), \xi^2(0))_H &= 0 \qquad\qquad \forall \xi^2 \in \bar{X}. \end{aligned} \tag{3.1.4}$$

Proof. The weak formulation (3.1.2) is equivalent to

$$(\tilde{y}_{tt}, \xi)_I + a(u, \tilde{y})(\xi) + (\tilde{y}_t(0) - y_1(u), \xi(0))_H + (y_0(u) - \tilde{y}(0), \xi_t(0))_H = (f, \xi)_I \quad \forall \xi \in X \tag{3.1.5}$$

with $\tilde{y} \in X$. We show the equivalence of (3.1.4) and (3.1.5):
" $\Rightarrow$ ": Set $\xi^2 = \xi_t^1$, apply partial integration in the second equation and obtain

$$\begin{aligned} -(y_{tt}^1, \xi^1)_I + (y_t^1(T), \xi^1(T)) - (y_t^1(0), \xi^1(0)) + (y_t^2, \xi^1)_I - (y^2(T), \xi^1(T)) \\ + (y^2(0), \xi^1(0)) - (y_0(u) - y^1(0), \xi_t^1(0))_H = 0 \quad \forall \xi^1 \in X. \end{aligned} \tag{3.1.6}$$

Since $(y_t^1(T), \xi^1(T)) - (y^2(T), \xi^1(T)) = 0$ vanishes, we obtain the assertion by replacing $(y_t^2, \xi^1)_I$ in the first equation using (3.1.6).
" $\Leftarrow$ ": Set

$$y^2 = \tilde{y}_t, \tag{3.1.7}$$

$y^1 = \tilde{y}$, $\xi^2 = \xi_t$ and $\xi^1 = \xi$ and test equation (3.1.7) with ξ^2 and integrate over Ω and the time interval $[0, T]$. □

Let the cost functional $J \colon U \times X \to \mathbb{R}$ (cf. (2.2.1)) be defined by using two three times Fréchet-differentiable functionals $J_1 \colon H \to \mathbb{R}$ and $J_2 \colon H \to \mathbb{R}$ by

$$J(u, y^1) = \int_0^T J_1(y^1(t))dt + J_2(y^1(T)) + \frac{\alpha}{2} \|u\|_U^2$$

with $\alpha > 0$ and $u \in U$, $y^1 \in X$.
Then, we can state the optimal control problem

$$\text{Minimize} \quad J(u, y^1) \text{ s.t. } (3.1.4), \quad (u, y^1) \in U \times X. \tag{P^{DWR}}$$

Remark 3.1.5. We only allow that the functional J depends on y^1 and not also on y^2. Otherwise, the right hand side of the corresponding adjoint equation (cf. (3.1.9), (3.1.11)) may be only in $L^2(V^*)$ and thus, the solution of the adjoint equation is in $C(H) \cap C(V^*)$ according to Theorem 2.3.3.

Remark 3.1.6. We assume that problem (P^{DWR}) admits a (locally) unique solution; cf. Proposition 2.2.6.

Remark 3.1.7. Further, in analogy to Meidner & Vexler [99], we assume that there exists a neighbourhood $D \subset U \times X$ of a local solution of (P^{DWR}), such that the linearized form $\tilde{a}'_{y^1}(u(t), y^1(t))(\cdot,\cdot)$ considered as a linear operator

$$\tilde{a}'_{y^1}(u(t), y^1(t)) \colon V \to V^*$$

is an isomorphism for all $(u, y^1) \in D$ and almost all $t \in (0, T)$. This allows all considered adjoint problems to be well-posed.

Let the reduced cost functional be defined as in (2.2.7). We assume that j is three times Fréchet-differentiable. Then, in a local solution u the first (directional) derivative of j vanishes, i.e.

$$j'(u)(\delta u) = 0 \quad \forall \delta u \in U.$$

Let the Lagrangian $\widetilde{\mathcal{L}} \colon U \times Y \times Y \to \mathbb{R}$ be defined by

$$\widetilde{\mathcal{L}}(u, y, p) = J(u, y^1) + (f - y^2_t, p^1)_I - a(u, y^1)(p^1) - (y^1_t - y^2, p^2)_I \\ - (y^2(0) - y_1(u), p^1(0))_H + (y_0(u) - y^1(0), p^2(0))_H$$

for $(u, y, p) \in U \times Y \times Y$ and $y = (y^1, y^2)$ as well as $p = (p^1, p^2)$.

Using the definition of the Lagrangian we can present an explicit representation of the first derivative of the functional j.

Theorem 3.1.8. *Let for a given control $u \in U$ the state $y^1 = S(u)$ satisfy the state equation*

$$\widetilde{\mathcal{L}}'_p(u, y, p)(\delta p) = 0 \quad \forall \delta p \in Y \tag{3.1.8}$$

for $y \in Y$ and if additionally $p \in Y$ is chosen as the solution of the adjoint equation

$$\widetilde{\mathcal{L}}'_y(u, y, p)(\delta y) = 0 \quad \forall \delta y \in Y, \tag{3.1.9}$$

then the following representation of the first derivative of the reduced cost functional holds:

$$j'(u)(\delta u) = \widetilde{\mathcal{L}}'_u(u, y, p)(\delta u) = \alpha(u, \delta u)_I - a'_u(u, y^1)(\delta u, p^1) \\ + (y'_1(u)(\delta u), p^1(0))_H + (y'_0(u)(\delta u), p^2(0))_H \quad \forall \delta u \in U. \tag{3.1.10}$$

The proof follows immediately with standard arguments, .

Remark 3.1.9. The optimality system of the control problem is determined by the derivatives of the Lagrangian, i.e. for a local solution (u, y) the optimality system is given by (3.1.8), (3.1.9) and the optimality condition

$$\widetilde{\mathcal{L}}'_u(u, y, p)(\delta u) = 0 \quad \forall \delta u \in U.$$

For given $y = (y^1, y^2) \in Y$ and $u \in U$ a function $p = (p^1, p^2) \in Y$ is a solution of the adjoint equation (3.1.9) if

$$\begin{aligned} -(\psi^1, p^2_t)_I + a'_{y^1}(u, y^1)(\psi^1, p^1) + (\psi^1(T), p^2(T))_H &= \int_0^T J'_{1,y^1}(y^1)(\psi^1) \\ &\quad + J'_{2,y^1}(y^1(T))(\psi^1(T)) \quad \forall \psi^1 \in X, \\ -(\psi^2, p^1_t)_I - (\psi^2, p^2)_I + (\psi^2(T), p^1(T))_H &= 0 \qquad \forall \psi^2 \in \bar{X}. \end{aligned} \tag{3.1.11}$$

Remark 3.1.10. Under the assumption of Remark 3.1.3 on the form a and for functionals

$$J_1(y^1) = \int_\Omega (y^1 - y_\mathrm{d})^2 dx, \quad J_2(y^1(T)) = \int_\Omega (y^1(T) - y_\mathrm{c})^2 dx$$

with given functions $y_\mathrm{d} \in L^2(H)$ and $y_\mathrm{c} \in V$, existence and uniqueness of a solution p in Y follows by Theorem 2.3.1.

Second derivatives

To formulate sufficient optimality conditions (cf. Proposition 2.2.11) and to apply Newton's method to solve the optimization problem (cf. (3.3.4)), we consider second derivatives of the reduced cost functional.

There holds the following relation

$$j(u) = \widetilde{\mathcal{L}}(u, y, p),$$

where $y \in Y$ is the solution of the state equation for given control $u \in U$ and $p \in Y$ arbitrary.

According to Meidner [97] we obtain for the second derivative of the reduced cost functional the following representation for $u \in U$, corresponding state y and adjoint state p, and directions $\delta u, \tau u \in U$

$$j''(u)(\delta u, \tau u) = \widetilde{\mathcal{L}}_{uu}(u, y, p)(\delta u, \tau u) + \widetilde{\mathcal{L}}_{yu}(u, y, p)(\delta y, \tau u) + \widetilde{\mathcal{L}}_{pu}(u, y, p)(\delta p, \tau u), \quad (3.1.12)$$

where $\delta y \in Y$ is a solution of the tangent equation

$$\widetilde{\mathcal{L}}''_{up}(u, y, p)(\delta u, \xi) + \widetilde{\mathcal{L}}''_{yp}(u, y, p)(\delta y, \xi) = 0 \quad \forall \xi \in Y \quad (3.1.13)$$

and $\delta p \in Y$ a solution of the additional adjoint equation

$$\widetilde{\mathcal{L}}''_{uy}(u, y, p)(\delta u, \psi) + \widetilde{\mathcal{L}}''_{yy}(u, y, p)(\delta y, \psi) + \widetilde{\mathcal{L}}''_{py}(u, y, p)(\delta p, \psi) = 0 \quad \forall \psi \in Y. \quad (3.1.14)$$

The explicit representations of (3.1.12), (3.1.13) and (3.1.14) read as follows with $y = (y^1, y^2)$ and $p = (p^1, p^2)$.

The second derivatives $j''(u)(\delta u, \tau u)$ for $\delta u, \tau u \in U$ are given by

$$\begin{aligned} j''(u)(\delta u, \tau u) = {} & \alpha(\delta u, \tau u)_U - a''_{uu}(u, y^1)(\delta u, \tau u, p) - a''_{y^1 u}(u, y^1)(\delta y, \tau u, p) \\ & - a'_u(u, y)(\tau u, \delta p) + (y'_0(u)(\tau u), \delta p_t(0)) + (y''_0(u)(\delta u, \tau u), p_t(0)) \\ & + (y'_1(u)(\tau u), \delta p(0)) + (y''_1(u)(\delta u, \tau u), p(0)). \end{aligned} \quad (3.1.15)$$

The tangent equation is given by

$$\begin{aligned} (\delta y^2_t, \xi^1)_I + a'_{y^1}(u, y^1)(\delta y^1, \xi^1) + (\delta y^2(0), \xi^1(0))_H &= -a'_u(u, y^1)(\delta u, \xi^1) \\ & \quad + (y'_1(u)(\delta u), \xi^1(0)) \quad \forall \xi^1 \in X, \\ (\delta y^1_t, \xi^2)_I - (\delta y^2, \xi^2)_I + (\delta y^1(0), \xi^2(0))_H &= (y'_0(u)(\delta u), \xi^2(0)) \qquad \forall \xi^2 \in \bar{X} \end{aligned} \quad (3.1.16)$$

and the additional adjoint by

$$
\begin{aligned}
&-(\psi^1,\delta p_t^2)_I + a'_{y^1}(u,y^1)(\psi^1,\delta p^1) + (\psi^1(T),\delta p^2(T))_H + a''_{y^1y^1}(u,y^1)(\delta y^1,\psi^1,p^1)\\
&\qquad + a''_{uy^1}(u,y^1)(\delta u,\psi^1,p^1) = \int_0^T J''_{1,y^1y^1}(y^1)(\delta y^1,\psi^1)dt\\
&\qquad\qquad + J''_{2,y^1y^1}(y^1(T))(\delta y^1(T),\psi^1(T)) \quad \forall \psi^1 \in X,\\
&-(\psi^2,\delta p_t^1)_I - (\psi^2,\delta p^2)_I + (\psi^2(T),\delta p^1(T))_H = 0 \quad \forall \psi^2 \in \bar{X}.
\end{aligned}
\tag{3.1.17}
$$

3.2 Discretization

In this section we discuss the discretization of the optimal control problem (P^{DWR}). We apply a finite element method for both the temporal and the spatial discretization. For the temporal discretization of the state equation we use a Petrov-Galerkin scheme with continuous ansatz functions and discontinuous (in time) test functions. For the spatial discretization we use usual conforming finite elements. This type of discretization, we apply here, is often referred to as the cG(r)cG(s) discretization. The $cG(r)$ method for time discretization is motivated by the fact that it implies conservation of energy of the homogeneous equation and thus reflects the behaviour on the continuous level.

First of all we formulate the semi-discretization in time, then the semi-discretization in space, and finally the discretization of the control. The approaches of optimize-then-discretize and discretize-then-optimize, which are different in general, coincide; see Becker, Meidner & Vexler [12], and Meidner [97]: Discretizing of the optimality system of the continuous problem leads to the same discrete system as deriving the optimality system of the discretized control problem. This results from the fact that we apply a Galerkin discretization.

Finite element discretizations of hyperbolic equations of second order are analyzed in many publications, see, e.g. Johnson [66], where the wave equation is discretized by discontinuous finite elements in time and continuous elements in space and Bangerth & Rannacher [9, 8], where the DWR method is applied to the wave equation; cf. also the references in Section 3.2.4.

In the first section we discretize the state equation in time, in the second we discretize in space and finally we discuss the discretization of the control space. At the end we make some remarks on the discretization concerning a priori estimates.

3.2.1 Time discretization

In this section we introduce the semi-discretization in time of the problem under consideration. Therefore, we consider a partition of the time interval $\bar{I} = [0, T]$ as

$$\bar{I} = \{0\} \cup I_1 \cup \cdots \cup I_M$$

with subintervals $I_m = (t_{m-1}, t_m]$ of size k_m and time points

$$0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T.$$

We define the time discretization parameter k as a piecewise constant function by setting $k|_{I_m} = k_m$ for $m = 1, \dots, M$.

Now, we can define the semi-discrete spaces:

$$X_k^r = \{\, v_k \in C(\bar{I}, H) \mid v_k|_{I_m} \in \mathcal{P}_r(I_m, V) \,\},$$
$$\widetilde{X}_k^r = \{\, v_k \in L^2(I, V) \mid v_k|_{I_m} \in \mathcal{P}_r(I_m, V) \text{ and } v_k(0) \in H \,\},$$

where $\mathcal{P}_r(I_m, V)$ denotes the space of all polynomials of degree smaller or equal to $r \in \mathbb{N}_0$ defined on I_m with values in V. Thus, the space X_k^r consists of continuous functions, whereas in $\widetilde{X}_k^r$ the functions can be discontinuous.

Using these spaces we can formulate the discrete state equation.

Definition 3.2.1. For given control $u_k \in U$ we call $y_k = (y_k^1, y_k^2) \in X_k^r \times X_k^r$ a solution of the semi-discrete state equation if

$$\begin{aligned}
\sum_{m=1}^{M} (\partial_t y_k^2, \xi^1)_{I_m} + a(u_k, y_k^1)(\xi^1) + (y_k^2(0) - y_1(u_k), \xi^1(0))_H &= (f, \xi^1)_I \quad \forall \xi^1 \in \widetilde{X}_k^{r-1}, \\
\sum_{m=1}^{M} (\partial_t y_k^1, \xi^2)_{I_m} - (y^2, \xi^2)_I - (y_0(u_k) - y_k^1(0), \xi^2(0))_H &= 0 \qquad\qquad \forall \xi^2 \in \widetilde{X}_k^{r-1}.
\end{aligned} \tag{3.2.1}$$

Remark 3.2.2. The semi-discrete state equation (3.2.1) is assumed to admit a unique solution. The existence can be shown directly for the case of a $cG(1)$ discretization in time if the form a is given by (3.1.3). The $cG(1)$ method can be written as a time stepping scheme, since the test functions are discontinuous. Let $(Y_m^1, Y_m^2) = y_k(t_m)$ for $m = 0, \dots, M$. Then, for all $\xi^1, \xi^2 \in V$ and $m = 1, \dots, M$ there holds

$$\begin{aligned}
-\frac{k_m}{2}\bar{a}(Y_m^1, \xi^1) - \frac{2}{k_m}(Y_m^1, \xi^1)_H &= -\frac{2}{k_m}(Y_{m-1}^1, \xi^1)_H - 2(Y_{m-1}^2, \xi^1)_H - (f, \xi^1)_{I_m} \\
&\quad + (\tau(u_k), \xi^1)_{I_m} + \frac{k_m}{2}\bar{a}(Y_{m-1}^1, \xi^1), \\
(Y_m^2, \xi^2)_H &= \frac{2}{k_m}(Y_m^1 - Y_{m-1}^1, \xi^2)_H - (Y_{m-1}^2, \xi^2)_H,
\end{aligned}$$

and for all $\xi \in H$

$$(Y_0^1, \xi)_H = (y_0(u_k), \xi)_H, \quad (Y_0^2, \xi)_H = (y_1(u_k)_H, \xi)_H.$$

In each time step an elliptic problem has to be solved, which has a unique solution. The $cG(1)$ method results in a Crank-Nicolson scheme when evaluating the temporal integrals by a trapedoizal rule up to terms of order $\mathcal{O}(k^2)$. The Crank-Nicolson scheme is known to be A-stable and of second order. An a priori analysis for the Crank-Nicolson scheme applied to optimal control of parabolic equations can be found in Meidner & Vexler [102].

Semi-discrete control problem

After these considerations we formulate the semi-discrete optimal control problem

$$\text{Minimize} \quad J(u_k, y_k^1), \quad (u_k, y_k^1) \in U \times X_k^r, \quad \text{s.t.} \quad (3.2.1). \qquad (P_k^{\mathrm{DWR}})$$

The semi-discrete optimal control problem is assumed to admit a (locally) unique solution. To prove existence on the discrete level one can apply the same techniques as on the continuous level; cf. Proposition 2.2.6.

As in the continuous case we define a Lagrangian by

$$\mathcal{L}\colon U \times \big(\mathrm{span}(X \cup X_k^r) \times \mathrm{span}(\bar{X} \cup X_k^r)\big) \times \Big(\mathrm{span}(X \cup \widetilde{X}_k^{r-1}) \times \mathrm{span}(\bar{X} \cup \widetilde{X}_k^{r-1})\Big) \longrightarrow \mathbb{R},$$

with

$$\begin{aligned}\mathcal{L}(u, y, p) = J(u, y^1) + (f, p^1)_I - \sum_{m=1}^{M} (\partial_t y^2, p^1)_{I_m} - a(u, y^1)(p^1) - \sum_{m=1}^{M} (\partial_t y^1, p^2)_{I_m} \\ + (y^2, p^2)_I - (y^2(0) - y_1(u), p^1(0))_H + (y_0(u) - y^1(0), p^2(0))_H \end{aligned} \qquad (3.2.2)$$

for $(u, y, p) \in U \times \big(\mathrm{span}(X \cup X_k^r) \times \mathrm{span}(\bar{X} \cup X_k^r)\big) \times \Big(\mathrm{span}(X \cup \widetilde{X}_k^{r-1}) \times \mathrm{span}(\bar{X} \cup \widetilde{X}_k^{r-1})\Big)$. Immediately, we derive $\widetilde{\mathcal{L}} = \mathcal{L}|_{U \times Y \times Y}$.

Before we formulate the semi-discrete adjoint equation, we introduce the following notations for functions $v \in \widetilde{X}_k^r$:

$$v_{k,m}^+ = \lim_{t \downarrow 0} v_k(t_m + t), \quad v_{k,m}^- = \lim_{t \downarrow 0} v_k(t_m - t) = v_k(t_m), \quad [v_k]_m = v_{k,m}^+ - v_{k,m}^-.$$

The semi-discrete adjoint equation is derived as in the continuous case as a derivative of the Lagrangian (3.2.2):

For given $y_k = (y_k^1, y_k^2) \in X_k^r \times X_k^r$ and $u_k \in U$ the function $p_k = (p_k^1, p_k^2) \in \widetilde{X}_k^{r-1} \times \widetilde{X}_k^{r-1}$ is a solution of the semi-discrete adjoint equation if

$$\begin{aligned} -\sum_{m=1}^{M} (\psi^1, \partial_t p_k^2)_{I_m} - \sum_{m=0}^{M-1} (\psi_m^1, [p_k^2]_m)_H + a'_{y^1}(u_k, y_k^1)(\psi^1, p_k^1) + (\psi_M^1, p_{k,M}^2)_H \\ = \int_0^T J'_{1,y^1}(y_k^1)(\psi^1) dt + J'_{2,y^1}(y_M^1)(\psi_M^1) \qquad \forall \psi^1 \in X_k^r, \\ -\sum_{m=1}^{M} (\psi^2, \partial_t p_k^1)_{I_m} - \sum_{m=0}^{M-1} (\psi_m^2, [p_k^1]_m)_H - (\psi^2, p_k^2)_I + (\psi_M^2, p_{k,M}^1)_H = 0 \qquad \forall \psi^2 \in X_k^r. \end{aligned}$$

3.2.2 Space discretization

In this section the spatial discretization is introduced and we begin with defining the discrete finite element spaces. Here, we assume that Ω is a polygonal and convex domain. For spatial discretization we will consider two- or three-dimensional regular meshes; see, e.g., Ern & Guermond [38]. A mesh consists of quadrilateral or hexahedral cells K, which constitute a non-overlapping cover of the computational domain Ω. In case of a domain with C^2-boundary we have to consider additional elements approximating the boundary, which we omit here. The corresponding mesh is denoted by $\mathcal{T}_h = \{K\}$, where we define the discretization parameter h as a cellwise function by setting $h|_K = h_K$ with the diameter h_K of the cell K. The mesh is called regular if the following conditions are satisfied.

Definition 3.2.3 (Regular mesh)**.** The triangulation $\mathcal{T}_h$ is regular if the following conditions are satisfied

1. $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$,
2. $K \cap \tilde{K} = \emptyset$ or $K = \tilde{K} \quad \forall K, \tilde{K} \in \mathcal{T}_h$,
3. any face of a cell $K \in \mathcal{T}_h$ is either a subset of $\partial\Omega$, or a face of another cell $\tilde{K} \in \mathcal{T}_h$.

Remark 3.2.4. We may weaken the last property in this chapter in the following way. Cells may have hanging nodes, but at most one is allowed for each face in two dimensions (lying on midpoints of faces of neighboring cells) and five in three dimensions.

We construct on the mesh $\mathcal{T}_h$ conforming finite element spaces $V_h^s \subset V$ in a standard way by

$$V_h^s = \{\, v \in V \mid v|_K \in (\mathcal{Q}^s(K))^n \text{ for } K \in \mathcal{T}_h \,\}$$

for $s \in \mathbb{N}$ and $n \in \mathbb{N}$. Here, $\mathcal{Q}^s(K)$ consists of shape functions obtained by bi- or trilinear transformations of polynomials in $\widehat{\mathcal{Q}}^s(\hat{K})$ defined on the reference cell $\hat{K} = (0,1)^d$, where

$$\widehat{\mathcal{Q}}^s(\hat{K}) = \operatorname{span}\left\{ \prod_{j=1}^{d} x_j^{k_j} \;\middle|\; k_j \in \mathbb{N}_0, k_j \le s \right\}$$

and n denotes the number of components of the discrete functions.

Remark 3.2.5. No degrees of freedom are associated to hanging nodes. The value of the finite element functions which corresponds to the hanging node is determined by pointwise interpolation of the neighboring nodes.

In analogy to Schmich & Vexler [123] we allow dynamic mesh change in time and keep the time steps k_m constant in space. We associate with each time point t_m a mesh $\mathcal{T}_h^m$ and a corresponding (spatial) finite element space $V_h^{s,m}$.

Let $\{\, \tau_0, \dots, \tau_r \,\}$ be a basis of $\mathcal{P}_r(I_m, \mathbb{R})$ with the following property:

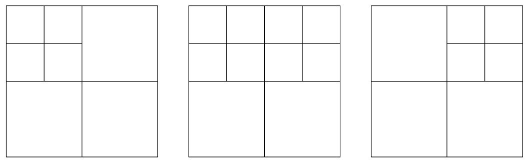
$$\tau_0(t_{m-1}) = 1, \quad \tau_0(t_m) = 0, \quad \tau_i(t_{m-1}) = 0, \quad i = 1, \dots, r.$$

We define

$$\begin{aligned} X_{k,h}^{r,s,m} &= \operatorname{span}\{\, \tau_i v_i \mid v_0 \in V_h^{s,m-1},\ v_i \in V_h^{s,m}, i = 1, \dots, r \,\} \subset \mathcal{P}_r(I_m, V), \\ X_{k,h}^{r,s} &= \{\, v_{kh} \in C(\bar{I}, H) \mid v_{kh}|_{I_m} \in X_{k,h}^{r,s,m} \,\} \subset X_k^r, \\ \widetilde{X}_{k,h}^{r,s} &= \left\{\, v_{kh} \in L^2(I, V) \;\middle|\; v_{kh}|_{I_m} \in \mathcal{P}_r(I_m, V_h^{s,m}) \text{ and } v_{kh}(0) \in V_h^{s,0} \,\right\} \subset \tilde{X}_k^r. \end{aligned}$$

The definition of $X_{k,h}^{r,s,m}$ implies the continuity of functions in $X_{k,h}^{r,s}$.

After this preparation we can formulate the discretized state equation:



(a) $\mathcal{T}_{m-1}$ (b) $\mathcal{T}_{m-\frac{1}{2}}$ (c) $\mathcal{T}_m$

Figure 3.1: Intermediate mesh

Definition 3.2.6. For given $u_{kh} \in U$ we call $y_{kh} = (y_{kh}^1, y_{kh}^2) \in X_{k,h}^{r,s} \times X_{k,h}^{r,s}$ a solution of the discrete state equation if

$$\begin{aligned}\sum_{m=1}^{M} (\partial_t y_{kh}^2, \xi^1)_{I_m} + a(u_{kh}, y_{kh})(\xi^1) + (y_{kh}^2(0) - y_1(u_{kh}), \xi^1(0))_H &= (f, \xi^1)_I \quad && \forall \xi^1 \in \widetilde{X}_{k,h}^{r-1,s}, \\ \sum_{m=1}^{M} (\partial_t y_{kh}^1, \xi^2)_{I_m} - (y_{kh}^2, \xi^2)_I - (y_0(u_{kh}) - y_{kh}^1(0), \xi^2(0))_H &= 0 && \forall \xi^2 \in \widetilde{X}_{k,h}^{r-1,s}.\end{aligned} \tag{3.2.3}$$

The discretized equation (3.2.3) is assumed to admit a unique solution; cf. Remark 3.2.2. Thus, we can state the optimal control problem discretized in time and space.

$$\text{Minimize} \;\; J(u_{kh}, y_{kh}^1), \quad u_{kh} \in U, \quad y_{kh}^1 \in X_{k,h}^{r,s} \quad \text{s.t.} \quad (3.2.3). \tag{P_{kh}^{DWR}}$$

The discretized control problem (P_{kh}^{DWR}) is assumed to admit a (locally) unique solution; cf. the semi-discrete case.

Remark 3.2.7. During the computation we have to evaluate terms as (φ_{m-1}, ψ_m) with $\varphi_{m-1} \in V_h^{s,m-1}$ and $\psi_m \in V_h^{s,m}$ living on different spatial meshes. To tackle this problem, we assume that all meshes $\mathcal{T}_h^m$, $m = 0, \ldots, M$, result from one original mesh $\overline{\mathcal{T}_h}$ by hierarchical refinement. Thus we build up a temporary mesh $\mathcal{T}_h^{m-\frac{1}{2}}$ as a common refinement of $\mathcal{T}_h^{m-1}$ and $\mathcal{T}_h^m$, see Figure 3.1, to evaluate these inner products. For a detail consideration of the practical realization we refer to Schmich & Vexler [123].

3.2.3 Discretization of the control

For the control discretization we introduce a finite dimensional subspace

$$U_d = U_{k_d,h_d}^{r_d,s_d}$$

of U with control discretization parameters r_d, s_d, k_d, h_d, where k_d and h_d are the temporal and spatial mesh parameters and r_d and s_d the maximal polynomial degrees of the temporal and spatial ansatz functions, respectively. In case of distributed control we may choose, e.g., $U_d = X_{k,h}^{0,1}$ with mesh parameters k and h as for the state discretization. If the control is a time dependent parameter with values in $\mathbb{R}^n$, $n \in \mathbb{N}$, we may discretize the control by piecewise constants in time with values in $\mathbb{R}^n$. For a discussion of these aspects in more detail cf. Meidner [97, pp. 37].

All formulations of the state and adjoint equation, the control problems, and the Lagrangian defined on the discrete state spaces and continuous control space can be directly transferred to the level with discretized control and state spaces. Thus the fully-discretized problem reads as

$$\text{Minimize} \quad J(u_\sigma, y_\sigma^1), \quad u_\sigma \in U_d, \quad y_\sigma^1 \in X_{k,h}^{r,s} \quad \text{s.t.} \quad (3.2.3). \tag{P_σ^{DWR}}$$

The discrete solutions are denoted with the index σ collecting the discretization parameters k, h and d. We assume that the corresponding solutions exist; cf. the semi-discrete case.

3.2.4 Remarks on a priori error estimates

In this section we recall some results from the literature about a priori analysis.

There exist many publications on a priori estimates for optimal control for elliptic and parabolic equations.

A priori error estimates for optimal control of elliptic state equations are derived e.g. in Casas & Tröltzsch [29], Arada, Casas & Tröltzsch [3], Casas & Raymond [28], Casas, Mateos & Tröltzsch [27], Casas & Mateos [26]. A variational approach to obtain convergence of second order is proposed in Hinze [57] without discretizing the control. In Meyer & Rösch [103] convergence of second order is shown when applying a post-processing step. A priori estimates for optimal control of an elliptic state equation with bilinear control are derived in Kröner & Vexler [77], Kröner [72] and for an optimal Dirichlet boundary control problem governed by an elliptic equation in May, Rannacher & Vexler [96], Casas & Raymond [28], and Hinze, Deckelnick & Günther [36].

A priori error estimates for optimal control of parabolic equations are shown in several publications, see, e.g., Malanowski [93], Winter [135] and Rösch [121]. In Meidner & Vexler [100, 101] a priori error estimates are derived for a dG(r)cG(s)-discretization of a linear parabolic state equation. In Neitzel & Vexler [111] these methods are transferred to optimal control of a semi-linear parabolic equation. Further results on a priori estimates for optimal control of linear parabolic equations can be found, e.g., in Meidner & Vexler [102], where a Crank-Nicolson scheme is used for time discretization and in Meidner, Rannacher & Vexler [98], where additional state constraints are given.

To the knowledge of the author there exist no results on a priori error estimates for optimal control of second order hyperbolic equations. Thus, here, we just present a short overview on a priori error estimates for the wave equation. These estimates have in common that they assume a lot of regularity on the data; see, e.g., French & Peterson [44], Karakashian & Makridakis [68], Hulbert & Hughes [60]. For a discontinuous Galerkin method for the wave equation see Grote, Schneebeli & Schötzau [48]. In Jenkins, Riviere & Wheeler [65] a priori error estimates for a mixed finite element method applied to the wave equation are derived. In Bales & Lasiecka [6] a priori error estimates for boundary value problems for wave equations are derived and in Bales & Lasiecka [5] for problems with homogeneous boundary conditions. Further a priori estimates can be found in Cowsar, Dupont & Wheeler [34] and Rauch [117].

In Larsson, Thomee & Wahlbin [81] a priori error estimates for the strongly damped wave equation are derived.

We recall an a priori estimate for the wave equation (2.3.15) with $u \equiv 0$ and $f \in L^2(Q)$ from French & Peterson [44]. Let r be the polynomial degree characterizing the discretization in time and s in space time, as described above; cf. Section 3.2.1 and 3.2.2. Further, let $\bar{s} = \max(s, 2)$. Then for $t \in [0, T]$ there holds for a sufficient smooth solution y and the corresponding semi-discrete solution y^1_{kh} (according to Section 3.2.2)

$$\begin{aligned} \left\| y^1_{kh} - y \right\|_{L^\infty(L^2(\Omega))} &\leq C(T+1)k^{r+1} \left(\left\| \partial_t^{r+2} y \right\|_{L^\infty(L^2(\Omega))} + \left\| \partial_t^{r+1} y \right\|_{L^\infty(H^1_0(\Omega))} \right) \\ &\quad + C(T+1)h^{s+1} \left(\| y_{tt} \|_{L^\infty(H^{\bar{s}}(\Omega))} + \| y \|_{L^\infty(H^{s+1})(\Omega)} \right). \end{aligned} \tag{3.2.4}$$

Further, there hold corresponding estimates with respect to the $L^\infty(H^1(\Omega))$-norm and for the first time derivative of the solution y.

3.3 Optimization algorithm

The discrete optimization problems (P^{DWR}_σ) are solved by a Newton method, as described in Meidner [97]. Here, we present a short overview about the main algorithmic aspects.

As on the continuous level the discrete state equation defines a discrete solution operator S_{kh} mapping a given control u_σ to the first component of the corresponding state y^1_σ. To simplify notations in this section we omit the subscript σ at all functions. With

$$j_{kh}(u) = J(u, S_{kh}(u)) \tag{3.3.1}$$

the discrete reduced optimization problem reads as

$$\text{Minimize} \quad j_{kh}(u) \quad \text{for} \quad u \in U_d. \tag{3.3.2}$$

For a given optimal control problem we consider the first necessary optimality condition, i.e. we solve

$$j'_{kh}(u)(\tau u) = 0 \quad \forall \tau u \in U_d$$

for $u \in U_d$. To solve this equation we apply Newton's method and obtain

$$j''_{kh}(u)(\delta u, \tau u) = -j'_{kh}(u)(\tau u) \quad \forall \tau u \in U_d, \quad u_{\text{new}} = \delta u + u_{\text{old}}. \tag{3.3.3}$$

Using Riesz representation theorem we have

$$\begin{aligned} (\nabla j_{kh}(u), \tau u)_U &= j'_{kh}(u)(\tau u) & \forall \tau u \in U_d, \\ (\nabla^2 j_{kh}(u)\delta u, \tau u)_U &= j''_{kh}(u)(\delta u, \tau u) & \forall \delta u, \tau u \in U_d \end{aligned}$$

with $\nabla j_{kh}(u) \in U_d$ and $\nabla^2 j_{kh}(u) \colon U_d \to U_d$. Thus the Newton equation (3.3.3) reads as

$$(\nabla^2 j_{kh}(u)\delta u, \tau u_i)_U = -(\nabla j_{kh}(u), \tau u_i)_U, \quad i = 1, \cdots, \dim U_d \tag{3.3.4}$$

with the basis $\{\tau u_i\}_{i=1}^{\dim U_d}$ of U_d. We can represent the right hand side of (3.3.4) by the coefficient vector $f \in \mathbb{R}^{\dim U_d}$ and the left hand side by the coefficient vector $d \in \mathbb{R}^{\dim U_d}$ as follows

$$(\nabla j_{kh}(u), \tau u_i)_U = \sum_{j=1}^{\dim U_d} f_j (\tau u_j, \tau u_i)_U,$$

$$(\nabla^2 j_{kh}(u)\delta u, \tau u_i)_U = \sum_{j=1}^{\dim U_d} d_j \left(\nabla^2 j_{kh}(u)\tau u_j, \tau u_i\right)_U,$$

i.e. f and d are given by

$$\begin{aligned} Gf &= ((\nabla j_{kh}(u), \tau u_i)_U)_{i=1}^{\dim U_d} = \left(j'_{kh}(u)(\tau u_i)\right)_{i=1}^{\dim U_d}, \\ Kd &= \left((\nabla^2 j_{kh}(u)\delta u, \tau u_i)_U\right)_{i=1}^{\dim U_d} = \left((j''_{kh}(u)\delta u, \tau u_i)\right)_{i=1}^{\dim U_d}, \end{aligned}$$

with the Gramian matrix G given by

$$G_{ij} = (\tau u_j, \tau u_i)_{U_d}$$

and the matrix K given by

$$K_{ij} = \left((\nabla^2 j_{kh}(u)\delta u_j, \tau u_i)_U\right)_{i=1}^{\dim U_d} = j''_{kh}(u)(\delta u_j, \tau u_i).$$

Hence, we obtain the Newton equation in the following form

$$Hd = -f$$

with the coefficient matrix $H = G^{-1}K$ of the Hessian $\nabla^2 j_{kh}(u)$.

If $\dim U_d$ is large, the computation of H is very costly. To avoid assembling the Hessian, we just compute the coefficient vector h of $\nabla^2 j_{kh}(u)\delta u \in U_d$ and obtain

$$(\nabla^2 j_{kh}(u)\delta u, \tau u_i)_U = \sum_{j=1}^{\dim U_d} h_j (\tau u_j, \tau u_i)_U,$$

where h is given by

$$Gh = \left((\nabla^2 j_{kh}(u)\delta u, \tau u_i)_U\right)_{i=1}^{\dim U_d} = \left(j''_{kh}(u)(\delta u, \tau u_i)\right)_{i=1}^{\dim U_d}.$$

The Newton equation (3.3.4) is the first order condition for the linear-quadratic optimization problem

$$\text{Minimize} \quad m(u, \delta u) = j_{kh}(u) + j'_{kh}(u)(\delta u) + \frac{1}{2} j''_{kh}(u)(\delta u, \delta u), \quad \delta u \in U_d. \tag{3.3.5}$$

Furthermore if $j''_{kh}(u)$ is positive definite, the solution of (3.3.5) is also a solution of (3.3.4).

Taking the consideration from above into account problem (3.3.5) can be written as

$$m(u, d) = j_{kh}(u) + (f, d)_G + \frac{1}{2}(Hd, d)_G \tag{3.3.6}$$

Algorithm 3.1: Optimization algorithm

1: Choose initial $u^0 \in U_d$ and $l = 0$.
2: **repeat**
3: Solve the fully discretized state equation and obtain y_l
4: Solve the fully discretized adjoint equation and obtain p_l
5: Assemble f by solving
$$Gf = (j'_{kh}(u^l)(\tau u_i))_{i=1}^{\dim U_d},$$
where $j'_{kh}(u^l)(\tau u_i)$ is evaluated by (3.1.10).
6: Solve
$$\text{Minimize} \quad m(u^l, d), \quad d \in \mathbb{R}^{\dim U_d}, \tag{3.3.7}$$
approximately using only matrix-vector products of the Hessian computed by Algorithm 3.2.
7: Choose ν_l depending on the behaviour of the algorithm.
8: Set $u^{l+1} = u^l + \nu_l \delta u$.
9: Set $l = l + 1$.
10: **until** $|f|_G = \left\|\nabla j_{kh}(u^l)\right\|_U < TOL$.

Algorithm 3.2: Computation of $\nabla^2 j(u^l)\delta u$

Require: y^l and p^l are already computed for given u^l
1: Solve the discrete tangent equation (3.1.16) and obtain δy^l
2: Solve the discrete additional adjoint equation (3.1.17) and obtain δp^l
3: Assemble the coefficient vector h by solving
$$Gh = (j''_{kh}(u)(\delta u, \tau u_i))_{i=1}^{\dim U_d},$$
where $j''_{kh}(u)(\delta u, \tau u_i)$ can be evaluated by (3.1.15).

and we derive the optimization algorithm as given in Algorithm 3.1 with

$$(a, b)_G = a^T G b, \quad |a|_G = \sqrt{(a, a)_G}$$

for coefficient vectors $a, b \in \mathbb{R}^{\dim U_d}$. Thereby, problem (3.3.7) is solved by a conjugate gradient method. The parameter ν in the Algorithm 3.1 is chosen by globalization techniques as line search. For a discussion in more detail and further references we refer to Meidner [97].

3.4 A posteriori error estimates

In this section we consider a posteriori error estimates for the solution (u_σ, y_σ^1) of the fully discretized optimal control problem with respect to J of the following type:

$$J(u, y^1) - J(u_\sigma, y_\sigma^1) \approx \eta_k + \eta_h + \eta_d, \tag{3.4.1}$$

where η_k, η_h, and η_d describe the errors which arise from space, time and control discretization. Thereby, we follow the argumentation in Meidner [97], where optimal control problems for parabolic problems are analyzed.

3.4.1 Dual weighted residual method

To separate the errors in (3.4.1) we split the error in the following way

$$J(u, y^1) - J(u_\sigma, y^1_\sigma) = (J(u, y^1) - J(u_k, y^1_k)) + (J(u_k, y^1_k) - J(u_{kh}, y^1_{kh})) \\ + (J(u_{kh}, y^1_{kh}) - J(u_\sigma, y^1_\sigma)),$$

where (u, y^1) is the solution of the continuous problem (P^{DWR}), (u_k, y^1_k) of the time discretized problem (P^{DWR}_k), (u_{kh}, y^1_{kh}) the solution of the time and space discretized problem (P^{DWR}_{kh}) and (u_σ, y^1_σ) is the solution of the fully discretized problem $(P^{\mathrm{DWR}}_\sigma)$.

To estimate these differences we recall an important theorem in the framework of DWR estimators:

Theorem 3.4.1 (Becker & Rannacher 2002, Meidner 2008)**.** *Let $L\colon Z \to \mathbb{R}$ be a three times Gâteaux differentiable functional for a given function space Z. Further, let $y_1 \in Z_1$, $Z_1 \subset Z$, be a stationary point of L on Z_1, i.e.*

$$L'(y_1)(\delta y_1) = 0 \quad \forall \delta y_1 \in Z_1.$$

This equation is approximated by a Galerkin method using a subspace $Z_2 \subset Z$. The approximative problem seeks $y_2 \in Z_2$ satisfying

$$L'(y_2)(\delta y_2) = 0 \quad \forall \delta y_2 \in Z_2.$$

If the continuous solution y_1 fulfills additionally

$$L'(y_1)(\hat{y}_2) = 0 \quad \forall \hat{y}_2 \in Z_2,$$

then we have for arbitrary $\hat{y}_2 \in Z_2$ the error representation

$$L(y_1) - L(y_2) = \frac{1}{2} L'(y_2)(y_1 - \hat{y}_2) + \mathcal{R}, \tag{3.4.2}$$

where the remainder term $\mathcal{R}$ is given by means of $e = y_1 - y_2$ by

$$\mathcal{R} = \frac{1}{2} \int_0^1 L'''(y_2 + se)(e, e, e) \cdot s \cdot (s - 1) ds.$$

For a proof we refer to Meidner [97] and Becker & Rannacher [13].

We have the following result for a posteriori error estimation of the discretization error, thereby we follow the argumentation in Meidner [97] and Schmich & Vexler [123].

Theorem 3.4.2. *Assume that (u, y, p), (u_k, y_k, p_k), (u_{kh}, y_{kh}, p_{kh}) and $(u_\sigma, y_\sigma, p_\sigma)$ are stationary points of $\mathcal{L}$ on the continuous and on the different levels of discretization, respectively, i.e.*

$$
\begin{aligned}
\mathcal{L}'(u, y, z)(\delta u, \delta y, \delta p) &= 0 \quad \forall(\delta u, \delta y, \delta p) \in U \times Y \times Y,\\
\mathcal{L}'(u_k, y_k, z_k)(\delta u_k, \delta y_k, \delta p_k) &= 0\\
&\qquad \forall(\delta u_k, \delta y_k, \delta p_k) \in U \times (X_k^r)^2 \times (\widetilde{X}_k^{r-1})^2,\\
\mathcal{L}'(u_{kh}, y_{kh}, z_{kh})(\delta u_{kh}, \delta y_{kh}, \delta p_{kh}) &= 0\\
&\qquad \forall(\delta u_{kh}, \delta y_{kh}, \delta p_{kh}) \in U \times (X_{k,h}^{r,s})^2 \times (\widetilde{X}_{k,h}^{r-1,s})^2,\\
\mathcal{L}'(u_\sigma, y_\sigma, z_\sigma)(\delta u_\sigma, \delta y_\sigma, \delta p_\sigma) &= 0\\
&\qquad \forall(\delta u_\sigma, \delta y_\sigma, \delta p_\sigma) \in U_d \times (X_{k,h}^{r,s})^2 \times (\widetilde{X}_{k,h}^{r-1,s})^2.
\end{aligned}
$$

Then, there holds for the errors with respect to the cost functional due to time, space, and control discretization:

$$
\begin{aligned}
J(u, y^1) - J(u_k, y_k^1) &= \frac{1}{2}\mathcal{L}'(u_k, y_k, p_k)(u - \hat{u}_k, y - \hat{y}_k, p - \hat{p}_k) + \mathcal{R}_k,\\
J(u_k, y_k^1) - J(u_{kh}, y_{kh}^1) &= \frac{1}{2}\mathcal{L}'(u_{kh}, y_{kh}, p_{kh})(u_k - \hat{u}_{kh}, y_k - \hat{y}_{kh}, p_k - \hat{p}_{kh}) + \mathcal{R}_h,\\
J(u_{kh}, y_{kh}^1) - J(u_\sigma, y_\sigma^1) &= \frac{1}{2}\mathcal{L}'(u_\sigma, y_\sigma, p_\sigma)(u_{kh} - \hat{u}_\sigma, y_{kh} - \hat{y}_\sigma, p_{kh} - \hat{p}_\sigma) + \mathcal{R}_d.
\end{aligned}
$$

Here $(\hat{u}_k, \hat{y}_k, \hat{p}_k) \in U \times (X_k^r)^2 \times (\widetilde{X}_k^{r-1})^2$, $(\hat{u}_{kh}, \hat{y}_{kh}, \hat{p}_{kh}) \in U \times (X_{k,h}^{r,s})^2 \times (\widetilde{X}_{k,h}^{r-1,s})^2$, $(\hat{u}_\sigma, \hat{y}_\sigma, \hat{p}_\sigma) \in U_d \times (X_{k,h}^{r,s})^2 \times (\widetilde{X}_{k,h}^{r-1,s})^2$ can be chosen arbitrarily and the terms $\mathcal{R}_k$, $\mathcal{R}_h$ and $\mathcal{R}_d$ have the same structure as given in Theorem 3.4.1.

Proof. We use the following identities which hold for the solutions of the control problems on the different levels:

$$
\begin{aligned}
J(u, y^1) - J(u_k, y_k^1) &= \mathcal{L}(u, y, p) - \mathcal{L}(u_k, y_k, p_k), && (3.4.3)\\
J(u_k, y_k^1) - J(u_{kh}, y_{kh}^1) &= \mathcal{L}(u_k, y_k, p_k) - \mathcal{L}(u_{kh}, y_{kh}, p_{kh}), && (3.4.4)\\
J(u_{kh}, y_{kh}^1) - J(u_\sigma, y_\sigma^1) &= \mathcal{L}(u_{kh}, y_{kh}, p_{kh}) - \mathcal{L}(u_\sigma, y_\sigma, p_\sigma). && (3.4.5)
\end{aligned}
$$

To apply the abstract error representation (3.4.2), we choose the spaces Z_1 and Z_2 in the following way:

$$
\begin{aligned}
\text{for (3.4.3)}: &\qquad Z_1 = U \times Y \times Y,\\
&\qquad Z_2 = U \times (X_k^r)^2 \times (\widetilde{X}_k^{r-1})^2,\\
\text{for (3.4.4)}: &\qquad Z_1 = U \times (X_k^r)^2 \times (\widetilde{X}_k^{r-1})^2,\\
&\qquad Z_2 = U \times (X_{k,h}^{r,s})^2 \times (\widetilde{X}_{k,h}^{r-1,s})^2,\\
\text{for (3.4.5)}: &\qquad Z_1 = U \times (X_{k,h}^{r,s})^2 \times (\widetilde{X}_{k,h}^{r-1,s})^2,\\
&\qquad Z_2 = U_d \times (X_{k,h}^{r,s})^2 \times (\widetilde{X}_{k,h}^{r-1,s})^2.
\end{aligned}
$$

For the second and third pairing we have $Z_2 \subset Z_1$ and we can choose $Z = Z_1$. In the first case we have $\widetilde{X}_k^{r-1} \not\subset X$, $\widetilde{X}_k^{r-1} \not\subset \bar{X}$ and $X_k^r \not\subset X$. Therefore, we set $Z = Z_1 \cup Z_2$ and have to verify

$$\mathcal{L}'_p(u,y,p)(\hat{p}_k) = 0 \quad \forall \hat{p}_k \in (\widetilde{X}_k^{r-1})^2, \tag{3.4.6}$$
$$\mathcal{L}'_y(u,y,p)(\hat{y}_k) = 0 \quad \forall \hat{y}_k \in (X_k^r)^2. \tag{3.4.7}$$

Equation (3.4.6) is equivalent to

$$\begin{aligned} (y_t^2, \hat{p}_k^1)_I + a(u,y^1)(\hat{p}_k^1) + (y^2(0) - y_1(u), \hat{p}_k^1(0))_H &= (f, \hat{p}_k^1)_I \quad \forall \hat{p}_k^1 \in \widetilde{X}_k^{r-1}, \\ (y_t^1, \hat{p}_k^2)_I - (y^2, \hat{p}_k^2)_I - (y_0(u) - y^1(0), \hat{p}_k^2(0))_H &= 0 \quad \forall \hat{p}_k^2 \in \widetilde{X}_k^{r-1} \end{aligned} \tag{3.4.8}$$

for $\hat{p}_k = (\hat{p}_k^1, \hat{p}_k^2)$. From the continuous equation (3.1.4) and since $V \subset H$ is dense, we have for all $w \in H$ the property $(y^2(0) - y_1(u), w)_H = 0$ and $(y_0(u) - y^1(0), w)_H = 0$, hence it remains to prove

$$\begin{aligned} (y_t^2, \hat{p}_k^1)_I + a(u,y^1)(\hat{p}_k^1) &= (f, \hat{p}_k^1)_I \quad \forall \hat{p}_k^1 \in \widetilde{X}_k^{r-1}, \\ (y_t^1, \hat{p}_k^2)_I - (y^2, \hat{p}_k^2)_I &= 0 \quad \forall \hat{p}_k^2 \in \widetilde{X}_k^{r-1}. \end{aligned} \tag{3.4.9}$$

Since $X \times \bar{X}$ is dense in $L^2(V) \times L^2(H)$ w.r.t. to the $L^2(V) \times L^2(H)$-norm, relation (3.4.9) holds true for all test functions $(\xi^1, \xi^2) \in L^2(V) \times L^2(H)$ instead of $(\hat{p}_k^1, \hat{p}_k^2)$ and hence for all functions $(\hat{p}_k^1, \hat{p}_k^2) \in \widetilde{X}_k^{r-1} \times \widetilde{X}_k^{r-1} \subset L^2(V) \times L^2(H)$. For the adjoint equation (3.4.7) the argument is the same. Thus, the assertion follows immediately from the previous Theorem 3.4.1. □

For

$$\begin{aligned} \hat{u}_k &= u \in U, & \hat{u}_{kh} &= u_k \in U, \\ \hat{p}_\sigma &= p_{kh} \in \widetilde{X}_{k,h}^{r-1,s} \times \widetilde{X}_{k,h}^{r-1,s}, & \hat{y}_\sigma &= y_{kh} \in X_{kh}^{r,s} \times X_{k,h}^{r,s}, \end{aligned}$$

we have

$$\begin{aligned} \mathcal{L}'_u(u_k, y_k, p_k)(u - \hat{u}_k) &= 0, & \mathcal{L}'_u(u_\sigma, y_{kh}, p_{kh})(u_k - \hat{u}_{kh}) &= 0, \\ \mathcal{L}'_y(u_\sigma, y_\sigma, p_\sigma)(y_{kh} - \hat{y}_\sigma) &= 0, & \mathcal{L}'_p(u_\sigma, y_\sigma, p_\sigma)(p_{kh} - \hat{p}_\sigma) &= 0. \end{aligned}$$

Hence, the statement of the theorem above can be formulated as

$$\begin{aligned} J(u,y^1) - J(u_k, y_k^1) &\approx \frac{1}{2}\left(\mathcal{L}'_y(u_k,y_k,p_k)(y - \hat{y}_k) + \mathcal{L}'_p(u_k,y_k,p_k)(p - \hat{p}_k)\right), \\ J(u_k, y_k^1) - J(u_{kh}, y_{kh}^1) &\approx \frac{1}{2}\big(\mathcal{L}'_y(u_{kh}, y_{kh}, p_{kh})(y_k - \hat{y}_{kh}) \\ &\qquad + \mathcal{L}'_p(u_\sigma, y_{kh}, p_{kh})(p_k - \hat{p}_{kh})\big), \\ J(u_{kh}, y_{kh}^1) - J(u_\sigma, y_\sigma^1) &\approx \frac{1}{2}\mathcal{L}'_u(u_\sigma, y_\sigma, p_\sigma)(u_{kh} - \hat{u}_\sigma). \end{aligned} \tag{3.4.10}$$

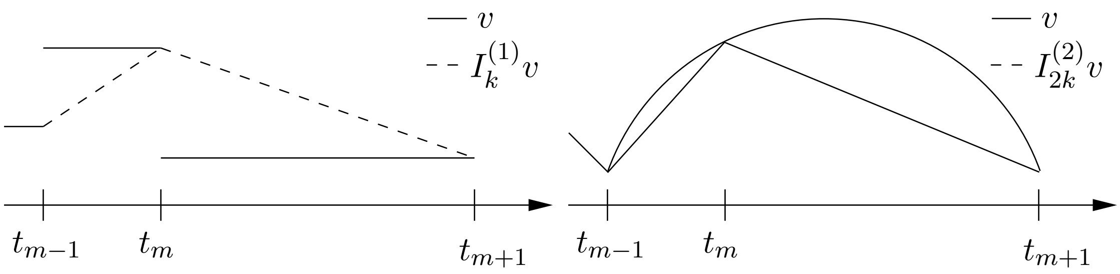


Figure 3.2: Linear and quadratic interpolation

3.4.2 Estimate of the weights

The error estimates presented in (3.4.10) contain the unknown state y and adjoint state p as well as their semi-discrete analogs and the control u_{kh}. In this section we present an approximation of the weights in (3.4.10) containing these unknown functions. There are several approaches how to treat these terms. We estimate them by interpolations in higher-order finite element spaces. There are several publications confirming that this approach works very well in the context of parabolic equations; see, e.g., Becker & Rannacher [13], Meidner & Vexler [99] and Schmich & Vexler [123]. Here, we consider the case with $r = s = 1$ and a discrete control space consisting of functions that are piecewise constant in time.

We introduce the following operators

$$P_k^{(1)} = \bar{I}_k^{(1)} - \mathrm{id}, \quad P_k^{(2)} = \bar{I}_{2k}^{(2)} - \mathrm{id}, \quad P_h^{(2)} = \bar{I}_{2h}^{(2)} - \mathrm{id},$$

with

$$\bar{I}_k^{(1)} = \begin{pmatrix} I_k^{(1)} & 0 \\ 0 & I_k^{(1)} \end{pmatrix}, \quad \bar{I}_{2k}^{(2)} = \begin{pmatrix} I_{2k}^{(2)} & 0 \\ 0 & I_{2k}^{(2)} \end{pmatrix}, \quad \bar{I}_{2h}^{(2)} = \begin{pmatrix} I_{2h}^{(2)} & 0 \\ 0 & I_{2h}^{(2)} \end{pmatrix}$$

and

$$I_k^{(1)} \colon \widetilde{X}_k^0 \to X_k^1, \quad I_{2k}^{(2)} \colon X_k^1 \to X_{2k}^2, \quad I_{2h}^{(2)} \colon \begin{cases} X_{k,h}^{1,1} \to X_{k,2h}^{1,2}, \\ \widetilde{X}_{k,h}^{0,1} \to \widetilde{X}_{k,2h}^{0,2}. \end{cases}$$

The action of the operators $I_k^{(1)}$ and $I_{2k}^{(2)}$ is presented in Figure 3.2. The action of the interpolation operator $I_{2h}^{(2)}$ can be computed for spatial meshes with a patch structure. A mesh has a patch structure in two (three) dimensions if we can combine four (eight) adjacent cells to a macrocell on which the biquadratic interpolation can be defined.

We replace the weights in the estimator (3.4.10) as follows

$$\begin{aligned} y - \hat{y}_k &\approx P_k^{(2)} y_k, & p - \hat{p}_k &\approx P_k^{(1)} p_k, & u_{kh} - \hat{u}_\sigma &\approx P_d u_\sigma, \\ y_k - \hat{y}_{kh} &\approx P_h^{(2)} y_{kh}, & p_k - \hat{p}_{kh} &\approx P_h^{(2)} p_{kh}, \end{aligned}$$

where the definition of P_d depends on the choice of U_d; cf. Remark 3.4.3.

For a discussion of these aspects in more detail cf. Meidner [97].

Now, in order to make the terms in the error estimator computable we replace the unknown solutions by the fully discretized ones. Thus, we obtain

$$J(u, y^1) - J(u_\sigma, y^1_\sigma) \approx \eta_h + \eta_k + \eta_d$$

with

$$\begin{aligned}
\eta_k &= \frac{1}{2}\left(\mathcal{L}'_y(u_\sigma, y_\sigma, p_\sigma)(P^{(2)}_k y_\sigma) + \mathcal{L}'_p(u_\sigma, y_\sigma, p_\sigma)(P^{(1)}_k p_\sigma)\right), \\
\eta_h &= \frac{1}{2}\left(\mathcal{L}'_y(u_\sigma, y_\sigma, p_\sigma)(P^{(2)}_h y_\sigma) + \mathcal{L}'_p(u_\sigma, y_\sigma, p_\sigma)(P^{(2)}_h p_\sigma)\right), \\
\eta_d &= \frac{1}{2}\mathcal{L}'_u(u_\sigma, y_\sigma, p_\sigma)(P_d u_\sigma).
\end{aligned} \tag{3.4.11}$$

Remark 3.4.3. In several cases the estimator η_d vanishes. If the control space U is finite dimensional, e.g. in the case of parameter estimation, we choose $P_d = 0$ because in this case we have $u_{kh} = u_\sigma$. Furthermore, in several cases there holds $\mathcal{L}'_u(u_\sigma, y_\sigma, p_\sigma)(\cdot) = 0$. This is often the case if the control enters linearly the right hand side or the boundary condition and if the control is discretized as the adjoint state or as the restriction of the adjoint state to the boundary. Then the optimality condition is also pointwise satisfied, and the derivative of the Lagrangian w.r.t. to the control vanishes; cf. Example 3.4.4. Nevertheless, to stabilize the algorithm it may be useful to discretize the control on a coarser time mesh as the adjoint state. Then $\mathcal{L}'_u(u_\sigma, y_\sigma, p_\sigma)(\cdot)$ does not vanish and we choose P_d as a modification of the operators P_k and P_h.

Example 3.4.4. We present an example where the estimator η_d vanishes and one where it does not vanish. Let the optimality condition be given by

$$(\alpha u_\sigma + p_\sigma, \delta u) = 0 \quad \forall \delta u \in U_d.$$

If $p_\sigma \in U_d$ there holds $\alpha u_\sigma + p_\sigma = 0$, which implies $\mathcal{L}'_u(u_\sigma, y_\sigma, p_\sigma)(\cdot) = 0$.

However, if the optimality condition is given by

$$(\alpha u_\sigma + y_\sigma p_\sigma, \delta u) = 0 \quad \forall \delta u \in U_d,$$

e.g. in optimal control problems with bilinear control, cf. Kröner & Vexler [77], the product $y_\sigma p_\sigma$ is in general not in U_d and we cannot expect $\mathcal{L}'_u(u_\sigma, y_\sigma, p_\sigma)(\cdot) = 0$.

To derive an explicit representation of the error estimators we set

$$\begin{aligned}
&Y_0 = y_\sigma(0), \quad Y_m = y_\sigma(t_m), \quad P_0 = p_\sigma(0), \quad P_m = p_\sigma|_{I_m}, \\
&U_0 = u_\sigma(0), \quad U_m = u_\sigma|_{I_m}
\end{aligned} \tag{3.4.12}$$

for $m = 1, \ldots, M$ and let

$$Y_m = (Y^1_m, Y^2_m), \quad P_m = (P^1_m, P^2_m) \tag{3.4.13}$$

for $Y^1_m, Y^2_m, P^1_m, P^2_m \in V^{1,m}_h$, $m = 0, \ldots, M$. We evaluate the time integrals on every interval $I_m = (t_{m-1}, t_m]$ by applying a box rule for all functions constant on I_m and by a Gaussian quadrature rule with Gauss points t^1_m, t^2_m or a trapedoizal rule for all other functions. We use the fact that $P^{(1)}_k p_\sigma$ is linear and $P^{(2)}_k y_\sigma$ is quadratic on I_m, so we can compute values of $P^{(1)}_k p_\sigma$ and $P^{(2)}_k y_\sigma$ exactly for every $t \in I_m$. In the following the derivatives of the Lagrangian are presented to determine η_h and η_k.

$$\mathcal{L}'_p(u_\sigma, y_\sigma, p_\sigma)(P_k p_\sigma) = \sum_{m=1}^{M}\sum_{i=1}^{2}\bigg\{\frac{k_m}{2}(f(t_m^i),(I_k^{(1)}p_\sigma^1)(t_m^i) - P_m^1)$$
$$-\frac{1}{2}(Y_m^2 - Y_{m-1}^2,(I_k^{(1)}p_\sigma^1)(t_m^i) - P_m^1) - \frac{k_m}{2}a(U_m, y_\sigma^1(t_m^i))(I_k^{(1)}p_\sigma^1(t_m^i) - P_m^1)$$
$$-\frac{1}{2}(Y_m^1 - Y_{m-1}^1,(I_k^{(1)}p_\sigma^2)(t_m^i) - P_m^2) + \frac{k_m}{2}(y_\sigma^2(t_m^i),(I_k^{(1)}p_\sigma^2)(t_m^i) - P_m^2)\bigg\},$$

$$\mathcal{L}'_y(u_\sigma, y_\sigma, p_\sigma)(P_k y_\sigma) = \sum_{m=1}^{M}\bigg\{\sum_{i=1}^{2}\frac{k_m}{2}(J'_{1,y^1}(y_\sigma^1(t_m^i))(I_{2k}^{(2)}y_\sigma^1(t_m^i)))$$
$$-\frac{k_m}{2}(J'_{1,y^1}(Y_m^1)(Y_m)+J'_{1,y^1}(Y_{m-1}^1)(Y_{m-1}))-\sum_{i=1}^{2}\frac{k_m}{2}a'_u(U_m, Y^1(t_i^*))((I_{2k}^{(2)}y_\sigma^1)(t_i^*), P_m^1)$$
$$+\frac{k_m}{2}\left(a'_u(U_m, Y_m^1)(Y_m^1, P_m^1) + a'_u(U_m, Y_{m-1}^1)(Y_{m-1}^1, P_m^1)\right)$$
$$+\sum_{i=1}^{2}\frac{k_m}{2}((I_{2k}^{(2)}y_\sigma^2(t_i^*), P_m^2)) - \frac{k_m}{2}(Y_m^2 + Y_{m-1}^2, P_m^2)\bigg\},$$

$$\mathcal{L}'_p(u_\sigma, y_\sigma, p_\sigma)(P_h p_\sigma) = \sum_{m=1}^{M}\bigg\{\frac{k_m}{2}(f(t_{m-1}) + f(t_m),\mathcal{I}P_m^1 - P_m^1)$$
$$-(Y_m^2 - Y_{m-1}^2,\mathcal{I}P_m^1 - P_m^1) - \frac{k_m}{2}a(U_m, Y_m^1)(\mathcal{I}P_m^1 - P_m^1) - \frac{k_m}{2}a(U_m, Y_{m-1}^1)(\mathcal{I}P_m^1 - P_m^1)$$
$$-(Y_m^1 - Y_{m-1}^1,\mathcal{I}P_m^2 - P_m^2) + \frac{k_m}{2}(Y_m^2,\mathcal{I}P_m^2 - P_m^2) + \frac{k_m}{2}(Y_{m-1}^2,\mathcal{I}P_m^2 - P_m^2)\bigg\}$$
$$-(Y_0^2 - y_1(u_\sigma),\mathcal{I}P_0^1 - P_0^1) + (y_0(u_\sigma) - Y_0^1,(\mathcal{I}P_0^2 - P_0^2)),$$

$$
\begin{aligned}
\mathcal{L}'_y(u_\sigma, y_\sigma, p_\sigma)(P_h y_\sigma) &= J'_{2,y^1}(Y_M)(\mathcal{I}Y_M^1 - Y_M^1) - (\mathcal{I}Y_M^1 - Y_M^1, P_M^2) \\
&+ \frac{k_M}{2} J'_{1,y^1}(Y_M)(\mathcal{I}Y_M^1 - Y_M^1) - \frac{k_M}{2} a'_y(U_M, Y_M^1)(\mathcal{I}Y_M^1 - Y_M^1, P_M^1) \\
&+ \sum_{m=1}^{M-1} \Bigg\{ \frac{k_m + k_{m+1}}{2} J'_{1,y^1}(Y_m)(\mathcal{I}Y_m^1 - Y_m^1) + (\mathcal{I}Y_m^1 - Y_m^1, P_{m+1}^2 - P_m^2) \\
&- \frac{k_{m+1}}{2} a'_y(U_{m+1}, Y_m^1)(\mathcal{I}Y_m^1 - Y_m^1, P_{m+1}^1) - \frac{k_m}{2} a'_y(U_m, Y_m^1)(\mathcal{I}Y_m^1 - Y_m^1, P_m^1) \Bigg\} \\
&+ \frac{k_1}{2} J'_{1,y^1}(Y_0)(\mathcal{I}Y_0^1 - Y_0^1) - (\mathcal{I}Y_0^1 - Y_0^1, P_1^2 - P_0^2) - \frac{k_m}{2} a'_y(U_1, Y_0^1)(\mathcal{I}Y_0^1 - Y_0^1, P_1^1) \\
&- (\mathcal{I}Y_M^2 - Y_M^2, P_M^1) - \frac{k_M}{2}(\mathcal{I}Y_M^2 - Y_M^2, P_M^2) + \sum_{m=1}^{M-1} \Bigg\{ (\mathcal{I}Y_m^2 - Y_m^2, P_{m+1}^1 - P_m^1) \\
&- \frac{k_{m+1}}{2}(\mathcal{I}Y_m^2 - Y_m^2, P_{m+1}^2) - \frac{k_m}{2}(\mathcal{I}Y_m^2 - Y_m^2, P_m^2) \Bigg\} - (\mathcal{I}Y_0^2 - Y_0^2, P_1^1 - P_0^1) \\
&- \frac{k_1}{2}(\mathcal{I}Y_0^2 - Y_0^2, P_1^2).
\end{aligned}
$$

3.4.3 Localization of error estimators

In this section we describe how we localize the error estimators η_k and η_h presented in the last section, cf. the similar case of a $cG(1)dG(0)$ discretization in Meidner [97]. The error estimator η_d can be localized in a similar way for concrete choices of the discretizations of the control space. Let η_k^m and η_h^m be given in terms of the time stepping residuals, i.e. we have for $m = 0, \ldots, M$

$$
\begin{aligned}
\eta_k^m &= \frac{1}{2}\left(\mathcal{L}_p^{m\prime}(u_\sigma, y_\sigma, p_\sigma)(P_k^{(1)} p_\sigma) + \mathcal{L}_y^{m\prime}(u_\sigma, y_\sigma, p_\sigma)(P_k^{(2)} y_\sigma) \right), \\
\eta_h^m &= \frac{1}{2}\left(\mathcal{L}_p^{m\prime}(u_\sigma, y_\sigma, p_\sigma)(P_h^{(2)} p_\sigma) + \mathcal{L}_y^{m\prime}(u_\sigma, y_\sigma, p_\sigma)(P_h^{(2)} y_\sigma) \right).
\end{aligned}
$$

The residuals $\mathcal{L}^m$ denote those parts of $\mathcal{L}$ which belong to the time interval I_m or to the initial time $t = 0$ for $m = 0$.

We split up the error estimators η_k and η_h into their contributions on each subinterval I_m by

$$
\eta_k = \sum_{m=1}^{M} \eta_k^m, \quad \eta_h = \sum_{m=0}^{M} \eta_h^m.
$$

In contrast to the temporal indicators η_k^m the spatial indicators η_h^m have to be further localized to indicators on each spatial mesh.

Remark 3.4.5. A direct localization of η_h^m by separating the contributions of each cell results in a large overestimation of the error due to the oscillatory behaviour of the residual terms; see Carstensen & Verfürht [25]. The localization is often done by using integration by parts in space; see Becker & Rannacher [13].

Here, we apply the following techniques introduced in Braack & Ern [20] to localize η_h^m. We define the following Lagrange nodal bases

$$\{\, \varphi_i^m \mid i = 1, \dots, N_m \,\}$$

of the space $V_h^{1,m}$ corresponding to the mesh $\mathcal{T}_h^m$ with $N_m = \dim V_h^{1,m}$, $m = 0, \dots, M$, and where φ_i^m is the nodal bases function associated with the node i. Accordingly, we obtain the biquadratic basis functions

$$\{\, \psi_i^m = I_{2h}^{(2)} \varphi_i^m \mid i = 1, 2, \dots, N_m \,\} \subset V_{2h}^{2,m}.$$

Moreover, let

$$\begin{aligned} \Psi_{m,i}^y &= \mathcal{L}_p^m(u_\sigma, y_\sigma, p_\sigma)(\psi_i^m - \varphi_i^m), \\ \Psi_{m,i}^p &= \mathcal{L}_y^m(u_\sigma, y_\sigma, p_\sigma)(\psi_i^m - \varphi_i^m). \end{aligned}$$

For the considered case of a $cG(1)cG(1)$ discretization y_σ is linear and p_σ constant in time on the interval I_m. Thus, we have

$$\begin{aligned} \text{for } m = 0, \dots, N\colon y_\sigma(t_m) &= \sum_{i=1}^{N_m} \varphi_i^m Y_i^m, \quad I_{2h} y_\sigma(t_m) = \sum_{i=1}^{N_m} \psi_i^m Y_i^m, \\ \text{for } m = 1, \dots, N\colon p_\sigma|_{I_m} &= \sum_{i=1}^{N_m} \varphi_i^m P_i^m, \quad I_{2h} p_\sigma|_{I_m} = \sum_{i=1}^{N_m} \psi_i^m P_i^m, \\ p_\sigma(0) &= \sum_{i=1}^{N_0} \varphi_i^0 P_i^0, \quad I_{2h} p_\sigma(0) = \sum_{i=1}^{N_0} \psi_i^0 P_i^0, \end{aligned}$$

where $Y^m \in (\mathbb{R} \times \mathbb{R})^{N_m}$ and $P^m \in (\mathbb{R} \times \mathbb{R})^{N_m}$ denotes the nodal vector of $y_\sigma(t_m)$ and of $p_\sigma(t_m)$, respectively. We obtain

$$\eta_h^m = \frac{1}{2} \left(\sum_{i=1}^{N_m} \Psi_{m,i}^y P_i^m + \sum_{i=1}^{N_m} \Psi_{m,i}^p Y_i^m \right).$$

Further, we introduce a filtering operator π given by

$$\pi = \mathrm{id} - \bar{I}_{2h}^{(1)} \quad \text{with} \quad \bar{I}_{2h}^{(1)}\colon \begin{cases} (\widetilde{X}_{k,h}^{0,1})^2 \to (\widetilde{X}_{k,2h}^{0,1})^2, \\ (X_{k,h}^{1,1})^2 \to (X_{k,2h}^{1,1})^2. \end{cases}$$

We denote the nodal vectors of the filtered solution $\pi y_\sigma(t_m)$ and of the adjoint solution $\pi p_\sigma(t_m)$ by $Y^{\pi,m}$ and $P^{\pi,m}$ defined by

$$\pi y_\sigma(t_m) = \sum_{i=1}^{N_m} \varphi_i^m Y_i^{\pi,m} \quad \text{and} \quad \pi p_\sigma(t_m) = \sum_{i=1}^{N_m} \varphi_i^m P_i^{\pi,m} \quad \text{for } m = 0, \dots, M.$$

There holds $V_{2h}^{1,m} \subset V_h^{1,m}$ and hence, $I_{2h}^{(1)}$ is the identity on $V_{2h}^{1,m}$. We derive

$$I_{2h}^{(1)}\pi\varphi_i^m - \pi\varphi_i^m = I_{2h}^{(1)}\varphi_i^m - \varphi_i^m = \psi_i^m - \varphi_i^m.$$

Thus, by linearity of the residuals with respect to the weights we have

$$\eta_h^m = \frac{1}{2}\sum_{i=1}^{N_m}\left(\Psi_{m,i}^y P_i^m + \Psi_{m,i}^p Y_i^m\right) = \frac{1}{2}\sum_{i=1}^{N_m}\left(\Psi_{m,i}^y P_i^{\pi,m} + \Psi_{m,i}^p Y_i^{\pi,m}\right)$$

and we can estimate

$$|\eta_h^m| \le \sum_{m=1}^{N_m} |\eta_{h,i}^m|$$

with

$$\eta_{h,i}^m = \frac{1}{2}\left(\Psi_{m,i}^y P_i^{\pi,m} + \frac{1}{2}\Psi_{m,i}^p, Y_i^{\pi,m}\right), \quad i = 1,\dots,N_m.$$

The estimator η_h^m depends linearly on the size of the time step k_m. To get rid of this dependence, the spatial estimators can be rescaled, for details we refer to Meidner [97].

3.5 Adaptive algorithm

In this section the principal steps of the utilized adaptive algorithm are presented, for details we refer to Meidner & Vexler [99] and Meidner [97]. The aim is to adapt the different types of discretizations in such a way that we obtain an equilibrated reduction of the corresponding discretization errors, i.e.

$$|\eta_k| \approx |\eta_h| \approx |\eta_d|.$$

Let (a, b, c) be a permutation of (k, h, d) with

$$|\eta_a| \ge |\eta_b| \ge |\eta_c|.$$

Then define

$$\gamma_{ab} = \frac{|\eta_a|}{|\eta_b|} \ge 1, \quad \gamma_{bc} = \frac{|\eta_b|}{|\eta_c|} \ge 1.$$

Thus, for a $d \in [1, 5]$ we apply Algorithm 3.3 to refine our discretizations until a given error tolerance TOL is reached. For every discretization to be adapted, we refine the meshes in dependence of the local error estimators. There exists several strategies, how to realize this; see Meidner [97].

Algorithm 3.3: Adaptive refinement algorithm

1: Choose an initial triple of discretizations $\mathcal{T}_{\sigma_0}$, $\sigma_0 = (k_0, h_0, d_0)$ and set $n = 0$.
2: Compute the solution $(u_{\sigma_n}, y^1_{\sigma_n})$.
3: Evaluate the estimators η_{k_n}, η_{h_n}, and η_{d_n}.
4: **if** $\eta_{k_n} + \eta_{h_n} + \eta_{d_n} \leq TOL$ **then**
5: Break
6: **else**
7: Determine, which discretizations have to be refined according to

$$\begin{cases} \gamma_{ab} \leq d \quad \wedge \quad \gamma_{bc} \leq d & : \quad a, b, c, \\ \gamma_{bc} > d & : \quad a, b, \\ \text{else} & : \quad a. \end{cases} \tag{3.5.1}$$

8: Refine $\mathcal{T}_{\sigma_n} \to \mathcal{T}_{\sigma_{n+1}}$ depending on the size of η_{k_n}, η_{h_n}, and η_{d_n} to equilibrate the three discretization errors.
9: Set $n = n + 1$.
10: **GOTO** 2.

3.6 Numerical examples

In this section we apply the techniques presented in the previous sections to three numerical examples. Let $\Omega = [0, 1]^2$ and $r = s = 1$, i.e. the state and adjoint equation are discretized by a $cG(1)cG(1)$ method. In the first example we consider an optimal control problem governed by the wave equation with distributed control, in the second one an optimal control problem with finite dimensional control and a nonlinear equation and in the third one an optimal control problem with distributed control and a nonlinear equation. For the computation we use the RoDoBo library [120], which incorporates the finite element toolkit Gascoigne [45]. For the visualization we use VisuSimple [133]. On a given discretization level let N_m denote the number of nodes of the mesh $\mathcal{T}_h^m$ for $m = 0, \dots, M$. We define

$$N_{\max} = \max_{m \in \{0, \dots, M\}} N_m,$$

where M denotes the number of time intervals and by dof the degrees of freedom of the discretization in space and time of the state. i.e.

$$dof = \sum_{m=0}^{M} N_m.$$

For simplifying the notation in this section we write y instead of y^1 here. To validate the error estimator we introduce the index

$$I_{\text{eff}} = \frac{J(u, y) - J(u_\sigma, y^1_\sigma)}{\eta_k + \eta_h + \eta_d},$$

for the solution (u, y) of (P^{DWR}) and (u_σ, y^1_σ) of the fully discretized problem (P^{DWR}_σ), which measures the efficiency of the estimator. Thereby, the exact solution is replaced by a discrete solution on a very fine mesh.

3.6.1 Distributed control of the wave equation

In this example we consider an optimal control problem of the wave equation with distributed control. We choose $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $U = L^2(L^2(\Omega))$ and consider the following control problem:

$$\begin{cases} \text{Minimize} \quad J(u,y) = \frac{1}{2}\|y\|^2_{L^2(L^2(\Omega))} + \frac{\alpha}{2}\|u\|^2_{L^2(L^2(\Omega))}, \quad u \in U,\ y \in X, \quad \text{s.t.} \\ \qquad y_{tt} - \Delta y = u \quad \text{in } Q, \\ \qquad y(0) = y_0 \quad \text{in } \Omega, \\ \qquad y_t(0) = y_1 \quad \text{in } \Omega, \\ \qquad y = 0 \quad \text{on } \Sigma, \end{cases} \tag{3.6.1}$$

with the data

$$y_0(x_1,x_2) = \begin{cases} 10^{11}(x_1-0.35)^3(x_2-0.35)^3(0.65-x_1)^3(0.65-x_2)^3, & 0.35 < x_1, x_2 < 0.65, \\ 0, & \text{else}, \end{cases}$$
$$y_1 = 0, \quad \alpha = 0.001, \tag{3.6.2}$$

for $(t, x_1, x_2) \in [0,T] \times \Omega = [0, 0.3] \times [0,1]^2$.

The discrete control space is chosen as $U_d = \widetilde{X}^{0,1}_{k,h}$, i.e. the discrete control space is equal to the discrete space of the adjoint state. As a consequence we have $\eta_d = 0$; cf. Remark 3.4.3.

In Table 3.1 the spatial and temporal error estimators as well es the effectivity indices for problem (3.6.1) are shown. Thereby, we denote by $\dim U_d$ the degrees of freedom of the discrete control space. The figure shows that the estimators are equilibrated and that we have a reduction of the error in the cost functional. Figure 3.3 shows the state and the

dof	$N_{\max}$	M	$\dim U_d$	η_h	η_k	$J(u,y) - J(u_\sigma, y^1_\sigma)$	I_{eff}
275	25	10	250	5.17e-02	-9.36e-04	-2.25e-02	-0.4
891	81	10	810	-4.82e-03	-6.84e-03	-1.38e-02	1.2
3757	289	12	3468	-1.58e-04	-3.81e-03	2.69e-04	-0.1
6647	289	22	6358	1.85e-05	-7.10e-04	1.66e-04	-0.2
11849	289	40	11560	1.17e-04	-1.26e-04	1.29e-04	-15.0
38731	1089	42	37674	-4.68e-06	-8.86e-05	-3.05e-05	0.3
40125	1089	44	39468	-5.22e-06	-6.89e-05	-1.73e-05	0.2
73777	1089	80	71760	-6.70e-06	-1.59e-05	-3.34e-05	1.5
207795	3897	82	127346	-6.89e-06	-1.13e-05	-1.09e-05	0.6
1208753	13257	160	524960	-2.89e-06	-1.91e-06	-3.90e-06	0.8

Table 3.1: Error estimators and effectivity indices for adaptive refinement for (3.6.1)

spatial meshes of the finest discretization presented in Table 3.1 at the time steps 0, 60, 120, 160. The figure confirms that the local refined parts of the spatial meshes move with the wave.

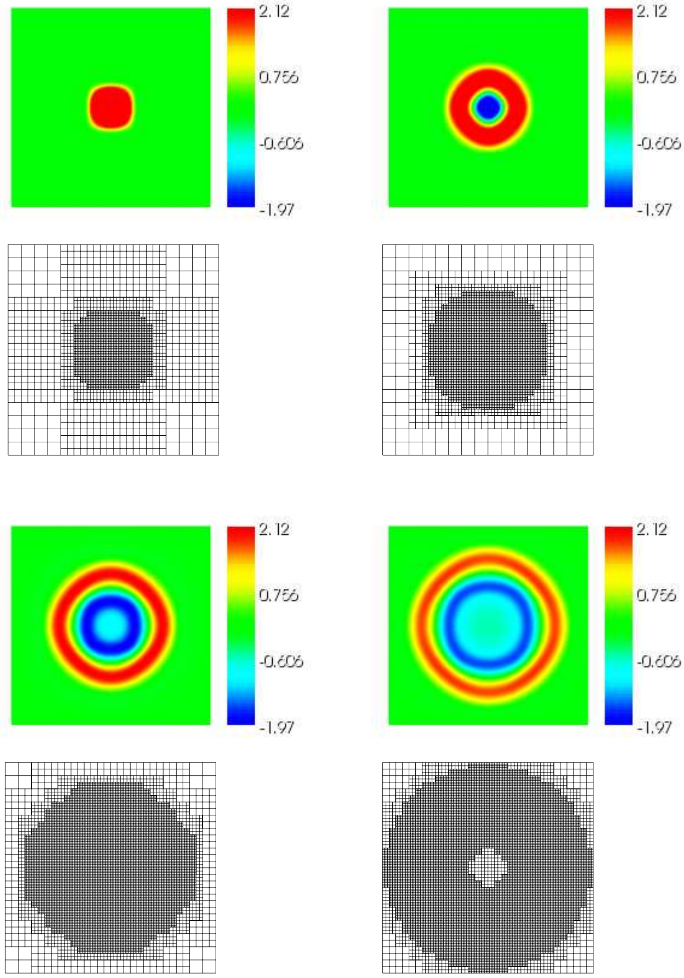


Figure 3.3: State and corresponding spatial meshes at time points t_m with $m \in \{0, 60, 120, 160\}$ for (3.6.1)

Discussion of independence of the estimators

Table 3.2 shows the temporal and spatial estimators for problem (3.6.1) on a fixed temporal and on locally refined spatial meshes as well as on locally refined temporal meshes and a fixed spatial mesh. The numerical example indicates that in this case the temporal and spatial estimators are nearly independent, which is a motivation for refining the discretizations separately, cf. the numerical example for optimal control of parabolic equations in Meidner & Vexler [99].

dof	$N_{\max}$	M	$\dim U_d$	η_h	η_k
4131	81	50	4050	-5.07e-03	-9.49e-05
14739	289	50	14450	1.36e-04	-6.63e-05
46563	1089	50	54450	-5.67e-06	-5.82e-05
127363	3801	50	190050	-7.11e-06	-4.95e-05
377459	13161	50	658050	-2.80e-06	-5.16e-05
10449	81	128	10368	-5.08e-03	-8.95e-06
10611	81	130	10530	-5.08e-03	-8.16e-06
10773	81	132	10692	-5.08e-03	-7.32e-06
20817	81	256	20736	-5.08e-03	-1.81e-06
20979	81	258	20898	-5.08e-03	-1.71e-06
21141	81	260	21060	-5.08e-03	-1.61e-06
41553	81	512	41472	-5.08e-03	-3.99e-07
41715	81	514	41634	-5.08e-03	-3.87e-07
83025	81	1024	82944	-5.08e-03	-9.19e-08
83187	81	1026	83106	-5.08e-03	-9.04e-08
165969	81	2048	165888	-5.08e-03	-2.21e-08
166131	81	2050	166050	-5.08e-03	-2.19e-08
331857	81	4096	331776	-5.08e-03	-5.41e-09

Table 3.2: Independence of the estimators for problem (3.6.1)

3.6.2 Optimal control of a nonlinear equation (I)

In this example we consider an optimal control problem with finite dimensional control and a nonlinear equation. We choose $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $U = \mathbb{R}^4$. Furthermore, let χ_A be the characteristic function with respect to a set $A \subset \mathbb{R}^2$. We consider the following control problem:

$$\begin{cases} \text{Minimize} \quad J(u,y) = \frac{1}{2}\left\|y-1\right\|^2_{L^2(L^2(\Omega))} + \frac{\alpha}{2}\left\|u\right\|^2_{\mathbb{R}^4}, \quad u \in U,\ y \in X, \quad \text{s.t.} \\ y_{tt} - \Delta y + y^3 = \sum_{i=1}^{4} \psi_i(x) u_i \quad \text{in } Q, \\ y(0) = y_0 \quad \text{in } \Omega, \\ y_t(0) = y_1 \quad \text{in } \Omega, \\ y = 0 \quad \text{in } \Sigma, \end{cases} \tag{3.6.3}$$

where

$$\psi_1 = \chi_{[0.0,0.5]\times[0.5,1.0]}, \quad \psi_2 = \chi_{[0.5,1.0]\times[0.5,1.0]},$$
$$\psi_3 = \chi_{[0,0.5]^2}, \quad \psi_4 = \chi_{[0.5,1.0]\times[0.0,0.5]},$$

and

$$y_0(x_1,x_2) = \begin{cases} -1, & \text{if } 0 < x_1 < 0.25 - \varepsilon, \varepsilon < x_2 < 1-\varepsilon \\ 0, & \text{if } x_1 \geq 0.25 \end{cases}, \quad 0 < \varepsilon < 10^{-5}, \quad y_0 \in V,$$
$$y_1(x_1,x_2) = -1$$

for $\alpha = 0.001$ and $(t,x_1,x_2) \in [0,T] \times \Omega = [0,0.3] \times [0,1]^2$. Thus, the control $u = (u_1,u_2,u_3,u_4)^T \in \mathbb{R}^4$ acts on four subdomains of the domain Ω, cf. Figure 3.4. The estimator η_d vanishes, since the control is a parameter, cf. Remark 3.4.3.

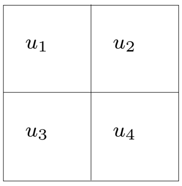


Figure 3.4: Domain Ω with the control acting on four subdomains

In Table 3.3 the spatial and temporal error estimators as well as the effectivity indices for (3.6.3) are shown. We see a reduction of the error in the cost functional and the effectivity indices confirm the quality of the estimator. Figure 3.5 shows the error corresponding to

dof	$N_{\max}$	M	η_h	η_k	$J(u,y) - J(u_\sigma, y_\sigma^1)$	I_{eff}
891	81	10	4.83e-05	2.64e-05	-8.16e-04	-10.9
2807	239	12	-6.74e-05	-1.29e-06	-4.07e-04	5.9
9401	805	12	-1.26e-04	-5.48e-05	-2.67e-04	1.5
49737	2591	20	-8.65e-05	-6.91e-05	-1.49e-04	1.0
286977	8911	36	-6.96e-05	-6.83e-05	-9.47e-05	0.7

Table 3.3: Error estimators and effectivity indices for adaptive refinement for (3.6.3)

the degrees of freedom in case of adaptive refinement in space and time in comparison to uniform refinement of the temporal and spatial meshes without equilibration. This confirms that we obtain a better accuracy of the discrete solution by local mesh refinement than by uniform refinement for a given number of degrees of freedom.

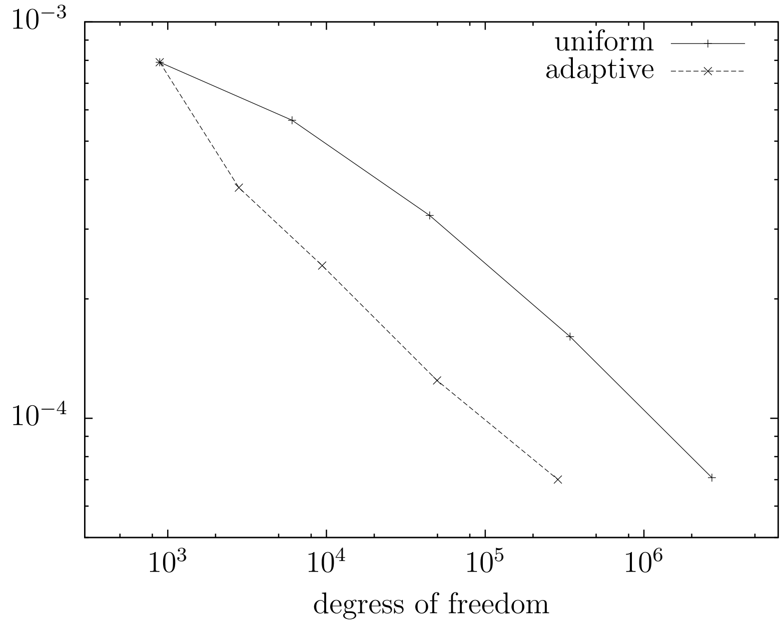


Figure 3.5: Error for uniform and adaptive refinement for (3.6.3)

3.6.3 Optimal control of a nonlinear equation (II)

In this numerical example we consider an optimal control problem for a nonlinear wave equation with distributed control. Let the control space be given by $U = L^2(L^2(\Omega))$. Then we consider the following control problem

$$\begin{cases} \text{Minimize} \quad J(u,y) = \frac{1}{2}\|y\|^2_{L^2(L^2(\Omega))} + \frac{\alpha}{2}\|u\|^2_{L^2(L^2(\Omega))}, \quad u \in U,\ y \in X, \quad \text{s.t.} \\ \qquad y_{tt} - \Delta y + y^2 = u + f \quad \text{in } Q, \\ \qquad y(0) = y_0 \quad \text{in } \Omega, \\ \qquad y_t(0) = y_1 \quad \text{in } \Omega, \\ \qquad y = 0 \quad \text{in } \Sigma, \end{cases} \tag{3.6.4}$$

with

$$f(t,x_1,x_2) = \begin{cases} 100, & \text{if } x_1 < 0.125,\ t < 0.05, \\ 0, & \text{else,} \end{cases} \qquad y_0 \equiv y_1 \equiv 0, \quad \alpha = 0.1$$

for $(t,x_1,x_2) \in [0,1] \times \Omega$. The control space is discretized as in the example in Section 3.6.1, i.e. we choose $U_d = \widetilde{X}^{0,1}_{k,h}$ and consequently, η_d vanishes, cf. Remark 3.4.3. Figure 3.6 shows the error for adaptive and uniform refinement. As in the previous example we see that we need less degrees of freedom for adaptive than for uniform refinement to reach a certain accuracy.

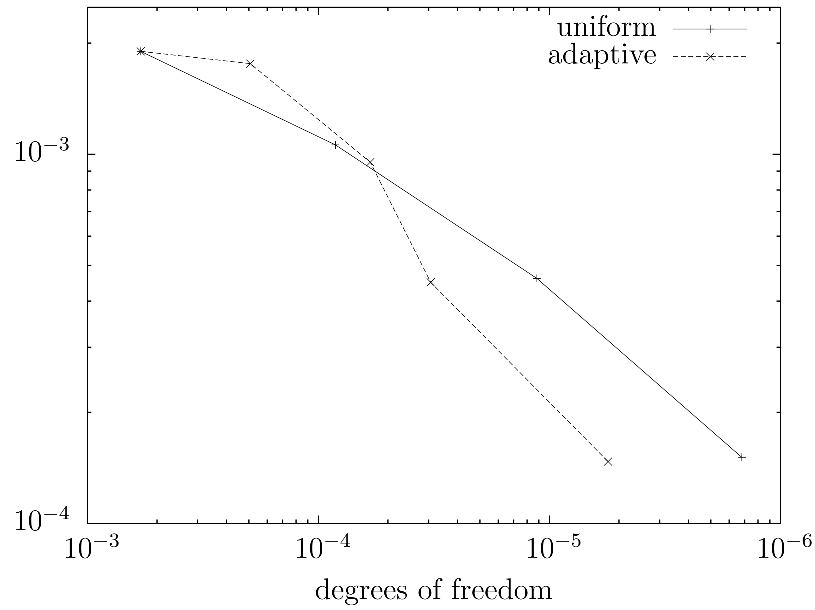


Figure 3.6: Error for adaptive and uniform refinement for (3.6.4).

3.7 Energy on dynamic meshes

It is well-known that the continuous homogeneous wave equation conserves the energy in time; cf. Proposition 2.4.2. To conserve this property on the discrete level, we discretize the wave equation by a $cG(r)$ method in time, cf. Section 3.2.1. However, on locally refined meshes this property might be lost. In this section we analyze the conservation of energy of the discrete system on meshes changing dynamically in time. We do not consider the corresponding control problem, since the control affects the energy and we cannot expect conservation of energy. The presented results are similar to those in Rademacher [116]; cf. also Bangerth, Geiger & Rannacher [7] and Eriksson et al. [37]. However, here we present a representation of the difference of the energy of the discrete system at two neighbouring time points and some numerical examples.

3.7.1 Behaviour of the energy in time

We consider the following system

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } Q, \\ y(0) = y_0 & \text{in } \Omega, \\ y_t(0) = y_1 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma \end{cases} \tag{3.7.1}$$

for $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$. The energy E of the system (3.7.1) defined in Definition (2.4.1) remains constant in time according to Proposition 2.4.2. In the following we analyze the energy of the discrete system corresponding to (3.7.1). We apply a $cG(1)cG(1)$ discretization (cf. Section 3.2) with $V = H_0^1(\Omega)$ and evaluate the arising time integrals by the trapedoizal rule, leading to a Crank-Nicolson scheme in time. We use the notations (3.4.12) and (3.4.13). The discrete solution $(Y_m^1, Y_m^2) \in V_h^{1,m} \times V_h^{1,m}$, $m = 0, \dots, M$, is given by

$$\begin{aligned}
(Y_0^1, \xi) = (y_0, \xi), \quad (Y_0^2, \xi) = (y_1, \xi) \qquad & \forall \xi \in V_h^{1,0}, \\
(Y_m^2, \xi^1) + \frac{k_m}{2}(\nabla Y_m^1, \nabla \xi^1) = (Y_{m-1}^2, \xi^1) - \frac{k_m}{2}(\nabla Y_{m-1}^1, \nabla \xi^1) \quad & \forall \xi^1 \in V_h^{1,m}, \\
(Y_m^1, \xi^2) - \frac{k_m}{2}(Y_m^2, \xi^2) = (Y_{m-1}^1, \xi^2) + \frac{k_m}{2}(Y_{m-1}^2, \xi^2) \qquad & \forall \xi^2 \in V_h^{1,m}
\end{aligned} \tag{3.7.2}$$

for $m = 1, \dots, M$.

Theorem 3.7.1. *Let $\pi_m \colon V_h^{1,m-1} \to V_h^{1,m}$ for $m = 1, \dots, M$. Then, for the discrete energy*

$$E_{k,h}(t_m) = \frac{1}{2}\left(\|Y_m^2\|^2 + \|\nabla Y_m^1\|^2\right), \quad m = 0, \dots, M,$$

of the discrete system (3.7.2) *there holds*

$$\begin{aligned}
E_{k,h}(t_m) = E_{k,h}(t_{m-1}) &- \frac{1}{k_m}(Y_{m-1}^1 - \pi_m Y_{m-1}^1, Y_m^2 - Y_{m-1}^2) \\
&- \frac{1}{k_m}(\pi_m Y_{m-1}^2 - Y_{m-1}^2, Y_m^1 - Y_{m-1}^1) - \frac{1}{2}(Y_{m-1}^2 - \pi_m Y_{m-1}^2, Y_m^2 + Y_{m-1}^2) \\
&- \frac{1}{2}(\nabla Y_m^1 + \nabla Y_{m-1}^1, \nabla(Y_{m-1}^1 - \pi_m Y_{m-1}^1)).
\end{aligned}$$

Proof. We can test (3.7.2) with

$$\xi^1 = \frac{Y_m^1 - \pi_m Y_{m-1}^1}{k_m} \in V_h^{1,m}, \quad \xi^2 = \frac{Y_m^2 - \pi_m Y_{m-1}^2}{k_m} \in V_h^{1,m}$$

for $m = 1, \dots, M$, and by addition of the equations we derive

$$\begin{aligned}
&\frac{1}{k_m}(Y_m^2, Y_m^1 - \pi_m Y_{m-1}^1) - \frac{1}{k_m}(Y_{m-1}^2, Y_m^1 - \pi_m Y_{m-1}^1) + \frac{1}{2}\left(\nabla Y_m^1, \nabla(Y_m^1 - \pi_m Y_{m-1}^1)\right) \\
&\quad + \frac{1}{2}\left(\nabla Y_{m-1}^1, \nabla(Y_m^1 - \pi_m Y_{m-1}^1)\right) - \frac{1}{k_m}(Y_m^2, Y_m^1 - Y_{m-1}^1) + \frac{1}{k_m}(\pi_m Y_{m-1}^2, Y_m^1 - Y_{m-1}^1) \\
&\quad + \frac{1}{2}(Y_m^2, Y_m^2 - \pi_m Y_{m-1}^2) + \frac{1}{2}(Y_{m-1}^2, Y_m^2 - \pi_m Y_{m-1}^2) = 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{1}{k_m}(Y_m^2, Y_{m-1}^1 - \pi_m Y_{m-1}^1) + \frac{1}{k_m}(Y_{m-1}^2, -Y_{m-1}^1 + \pi_m Y_{m-1}^1) \\
&\quad + \frac{1}{k_m}(\pi_m Y_{m-1}^2 - Y_{m-1}^2, Y_m^1 - Y_{m-1}^1) + \frac{1}{2}\|Y_m^2\|^2 - \frac{1}{2}(Y_m^2, \pi_m Y_{m-1}^2) - \frac{1}{2}\|Y_{m-1}^2\|^2 \\
&\quad + \frac{1}{2}(Y_{m-1}^2, Y_{m-1}^2) + \frac{1}{2}(Y_{m-1}^2, Y_m^2 - \pi_m Y_{m-1}^2) + \frac{1}{2}(\nabla Y_m^1, \nabla(Y_{m-1}^1 - \pi_m Y_{m-1}^1)) \\
&\quad + \frac{1}{2}\|\nabla Y_m^1\|^2 - \frac{1}{2}\|\nabla Y_{m-1}^1\|^2 + \frac{1}{2}(\nabla Y_{m-1}^1, \nabla(Y_{m-1}^1 - \pi_m Y_{m-1}^1)) = 0
\end{aligned}$$

and thus, the assertion follows. □

In the adaptive Algorithm 3.3 we start with identical uniform meshes at all time points. Then, according to the estimators the temporal and spatial meshes are refined and we obtain a new discretization level, on which the solution and the estimators are computed again. Then we repeat this process. That means, from one discretization level to the next, we have only refinement. However, on a fixed discretization level we may have refinement or coarsening of the spatial meshes from one time point to the next. To clarify this point, we consider Figure 3.7. It presents the way of refinement in a schematic order neglecting some technical issues as well as the fact that we claim a patch structure of the spatial meshes. On Level 1 we have two time intervals and three spatial meshes associated with the three time points, on Level 2 we have three time intervals and four spatial meshes and on Level 3 four time intervals and five spatial meshes. We start the algorithm with the same spatial mesh in each time point on Level 1. The estimators tell us that we have to refine the last time interval and the spatial meshes as shown in the figure. We associate with the new discrete time point the spatial mesh of the third time point. This process is repeated.

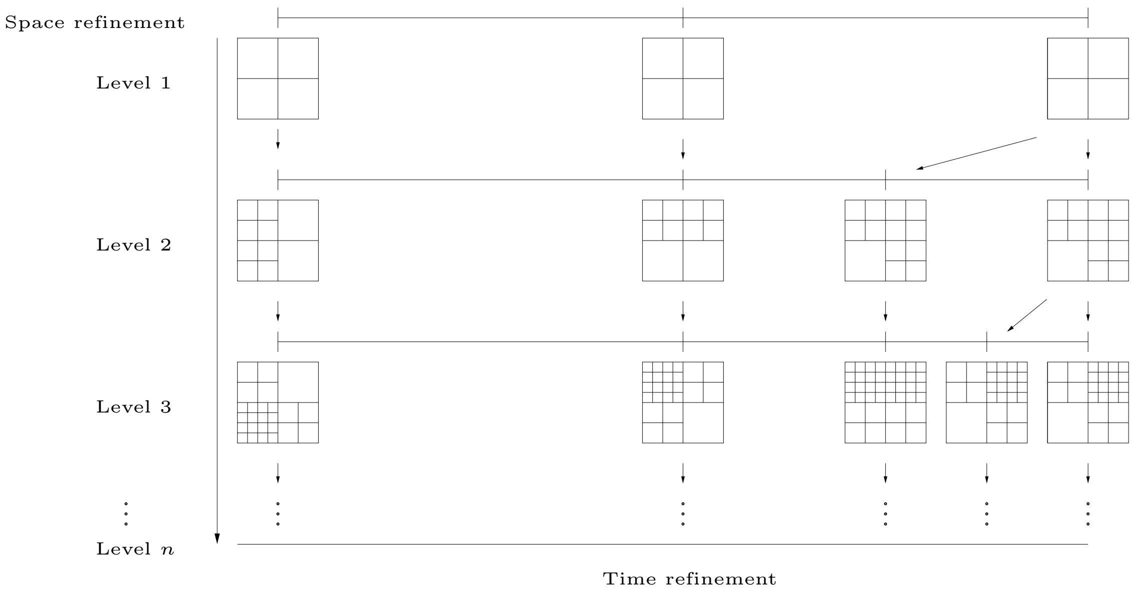


Figure 3.7: Meshes on different levels of discretization

In this sense, we obtain the following corollary.

Corollary 3.7.2. *On a given discretization level the energy remains constant in time independent of the size of* k_m *if for any step from* t_m *to* t_{m+1} $(m = 0, \dots, M-1)$ *the spatial mesh is only refined and not coarsened.*

Proof. Since we only allow refinement and no coarsening in space in each time step we have $V_h^{1,m-1} \subset V_h^{1,m}$ for all $m = 1, \dots, M$. Thus, let $\pi_m = \text{id}$ be the identity for $m = 1, \dots, M$ in Theorem 3.7.1. Then π_m is well-defined and we obtain $E_{k,h}(t_m) = E_{k,h}(t_{m+1})$ for $m = 0, \dots, M-1$. $\square$

3.7.2 Numerical example

In this section we present some numerical examples confirming the theoretical results from the previous section. We start with a comparison of the Euler and the Crank-Nicolson scheme. Then we consider the energy on meshes changing dynamically in time.

Discrete energy for Crank-Nicolson and implicit Euler scheme

Let initial data be given on the domain $\Omega = [0,1]^2$ by

$$y_0(x_1,x_2) = \sin(\pi x_1)\sin(\pi x_2), \quad y_1(x_1,x_2) = (1-x_1)(1-x_2)x_1x_2 \tag{3.7.3}$$

for $x = (x_1,x_2) \in \Omega$ and let $T = 1$. Table 3.4 shows the discrete energy w.r.t. (3.7.1) when applying the Crank-Nicolson as well as the implicit Euler scheme on uniform temporal and spatial meshes with 11 and 1089, respectively, nodes. We see that in case of the Crank-Nicolson scheme (CN) the energy remains constant in contrast to the implicit Euler scheme, where we have damping resulting in a reduction of the energy.

time	CN	implicit Euler
0.0	2.4699	2.4699
0.1	2.4699	2.0625
0.2	2.4699	1.7223
0.3	2.4699	1.4382
0.4	2.4699	1.2009
0.5	2.4699	1.0028
0.6	2.4699	0.8374
0.7	2.4699	0.6992
0.8	2.4699	0.5839
0.9	2.4699	0.4876
1.0	2.4699	0.4071

Table 3.4: Energy for Crank-Nicolson and implicit Euler scheme

Energy on dynamic meshes

We consider the homogeneous wave equation (3.7.1) with the initial data (3.7.3) on the time-space cylinder $[0,T] \times \Omega = [0,1] \times [0,1]^2$. A direct calculation shows that for the exact energy there holds

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1\int_0^1 \Big(2\pi^2\cos(\pi x_1)^2\sin(\pi x_2)^2 + ((1-x_1)(1-x_2)x_1x_2)^2\Big)\,dxdy \\ &= \frac{\pi^2}{4} - \frac{1}{1800} \approx 2.4668 \end{aligned}$$

for $x = (x_1,x_2) \in \Omega$ and $t \in [0,T]$.

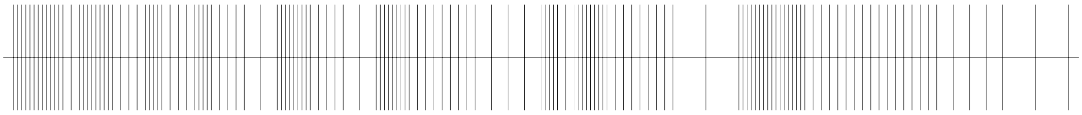
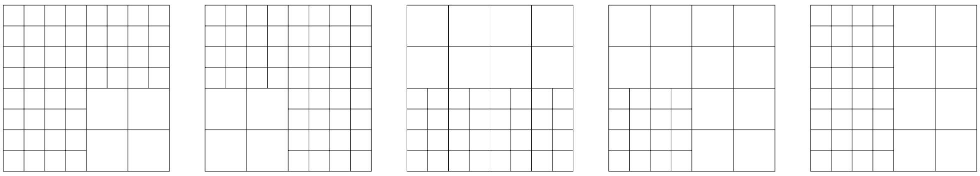


Figure 3.8: Time mesh - 140 time steps

We compute the solution on a temporal mesh with 141 nodes, cf. Figure 3.8, and identical uniform spatial meshes in every time step with 1089 nodes in each case. From the discrete solution we obtain the discrete energy $E_{k,h}(t_m) = 2.4699$ for all $m \in \{0, \dots, 140\}$. Thus, the error between the exact energy and the discrete one, depends only on the mesh size of the spatial mesh. This confirms our theoretical results of Section 3.7.1.

Table 3.5 shows the energy for the state equation discretized using a uniform temporal mesh with 11 nodes and different spatial meshes $\mathcal{T}_1, \dots, \mathcal{T}_5$, cf. Figure 3.9. This confirms that the energy is only affected if the spatial mesh is coarsened.



(a) $\mathcal{T}_1$ (b) $\mathcal{T}_2$ (c) $\mathcal{T}_3$ (d) $\mathcal{T}_4$ (e) $\mathcal{T}_5$

Figure 3.9: Spatial meshes

Time point	t_0	t_1	t_2	t_3	t_4	t_5	t_6
Mesh	$\mathcal{T}_1$	$\mathcal{T}_1$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_3$
Energy	2.5327	2.5327	2.5327	2.5361	2.5361	2.5346	2.5346

Time point	t_7	t_8	t_9	t_{10}
Mesh	$\mathcal{T}_4$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_5$
Energy	2.5441	2.5441	2.5441	2.5441

Table 3.5: Energy on a sequence of spatial meshes

3.8 Outlook

There are several interesting questions for future research on this topic.

- In this Chapter we developed estimators with respect to the cost functional. This could be extended to optimal control problems with a given additional quantity of interest, as in Meidner & Vexler [99], where optimal control of parabolic equations is considered.
- In this Chapter we assume $U = U_{\mathrm{ad}}$. Nevertheless, in many applications we have constraints on the controls. Thus, it is interesting to extend the presented techniques to the case $U_{\mathrm{ad}} \subsetneq U$. In this case the optimal control is not very smooth, because of the constraints and to estimate the weights in the estimator by higher order interpolations we may have to apply a post-processing step; cf. the discussion in Vexler & Wollner [132], where optimal control problems with elliptic equations are considered subject to control constraints and also the results presented in Chapter 4.
- As initial data we prescribe the state and velocity. Thus, it is worth to consider optimal control problems with a functional J depending on the control u, the state $y(T)$ and the velocity $y_t(T)$ at time T; cf. Lions [87, pp. 314].
- Chapter 2 shows that for inhomogeneous boundary value problems for the wave equation, the corresponding solution has low regularity, otherwise compatibility conditions have to be satisfied. Thus, it may be interesting to derive a posteriori error estimates for optimal control problems of hyperbolic equations with a state equation given in a very weak form.
- In Section 3.2.4 we recall results on a priori error estimates for optimal control of elliptic and parabolic equations. It is an interesting problem to prove a priori error estimates for optimal control of hyperbolic equations.
- In Section 3.7 numerical examples show that the energy associated with the discrete wave equation does not remain constant in time if we allow dynamically in time changing spatial meshes. Thus, we may develop methods which conserve the energy.

4 Semi-smooth Newton methods

In this chapter we consider semi-smooth Newton methods for solving optimal control problems governed by wave equations and subject to pointwise inequality control constraints. We discuss three different control actions: distributed control, Neumann boundary control and Dirichlet boundary control and analyze the convergence of the semi-smooth Newton method.

We consider general (linear-quadratic) optimal control problems, with the control space $U = L^2(\omega)$ and ω being either $\omega = Q$ or $\omega = \Sigma$ and the state space $L^2(Q)$. According to (2.2.2) let $S\colon U \to L^2(Q)$ be the control-to-state operator and we assume that S is injective and affine-linear with

$$S(u) = Tu + \bar{y}, \tag{4.0.1}$$

where $T \in \mathcal{L}(U, L^2(Q))$ and $\bar{y} \in L^2(Q)$. The cost functional J let be defined by

$$J(u,y) = \mathcal{G}(y) + \frac{\alpha}{2} \|u\|_{L^2(\omega)}, \tag{4.0.2}$$

where the operator

$$\mathcal{G}\colon L^2(Q) \to \mathbb{R} \tag{4.0.3}$$

is assumed to be quadratic with $\mathcal{G}'$ being an affine operator from $L^2(Q)$ to itself, and $\mathcal{G}''$ is assumed to be non-negative and $\alpha > 0$. In contrast to the previous chapter, here we assume additional constraints on the control. The set of admissible controls U_{ad} is given by bilateral box constraints

$$U_{\mathrm{ad}} = \{\, u \in U \mid u_a \le u \le u_b \,\} \quad \text{with } \ u_a, u_b \in U. \tag{4.0.4}$$

In the case of distributed control the state equation defining the operator S is given as

$$\begin{aligned} &y_{tt} - \Delta y = u \text{ in } Q, \\ &y(0) = y_0,\ y_t(0) = y_1 \text{ in } \Omega,\ y = 0 \text{ on } \Sigma, \end{aligned} \tag{4.0.5}$$

in the case of the Neumann boundary control we have

$$\begin{aligned} &y_{tt} - \Delta y = f \text{ in } Q, \\ &y(0) = y_0,\ y_t(0) = y_1 \text{ in } \Omega,\ \partial_n y = u \text{ on } \Sigma, \end{aligned} \tag{4.0.6}$$

and in the case of the Dirichlet boundary control

$$\begin{aligned} &y_{tt} - \Delta y = f \text{ in } Q, \\ &y(0) = y_0,\ y_t(0) = y_1 \text{ in } \Omega,\ y = u \text{ on } \Sigma. \end{aligned} \tag{4.0.7}$$

For this class of optimal control problems we will discuss a proper functional analytic setting, which is suitable for application of the semi-smooth Newton methods. These methods have proven their efficiency for a large class of optimization problems with partial differential equations, see, e. g. Ito & Kunisch [64, 63], Ulbrich [127, 128, 129], Hintermüller, Ito & Kunisch [56] and Kunisch & Vexler [78]. It is well-known that semi-smooth Newton methods are equivalent to primal dual active set strategies (PDAS), cf. Hintermüller, Ito & Kunisch [56], which exploit pointwise information from Lagrange multipliers for updating active sets. Here it is essential that the Lagrange multipliers are L^2-functions rather than measures, which can be achieved by setting $U = L^2(Q)$ for distributed control and $U = L^2(\Sigma)$ for both Neumann and Dirichlet boundary control problems, cf. the discussion in Kunisch & Vexler [78].

The aim of this chapter is to analyze semi-smooth Newton methods for optimal control problems governed by the wave equation with respect to superlinear convergence. These results are already published in Kröner, Kunisch & Vexler [76, 75]. To prove superlinear convergence we analyze whether a smoothing property of the operator mapping the control variable u to the adjoint state p or to a trace of p is given. For distributed and Neumann boundary control we will establish this smoothing property and prove superlinear convergence. For the case of Dirichlet boundary control we will provide an example illustrating the fact that such a property can not hold in general. In addition we will consider a Dirichlet boundary control problem governed by the strongly damped wave equation given as

$$\begin{aligned} &y_{tt} - \Delta y - \rho \Delta y_t = f \text{ in } Q, \\ &y(0) = y_0,\ y_t(0) = y_1 \text{ in } \Omega,\ y = u \text{ on } \Sigma, \end{aligned} \tag{4.0.8}$$

with a positive damping parameter $\rho > 0$. This equation appears often in models with loss of energy, e.g., it arises in the modeling of longitudinal vibrations in a homogeneous bar, in which there are viscous effects, cf. Massatt [95]. The corresponding optimal control problem (with small ρ) can also be regarded as regularization of the Dirichlet boundary control problem for the wave equation. For the resulting optimal control problem we will establish the required smoothing property and prove superlinear convergence of the semi-smooth Newton method.

For numerical realization the infinite dimensional problems are discretized following space-time finite element methods as in Chapter 3.

The chapter is organized as follows. In Section 4.1 we summarize known results for semi-smooth Newton methods, which are relevant for the analysis in this chapter. Moreover, we provide a set of assumptions for superlinear convergence of an abstract optimal control problem with control constraints. In Section 4.2 we introduce a distributed, Neumann boundary, and Dirichlet boundary control problem for the wave equation, as well as a Dirichlet boundary control problem for the strongly damped wave equation and derive optimality systems. In Section 4.3 we will check the assumptions from Section 4.1 for these problems, in Section 4.4 we discretize the problems, in Section 4.5 we present some numerical examples illustrating our theoretical results, and in Section 4.6 we give an outlook.

4.1 Semi-smooth Newton methods

Let E and Z be Banach spaces and let $F\colon D \subset E \to Z$ be a (nonlinear) mapping with open domain D.

We introduce the notion of Newton differentiability and semi-smoothness, cf. Ito & Kunisch [64].

Definition 4.1.1 (Newton differentiable)**.** The mapping $F\colon D \subset E \to Z$ is called Newton differentiable in the open subset $U \subset D$ if there exists a family of generalized derivatives $G\colon U \to \mathcal{L}(E, Z)$ such that

$$\lim_{h \to 0} \frac{1}{\|h\|_E} \ \|F(x+h) - F(x) - G(x+h)h\|_Z = 0 \tag{4.1.1}$$

for every $x \in U$.

Definition 4.1.2 (Semi-smoothness)**.** The mapping $F\colon D \subset E \to Z$ is called semi-smooth at x if it is Newton differentiable at x and

$$\lim_{t \to 0^+} G(x+th)h \quad \text{exists uniformly in } \ \|h\|_E = 1.$$

Example 4.1.3. Let E be a Hilbert space. Then the norm-functional $F(x) = \|x\|_E$ on E is Newton differentiable, even semi-smooth, with generalized derivative

$$G(x+h)h = \left(\frac{x+h}{\|x+h\|_E}, h \right)_E, \quad G(0)h = (\lambda, h)_E$$

for some $\lambda \in E$; cf. Ito & Kunisch [64].

There holds the following relation between semi-smooth and directionally differentiable functions; cf. Ito & Kunisch [64].

Lemma 4.1.4. *Let $F\colon D \subset E \to Z$ be Newton differentiable at $x \in D$ with Newton derivative G. Then, F is directionally differentiable at x if and only if F is semi-smooth. In this case there holds*

$$\lim_{t \to 0^+} G(x+th)h = \lim_{t \to 0^+} \frac{F(x+th) - F(x)}{t}.$$

Further, we have the following relation between Newton differentiable and Fréchet differentiable functions.

Lemma 4.1.5. *Every continuously Fréchet-differentiable function $f\colon D \subset E \to Z$ is also Newton differentiable.*

Proof. Using the triangular inequality we obtain directly

$$\frac{\|F(x+h) - F(x) - F'(x+h)h\|_Z}{\|h\|_E} = \frac{\|F(x+h) - F(x) - F'(x)h\|_Z}{\|h\|_E} + \frac{\|F'(x)h - F'(x+h)h\|_Z}{\|h\|_E}$$

and the assertion follows. □

The following theorem provides a result on superlinear convergence for semi-smooth Newton methods.

Theorem 4.1.6. *Suppose that $x^* \in D$ is a solution to $F(x) = 0$ and that F is Newton differentiable with Newton derivative G in an open neighborhood U containing x^* and that*

$$\{ \left\|G(x)^{-1}\right\|_{\mathcal{L}(Z,E)} \mid x \in U \}$$

is bounded. Then for $x_0 \in D$ the Newton iteration

$$x_{k+1} = x_k - G(x_k)^{-1} F(x_k), \quad k \in \mathbb{N}_0,$$

converges superlinearly to x^* provided that $\|x_0 - x^*\|_E$ is sufficiently small.

Proof. We recall the proof from Hintermüller, Ito & Kunisch [56]. The Newton iterates satisfy the following inequality

$$\left\|x^{k+1} - x^*\right\|_E \leq \left\|G(x^k)^{-1}\right\|_{\mathcal{L}(Z,E)} \left\|F(x^k) - F(x^*) - G(x^k)(x^k - x^*)\right\|_Z \tag{4.1.2}$$

for $x^k \in U$. Let $B_r(x^*) \subset U$ be a ball of radius r centered at x^* and choose $M > 0$, such that

$$\left\|G(x)^{-1}\right\|_{\mathcal{L}(Z,E)} \leq M$$

for all $x \in B(x^*, r)$. For arbitrary $\eta \in (0,1]$ there exists a $\rho \in (0, r)$ such that

$$\|F(x^* + h) - F(x^*) - G(x^* + h)h\|_Z < \frac{\eta}{M} \|h\|_E \leq \frac{1}{M} \|h\|_E \tag{4.1.3}$$

for all $\|h\|_E < \rho$, $h \in E$. Here, we used (4.1.1). Consequently, if we choose x^0, such that

$$\left\|x^0 - x^*\right\|_E \leq \rho,$$

then using (4.1.2), (4.1.3) we obtain by an induction argument with $h = x^k - x^*$ that

$$\left\|x^{k+1} - x^*\right\|_E < \rho$$

and so $x^{k+1} \in B_\rho(x^*)$. This implies that all iterates are well-defined. Since $\eta \in (0,1]$ is chosen arbitrarily the iterations x^k converge superlinearly to x^*. □

In the following we need Newton differentiability of the max-operator. For this purpose let $\mathcal{E} = \{ v \colon \omega \to \mathbb{R} \}$ denote a function space of real-valued functions on an open domain $\omega \subset \mathbb{R}^n$ and let $\max(0, v)$ denote the pointwise max-operation for $v \in \mathcal{E}$. Then candidates for the generalized derivative are given by

$$G_{m,\delta}(v)(x) = \begin{cases} 1 & \text{if } v(x) > 0, \\ 0 & \text{if } v(x) < 0, \\ \delta & \text{if } v(x) = 0, \end{cases} \tag{4.1.4}$$

for $v \in \mathcal{E}$ and $\delta \in \mathbb{R}$ arbitrary.

Proposition 4.1.7. *There hold the following two properties on Newton differentiability.*

1. *The mapping* $\max(0,\cdot)\colon L^q(\omega) \to L^p(\omega)$ *with* $1 \le p < q < \infty$ *is Newton differentiable on* $L^q(\omega)$ *and* $G_{m,\delta}$ *is a generalized derivative.*

2. $G_{m,\delta}$ *can in general not serve as a Newton derivative for* $\max(0,\cdot)\colon L^p(\Omega) \to L^p(\Omega)$, *for* $1 \le p \le \infty$.

Proof. 1. For the proof we refer to Ito & Kunisch [64] and Hintermüller, Ito & Kunisch [56].

2. This can be found in Ito & Kunisch [64]. Here, we recall the proof for the case $1 \le p < \infty$. Let

$$x(s) = -|s|, \quad h_n(s) = \frac{1}{n} \cdot \chi_{\mathcal{K}}, \quad s \in \mathcal{K} = (-1,1),$$

where $\chi_{\mathcal{K}}$ is the characteristic function w.r.t. to $\mathcal{K}$. There holds

$$\|h_n\|^p_{L^p(\mathcal{K})} = \frac{2}{n^{p+1}}$$

and

$$\int_{-1}^{1} |\max(x+h_n) - \max(x) - G_m(x+h_n)h_n|^p ds = \int_{-\frac{1}{n}}^{\frac{1}{n}} |x(s)|^p ds = \frac{2}{p+1}\frac{1}{n^{p+1}}.$$

Since

$$\frac{1}{\|h\|^p_{L^p(\mathcal{K})}} \int_{-1}^{1} |\max(x+h_n) - \max(x) - G_m(x+h_n)h_n|^p ds = \frac{1}{p+1} \neq 0$$

for $1 \le p < \infty$, the assertion follows at once. □

We also have the following chain rule; cf. Ito & Kunisch [63] and Ulbrich [129].

Lemma 4.1.8 (Chain rule)**.** *Let* E_1, E_2 *be Banach spaces and* $\psi\colon D \subset E_1 \to E_2$ *be continuously Fréchet differentiable at* $y^* \in D$ *and let* $\varphi\colon E_2 \to E_1$ *be Newton differentiable at* $\psi(y^*)$ *with a generalized derivative* G. *Then*

$$F = \varphi \circ \psi\colon D \subset E_1 \to E_1$$

is Newton differentiable at y^* *with a generalized derivative given by* $(G \circ \psi)\psi' \in \mathcal{L}(E_1, E_1)$.

Proof. We recall the proof from Ito & Kunisch [64]. Let $V \subset D$ be a convex neighbourhood of $x \in D$, such that $\psi' \in \mathcal{L}(E_1, E_2)$ is continuous in V and $\psi(V) \subset U(\psi(x))$, where $U(\psi(x))$ is defined according to Newton-differentiability of φ at $\psi(x)$. Further, let $h \in E_1$ with $x + h \in V$. Since $\psi' \in \mathcal{L}(E_1, E_2)$ is continuous at x there holds

$$\left\| \int_0^1 \psi'(x+\theta h)d\theta - \psi'(x+h) \right\|_{E_2} \longrightarrow 0 \quad \text{for } \|h\|_{E_1} \to 0, \tag{4.1.5}$$

and further,

$$\psi(x+h) = \psi(x) + \int_0^1 \psi'(x+\theta h)hd\theta. \tag{4.1.6}$$

Since φ is Newton differentiable at $\psi(x)$ and using (4.1.6) we have

$$\lim_{\|h\|_{E_1}} \frac{1}{\|h\|_{E_1}} \left\| \varphi(\psi(x+h)) - \varphi(\psi(x)) - G(\psi(x+h)) \int_0^1 \psi'(x+\theta h)hd\theta \right\|_{E_1} = 0.$$

With (4.1.5) we obtain

$$\lim_{\|h\|_{E_1}} \frac{1}{\|h\|_{E_1}} \left\| \varphi(\psi(x+h)) - \varphi(\psi(x)) - G(\psi(x+h))\psi'(x+\theta h)h \right\|_{E_1} = 0.$$

This implies the Newton differentiability of $F = \varphi \circ \psi$ in x. □

For a further discussion of Newton differentiable mappings and their properties we refer the reader to Ito & Kunisch [64] and Ulbrich [127, 129].

According to (2.2.7) the reduced cost functional is given by

$$j\colon U \to \mathbb{R}, \quad j(u) = \mathcal{G}(S(u)) + \frac{\alpha}{2} \|u\|_U^2$$

with S defined as in (4.0.1). Thus the reduced problem is given by

$$\text{Minimize } \ j(u), \quad u \in U_{\text{ad}}. \tag{4.1.7}$$

Proposition 4.1.9. *There exists a unique global solution of the optimal control problem* (4.1.7)*.*

Proof. Since G is strictly convex and continuous, this follows immediately from Proposition 2.2.6 and Remark 2.2.8. □

Next, we formulate the first derivative of the reduced cost functional.

Lemma 4.1.10. *The first (directional) derivative of j is given as*

$$j'(u)(\delta u) = (\alpha u - q(u), \delta u)_\omega$$

for $\delta u \in U_{ad}$, where the operator $q\colon U \to U$ is given by

$$q(u) = -T^* \mathcal{G}'(S(u)) \tag{4.1.8}$$

and $(\cdot,\cdot)_\omega$ denotes the inner product in $U = L^2(\omega)$.

Proof. This follows immediately by chain rule. □

Proposition 4.1.11. *Let the above assumptions be fulfilled. Then the necessary and sufficient optimality conditions for* (4.1.7) *can be expressed as the variational inequality*

$$(\alpha u - q(u), \delta u - u)_\omega \geq 0 \quad \text{for all } \delta u \in U_{ad}. \tag{4.1.9}$$

This can alternatively be expressed as an optimality system for the control $u \in U$ and the Lagrange multiplier $\lambda \in U$ as

$$\begin{cases} \alpha u + \lambda = q(u) \\ \lambda = \max(0, \lambda + c(u - u_b)) + \min(0, \lambda + c(u - u_a)) \end{cases} \tag{4.1.10}$$

with an arbitrary $c > 0$.

This follows by standard arguments; cf. Ito & Kunisch [64].

Using (4.1.10) inequality (4.1.9) can be equivalently formulated as an operator equation, which can be solved by the semi-smooth Newton method. We set $c = \alpha$, eliminate the Lagrange multiplier λ and obtain an equivalent formulation.

Lemma 4.1.12. *Condition* (4.1.10) *is equivalent to*

$$\mathcal{F}(u) = 0, \tag{4.1.11}$$

with the operator $\mathcal{F}\colon L^2(\omega) \to L^2(\omega)$ defined by

$$\mathcal{F}(u) = \alpha(u - u_b) + \max(0, \alpha u_b - q(u)) + \min(0, q(u) - \alpha u_a). \tag{4.1.12}$$

Proof. Setting $c = \alpha$ and using the fact that $\max(0, x) - x = \max(0, -x)$ for $x \in \mathbb{R}$ we obtain that (4.1.11) is equivalent to

$$0 = \alpha u - q(u) + \max(0, q(u) - \alpha u_b) + \min(0, q(u) - \alpha u_a),$$

which corresponds to

$$\lambda(u) = \max(0, \lambda + c(u - u_b)) + \min(0, \lambda + c(u - u_a)), \quad \alpha u + \lambda = q(u). \qquad \square$$

We will use the generalized derivatives of max- and min-operators, see (4.1.4), chosen as

$$(G_{\max}(v)\phi)(x) = \begin{cases} \phi(x) & \text{if } v(x) \geq 0, \\ 0 & \text{if } v(x) < 0 \end{cases} \quad \text{and} \quad (G_{\min}(v)\phi)(x) = \begin{cases} \phi(x) & \text{if } v(x) \leq 0, \\ 0 & \text{if } v(x) > 0 \end{cases}$$

for $v, \phi \in L^2(\omega)$.

The following assumption will insure the superlinear convergence of the semi-smooth Newton method applied to (4.1.11).

Assumption 4.1.13. We assume that the operator q defined in (4.1.8) is a continuous affine-linear operator $q\colon L^2(\omega) \to L^r(\omega)$ for some $r > 2$.

In the following sections we will check Assumption 4.1.13 for optimal distributed, Neumann boundary and Dirichlet boundary control problems, since this implies Newton differentiability.

Lemma 4.1.14. *Let Assumption 4.1.13 be fulfilled and $u_a, u_b \in L^r(\omega)$ for some $r > 2$. Then the operator $\mathcal{F}\colon L^2(\omega) \to L^2(\omega)$ is Newton differentiable and a generalized derivative $G_{\mathcal{F}}(u) \in \mathcal{L}(L^2(\omega), L^2(\omega))$ is given as*

$$G_{\mathcal{F}}(u)h = \alpha h + G_{\max}(\alpha u_b - q(u))\, T^*\mathcal{G}''(S(u))Th - G_{\min}(q(u) - \alpha u_a)\, T^*\mathcal{G}''(S(u))Th.$$

Proof. The statement follows from the chain rule in Lemma 4.1.8, the Newton differentiability of max- and min-operators and from Assumption 4.1.13. □

For the operators $G_{\mathcal{F}}(u)$ we have the following lemma.

Lemma 4.1.15. *There exists a constant C_G, such that*

$$\|G_{\mathcal{F}}(u)^{-1}(w)\|_{L^2(\omega)} \le C_G \,\|w\|_{L^2(\omega)} \tag{4.1.13}$$

for all $w \in L^2(\omega)$ and for each $u \in L^2(\omega)$.

Proof. Let χ_I denote the characteristic function of the set

$$I = \{x \in \omega : \alpha u_a(x) \le q(u)(x) \le \alpha u_b(x)\},$$

and analogously let χ_A be the characteristic function of $A = \omega \setminus I$. Let $h \in L^2(\omega)$ and set

$$w = G_{\mathcal{F}}(u)(h). \tag{4.1.14}$$

On A there holds

$$G_{\mathcal{F}}(u)(h) = \alpha h$$

and on I

$$G_{\mathcal{F}}(u)(h) = \alpha h + T^*\mathcal{G}''(S(u))Th.$$

Hence, we deduce

$$\|h\chi_A\|_{L^2(\omega)} \le \frac{1}{\alpha}\,\|w\chi_A\|_{L^2(\omega)} \tag{4.1.15}$$

and taking the inner product of (4.1.14) with $h\chi_I$ we find

$$\alpha\,\|h\chi_I\|^2_{L^2(\omega)} + (\mathcal{G}''(S(u))Th,\, Th\chi_I) = (w,\, h\chi_I).$$

This implies that

$$\alpha\,\|h\chi_I\|^2_{L^2(\omega)} + (\mathcal{G}''(S(u))Th\chi_I,\, Th\chi_I) = (w,\, h\chi_I) - (\mathcal{G}''(S(u))Th\chi_A,\, Th\chi_I).$$

Thus, since $\mathcal{G}''$ is non-negative and $\mathcal{G}$ quadratic we deduce further

$$\alpha\,\|h\chi_I\|^2_{L^2(\omega)} \le \|w\chi_I\|_{L^2(\omega)}\,\|h\chi_I\|_{L^2(\omega)} + K\|h\chi_A\|_{L^2(\omega)}\,\|h\chi_I\|_{L^2(\omega)},$$

for a constant K independent of h and u. Consequently,

$$\alpha\,\|h\chi_I\|_{L^2(\omega)} \le \|w\chi_I\|_{L^2(\omega)} + K\|h\chi_A\|_{L^2(\omega)} \le \|w\chi_I\|_{L^2(\omega)} + \frac{K}{\alpha}\,\|w\chi_A\|_{L^2(\omega)}. \tag{4.1.16}$$

Combining (4.1.15) and (4.1.16) the desired result follows. □

After these considerations we can formulate the following theorem.

Theorem 4.1.16. *Let Assumption 4.1.13 be fulfilled and suppose that $u^* \in L^2(\omega)$ is a solution to the optimal control problem under consideration. Then, for $u_0 \in L^2(\omega)$ with $\|u_0 - u^*\|_{L^2(\omega)}$ sufficiently small, the semi-smooth Newton method*

$$G_{\mathcal{F}}(u_k)(u_{k+1} - u_k) + \mathcal{F}(u_k) = 0, \quad k = 0, 1, 2, \dots, \tag{4.1.17}$$

converges superlinearly.

Proof. This follows from Theorem 4.1.6, and the Lemmas 4.1.14 and 4.1.15. □

Primal-dual active set algorithm

The semi-smooth Newton method (4.1.17) is known to be equivalent to a primal-dual active set method (PDAS); cf. Hintermüller, Ito & Kunisch [56] and Proposition 4.1.18. PDAS is used to treat inequality constraints. Algorithm 4.1 gives a sketch overview on the continuous level of the algorithm. On the discrete level the algorithm works analogously.

Algorithm 4.1: Primal-dual active set method

1: Choose u_0 and set $\lambda_0 = q(u_0) - \alpha u_0$.

2: Given (u_k, λ_k) determine

$$\begin{aligned}
\mathcal{A}^b_{k+1} &= \{ x \in \omega \mid \lambda_k(x) + \alpha(u_k - u_b)(x) > 0 \}, \\
\mathcal{A}^a_{k+1} &= \{ x \in \omega \mid \lambda_k(x) + \alpha(u_k - u_a)(x) < 0 \}, \\
\mathcal{I}_{k+1} &= \omega \backslash (\mathcal{A}^b_{k+1} \cup \mathcal{A}^a_{k+1}).
\end{aligned}$$

3: Determine u_{k+1} as the solution to

$$\begin{cases} \text{Minimize} & j(u_{k+1}), \quad u_{k+1} \in U, \\ \text{subject to} & u_{k+1} = u_b \text{ on } \mathcal{A}^b_{k+1}, \quad u_{k+1} = u_a \text{ on } \mathcal{A}^a_{k+1}. \end{cases}$$

4: Update λ_{k+1} according to

$$\lambda_{k+1} = q(u_{k+1}) - \alpha u_{k+1}.$$

5: Update $k = k + 1$.

Remark 4.1.17. If the algorithm finds two successive active sets, for which $\mathcal{A}_k = \mathcal{A}_{k+1}$, then u_k is the solution of the problem. We apply this condition as a stopping criterion.

The equivalence of PDAS and the semi-smooth Newton method can be shown directly.

Proposition 4.1.18. *The primal dual active set strategy and the semi-smooth Newton method are equivalent.*

Proof. The Newton iteration (4.1.17) and the PDAS method can equivalently be expressed as

$$\begin{aligned}\alpha(u_{k+1}-u_b) - G_{\max}(\alpha u_b - q(u_k))(q(u_{k+1})-q(u_k)) \\ + G_{\min}(q(u)-\alpha u_a)(q(u_{k+1})-q(u_k)) \\ + \max(0,\, \alpha u_b - q(u_k)) + \min(0,\, q(u_k)-\alpha u_a) = 0. \quad \square\end{aligned}$$

4.2 Optimal control problems

The semi-smooth Newton method introduced in the previous section is applied to optimal control problems of wave equations with different types of control action. We consider distributed control and Neumann boundary control for the wave equation and Dirichlet boundary control for the wave equation as well as the strongly damped wave equation. The control problems are formulated and optimality systems are derived.

4.2.1 Distributed control

The optimal control problem in case of distributed control reads as

$$\begin{cases} \text{Minimize} \quad J(u,y) = \mathcal{G}(y) + \dfrac{\alpha}{2}\,\|u\|^2_{L^2(Q)}, \quad y \in L^2(Q),\ u \in L^2(Q), \quad \text{s.t.} \\ \qquad y_{tt} - \Delta y = u \quad \text{in } Q, \\ \qquad y(0) = y_0 \quad \text{in } \Omega, \\ \qquad y_t(0) = y_1 \quad \text{in } \Omega, \\ \qquad y = 0 \quad \text{on } \Sigma, \\ \qquad u_a \le u \le u_b \quad \text{a.e. in } Q, \end{cases} \tag{4.2.1}$$

where $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$ and the state equation is understood in the sense of Theorem 2.3.1. Further we assume that u_a, u_b are in $L^r(Q)$ for some $r > 2$.

The optimality system can be derived by standard techniques, see Lions [87, pp. 296] and cf. Remark 3.1.9.

Theorem 4.2.1 (Optimality system - distributed control)**.** *The optimality system for* (4.2.1) *is given by*

$$\begin{cases} y_{tt} - \Delta y = u, \\ y(0) = y_0, \quad y_t(0) = y_1, \quad y|_\Sigma = 0, \\ p_{tt} - \Delta p = -\mathcal{G}'(y), \\ p(T) = 0, \quad p_t(T) = 0, \quad p|_\Sigma = 0, \\ \alpha u + \lambda = p, \\ \lambda = \max(0, \lambda + c(u-u_b)) + \min(0, \lambda + c(u-u_a)) \end{cases} \tag{4.2.2}$$

for any $c > 0$, $\lambda \in L^2(Q)$ *and* $p \in C(H^1(\Omega)) \cap C^1(L^2(\Omega))$.

4.2.2 Neumann boundary control

The optimal control problem in case of Neumann boundary control reads as

$$\begin{cases} \text{Minimize} \quad J(u,y) = \mathcal{G}(y) + \frac{\alpha}{2}\, \|u\|^2_{L^2(Q)}\,, \quad y \in L^2(Q),\ u \in L^2(\Sigma), \quad \text{s.t.} \\[1ex] \qquad\qquad y_{tt} - \Delta y = f \quad \text{in } Q, \\ \qquad\qquad y(0) = y_0 \quad \text{in } \Omega, \\ \qquad\qquad y_t(0) = y_1 \quad \text{in } \Omega, \\ \qquad\qquad \partial_n y = u \quad \text{on } \Sigma, \\ \qquad\qquad u_a \le u \le u_b \quad \text{a.e. in } \Sigma, \end{cases} \tag{4.2.3}$$

where $y_0 \in L^2(\Omega)$, $y_1 \in (H^1(\Omega))^*$, $f \in L^1((H^1(\Omega))^*)$, $u_a, u_b \in L^r(\Sigma)$ with some $r > 2$ and the state equation is understood in the sense of Theorem 2.3.6.

Theorem 4.2.2 (Optimality system - Neumann boundary control)**.** *The optimality system for* (4.2.3) *is given by*

$$\begin{cases} y_{tt} - \Delta y = f, \\ y(0) = y_0, \quad y_t(0) = y_1, \quad \partial_n y|_\Sigma = u, \end{cases} \tag{4.2.4a}$$

$$\begin{cases} p_{tt} - \Delta p = -\mathcal{G}'(y), \\ p(T) = 0, \quad p_t(T) = 0, \quad \partial_n p|_\Sigma = 0, \end{cases} \tag{4.2.4b}$$

$$\begin{cases} \alpha u + \lambda = p|_\Sigma, \\ \lambda = \max(0, \lambda + c(u - u_b)) + \min(0, \lambda + c(u - u_a)) \end{cases} \tag{4.2.4c}$$

for any $c > 0$, $\lambda \in L^2(\Sigma)$ *and* $p \in C(H^1(\Omega)) \cap C^1(L^2(\Omega))$.

Proof. We recall the proof from Lions [87, pp. 321]. The optimal control u is characterized by

$$(\mathcal{G}'(S(u)), Tv - Tu)_I + \alpha\langle u, v - u\rangle_I \ge 0 \quad \forall v \in U_{\text{ad}}. \tag{4.2.5}$$

From the very weak formulation (2.3.13) we deduce for all $v \in U_{\text{ad}}$

$$(Tv - Tu, g)_I = \langle v - u, \zeta\rangle_I, \tag{4.2.6}$$

where $\zeta = \zeta_g$ is the solution to

$$\begin{cases} \zeta_{tt} - \Delta\zeta = g & \text{in } Q, \\ \zeta(T) = 0 & \text{in } \Omega, \\ \zeta_t(T) = 0 & \text{in } \Omega, \\ \partial_n \zeta = 0 & \text{on } \Sigma \end{cases}$$

for arbitrary $g \in L^2(Q)$. Let $p(u)$ the solution of the adjoint equation (4.2.4b) corresponding to the control u. Then with $g = -\mathcal{G}'(S(u))$ we have

$$\big(Tv - Tu, -\mathcal{G}'(S(u))\big)_I = \langle v - u, p(u)\rangle_I. \tag{4.2.7}$$

Here, $-\mathcal{G}'(S(u))$ denotes its $L^2(Q)$ representative with respect to the Riesz representation theorem. Finally, together with (4.2.5) we deduce

$$\langle \alpha u - p(u), v - u\rangle_I \geq 0 \quad \forall v \in U_{\mathrm{ad}}. \qquad \square$$

4.2.3 Dirichlet boundary control

In case of Dirichlet boundary control we consider two different state equations, i.e. we analyze control of the wave equation as well as control of the strongly damped wave equation.

Dirichlet boundary control for the wave equation

The optimal control problem for the wave equation in case of Dirichlet boundary control reads as

$$\begin{cases} \text{Minimize} \quad J(u,y) = \mathcal{G}(y) + \dfrac{\alpha}{2}\,\|u\|^2_{L^2(Q)}, \quad y \in L^2(Q),\ u \in L^2(\Sigma), \quad \text{s.t.} \\[2ex] \qquad y_{tt} - \Delta y = f \quad \text{in } Q, \\ \qquad y(0) = y_0 \quad \text{in } \Omega, \\ \qquad y_t(0) = y_1 \quad \text{in } \Omega, \\ \qquad y = u \quad \text{on } \Sigma, \\ \qquad u_a \leq u \leq u_b \quad \text{a.e. in } \Sigma, \end{cases} \tag{4.2.8}$$

where $y_0 \in L^2(\Omega)$, $y_1 \in H^{-1}(\Omega)$, $f \in L^1((H^{-1}(\Omega))$, $u_a, u_b \in L^r(\Sigma)$ with some $r > 2$ and the state equation is understood in the sense of Theorem 2.3.12.

Theorem 4.2.3 (Optimality system - Dirichlet boundary control)**.** *The optimality system is given by*

$$\begin{cases} y_{tt} - \Delta y = f, \\ y(0) = y_0, \quad y_t(0) = y_1, \quad y|_\Sigma = u, \end{cases} \tag{4.2.9a}$$

$$\begin{cases} p_{tt} - \Delta p = -\mathcal{G}'(y), \\ p(T) = 0, \quad p_t(T) = 0, \quad p|_\Sigma = 0, \end{cases} \tag{4.2.9b}$$

$$\begin{cases} \alpha u + \lambda = -\partial_n p|_\Sigma, \\ \lambda = \max(0, \lambda + c(u - u_b)) + \min(0, \lambda + c(u - u_a)) \end{cases} \tag{4.2.9c}$$

for $c > 0$, $\lambda \in L^2(\Sigma)$ and $p \in C(H^1(\Omega)) \cap C^1(L^2(\Omega))$.

Proof. The proof is similar to that one in the Neumann case, see Theorem 4.2.2. Here, we deduce from the very weak formulation (2.3.20) for the optimal control u and arbitrary $v \in U_{\mathrm{ad}}$

$$(Tv - Tu, g)_I = \langle v - u, -\partial_n \zeta\rangle_I, \tag{4.2.10}$$

where $\zeta = \zeta_g$ is the solution to

$$\begin{cases} \zeta_{tt} - \Delta\zeta = g & \text{in } Q, \\ \zeta(T) = 0 & \text{in } \Omega, \\ \zeta_t(T) = 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \Sigma \end{cases}$$

for arbitrary $g \in L^2(Q)$. Let $p(u)$ the solution of the adjoint equation (4.2.9b) corresponding to the control u. Then with $g = -\mathcal{G}'(S(u))$ we have

$$\big(Tv - Tu, -\mathcal{G}'(S(u))\big)_I = \langle v - u, -\partial_n p(u)\rangle_I,$$

which implies

$$\langle \alpha u - \partial_n p(u), v - u\rangle_I \geq 0 \quad \forall v \in U_{\mathrm{ad}}. \qquad \square$$

Dirichlet boundary control for the strongly damped wave equation

The optimal control problem for the strongly damped wave equation in case of Dirichlet boundary control reads as

$$\begin{cases} \text{Minimize} \quad J(u,y) = \mathcal{G}(y) + \dfrac{\alpha}{2}\, \|u\|^2_{L^2(Q)}, \quad y \in L^2(Q),\ u \in L^2(\Sigma), \quad \text{s.t.} \\[1ex] \begin{aligned} y_{tt} - \Delta y - \rho \Delta y_t &= f && \text{in } Q, \\ y(0) &= y_0 && \text{in } \Omega, \\ y_t(0) &= y_1 && \text{in } \Omega, \\ y &= u && \text{on } \Sigma, \\ u_a \leq u &\leq u_b && \text{a.e. in } \Sigma, \end{aligned} \end{cases} \tag{4.2.11}$$

where $\rho > 0$, $f \in L^2(L^2(\Omega))$, $y_0 \in H^1(\Omega)$, $y_1 \in L^2(\Omega)$, $u_a, u_b \in L^r(\Sigma)$ with some $r > 2$ and the state equation is understood in the sense of Theorem 2.3.20.

Theorem 4.2.4 (Optimality system - Dirichlet boundary control for damped equation)**.** *The optimality system is given by*

$$\begin{cases} y_{tt} - \Delta y - \rho \Delta y_t = f, \\ y(0) = y_0, \quad y_t(0) = y_1, \quad y|_\Sigma = u, \\[1ex] p_{tt} - \Delta p + \rho \Delta p_t = -\mathcal{G}'(y), \\ p(T) = 0, \quad p_t(T) = 0, \quad p|_\Sigma = 0, \\[1ex] \alpha u + \lambda = -\partial_n p|_\Sigma + \rho \partial_n p_t|_\Sigma, \\ \lambda = \max(0, \lambda + c(u - u_b)) + \min(0, \lambda + c(u - u_a)) \end{cases} \tag{4.2.12}$$

for $c > 0$, $\lambda \in L^2(\Sigma)$ *and* $p \in H^2(L^2(\Omega)) \cap C^1(H^1_0(\Omega)) \cap H^1(H^2(\Omega))$.

The proof follows the argumentation as in the case without damping.

4.3 Convergence of the semi-smooth Newton method

The optimal control problems considered in Section 4.2 are solved by the semi-smooth Newton method formulated in Section 4.1. We analyze the convergence of this method. Therefore, we check Assumption 4.1.13 for each problem to verify if superlinear convergence of the semi-smooth Newton is given according to Theorem 4.1.16. Furthermore, we formulate some regularity results for the optimal controls and the optimal states.

In Section 4.3.1 we consider distributed control, in Section 4.3.2 Neumann boundary control and in Section 4.3.3 Dirichlet boundary control. We start with some results from interpolation theory and a trace theorem.

For a Banach space A and two Banach spaces $A_1, A_2 \subset A$ we call $\{A_1, A_2\}$ an interpolation couple. The following results can be found in Triebel [124, pp. 128].

Proposition 4.3.1. *Let* $1 \le p_0, p_1 < \infty$, $0 \le \theta \le 1$, *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Further, let $\{A_1, A_2\}$ *be an interpolation couple,* $A_1, A_2 \in A$, A *Banach space. Then*

$$[L^{p_0}(A_1), L^{p_1}(A_2)]_\theta = L^p([A_1, A_2]_\theta)$$

(for a definition of the interpolation space $[\cdot,\cdot]_\theta$ *see Triebel [124, pp. 58]. Especially, there holds*

$$[L^{p_0}(A), L^{p_1}(A)]_\theta = L^p(A).$$

To shorten notations we introduce the Hilbert spaces

$$\mathcal{H}^{r,s}(Q) = L^2(H^r(\Omega)) \cap H^s(L^2(\Omega)),$$
$$\mathcal{H}^{r,s}(\Sigma) = L^2(H^r(\partial\Omega)) \cap H^s(L^2(\partial\Omega))$$

for $r, s \ge 0$, $r, s \in \mathbb{R}$. There exists the following trace result; see Lions & Magenes [92, pp. 9].

Proposition 4.3.2 (Trace)**.** *Let* $v \in \mathcal{H}^{r,s}(Q)$ *with* $r > \frac{1}{2}$, $s \ge 0$. *Then for*

$$j \ge 0, \quad j < r - \frac{1}{2}, \quad \frac{\mu_j}{r} = \frac{\nu_j}{s} = \frac{r - j - \frac{1}{2}}{r} \quad (\nu_j = 0 \text{ if } s = 0)$$

the mapping

$$\mathcal{H}^{r,s}(Q) \to \mathcal{H}^{\mu_j,\nu_j}(\Sigma), \quad v \mapsto \frac{\partial^j v}{\partial \nu^j}$$

is continuous linear, where $\frac{\partial}{\partial\nu}$ *denotes the normal derivate on* Σ *oriented to the interior of* Σ.

Later, we need the property of the max operator to conserve H^s-regularity for $0 \le s \le 1$.

Proposition 4.3.3. *Let* D *be a domain in* $\mathbb{R}^d$, $d \ge 2$, *having the uniform 1-smooth regularity property (cf. Adams [1]) and a bounded boundary, and let* $s \in [0, 1]$.

1. *If* $v \in H^s(D)$, *then* $\max(0, v) \in H^s(D)$ *and*

$$\|\max(0, v)\|_{H^s(D)} \le \|v\|_{H^s(D)}.$$

2. *If* $v \in H^s(L^2(D))$, *then* $\max(0, v) \in H^s(L^2(D))$ *and*

$$\|\max(0, v)\|_{H^s(L^2(D))} \le \|v\|_{H^s(L^2(D))}.$$

For a proof we refer to Kunisch & Vexler [78, Lemma 3.3].

After these preparations we continue with the control problems.

4.3.1 Distributed control

The semi-smooth Newton method in case of distributed control converges superlinearly and the optimal control and optimal state have an improved regularity.

To prove the result on convergence we have to verify Assumption 4.1.13.

Theorem 4.3.4. *In the case of distributed control the operator q defined in* (4.1.8) *is a continuous affine-linear operator*

$$q\colon L^2(Q) \to L^r(Q)$$

with some $r > 2$.

Proof. A direct comparison between the general optimality system (4.1.10) and (4.2.2) shows that in this case for a given control $u \in L^2(Q)$ we have

$$q(u) = p,$$

where p is the solution of the corresponding adjoint equation. From Theorem 2.3.3 we deduce that in particular $p \in C(H^1(\Omega))$ and hence, for $d = 2$ we have

$$p \in L^r(Q)$$

for all $1 \le r < \infty$ and for $d \ge 3$ we have

$$p \in L^{\frac{2d}{d-2}}(Q),$$

which proves the assertion. □

From Theorem 4.1.16 we obtain immediately the following corollary.

Corollary 4.3.5. *The semi-smooth Newton method applied to* (4.2.1) *converges superlinearly.*

As a further consequence of Theorem 4.3.4 we obtain the following improved regularity results for the optimal control and the optimal state.

Corollary 4.3.6. *Let $u_a, u_b \in \mathcal{H}^{1,1}(Q)$. Then, there holds for the optimal control u*

$$u \in \mathcal{H}^{1,1}(Q).$$

Proof. According to Section 4.1 the optimality condition is equivalent to

$$\alpha(u - u_b) + \max(0, \alpha u_b - q(u)) + \min(0, q(u) - \alpha u_a) = 0$$

with $q(u) = p$. From Proposition 4.3.3 we deduce that the regularity of $q(u) \in \mathcal{H}^{1,1}(Q)$ is transferred to $\max(0, \alpha u_b - q(u))$ and $\min(0, q(u) - \alpha u_a)$ and therefore also to u. □

Further, we can formulate some improved regularity result for the optimal state. Therefore we only need $u \in H^1(L^2(\Omega))$.

Corollary 4.3.7 (Regularity optimal control)**.** *For $y_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $y_1 \in H^1_0(\Omega)$ and $u_a, u_b \in H^1(L^2(\Omega))$, there holds for the optimal state*

$$y \in C(H^2(\Omega)), \quad y_t \in C(H^1_0(\Omega)), \quad y_{tt} \in C(L^2(\Omega)).$$

Proof. With a similar argumentation as in Corollary 4.3.6 we obtain $u \in H^1(L^2(\Omega))$ and thus, the assertion follows with Theorem 2.3.8. □

4.3.2 Neumann boundary control

The semi-smooth Newton method applied to the Neumann boundary control problem (4.2.3) converges superlinearly. We derive that in this case Assumption 4.1.13 is given and prove a regularity result for the optimal control and optimal state.

Theorem 4.3.8. *In the case of Neumann boundary control the operator q defined in* (4.1.8) *is a continuous affine-linear operator*

$$q\colon L^2(\Sigma) \to L^r(\Sigma)$$

with some $r > 2$.

Proof. A direct comparison between the general optimality system (4.1.10) and (4.2.4) shows that in this case for a given control $u \in L^2(\Sigma)$ we have

$$q(u) = p|_{\Sigma},$$

where p is the solution of the corresponding adjoint equation. From Theorem 2.3.1 we deduce that $p \in \mathcal{H}^{1,1}(Q)$ and hence by Proposition 4.3.2,

$$p \in \mathcal{H}^{\frac{1}{2},\frac{1}{2}}(\Sigma).$$

By Adams [1, pp. 218] we have $H^{\frac{1}{2}}(L^2(\partial\Omega)) \hookrightarrow W^{\frac{1}{r},r}(L^2(\partial\Omega)) \hookrightarrow L^r(L^2(\partial\Omega))$ for all $2 \le r < \infty$. Consequently, we deduce

$$p \in L^2(H^{\frac{1}{2}}(\partial\Omega)) \cap L^r(L^2(\partial\Omega))$$

for all $2 \le r < \infty$ and hence interpolation, cf. Proposition 4.3.1, implies that

$$p \in L^{r_s}([H^{\frac{1}{2}}(\partial\Omega), L^2(\partial\Omega)]_s), \quad \text{where} \quad \frac{1}{r_s} = \frac{(1-s)}{2} + \frac{s}{r}, \quad s \in [0,1].$$

For $d \ge 3$ we use $H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^{\frac{2d-2}{d-2}}(\partial\Omega)$ and get

$$[H^{\frac{1}{2}}(\partial\Omega), L^2(\partial\Omega)]_s \hookrightarrow L^{q_s}(\partial\Omega), \quad \text{where} \quad \frac{1}{q_s} = \frac{(1-s)(d-2)}{2d-2} + \frac{s}{2}, \quad s \in [0,1].$$

We choose s in such a way that $r_s = q_s$. This implies

$$s = \frac{r}{2 + dr - 2d}, \quad r \ge 2$$

and hence

$$q_s = \frac{8d - 4d^2 - 4 + 2d^2 r - 2dr}{6d - 4 - 2d^2 + d^2 r - 2dr + r}.$$

q_s is monotonic increasing in r and hence we deduce $p \in L^{\frac{2d}{d-1}-\varepsilon}(\Sigma)$ for all $\varepsilon > 0$.

For $d = 2$ we have $H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^q(\partial\Omega)$ for all $q < \infty$ and hence,

$$[H^{\frac{1}{2}}(\partial\Omega), L^2(\partial\Omega)]_s \hookrightarrow L^{t_s}(\partial\Omega), \quad \text{where} \quad \frac{1}{t_s} = \frac{(1-s)}{q} + \frac{s}{2}, \quad s \in [0,1].$$

We choose s in such a way that $r_s = t_s$, i.e.

$$r_s = \frac{2r}{(1-s)r + 2s} = \frac{2q}{2(1-s) + qs} = t_s$$

and obtain by a direct computation

$$s = \frac{qr - 2r}{-2r + 2qr - 2q},$$

which leads to

$$t_s = \frac{-4\frac{1}{q} + 4 - \frac{4}{r}}{1 - \frac{4}{rq}} \longrightarrow 0 \quad (r, q \longrightarrow \infty).$$

This implies $p \in L^{4-\varepsilon}(\Sigma)$ for all $\epsilon > 0$. □

The next assertions follow immediately from Theorem 4.1.16 and the previous consideration.

Corollary 4.3.9. *The semi-smooth Newton method applied to* (4.2.3) *converges superlinearly.*

We obtain additional regularity results for the optimal control and the optimal state.

Corollary 4.3.10. *Let $u_a, u_b \in \mathcal{H}^{\frac{1}{2},\frac{1}{2}}(\Sigma)$. Then, the optimal control satisfies*

$$u \in \mathcal{H}^{\frac{1}{2},\frac{1}{2}}(\Sigma).$$

Additionally, let $f \in L^2(L^2(\Omega))$, $y_0 \in H^1(\Omega)$, and $y_1 \in L^2(\Omega)$, then the optimal state satisfies

$$y \in C(H^1(\Omega)) \cap C^1(L^2(\Omega)) \cap H^2(H^1(\Omega)^*).$$

Proof. We consider the equation

$$y_{tt} - \Delta y = 0, \quad y(0) = 0, \quad y_t(0) = 0, \quad \partial_n y|_\Sigma = g$$

with $g \in L^2(H^{\frac{1}{2}}(\partial\Omega))$. This equation admits a solution $y \in C(H^1(\Omega)) \cap C^1(L^2(\Omega))$; see Proposition 2.3.7. With $g = u \in L^2(H^{\frac{1}{2}}(\partial\Omega))$ and by Theorem 2.3.3 we obtain for the optimal state y of (4.2.3) that $y \in C(H^1(\Omega)) \cap C^1(L^2(\Omega)) \cap H^2(H^1(\Omega)^*)$, which proves the assertion. □

As a direct consequence we deduce that under the assumptions of Corollary 4.3.10 the very weak solution y of the state equation which corresponds to the optimal control u is in fact a variational solution in the sense that $y \in C(H^1(\Omega)) \cap C^1(L^2(\Omega)) \cap H^2(H^1(\Omega)^*)$ and

$$(y_{tt}, \zeta_t)_I + (\nabla y, \nabla \zeta)_I - \langle u, \zeta \rangle_I + (y_0 - y(0), \zeta_t(0)) + (y_t(0) - y_1, \zeta(0)) = (f, \zeta)_I$$

for all $\zeta \in X$ (cf. (2.3.9)), where $(y_{tt}, \zeta_t)_I$ is understood in the sense of Section 2.3.1. This variational formulation is important for numerical realizations, see the corresponding discussion in Kunisch & Vexler [78].

4.3.3 Dirichlet boundary control

The Dirichlet boundary control problem is more involved. We consider the Dirichlet boundary control problems (4.2.8) and (4.2.11) and analyze if the operator q has some smoothing property. We will obtain no smoothing of the mapping q in case of control of the wave equation in contrast to control of the strongly damped wave equation.

Dirichlet boundary control of the wave equation

In the case of Dirichlet boundary control of the wave equation the operator q defined in (4.1.8) is given by

$$q(u) = -\partial_n p,$$

where p is the solution of the corresponding adjoint equation (4.2.9b). From the hidden regularity result, cf. Theorem 2.3.10 we obtain that $\partial_n p \in L^2(\Sigma)$ and the operator q is a continuous affine-linear operator

$$q\colon L^2(\Sigma) \to L^2(\Sigma).$$

In the following we provide a one-dimensional example showing that in general the operator q does not have any smoothing properties in the sense that any control $u \in L^2(\Sigma)$ is mapped in $L^r(\Sigma)$ with $r > 2$. Therefore, Assumption 4.1.13 is not fulfilled in the case of Dirichlet boundary control.

We consider the one dimensional wave equation with Dirichlet boundary control

$$\begin{cases} y_{tt} - y_{xx} = 0 & \text{in } (0,1) \times (0,1), \\ y(0,x) = 0 & \text{in } (0,1), \\ y_t(0,x) = 0 & \text{in } (0,1), \\ y(t,0) = u(t) & \text{in } (0,1), \\ y(t,1) = 0 & \text{in } (0,1) \end{cases} \tag{4.3.1}$$

with $u \in L^2(0,1)$.

Lemma 4.3.11. *Let u be the solution of the Dirichlet boundary control problem* (4.2.8) *for the one dimensional wave equation given in* (4.3.1)*. Then there holds*

$$q(u)(t) = -p_x(t,0) = -u(t)(1-t),$$

where p is the solution of the corresponding adjoint equation

$$\begin{cases} p_{tt} - p_{xx} = y & \textit{in } (0,1) \times (0,1), \\ p(0,x) = 0 & \textit{in } (0,1), \\ p_t(0,x) = 0 & \textit{in } (0,1), \\ p(t,0) = 0 & \textit{in } (0,1), \\ p(t,1) = 0 & \textit{in } (0,1). \end{cases} \tag{4.3.2}$$

Proof. We denote

$$\xi = t + x,\ \xi \in [0,2], \qquad \eta = t - x,\ \eta \in [-1,1].$$

and obtain

$$y(\xi,\eta) = \begin{cases} 0, & \eta < 0, \\ u(\eta), & \eta \geq 0. \end{cases}$$

Let

$$p(x,t) = \tilde{p}(\xi(x,t), \eta(x,t)).$$

Thus, we obtain

$$\begin{aligned} p_t &= \tilde{p}_\xi + \tilde{p}_\eta, \quad p_{tt} = \tilde{p}_{\xi\xi} + \tilde{p}_{\xi\eta} + \tilde{p}_{\eta\xi} + \tilde{p}_{\eta\eta}, \\ p_x &= \tilde{p}_\xi - \tilde{p}_\eta, \quad p_{xx} = \tilde{p}_{\xi\xi} - \tilde{p}_{\eta\xi} - \tilde{p}_{\xi\eta} + \tilde{p}_{\eta\eta}, \end{aligned}$$

i.e. for the differential equation there holds

$$p_{tt} - p_{xx} = 4\tilde{p}_{\eta\xi}. \tag{4.3.3}$$

This implies the following representation of the function $\tilde{p}$, which we prove subsequently:

$$\tilde{p}(\xi,\eta) = \frac{1}{4}\begin{cases} U(\eta)\xi - (2-\eta)U(\eta) - \hat{U}(\eta) + (2-2\xi)U(\xi) + \hat{U}(\xi), & \eta \geq 0,\ \xi < 1, \\ U(\eta)\xi - (2-\eta)U(\eta) - \hat{U}(\eta) + \hat{U}(2-\xi), & \eta \geq 0,\ \xi \geq 1, \\ -\hat{U}(-\eta) + \hat{U}(2-\xi), & \eta < 0,\ \xi \geq 1, \\ -\hat{U}(-\eta) + (2-2\xi)U(\xi) + \hat{U}(\xi), & \eta < 0,\ \xi < 1, \end{cases} \tag{4.3.4}$$

where

$$U(z) = \int_0^z g(s)ds \quad \text{and} \quad \hat{U}(z) = \int_0^z U(s)ds, \quad z \in [-1,1]$$

for

$$g(s) = \begin{cases} 0, & s \in [-1,0), \\ u(s), & s \in [0,1]. \end{cases}$$

The representation of $\tilde{p}$ given in (4.3.4) can be derived by integrating

$$4\tilde{p}_{\eta\xi} = g$$

and choosing all unknowns in such a way that the boundary and initial conditions are satisfied. The function $\tilde{p}$ satisfies (4.3.2) since there holds (4.3.3) and we have

$$4\tilde{p}_{\xi\eta}(\xi,\eta) = U'(\eta) = g(\eta) \quad \text{a.e. in } [0,2]\times[-1,1],$$

and for the boundary values we have

$$p(t,0) = \tilde{p}(t,t) = U(t)t - (2-t)U(t) - \hat{U}(t) + (2-t)U(t) - U(t)t + \hat{U}(t) = 0,$$

$$p(t,1) = \tilde{p}(t+1,t-1) = \hat{U}(1-t) - \hat{U}(1-t) = 0,$$

and for the initial data

$$p(1,x) = \tilde{p}(1+x, 1-x) = U(1-x)(1+x) - (1+x)U(1-x) - \hat{U}(1-x) + \hat{U}(1-x) = 0,$$

$$p_t(1,x) = \tilde{p}_\xi + \tilde{p}_\eta = U(\eta) - U(2-\xi) + u(\eta)\xi + U(\eta) - (2-\eta)u(\eta) - U(\eta)) = 0.$$

To consider the regularity of the normal derivative p_x of the adjoint state we argue as follows

$$\begin{aligned} p_x(t,0) &= \tilde{p}_\xi(\xi,\eta) - \tilde{p}_\eta(\xi,\eta) \\ &= \frac{1}{4}(U(\eta)U'(\xi)(2-\xi) - U(\xi) - U'(\xi)\xi - U(\xi) + U(\xi) \\ &\quad - U'(\eta)\xi + U'(\eta)(2-\eta) - U(\eta) + U(\eta)) \\ &= u(t)(1-t). \end{aligned}$$

Latter we used the fact that $\eta = \xi = t$. Thus, for a general control $u \in L^2(0,1)$ there holds

$$q(u)(t) = -p_x(t,0) = -u(t)(1-t). \qquad \square$$

As a direct consequence, we obtain the next theorem.

Theorem 4.3.12. *The optimal control of* (4.2.8) *for* $d = 1$ *does not have an improved regularity* $q(u) \in L^r(0,1)$ *with some* $r > 2$.

Remark 4.3.13. *This lack of additional regularity is due to the nature of the wave equation. In the elliptic and parabolic cases, the corresponding operator q possess the required regularity for Dirichlet boundary control; cf. Kunisch & Vexler [78].*

Dirichlet control of the strongly damped wave equation

The previous consideration motivates to consider the Dirichlet problem for the strongly damped wave equation as a regularization of the Dirichlet problem of the wave equation. Here, Assumption 4.1.13 is fulfilled as we show next.

Theorem 4.3.14. *In the case of Dirichlet boundary control problem* (4.2.11) *with* $\rho > 0$, *the operator q defined in* (4.1.8) *is a continuous affine-linear operator*

$$q \colon L^2(\Sigma) \to L^r(\Sigma)$$

with some $r > 2$.

Proof. By a direct comparison of the optimality systems (4.1.10) and (4.2.12) we obtain $q(u) = -\partial_n p + \rho \partial_n p_t$.

At first we verify that $\partial_n p_t \in L^p(Q)$ for some $p > 2$. From Theorem 2.3.15 we obtain that p_t in particular fulfills

$$p_t \in \mathcal{H}^{2,1}(Q).$$

By Proposition 4.3.1 we get

$$\partial_n p_t \in \mathcal{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma).$$

Now, we follow the argumentation in [78, Theorem 3.2]. Since

$$H^{\frac{1}{4}}(L^2(\partial\Omega)) \hookrightarrow L^4(L^2(\partial\Omega));$$

cf. [1, Thm. 7.58], we have

$$\partial_n p_t \in L^2(H^{\frac{1}{2}}(\partial\Omega)) \cap L^4(L^2(\partial\Omega)).$$

Now, using Proposition 4.3.1 we obtain

$$\partial_n p_t \in L^{r_s}[H^{\frac{1}{2}}(\partial\Omega), L^2(\partial\Omega)]_s, \quad \frac{1}{r_s} = \frac{1-s}{2} + \frac{s}{4}, \quad s \in [0,1].$$

For $d \geq 3$ we use $H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^{\frac{2d-2}{d-2}}(\partial\Omega)$ and get

$$[H^{\frac{1}{2}}(\partial\Omega), L^2(\partial\Omega)]_s \hookrightarrow L^{q_s}(\partial\Omega), \quad \text{where} \quad \frac{1}{q_s} = \frac{(1-s)(d-2)}{2d-2} + \frac{s}{2}, \quad s \in [0,1].$$

We choose s in such a way that $r_s = q_s$, i.e.

$$r_s = \frac{8}{4-2s} = \frac{2d-2}{d+s-2} = q_s.$$

This implies

$$s = \frac{2}{1+d}, \quad r \geq 2.$$

Thus, we obtain

$$\partial_n p_t \in L^{\frac{2(d+1)}{d}}(\Sigma) \quad \text{for} \;\; d \geq 3.$$

For $d = 2$ we use $H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^r(\partial\Omega)$, $r < \infty$, and get

$$[H^{\frac{1}{2}}(\partial\Omega), L^2(\partial\Omega)]_s \hookrightarrow L^{q_s}(\partial\Omega), \quad \text{where} \quad \frac{1}{q_s} = \frac{(1-s)}{r} + \frac{s}{2}, \quad s \in [0,1].$$

We choose s in such a way that $r_s = q_s$, i.e.

$$r_s = \frac{4}{2-s} = \frac{2r}{2-2s+sr} = q_s.$$

This implies

$$s = \frac{2}{1+n}, \quad r \geq 2$$

and thus,

$$r_s = \frac{3r-4}{r-1}.$$

We conclude

$$\partial_n p_t \in L^{3-\frac{1}{r-1}}(\Sigma) \quad \text{for} \;\; d = 2.$$

Since we have

$$\partial_n p \in \mathcal{H}^{\frac{1}{2},\frac{1}{2}}(\Sigma),$$

we obtain by a similar argumentation as above that in particular

$$\partial_n p \in L^{\frac{2(d+1)}{d}}(\Sigma) \quad \text{for} \;\; d \geq 3,$$
$$\partial_n p \in L^{3-\frac{1}{r-1}}(\Sigma) \quad \text{for} \;\; d = 2. \qquad \square$$

Corollary 4.3.15. *The semi-smooth Newton method applied to* (4.2.11) *converges superlinearly.*

Remark 4.3.16. From the previous consideration we deduce that the optimal control is an element in $\mathcal{H}^{\frac{1}{2},\frac{1}{4}}(\Sigma)$. For a further discussion of regularity results for this control problem we refer to Bucci [23] and Lasiecka, Pandolfi, Triggiani [82].

4.4 Discrete problems

The optimal control problems (4.2.1), (4.2.3), (4.2.8), and (4.2.11) are discretized by finite elements similar as in Chapter 3; cf. also Kunisch & Vexler [78]. Using the definitions from Section 3.2.2 we consider uniform temporal and spatial meshes and let $V_h^{s,m} = V_h^{s,n}$ for all $0 \le m, n \le N$, that means the discrete spatial ansatz spaces are the same at each time point. We set

$$\begin{aligned} V_h &= V_h^1 && && \text{if } V = H^1(\Omega),\\ V_h^0 &= V_h^1 && && \text{if } V = H_0^1(\Omega),\\ \tilde{X}_{k,h}^{r,s,a} &= \tilde{X}_{k,h}^{r,s} && \text{and} \quad X_{k,h}^{r,s,a} = X_{k,h}^{r,s} && \text{if } V = H^1(\Omega),\\ \tilde{X}_{k,h}^{r,s,b} &= \tilde{X}_{k,h}^{r,s} && \text{and} \quad X_{k,h}^{r,s,b} = X_{k,h}^{r,s} && \text{if } V = H_0^1(\Omega). \end{aligned}$$

For the definition of the discrete control space in the case of boundary control, we introduce the space of traces of functions in V_h

$$W_h = \left\{ w_h \in H^{\frac{1}{2}}(\partial\Omega) \;\middle|\; w_h = \gamma(v_h), v_h \in V_h \right\},$$

where $\gamma\colon H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ denotes the trace operator.

Based on the equivalent formulation of the state equations as first-order systems we introduce a Galerkin finite element formulation of the state equations. We define a bilinear form $a_\rho\colon X_{k,h}^{1,1,a} \times X_{k,h}^{1,1,a} \times \widetilde{X}_{k,h}^{0,1,a} \times \widetilde{X}_{k,h}^{0,1,a} \to \mathbb{R}$ by

$$\begin{aligned} a_\rho(y,\xi) = a_\rho(y^1,y^2,\xi^1,\xi^2) = (\partial_t y^2, \xi^1)_I + (\nabla y^1, \nabla \xi^1)_I + \rho(\nabla y^2, \nabla \xi^1)_I \\ + (\partial_t y^1, \xi^2)_I - (y^2, \xi^2)_I + (y^2(0), \xi^1(0)) - (y^1(0), \xi^2(0)) \end{aligned}$$

with $y = (y^1, y^2)$ and $\xi = (\xi^1, \xi^2)$ and with a real parameter $\rho \ge 0$.

In the following we use the subscripts k and h also for the discrete control to indicate that the discrete control and discrete state are defined on the same spatial and temporal meshes.

The discrete problems for the cases of distributed, Neumann boundary and Dirichlet boundary control are formulated in the sequel.

4.4.1 Distributed control

For the distributed control problem we choose the discrete control space $U_{k,h}^D = X_{k,h}^{1,1,a}$. The discretized optimization problem is then formulated as follows:

$$\text{Minimize} \;\; J(u_{kh}, y_{kh}^1)$$

for $u_{kh} \in U^D_{k,h} \cap U_{\mathrm{ad}}$ and $y_{kh} \in X^{1,1,b}_{k,h} \times X^{1,1,a}_{k,h}$ subject to

$$a_0(y_{kh}, \xi) = (u_{kh}, \xi^1)_I + (y_1, \xi^1(0)) - (y_0, \xi^2(0)) \quad \text{for all} \;\; \xi \in \widetilde{X}^{0,1,b}_{k,h} \times \widetilde{X}^{0,1,a}_{k,h}. \tag{4.4.1}$$

Remark 4.4.1. Here, we allow for the second component to be nonzero on the boundary in contrast to Chapter 3. For smooth solutions of the continuous problem, we have $y_t|_\Sigma = 0$, but if y_t is only an element in $L^2(\Omega)$ there exist no boundary values.

4.4.2 Neumann boundary control

For the Neumann boundary control problem we choose the discrete control space as

$$U^B_{k,h} = \left\{ \, v \in C(\bar{I}, W_h) \;\middle|\; v|_{I_m} \in \mathcal{P}^1(I_m, W_h) \, \right\}.$$

The corresponding discrete optimization problem is formulated as follows:

$$\text{Minimize} \;\; J(u_{kh}, y^1_{kh})$$

for $u_{kh} \in U^B_{k,h} \cap U_{\mathrm{ad}}$ and $y_{kh} \in X^{1,1,a}_{k,h} \times X^{1,1,a}_{k,h}$ subject to

$$a_0(y_{kh}, \xi) = \langle u_{kh}, \xi^1 \rangle_I + (f, \xi^1)_I + (y_1, \xi^1(0)) - (y_0, \xi^2(0)) \text{ for all } \xi \in \widetilde{X}^{0,1,a}_{k,h} \times \widetilde{X}^{0,1,a}_{k,h}. \tag{4.4.2}$$

4.4.3 Dirichlet boundary control

For the Dirichlet boundary control problems we choose the discrete control space as in the Neumann case. For a function $u_{kh} \in U^B_{k,h}$ we define an extension $\widehat{u}_{kh} \in X^{1,1,a}_{k,h}$ such that

$$\gamma(\widehat{u}_{kh}(t, \cdot)) = u_{kh}(t, \cdot) \text{ and } \widehat{u}_{kh}(t, x_i) = 0 \tag{4.4.3}$$

on all interior nodes x_i of $\mathcal{T}_h$ and for all $t \in \bar{I}$.

The discrete optimization problem is formulated as follows:

$$\text{Minimize} \;\; J(u_{kh}, y^1_{kh})$$

for $u_{kh} \in U^{\mathrm{B}}_{k,h} \cap U_{\mathrm{ad}}$ and $y_{kh} \in (\widehat{u}_{kh} + X^{1,1,b}_{k,h}) \times X^{1,1,a}_{k,h}$ subject to

$$a_\rho(y_{kh}, \xi) = (f, \xi^1)_I + (y_1, \xi^1(0)) - (y_0, \xi^2(0)) \text{ for all } \xi \in \widetilde{X}^{0,1,b}_{k,h} \times \widetilde{X}^{0,1,a}_{k,h}. \tag{4.4.4}$$

Derivatives of the discrete reduced cost functional in case of Dirichlet control

As in Section 3.3 each of the discrete state equations (4.4.1), (4.4.2) and (4.4.4) defines a corresponding discrete solution operator S_{kh} mapping a given control u_{kh} to the first component of the state y^1_{kh} and the discrete reduced cost functional is given by

$$j_{kh}(u_{kh}) = J(u_{kh}, S_{kh}(u_{kh})). \tag{4.4.5}$$

Thus, we obtain the discrete reduced optimization problem as

$$\text{Minimize } \; j_{kh}(u_{kh}) \quad \text{for } \; u_{kh} \in U_{k,h} \cap U_{\text{ad}},$$

where the discrete control space is $U_{k,h} = U_{k,h}^D$ for distributed control and $U_{k,h} = U_{k,h}^B$ for boundary control. This optimization problem is solved using the PDAS-algorithm (semi-smooth Newton method) as described in Section 4.1 for the continuous problem.

To realize this method on the discrete level we have to specify the operator q_{kh} corresponding to the operator q in (4.1.8) on the continuous level and the solution of the equality constrained optimization problem in step (iii) of the PDAS-algorithm on the discrete level. The latter problem is solved using Newton-method utilizing the derivatives $j_{kh}'(u_{kh})(\delta u_{kh})$ and $j_{kh}''(u_{kh})(\delta u_{kh}, \tau u_{kh})$ in directions $\delta u_{kh}, \tau u_{kh} \in U_{k,h}$ according to Algorithm 3.1.

Remark 4.4.2. *For quadratic functionals $\mathcal{G}(\cdot)$ the Newton method for the equality constrained optimization problem in step (iii) of the PDAS-algorithm converges in one iteration.*

For distributed and Neumann control the required derivatives of j_{kh} can be represented as on the continuous level using adjoint and linearized (tangent) discrete equations, cf. Becker, Meidner & Vexler [12] and Meidner [97]. Since the case of Dirichlet boundary conditions is more involved, we discuss it in the sequel. In all three cases the operator q_{kh} is defined in such a way that the derivative of the discrete reduced cost functional can be expressed by

$$j_{kh}'(u_{kh})(\delta u_{kh}) = (\alpha u_{kh} - q_{kh}(u_{kh}), \delta u_{kh})_\omega.$$

In the case of Dirichlet control of the wave equation the derivative $j'(u)(\delta u)$ on the continuous level is given as

$$j'(u)(\delta u) = (\alpha u + \partial_n p, \delta u)_\Sigma,$$

where p is the solution of the adjoint equation for given control u, cf. the optimality system (4.2.9). A direct discretization of the term $\partial_n p$ does not lead in general to the derivative of the discrete cost functional j_{kh}. Therefore, we establish another representation using a residual of the adjoint equation, cf. Vexler [131] and Kunisch & Vexler [78].

Proposition 4.4.3. *Let the discrete reduced cost functional j_{kh} be defined as in (4.4.5) with the solution operator $S_{kh}\colon U_{k,h}^B \to (\widehat{u}_{kh} + X_{k,h}^{1,1,b})$ for the discrete state equation (4.4.4) in the Dirichlet case. Then the following representations hold:*

1. *The first directional derivative in direction $\delta u_{kh} \in U_{k,h}^B$ can be expressed as*

$$\begin{aligned} j_{kh}'(u_{kh})(\delta u_{kh}) = (\mathcal{G}'(y_{kh}^1), \widehat{\delta u}_{kh})_I + (\partial_t \widehat{\delta u}_{kh}, p_{kh}^1)_I + (\nabla \widehat{\delta u}_{kh}, \nabla p_{kh}^1)_I \\ + \alpha \langle u_{kh}, \delta u_{kh} \rangle_I, \end{aligned} \tag{4.4.6}$$

where $y_{kh}^1 = S_{kh}(u_{kh})$, $\widehat{\delta u}_{kh}$ is the extension of δu_{kh} defined as in (4.4.3), and $p_{kh} = (p_{kh}^1, p_{kh}^2) \in \widetilde{X}_{k,h}^{0,1,b} \times \widetilde{X}_{k,h}^{0,1,a}$ is the solution to the discrete adjoint equation

$$a_0(\eta, p_{kh}) = -J_y'(u_{kh}, y_{kh}^1)(\eta^1) \quad \text{for all } \eta \in X_{k,h}^{1,1,b} \times X_{k,h}^{1,1,a}. \tag{4.4.7}$$

2. *The second derivative of j_{kh} in directions $\delta u_{kh}, \tau u_{kh} \in U^B_{k,h}$ can be expressed as*

$$j''_{kh}(u_{kh})(\delta u_{kh}, \tau u_{kh}) = \mathcal{G}''(y^1_{kh})(\delta y^1_{kh}, \widehat{\tau u}_{kh}) + (\partial_t \widehat{\tau u}_{kh}, \delta p^1_{kh})_I + (\nabla \widehat{\tau u}_{kh}, \nabla \delta p^1_{kh})_I \\ + \alpha \langle \delta u_{kh}, \tau u_{kh} \rangle_I,$$

where $\delta y_{kh} = (\delta y^1_{kh}, \delta y^2_{kh}) \in (\widehat{\delta u_{kh}} + X^{1,1,b}_{k.h}) \times X^{1,1,a}_{k,h}$ is the solution of the discrete tangent equation

$$a_0(\delta y_{kh}, \xi) = 0 \quad \text{for all } \; \xi \in \widetilde{X}^{0,1,b}_{k,h} \times \widetilde{X}^{0,1,a}_{k,h} \tag{4.4.8}$$

and $\delta p_{kh} \in \widetilde{X}^{0,1,b}_{k,h} \times \widetilde{X}^{0,1,a}_{k,h}$ is given by

$$a_0(\eta, \delta p_{kh}) = -J''_{y^1 y^1}(u_{kh}, y^1_{kh})(\delta y^1_{kh}, \eta^1) \quad \text{for all } \; \eta \in X^{1,1,b}_{kh} \times X^{1,1,a}_{kh}. \tag{4.4.9}$$

Proof. Using the solution δy_{kh} of the discretized tangent equation (4.4.8), we obtain

$$j'_{kh}(u_{kh})(\delta u_{kh}) = J'_{y^1}(u_{kh}, y^1_{kh})(\delta y^1_{kh}) + J'_u(u_{kh}, y^1_{kh})(\delta u_{kh}),$$

rewriting the first term using (4.4.7) and (4.4.8) we get:

$$\begin{aligned} J'_{y^1}(u_{kh}, y^1_{kh})(\delta y^1_{kh}) &= J'_{y^1}(u_{kh}, y^1_{kh})(\delta y^1_{kh} - \widehat{\delta u}_{kh}) + J'_{y^1}(u_{kh}, y^1_{kh})(\widehat{\delta u}_{kh}) \\ &= -(\partial_t(\delta y^1_{kh} - \widehat{\delta u}_{kh}), p^1_{kh})_I - (\nabla(\delta y^1_{kh} - \widehat{\delta u}_{kh}), \nabla p^1_{kh})_I \\ &\quad + (\mathcal{G}'(y^1_{kh}), \widehat{\delta u}_{kh})_I \\ &= (\partial_t \widehat{\delta u}_{kh}, p^1_{kh})_I + (\nabla \widehat{\delta u}_{kh}, \nabla p^1_{kh})_I + (\mathcal{G}'(y^1_{kh}), \widehat{\delta u}_{kh})_I. \end{aligned}$$

This gives the desired representation (4.4.6). The representation of the second derivative is obtained in a similar way. □

Remark 4.4.4. *For the state equation* (4.4.4) *and the tangent equation* (4.4.8) *the discrete solutions are continuous piecewise linear in time functions, i.e. the ansatz space is $X^{1,1,b}_{k,h} \times X^{1,1,a}_{k,h}$ and the test space consists of discontinuous piecewise constant (in time) functions, i.e. the test space is $\widetilde{X}^{0,1,b}_{k,h} \times \widetilde{X}^{0,1,a}_{k,h}$. For the adjoint equations* (4.4.7) *and* (4.4.9) *the ansatz and the test spaces are exchanged. The ansatz functions are discontinuous and piecewise constant (in time) and test functions are continuous piecewise linear in time. This allows for a consistent formulation, cf. Becker, Meidner & Vexler [12] and Meidner [97].*

4.4.4 Time stepping formulations for Dirichlet control

The discrete state equation (4.4.4) as well as the discrete tangent (4.4.8) and adjoint (4.4.7), (4.4.9) equations are formulated globally in time, nevertheless they result in time stepping schemes; cf. Remark 3.2.2. This is due to the fact that for all these equations either the ansatz or the test functions are discontinuous in time. Applying the trapezoidal rule piecewise for approximation of time integrals, the considered time discretization results in a Crank-Nicolson scheme. In the following we present the time stepping schemes for equations (4.4.4),(4.4.7), (4.4.8), and (4.4.9) explicitly, cf. Kunisch & Vexler [78] for Dirichlet boundary control of the heat equation discretized by a discontinuous Galerkin variant of the implicit Euler scheme. Thereby, we assume that the functional $\mathcal{G}$ can be represented as

$$\mathcal{G}(y) = \int_0^T g(y(t))\,dt$$

with a functional $g \in C^2(L^2(\Omega), \mathbb{R})$.

We define for $m = 0, \ldots, M$

$$U_m = u_{kh}(t_m),\ Y_m^1 = y_{kh}^1(t_m),\ Y_m^2 = y_{kh}^2(t_m),$$

for $m = 1, \ldots, M$

$$P_m^1 = p_{kh}^1|_{I_m},\ P_m^2 = p_{kh}^2|_{I_m},$$

and

$$P_0^1 = p_{kh}^1(0),\ P_0^2 = p_{kh}^2(0).$$

The discrete state equation for $Y_0^1,\ Y_0^2 \in V_h$ and $Y_m^1 \in \widehat{U}_m + V_h^0,\ Y_m^2 \in V_h$ for $m = 1, \ldots, M$ is given as follows:

$\boldsymbol{m = 0}$**:**

$$(Y_0^2, \varphi^1) + (Y_0^1, \varphi^2) = (y_1, \varphi^1) + (y_0, \varphi^2) \quad \text{for all } \varphi^1,\ \varphi^2 \in V_h,$$

$\boldsymbol{m = 1, \ldots, M}$**:**

$$\begin{aligned}(Y_m^2, \varphi^1) + (Y_m^1, \varphi^2) &+ \frac{k_m}{2}(\nabla Y_m^1, \nabla\varphi^1) + \rho\frac{k_m}{2}(\nabla Y_m^2, \nabla\varphi^1) - \frac{k_m}{2}(Y_m^2, \varphi^2)\\ &= (Y_{m-1}^2, \varphi^1) + (Y_{m-1}^1, \varphi^2) - \frac{k_m}{2}(\nabla Y_{m-1}^1, \nabla\varphi^1) - \rho\frac{k_m}{2}(\nabla Y_{m-1}^2, \nabla\varphi^1)\\ &\quad + \frac{k_m}{2}(Y_{m-1}^2, \varphi^2) + \frac{k_m}{2}(f(t_{m-1}), \varphi^1) + \frac{k_m}{2}(f(t_m), \varphi^1)\\ &\qquad\qquad \text{for all } \varphi_1 \in V_h^0,\ \varphi_2 \in V_h.\end{aligned}$$

The discrete adjoint equation for $P_0^1, P_0^2 \in V_h$ and $P_m^1 \in V_h^0,\ P_m^2 \in V_h$ for $m = 1, \ldots, M$ is given as follows:

$\boldsymbol{m = M}$**:**

$$\begin{aligned}(\eta^2, P_M^1) + (\eta^1, P_M^2) + \frac{k_M}{2}(\nabla\eta^1, \nabla P_M^1) - \rho\frac{k_M}{2}(\nabla\eta^2, \nabla P_M^1) - \frac{k_M}{2}(\eta^2, P_M^2)&\\ = -\frac{k_M}{2}g'(Y_M^1)(\eta^1) \quad \text{for all } \eta^1 \in V_h^0,\ \eta^2 \in V_h,&\end{aligned}$$

$m = M - 1, \dots, 1$:

$$\begin{aligned}(\eta^2, P_m^1) + (\eta^1, P_m^2) + \frac{k_m}{2}(\nabla\eta^1, \nabla P_m^1) - \rho\frac{k_m}{2}(\nabla\eta^2, \nabla P_m^1) - \frac{k_m}{2}(\eta^2, P_m^2)\\ = (\eta^2, P_{m+1}^1) + (\eta^1, P_{m+1}^2) - \frac{k_{m+1}}{2}(\nabla\eta^1, \nabla P_{m+1}^1) + \rho\frac{k_{m+1}}{2}(\nabla\eta^2, \nabla P_{m+1}^1)\\ + \frac{k_{m+1}}{2}(\eta^2, P_{m+1}^2) - \frac{k_m + k_{m+1}}{2}g'(Y_m^1)(\eta^1) \quad \text{for all } \eta^1 \in V_h^0,\ \eta^2 \in V_h,\end{aligned}$$

$m = 0$:

$$\begin{aligned}(\eta^2, P_0^1) + (\eta^1, P_0^2) = (\eta^2, P_1^1) + (\eta^1, P_1^2) - \frac{k_1}{2}(\nabla\eta^1, \nabla P_1^1) + \rho\frac{k_1}{2}(\nabla\eta^2, \nabla P_1^1)\\ + \frac{k_1}{2}(\eta^2, P_1^2) - \frac{k_1}{2}g'(Y_0^1)(\eta^1) \text{ for all } \eta^1, \eta^2 \in V_h.\end{aligned}$$

Next we describe the equations (4.4.8) and (4.4.9). Therefore, we define for $i = 0, \dots, M$:

$$\delta U_m = \delta u_\sigma(t_m),\ \delta Y_m^1 = \delta y_{kh}^1(t_m),\ \delta Y_m^2 = \delta y_{kh}^2(t_m),$$

for $i = 1, \dots, m$

$$\delta P_m^1 = \delta p_{kh}^1|_{I_m},\ \delta P_m^2 = \delta p_{kh}^2|_{I_m}$$

and

$$\delta P_0^1 = \delta p_{kh}^1(0),\ \delta P_0^2 = \delta p_{kh}^2(0).$$

The discrete tangent equation for $\delta Y_0^1, \delta Y_0^2 \in V_h$ and $\delta Y_m^1 \in \widehat{\delta U}_m + V_h^0,\ \delta Y_m^2 \in V_h$ for $m = 1, \dots, M$ is given as follows:

$m = 0$:

$$\delta Y_0^1 = \delta Y_0^2 = 0,$$

$m = 1, \dots, M$:

$$\begin{aligned}(\delta Y_m^2, \varphi^1) + (\delta Y_m^1, \varphi^2) + \frac{k_m}{2}(\nabla\delta Y_m^1, \nabla\varphi^1) + \rho\frac{k_m}{2}(\nabla\delta Y_m^2, \nabla\varphi^1) - \frac{k_m}{2}(\delta Y_m^2, \varphi^2)\\ = (\delta Y_{m-1}^2, \varphi^1) + (\delta Y_{m-1}^1, \varphi^2) - \frac{k_m}{2}(\nabla\delta Y_{m-1}^1, \nabla\varphi^1) - \rho\frac{k_m}{2}(\nabla\delta Y_m^2, \nabla\varphi^1)\\ + \frac{k_m}{2}(\delta Y_{m-1}^2, \varphi^2) \text{ for all } \varphi^1 \in V_h^0,\ \varphi^2 \in V_h.\end{aligned}$$

The additional adjoint equation for $\delta P_0^1, \delta P_0^2 \in V_h$ and $\delta P_m^1 \in V_h^0,\ \delta P_m^2 \in V_h$ for $m = 1, \dots, M$ is given as follows:

$\boldsymbol{m = M}$:

$$(\eta^2, \delta P_M^1) + (\eta^1, \delta P_M^2) + \frac{k_M}{2}(\nabla\eta^1, \nabla\delta P_M^1) - \rho\frac{k_M}{2}(\nabla\eta^2, \nabla\delta P_M^1) \\ - \frac{k_M}{2}(\eta^2, \delta P_M^2) = -\frac{k_M}{2} g''(Y_M)(\delta Y_M, \eta^1) \\ \text{for all } \eta^1 \in V_h^0, \eta^2 \in V_h.$$

$\boldsymbol{m = M-1, \dots, 1}$:

$$(\eta^2, \delta P_m^1) + (\eta^1, \delta P_m^2) + \frac{k_m}{2}(\nabla\eta^1, \nabla\delta P_m^1) - \rho\frac{k_m}{2}(\nabla\eta^2, \nabla\delta P_M^1) - \frac{k_m}{2}(\eta^2, \delta P_m^2) \\ = (\eta^2, \delta P_{m+1}^1) + (\eta^1, \delta P_{m+1}^2) - \frac{k_{m+1}}{2}(\nabla\eta^1, \nabla\delta P_{m+1}^1) + \rho\frac{k_{m+1}}{2}(\nabla\eta^2, \nabla\delta P_M^1) \\ + \frac{k_{m+1}}{2}(\eta^2, \delta P_{m+1}^2) - \frac{k_m + k_{m+1}}{2} g''(Y_m)(\delta Y_m, \eta^1) \\ \text{for all } \eta^1 \in V_h^0,\ \eta^2 \in V_h,$$

$\boldsymbol{m = 0}$:

$$(\eta^2, \delta P_0^1) + (\eta^1, \delta P_0^2) = (\eta^2, \delta P_1^1) + (\eta^1, \delta P_1^2) - \frac{k_1}{2}(\nabla\eta^1, \nabla\delta P_1^1) \\ + \frac{k_1}{2}(\nabla\eta^2, \nabla\delta P_1^1) + \frac{k_1}{2}(\eta^2, \delta P_1^2) - \frac{k_1}{2} g''(Y_0)(\delta Y_0, \eta^1) \\ \text{for all } \eta^1 \in V_h^0,\ \eta^2 \in V_h.$$

4.5 Numerical examples

In this section we discuss numerical examples illustrating our theoretical results for the optimal control problems (4.2.1), (4.2.3), (4.2.8) and (4.2.11). We present a comparison of the numbers of PDAS iterations for different discretization levels as well as some results illustrating the error behavior on a fixed mesh. On the discrete level (for fixed temporal and spatial meshes) the PDAS-method typically converges in a finite number of steps (cf. the stopping criterion in Remark 4.1.17), which is better than superlinear convergence. The examples indicate superlinear convergence also before the PDAS method stops finding the optimal discrete solution.

All computations are done using the optimization library RoDoBo [120] and the finite element toolkit Gascoigne [45].

In the following we consider distributed, Neumann boundary and Dirichlet boundary control with and without damping on the unit square $\Omega = (0,1)^2 \subset \mathbb{R}^2$. Here, we specify the functional $\mathcal{G}$ in the following way: For a given function $y_d \in L^2(Q)$ we define

$$\mathcal{G}(y) = \frac{1}{2}\|y - y_{\mathrm{d}}\|_{L^2(Q)}^2 .$$

4.5.1 Distributed control

We compute the distributed optimal control problem (4.2.1) with the following data:

$$\alpha = 0.01, \quad u_a = -0.6, \quad u_b = 2, \quad T = 1,$$
$$y_\mathrm{d}(t,x) = \begin{cases} 10x_2, & \text{if } x_1 < 0.5, \\ 1, & \text{else,} \end{cases} \qquad y_0(x) = \sin(\pi x_1)\sin(\pi x_2), \quad y_1(x) = 0$$

for $t \in [0,T]$ and $x = (x_1, x_2) \in \Omega$.

Level	N	M	PDAS steps
1	16	2	5
2	64	4	4
3	256	8	5
4	1024	16	4
5	4096	32	4
6	16384	64	5

Table 4.1: Numbers of PDAS iterations on a sequence of uniformly refined meshes for distributed control

This optimal control problem is discretized by space-time finite elements as described above. The resulting finite-dimensional problem is solved by the PDAS method. In Table 5.2 the numbers of iterations is shown for a sequence of uniformly refined discretizations. Here, N denotes the number of cells in the spatial mesh $\mathcal{T}_h$ and M denotes the number of time intervals. The results indicate a mesh-independent behavior of the PDAS-algorithm.

To analyze the convergence behavior of the PDAS method we define the PDAS iteration error

$$e_i = \|u_{kh}^{(i)} - u_{kh}\|_{L^2(\omega)},$$

where $u_{kh}^{(i)}$ denotes the ith iterate and u_{kh} the optimal discrete solution. For a fixed discretization with $N = 16384$ cells and $M = 64$ time steps Table 5.3 depicts the rate of convergence of the PDAS-iteration. The results presented demonstrate superlinear convergence.

i	1	2	3	4
e_i	$3.6 \cdot 10^{-2}$	$9.7 \cdot 10^{-4}$	$2.1 \cdot 10^{-5}$	0
e_{i+1}/e_i	$2.7 \cdot 10^{-2}$	$2.2 \cdot 10^{-2}$	0	-

Table 4.2: Superlinear convergence of the PDAS-method for distributed control - PDAS-iteration error

4.5.2 Neumann boundary control

We consider the Neumann boundary control problem (4.2.3) with the following data:

$$f(t,x) = \begin{cases} 1, & \text{if } x_1 > 0.25, \\ -1, & \text{else} \end{cases}, \quad \alpha = 0.01, \quad u_a = -0.8, \quad u_b = 1, \quad T = 1,$$

$$y_{\mathrm{d}}(t,x) = \begin{cases} -x_1, & \text{if } x_1 > 0.05, \\ 2, & \text{else,} \end{cases} \quad y_0(x) = \sin(\pi x_1)\sin(\pi x_2), \quad y_1(x) = 0$$

for $t \in [0,T]$ and $x = (x_1, x_2) \in \Omega$.

As in the previous example we see in Table 4.3 that the number of PDAS iterations is mesh-independent under uniformly refinement of the discretizations.

For a fixed discretization with $N = 16384$ cells and $M = 64$ time steps Table 4.4 shows the rate of convergence of the PDAS-iteration illustrating superlinear convergence.

Level	N	M	PDAS steps
1	16	2	5
2	64	4	5
3	256	8	3
4	1024	16	4
5	4096	32	4
6	16384	64	5

Table 4.3: Numbers of PDAS iterations on a sequence of uniformly refined meshes for Neumann boundary control

i	1	2	3	4
e_i	$3.0 \cdot 10^{-2}$	$9.7 \cdot 10^{-4}$	$2.8 \cdot 10^{-5}$	0
e_{i+1}/e_i	$3.2 \cdot 10^{-2}$	$2.9 \cdot 10^{-2}$	0	-

Table 4.4: Superlinear convergence of the PDAS-method for Neumann boundary control - PDAS-iteration error

4.5.3 Dirichlet boundary control

This is a Dirichlet optimal control problems (4.2.8) and (4.2.11) with the following data:

$$f(t,x) = \begin{cases} 1, & x_1 > 0.5, \\ x_1, & \text{else} \end{cases}, \quad u_a = -0.18, \quad u_b = 0.2, \quad T = 1,$$

$$y_{\mathrm{d}}(t,x) = \begin{cases} x_1 & x_1 > 0.5 \\ -x_1 & \text{else} \end{cases}, \quad y_0(x) = \sin(\pi x_1)\sin(\pi x_2), \quad y_1(x) = 0$$

for $t \in [0,T]$ and $x = (x_1, x_2) \in \Omega$.

			$\alpha = 10^{-4}$			$\alpha = 10^{-2}$		
Level	N	M	$\rho = 0$	$\rho = 0.1$	$\rho = 0.7$	$\rho = 0$	$\rho = 0.1$	$\rho = 0.7$
1	16	2	4	3	5	4	4	5
2	64	4	5	4	3	4	4	3
3	256	8	5	5	4	5	4	4
4	1024	16	6	6	6	5	7	5
5	4096	32	11	7	7	9	6	5
6	16384	64	13	9	7	10	8	5

			$\alpha = 1$		
Level	N	M	$\rho = 0$	$\rho = 0.1$	$\rho = 0.7$
1	16	2	3	3	2
2	64	4	3	3	1
3	256	8	4	3	1
4	1024	16	4	2	1
5	4096	32	3	3	1
6	16384	64	3	4	1

Table 4.5: Numbers of PDAS-iterations on a sequence of uniformly refined meshes for different parameters α and ρ for optimal Dirichlet boundary control

Table 4.5 illustrates the effect of damping introduced by the term $-\rho\Delta y_t$ on the number of PDAS steps. For $\alpha = 0.01$ and $\rho = 0$ we observe a mesh-dependence of the algorithm. Moreover, the number of PDAS steps declines for increasing value of ρ and stays mesh independent for $\rho > 0$. Furthermore, we consider the effect of α on the number of PDAS steps. As expected the number of iterations declines also for increasing α.

In Table 4.6 and in Table 4.7 we consider the PDAS-iteration error for the discretization with $N = 16384$ cells and $M = 64$ time steps, where we choose $\rho = 0$ and $\rho = 0.1$, respectively, and $\alpha = 0.01$. These tables indicate that we only have superlinear convergence for $\rho > 0$.

i	1	2	3	4	5	6	7
e_i	$2.3 \cdot 10^{-2}$	$2.2 \cdot 10^{-2}$	$4.5 \cdot 10^{-3}$	$1.9 \cdot 10^{-3}$	$7.2 \cdot 10^{-4}$	$3.8 \cdot 10^{-4}$	$1.2 \cdot 10^{-4}$
e_{i+1}/e_i	$9.5 \cdot 10^{-1}$	$2.0 \cdot 10^{-1}$	$4.2 \cdot 10^{-1}$	$3.8 \cdot 10^{-1}$	$5.2 \cdot 10^{-1}$	$3.1 \cdot 10^{-1}$	$4.1 \cdot 10^{-1}$

i	7	8	9
e_i	$4.8 \cdot 10^{-5}$	$1.4 \cdot 10^{-5}$	0
e_{i+1}/e_i	$3.0 \cdot 10^{-1}$	0	-

Table 4.6: Equation without damping, $\rho = 0$ - PDAS-iteration error

i	1	2	3	4	5	6	7
e_i	$3.8 \cdot 10^{-1}$	$5.2 \cdot 10^{-2}$	$1.0 \cdot 10^{-2}$	$1.5 \cdot 10^{-3}$	$1.8 \cdot 10^{-4}$	$1.7 \cdot 10^{-5}$	0
e_{i+1}/e_i	$1.3 \cdot 10^{-1}$	$1.9 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$	$1.2 \cdot 10^{-1}$	$9.3 \cdot 10^{-2}$	0	-

Table 4.7: Equation with damping $\rho = 0.1$ - PDAS-iteration error

4.6 Outlook

There are several question which are worth to analyze in future research.

- The convergence of the semi-smooth Newton method can be analyzed for nonlinear wave equations.
- In this chapter we proved local superlinear convergence. Thus, the next step is to formulate conditions under which we obtain global convergence; cf. Ulbrich [129].
- In this thesis it remains open to prove an improved regularity result of the optimal state for Dirichlet control of the strongly damped wave equation in case of control constraints.
- Finally, it is interesting to analyze the behaviour of the solution of the optimal control problem of the strongly damped wave equation and its discrete analogon for $\rho \to 0$.

5 Application to the dynamical Lamé system

In this chapter we apply the techniques developed in the Chapters 3 and 4, i.e. adaptive finite element and semi-smooth Newton methods, to optimal control problems governed by the dynamical Lamé system. The dynamical Lamé system describes the phenomena when waves propagate in solid materials. For an introduction we refer the reader to Hughes [59] and for the static case to Braess [21]. We present a numerical example using adaptive finite elements and analyze the convergence of the semi-smooth Newton method subject to different types of control action.

There exist some related publications to these topics. An adaptive finite element method for an inverse problem governed by the elastic wave equation is considered in Beilina [14] with Dirac measures in the quantity of interest. In Belishev & Lasiecka [15] regularity results for controllability of the Lamé system are derived.

Further, in Nestler [112], the optimal design of a cylinder basin is considered, which can be seen as an application of optimal control in linear elasticity. Acoustic problems as noise suppression is analyzed in Banks, Keeling & Silcox [10]. Although in the latter publication the classical wave equation is considered, this is an interesting problem also with respect to optimal control of the dynamical Lamé system.

The chapter is organized as follows. In Section 5.1 we introduce the mathematical setting of the dynamical Lamé system and recall the physical background, in Section 5.2 we apply the adaptive finite element method described in Chapter 3 to an optimal control problem of the dynamical Lamé system and in Section 5.3 we apply the semi-smooth Newton method from Chapter 4 on optimal control problems of the dynamical Lamé system and evaluate its convergence.

5.1 The dynamical Lamé system

The dynamical Lamé system describes the propagation of elastic waves in an elastic medium. The elasticity of the material provides the restoring force of the wave. Most solid materials are elastic, so this equation can be seen as a model for such phenomena as seismic waves in the earth and acoustic waves in solid materials. The system is given by

$$\begin{cases} y_{tt} - \operatorname{div} \sigma(y) = f & \text{in } Q, \\ y(0) = y_0 & \text{in } \Omega, \\ y_t(0) = y_1 & \text{in } \Omega, \end{cases} \tag{5.1.1}$$

and homogeneous boundary condition

$$\begin{cases} \sigma(y) \cdot n = 0 & \text{on } \Sigma \quad \text{or} \\ y = 0 & \text{on } \Sigma, \end{cases} \tag{5.1.2}$$

with the function of displacement $y\colon (0,T)\times\Omega\to\mathbb{R}^d$, stress tensor

$$\sigma_{ij} = \lambda\delta_{ij}\,\mathrm{tr}(\varepsilon) + 2\mu\varepsilon_{ij} \tag{5.1.3}$$

($\mathrm{tr}\colon \mathbb{R}^{d\times d}\to\mathbb{R}$ denotes the usual trace operator, cf. Chapter 7) with Lamé parameters $\lambda,\mu>0$, strain tensor

$$\varepsilon_{ij}(v) = \frac{1}{2}\left(\partial_j v_i + \partial_i v_j\right),$$

for $i,j\in\{1,2,\ldots,d\}$, $v\in H^1(\Omega)^d$, a given force $f\in L^2(L^2(\Omega)^d)$, and outer normal n for $d=2,3$. Material with these properties is called *St. Venant-Kirchhoff-material.* In the following we write

$$\mathcal{D}v = \varepsilon(v).$$

The relation between the stain and stress tensor can be derived by the general Navier-Lamé system after some linearizations and assuming that the material is homogeneous and isotropic; cf. Braess [21], Hughes [59], Beilina [14].

Remark 5.1.1. We present a short physical motivation for this system, cf. Evans [40, pp. 66]. Let V represent any smooth subregion of $\mathbb{R}^d$, $d=1,2,3$. The acceleration within V is then

$$\frac{d^2}{dt^2}\int_V y\,dx = \int_V y_{tt}\,dx$$

and

$$-\int_{\partial V}\sigma\cdot n\,dS, \tag{5.1.4}$$

where (5.1.4) describes the force acting on V through ∂V and the mass density is taken to be unit. Newton's law implies the mass times the acceleration equals the force

$$\int_V y_{tt}\,dx = -\int_{\partial V}\sigma\cdot n\,dS.$$

This identity obtains for each subregion V and so

$$y_{tt} = \mathrm{div}\sigma.$$

Remark 5.1.2. In many cases the Poisson's ratio ν and Young's modulus E are given instead of the Lamé coefficients λ and μ. There holds the following relation between these quantities, cf. Braess [21],

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},\quad \mu = \frac{E}{2(1+\nu)}.$$

System (5.1.1) can be equivalently written as

$$\begin{cases} y_{tt} - \lambda\nabla\,\mathrm{div}\,y - 2\mu\,\mathrm{div}\,\mathcal{D}y = f & \text{in } Q, \quad (5.1.5a)\\ y(0) = y_0 & \text{in } \Omega, \quad (5.1.5b)\\ y_t(0) = y_1 & \text{in } \Omega. \quad (5.1.5c)\end{cases}$$

Remark 5.1.3. By a direct calculation the strain tensor can be eliminated for sufficiently smooth functions, i.e. (5.1.5a) can be written as

$$y_{tt} - (\lambda + \mu)\nabla \operatorname{div} y - \mu \Delta y = f. \tag{5.1.6}$$

This equation is often called the elastic wave equation. However, we have to distinguish carefully the variational formulations associated to (5.1.5a) and (5.1.6), in case of inhomogeneous Neumann boundary conditions, which we obtain formally by testing with some test function and integration by parts, cf. also (5.1.9).

5.1.1 Physical background

The elastic wave equation are used to model several physical phenomena. In the following we will present two of them. In the first we consider the equation as a model for seismic waves and in the second as a model for noise emission problems.

The elastic wave equation as a model for seismic waves

The elastic wave equation (5.1.6) can be interpreted as a model equation for seismic waves. Seismic waves are caused by earthquakes. For numerical methods to solve these equations and inverse problems related to seismic waves we refer to the publications Komatitsch, Liu & Tromp [69] and Komatitsch & Tromp [70]. Seismic waves can be decomposed into p-waves (primary or pressure waves) and s-waves (second or shear wave). P-waves are longitudinal waves, i.e., the oscillations occur in the same direction (and opposite) direction of wave propagation. S-waves are perpendicular to the direction of propagation. P-waves travel faster in rock as s-waves, thus the s-wave is the second wave arriving at a point arising from a earthquake, after the p-wave.

In the following physical interpretation of the elastic wave equation we assume that all functions are sufficiently smooth. We can reformulate equation (5.1.6) as

$$y_{tt} - (\lambda + 2\mu)\nabla \operatorname{div} y - \mu \nabla \times (\nabla \times y) = f \tag{5.1.7}$$

for $d = 3$. Let $\theta = \operatorname{div} y$ and apply the divergence operator on (5.1.7). Then using the identity

$$\operatorname{div}(\nabla \times \Psi) = 0$$

for functions $\Psi \colon \Omega \to \mathbb{R}^3$, we obtain the acoustic wave equation describing p-waves

$$\theta_{tt} - \frac{1}{\alpha^2}\Delta\theta = f$$

traveling in the direction of propagation with velocity

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}.$$

Let $\varphi = \nabla \times y$ and apply the curl-operator on (5.1.7). We obtain

$$\varphi_{tt} - \frac{1}{\beta^2}\Delta\varphi = f$$

describing the transversal movements with velocity β given by

$$\beta = \sqrt{\frac{\mu}{\rho}}.$$

S-waves are divergence-free, since $\operatorname{div}\operatorname{curl} y = 0$. For details we refer the reader to Pujol [115].

The elastic wave equation as a model for acoustic noise emission problems

Acoustic noise emission is defined as emission of elastic waves by structural change of a material being under pressure. These waves propagate as radial symmetric space waves through the media and can be registered by sensors. By these signals one can draw conclusions of the reason for this deformation and the state of the material. The permanent deformations of the material is a requirement for the sound emission technique. One can create this by different methods, as for e.g. by stressing the material mechanically or thermically, putting it under pressure by gas or water or by exposing it to an acoustic field. Here, we are interested in the last case. To create a certain acoustic field in the interior of the material you have to know, how to control the transmitters on the boundary. In Schechinger [122] the technical issues are considered.

5.1.2 Existence and uniqueness

In this section we recall some basic results on existence and regularity of the solution of the dynamical Lamé system (5.1.1), (5.1.2) with σ given by (5.1.3). Let $V = H_0^1(\Omega)^d$ (homogeneous Dirichlet condition) or $V = H^1(\Omega)^d$ (homogeneous Neumann condition). In the usual way we extend the definitions of $(\cdot,\cdot)$ and $\|\cdot\|$ to functions in $L^2(\Omega)^d$, of $\langle\cdot,\cdot\rangle$ to functions in $L^2(\partial\Omega)^d$, of $(\cdot,\cdot)_I$ to functions in $L^2(L^2(\Omega)^d)$, and of $\langle\cdot,\cdot\rangle_I$ to functions in $L^2(L^2(\partial\Omega)^d)$.

To obtain a variational formulation we assume the solution is smooth, test the equation with $v \in V$, integrate in space obtaining

$$(y_{tt}, v) + (\operatorname{div}\sigma(y), v) = (f, v) \quad \forall v \in V$$

and apply integration by parts. Thereby, we use the following identities for $v, w \in V$ and the unit matrix $\mathbb{1}_d$ in $\mathbb{R}^{d\times d}$

$$\begin{aligned}
\mathbb{1}_d : \mathcal{D}v &= \operatorname{tr}\mathcal{D}v,\\
\sigma : \nabla v &= \sigma : \mathcal{D}v,\\
\sigma : \mathcal{D}v &= (\lambda \operatorname{tr}(\mathcal{D}v)I + 2\mu\mathcal{D}v) : \mathcal{D}v = \lambda(\operatorname{tr}(\mathcal{D}v))^2 + 2\mu\mathcal{D}v : \mathcal{D}v
\end{aligned}$$

with the product

$$A : B = \operatorname{tr}(A^T B)$$

for matrices $A, B \in \mathbb{R}^{\nu\times\nu}$, $\nu \in \mathbb{N}$; cf. Braess [21, pp. 277] and Green's formula

$$(\operatorname{div}\sigma(w), v) - (w, \operatorname{div}\sigma(v)) = \langle \sigma(w)\cdot n, v|_{\partial\Omega}\rangle - \langle w|_{\partial\Omega}, \sigma(v)\cdot n\rangle. \qquad (5.1.8)$$

This leads to the following variational formulation: We look for a solution $y \in X$ of

$$\begin{cases} (y_{tt}, \xi) + \lambda(\operatorname{div} y, \operatorname{div} \xi) + 2\mu(\mathcal{D}y : \mathcal{D}\xi) = (f, \xi) & \forall \xi \in V, \\ y(0) = y_0 & \text{in } \Omega, \\ y_t(0) = y_1 & \text{in } \Omega \end{cases} \tag{5.1.9}$$

for given initial data $y_0 \in V$, $y_1 \in H = L^2(\Omega)^d$, force $f \in L^2(L^2(\Omega)^d)$ and the corresponding space X defined by (2.3.9). We introduce the form

$$\bar{a}\colon V \times V \to \mathbb{R}, \quad \bar{a}(v, w) = \lambda(\operatorname{div} v, \operatorname{div} w) + 2\mu(\mathcal{D}v : \mathcal{D}w) \tag{5.1.10}$$

to simplify notations. To apply Theorem 2.3.1 on equation (5.1.9) we have to verify the coercivity and boundedness of the form $\bar{a}$ in (5.1.10). Therefore we recall Korn's first and second inequality. For the proofs of these inequalities we refer to Braess [21] and the references therein.

Proposition 5.1.4 (First Korn's inequality)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open, bounded set with piecewise smooth boundary. Then there exists a constant $c > 0$, such that*

$$\int_\Omega \mathcal{D}y : \mathcal{D}y dx + \|y\|^2_{L^2(\Omega)^d} \geq C \|y\|^2_{H^1(\Omega)^d} \quad \forall y \in H^1(\Omega)^d. \tag{5.1.11}$$

If homogeneous Dirichlet boundary conditions are prescribed we obtain an improved estimate.

Proposition 5.1.5 (Second Korn's inequality)**.** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open, bounded set with piecewise smooth boundary. Then there exists a constant $c > 0$, such that*

$$\int_\Omega \mathcal{D}y : \mathcal{D}y dx \geq C \|y\|^2_{H^1_0(\Omega)^d} \quad \forall y \in H^1_0(\Omega)^d. \tag{5.1.12}$$

Now, we can prove the coercivity and boundedness of the form $\bar{a}$.

Lemma 5.1.6. *There holds for all $v, y \in V$ with $V = H^1(\Omega)^d$ or $V = H^1_0(\Omega)^d$ the following inequalities:*

- *The strain tensor is bounded, i.e.*

$$\|\mathcal{D}y\|_{L^2(\Omega)^{d\times d}} \leq \|y\|_V \,. \tag{5.1.13}$$

- *The form $\bar{a}$ is continuous, i.e.*

$$\bar{a}(y, v) \leq (\lambda + 2\mu) \|y\|_V \|v\|_V \,. \tag{5.1.14}$$

- *For $y \in H^1(\Omega)^d$ there holds*

$$\bar{a}(y, y) \geq \lambda \|\operatorname{div} y\|^2_{L^2(\Omega)} + 2\mu \int_\Omega \mathcal{D}y : \mathcal{D}y dx \geq C \|y\|^2_{H^1(\Omega)^d} - 2\mu \|y\|^2_{L^2(\Omega)^d} \,. \tag{5.1.15}$$

- *For $y \in H_0^1(\Omega)^d$ there holds*

$$\bar{a}(y,y) \geq \lambda \|\operatorname{div} y\|_{L^2(\Omega)}^2 + 2\mu \int_\Omega \mathcal{D}y : \mathcal{D}y dx \geq C \|y\|_{H_0^1(\Omega)^d}^2 . \tag{5.1.16}$$

Proof. Inequality (5.1.13) follows by a direct calculation and implies (5.1.14). The inequalities (5.1.15) and (5.1.16) follow with Korn's first and second inequalities (5.1.11) and (5.1.12), respectively. □

After these preparation we can formulate an existence and regularity result.

Theorem 5.1.7. *There exists a unique solution $y \in X$ of* (5.1.9).

Proof. This follows from Lemma 5.1.6 and Theorem 2.3.1. □

5.2 Adaptive finite element methods

In this section we apply the adaptive finite element method considered in Chapter 3 to an optimal control problem of the dynamical Lamé system, which reads as

$$\begin{cases} \text{Minimize} \quad J(u,y) = \dfrac{1}{2} \|y - y_\mathrm{d}\|_{L^2(L^2(\Omega)^d)}^2 + \dfrac{\alpha}{2} \|u\|_{L^2(\mathbb{R}^l)}^2 , \quad u \in U,\ y \in X, \quad \text{s.t.} \\ \qquad y_{tt} - \operatorname{div} \sigma(y) = f + \mathcal{B}u \quad \text{in } Q, \\ \qquad y(0) = y_0 \quad \text{in } \Omega, \\ \qquad y_t(0) = y_1 \quad \text{in } \Omega, \\ \qquad y = 0 \quad \text{in } \Sigma \end{cases} \tag{5.2.1}$$

for initial data $y_0 \in H_0^1(\Omega)^d$, $y_1 \in L^2(\Omega)^d$, $f \in L^2(L^2(\Omega)^d)$, stress tensor σ given by (5.1.3), control space $U = L^2(\mathbb{R}^l)$ and operator

$$\mathcal{B} \colon U \to L^2(L^2(\Omega)^d), \quad \mathcal{B}u = \sum_{i=1}^{l} u_i(t) g_i(x)$$

for given functions $g_i \in L^2(\Omega)^d$, $i = 1, \dots, l$, $l \in \mathbb{N}$. Thus, in this example the control is time-dependent with values in $\mathbb{R}^l$.

For the discretization we proceed as in Section 3.2. The discrete control space is chosen as

$$U_d = \{\, u \in L^2(\mathbb{R}^l) \mid u|_{I_m} \in \mathcal{P}_{r_d}(I_m, \mathbb{R}^l), \quad u(0) \in \mathbb{R}^l \,\} ,$$

where the time intervals I_m are the same as used for the discretization of the state and let $r_d = r - 1$, where r is the polynomial degree of the ansatz functions in time used for the discretization of the state.

We verify that the estimator η_d (cf. (3.4.11)) vanishes in this case. Therefore, we introduce the adjoint operator $\mathcal{B}^*$ given by

$$\mathcal{B}^* \colon L^2(L^2(\Omega)^d) \to U, \quad (\mathcal{B}^* q)(t)_i = (g_i, q)_{L^2(\Omega)^d} \quad (i = 1, \dots, l),$$

since

$$(u, \mathcal{B}^*q)_U = \int_0^T \sum_{i=1}^l u_i(\mathcal{B}^*q)(t)_i dt = \int_0^T \sum_{i=1}^l u_i(t)(g_i, q(t))_{L^2(\Omega)^d} dt = (\mathcal{B}u, q)_I$$

for $q \in L^2(L^2(\Omega)^d)$.

Lemma 5.2.1. *Under the assumptions from above, the estimator η_d vanishes.*

Proof. Since

$$j'(u_\sigma)(\delta u) = (\alpha u_\sigma, \delta u)_U + (p_\sigma, \mathcal{B}\delta u)_I \quad \forall \delta u \in U_d$$

the optimality condition reads as

$$(\alpha u_\sigma + \mathcal{B}^* p_\sigma, \delta u)_U = 0 \quad \forall \delta u \in U_d. \tag{5.2.2}$$

There holds $\mathcal{B}^* p_\sigma \in U_d$ for all $p_\sigma \in X_{k,h}^{r,s}$, since

$$p_\sigma(t) = \sum_{k=0}^{r_d} p_{\sigma,k} t^k, \quad p_{\sigma,k} \in V_h^s, \quad t \in I_m$$

and so

$$(\mathcal{B}^* p_\sigma)_i|_{I_m} = \sum_{k=0}^{r_d} \left(\int_\Omega g_i p_{\sigma,k} dx \right) t^k \in \mathcal{P}_{r_d}(I_m, \mathbb{R}^l).$$

Thus, we can choose $\delta u = \alpha u_\sigma + \mathcal{B}^* p_\sigma$ in (5.2.2) and obtain $\mathcal{L}'_u(u_\sigma, y_\sigma, p_\sigma)(\cdot) = 0$. □

Remark 5.2.2. The previous lemma can be generalized to the case of different temporal meshes for the control and state discretization if the set of time points of the state discretization is a subset of the time points of the control discretization.

For the computations we choose the polynomial degree of the spatial and temporal ansatz functions for the state as $r = s = 1$, the degree of the ansatz functions for the control as $r_d = 0$ and the data as follows

$$y_0(x) = \begin{cases} (\sin(8\pi(x_1 - 0.125)) \sin(8\pi(x_2 - 0.125)), 0)^T, & 0.125 < x_1, x_2 < 0.25, \\ (0,0)^T, & \text{else,} \end{cases}$$

$$y_1(x) = (0,0)^T,$$

$$y_\mathrm{d}(t,x) = 0, \quad f(t,x) = (0,0)^T,$$

$$g_1(x) = \begin{cases} (1,1)^T, & \text{for } x_1 < 0, \\ (0,0)^T, & \text{else} \end{cases}$$

$$g_2(x) = \begin{cases} (1,1)^T, & \text{for } x_1 > 0, \\ (0,0)^T, & \text{else} \end{cases}$$

$$\alpha = 0.001, \quad d = 2, \quad l = 2, \quad \lambda = 1, \quad \mu = 1 \tag{5.2.3}$$

for $(t, x) = (t, x_1, x_2) \in [0, T] \times \Omega = [0, 0.5] \times [-1, 1]^2$.

In Figure 5.1 we present a comparison of the error in the cost functional for adaptive and uniform refinement. It illustrates that in case of adaptive refinement we need less degrees of freedom than in case of uniform refinement to reach a given error tolerance. Further, in Table 5.1 we compare the CPU time and the degrees of freedom to reach an error less than $6.5 \cdot 10^{-8}$ normalizing the values for uniform refinement to 100%. We have an essential gain in time and number of unknowns in case of adaptive refinement. In Figure 5.2-5.6 the spatial meshes at different time points are given.

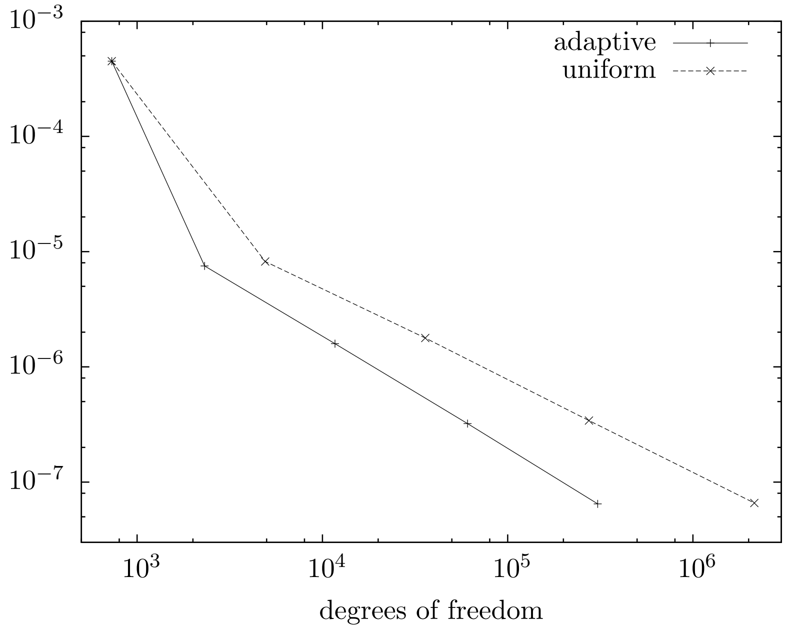


Figure 5.1: Error for adaptive and uniform refinement for (5.2.1)

refinement	CPU-time	dof	error
uniform	100%	100%	$6.6 \cdot 10^{-8}$
adaptive	34%	15 %	$6.5 \cdot 10^{-8}$

Table 5.1: Comparison of the CPU-time for uniform and adaptive refinement for (5.2.1)

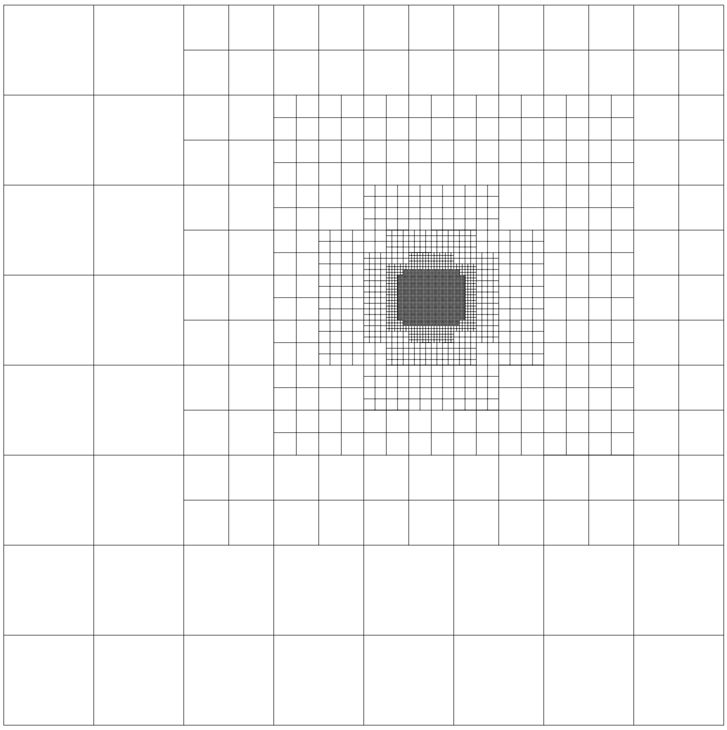


Figure 5.2: Spatial mesh at time $t = 0$ for (5.2.1)

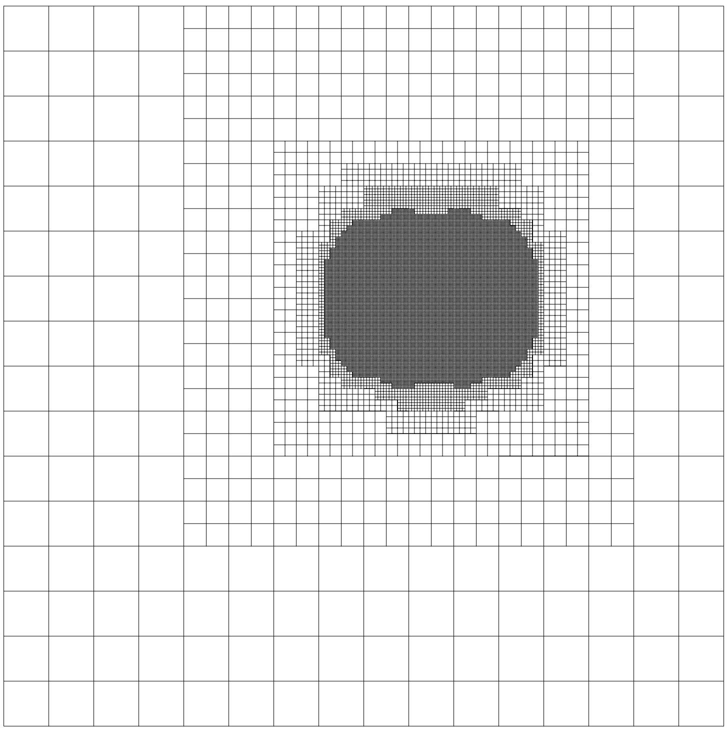


Figure 5.3: Spatial mesh at time $t = 0.25$ for (5.2.1)

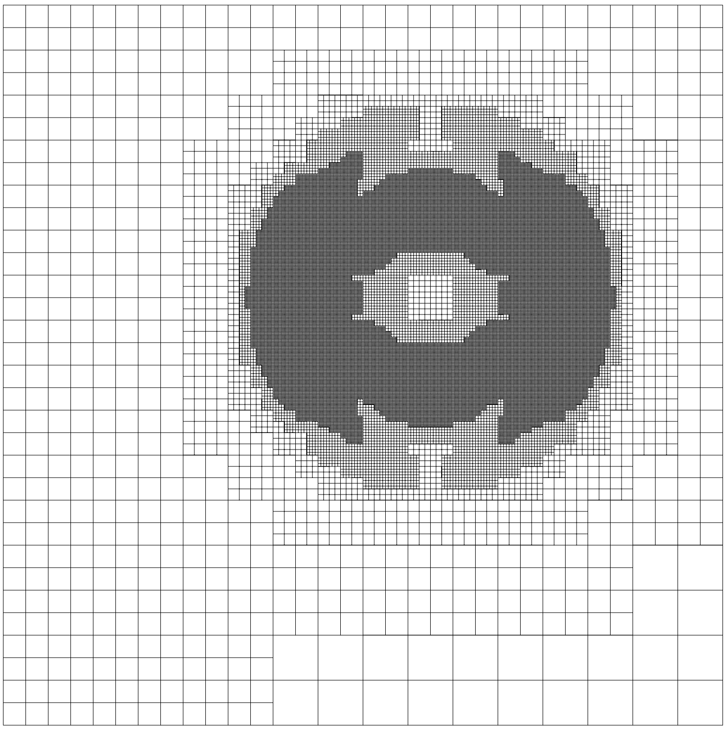


Figure 5.4: Spatial mesh at time $t = 0.5$ for (5.2.1)

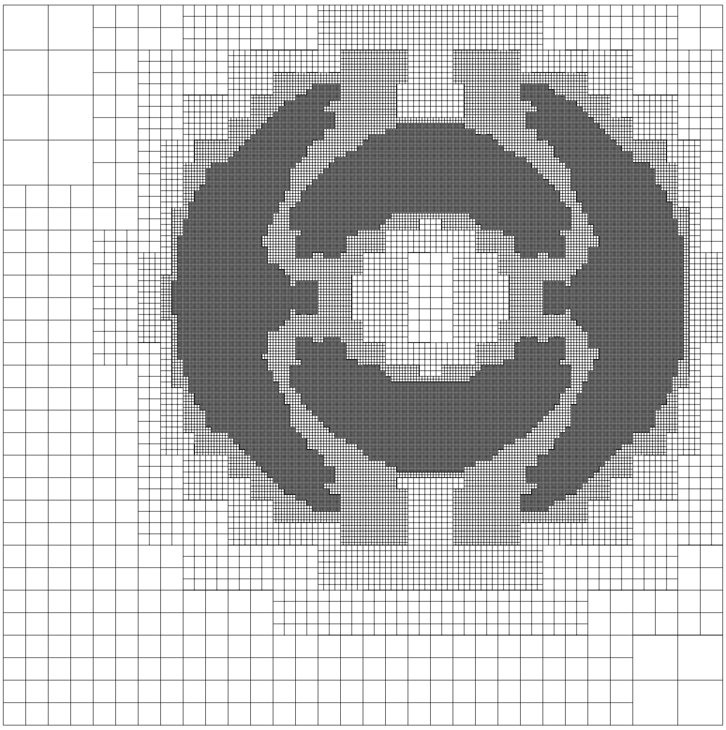


Figure 5.5: Spatial mesh at time $t = 0.75$ for (5.2.1)

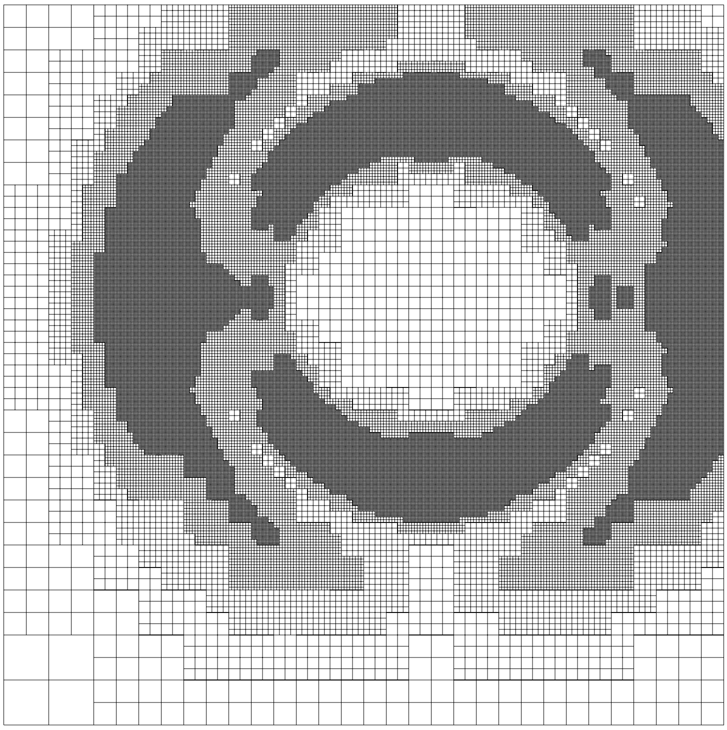


Figure 5.6: Spatial mesh at time $t = 1$ for (5.2.1)

5.3 Semi-smooth Newton methods

In this section we consider semi-smooth Newton methods applied to optimal control problems governed by the dynamical Lamé system with constraints on the control and apply the techniques developed in Chapter 4. We consider the cases of distributed, Neumann boundary and Dirichlet boundary control and analyze the convergence of the semi-smooth Newton method.

To apply the framework developed in Chapter 4, we have to extend the definitions in (4.0.1)-(4.0.3) to systems with d components. We define

$$\mathcal{G}\colon L^2(L^2(\Omega)^d) \to \mathbb{R},$$

and assume that the functional is quadratic with $\mathcal{G}'$ being an affine operator from $L^2(L^2(\Omega)^d)$ to itself, and that $\mathcal{G}''$ is non-negative, i.e. $(\mathcal{G}''(x)\delta x, \delta x) \geq 0$ for all $x, \delta x \in L^2(L^2(\Omega)^d)$. Let $U = L^2(\omega)^d$, the state space be given by $Y = L^2(L^2(\Omega)^d)$ and the control-to-state operator $S\colon U \to Y$ affine-linear with

$$S(u) = Tu + \bar{y} \tag{5.3.1}$$

for $T \in \mathcal{L}(U, Y)$ and $\bar{y} \in Y$. Further, let the operator $q\colon U \to U$ be given by

$$q(u) = -T^*\mathcal{G}'(S(u)).$$

We define the generalized derivative for functions in $\{\, v\colon \omega \to \mathbb{R}^d \,\}$ in analogy to (4.1.4) by components and derive the boundedness of the corresponding inverse generalized derivative according to Lemma 4.1.15; we only need a slightly modification of the proof. We set for $i = 1, \ldots, d$

$$\begin{aligned} I_i &= \{x \in \omega : \alpha u_a(x) \leq q(u)_i(x) \leq \alpha u_b(x)\}, \\ A_i &= \omega \setminus I_i, \end{aligned}$$

where the inequalities are understood by components and $u_a, u_b \in U$. Then, we can follow the arguments in Lemma 4.1.15 considering the inner products with

$$\chi_A = \begin{pmatrix} \chi_{A_1} \\ \vdots \\ \chi_{A_d} \end{pmatrix}, \quad \chi_I = \begin{pmatrix} h_1\chi_{I_1} \\ \vdots \\ h_d\chi_{I_d} \end{pmatrix}$$

for $h = (h_1, \ldots, h_d)^T \in L^2(\omega)^d$.

Finally, we need the Newton differentiability of $\max(0, \cdot)\colon L^q(\omega)^d \to L^p(\omega)^d$ for exponents $1 \leq p < q < \infty$, where the max-operator is understood by components. This property follows directly from the Definition 4.1.1, since Newton-differentiability is given with respect to every component.

Thus, according to Section 4.3 the main issue remains to verify that the operator mapping the control to the adjoint state or a trace of the adjoint state, respectively, has some smoothing property.

We start our consideration with distributed control.

5.3.1 Distributed control

The optimal distributed control problem of the Lamé system reads as

$$\begin{cases} \text{Minimize} \quad J(u,y) = \mathcal{G}(y) + \dfrac{\alpha}{2}\, \|u\|^2_{L^2(L^2(\Omega)^d)}\,, \; u \in L^2(L^2(\Omega)^d), \quad y \in L^2(L^2(\Omega)^d), \quad \text{s.t.} \\ \qquad\qquad y_{tt} - \operatorname{div}\sigma(y) = u \quad \text{in } Q, \\ \qquad\qquad y(0) = y_0 \quad \text{in } \Omega, \\ \qquad\qquad y_t(0) = y_1 \quad \text{in } \Omega, \\ \qquad\qquad y = 0 \quad \text{on } \Sigma, \\ \qquad\qquad u_a \le u \le u_b \quad \text{a.e. in } \Sigma \end{cases} \tag{5.3.2}$$

for $y_0 \in H_0^1(\Omega)^d$, $y_1 \in L^2(\Omega)^d$, $f \in L^2(\Omega)^d$ stress tensor σ given by (5.1.3), $u_a, u_b \in L^r(L^r(\Omega)^d)$, $r > 2$, and $\alpha > 0$.

The existence of a solution of (5.3.2) follows from Theorem 5.1.7 and Proposition 2.2.6. Thus we can directly formulate the result on superlinear convergence.

Theorem 5.3.1. *The semi-smooth Newton method applied to the distributed control problem* (5.3.2) *converges superlinearly.*

Proof. The proof follows similar arguments as in the proof of Theorem 4.3.4. From Theorem 5.1.7 we deduce that the adjoint state is in particular an element in

$$L^2(H^1(\Omega)^d) \cap H^1(L^2(\Omega)^d) \hookrightarrow L^p(L^p(\Omega)^d)$$

for all $1 \le p < \infty$ for $d = 2$ and all $1 \le p \le 6$ for $d = 3$, which implies the assertion. □

5.3.2 Neumann boundary control

The optimal Neumann boundary control problem of the Lamé system reads as

$$\begin{cases} \text{Minimize} \quad J(u,y) = \mathcal{G}(y) + \dfrac{\alpha}{2}\, \|u\|^2_{L^2(L^2(\partial\Omega)^d)}\,, \; u \in L^2(L^2(\partial\Omega)^d), \quad y \in L^2(L^2(\Omega)^d), \\ \qquad \text{s.t.} \\ \qquad\qquad y_{tt} - \operatorname{div}\sigma(y) = f \quad \text{in } Q, \\ \qquad\qquad y(0) = y_0 \quad \text{in } \Omega, \\ \qquad\qquad y_t(0) = y_1 \quad \text{in } \Omega, \\ \qquad\qquad \sigma(y)\cdot n = u \quad \text{on } \Sigma, \\ \qquad\qquad u_a \le u \le u_b \quad \text{a.e. in } \Sigma \end{cases} \tag{5.3.3}$$

for $y_0 \in L^2(\Omega)^d$, $y_1 \in ((H^1(\Omega))^*)^d$, $f \in L^1((H^1(\Omega)^*)^d)$, stress tensor σ given by (5.1.3), $u_a, u_b \in L^r(L^r(\partial\Omega)^d)$, $r > 2$, $\alpha > 0$, and outer normal n.

There exists a unique solution of the state equation.

Lemma 5.3.2. *For $u \in L^2(L^2(\partial\Omega)^d)$ there exists a very weak solution $y \in L^2(L^2(\Omega)^d)$ of the state equation of problem* (5.3.3) *satisfying*

$$(y, g)_I = (f, \xi)_I - (y_0, \xi_t(0)) + (y_1, \xi(0)) + \langle u, \xi\rangle_I \tag{5.3.4}$$

where $\xi = \xi_g$ is the solution of

$$\begin{cases} \xi_{tt} - 2\mu \operatorname{div} \mathcal{D}\xi - \lambda\nabla \operatorname{div} \xi = g & \text{in } Q, \\ \xi(0) = 0 & \text{in } \Omega, \\ \xi_t(0) = 0 & \text{in } \Omega, \\ \sigma(\xi) \cdot n = 0 & \text{on } \Sigma \end{cases} \tag{5.3.5}$$

for all $g \in L^2(L^2(\Omega)^d)$.

Proof. From Theorem 5.1.7 we obtain the boundedness of the right side in (5.3.4). Thus, the assertion follows by Riesz representation theorem. □

The existence of a solution of the control problem is given by Proposition 2.2.6.

Superliner convergence of the semi-smooth Newton method applied to the Neumann boundary control problem follows by the smoothing property of the control-to-adjoint state mapping.

Theorem 5.3.3. *The semi-smooth Newton method applied to the Neumann boundary control problem* (5.3.3) *converges superlinearly.*

Proof. The solution of the corresponding adjoint state equation is an element in

$$L^2(H^1(\Omega)^d) \cap H^1(L^2(\Omega)^d)$$

by Theorem 5.1.7. Thus, in analogy to Theorem 4.3.8 we obtain superlinear convergence. □

5.3.3 Dirichlet boundary control

The optimal Dirichlet boundary control problem for the Lamé system reads as

$$\begin{cases} \text{Minimize} \quad J(u, y) = \mathcal{G}(y) + \dfrac{\alpha}{2} \|u\|^2_{L^2(L^2(\Omega)^d)}, \; u \in L^2(L^2(\partial\Omega)^d), \quad y \in L^2(L^2(\Omega)^d), \\ \quad \text{s.t.} \\ \quad y_{tt} - \operatorname{div} \sigma(y) = f \quad \text{in } Q, \\ \quad y(0) = y_0 \quad \text{in } \Omega, \\ \quad y_t(0) = y_1 \quad \text{in } \Omega, \\ \quad y = u \quad \text{on } \Sigma, \\ \quad u_a \le u \le u_b \quad \text{a.e. in } \Sigma \end{cases} \tag{5.3.6}$$

where $y_0 \in L^2(\Omega)^d$, $y_1 \in H^{-1}(\Omega)^d$, $f \in L^1(H^{-1}(\Omega)^d)$, stress tensor σ given by (5.1.3), $u_a, u_b \in L^r(L^r(\partial\Omega)^d)$, $r > 2$ and $\alpha > 0$.

First of all, we have to prove existence of a solution of the state equation of (5.3.6). Following the argumentation in the proof of Theorem 2.3.12 we begin with considering the equation with homogeneous Dirichlet boundary condition, for which we need some hidden regularity result for the Neumann trace of the solution. This can not be directly obtained from the theory for linear hyperbolic equations, in Theorem 2.3.10, since here we have a coupled system. Nevertheless, in Belishev & Lasiecka [15] the technique to prove a hidden regularity result for wave equations (cf. Lasiecka, Lions & Triggiani [85]), was extended to the Lamé system. Further regularity results for traces are derived in Bucci & Lasiecka [24].

Lemma 5.3.4. *For $u \in L^2(L^2(\partial\Omega)^d)$ there exists a very weak solution $y \in L^2(L^2(\Omega)^d)$ of the state equation of* (5.3.6) *satisfying*

$$(y, g)_I = (f, \xi)_I - (y_0, \xi_t(0)) + (y_1, \xi(0)) - \langle u, \sigma(\xi) \cdot n \rangle_I, \tag{5.3.7}$$

where $\xi = \xi_g$ is the solution of

$$\begin{cases} \xi_{tt} - 2\mu \operatorname{div} \mathcal{D}\xi - \lambda \nabla \operatorname{div} \xi = g & \text{in } Q, \\ \xi(T) = 0 & \text{in } \Omega, \\ \xi_t(T) = 0 & \text{in } \Omega, \\ \xi = 0 & \text{on } \Sigma \end{cases} \tag{5.3.8}$$

with $g \in L^2(L^2(\Omega)^d)$.

Proof. We follow the argumentation in the proof of Theorem 2.3.12. To show that the right part of (5.3.7) is bounded, the main task is, to verify that for the solution ξ of the system (5.3.8) the normal derivative $\sigma(\xi) \cdot n$ has some hidden regularity, i.e. we have to show that $\sigma(\xi) \cdot n \in L^2(L^2(\partial\Omega)^d)$. The boundedness of the other terms follows by Theorem 5.1.7. The hidden regularity is shown in Belishev & Lasiecka [15, Proof of Proposition 1]. They consider the case $d = 3$, but the results hold also true for $d = 2$. Thus, the stated regularity follows by the Riesz representation theorem. □

Now, we return to the optimal control problem. The existence of a solution of the control problem follows by Proposition 2.2.6.

To study the behaviour of convergence of the semi-smooth Newton method applied to (5.3.6) we analyze whether the operator

$$q : U \to U, \quad q(u) = -T^* \mathcal{G}'(S(u)) = -\sigma(p(u)) \cdot n$$

mapping the control u to the Neumann trace of the corresponding adjoint state $p(u)$ has some smoothing property. In the one dimensional case, $d = 1$, the Lamé system reads as

$$y_{tt} - \lambda y_{xx} - 2\mu y_{xx} = f.$$

Thus for $\lambda + 2\mu = 1$ we obtain the classical wave equation with velocity $c = 1$, considered in Theorem 4.3.12, i.e. in this case there is no smoothing of the operator q given.

As in the case of optimal Dirichlet boundary control for the wave equation this motivates to consider the strongly damped dynamical Lamé system for some $\rho \in \mathbb{R}$ with $0 < \rho < \rho_0$, $\rho_0 \in \mathbb{R}^+$, given by

$$\begin{cases} y_{tt} - \operatorname{div}\sigma(y) - \rho \operatorname{div}\sigma(y_t) = f & \text{in } Q, \\ y(0) = y_0 & \text{in } \Omega, \\ y_t(0) = y_1 & \text{in } \Omega, \\ y = u & \text{on } \Sigma \end{cases} \tag{5.3.9}$$

with $f \in L^2(L^2(\Omega)^d)$, $y_0 \in H^1(\Omega)^d$, and $y_1 \in L^2(\Omega)^d$ and the corresponding optimal control problem

$$\begin{cases} \text{Minimize} \quad J(u,y) = \mathcal{G}(y) + \dfrac{\alpha}{2}\,\|u\|^2_{L^2(L^2(\partial\Omega)^d)}\,, \; u \in L^2(L^2(\partial\Omega)^d), \quad y \in L^2(L^2(\Omega)^d), \\ \qquad \text{s.t.} \\ \qquad\qquad (5.3.9) \text{ with } u_a \le u \le u_b \quad \text{a.e. in } \Sigma \end{cases} \tag{5.3.10}$$

for $u_a, u_b \in L^r(L^r(\partial\Omega)^d)$, $r > 2$. According to optimal Dirichlet boundary control of the wave equation we prove a regularity result for the adjoint strongly damped Lamé system given as follows (using the fact that it is reversible in time)

$$\begin{cases} p_{tt} - \lambda\nabla\operatorname{div} p - 2\mu\operatorname{div}\mathcal{D}p - \rho(\lambda\nabla\operatorname{div} p_t + 2\mu\operatorname{div}\mathcal{D}p_t) = g & \text{in } Q, \\ p(0) = p_0 & \text{in } \Omega, \\ p_t(0) = p_1 & \text{in } \Omega, \\ p = 0 & \text{on } \Sigma \end{cases} \tag{5.3.11}$$

for $g \in L^2(L^2(\Omega)^d)$. Assuming all terms are well-defined we obtain the following equivalent formulation

$$\begin{cases} p_{tt} - (\mu+\lambda)\nabla\operatorname{div} p - \mu\Delta p - \rho((\lambda+\mu)\nabla\operatorname{div} p_t + \mu\Delta p_t) = g & \text{in } Q, \\ p(0) = p_0 & \text{in } \Omega, \\ p_t(0) = p_1 & \text{in } \Omega, \\ p = 0 & \text{on } \Sigma. \end{cases} \tag{5.3.12}$$

Theorem 5.3.5 (Regularity for the homogeneous strongly damped Lamé system)**.** *For $f \in L^2(L^2(\Omega)^d)$, $p_0 \in H^1_0(\Omega)^d \cap H^2(\Omega)^d$, and $p_1 \in H^1_0(\Omega)^d$, there exists a unique weak solution of* (5.3.12)

$$p \in H^2(L^2(\Omega)^d) \cap C^1(H^1_0(\Omega)^d) \cap H^1(H^2(\Omega)^d) \tag{5.3.13}$$

defined by $p(0) = p_0$, $p_t(0) = p_1$ and

$$(p_{tt}(s),\phi) + (\lambda+\mu)(\operatorname{div} p(s),\operatorname{div}\phi) + \mu(\nabla p(s) : \nabla\phi) + \rho(\lambda+\mu)(\operatorname{div} p_t(s),\operatorname{div}\phi) + \rho\mu(\nabla p_t(s) : \nabla\phi) = (f(s),\phi) \quad \forall\phi \in H^1_0(\Omega)^d \text{ a.e. in } (0,T). \tag{5.3.14}$$

Moreover, the a priori estimate

$$\|p\|_{H^2(L^2(\Omega)^d)\cap C^1(H_0^1(\Omega)^d)\cap H^1(H^2(\Omega)^d)} \le C\Big(\|g\|_{L^2(L^2(\Omega)^d)} + \|\nabla p_0\| + \|\operatorname{div} p_0\| + \|\Delta p_0\| + \|\nabla \operatorname{div} p_0\| + \|\nabla p_1\| + \|\operatorname{div} p_1\|\Big), \quad (5.3.15)$$

holds, where the constant $C = C(\rho)$ tends to infinity as ρ tends to zero.

To prove this theorem we argue as in Chapter 2 and apply a Galerkin method. Therefore we derive a priori estimates for the strongly damped Lamé system according to the Lemmas 2.3.16 - 2.3.19.

Proof of Theorem 5.3.5. The proof is presented in four steps.

1. We test (5.3.14) with p_t. Then there holds

$$\|p_t(t)\|^2 + (\lambda+\mu)\|\operatorname{div} p(t)\|^2 + \mu\|\nabla p(t)\|^2 + \rho(\lambda+\mu)\int_0^t \|\operatorname{div} p_t(s)\|^2 ds + \rho\mu\int_0^t \|\nabla p_t(s)\|^2 ds \le C\left(\|\nabla p_0\|^2 + \|p_1\|^2 + \|\operatorname{div} p_0\|^2 + \|g\|^2_{L^2(L^2(\Omega)^d)}\right). \quad (5.3.16)$$

2. Let $e(p) = -(\lambda+\mu)\nabla \operatorname{div} p - \mu\Delta p$. Then we test (5.3.14) with $\phi = -e(p)$. There holds

$$-(p_{tt}(s), e(p)(s)) + \|e(p)(s)\|^2 + \rho(e(p_t)(s), e(p)(s)) = -(g(s), e(p)(s))$$

   or equivalently

$$-(p_{tt}(s), e(p)(s)) + \|e(p)(s)\|^2 + \frac{\rho}{2}\frac{d}{dt}\|e(p)(s)\|^2 = -(g(s), e(p)(s))\ .$$

   Integrating in time from 0 to t implies that

$$-\int_0^t (p_{tt}(s), e(p)(s))\, ds + \int_0^t \|e(p)(s)\|^2\, ds + \frac{\rho}{2}\|e(p)(t)\|^2 \le \frac{1}{2}\|g\|^2_{L^2(L^2(\Omega))} + \frac{1}{2}\int_0^t \|e(p)(s)\|^2\, ds + \frac{\rho}{2}\|\mu\Delta p_0 + (\lambda+\mu)\nabla \operatorname{div} p_0\|^2.$$

   For the first term on the left-hand side we get for almost every $t \in (0,T)$

$$-\int_0^t (p_{tt}(s), e(p)(s))\, ds = \int_0^t (p_t(s), e(p_t)(s)))\, ds - (p_t(t), e(p)(t)) + (p_t(0), e(p)(0)) = -(\lambda+\mu)\int_0^t \|\operatorname{div} p_t(s)\|^2\, ds - \mu\int_0^t \|\nabla p_t(s)\|^2\, ds - (p_t(t), e(p)(t)) + (p_1, (\lambda+\mu)\nabla \operatorname{div} p_0 + \mu\Delta p_0).$$

Here, we have used the fact that $p_{tt} = p_t = 0$ on Σ and $p_1 = 0$ on $\partial\Omega$. This yields

$$\int_0^t \|e(p)(s)\|^2\,ds + \frac{\rho}{2}\|e(p)(t)\|^2 \le \frac{1}{2}\|g\|^2_{L^2(L^2(\Omega))} + \frac{1}{2}\int_0^t \|e(p)(s)\|^2\,ds$$
$$+ \frac{\rho}{2}\|(\lambda+\mu)\nabla \operatorname{div} p_0 + \mu\Delta p_0\|^2 + (\lambda+\mu)\int_0^t \|\operatorname{div} p_t(s)\|^2\,ds$$
$$+\mu\int_0^t \|\nabla p_t(s)\|^2\,ds + \frac{1}{\rho}\|p_t(t)\|^2 + \frac{\rho}{4}\|e(p)(t)\|^2 + \frac{1}{2}\|p_1\|^2 + \frac{1}{2}\|(\lambda+\mu)\nabla \operatorname{div} p_0 + \mu\Delta p_0\|^2.$$

Absorbing terms we obtain

$$\frac{1}{2}\int_0^t \|e(p)(s)\|^2\,ds + \frac{\rho}{4}\|e(p)(s)\|^2 \le \frac{1}{2}\|g\|^2_{L^2(L^2(\Omega))} + \frac{\rho+1}{2}\|(\lambda+\mu)\nabla \operatorname{div} p_0 + \mu\Delta p_0\|^2$$
$$+ (\lambda+\mu)\int_0^t \|\operatorname{div} p_t(s)\|^2\,ds + \mu\int_0^t \|\nabla p_t(s)\|^2\,ds + \frac{1}{\rho}\|p_t(t)\|^2 + \frac{1}{2}\|p_1\|^2.$$

Using (5.3.16) we obtain the desired estimate

$$\begin{aligned}&\int_0^t \|e(p)(s)\|^2 ds + \rho\|e(p)(t)\|^2\\ &\le \frac{C}{\rho}\Big(\|\operatorname{div} p_0\|^2 + \|\nabla p_0\|^2 + \|(\lambda+\mu)\nabla \operatorname{div} p_0 + \mu\Delta p_0\|^2 + \|p_1\|^2 + \|g\|^2_{L^2(L^2(\Omega)^d)}\Big).\end{aligned} \tag{5.3.17}$$

3. We test (5.3.14) with $\phi = e(p_t)$. Then there holds

$$-(p_{tt}(s), e(p_t)(s)) + (e(p)(s), e(p_t)(s)) + \rho\|e(p_t)(s)\|^2 = -(g(s), e(p_t)(s)).$$

We integrate by parts in the first term and obtain for almost every s

$$(\lambda+\mu)\frac{1}{2}\frac{d}{dt}\|\operatorname{div} p_t(s)\|^2 + \mu\frac{1}{2}\frac{d}{dt}\|\nabla p_t(s)\|^2 + \frac{1}{2}\frac{d}{dt}\|e(p)(s)\|^2$$
$$+ \rho\|e(p_t)(s)\|^2 = -(g(s), e(p_t)(s)).$$

Integrating in time from 0 to t we obtain:

$$(\lambda+\mu)\frac{1}{2}\|\operatorname{div} p_t(t)\|^2 + \mu\frac{1}{2}\|\nabla p_t(t)\|^2 + \frac{1}{2}\|e(p)(t)\|^2 + \rho\int_0^t \|e(p_t)(s)\|^2\,ds$$
$$\le \frac{1}{2\rho}\|g\|^2_{L^2(L^2(\Omega))} + \frac{\rho}{2}\int_0^t \|e(p_t)(s)\|^2\,ds + (\lambda+\mu)\frac{1}{2}\|\operatorname{div} p_1\|^2 + \mu\frac{1}{2}\|\nabla p_1\|^2$$
$$+ \frac{1}{2}\|(\lambda+\mu)\nabla \operatorname{div} p_0 + \mu\Delta p_0\|^2.$$

This implies the desired estimate

$$\begin{aligned}&(\lambda+\mu)\|\operatorname{div} p_t(t)\|^2 + \mu\|\nabla p_t(t)\|^2 + \|e(p)(t)\|^2 + \rho\int_0^t \|e(p_t)(s)\|^2\,ds\\ &\le \frac{C}{\rho}\left(\|g\|^2_{L^2(L^2(\Omega)^d)} + \|\nabla p_1\|^2 + \|\operatorname{div} p_1\|^2 + \|\Delta p_0\|^2 + \|\nabla \operatorname{div} p_0\|^2\right).\end{aligned} \tag{5.3.18}$$

4. We test (5.3.14) with $\phi = p_{tt}$. This yields
$$\|p_{tt}(s)\|^2 - (e(p)(s), p_{tt}(s)) - \rho(e(p_t), p_{tt}(s)) = (g(s), p_{tt}(s)).$$
Hence,
$$\int_0^t \|p_{tt}(s)\|^2\, ds + \int_0^t (e(p_t)(s), p_t(s))\, ds - (e(p)(t), p_t(t))$$
$$+ ((\lambda+\mu)\nabla \operatorname{div} p(0) + \mu \Delta p(0), p_t(0)) = \int_0^t (g, p_{tt}) ds + \rho \int_0^t (e(p_t)(s), p_{tt}(s))$$
and thus, we obtain
$$\int_0^t \|p_{tt}(s)\|^2\, ds \le \|g\|^2_{L^2(L^2(\Omega)^n)} + \frac{1}{4}\int_0^t \|p_{tt}(s)\|^2 ds + \frac{\rho^2}{2}\int_0^t \|e(p_t)(s)\|^2 ds$$
$$+ \frac{1}{2}\int_0^t \|p_{tt}(s)\|^2 ds + (\lambda+\mu)\int_0^t \|\operatorname{div} p_t(s)\|^2 ds + \mu \int_0^t \|\nabla p_t(s)\|^2 ds$$
$$+ \frac{1}{2}(\lambda+\mu)\|\operatorname{div} p(t)\|^2 + \frac{1}{2}(\lambda+\mu)\|\operatorname{div} p_t(t)\|^2 + \frac{1}{2}\mu\|\nabla p(t)\|^2 + \frac{1}{2}\mu\|\nabla p_t(t)\|^2$$
$$+ \frac{1}{2}\|(\lambda+\mu)\nabla \operatorname{div} p_0 + \mu \Delta p_0\|^2 + \frac{1}{2}\|p_1\|^2.$$
Absorbing terms and using (5.3.16) and (5.3.18) we obtain the desired estimate
$$\int_0^t \|p_{tt}(s)\|^2 ds \le \frac{C}{\rho}\Big(\|g\|^2_{L^2(L^2(\Omega)^d)} + \|\nabla p_0\|^2 + \|\operatorname{div} p_0\|^2 + \|\Delta p_0\|^2 + \|\nabla \operatorname{div} p_0\|^2$$
$$+ \|\nabla p_1\|^2 + \|\operatorname{div} p_1\|^2 \Big).$$

Finally, we use an estimate following from elliptic theory, cf. Brenner & Sung [22, Lemma 2.2]
$$\|p(t)\|_{H^2(\Omega)^d} \le C\|g(t) - p_{tt}(t) + e(p_t)(t)\| = C\|e(p)(t)\|, \quad t \in (0,t).$$
Using a Galerkin method and proceeding as in the proof of Theorem 2.3.15 we obtain the assertion. □

Now, we return to the inhomogeneous equation and introduce the following very weak formulation
$$(y,g)_I = -(y_0, \zeta_t(0)) + (y_1, \zeta(0)) - \langle u, \sigma(\zeta)\cdot n\rangle_I$$
$$+ \rho\langle u, \sigma(\zeta_t)\cdot n\rangle_I - \rho(y_0, \operatorname{div}\sigma(0)) + \rho\langle y_0, \sigma(\zeta(0))\cdot n\rangle + (f,\zeta)_I, \quad (5.3.19)$$
with the solution $\zeta = \zeta_g$ of
$$\begin{cases} \zeta_{tt} - \operatorname{div}\sigma(\zeta) - \operatorname{div}\sigma(\zeta_t) = g & \text{in } Q,\\ \zeta(T) = 0 & \text{in } \Omega,\\ \zeta_t(T) = 0 & \text{in } \Omega,\\ \zeta = 0 & \text{on } \Sigma \end{cases} \quad (5.3.20)$$
and arbitrary $g \in L^2(L^2(\Omega)^d)$.

Theorem 5.3.6. *For $u \in L^2(L^2(\partial\Omega)^d)$, $f \in L^2(L^2(\Omega)^d)$, $y_0 \in H^1(\Omega)^d$ and $y_1 \in L^2(\Omega)^d$, equation* (5.3.9) *possess a unique very weak solution defined by* (5.3.19) *and there the following estimate*

$$\|y\|_{L^2(L^2(\Omega)^d)} \leq C\left(\|u\|_{L^2(L^2(\partial\Omega)^d)} + \|y_0\|_{H^1(\Omega)^d} + \|y_1\| + \|f\|_{L^2(L^2(\Omega)^d)}\right)$$

holds, where the constant $C = C(\rho)$ tends to infinity as ρ tends to zero.

Proof. The right hand side of (5.3.19) defines a linear functional $G(g)$ on $L^2(L^2(\Omega)^d)$. This functional is bounded. In fact as a consequence of Theorem 5.3.5 we have

$$\begin{aligned}\|\zeta_t(0)\| + \|\zeta(0)\| + \|\operatorname{div}\sigma(\zeta(0))\| + \|\sigma(\zeta(0))\cdot n\|_{L^2(\partial\Omega)^d}\\ + \|\sigma(\zeta)\cdot n\|_{L^2(L^2(\partial\Omega)^d)} + \|\sigma(\zeta_t)\cdot n\|_{L^2(L^2(\partial\Omega)^d)} + \|\zeta\|_{L^2(L^2(\Omega)^d)} \leq C\|g\|_{L^2(L^2(\Omega)^d)}.\end{aligned}$$

The representative of this functional in $L^2(L^2(\Omega)^d)$ is y. This implies the desired result. □

The existence of a solution of the control problem (5.3.10) follows by Proposition 2.2.6.
Further, we obtain superlinear convergence in case of optimal Dirichlet boundary control of the strongly damped Lamé system.

Theorem 5.3.7. *The semi-smooth Newton method applied to the optimal Dirichlet boundary control problem* (5.3.10) *of the strongly damped Lamé system converges superlinearly.*

Proof. In this case there holds

$$q(u) = -\sigma(p(u))\cdot n + \rho\sigma(p(u)_t)\cdot n,$$

where $p = p(u)$ is the solution of the corresponding adjoint equation for given control u. From Theorem 5.3.5 we obtain

$$p_t \in H^1(L^2(\Omega)^d) \cap L^2(H^2(\Omega)^d) \tag{5.3.21}$$

and hence,

$$p_t \in L^r(L^2(\Omega)^d) \cap L^2(H^2(\Omega)^d)$$

for $1 \leq r \leq \infty$. Thus, from Proposition 4.3.1 we further derive

$$p_t \in L^{q_s}([L^2(\Omega)^d, H^2(\Omega)^d]_s) = L^{q_s}(H^{2s}(\Omega)^d), \quad \frac{1}{q_s} = \frac{s}{2} + \frac{1-s}{r}, \quad s \in [0,1],$$

where the interpolation is understood by components. Let $s \in \left(\frac{3}{4}, 1\right]$, then we have

$$\partial_i p_t \in L^{q_s}(H^{2s-1}(\Omega)^d), \quad i = 1, \ldots, d,$$

and on the boundary

$$\partial_i p_t|_\Sigma \in L^{q_s}(H^{2s-\frac{3}{2}}(\partial\Omega)^d), \quad i = 1, \ldots, d.$$

According to Adams [1, Thm. 7.58] there holds the following embedding for $s \in \left(\frac{3}{4}, 1\right]$

$$H^{2s-\frac{3}{2}}(\partial\Omega) \hookrightarrow L^{\frac{2d-2}{d-4s+2}}(\partial\Omega) \quad \text{for } d \geq 3, \tag{5.3.22}$$

i.e. for $d = 3$ we have

$$\partial_i p_t|_\Sigma \in L^{q_s}\left(L^{\frac{4}{5-4s}}(\partial\Omega)^3\right).$$

From the condition

$$q_s = \frac{2r}{sr + 2(1-s)} = \frac{4}{5-4s}, \quad r < \infty,$$

we have

$$s = \frac{10r-8}{12r-8} > \frac{3}{4}$$

for $2 < r < \infty$, which implies

$$q_s = \frac{12r-8}{5r-2} \to \frac{12}{5} \quad (r \to \infty).$$

So, we obtain

$$\sigma(p_t) \cdot n \in L^q(L^q(\partial\Omega)^3) \tag{5.3.23}$$

for $2 \leq q < \frac{12}{5}$. For $d = 2$ there holds, cf. Adams [1, Thm. 7.58]

$$H^{2s-\frac{3}{2}}(\partial\Omega) \hookrightarrow L^{\frac{1}{2-2s}}(\partial\Omega), \quad s \in \left(\frac{3}{4}, 1\right).$$

Further

$$q_s = \frac{2r}{sr + 2(1-s)} = \frac{1}{2-2s}$$

implies

$$s = \frac{4r-2}{5r-2} > \frac{3}{4},$$

for $2 < r < \infty$ and hence,

$$q_s = \frac{5r-2}{2r} \to \frac{5}{2} \quad (r \to \infty).$$

So, we finally obtain

$$\sigma(p_t) \cdot n \in L^q(L^q(\partial\Omega)^2) \tag{5.3.24}$$

for $2 \leq q < \frac{5}{2}$.

Accordingly, we derive that $\sigma(p) \cdot n$ has in particular the regularity as $\sigma(p_t) \cdot n$ presented in (5.3.23) and (5.3.24) for $d \geq 3$ and $d = 2$, respectively.

In conclusion, we derive superlinear convergence as in Corollary 4.3.15 for $d = 2, 3$. □

5.3.4 Discretization

We discretize the three problems (5.3.2), (5.3.3), (5.3.6), and (5.3.10) according to Section 4.4.

Let

$$\begin{aligned} V_h &= V_h^1 && \text{if } V = H^1(\Omega)^d, \\ X_{kh}^{r,s,a} &= X_{kh}^{r,s} && \text{if } V = H^1(\Omega)^d, \\ X_{kh}^{r,s,b} &= X_{kh}^{r,s} && \text{if } V = H_0^1(\Omega)^d. \end{aligned}$$

Further, we set

$$W_h = \left\{ w_h \in H^{\frac{1}{2}}(\partial\Omega)^d \;\middle|\; w_h = \gamma(v_h), v_h \in V_h \right\}$$

with the trace operator $\gamma\colon H^1(\Omega)^d \to H^{\frac{1}{2}}(\partial\Omega)^d$. We introduce the bilinear form

$$\begin{aligned} &a_\rho\colon X_{k,h}^{1,1,a} \times X_{k,h}^{1,1,a} \times \widetilde{X}_{k,h}^{0,1,a} \times \widetilde{X}_{k,h}^{0,1,a} \longrightarrow \mathbb{R} \\ &a_\rho(y,\xi) = a_\rho(y^1,y^2,\xi^1,\xi^2) = (\partial_t y^2, \xi^1)_I + \lambda(\operatorname{div} y^1, \operatorname{div} \xi^1)_I + 2\mu(\mathcal{D}y^1 : \mathcal{D}\xi^1)_I \\ &\quad + \rho\lambda(\operatorname{div} y^2, \operatorname{div} \xi^1)_I + 2\rho\mu(\mathcal{D}y^2 : \mathcal{D}\xi^1)_I \\ &\quad + (\partial_t y^1, \xi^2)_I - (y^2, \xi^2)_I + (y^2(0), \xi^1(0)) - (y^1(0), \xi^2(0)) \end{aligned}$$

with $y = (y^1, y^2)$ and $\xi = (\xi^1, \xi^2)$ and $\rho \geq 0$. Then the discrete problems are given as in Sections 4.4.1 - 4.4.3.

5.3.5 Numerical examples

We present examples for distributed, Neumann boundary and Dirichlet boundary control. Thereby, we consider the case $d = 2$ on the unit square $\Omega = [0,1]^2$.

Distributed control

In this numerical example we consider the distributed optimal control problem (5.3.2). Let the data be given as follows

$$y_0(x) = \begin{pmatrix} \sin(\pi x_1)\sin(\pi x_2) \\ 0 \end{pmatrix}, \qquad y_1(x) = \begin{pmatrix} x_1 x_2 (1-x_1)(1-x_2) \\ 0 \end{pmatrix},$$

$$y_{\mathrm{d}}(t,x) = 1, \qquad f(t,x) = \begin{cases} (0, 0.5)^T, & x_2 < 0.5, t < 0.5, \\ (1, 0.5)^T, & x_2 > 0.5, t > 0.5, \\ (0,0)^T, & \text{else} \end{cases}$$

$$\alpha = 3 \cdot 10^{-4}, \quad T = 1, \qquad u_a = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \; u_b = \begin{pmatrix} 2.1 \\ 2.1 \end{pmatrix}, \quad \mu = 1, \quad \lambda = 1$$

for $(t,x) = (t, x_1, x_2) \in [0,T] \in \Omega$.

In Table 5.2 the numbers of PDAS-iterations are shown for a sequence of uniformly refined meshes. Here, N denotes the number of cells in the spatial mesh $\mathcal{T}_h$ and M denotes

Level	N	M	PDAS steps
1	16	4	7
2	64	8	6
3	256	16	6
4	1024	32	6
5	4096	64	5

Table 5.2: Numbers of PDAS iterations on a sequence of uniformly refined meshes for control problem (5.3.2)

the number of time intervals. The results indicate a mesh-independent behavior of the PDAS-algorithm.

To analyze the convergence behavior of the PDAS method we proceed as in Section 4.5.1 For the fixed discretization with 64 intervals and a spatial mesh with 4096 cells at each time node Table 5.3 depicts the error of the PDAS-iteration. The results presented demonstrate superlinear convergence.

i	1	2	3	4	5
e_i	$2.1 \cdot 10^{-1}$	$6.3 \cdot 10^{-2}$	$7.0 \cdot 10^{-3}$	$2.3 \cdot 10^{-4}$	0
e_{i+1}/e_i	$3.0 \cdot 10^{-1}$	$1.1 \cdot 10^{-1}$	$3.4 \cdot 10^{-2}$	0	-

Table 5.3: Superlinear convergence of the PDAS-method for distributed control - PDAS-iteration error

Neumann boundary control

In this numerical example we consider the Neumann boundary control problem (5.3.3). Let the data be given as follows.

$$y_0(x) = \begin{pmatrix} \sin(\pi x_1)\sin(\pi x_2) \\ 1 \end{pmatrix}, \qquad y_1(x) = \begin{pmatrix} \sin(\pi x_1)\sin(\pi x_2) \\ x_1 \end{pmatrix},$$

$$y_\mathrm{d}(t,x) = \begin{cases} 1, & x_1 > 0.5, \\ 0, & \text{else}, \end{cases} \qquad f(t,x) = (0,0)^T,$$

$$\alpha = 10^{-2}, \quad T = 1, \qquad u_a = \begin{pmatrix} -0.8 \\ -0.8 \end{pmatrix},\ u_b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mu = 1, \quad \lambda = 1$$

for $(t,x) = (t,x_1,x_2) \in [0,T] \in \Omega$. Table 5.4 shows the numbers of PDAS steps on a sequence of uniform refined meshes.

On a time mesh with 32 intervals and a spatial mesh at each time point with 4096 spatial nodes the development of the error is presented in Table 5.5 confirming superlinear convergence.

Level	N	M	PDAS steps
1	16	2	5
2	64	4	5
3	256	8	4
4	1024	16	5
5	4096	32	5

Table 5.4: Numbers of PDAS iterations on a sequence of uniformly refined meshes for control problem (5.3.2)

i	1	2	3	4	5
e_i	$4.9 \cdot 10^{-2}$	$9.5 \cdot 10^{-3}$	$2.3 \cdot 10^{-3}$	$3.6 \cdot 10^{-4}$	0
e_{i+1}/e_i	$1.9 \cdot 10^{-1}$	$2.4 \cdot 10^{-1}$	$1.6 \cdot 10^{-1}$	0	-

Table 5.5: Superlinear convergence of the PDAS-method for Neumann boundary control - PDAS iteration error

Dirichlet boundary control

In this numerical example we consider the Dirichlet optimal control problems (5.3.6) and (5.3.10). Let the data be given as follows

$$y_0(x) = y_1(x) = (0,0)^T \qquad y_{\mathrm{d}}(t,x) = \begin{cases} x_1, & x_1 > 0.5, \\ -x_1, & \text{else}, \end{cases} \qquad f(t,x) = (x_1^2, t)^T,$$

$$\alpha = 10^{-3}, \quad T = 1, \qquad u_a = \begin{pmatrix} -0.18 \\ -0.18 \end{pmatrix}, \quad u_b = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix}, \qquad \mu = 1, \quad \lambda = 1$$

for $(t,x) = (t, x_1, x_2) \in [0,T] \in \Omega$.

Table 5.6 shows the numbers of PDAS steps on a sequence of uniform refined meshes. On a time mesh with 32 intervals and a spatial mesh at each time point with 4096 nodes

Level	N	M	$\rho = 0$	$\rho = 0.1$
1	16	2	5	4
2	64	4	4	5
3	256	8	6	3
4	1024	16	9	4
5	4096	32	12	5

Table 5.6: Numbers of PDAS iterations on a sequence of uniformly refined meshes for control problem (5.3.2)

the development of the error for $\rho = 0$ and $\rho = 0.1$ is presented in Table 5.7 and Table 5.8. Comparing the control problems with and without damping we see a reduction of the numbers of PDAS steps in case of $\rho > 0$ which corresponds to the results for Dirichlet control of the wave equation, cf. Section 4.5.3.

i	1	2	3	4	5	6
e_i	$5.0 \cdot 10^{-2}$	$7.2 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$	$9.9 \cdot 10^{-3}$	$5.5 \cdot 10^{-3}$	$4.2 \cdot 10^{-3}$
e_{i+1}/e_i	1.4	$2.6 \cdot 10^{-1}$	$5.4 \cdot 10^{-1}$	$5.5 \cdot 10^{-1}$	$7.7 \cdot 10^{-1}$	$8.0 \cdot 10^{-1}$

i	7	8	9	19	11	12
e_i	$3.4 \cdot 10^{-3}$	$2.5 \cdot 10^{-3}$	$1.8 \cdot 10^{-3}$	$1.1 \cdot 10^{-3}$	$3.1 \cdot 10^{-4}$	0
e_{i+1}/e_i	$7.4 \cdot 10^{-1}$	$7.0 \cdot 10^{-1}$	$6.0 \cdot 10^{-1}$	$3.0 \cdot 10^{-1}$	0	-

Table 5.7: Dirichlet boundary control without damping, $\rho = 0$ - PDAS-iteration error

i	1	2	3	4	5
e_i	$3.1 \cdot 10^{-1}$	$5.1 \cdot 10^{-2}$	$8.9 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$	0
e_{i+1}/e_i	$1.6 \cdot 10^{-1}$	$1.8 \cdot 10^{-1}$	$1.3 \cdot 10^{-1}$	0	-

Table 5.8: Superlinear convergence of the PDAS-method for Dirichlet boundary control with $\rho = 0.1$ - PDAS iteration error

6 Controllability of wave equations

In this chapter we consider an exact controllability problem for the wave equation. Roughly speaking exact controllability of the wave equation means, to analyze whether the solution of the wave equation can be driven to a final target exactly by a control acting on the boundary or a subdomain of the domain the equation evolves in. This is different to the corresponding optimal control problem, where the requirement of achieving the final target exactly is relaxed due to the term describing control costs.

In contrast to optimal control of wave equations, there exist many publications on controllability of the wave equation. For an overview we refer the reader to the review article by Zuazua [141] and for an introduction to the topic to the monograph by Lions [90].

In this chapter we recall some main results from the literature on the numerical treatment of an exact controllability problem and confirm by a numerical example that the discretization of this problem may lead to spurious solutions. These effects also appear when solving certain optimal control problems for wave equations with small Tikhonov parameter. Therefore, it is important to bear these effects in mind when considering optimal control problems for wave equations.

We start the consideration with an introduction of the continuous exact controllability problem and proceed with the relation of this problem to the corresponding optimal control problem for the Tikhonov parameter tending to zero. Further, we recall that the numerical approximation schemes, which are stable for solving the initial-boundary value problem, may lead to instabilities when they are applied to the exact controllability problem. The reason for this, is the fact that the spurious high frequency discrete solutions cannot be controlled uniformly as the mesh parameter tends to zero; cf. Zuazua [141, 139, 138, 140]. Different methods have been developed to tackle these difficulties as Fourier filtering, bi-grid or mixed finite elements, (cf. Zuazua [141], Glowinski [47]) . The convergence of the bi-grid method for finite difference methods was considered in Ignat & Zuazua [61] and for finite element methods in Negreanu & Zuazua [110], Negreanu [109]. In the end of this chapter we compare the numerical solution of the exact controllability problem with the solution of the corresponding optimal control problem.

This chapter is organized as follows. In Section 6.1 we introduce the continuous problem. In Section 6.2 we formulate the optimality system, in Section 6.3 we discuss the relation between exact controllability and optimal control, in Section 6.4 we discretize the problem and recall the difficulties arising from the discretization, in Section 6.5 we present some numerical examples, and in Section 6.6 we give an outlook.

6.1 Continuous problem

In this section we recall some definitions and basic results on controllability; cf. Lions [87], Micu & Zuazua [104], Lasiecka & Triggiani [84], Mariegaard [94].

Let $\Gamma_0 \subset \partial\Omega$ be open and nonempty. We consider solutions $y \in C(L^2(\Omega)) \cap C(H^{-1}(\Omega))$ of the following equation

$$\begin{cases} y_{tt} - \Delta y = 0, & \text{in } Q, \\ y(0) = z_0, & \text{in } \Omega, \\ y_t(0) = z_1, & \text{in } \Omega, \\ y = 0, & \text{on } I \times \partial\Omega \setminus \Gamma_0, \\ y = v, & \text{on } I \times \Gamma_0 \end{cases} \tag{6.1.1}$$

with initial data $z_0 \in L^2(\Omega)$ and $z_1 \in H^{-1}(\Omega)$ and boundary control $v \in L^2(\Omega)$. From Chapter 2 we directly obtain that there exists a uniquely determined solution.

Further, we define

$$E = H^1(\Omega) \times L^2(\Omega), \quad E^* = L^2(\Omega) \times H^{-1}(\Omega).$$

To distinguish three different forms of controllability we introduce the set of all values of the state and velocity at the final datum T

$$\mathcal{R}(T; (z_0, z_1)) = \{ (y(T), y_t(T)) : y \text{ is solution of (6.1.1) with } v \in L^2(L^2(\Gamma_0)) \} .$$

Now, we can make the following definitions: Equation (6.1.1) is called

- approximately controllable in time T if
$$\forall (z_0, z_1) \in L^2(\Omega) \times H^{-1}(\Omega) : \mathcal{R}(T; (y_0, y_1)) \text{ dense in } L^2(\Omega) \times H^{-1}(\Omega). \tag{6.1.2}$$
- exactly controllable in time T if
$$\forall (z_0, z_1) \in L^2(\Omega) \times H^{-1}(\Omega) : \mathcal{R}(T; (y_0, y_1)) = L^2(\Omega) \times H^{-1}(\Omega). \tag{6.1.3}$$
- null controllable in time T if
$$\forall (z_0, z_1) \in L^2(\Omega) \times H^{-1}(\Omega) : (0, 0) \in \mathcal{R}(T; (y_0, y_1)). \tag{6.1.4}$$

Remark 6.1.1. Since the system (6.1.1) is linear and reversible in time null controllability is equivalent to exact controllability, cf. Zuazau [139].

Thus we can formulate the exact controllability problem as follows.

Definition 6.1.2 (Exact controllability problem)**.** Find for given initial data $z_0 \in L^2(\Omega)$, $z_1 \in H^{-1}(\Omega)$ a control $v \in L^2(\Sigma)$, such that for the solution y of (6.1.1) holds

$$y(T) = y_t(T) = 0.$$

Remark 6.1.3. If $y(t_0) = y_t(t_0) = 0$ for some $t_0 \geq 0$, then we can choose $v(t) = 0$ for all $t \geq t_0$ and obtain

$$y(t) = y_t(t) = 0, \quad \forall t \geq t_0.$$

Remark 6.1.4. Since the propagation of waves is finite, we can only expect exact controllability for $T > 0$ sufficiently large.

To formulate an optimization problem, whose solution solves the exact controllability problem, we introduce the adjoint system.

Adjoint system

Let for $(p^0, p^1) \in E$ the function $p \in C(H^1(\Omega)) \cap C^1(L^2(\Omega))$ be the solution of the following equation

$$\begin{cases} p_{tt} - \Delta p = 0 & \text{in } Q, \\ p(0) = p_0 & \text{in } \Omega, \\ p_t(0) = p_1 & \text{in } \Omega, \\ p = 0 & \text{on } \Sigma. \end{cases} \tag{6.1.5}$$

The following results are taken from Micu & Zuazua [104].

Theorem 6.1.5. *The initial data $(z_0, z_1) \in E^*$ is controllable to zero if and only if there exists $v \in L^2((0,T) \times \Gamma_0)$, such that*

$$\int_0^T \int_{\Gamma_0} (\partial_n p\, v) d\sigma dt + \int_\Omega (z_0 p_t(0)) dx - \langle z_1, p(0) \rangle_{H^{-1}(\Omega), H^1(\Omega)} = 0, \tag{6.1.6}$$

for all $(p_0^T, p_1^T) \in E$, where p is the solution of the backward equation

$$\begin{cases} p_{tt} - \Delta p = 0 & \textit{in } Q, \\ p(T) = p_0^T & \textit{in } \Omega, \\ p_t(T) = p_1^T & \textit{in } \Omega, \\ p = 0 & \textit{on } \Sigma. \end{cases} \tag{6.1.7}$$

Proof. By integration by parts we obtain for smooth functions

$$\begin{aligned} 0 &= \int_0^T \int_\Omega p(y_{tt} - \Delta y) dx dt, \\ &= \int_\Omega (p y_t - p_t y) dx \Big|_0^T + \int_0^T \int_{\Gamma_0} \left(-(\partial_n y) p + (\partial_n p) y \right) d\sigma dt \\ &= \int_0^T \int_{\Gamma_0} (\partial_n p y) d\sigma dt + \int_\Omega (p_0^T y_t(T) - p_1^T y(T)) dx - \int_\Omega (p(0) z_1 - p_t(0) z_0) dx. \end{aligned}$$

As a direct consequence we have that if (z^0, z^1) is controllable to zero, then there holds relation (6.1.6). Conversely if (6.1.6) is true, then we can drive (z^0, z^1) to zero by choosing $\partial_n p$ as the control. □

Since the wave equation is reversible in time we can draw the following conclusion.

Corollary 6.1.6. *The initial data $(z_0, z_1) \in E^*$ is controllable to zero if and only if there exists $v \in L^2((0,T) \times \Gamma_0)$ such that*

$$\int_0^T \int_{\Gamma_0} (\partial_n p v) d\sigma dt + \int_\Omega (z_0 p_1) dx - \langle z_1, p_0 \rangle_{H^{-1}(\Omega), H^1(\Omega)} = 0,$$

for all $(p_0, p_1) \in E$, where p is the solution of (6.1.5).

We define the functional $J\colon H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ with

$$J(p_0, p_1) = \frac{1}{2} \int_0^T \int_{\Gamma_0} |\partial_n p|^2 \, d\sigma dt + \langle (z_0, z_1), (p_0, p_1) \rangle_{E^*,E}, \tag{6.1.8}$$

where p is the solution of (6.1.5) with initial data $(p^0, p^1) \in E$ and

$$\langle (z_0, z_1), (p_0, p_1) \rangle_{E^*,E} = \int_\Omega (z_0 p_1) dx - \langle z_1, p_0 \rangle_{H^{-1}(\Omega), H^1(\Omega)}.$$

By the hidden regularity (cf. Theorem 2.3.10) the normal derivative $\partial_n p$ is well-defined.

Definition 6.1.7 (Observability)**.** Let $T \geq 0$. Then for equation (6.1.5) observability in time T is given if there exists a positive constant $C(T) > 0$, such that for all $(p_0, p_1) \in E$

$$\|p_0\|^2_{H^1(\Omega)} + \|p_1\|^2_{L^2(\Omega)} \leq C(T) \int_0^T \int_{\Gamma_0} |\partial_n p|^2 \, d\sigma dt. \tag{6.1.9}$$

Remark 6.1.8. The inequality (6.1.9) means that the energy can be estimated by observations on the boundary during the time interval and uniformly in the whole class of solutions.

Observability implies the existence of a minimizer of J.

Theorem 6.1.9. *Let the system* (6.1.5) *be observable in time* T *and* $(z_0, z_1) \in E^*$*, then* J *has a unique minimizer* $(\bar{p}_0, \bar{p}_1) \in E$.

Proof. The functional J is continuous and strictly convex. Further, J is coercive, i.e.

$$\lim_{\|(p_0,p_1)\|_E \to \infty} \mathcal{J}(p_0, p_1) = \infty,$$

since

$$\begin{aligned} \mathcal{J}(p_0, p_1) &\geq \frac{1}{2} \left(\int_0^T \int_{\Gamma_0} |\partial_n p|^2 d\sigma - \|(z_0, z_1)\|_{E^*} \|(p_0, p_1)\|_E \right) \\ &\geq \frac{C}{2} \|(p_0, p_1)\|^2_E - \frac{1}{2} \|(z_0, z_1)\|_{E^*} \|(p_0, p_1)\|_E. \end{aligned}$$

Hence, $\mathcal{J}$ has a minimizer.

The operator mapping the initial data (p_0, p_1) to the normal derivative of the corresponding solution p is injective, since the system is observable. Thus, the functional is strict convex and we obtain uniqueness of the minimizer. □

There holds the following relation between exact controllability and observability.

Proposition 6.1.10. *Let for given* $(z^0, z^1) \in E^*$ *the pair* $(\bar{p}^0, \bar{p}^1) \in E$ *be a minimizer of* J *and* $\bar{p}$ *be the solution with initial values* $(\bar{p}^0, \bar{p}^1)$*, then*

$$v = \partial_n \bar{p} \quad \text{on } \Gamma_0$$

is a control which leads (z^0, z^1) *to zero in time* T.

Proof. The functional J takes its minimum in $(\bar{p}_0, \bar{p}_1)$, hence there holds

$$0 = \lim_{h \to 0} \frac{1}{h}(J((\bar{p}_0, \bar{p}_1) + h(p_0, p_1)) - J((\bar{p}_0, \bar{p}_1))$$
$$= \frac{1}{2} \int_0^T \int_{\Gamma_0} \partial_n \bar{p} \partial_n p d\sigma dt + \langle (z_0, z_1), (p_0, p_1) \rangle_{E^*, E},$$

for arbitrary $(p_0, p_1) \in E$ and the corresponding solution p of (6.1.5). Corollary 6.1.6 implies that $v = \partial_n \bar{p}$ is a control leading (z_0, z_1) to zero in time T. □

The control, which is determined by minimizing the functional (6.1.8), is that one with minimal $L^2((0,T) \times \Gamma_0)$-norm and is called the HUM control; see. [104].

Theorem 6.1.11. *Let $v \in L^2(L^2(\Gamma_0))$ be the control given by minimizing the functional J, then for any other control $g \in L^2(L^2(\Gamma_0))$ driving to zero the initial data (z_0, z_1) in time T there holds*

$$\|v\|_{L^2(L^2(\Gamma_0))} \leq \|g\|_{L^2(L^2(\Gamma_0))}.$$

For the remaining part of this section, we want to consider the one-dimensional case, i.e. $d = 1$, $\Omega = (0,1)$ and let the control only act on the right end ($x = 1$) of the spatial interval. Then there holds the following result concerning observability; see [141].

Proposition 6.1.12. *For any $T \geq 2$, $d = 1$, system (6.1.5) is observable, i.e. for any $T \geq 2$ there exists $C(T) > 0$ such that the observability condition holds. Conversely, if $T < 2$, the system is not observable, or, equivalently,*

$$\sup_{(p_0,p_1) \in E} \left\{ \frac{\left(\|p_0\|^2_{H^1(\Omega)} + \|p_1\|^2_{L^2(\Omega)} \right)}{\|p_x(1,t)\|^2_{L^2(0,T)}} \right\} = \infty.$$

Remark 6.1.13. When considering control problems in several space dimensions, one has to take into account the *geometric control condition*; see Zuazua [141], which says that all rays of geometric optics propagating in Ω and being reflected on the boundary $\partial\Omega$ enter the control domain Γ in time less than T.

Remark 6.1.14. The minimal time T, for which observability is given, is called the characteristic time. We use this expression also in case of several space dimensions.

For T too small (i.e. here $T < 2$) observability is not given; see Proposition 6.1.12. This can be seen in the next example, taken from Ervedoza & Zuazua [39].

Example 6.1.15. For $0 \leq T < 2$ the observability condition (6.1.9) does not hold. Let $T = 2 - 2\delta$ with $\delta \in (0,2)$ and consider

$$p_{tt} - p_{xx} = 0, \quad (t,x) \in (0,T) \times (0,1), \quad p(t,0) = p(t,1) = 0, \quad 0 < t < T$$

with given data for $p\left(\frac{T}{2} = 1 - \delta, x\right)$ and $x \in (0, \delta)$. We have

$$y_x(t,1) = 0$$

for $0 < t < T = 2 - 2\delta$, since the sets $\{ (t,1),\ t \in (0,T) \}$ and $\{ (\frac{T}{2}, x),\ x \in (0,\delta) \}$ are disjunct.

6.2 The Hilbert uniqueness method

Lions proposed the Hilbert Uniqueness Methods (HUM), to solve exact controllability problems. We recall this approach for several space dimensions.

The Hilbert Uniqueness Method

There exist a lot of publications considering different aspects of the HUM approach, we refer the reader to Glowinski [47], Lions [90, 89] and Zuazua [141].

We introduce the following operator

$$\Lambda \colon E \longrightarrow E^*, \quad e \longmapsto (-y(0), y_t(0)),$$

where y is the solution of

$$\begin{cases} y_{tt} - \Delta y = 0 & \text{in } Q, \\ y(T) = 0 & \text{in } \Omega, \\ y_t(T) = 0 & \text{in } \Omega, \\ y = 0 & \text{in } I \times \partial\Omega \setminus \Gamma_0, \\ y = \partial_n p & \text{on } I \times \Gamma_0 \end{cases} \tag{6.2.1}$$

for given p determined as the solution of

$$\begin{cases} p_{tt} - \Delta p = 0 & \text{in } Q \\ p(0) = p_0 & \text{in } \Omega, \\ p_t(0) = p_1 & \text{in } \Omega, \\ p = 0 & \text{on } \Sigma \end{cases} \tag{6.2.2}$$

for $e = (p_0, p_1)$.

That means, for given $f = (-z_0, z_1)$ we look for a pair $e = (p_0, p_1) \in E$ with

$$\Lambda e = f. \tag{6.2.3}$$

The operator Λ has the following properties; cf. Lions [90, 89].

Theorem 6.2.1. *The operator $\Lambda \colon E \to E^*$ is linear and continuous. For T larger than the characteristic time (cf. Remark 6.1.14) Λ is an isomorphism.*

Remark 6.2.2. The HUM approach applied to an exact boundary controllability problem bases on the following steps:

- Suppose $(z_0, z_1) \in E^*$. Take $f = (-z_0, z_1)$.
- Solve (6.2.3) and obtain e.
- Solve the adjoint wave equation (6.2.2) for initial conditions e.
- Solve the associated wave equation (6.2.1) and obtain $y(0) = z_0$, $y_t(0) = z_1$.

Remark 6.2.3. Equation (6.2.3) has been considered for as the basis for numerical discretizations of this problem, cf. Glowinski [47].

According to Section 6.1 the equations (6.2.1) and (6.2.2) can be derived as the optimality system of the following optimization problem

$$\begin{cases} \text{Minimize} \quad J(u_0,u_1) = \dfrac{1}{2}\,\|\partial_n p\|^2_{L^2(L^2(\Gamma_0))} + \langle (z_0,z_1),(p_0,p_1)\rangle_{E^*,E}, \quad (u_0,u_1)\in E, \quad \text{s.t.} \\ \qquad p_{tt} - \Delta p = 0 \quad \text{in } Q, \\ \qquad p(0) = u_0 \quad \text{in } \Omega, \\ \qquad p_t(0) = u_1 \quad \text{in } \Omega, \\ \qquad p = 0 \quad \text{on } \Sigma. \end{cases} \tag{6.2.4}$$

Thus, for sufficiently smooth solutions we obtain the optimality system as follows

$$\begin{cases} p_{tt} - \Delta p = 0, \\ p(0) = u_0, \quad p_t(0) = u_1 \quad p|_\Sigma = 0, \\ y_{tt} - \Delta y = 0, \\ y(T) = 0, \quad y_t(T) = 0, \quad y|_{I\times\partial\Omega\setminus\Gamma_0} = 0, \quad y = \partial_n p|_{I\times\Gamma_0}, \\ y(0) = z_0, \quad y_t(0) = z_1. \end{cases}$$

6.3 Relation to optimal control

Now, we consider an optimal Dirichlet boundary control problem and analyze its relation to the exact controllability problem.

The optimal Dirichlet boundary control problem is given by

$$\begin{cases} \text{Minimize} \quad J(u,y) = \dfrac{1}{2}\left(\|y(T)\|^2_{L^2(\Omega)} + \|y_t(T)\|^2_{H^{-1}(\Omega)}\right) + \dfrac{\alpha}{2}\,\|u\|^2_{L^2(L^2(\Gamma_0))} \\ \qquad y \in C(L^2(\Omega)) \cap C^1(H^{-1}(\Omega)),\ u \in L^2(L^2(\Gamma_0)), \quad \text{s.t.} \\ \qquad y_{tt} - \Delta y = 0 \quad \text{in } Q, \\ \qquad y(0) = z_0 \quad \text{in } \Omega, \\ \qquad y_t(0) = z_1 \quad \text{in } \Omega, \\ \qquad y = 0 \quad \text{on } I\times\partial\Omega\setminus\Gamma_0. \\ \qquad y = u \quad \text{on } I\times\Gamma_0. \end{cases} \tag{6.3.1}$$

The reduced problem is given by

$$\text{Minimize} \ \ j(u), \quad u \in L^2(L^2(\Gamma_0))$$

with the convex, continuous and coercive reduced functional $j\colon U \to \mathbb{R}$, $j(u) = J(u, S(u))$. Thus, existence of a solution for $\alpha > 0$ follows immediately by Proposition 2.2.6. The optimality system in a strong formulation reads as

$$\begin{cases} y_{tt} - \varDelta y = 0, \\ y(0) = z_0, \quad y_t(0) = z_1, \quad y|_{I\times\partial\varOmega\setminus\varGamma_0} = 0, \quad y|_{I\times\varGamma_0} = u. \\ p_{tt} - \varDelta p = 0, \\ p(T) = (-\varDelta)^{-1} y_t(T), \quad p_t(T) = y(T), \quad p|_{\varSigma} = 0, \\ \alpha u = -\partial_n p|_{I\times\varGamma_0}. \end{cases}$$

For α tending to zero there holds the following relation between the exact controllability problem and the optimal control problem.

Theorem 6.3.1. *Let* (6.1.1) *be exact controllable in the sense of* (6.1.3)*. Let y_α denote the solution of* (6.3.1) *corresponding to the parameter α. Then for every sequence $(\alpha_k)_{k\in N}$ with $\alpha_k \to 0$ for $k \to \infty$, we can select a subsequence $(\alpha_k)_{k\in N}$, such that*

$$y_{\alpha_k}(T) \rightharpoonup 0 \quad \textit{weakly in } L^2(\varOmega), \quad y_{\alpha_k,t}(T) \rightharpoonup 0 \quad \textit{weakly in } H^{-1}(\varOmega)$$

for α_k tending to 0*.*

Proof. Here we use techniques from Fernández & Zuazua [42], where distributed control problems for parabolic equations are considered. To shorten notations we write α instead of α_k. Let us consider the reduced cost functional in dependence of $\alpha > 0$

$$j_\alpha(u) = \frac{1}{2}\left(\|y(T)\|^2_{L^2(\varOmega)} + \|y_t(T)\|^2_{H^{-1}(\varOmega)}\right) + \frac{\alpha}{2}\|u\|^2_{L^2(L^2(\varGamma_0))}, \quad u \in L^2(L^2(\varGamma_0)).$$

In the following let u_α denote the optimal control. Then u_α satisfies for all $\delta u \in L^2(L^2(\varGamma_0))$

$$j'_\alpha(u_\alpha)(\delta u) = (y_\alpha(T), \delta y(T)) + (y_{\alpha,t}(T), \delta y_t(T))_{H^{-1}(\varOmega)} + \alpha\langle u_\alpha, \delta u\rangle_{L^2(L^2(\varGamma_0))} = 0, \quad (6.3.2)$$

where δy solves

$$\begin{cases} \delta y_{tt} - \varDelta\delta y = 0 & \text{in } Q, \\ \delta y(0) = 0 & \text{in } \varOmega, \\ \delta y_t(0) = 0 & \text{in } \varOmega, \\ \delta y = 0 & \text{on } I\times\partial\varOmega\setminus\varGamma_0, \\ \delta y = \delta u & \text{on } I\times\varGamma_0. \end{cases}$$

Further, since u_α is optimal we have

$$j_\alpha(u_\alpha) \le j_\alpha(0)$$

with $j_\alpha(0)$ independent of α. Hence, we obtain by selecting a subsequence if necessary

$$\begin{aligned} y_\alpha(T) &\rightharpoonup \psi \text{ weakly in } L^2(\varOmega), \\ y_{\alpha,t}(T) &\rightharpoonup \phi \text{ weakly in } H^{-1}(\varOmega), \\ \sqrt{\alpha}u_\alpha &\rightharpoonup \bar{u} \text{ weakly in } L^2(\varGamma_0) \end{aligned}$$

for $\psi \in L^2(\Omega)$, $\phi \in H^{-1}(\Omega)$ and $\bar{u} \in L^2(L^2(\Gamma_0))$. For $\alpha \to 0$ we derive from (6.3.2)

$$(\psi, \delta y(T)) + (\phi, \delta y_t(T))_{H^{-1}(\Omega)} = 0 \quad \forall \delta u \in L^2(L^2(\Gamma_0)). \tag{6.3.3}$$

Choose δu so, that $\delta y(T) = \psi$ and $\delta y_t(T) = \phi$; this is possible, since (6.1.1) is exact controllable. Consequently, we obtain

$$\psi = 0 \quad \text{in } L^2(\Omega), \quad \phi = 0 \quad \text{in } H^{-1}(\Omega). \qquad \square$$

6.4 Discretized problem

The problem (6.2.4) is discretized by piecewise linear finite elements in space resulting in a semi-discrete formulation. Here, we only consider the one-dimensional case on the unit interval. We recall briefly some fundamental aspect concerning the difficulties arising from this numerical approximation, for details see Glowinski [47], Zuazua [141], Negreanu [109] and Infante & Zuazua [62].

Let the spatial interval $\Omega = (0,1)$ be divided in $N+1$, $N \in \mathbb{N}$, intervals of the length $h = \frac{1}{N+1}$. Then the semi-discrete problem reads as

$$\int_0^T \partial_{tt} p_h \psi dx = \int_0^1 \partial_x p_h \partial_x \psi dx, \quad 0 < t < T, \quad \forall \psi \in V_h^1 \tag{6.4.1}$$

for

$$p_h(t,x) = \Sigma_{j=1}^N p_j(t)\psi_j(x)$$

with the nodal basis functions ψ_j of V_h^1, which is defined according to Section 3.2.2 for $V = H_0^1(0,1)$. The adjoint semi-discrete equation is given by

$$\begin{aligned} \frac{2}{3}p_{tt,j}(t) + \frac{1}{6}p_{tt,j+1}(t) + \frac{1}{6}p_{tt,j-1}(t) &= \frac{p_{j+1}(t) + p_{j-1}(t) - 2p_j(t)}{h^2}, && 0 < t < T,\ j = 1, \ldots, N, \\ p_0 = p_N &= 0, && 0 < t < T, \\ p_j(T) = p_{0,j}, \quad p_{t,j}(T) &= p_{1,j} && j = 1, \ldots, N \end{aligned} \tag{6.4.2}$$

with $p_j(t) = p_h(t, jh)$, $p_{0,j} = p_0(jh)$ and $p_{1,j} = p_1(jh)$ for $j = 0, \ldots, N+1$. With the semi-discrete system (6.4.2) we associate the semi-discrete energy

$$E_h(t) = \frac{h}{6}\sum_{j=1}^N |p_{t,j}(t)|^2 + \frac{h}{12}\sum_{j=0}^N |p_{t,j}(t) + p_{t,j+1}(t)|^2 + \frac{h}{2}\sum_{j=0}^N \left|\frac{p_{j+1}(t) - p_j(t)}{h}\right|^2.$$

The discrete observability condition is given by

$$E_h(t) \le C_h(T) \int_0^T \left|\frac{p_h(t, Nh)}{h}\right| dt,$$

where p_h is the solution of (6.4.2).

Following Negreanu [109] we recall that on the semi-discrete level non-uniform observability is given. The eigenvalue problem arising from the spatial discretization reads as

$$\begin{aligned} -\frac{\varphi_{k+1}+\varphi_{k-1}-2\varphi_k}{h^2} &= \mu\left(\frac{2}{3}\varphi_k+\frac{1}{6}\varphi_{k+1}+\frac{1}{6}\varphi_{k-1}\right), \quad k=1,\dots,N \\ \varphi_0 &= \varphi_{N+1}=0 \end{aligned} \tag{6.4.3}$$

and the eigenvalues of the discrete operator are given by

$$\mu_k^h = \frac{6}{h^2}\frac{1-\cos(k\pi h)}{2+\cos(k\pi h)}, \quad k=1,\dots,N$$

with eigenvectors

$$w_k^h = (w_{k,1},\dots,w_{k,N})^T, \quad w_{k,j}=\sin(k\pi jh), \quad k,j=1,\dots,N.$$

For the continuous problem the eigenvalues are given by

$$\lambda_k=(k\pi)^2, \quad k\in\mathbb{N}$$

with eigenvectors

$$w_k=\sin(k\pi x), \quad k\in\mathbb{N}.$$

There holds

$$\mu_k^h\to\lambda_k \quad (h\to 0)$$

for fixed k. The observability inequality is uniform if the constant $C_h(T)$ is bounded uniformly in h for $h\to 0$. However, in Infante & Zuazua [62] it was proved using spectral analysis that for all $T>0$ the best constant C_h tends to infinity for $h\to 0$. The highly oscillatory components of the solution lead to this non-uniformity. To remedy this effect one can look for solutions in the set of filtered solutions without the highly oscillatory components. Therefore, the solutions are developed in Fourier series

$$y=\sum_{k=1}^{N}\left(a_k\cos\left(\sqrt{\mu_k^h}t\right)+\frac{b_k}{\sqrt{\mu_k^h}}\sin\left(\sqrt{\mu_k^h}t\right)\right)w_k^h, \tag{6.4.4}$$

where a_k and b_k are given by the initial data

$$y_0=\sum_{k=1}^{N}a_kw_k^h, \quad y_1=\sum_{k=1}^{N}b_kw_k^h \tag{6.4.5}$$

and one looks only for solutions being in the set

$$C_\delta(h)=\left\{p \text{ solution of (6.4.2) s.t. } y=\sum_{k=1}^{[\delta/h]}a_k\cos\left(\sqrt{\mu_k^h}t\right)+\frac{b_k}{\sqrt{\mu_k^h}}\sin\left(\sqrt{\mu_k^h}t\right)w_k^h\right\}$$

for $0<\delta<1$. Then, for any $\delta>0$ there exits a $T(\delta)>0$, such that for all $T>T(\delta)$ there exits a constant $C(T,\delta)$ for which uniform observability is given, see Zuazua [141].

Alternatively, other methods can be applied as bi-grid, mixed finite elements and Tikhonov regularization to get convergent methods. For a discussion in detail we refer to Zuazua [141].

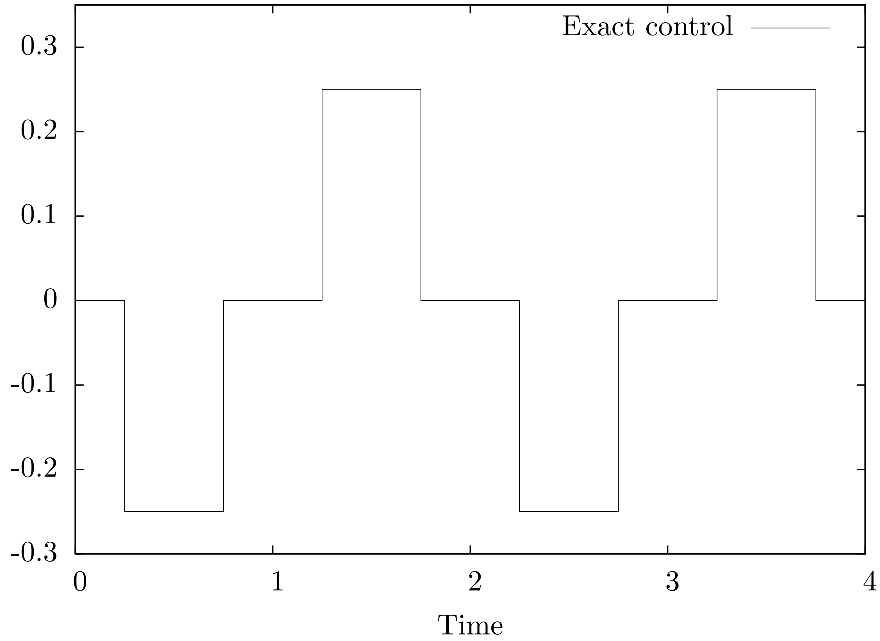


Figure 6.1: Exact control v at $x = 1$ w.r.t. the control problem given in Definition 6.1.2.

6.5 Numerical examples

In this section we solve the exact controllability problem formulated in Definition 6.1.2 numerically. We look for a solution of the control problem with

$$z_0(x) = \begin{cases} -1, & 0.25 \leq x \leq 0.75, \\ 0, & \text{else,} \end{cases} \qquad z_1(x) = 0 \tag{6.5.1}$$

for $x \in \Omega = (0, 1)$ and $T = 4$. This configuration in (6.5.1) is taken from Zuazua [141]. The exact control can be computed explicitly. We consider the extension of the initial data on the whole space $\mathbb{R}$ using d'Alembert's formula taking into account that we have reflection of the waves on the boundaries, i.e. the sign of the solution changes at the boundaries. Then using d'Alembert's formula again we can compute the exact solution and consequently its trace on the boundary, the exact control, which is shown in Figure 6.1.

To solve the exact controllability problem numerically we consider two approaches given by the problems (6.2.4) and (6.3.1). In the latter one we choose $\alpha = 1$. As in the previous chapters we apply a Crank-Nicolson scheme for time discretization.

In Figure 6.2 and 6.3 we compare the normal derivative $\partial_x p_h$ with respect to the discretized problem (6.2.4) at the right end of the spatial interval with the semi-discrete optimal control u_h of the discretized problem (6.3.1). In both cases, the state is computed on a spatial mesh with 4 and 16 cells, respectively, and for the temporal discretization we use 100 time steps. According to the applied discretization, the normal derivative $\partial_x p_h$ is a piecewise linear function in time, whereas u_h is a piecewise constant function in time. To discretize the $H^{-1}(\Omega)$-norm we use the following identity, which holds in arbitrary dimensions d:

$$\|y_t(t)\|_{H^{-1}(\Omega)} = \left\|\nabla(-\Delta)^{-1}y_t(t)\right\|_{L^2(\Omega)} = \|\nabla z\|_{L^2(\Omega)}$$

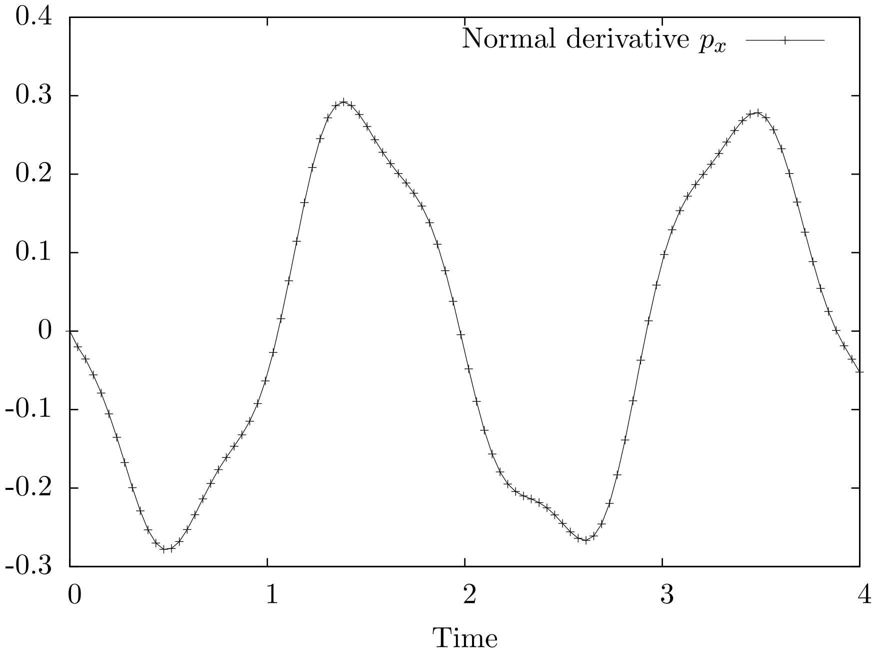
for all $t \in [0,1]$, where z is defined as the solution of

$$\begin{aligned} -\Delta z &= y_t(t) && \text{in } \Omega, \\ z &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{6.5.2}$$

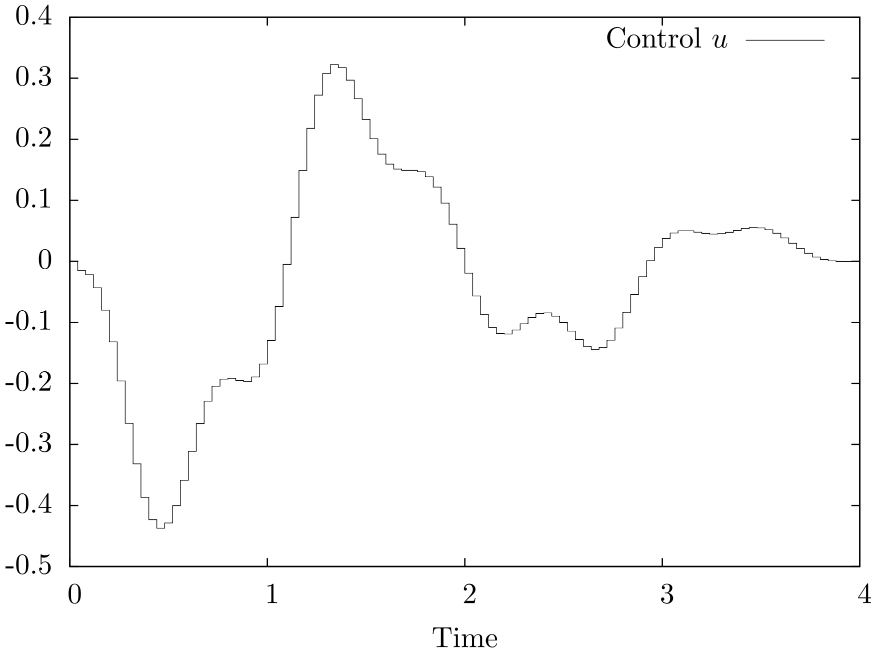
Thus, in this case the reduced cost functional of optimal control problem (6.3.1) is equivalent to

$$j(u) = \|y(T)\|^2_{L^2(\Omega)} + \|\partial_x z\|^2_{L^2(\Omega)} + \alpha\,\|u\|^2_{L^2(L^2(\Gamma_0))} \tag{6.5.3}$$

with z defined as in (6.5.2) for $t = T$. The functions z are discretized as the state y of problem (6.2.4) in space. As in Zuazua [141] we see oscillations of the semi-discrete normal derivative p_x at $x = 1$ with respect to (6.2.4) which rise with the number of spatial nodes. Moreover, similar effects occur for the semi-discrete optimal control of the optimal control problem (6.3.1). However, in this case the exact controllability condition is relaxed by the term describing control costs, i.e. we do not have to drive the oscillations to zero exactly.

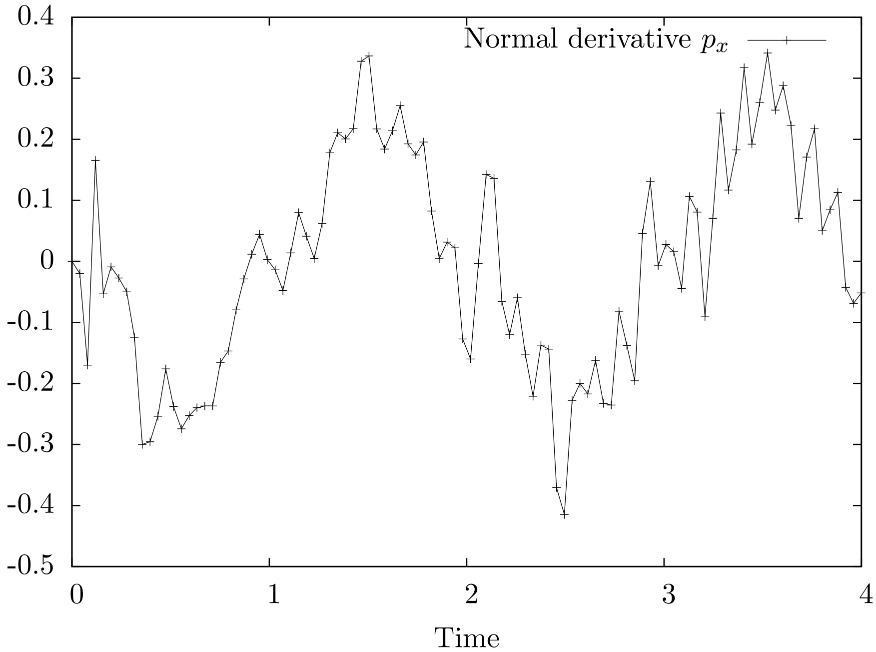


(a) Normal derivative of the adjoint state w.r.t. (6.2.4) at $x = 1$.

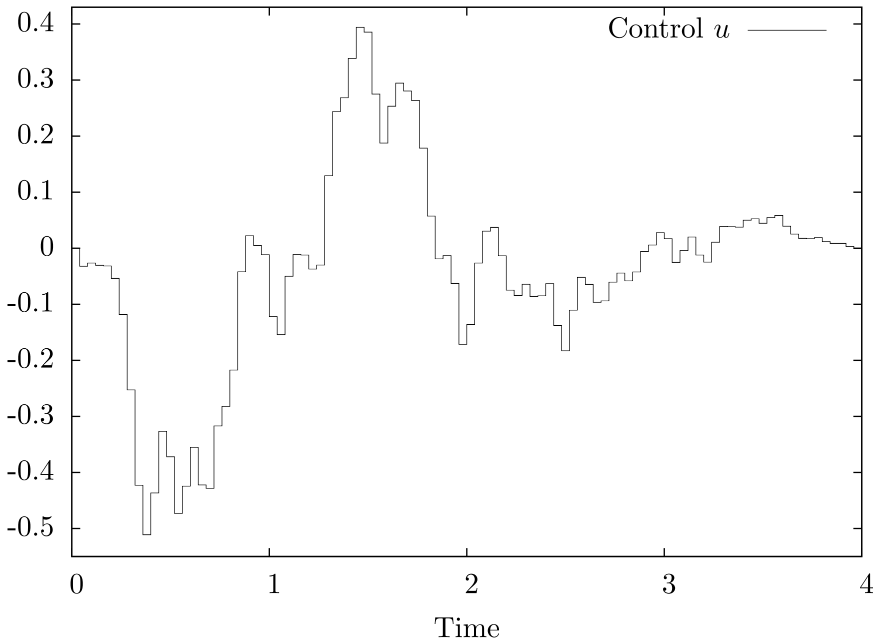


(b) Optimal control of (6.3.1) at $x = 1$.

Figure 6.2: Comparison of the exact controllability problem and the optimal control problem on a mesh with 4 spatial cells.



(a) Normal derivative of the adjoint state w.r.t. (6.2.4) at $x = 1$.



(b) Optimal control of (6.3.1) at $x = 1$.

Figure 6.3: Comparison of the exact controllability problem and the optimal control problem on a mesh with 16 spatial cells.

6.6 Outlook

There are several interesting directions for future research on this topic:

- The optimal control problem (6.3.1) is a relaxed exact controllability problem. It is interesting to improve the assertion of Theorem 6.3.1 and possibly, to derive convergence rates with respect to the parameter α.
- Further, it is interesting to consider the behaviour of the semi-discrete solution and in particular the spurious oscillations in dependence of the number of time steps for the temporal discretization.
- The previous discussions motivate the question for which given initial data z_0 and z_1 it is necessary to apply a method as Fourier filtering or bi-grid to obtain a semi-discrete observability inequality uniformly in h; it may be possible to develop an adaptive algorithm, which decides in dependence of the initial data z_0 and z_1 when some additional smoothing methods have to be applied.
- Considering exact controllability problems the question for time optimality arise, cf. Kunisch and Wachsmuth [134], where a time optimal control problem for the wave equation is considered and regularization techniques as well as its numerical realization is presented.

7 Notation

In this thesis we use the following notations; cf. [91, 92, 119]:

Sets

- $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$
- $I = (0, T)$, $T > 0$
- $Q = I \times \Omega$
- $\Sigma = I \times \partial\Omega$
- $B_r(x)$: ball around x with radius r

Spaces

Let W, Z be Banach spaces, $k \geq 0$, $k \in \mathbb{N}_0$, $1 \leq p \leq \infty$, $s \geq 0$, and $d \in \mathbb{N}$.

- Product space: $W^2 = W \times W$
- $\mathbb{N}$ natural numbers without 0
- $\mathcal{L}(W, Z)$: set of all linear and continuous mappings from W to Z.
- $L^p(\Omega)$: space of all measurable functions f on Ω such that $\int_\Omega |f(x)|^p dx < \infty$ if p is finite and $\operatorname{ess\,sup}_{x\in\Omega} |f(x)| < \infty$ if $p = \infty$
- $W^{k,p}(\Omega)$: space of all $f \in L^p(\Omega)$ whose derivatives through order k are in $L^p(\Omega)$
- $W_0^{k,p}(\Omega)$: completion of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ for $1 \leq p < \infty$
- $W^{-k,p}(\Omega)$: dual space of $W_0^{k,p}(\Omega)$
- $H^k(\Omega) = W^{k,2}(\Omega)$
- $H^s(\Omega) = [H^m(\Omega), L^2(\Omega)]_{1-\frac{s}{m}}$ with integer $m \geq s \geq 0$, $s \in \mathbb{R}$ and interpolation spaces $[\cdot,\cdot]$; cf. [91]
- $H_0^s(\Omega)$: closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$
- $H^{-s}(\Omega)$: dual space of $H^s(\Omega)$
- $L^p(W) = L^2(0, T; W)$: space of all measurable functions $f\colon I \to W$, such that $\int_I \|f(t)\|_W \, dt < \infty$ if p is finite and $\operatorname{ess\,sup}_{t\in I} \|f(t)\|_W < \infty$ if $p = \infty$

- $W^{k,p}(I, W) = W^{k,p}(W)$: space of all $f \in L^p(W)$ whose derivative through order k are in $L^p(W)$
- $H^m(W) = W^{m,2}(W)$
- $H^s(W) = [H^m(W), L^2(W)]_{1-\frac{s}{m}}$ with integer $m \geq s \geq 0$, $s \in \mathbb{R}$
- $\mathcal{H}^{r,s}(Q) = L^2(H^r(\Omega)) \cap H^s(L^2(\Omega))$
- $\mathcal{H}^{r,s}(\Sigma) = L^2(H^r(\partial\Omega)) \cap H^s(L^2(\partial\Omega))$
- $C^k(W) = C^k([0,T]; W)$: set of continuous differentiable functions $f\colon [0,T] \to W$ through order k

Norms

- $|\cdot|$ absolute value
- $\|\cdot\| = \|\cdot\|_{L^2(\Omega)^d}$ for $d = 1, 2, 3$

Scalar products

For $d = 1, 2, 3$ we use the notation

- $(\cdot, \cdot)$ for the $L^2(\Omega)^d$-inner product,
- $\langle \cdot, \cdot \rangle$ the $L^2(\partial\Omega)^d$-inner product,
- $(\cdot, \cdot)_I$ inner product in $L^2(L^2(\Omega)^d)$
- $\langle \cdot, \cdot \rangle_I$ inner product in $L^2(L^2(\Sigma)^d)$,
- $A : B = \mathrm{tr}(A^T B)$
- $(u, v)_I = \int_0^T (u(t), v(t))_H dt$ for the Hilbert space H; cf. Chapter 2

Miscellaneous

- $\mathrm{tr}\, A = \sum_{i=1}^n a_{ii}$ for $A = (a_{ij})_{ij} \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$
- $C > 0$ a generic constant
- $\mathbb{1}_d$ unit matrix in $\mathbb{R}^{d \times d}$

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