# Oberwolfach Preprints

OWP 2014 - 09

Anatoli F. Ivanov and Erik I. Verriest

## Square Wave Periodic Solutions of a Differential Delay Equation

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

## Oberwolfach Preprints (OWP)

Starting in 2007, the MFO publishes a preprint series which mainly contains research results related to a longer stay in Oberwolfach. In particular, this concerns the Research in Pairs-Programme (RiP) and the Oberwolfach-Leibniz-Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it on its website as well as by hard copy. Every RiP group or Oberwolfach-Leibniz-Fellow may receive on request 30 free hard copies (DIN A4, black and white copy) by surface mail.

Of course, the full copy right is left to the authors. The MFO only needs the right to publish it on its website *www.mfo.de* as a documentation of the research work done at the MFO, which you are accepting by sending us your file.

In case of interest, please send a **pdf file** of your preprint by email to *rip@mfo.de* or *owlf@mfo.de*, respectively. The file should be sent to the MFO within 12 months after your stay as RiP or OWLF at the MFO.

There are no requirements for the format of the preprint, except that the introduction should contain a short appreciation and that the paper size (respectively format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX - XX).

We cordially invite the researchers within the RiP or OWLF programme to make use of this offer and would like to thank you in advance for your cooperation.

## Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

# SQUARE WAVE PERIODIC SOLUTIONS OF A DIFFERENTIAL DELAY EQUATION

Anatoli F. Ivanov[1] and Erik I. Verriest[2]

[1]Department of Mathematics, Pennsylvania State University
P.O. Box PSU, Lehman, PA 18627, USA
Corresponding author email: afi1@psu.edu

[2]School of Electrical and Computer Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0250, USA

Email: erik.verriest@ece.gatech.edu

**Abstract.** We prove the existence of periodic solutions of the differential delay equation

$$\varepsilon\dot{x}(t) + x(t) = f(x(t-1)), \qquad \varepsilon > 0$$

under the assumptions that the continuous nonlinearity $f(x)$ satisfies the negative feedback condition, $x \cdot f(x) < 0,\ x \neq 0$, has sufficiently large derivative at zero $|f'(0)|$, and possesses an invariant interval $I \ni 0, f(I) \subseteq I$, as a one-dimensional map. As $\varepsilon \to 0+$ we show the convergence of the periodic solutions to a discontinuous square wave function generated by the globally attracting 2-cycle of the map $f$.

**Keywords.** Singular differential equations with delay; Oscillation and instability; Existence of periodic solutions; Schauder fixed point theorem; Interval maps; Globally attracting cycles; Asymptotic shape of periodic solutions.

**AMS (MOS) subject classification:** Primary: 34K13; Secondary: 34K26

# 1 Introduction

Consider the differential delay equation

$$\varepsilon\dot{x}(t) + x(t) = f(x(t-1)) \tag{1}$$

with the (small) parameter $\varepsilon > 0$ and continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying the negative feedback condition

$$x \cdot f(x) < 0, \quad x \neq 0. \tag{2}$$

We shall additionally assume throughout the paper that the nonlinearity $f$ as a one-dimensional map has a closed bounded invariant interval $I := [\alpha, \beta]$,

$$f(x) \in I \quad \text{for all} \quad x \in I, \tag{3}$$

which contains the fixed point $x = 0, 0 \in I$.

Equation (1) serves as a mathematical model of a large variety of real life phenomena, see e.g. [3, 4, 8, 14, 15, 18] and further references therein. Of particular importance are models from physics and natural sciences where small viscosity or capacity effects lead to singular equations of type (1) with a small parameter $\varepsilon \ll 1$ [10, 15, 20, 24]. Equation (1) is also equivalent to the differential delay equation

$$x'(s) + x(s) = f(x(s - \tau)) \tag{4}$$

via time rescaling $s = \tau t$ with $\varepsilon = 1/\tau$. Large delay $\tau$ in equation (4) corresponds to the small parameter $\varepsilon$ in equation (1) with the normalized delay $\tau = 1$.

Criteria for existence of periodic solutions of equations (1) and (4) are well-known [4, 8, 16, 22]; for equations of form (1) it was first established by Hadeler and Tomiuk in [6]. They have shown that equation (1) has a slowly oscillating periodic solution provided the nonlinearity $f$ is one-sided bounded, and the linearization of (1) about the trivial solution $x(t) \equiv 0$, the linear differential delay equation

$$\varepsilon \dot{y}(t) + y(t) = f_0\, y(t-1), \quad f_0 := f'(0) < 0 \tag{5}$$

is unstable (i.e., its zero solution $y(t) \equiv 0$ is Liapunov unstable). The slow oscillation means that the solution's consecutive zeros are separated by a distance greater than the delay $\tau = 1$.

The instability of the linear differential delay equation (5) is completely described in terms of its characteristic equation

$$\varepsilon\, \lambda + 1 = f_0 \exp(-\lambda). \tag{6}$$

If (6) has a solution $\lambda = \alpha + i\,\beta$ with a positive real part $\alpha > 0$ then the differential delay equation (5) is unstable [4, 8].

Note that the assumption of one-sided boundedness of nonlinearity $f$ implies the existence of an invariant interval $I = [\alpha, \beta]$ for the map $f$. Indeed, in the case when $f(x) \geq -M$ $\forall x \in \mathbb{R}$ and some $M > 0$ one can choose $\alpha := -M$ and $\beta := \max\{f(x), x \in [-M, 0]\}$. Therefore, the assumption (3) is less restrictive then the one-sided boundedness.

The classical and standard by now proof of the existence of slowly oscillating periodic solutions to scalar differential delay equations and systems is the *ejective fixed point* techniques. It is based on the ejective fixed point Theorem due to F. Browder [2] (see also monographs [4, 8] for details of the theory and further references). With regard to equation (1) the theory requires that the zero solution of its linearized equation (5) is unstable and the nonlinearity $f$ is one-sided bounded: $f(x) \geq -M$ or $f(x) \leq M$ for all $x \in \mathbb{R}$ and some $M > 0$. The ejective fixed point theory has also been applied to other classes of scalar equations of first and higher order, see selected references [1, 7, 11, 12, 13, 21, 28, 29, 30], as well as to systems [9, 14].

In this paper we use a different approach to derive periodic solutions. We construct a bounded convex subset $S_\delta$ of the phase space of initial functions $C([-1, 0], \mathbb{R})$ for the differential delay equation (1) which elements are bounded away from $x(t) \equiv 0$: $||\varphi|| \geq \delta$ for some fixed $\delta > 0$ and all $\varphi \in S_\delta$. By choosing sufficiently small $\varepsilon > 0$, we show that the translation operator along solutions of equation (1) maps $S_\delta$ into itself. This allows us to use the Schauder fixed point theorem to prove the existence of slowly oscillating periodic solutions. Note that a similar approach was used in [23] as applied to another class of differential delay equations. Also, paper [27] has used Schauder fixed point theorem applied to the simplest scalar differential delay equation of the form $x' = g(x(t-1))$.

We also prove the asymptotic shape as $\varepsilon \to +0$ of any such slowly oscillating periodic solution in the case when map $f$ has a globally attracting cycle of period two. The periodic solutions converge to the piece-wise constant 2-periodic function $w_{ab}(t)$ defined by the globally attracting cycle $\{a, b\} \subset I$ of the map $f$: $w_{ab}(t) = a$ for $t \in [0, 1)$, $w_{ab}(t) = b$ for $t \in [1, 2)$, and

$w_{ab}(t+2) = w_{ab}(t) \quad \forall t \in \mathbb{R}$. The tools we use to prove the convergence are the closeness results on $\varepsilon$ between solutions of differential delay equation (1) and its limiting case when $\varepsilon = 0$, the difference equation with continuous variable $t$:

$$x(t) = f(x(t-1)). \tag{7}$$

Properties of the latter are largely determined by the dynamics of interval map $f$. Most of the closeness results on the parameter $\varepsilon > 0$ can be found in the review paper [10]. We use some of those in this paper by accordingly adjusting them for our needs. We would also like to note that a different approach to establish the asymptotic shape of the slowly oscillating periodic solutions, through time rescaling and deriving a related transition layer system, was used in [16].

## 2 Preliminaries

This section contains some basic information and facts for differential delay equation (1), which are established elsewhere and which we will need to use in this paper.

For every initial function $\varphi \in \mathcal{C} := C([-1,0],\mathbb{R})$ the corresponding solution $x(t) = x(t,\varphi,\varepsilon)$ to equation (1) is easily found by successive integration for all $t \geq 0$. At every time $t \geq 0$ the solution $x(t,\varphi,\varepsilon)$ can be viewed as an element $x_t(s)$ of the state space $\mathcal{C}$ by $x(t+s,\varphi,\varepsilon) := x_t(s), s \in [-1,0]$. $\mathcal{C}$ is a complete normed space with the norm $||\varphi|| = \sup\{|\varphi(s)|, s \in [-1,0]\}$. For basics of the theory of differential delay equations see monographs [4, 8].

Let $\mathcal{C}_I := C([-1,0],I)$ be a subset of $\mathcal{C}$ of those initial functions $\varphi$ whose range is contained in $I$. The following invariance property holds for equation (1) [10].

**Lemma 2.1** *(Invariance) For arbitrary $\varphi \in \mathcal{C}_I$ and every $\varepsilon > 0$ the corresponding solution $x(t) = x(t,\varphi,\varepsilon)$ of equation (1) satisfies $x(t) \in I$ for all $t \geq 0$.*

This lemma guarantees that every solution of equation (1) remains within the invariant interval $I$ of the map $f$ for all $t \geq 0$ provided the range of its initial function is within $I$ (for any value $\varepsilon > 0$). That is, $x_t(s) \in \mathcal{C}_I \;\; \forall t \geq 0$.

In the limit as $\varepsilon \to 0+$ equation (1) formally results in the difference equation (7). Equation (7) can be viewed as a difference equation with the continuous variable $t \geq 0$. For every initial function $\psi \in \mathcal{C}^0 := C([-1,0), \mathbb{R})$ the corresponding solution $x = x(t, \psi)$ of equation (7) exists for all $t \geq 0$; it is easily found by successive iterations.

Note that typically such a solution of difference equation (7) is discontinuous at every integer value $t_i = i, i \in \mathbb{N}^+ = \{0, 1, 2, 3, \dots\}$. In order for the solution $x(t, \psi)$ to be continuous for all $t > 0$ one has to assume that the following *contiguity* condition

$$\lim_{t \to 0^-} \psi(t) = f(\psi(0)) \tag{8}$$

is satisfied for the initial function $\psi \in \mathcal{C}^0$

We shall restrict our attention in this paper to the subset $\mathcal{C}^0_I$ of the set $\mathcal{C}^0$ of initial functions for the continuous time difference equation (7) defined by $\mathcal{C}^0_I := \{\psi \in \mathcal{C}^0 | \psi(s) \in I \;\; \forall s \in [-1, 0)\}$. Due to the invariance of $f$ on the interval $I$ this implies that any solution $x = x(t, \psi)$ of difference equation (7) satisfies $x(t) \in I \;\; \forall t \geq 0$ provided $\psi \in \mathcal{C}^0_I$.

Properties of solutions of difference equation (7) are largely determined by relevant properties of the interval map $f : I \to I$. They are completely described in monograph [24]; description of those properties related to aspects of this paper can also be found in the review paper [10]. For basics of the theory of interval maps one can use the monograph [25]

Given $\varphi \in \mathcal{C}_I$ and $\psi \in \mathcal{C}^0_I$ we will need to measure their closeness on the common interval of their definition $[-1, 0)$. One way to do this is to introduce the distance between the two by $||\varphi - \psi|| := \sup_s\{|\varphi(s) - \psi(s)|, s \in [-1, 0)\}$. Another way to do it is to set $\psi(0) := \lim_{t \to 0^-} \psi(s)$ and then define $||\varphi - \psi|| := \sup_s\{|\varphi(s) - \psi(s)|, s \in [-1, 0]\}$. Due to the continuity of the extended $\psi$ on the entire interval $[-1, 0]$ these two definitions are equivalent. Likewise, sometimes we will need a solution to differential delay equation (1) through an initial function $\psi \in \mathcal{C}^0_I$. We shall use then the above extension of $\psi$ on the entire interval $[-1, 0]$ given by $\psi(0) := \lim_{t\to 0^-} \psi(s)$.

In this paper we are primarily interested in the so-called slowly oscillating solutions to differential delay equation (1). As usual, a nontrivial solution $x(t)$ is called oscillatory (about the trivial solution $x(t) \equiv 0$) if it has a sequence of zeros $\{t_k, k \in \mathbb{N}\}$ such that $t_k \to +\infty$ as $k \to \infty$. Conditions for the oscillation of all solutions of equation (1) are well known (see e.g. [5]). If the characteristic equation (6) of the corresponding linearized equation (5) does not have real solutions then all solutions to both equations (1) and (5) oscillate. In particular, this holds true when the characteristic equation (6) has a complex solution with a positive real part.

**Definition 2.2** *A solution $x(t) = x(t, \varphi, \varepsilon)$ to equation (1) is called slowly oscillating if the distance between its any two zeros $t_1 < t_2$ is greater than the delay $\tau = 1$: $t_2 - t_1 > 1$.*

As our primary interest in this paper is the existence of slowly oscillating periodic solutions under the assumption of smallness of $\varepsilon$ and $f'(0) < -1$ we shall use the following simple sufficient condition for all solutions of equation (1) to oscillate.

**Proposition 2.3** *If the inequality $0 < \varepsilon < |f'(0)|$ is satisfied then all solutions to both differential delay equations (1) and (5) oscillate.*

*Proof.* The proof of this statement is straightforward, and can be found elsewhere. We provide it here for the sake of completeness.

Let $x(t) > 0, t \geq 0$, be a non-oscillatory solution to equation (1). Then it is monotone decreasing for $t \geq 0$ with $\lim_{t\to\infty} x(t) = 0$. Let $\sigma > 0$ be arbitrary and sufficiently small (so that $|f_0+\sigma| < \varepsilon$), and $T$ be sufficiently large. Integration of (1) over the time interval $[t, t+1], t \geq T$, yields: $\varepsilon x(t+1) - \varepsilon x(t) + \int_t^{t+1} x(s)\, ds \leq (f_0 + \sigma) \int_t^{t+1} x(s-1)\, ds \leq (f_0 + \sigma) x(t)$. The later results in $x(t+1) \leq x(t)[1 + \frac{f_0+\sigma}{\varepsilon}]$, contradicting the positiveness of $x(t)$.

Throughout the reminder of the paper we shall assume that the singular parameter $\varepsilon$ satisfies the inequality $0 < \varepsilon < |f'(0)|$, so that any solution of equation (1) is oscillatory.

Consider a subset $S_0$ of the set $\mathcal{C}$ of all initial functions defined by

$$S_0 := \{\varphi \in \mathcal{C} \mid \varphi(0) = 0 \quad \text{and} \quad \varphi(s) > 0 \ \ \forall s \in [-1, 0)\}. \qquad (9)$$

For every $\varphi \in S_0$ the corresponding solution $x = x(t, \varphi, \varepsilon)$ of equation (1) is slowly oscillating. It is a well known fact which follows from simple considerations. Indeed, rewritten equation (1), $\varepsilon x'(t) = -x(t) + f(x(t-1))$, and the negative feedback assumption (2), imply that the solution $x$ remains negative on the entire interval $(0, 1]$. One also sees that $x'(1) > 0$, so the solution $x$ is increasing in some right neighborhood of $t = 1$. It will continue to increase until its first positive zero $z_1 > 0$. Thus, the solution $x(t)$ has its first two zeros $z_0 = 0$ and $z_1 > 1$ such that $x(t) < 0 \ \ \forall t \in (z_0, z_1)$. Likewise, it can be shown that there exists next zero $z_2$ of the solution $x$ with $x(t) > 0 \ \ \forall t \in (z_1, z_2)$ and $z_2 - z_1 > 1$. This way a sequence of zeros $\{z_k, k = 0, 1, 2, \dots\}$ can be identified for the solution $x(t, \varphi, \varepsilon)$, generated by any initial function $\varphi \in S_0$, with the following properties

$$x(t) < 0 \ \ \forall t(z_{2k}, z_{2k+1}), \ \ x(t) > 0 \ \ \forall t(z_{2k+1}, z_{2k+2}), \ \ z_{k+1} - z_k > 1.$$

Let $\phi(s) := x(z_2 + s, \varphi, \varepsilon), s \in [-1, 0]$ be a segment of the solution through the initial function $\varphi \in S_0$ considered on the interval $[z_2, z_2 - 1]$ as an element of the phase space $\mathcal{C}$. By the construction one has that $\phi \in S_0$. Define now the mapping on $\mathcal{F}$ on $S_0$ by

$$\mathcal{F}(\varphi(s)) := \phi(s), \quad s \in [-1, 0]. \qquad (10)$$

By the continuity, one should also define $\mathcal{F}(\mathbf{0}) = \mathbf{0}$, where $\mathbf{0} := 0 \ \ \forall s \in [-1, 0]$ is the trivial fixed point of $\mathcal{F}$ corresponding to the zero solution $x(t) \equiv 0$ of equation (1).

The following proposition is obvious.

**Proposition 2.4** *Differential delay equation (1) has a slowly oscillating periodic solution if and only if the mapping $\mathcal{F}$ has a nontrivial fixed point on $S_0$.*

The difficulty in applying Proposition 2.4 to prove the existence of periodic solutions is that the zero solution of differential delay equation (1) corresponds to the trivial fixed point $\varphi \equiv 0$ of the map $\mathcal{F}$. One needs to establish the existence of fixed points different from the trivial one. The ejective fixed point theory does exactly that. As already mentioned, the details of the theory as applied to differential delay equations can be found in [4, 8].

We employ a different approach, which eventually allows us to us the Schauder fixed point theorem. To facilitate the proof of our main existence result in Section 3 we construct a convex bounded subset $S_\delta$ of the set $S_0$, all of which elements are bounded away from $\varphi \equiv 0$: $||\varphi|| \geq \delta \ \ \forall \varphi \in S_\delta$ for some $\delta > 0$. We then show that the above defined compact mapping $\mathcal{F}$ maps $S_\delta$ into itself for all sufficiently small $\varepsilon > 0$. Thus, the existence of a slowly oscillating periodic solution follows by the Schauder fixed point theorem.

# 3 Main Results

## 3.1 Existence of Periodic Solutions

In addition to the basic assumptions of the negative feedback (2) and existence of an invariant interval (3) for the nonlinearity $f$, we shall assume throughout this section the following hypothesis to hold:

(**H1**) There exist a constant $\lambda_0 < 0$ and a positive number $\delta_0$ such that the following inequalities hold

$$f(x) > \lambda_0 x \ \ \forall x \in (-\delta_0, 0) \quad \text{and} \quad f(x) < \lambda_0 x \ \ \forall x \in (0, \delta_0). \quad (11)$$

Inequalities (11) mean that function $f(x)$ is *super-linear* in some $\delta_0$-neighborhood of $x = 0$ with respect to the linear function $y = \lambda_0 x$. In the case when $f(x)$ is continuously differentiable at $x = 0$ with $f'(0) := f_0 < 0$ one can choose $\lambda_0$ to be any negative number with $\lambda_0 > f_0$. For any such choice of $\lambda_0$ the corresponding value of $\delta_0 = \delta_0(\lambda_0, f)$ can be determined based on further properties of nonlinearity $f$ near $x = 0$.

**Theorem 3.1** *Assume conditions (2) and (3), and let (H1) be satisfied with* $\lambda_0 < -(3+\sqrt{17})/2$. *There exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$ *equation (1) has a slowly oscillating periodic solution with period* $T = 2 + O(\varepsilon)$. *Moreover, there are positive constants* $K$ *and* $L$ *such that if* $z_1 \geq 0, z_2 > z_1+1$, *and* $z_3 > z_2+1$ *are three consecutive zeros of the periodic solution then one has*

$$T = z_3, \;\; p(t) < 0 \; for \, t \in (z_1, z_2), \;\; p(t) > 0 \; for \, t \in (z_2, z_3)$$

$$K\varepsilon \leq z_2 - 1 \leq L\varepsilon \; and \; K\varepsilon \leq z_3 - 2 \leq L\varepsilon. \tag{12}$$

*Proof.* The proof uses the Schauder fixed point theorem.

Consider the set of initial functions

$$S_\delta := \{\varphi \in \mathcal{C}_I : \varphi(0) = \delta \;\text{ and }\; \delta \leq \varphi(s) \leq \beta, \; \forall s \in [-1, 0]\} \tag{13}$$

where $\delta > 0$ is sufficiently small.

For arbitrary $0 < \varepsilon < |f_0|$ any initial function $\varphi \in S_\delta$ gives rise to a slowly oscillating solution of equation (1) (Proposition 2.3). Let $0 < z_1 < z_2 < z_3$ be the solution's first three consecutive zeros. We shall show next that for any such solution there are values $\eta_1$ and $\eta_2$, $z_1 < \eta_1 < \eta_2 < z_2$, such that

$$x(\eta_1) = x(\eta_2) = -\delta, \;\; x(t) \leq -\delta \quad \forall t \in [\eta_1, \eta_2], \quad \text{and} \quad \eta_2 - \eta_1 > 1.$$

Consider next the set of initial functions

$$S_{-\delta} := \{\phi \in \mathcal{C}_I : \phi(0) = -\delta \text{ and } \alpha \leq \varphi(s) \leq -\delta, \; \forall s \in [-1, 0]\}. \tag{14}$$

For every $\varphi \in \mathcal{S}_\delta$ there exists a sequence of zeros $0 < z_1 < z_2 < z_3 < \cdots < z_k < z_{k+1} < \cdots \to \infty$ such that $z_{k+1} - z_k > 1, k \in \mathbb{N}$ and

$$x(t) < 0 \;\; \forall t \in (z_{2k-1}, z_{2k}), \quad x(t) > 0 \;\; \forall t \in (z_{2k}, z_{2k+1}).$$

For the solution $x = x(t, \varphi, \varepsilon)$ on the interval $[z_1, z_2]$, we shall construct an upper bound function $x^u(t)$ and a lower bound function $x^l(t)$ such that uniformly for all $\varphi \in S_\delta$ one has

$$x^l(t) \leq x(t, \varphi, \varepsilon) \leq x^u(t), \quad \forall t \in [z_1, z_2] \tag{15}$$

for all $0 < \varepsilon < \varepsilon_0$ and some sufficiently small $\varepsilon_0$.

In the computational estimates below we shall use two simple facts:

a) The exponential solution of the initial value problem $\varepsilon x'(t) + x(t) = A$ – constant, $x(t_0) = x_0$ is given by the formula $x(t) = A + (x_0 - A)\exp(-\frac{t-t_0}{\varepsilon})$;

b) If $f(x(t-1)) \le A$ for $t \in [t_0, T]$ then the solution of the initial value problem $\varepsilon x'(t)+x(t) = f(x(t-1),\ x(t_0) = x_0$ satisfies $x(t) \le A+(x_0 - A)\exp(-\frac{t-t_0}{\varepsilon})\ \ \forall t \in [t_0, T]$. Likewise, one has the inequality $x(t) \ge A+(x_0 - A)\exp(-\frac{t-t_0}{\varepsilon})\ \ \forall t \in [t_0, T]$ if the nonlinearity $f$ satisfies $f(x(t-1)) \ge A$.

Let $\varphi \in S_\delta$ be arbitrary. We first derive an upper and lower bounds for the value of the first zero $z_1$ of the solution $x = x(t, \varphi, \varepsilon)$. This will also establish the existence of constants $K$ and $L$ in the estimates on the magnitudes of $z_2 - 1$ and $z_3 - 2$ (see the statement of the theorem).

In view of the hypothesis (H1) and the invariance of $f$ on the interval $I = [\alpha, \beta]$ one has

$$\sup\{f(x),\ x \in [\delta, \gamma]\} \le \lambda\delta, \quad \inf\{f(x),\ x \in [\delta, \gamma]\} \le \alpha.$$

Therefore, on the time interval $[0, 1]$ the corresponding solution $x = x(t, \varphi, \varepsilon)$ can be estimated as

$$x^l := \alpha+(\delta-\alpha)\exp(-t/\varepsilon) \le x(t) \le x^u := \lambda\delta+(\delta-\lambda\delta)\exp(-t/\varepsilon).$$

The latter implies that the first zero $z_1$ can be estimated as

$$z_1^l = \varepsilon \ln \frac{\alpha - \delta}{\alpha} \le z_1 \le z_1^u = \varepsilon \ln \frac{\lambda - 1}{\lambda}.$$

If $\varepsilon > 0$ is sufficiently small, say $\varepsilon < \varepsilon_1$, then there exists first time $t = \eta_1 > z_1$ such that $x(\eta_1) = -\delta, \eta_1 < 1$, and $x(t)$ is strictly decreasing in $[0, \eta_1]$. Moreover, $x(t) < -\delta$ for all $t \in (\eta_1, 1]$. Indeed, since $x(t) \le x^u = \lambda\delta + (\delta - \lambda\delta)\exp(-t/\varepsilon),\ \ \forall t \in [0, 1]$ one sees that $\eta_1 \le \eta_1^u$, where $\eta_1^u$ is such value of $t$ where $x^u(t) = -\delta$. It is easily found that $\eta_1^u = \varepsilon \ln \frac{\lambda+1}{\lambda-1}$.

Define the values: $x_1 = x(1, \varphi, \varepsilon)$ and $x_1^u = x^u(1) = \lambda\delta + (\delta - \lambda\delta)\exp(-1/\varepsilon)$. It is obvious that values $x_1 < x_1^u$ and $x_1^u$ can be made as close to $\lambda\delta$ as desired by choosing $\varepsilon$ to be small enough (say $0 < \varepsilon < \varepsilon_2$).

We construct next an upper estimate $x^u(t)$ for the solution $x(t, \varphi, \varepsilon)$ on the interval $[1, 1 + \eta_1]$. Since $x(t) \in [0, \delta]$ for all $t$ on the interval $[0, z_1]$ one has that for $t \in [1, 1 + z_1]$ the inequality $f(x(t - 1) \leq 0$ holds. Therefore, $x(t) \leq x^u(t) = x_1^u \exp\{-(t - 1)/\varepsilon\}, t \in [1, 1+z_1]$, where $x^u(t)$ is the solution of the initial value problem $\varepsilon x'(t) + x(t) = 0,\ x(1) = x_1^u$. Set $x_2^u := x^u(1 + z_1)$.

Since $x(t) \in [-\delta, 0]\ \forall t \in [z_1, \eta_1]$ one has that for $t \in [1 + z_1, 1 + \eta_1]$ the inequality holds $f(x(t - 1)) \leq -\lambda\delta$. Therefore, for $t \in [1 + z_1, 1 + \eta_1]$ one has

$$x(t) \leq x^u(t) = -\lambda\delta + (x_2^u + \lambda\delta)\exp\{-[t - (1 + z_2)]/\varepsilon\}.$$

Let $x_3^u = x^u(1 + \eta_1)$. We shall show next that there exists $\varepsilon_3 > 0$ such that $x_3^u < -\delta$ for all $0 < \varepsilon < \varepsilon_3$.

Since $x_1^u = \lambda\delta + (1 - \lambda)\delta\exp(-1/\varepsilon)$ is close to $\lambda\delta$ for small $\varepsilon > 0$, consider on the interval $[1, 1+z_1]$ the following initial value problem $\varepsilon x'(t) + x(t) = 0,\ x(1) = \lambda\delta$. Its solution is given by $x^*(t) = \lambda\delta\exp\{-(t - 1)/\varepsilon\}$. In view of the estimate $z_1 \leq \varepsilon \ln \frac{\lambda-1}{\lambda}$ one easily sees that $x^*(1 + z_1) \leq \frac{\lambda^2}{\lambda-1}\delta := x_2^*$.

Consider on the interval $[1 + z_1, 1 + \eta_1]$ the following initial value problem

$$\varepsilon x'(t) + x(t) = -\lambda\delta,\ x(1 + z_2) = x_2^*.$$

Its solution is given by

$$x_*^u(t) = -\lambda\delta + (x_2^* + \lambda\delta)\exp\{-\frac{t - (1 + x_2)}{\varepsilon}\}.$$

Since $\eta_2 - z_1 \leq \varepsilon \ln \frac{\lambda}{\lambda+1}$ one sees that

$$x_*^u(1+\eta_1) \geq -\lambda\delta+(x_2^*+\lambda\delta)\exp\{-\frac{\eta_2 - z_1}{\varepsilon}\} \geq \delta\frac{\lambda^2 + 2\lambda - 1}{\lambda - 1} := x_3^*.$$

We will show next that $x_3^* < -\delta$ provided $\lambda < \lambda_0 = -\frac{3+\sqrt{17}}{2}$. Indeed, the desired inequality $x_3^* < -\delta$ follows from the assumption $(\lambda^2+2\lambda-1)/(\lambda-1) < -1$, which is equivalent to $\lambda^2+3\lambda-2 > 0$. This results in $\lambda < -(3 + \sqrt{17})/2 := \lambda_0$.

Finally we have to show that $x_3^u < -\delta$ provided $\varepsilon > 0$ is small enough. This follows from the continuous dependence of the solution $x(t)$ of equation (1) on the initial conditions and the fact that the value $x_1^u = \lambda\delta + (1-\lambda)\delta\exp(-1/\varepsilon)$ is close to $x_1^*$ for all sufficiently small $\varepsilon$: $0 < \varepsilon < \varepsilon_4$ for some $\varepsilon_4 > 0$. Therefore, the solution $x = x(t, \varphi, \varepsilon)$ satisfies $x(t) \leq -\delta$ for all $t \in [\eta_1, \eta_1 + 1$ provided $\lambda < \lambda_0 = -(3+\sqrt{17})/2$ and $0 < \varepsilon \leq \varepsilon_5 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$.

Since $x(t) < 0$ for $t \in [\eta_1, \eta_1 + 1]$ the solution is increasing in some right neighborhood of $t = 1 + \eta_1$. Therefore, there exists first value $\eta_2 > 1 + \eta_1$ such that $x(\eta_2, \varphi, \varepsilon) = -\delta$.

Define next $\phi_1(s) := x(s + \eta_2, \varphi, \varepsilon), s \in [-1, 0]$ as an element of $\mathcal{C}_I$. In view of the above considerations $\phi_1(s) \in S_{-\delta}$. Similar to the previous steps of the proof one can show that, with all the assumptions in place, there exist values $z_2 < \eta_3 < \eta_4$ with $\eta_4 - \eta_3 > 1$ and such that $x(z_2) = 0, x(\eta_3) = x(\eta_4) = \delta$, and $x(t) \geq \delta$ for all $t \in [\eta_3, \eta_4]$, provided $0 < \varepsilon \leq \varepsilon_6$ for some $\varepsilon_6 > 0$.

Let $\phi(s) := x(s + \eta_4, \varphi, \varepsilon), s \in [-1, 0]$ and define the mapping $\mathcal{F} : S_\delta \to S_\delta$ by $\mathcal{F}(\varphi) = \phi$. It is a standard observation that $\mathcal{F}$ is a compact mapping of the convex bounded set $S_\delta \subset \mathcal{C}$ into itself. Therefore, by the Schauder fixed point Theorem, it has a fixed point $\varphi_0 \in S_\delta$, $\mathcal{F}(\varphi_0) = \varphi_0$. Clearly, by the construction, the corresponding solution $x = x(t, \varphi_0, \varepsilon)$ is a nontrivial periodic slowly oscillating solution for all $0 < \varepsilon < \varepsilon_0 := \min\{\varepsilon_5, \varepsilon_6\}$. This completes the proof.

### 3.2 Asymptotic Shape as $\varepsilon \to +0$

Let $\{a_1, a_2, \ldots, a_k\}$ be a $k$-cycle of the map $f$:

$$f(a_i) = a_{i+1}, i = 1, 2, \ldots, k-1 \text{ and } f(a_k) = a_1.$$

Introduce the following piece-wise constant $k$-periodic function $w_{a_1 a_2 \ldots a_k}(t)$

$$w_{a_1 a_2 \ldots a_k}(t) := a_i, \quad t \in [i-1, i), 1 \leq i \leq k, \tag{16}$$

and

$$w_{a_1 a_2 \ldots a_k}(t) = w_{a_1 a_2 \ldots a_k}(t + k), \quad \forall t \in \mathbb{R}.$$

We say that a continuous function $x = x(t,\varepsilon) \in C([0,k],\mathbb{R})$ converges to $w_{a_1a_2\ldots a_k}(t)$ as $\varepsilon \to 0+$ if for arbitrary $\eta > 0, \sigma > 0$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ one has

$$\sup_t\{|x(t) - w_{a_1a_2\ldots a_k}(t)|,\ t \in [0,k] \setminus \cup_{i=0}^k U_\sigma(i)\} < \eta,$$

where $U_\sigma(i)$ is a $\sigma$-neighborhood of point $t = i$. Functions $x(t)$ and $w_{a_1a_2\ldots a_k}(t)$ are $\sigma$-close on the interval $[0,k]$ everywhere except small $\sigma$-neighborhoods of integer value points $t = i$.

Principal auxiliary facts used in this subsection are a series of results on the continuous dependence on parameter $\varepsilon$ between solutions of equations (1) and (7) obtained in [10], and their modifications.

Given $T > 0$ and $\sigma > 0$ let $J_T^\sigma$ be the following set:

$$J_T^\sigma = [0,T] \setminus \{\cup_{i\geq 0} U_\sigma(i)\},$$

where $U_\sigma(i)$ is the $\sigma$-neighborhood of point $t = i$. $J_T^\sigma$ is the closed finite interval $[0,T]$ of which the $\sigma$-neighborhoods of all integer points $t = i$ are taken out.

In order to prove our main result of this subsection, Theorem 3.6 stated below, we need a series of auxiliary results on the closeness between solutions of differential delay equation (1) and continuous time difference equation (7). For this purpose we adopt and modify some of the relevant closeness results from the review paper [10].

The following statement shows the continuous dependence of solutions of equation (1) on the initial data and the singular parameter $\varepsilon$.

**Lemma 3.2** *For every $\varphi \in C_I$ and arbitrary $\sigma > 0$ there exists $\delta = \delta(\varphi,\sigma) > 0$ such that $||x(t,\varphi,\varepsilon) - x(t,\phi,\varepsilon)||_{[0,1]} \leq \sigma$ for all $\varepsilon > 0$ provided $\phi \in C_I$ and $||\varphi - \phi||_{[-1,0]} \leq \delta$.*

*Proof.* By using the following integral form of the differential delay equation (1)

$$x(t) = x(0)\exp\left(-\frac{t}{\varepsilon}\right) + \frac{1}{\varepsilon}\int_0^t \exp\left(\frac{s-t}{\varepsilon}\right) f(x(s-1))\,ds, \quad t \geq 0 \tag{17}$$

one derives this estimate for $t \in [0,1]$

$$|x(t,\varphi,\varepsilon) - x(t,\phi,\varepsilon)| \le |\varphi(0) - \phi(0)| \exp\left(-\frac{t}{\varepsilon}\right) +$$

$$+\frac{1}{\varepsilon}\int_0^t \exp\left(\frac{s-t}{\varepsilon}\right) |f(\varphi(s-1)) - f(\phi(s-1))|\, ds.$$

Since $f$ is uniformly continuous on $I$, for arbitrary $\sigma > 0$ there exists $\delta > 0$ such that if $|\varphi(u) - \phi(u)| \le \delta$ then $|f(\varphi(u)) - f(\phi(u))| \le \sigma$. In view of this the previous inequality can be continued as

$$|x(t,\varphi,\varepsilon) - x(t,\phi,\varepsilon)| \le \delta \exp\left(-\frac{t}{\varepsilon}\right) - \sigma\left[1 - \exp\left(-\frac{t}{\varepsilon}\right)\right] \le$$

$$\le \max\{\delta, \sigma\},$$

which completes the proof.

Next statement shows the closeness between solutions of differential delay equation (1) and continuous time difference equation (7) generated by the same initial function. The closeness takes place everywhere on the interval $[0,1]$ except a small right vicinity of point $t = 0$.

**Lemma 3.3** *For every $\varphi \in C_I$ and arbitrary $\sigma > 0$ and $\kappa > 0$ there exists $\varepsilon_0 = \varepsilon_0(\sigma,\kappa,\varphi) > 0$ such that for all $0 < \varepsilon \le \varepsilon_0$ one has $||x(t,\varphi,\varepsilon) - x(t,\varphi)||_{[\kappa,1]} \le \sigma$.*

*Proof.* Taking into account the identity

$$1 - \exp\left(-\frac{t}{\varepsilon}\right) = \frac{1}{\varepsilon}\int_0^t \exp\left(\frac{s-t}{\varepsilon}\right) ds$$

we have that for all $t \in [0,1]$ the following inequality holds

$$|x(t,\varphi,\varepsilon) - x(t,\varphi)| \le |\varphi(0) - f(\varphi(s-1))| \exp\left(-\frac{t}{\varepsilon}\right) +$$

$$+\frac{1}{\varepsilon}\int_0^t \exp\left(\frac{s-t}{\varepsilon}\right) |f(\varphi(s-1)) - f(\varphi(t-1))|\, ds.$$

Since $f(\varphi(\cdot))$ is uniformly continuous for $t \in [-1, 0]$ it follows that for any $\sigma_1 > 0$ there exists $\delta_1 > 0$ such that $|f(\varphi(t_1)) - f(\varphi(t_2))| \leq \sigma_1$ provided $|t_1 - t_2| \leq \delta_1$. Suppose now that $t \in [\delta_1, 1]$. Then

$$|\varphi(0) - f(\varphi(s-1))| \exp\left(-\frac{t}{\varepsilon}\right) \leq \operatorname{diam} I \exp\left(-\frac{\delta_1}{\varepsilon}\right) \leq \sigma_1$$

for all sufficiently small $\varepsilon$, say $0 < \varepsilon \leq \varepsilon_1$. On the other hand

$$\frac{1}{\varepsilon}\int_0^t \exp\left(\frac{s-t}{\varepsilon}\right) |f(\varphi(s-1)) - f(\varphi(t-1))|\, ds = \frac{1}{\varepsilon}\left(\int_0^{t-\delta_1} ...ds + \right.$$

$$\left. + \int_{t-\delta_1}^t ...ds\right) \leq \operatorname{diam} I \exp\left(-\frac{\delta_1}{\varepsilon}\right) + \sigma_1\left[1 - \exp\left(-\frac{\delta_1}{\varepsilon}\right)\right] \leq 2\sigma_1$$

for all sufficiently small $\varepsilon$, say $0 < \varepsilon \leq \varepsilon_2$. Therefore, setting $\varepsilon_3 := \min\{\varepsilon_1, \varepsilon_2\}$ we have

$$\frac{1}{\varepsilon}\int_0^t \exp\left(\frac{s-t}{\varepsilon}\right) |f(\varphi(s-1)) - f(\varphi(t-1))|\, ds \leq 2\sigma_1$$

for all $0 < \varepsilon \leq \varepsilon_3$. This implies $|x(t, \varphi, \varepsilon) - x(t, \varphi)| \leq 3\sigma_1 := \sigma$ for all $0 < \varepsilon \leq \varepsilon_3$ and $t \in [\delta_1, 1]$ with $\sigma_1 = \frac{\sigma}{3}$. The proof is complete.

The following statement is an easy consequence of Lemmas 3.2 and 3.3.

**Lemma 3.4** *For every $\varphi \in C_I$ and arbitrary $\sigma > 0, \kappa > 0$ there exists $\delta = \delta(\varphi, \sigma, \kappa) > 0$ and $\varepsilon_0 = \varepsilon(\varphi, \sigma, \kappa) > 0$ such that $||x(t, \varphi, \varepsilon) - x(t, \psi)||_{[\kappa, 1]} \leq \sigma$ for all $0 < \varepsilon \leq \varepsilon_0$ provided $||\varphi - \phi||_{[-1,0]} \leq \sigma$.*

*Proof.* It is straightforward from the triangle inequality for the norm and Lemmas 3.2 and 3.3.

The next statement shows the closeness of solutions between equations (1) and (7) on any finite time imterval.

**Lemma 3.5** *For every $\varphi \in C_I$ and arbitrary $T > 0, \sigma > 0$, and $\kappa > 0$ there exist $\delta = \delta(\varphi, \sigma, \kappa, T)$ and $\varepsilon_0 = \varepsilon_0(\varphi, \sigma, \kappa, T) > 0$ such that $||x(t, \varphi, \varepsilon) - x(t, \psi)||_{J_T^\kappa} \leq \sigma$ for all $0 < \varepsilon \leq \varepsilon_0$ provided $||\varphi - \phi||_{[-1,0]} \leq \sigma$.*

*Proof.* It easily follows by induction by applying Lemmas 3.2, 3.3, 3.4, and 3.5. We leave details to the reader.

**Theorem 3.6** *Suppose that $f$ is continuously differentiable on $I$, all assumptions of Theorem 3.1 are satisfied, and $\{a, b\}$ is a globally attracting cycle of the map $f$ on the interval $I$. Then any periodic solution $x = p(t, \varepsilon)$ guaranteed by it converges to the 2-periodic square wave function $w_{ab}(t)$ as $\varepsilon \to 0+$.*

*Proof.* Let $\varphi_0(s) \in S_\delta$ be an initial function which generates a slowly oscillating periodic solution guaranteed by Theorem 3.1. Consider corresponding solutions, $x(t, \varphi_0, \varepsilon)$ of differential delay equation (1), and $x(t, \varphi_0)$ of difference equation (7), respectively. Since $f'(0) < -1$ the fixed point $x = 0$ of the map $f$ is repelling. Let $\{a, b\}$ be the globally attracting 2-cycle of $f$ with $a < 0 < b$. The negative feedback condition (2) implies that $x = b$ is an attracting fixed point of the map $f^2 = f \circ f$ with the domain $(0, \beta)$ of immediate attraction. Therefore, for arbitrary $\sigma > 0$ and any $\delta > 0$ there exists a positive integer $n_0 = n_0(\sigma, \delta, f)$ such that $|f^{2n}(x) - b| < \sigma$ for all $x \in [\delta, \beta]$ and all $n \geq n_0$. This implies that $||x(t, \varphi_0) - b||_{J_\kappa^{n_0}} \leq \sigma$ where $J_\kappa^{n_0} = [2n_0 + \kappa, 2n_0 + 1 - \kappa]$. On the other hand, in view of Lemmas 3.2, 3.3, 3.4, and 3.5 above, $||x(t, \varphi_0, \varepsilon) - x(t, \varphi_0)|| \leq \sigma, \quad \forall t \in J_\kappa^{n_0}$ and any $0 < \varepsilon \leq \varepsilon_0$. Therefore

$$||x(t, \varphi_0, \varepsilon) - b|| \leq ||x(t, \varphi_0, \varepsilon) - x(t, \varphi_0)|| + ||x(t, \varphi_0) - b)|| \leq 2\sigma,$$

which proves the convergence of $x(t, \varphi_0, \varepsilon)$ to $x(t) \equiv b$ on intervals $J_\kappa^{n_0}$. Likewise, one can show that the periodic solution $x(t, \varphi_0, \varepsilon)$ converges to $x(t) \equiv a$ on intervals $J_\kappa^{n_0+1} := [2n_0 + 1 + \sigma, 2n_0 + 2 - \sigma]$. This completes the proof of Theorem 3.6.

Note that neither the stability nor the uniqueness of the periodic solution can be deduced under the assumptions of theorem 3.6. In fact, examples are constructed elsewhere which show the complex structure of the set of periodic solutions and the dynamics in general on the set of slowly oscillating solutions of equation (1) (see works [10, 17, 24] an further references therein). In the next section we briefly describe an example which demonstrates those complexities.

# 4 Further Discussion and Open Questions

One drawback of the existence result, Theorem 3.1, is that the assumption (H1) requires the constant $\lambda_0$ to be large enough, $|\lambda_0| > (3 + \sqrt{17})/2$. On the other hand, this assumption allows us to relatively easy establish the asymptotic shape of the periodic solutions as $\varepsilon \to 0+$. This is done by using the continuous dependence results on $\varepsilon$ between solutions of equations (1) and (7), and the fact that initial functions for periodic solutions belong to the set $S_\delta$ which is bounded away from zero solution.

Since the existence of periodic solutions is known under more general assumptions (see the Introduction), it is desirable to extend our approaches and result to more general situations. In particular to the case of any value of $f'(0) < -1$ and all sufficiently small $\varepsilon$. It is a known fact that for every $f'(0) < -1$ there exists $\varepsilon_0$ such that for all $0 < \varepsilon < \varepsilon_0$ the linearized differential delay equation (5) is unstable, and therefore nonlinear equation (1) has a slowly oscillating periodic solution. While the linear equation (5) is stable for all $\varepsilon > \varepsilon_0$. The principal difficulty here would be to construct an appropriate closed convex subset $\mathcal{C}_\delta \subset \mathcal{C}_I$ of initial functions which is bounded away from zero $\varphi \equiv 0$ and is mapped into itself by a shift operator along solutions of differential delay equation (1).

As it was mentioned earlier, generally there is no stability or uniqueness of the periodic solutions under the assumptions of theorem 3.6. In fact, there can be multiple periodic solutions and even chaotic dynamics in the case of globally attracting 2-cycle of the map $f$. We provide below a brief summary of an example from [24].

Let $f(x)$ be a piece-wise constant two-parameter function given by

$$f(x) = \begin{cases} -1, & \text{if} \quad x \geq h \quad (0 < h < 1); \\ -A, & \text{if} \quad x \in [0, h) \quad (1 < A); \\ -f(-x), & \text{if} \quad x < 0. \end{cases} \tag{18}$$

It is easily seen that the corresponding one-dimensional map $f$

has an invariant interval $[-A, A]$ which contains the globally attracting cycle $\{-1, 1\}$ of period two.

For a one parameter subset of initial functions $\mathcal{C}_\lambda$ of the phase space $\mathcal{C}_I$ there is an exact reduction of the dynamics of slowly oscillating solutions of equation (1) to the dynamics of a one-dimensional map $F_\lambda$ of a closed interval $I_\lambda := [\lambda_1, \lambda_2]$ into itself. It is shown that for an open subset of the parameters values $(h, A)$, the map $F_\lambda : I_\lambda \to I_\lambda$ can be as complex and dynamically rich as a general interval map can be. In particular, it can have multiple periodic solutions, including those with periods not a power of two, it can be chaotic, and it can have an absolutely continuous invariant measure. Thus, the dynamics of the corresponding solutions of the differential delay equation (1) can accordingly be arbitrarily complex, even though the dynamics of the map $f$ is simple: it only has a globally attracting cycle of period two. This example can be modified to a case of continuous/smooth $f$ by removing the jump discontinuities in small neighborhoods of points $\{-1, 0, 1\}$. See details in [24].

It is known that dynamics of differential delay equations of form (1) are relatively simple when the nonlinearity $f$ is monotone (strictly decreasing). More specifically, the work by Mallet-Paret and Walther [19, 26] shows that in this case of monotone $f$ the slow oscillation is typical. That is, the subset $\mathcal{C}_{so} \subset \mathcal{C}_I$ of initial functions $\varphi \in \mathcal{C}_I$ which give rise to eventually slowly oscillating solutions is open and dense in $\mathcal{C}_I$. We believe that the simplifying assumption of the monotonicity on $f$ in equation (1) should lead to the uniqueness of the slowly oscillating periodic solution as well as to its stability.

In conclusion, we would like to state a conjecture concerning the existence, asymptotic shape, uniqueness and stability of periodic solutions of equation (1). This would be a generalization of both Theorems 3.1 and 3.6 along the lines discussed above. We believe the conjecture can be proved by using an adequate construction of the above mentioned set $\mathcal{C}_\delta$ and some appropriate modifications of the continuous dependence on the parameter $\varepsilon$ results.

**Theorem 4.1** *(Conjecture) Suppose that all assumptions of Theorem 3.1 are satisfied with* $\lambda_0 = -1$. *Then*

*(i) There exists* $\varepsilon_0 > 0$ *such that equation (1) has a slowly oscillating periodic solution* $x = p(t)$ *for every* $0 < \varepsilon < \varepsilon_0$*;*

*(ii) If, in addition, map* $f$ *has a globally attracting cycle* $\{a, b\} \subset I$ *of period two, then any such periodic solution* $p(t)$ *converges to* $w_{ab}(t)$ *as* $\varepsilon \to 0+$*;*

*(iii) If, in addition to assumptions in (i) and (ii), function* $f$ *is strictly decreasing on* $I$*, then there is only one such slowly oscillating periodic solution, which is also asymptotically stable with the asymptotic phase.*

# 5 Acknowledgements

This work was made possible through the RIP Programme at the Mathematisches Forschunginstitut Oberwolfach, Germany. It was initiated during the authors' research stay and joint work at the MFO over the period 10-23 March 2013.

This is a preliminary version of a research paper to be submitted for publication elsewhere.

# 6 References

[1] U. an der Heiden, Periodic solutions of a nonlinear second order differential equation with delay. *J. Math. Anal. Appl.* **70** (1979), 599–609.

[2] F.E. Browder, A further generalization of the Schauder fixed point theorem. *Duke Math. J.* **32** (1965), 575–578.

[3] S.N. Chow and J. Mallet-Paret, Singularly perturbed delay-differential equations. *Coupled nonlinear oscillators (Los Alamos, N. M., 1981), 7-12, North-Holland Math. Stud.* **80**, *North-Holland, Amsterdam,* 1983.

[4] O. Diekmann, S. van Gils, S.Verdyn Lunel, and H.O.Walther, Delay Equations: Complex, Functional, and Nonlinear Analysis. *Springer-Verlag, New York,* 1995.

[5] I. Györy and G. Ladas, Oscillation Theory of Delay Differential Equations. *Oxford Science Publications, Clarendon Press, Oxford*, 1991, 368 pp.

[6] K.P. Hadeler and J. Tomiuk, Periodic solutions of difference differential equations. *Arch. Rat. Meck. Anal.* **65** (1977), 87–95.

[7] J.K. Hale and A.F. Ivanov, On a high order differential delay equation. *J. Math. Anal. Appl.* **173** (1993), 505–514.

[8] J.K. Hale and S.M. Verduyn Lunel, Introduction to Functional Differential Equations. *Springer Applied Mathematical Sciences*, vol. 99, 1993.

[9] A.F. Ivanov and B. Lani-Wayda, Periodic solutions for three-dimensional non-monotone cyclic systems with time delays. *Discrete and Continuous Dynamical Systems* **11** (2004), nos. 2&3, 667–792.

[10] A.F. Ivanov and A.N. Sharkovsky, Oscillations in singularly perturbed delay equations. *Dynamics Reported (New Series), Springer Verlag* **1** (1991), 165–224.

[11] G. Jones, The existence of periodic solutions of $f'(x) = -\alpha f(x-1)[1+f(x)]$ . *J. Math. Anal. Appl.* **5** (1962), 435–450.

[12] G. Jones, On the nonlinear differential difference equation $f'(x) = -\alpha f(x-1)[1+f(x)]$ . *J. Math. Anal. Appl.* **4** (1962), 440–469.

[13] G. Jones, Periodic motions in Banach space and applications to functional differential equations. *Contributions to Differential Equations* **3** (1964), 75-106.

[14] J. Mahaffy, Periodic solutions of certain protein synthesis models. *J. Math. Anal. Appl.* **74** (1980), 72–105.

[15] R.N. Madan (ed.), Chuas's Circuits: A Paradigm for Chaos. *World Scientific Series on Nonlinear Science, Series B*, vol. 1 (1993), 1043 pp.

[16] J. Mallet-Paret and R.D. Nussbaum, Global continuation and asymptotic behaviour for periodic solutions of a differential-delay equation. *Ann. Mat. Pura Appl.* **145** (1986), 33–128.

[17] J. Mallet-Paret and R.D. Nussbaum, A bifurcation gap for a singularly perturbed delay equation. *Chaotic Dynamics and Fractals* **2** (1986), 263–287, (ed. by M.Barnsley and S.Demko, Academic Press, New York).

[18] J. Mallet-Paret and R.D. Nussbaum, A differential delay equation arising in optics and physiology. *SIAM J. Math. Anal.* **20** (1989), 249–292.

[19] J. Mallet-Paret and H.-O. Walther, Rapid oscillations are rare in scalar systems governed by monotone negative feedback with a time lag. *Preprint*, 1994, 35pp.

[20] J. Nagumo and M. Shimura, Self-oscillation in a transmission line with a tunnel diode. *Proc. IRE* (1961), 1281–1291.

[21] R.D. Nussbaum, Periodic solutions of some nonlinear autonomous functional differential equations. *Ann. Mat. Pura Appl.* **101** (1974), 263–306.

[22] R.D. Nussbaum, Periodic solutions of nonlinear autonomous functional differential equations. *Functional differential equations and approximation of fixed points* (Proc. Summer School and Conf., Univ. Bonn, Bonn, 1978), pp. 283–325, Lecture Notes in Math., **730**, Springer, Berlin, 1979.

[23] Ya.B. Pesin, Behavior of solutions of a strongly nonlinear differential equation with delay argument. *Differential Equations* **10**, no. 6 (1974), 789–798.

[24] A.N. Sharkovsy, Yu.L. Maistrenko and E.Yu. Romanenko, Difference Equations and Their Perturbations. *Kluwer Academic Publishers*, 1993, vol. 250, 358 pp.

[25] A.N. Sharkovsky, S.F. Kolyada, A.G. Sivak and V.V. Fedorenko, "Dynamics of One-dimensional Maps", Kluwer Academic Publishers, Ser.: Mathematics and Its Application, **407**, 1997, 261 pp.

[26] H.-O. Walther, Density of slowly oscillating solutions of $\dot{x}(t) = -f(x(t-1))$. *J. Math. Anal. Appl.* **79** (1981), no. 1, 127–140.

[27] H.-O. Walther, On instability, $\omega$-limit sets and periodic solutions of nonlinear autonomous differential delay equations. *Springer Lecture Notes in Mathematics* **730** (1979), 489–502.

[28] H.-O. Walther, Existence of a non-constant periodic solution of a nonlinear autonomous functional differential equation representing the growth of a single species population. *J. Math. Biol.* **1** (1974/75), no. 3, 227–240.

[29] E.M. Wright, A non-linear differential-difference equation. *J. Reine Angew. Math.* **194** (1955), 66-87.

[30] X. Xie, The multiplier equation and its application to S-solutions of a differential delay equation. *J. Differential Equations.* **95** (1992), 259–271.