Contributions in Surface Dynamics

A Classification of Minimal Sets of Homeomorphisms and Aspects of the Rotation Theory on the Torus.

Der Naturwissenschaftlichen Fakultät
der Friedrich-Alexander-Universität
Erlangen-Nürnberg
zur
Erlangung des Doktorgrades Dr. rer. nat.
vorgelegt von
Alejandro Passeggi Diaz Robles
aus Montevideo (Uruguay).

Als Dissertation genehmigt
von der Naturwissenschaftlichen Fakultät
der Friedrich-Alexander-Universität Erlangen-Nürnberg
Tag der mündlichen Prüfung: 11. November 2013.
Vorsitzende/r des Promotionsorgans: Prof. Johannes Barth, PhD
Gutachter/in:

Prof. Dr. Gerhard Keller

Prof. Dr. Andreas Knauf

Acknowledgments.

I deeply thank Tobias Oertel-Jäger for the opportunity of having this wonderful experience in Germany, and for his great advising during my PhD. He has introduced me to very interesting problems, has shown me very deep mathematical ideas, and has been very patient with different aspects of my development as a mathematician. Further, I want to thank him for taking care of me not only with respect to mathematical aspects, but also for my wellbeing during this time, allowing me to experience something far beyond the mathematical achievements made during my PhD.

I want to thank Martin Sambarino who played a fundamental role in making this experience possible. He was and still is a very inspiring person for my mathematical carrier, who has shown me (and several other young Uruguayan mathematicians) a very beautiful way of doing mathematics. I want to recognize here his awesome work in Uruguay, which inspires a lot of young students, and has created a wonderful community. Finally, I want to thank him for our friendship.

I want to thank Gabriel Fuhrmann and Maik Gröger, whose friendship is a treasure I have found during this experience. They are two incredible persons with whom I have shared a lot of great moments, discussions (mathematical ones included), thoughts, feelings and Caipiriña. Further, I want to acknowledge them for helping me with a lot of very annoying issues during this time. I also want to thank Vasso Anagnostopoulou and Josephine Ulrich for the very nice moments we spend in Dresden and London.

During the time of my PhD I had the opportunity of discuss mathematics with several people. I want to thank Gerhard Keller, Andreas Knauf, Patrice Le Calvez, Martin Sambarino, Rafael Potrie, Andres Sambarino, Maik Gröger, Gabriel Fuhrmann, Juliana Xavier, Pablo Davalos, Fabio Tal, Vasso Anagnostopoulou and Ferry Kwakkel for the very interesting discussions and all the profit I got from them. I also want to thank here Jorge Lewowicz, for his very inspiring insights, which go far beyond the scope of mathematics.

I thank very much my family, my mother, my father and my sister, for their love, their education, and for the gift of passion which leads me to wonderful places. I want to remember at this point my grandfathers Juan and Carlos, whom were extremely inspiring persons to me. I also want to acknowledge my family for supporting me in a very nice way during this experience.

I want to thank Rafael Potrie and Andres Sambarino for their friendship, which is an incredible source of inspiration, fun, and knowledge, among other outstanding features. They had a substantial part making this PhD being possible. Further, I want to thank Marcelo Cerminara, who also helped a lot with respect to this process.

Finally, I want to thank Valeria. She is an incredible person, full of beauty and generosity, and sharing this experience with her was a source of inspiration and strength. Above all, I want to thank Valeria for the love we share.

Claim

The results presented in this thesis are contained in the articles [JKP13], [PX], [Pas13], and [JP] listed below. These articles were written during my PhD studies, and have been published in a renowned mathematical journal, or are already submitted for publication.

1) The article [JKP13] is a joint work with T. Jäger and F. Kwakkel.

2) The article [PX] is a joint work with J. Xavier.

3) The article [Pas13] is a solo article.

4) The article [JP] is a joint work with T. Jäger.

A declaration of the co-authors of [JKP13, PX, JP] concerning the contributions I have done for the corresponding projects, will be handed in together with the thesis.

In this thesis I merge all the results of the articles in a single document, with an unified notation and a comprehensive introduction which serves as a common context for all of them.

Contents

1 Introduction. 1
1.1 Minimal sets of homeomorphisms on surfaces. 1
1.2 Rotation Theory of $\mathbb{T}^2$. 13

2 Classification of minimal sets of torus homeomorphisms. 23
2.1 Statements of the results. 23
2.2 Topological background 24
2.3 Classification of minimal sets 30
2.4 Special Cases and Applications 42
2.5 Examples. 45

3 Classification of minimal sets of surfaces homeomorphisms. 47
3.1 Statements of the results. 47
3.2 A preliminary classification. 49
3.3 Periodic disks in $\Pi_0(\mathcal{M}^c)$ 53
3.4 Type 2 minimal sets. 55
3.5 Type 3 minimal sets. 60
3.6 Proof of Theorem 3.1.1 and Theorem 3.1.2. 62
3.7 Classification Theorem in the non-wandering case. 64
3.8 Classification and Nielsen-Thurston theory. 65
3.9 Examples. 68

4 Rotation sets of generic homeomorphisms and axiom A diffeomorphisms. 71
4.1 Statements of the results. 71
4.2 Symbolic representation and rotation set. 72
4.3 Construction of the set $\mathcal{D}$. 76
4.4 Extension to any axiom A diffeomorphism. 81
4.5 Minimal sets with rotation set given by a non trivial continuum. 88

5 Toral homeomorphisms semiconjugate to irrational rotations. 95
5.1 Statements of the results. 95
5.2 Notation and preliminaries 96
5.3 Semiconjugacy to an irrational rotation 100

5.4 On the uniqueness of the semiconjugacy. 102
5.5 Fibred rotation number for foliations of circloids 104
5.6 Comments on the Topology and Rotation Sets of thin annular continua. 109

Chapter 1

Introduction.

In this thesis we present a series of results in the area of Surface Dynamic obtained during the PhD. These results are related to the study of dynamics in surfaces given by homeomorphisms without any assumptions about their regularity. The obtained results belong to the two following directions: *classification of the minimal sets of homeomorphisms of closed and orientable surfaces with genus* and *rotation theory of the two-dimensional torus*. We dedicate the following two Sections to introduce these two topics.

1.1 Minimal sets of homeomorphisms on surfaces.

In this paragraph we provide some of the most relevant results concerning *minimal sets* of homeomorphisms and flows, and present our contributions to this topic.

1.1.1 State of the art.

For any group G acting on a compact topological space $\mathcal{X}$, a compact G-invariant subspace $\mathcal{M} \subset \mathcal{X}$ is said to be minimal iff for every $x \in \mathcal{M}$ the orbit of x is dense in $\mathcal{M}$. In this general context Zorn's Lemma implies that these sets always exist, and it is natural to consider them as the basic bricks of the dynamics. They are also a natural generalization of periodic orbits, which unlike the latter has the strong property of existence for any dynamical system defined on a compact space. Moreover, they are often considered to be the topological analogue to ergodic measures.

As minimal sets relate naturally to many other dynamical notions, great effort has been devoted to the description of minimal sets and their intrinsic structure. This led to the identification of important subclasses, like almost

periodic or almost automorphic minimal sets (see, for example, [Aus88, Vee65, SY98] and references therein).

However, there exist only very few situations in which a complete classification of the possible structure of minimal sets in a given manifold is available. One of the most important cases are orientation-preserving homeomorphisms of the circle, whose minimal sets classify into either periodic orbits, Cantor sets or the whole circle. In dimension two, by means of a suitable Poincaré section, this also provides a classification of minimal sets of flows on the two-torus generated by fixed point free vector fields, which are a suspension of one of the three types occurring for circle homeomorphisms. An extensive study of flows on closed hyperbolic surfaces given in [ABZ96] yields a classification of the minimal sets in this context. The Poincaré-Bendixon Theorem for planar flows or Aubry-Mather Theory for twist maps provide further classical examples (see e.g. [KH97]). Further, it is known that the unique surfaces that can be minimal sets of some homeomorphisms are either the torus or the Klein bottle [KS09].

In dimension higher than two, relatively few results exist concerning the Topological Dynamics. It can be said that in this context the theory deals with very general problems which often have to do only with the local aspect of manifolds, since the global topology of the spaces has no general classification, unlike in the two dimensional case [1]. In particular, for minimal sets in manifolds of dimension higher than two, there are general results by Katok [Kat72] and Fathi and Herman [FH77] that provide a sufficient condition for a manifold to be the minimal set of some homeomorphism. This conditions is the existence of a locally free action of the circle $\mathbb{T}^1$ in the manifold[2]. This implies in particular that all the odd-dimensional spheres admit minimal diffeomorphisms. An eventual classification of compact n-manifolds with n higher than two, that admit a minimal map is still an open problem (see e.g. [KS09]).

There exists a class of results in which the question about the existence of minimal homeomorphisms (or maps in general) is considered for topological spaces which are no manifolds. In particular, there are several results concerning spaces of dimension between zero and one (see e.g [KS09]), and planar continua. For instance, it is proved in [Han82] that the so-called *pseudo-circle* does admit a minimal homeomorphism. Furthermore, such a homeomorphism can be considered as the restriction of a C^∞ diffeomorphism defined in the two sphere to the pseudo-circle.

Back to dimension two, recently, a classification of minimal sets of homeomorphisms of the two torus having a unique totally irrational rotation vector was given by F. Kwakkel in [Kwa11]. The proposed classification is obtained

[1] Recently was obtained a classification of the three manifolds with the renowned works of authors as Thurston, Perelman and Agol among many others.

[2] An action of $\mathbb{T}^1$ in a manifold M is locally free, if there exists a neighborhood U of the identity $e \in \mathbb{T}^1$ such that if $g \cdot x = x$ for some $g \in U$ and $x \in M$, then $g = e$.

by the study of the family of connected components in the complement of a minimal set for this kind of homeomorphism. It turns out to be a complete classification, in which the global topology of the space and the dynamics of these special homeomorphisms are highly involved.

1.1.2 Presentation of the results

The first result that we present is a classification of minimal sets of homeomorphisms on the torus, which generalizes the classification given by F. Kwakkel [Kwa11]. We remove the conditions of *being in the isotopy class of the identity* and *having a unique totally irrational rotation vector*, and we obtain a complete and detailed topological classification for the minimal sets of homeomorphisms of the torus. We also study the classification under the non-wandering assumption (which is the topological generalization of the area preserving condition), in which strong restrictions are obtained. Further, the given classification is studied in each homotopy class, and restrictions are obtained for the Anosov classes. In the homotopy class of the identity, additional results that relate the classification and Rotation Theory are obtained.

After this, in a second result we extend the classification to general closed (i.e. compact without boundary) hyperbolic surfaces. Thus, in light of the Classification Theorem of Surfaces, these two results yields a first topological classification of minimal sets of homeomorphisms on closed and orientable surfaces with positive genus.

As in the case of the torus, we study the classification for the non-wandering case and in each homotopy class in light of the so-called *Nielsen-Thurston Theory*, where restrictions are found in the case that f is homotopic to a reducible map with pseudo-Anosov components.

1.1.3 Some preliminaries.

Before presenting the results, we want to state some preliminaries concerning surface topology. As the general idea behind the classification results, is to describe the possible families of connected components in the complement of the minimal sets, we want to characterize the possible components through certain topological properties.

A *surface* is a two dimensional manifold. We say that a surface is closed if it is compact and boundaryless. The two dimensional torus is defined as $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, and the hyperbolic surfaces as $S := \mathbb{D}/G$ where $\mathbb{D}$ is the open unit disk equipped with the Poincaré metric and G a group of covering transformations given by isometries on $\mathbb{D}$. The projections are denoted by $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ in the first case and by $p : \mathbb{D} \to S$ in the second case. All the considered surfaces in this thesis are assumed to be oriented, and for a given closed surface S we denote by $g(S)$ its genus. The Classification Theorem of Surfaces states that

any closed and oriented surface with positive genus is homeomorphic to either the torus or to a closed hyperbolic surface.

We say that $U \subset S$ is a *domain* if it is an open and connected set. Further, we say that $U \subset S$ is a *disk* whenever there exists a homeomorphism between U and the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and that it is an *annulus* whenever there exists a homeomorphism between U and $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$. If an annulus $U \subset S$ contains some non-homotopically trivial loop γ of S, we say that it is an *essential* annulus.

Given a domain $U \subset S$ we denote the homotopy map induced by the inclusion as $i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$. A domain U in a surface S is said to be

(1) *trivial* if $\mathrm{Im}(i_*) = \{0\}$,

(2) *essential* if $\{0\} \neq \mathrm{Im}(i_*) \neq \pi_1(S, x_0)$,

(3) *fully essential* (or doubly essential if S is the torus) if $\mathrm{Im}(i_*) = \pi_1(S, x_0)$.

Notice that the cases (1), (2) and (3) are complementary. As easy examples to have in mind for each of the cases we can consider:

(1) a disk and a disk minus a number of points;

(2) an essential annulus, an essential annulus minus a number of points, and the complement of any essential loop in a closed surface with positive genus;

(3) the whole surface minus a number of points and the whole surface minus a Cantor set.

A domain $U \subset S$ is said to be *locally simple connected* or *locally homotopically trivial* whenever i_* is injective. Note that a disk minus a point or an annulus minus a point are not locally simply connected, and that an essential annulus is locally simply connected while a trivial annulus is not. It is a well known fact due to the Riemann Mapping Theorem that given a domain $U \subset S$, if it is trivial and locally simply connected then it is a disk, and if it is essential with $\mathrm{Im}(i_*) \cong \mathbb{Z}$ and locally simple connected then it is an essential annulus. A proof is given in [Kwa11] for the torus, and easily extends to any surface. This implies in particular, that any essential locally connected domain in $\mathbb{T}^2$ is an essential annulus.

A domain $U \subset \mathbb{T}^2$, respectively $U \subset S$, is called *bounded* if any connected components of $\pi^{-1}(U)$, respectively $p^{-1}(U)$, is pre-compact (i.e. its closure is compact).

A *continuum* X is a compact and connected topological space. A continuum C contained in either $\mathbb{T}^2$ or a hyperbolic surface S, is said to be *bounded* or *contractible* if it is contained in some bounded disk. Whenever

a domain, respectively continuum, in either the torus or a hyperbolic surface is not bounded, we say that it is *unbounded*.

We now are interested in certain continua which will help us to describe some of the cases in the classification of minimal sets. We say that a compact set $E \subset \mathbb{A}$ is *essential* if $\mathbb{A} \setminus E$ has two unbounded connected components. We say that an essential continuum A is an *annular* continuum if the connected components of $\mathbb{A} \setminus A$ are exactly the two unbounded components. A *circloid* $C \subset \mathbb{A}$ is an annular continuum that is minimal with respect to the inclusion (i.e. does not strictly contain any other annular continuum). These definitions extend to every topological annulus.

More general, given a surface S we say that $C \subset S$ is a *annular* continuum, respectively *circloid*, if there exists an annulus $\mathcal{A} \subset S$ such that C is an annular continuum of $\mathcal{A}$, respectively circloid of $\mathcal{A}$. Moreover, we say that C is essential if $\mathcal{A}$ is essential. Notice that there exist annular continua which are trivial, as to be any annular continuum contained in a disk. As basic example of essential circloids one may consider any essential loop in a surface. However, the topology of these continua can show high complexity as in the case of the so called *pseudo-circle*, and can even have non-empty interior. Circloids with complicate topology are shown to appear as invariant (and even minimal) sets of homeomorphisms defined in manifolds [Han82, LC88]. Moreover, these continua appear frequently in the theory of torus and annular homeomorphisms [Han82, LC88, Her86, FLC03, KT13, Mat10, Jäg09b], and in this thesis they will be involved in several of the different results.

A continuum $C \subset S$ is said to be non-separating if its complement contains only one connected component. The set of connected components of any topological space $\mathcal{X}$ is denoted by $\Pi_0(\mathcal{X})$. Further, if $\mathcal{X}$ is a compact set in a surface S we denote the set of essential components of $S \setminus \mathcal{X}$ as $\mathcal{E}(\mathcal{X}^c)$. It is not difficult to observe that whenever $\mathcal{E}(\mathcal{X}^c) \neq \emptyset$, then there is no fully essential component in $\mathcal{X}^c$ (a proof is given in the chapters 2 and 3 for the case of the torus and the closed hyperbolic surfaces respectively).

Given a metric space (X, d) and $C, D \subset X$, the Hausdorff distance is defined as

$$d_{\mathcal{H}}(C, D) = \max\{\sup_{x \in C} d(x, D), \sup_{y \in D} d(y, C)\}.$$

The convergence of a sequence $\{C_n\}_{n \in \mathbb{N}}$ of subsets in X to $A \subset X$ in this distance is denoted either by $C_n \to_{\mathcal{H}} A$ or by $\lim^{\mathcal{H}}_{n \to \infty} C_n = A$. Note that $d_{\mathcal{H}}(C, D) < \varepsilon$ if and only if $C \subseteq B_\varepsilon(D)$ and $D \subseteq B_\varepsilon(C)$, and that the Hausdorff distance defines a metric if one restricts to compact subsets.

We now want to state some dynamical definitions. We denote the set of homeomorphisms on a surface S by $\mathrm{Homeo}(S)$. Let f and $\widetilde{f}$ be homeomorphisms in $\mathrm{Homeo}(S)$ with minimal sets $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ respectively. We say $(f, \mathcal{M})$ is an *extension* of $(\widetilde{f}, \widetilde{\mathcal{M}})$ if there exists a continuous map $\Phi : S \to S$ such

that:

(1) $\Phi \circ f = \widetilde{f} \circ \Phi$ and $\Phi(\mathcal{M}) = \widetilde{\mathcal{M}}$;

(2) Φ is homotopic to the identity;

When $\widetilde{\mathcal{M}}$ is finite we simply say $\mathcal{M}$ is an *extension of a periodic orbit*, and when $\widetilde{\mathcal{M}}$ is a Cantor set we say $\mathcal{M}$ is an *extension of a Cantor set*. The idea behind of this notion is that the "system" $(f, \mathcal{M})$ is *simplified* by the "system" $(\widetilde{f}, \widetilde{\mathcal{M}})$, via a semiconjugacy Φ which has to verify (2). This last implies some restriction to the *simplification*, so that we do not lose too much information about global dynamic of the original system f. We point out here the topological fact that whenever a minimal set $\mathcal{M}$ is an extension of either a periodic orbit or a Cantor set, then $\Pi_0(\mathcal{M}^c)$ contains a fully essential component given by $\Phi^{-1}\left(\widetilde{\mathcal{M}}^c\right)$.

As a simple example of a minimal set $\mathcal{M}$ which is an extension of a periodic orbit, we can consider the following. Consider $f \in \text{Homeo}(S)$ which leaves invariant a disk $D \subset S$. Further we assume that $\mathcal{M} = \partial D$ is a trivial loop in S, so that $f_{|_{\mathcal{M}}}$ is conjugate to an irrational rotation. Then if we consider $S' = S/\sim$, where $x \sim y$ iff $x, y \in D \cup \mathcal{M}$, we have that S' is a surface homeomorphic to S, and the induced map $\tilde{f} \in \text{Homeo}(S')$ by f is well defined. Furthermore, we have for $q : S \to S$ that $\widetilde{M} = q(D \cup \mathcal{M}) = q(\mathcal{M})$ is a fixed point of $\tilde{f}$, and that verifies (1) and (2). So $\mathcal{M}$ is an extension of a periodic orbit. Non-trivial examples of extensions of minimal Cantor sets are difficult to construct. We quote [Kwa11] for these kind of examples.

Given a surface S and a minimal set $\mathcal{M} \subset S$ of $f \in \text{Homeo}(S)$, we say that $U \in \Pi_0(\mathcal{M}^c)$ is periodic if there exists $n \in \mathbb{N}$ such that $f^n(U) = U$. Otherwise we say that it is wandering. Notice that a wandering component verifies $f^n(U) \cap U = \emptyset$ for every $n \in \mathbb{Z} \setminus \{0\}$.

Given a map $f \in \text{Homeo}(S)$, the non-wandering set is defined as follows:

$$\Omega(f) := \{x \in S \mid \text{ for every neighborhood } U_x \text{ of } x \; \exists \; n \in \mathbb{N} : f^n(U_x) \cap U_x \neq \emptyset\}$$

The map f is said to be non-wandering if $\Omega(f) = S$. The non-wandering condition for a map f can be thought of as the topological version of the area-preserving condition.

1.1.4 General idea of the classification.

The classification of minimal sets in closed and oriented surfaces with positive genus is given by two parts. Firstly, we do the classification in the torus, and secondly in closed hyperbolic surfaces. Although the classifications in these two cases show differences, which are due to the global topology of the spaces, the idea behind the classifications is the same in both cases.

This idea is to consider for a minimal set $\mathcal{M} \subset S$ of $f \in \mathrm{Homeo}(S)$ the following three complementary cases:

(1) every element in $\Pi_0(\mathcal{M}^c)$ is trivial;

(2) there is at least one element in $\Pi_0(\mathcal{M}^c)$ which is essential;

(3) there is an element in $\Pi_0(\mathcal{M}^c)$ which is fully essential.

This first thrichotomy leads then to a precise description of the possible sets $\Pi_0(\mathcal{M}^c)$, which has the following general features:

- In case (1) every element in $\Pi_0(\mathcal{M}^c)$ is a disk;
- in case (2) every element in $\Pi_0(\mathcal{M}^c)$ is locally simply connected. Then given $U \in \Pi_0(\mathcal{M}^c)$ we have that either U is a disk or an essential component which is locally simply connected (an annulus if $S = \mathbb{T}^2$);
- in case (3) every element in $\Pi_0(\mathcal{M}^c)$ different of the fully essential component is a disk.

A detailed description concerning the dynamics of the elements in $\Pi_0(\mathcal{M}^c)$ and the structure of the minimal sets is then achieved once we restrict to the cases of the torus and of the hyperbolic surfaces. These two cases lead to particular dynamical phenomena related to the global topology of the spaces, which distinguish them. The differences arise by following the cases (1) and (2) of the first trichotomy.

For the case (3), the same is done for both the torus and the hyperbolic surfaces: we use the existence of a fully essential component in $\Pi_0(\mathcal{M}^c)$ in order to apply a result by Moore on *plane decomposition by continua*, which allows to construct a continuous map $\Phi : S \to S$ verifying:

(a) Φ is homotopic to the identity and one-to-one in the fully essential component of $\Pi_0(\mathcal{M}^c)$;

(b) $\widetilde{\mathcal{M}} := \Phi(\mathcal{M})$ is totally disconnected;

(c) for every $y \in \widetilde{\mathcal{M}}$ there exists a unique connected component C of $\mathcal{M}$ such that $C = \Phi^{-1}(y) \cap \mathcal{M}$.

The existence of this map leads us to the conclusion that the minimal set $\mathcal{M}$ has to be an extension of either a periodic orbit or a minimal Cantor set, where the semiconjugacy which defines this extension is the map Φ. Properties (a), (b) and (c) give important restrictions to the loss of dynamical information by means of the semiconjugacy.

We finish this subsection with a list of mental pictures to have in mind, for the cases (1), (2), and (3).

(1) In the torus one can consider a map f with a bounded wandering disk D, such that $\mathcal{M} := \left(\bigcup_{n\in\mathbb{Z}} f^n(D)\right)^c$ is a minimal set. On the other hand one can consider the time one map f of a flow ϕ_g given by the suspension of a Denjoy example g on the circle (see, e.g. [DV93]). The map f leaves invariant an unbounded disk D given by the suspension of the wandering interval of g, and verifies that $\mathcal{M} = D^c$ is a minimal set. Notice that while in the first case the disks in $\Pi_0(\mathcal{M}^c)$ are all bounded, in the second case $\Pi_0(\mathcal{M}^c)$ has a unique disk D which is unbounded.

In hyperbolic surfaces the construction is more subtle. For this we can make use of a full geodesic lamination Λ in S of genus g, as constructed in [CB88] such that $S\backslash\Lambda$ is the union of a collection of disks that lift to the universal covering space to ideal $2g$-gones with alternating orientations on its sides (see Figure 3.1.2). This last condition allows to complete the lamination to a flow ϕ in S with a fixed point in each disk. Moreover, the lamination is a minimal set for the flow, and one can show that there exists $t \in \mathbb{R}$ such that Λ is minimal for the time t of this flow.

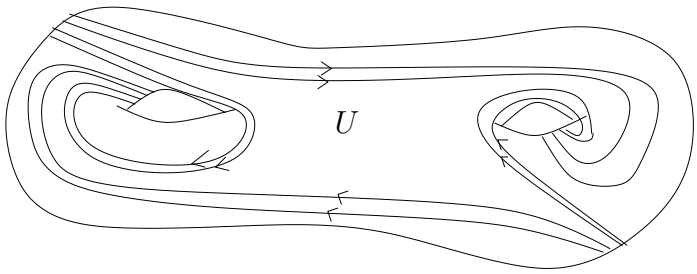


Figure 1.1.1: A minimal set in case (1) constructed by means of a geodesic lamination. The set U is a disk in $\Pi_0(\mathcal{M}^c)$.

(2) Any essential loop in the surface S whose dynamics is conjugate to an irrational rotation of the circle;

(3) A fixed point in a surface S;

Detailed constructions for all these examples and additional examples are presented in the sections 2.5 and 3.9.

1.1.5 Classification in the Torus.

The classification of minimal sets of torus homeomorphisms is given by the following theorem.

Theorem 1 ([JKP13], Theorem 1)**.** *Suppose* $f \in \mathrm{Homeo}(\mathbb{T}^2)$ *and* $\mathcal{M} \neq \mathbb{T}^2$ *is a minimal set. Then the elements of* $\Pi_0(\mathcal{M}^c)$ *consist of either:*

(1) *topological disks;*

(2) *at least one essential annulus and topological disks, where either:*

   (i) *the essential annuli in $\mathcal{M}^c$ are periodic and $\mathcal{M}$ is the orbit of the boundary of an essential periodic circloid, or*

   (ii) *every connected component in $\mathcal{M}^c$ is wandering and f is semi-conjugate to a one-dimensional irrational rotation;*

(3) *exactly one doubly essential component and a number of bounded topological disks, where either:*

   (i) *$\mathcal{M}$ is an extension of a periodic orbit, or*

   (ii) *$\mathcal{M}$ is an extension of a Cantor set.*

Moreover, in case (1) bounded periodic disks cannot occur and in case (2) only essential annuli can be periodic.

We say a minimal set $\mathcal{M}$ is of type N with $N = 1, 2, 3$ if it belongs to case N in the above classification. As we claimed before, examples for the different cases of this classification are given in Section 2.5. The classification provides a basic framework for a more precise study under different additional assumptions.

For non-wandering maps on surfaces, a result by Koropecki [Kor10] on aperiodic invariant continua allows to exclude unbounded disks in type 1 of the Classification Theorem. Together with some more elementary results on non-wandering homeomorphisms, this leads to the following more restrictive classification.

Theorem 2 ([JKP13], Theorem 4)**.** *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is non-wandering and $\mathcal{M} \neq \mathbb{T}^2$ is a minimal set. Then one of the following holds:*

(1^{nw}) *$\mathcal{M}$ is a periodic orbit;*

(2^{nw}) *$\mathcal{M}$ is the orbit of a periodic circloid;*

(3^{nw}) *$\mathcal{M}$ is the extension of a Cantor set, with all connected components non-separating.*

Note that (1^{nw}) and (3^{nw}) belong to type 3 in Theorem 2. (2^{nw}) belongs either to 2 or 3, depending on whether the circloid is essential or not, since the orbit of a homotopically trivial periodic circloid is a periodic orbit extension.

Regarding the classification in the different homotopy classes, there are two important situations in which type 1 and type 2 minimal sets can be excluded. The first is the case where $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is homotopic to an Anosov homeomorphism. Using classical results on Anosov homeomorphisms(see e.g. [Bow70a, Man74, Wal70]) one obtains the following.

Corollary 1 ([JKP13], Corollary 2)**.** *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is homotopic to an Anosov homeomorphism. Then any minimal set of f is of type 3.*

The second situation is more intricate and concerns the case where f is homotopic to the identity. For homeomorphisms in this homotopy class, given a fixed lift F of f we can consider its rotation set $\rho(F)$ defined as

$$\rho(F) := \left\{ \rho \in \mathbb{R}^2 \mid \exists x_i \in \mathbb{R}^2, n_i \nearrow +\infty : \lim_{i\to\infty} (F^{n_i}(x_i) - x_i)/n_i = \rho \right\}.$$

As we will see in Section 1.2 this is always a compact and convex set of the plane. The next results relate the given classification to the Rotation Theory on the torus.

Corollary 2 ([JKP13], Corollary 3)**.** *Suppose the rotation set of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ has non-empty interior. Then any minimal set is of type 3.*

A particular result by Misiurewicz and Ziemian [MZ91] in Rotation Theory states that for all $\rho \in \mathrm{int}(\rho(F))$ there exists a minimal set M_ρ such that ρ is the unique rotation vector on M_ρ. Corollary 2 implies that for all non-rational ρ these are extensions of Cantor sets.

Further information can be deduced for pseudo-rotations $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ (i.e. $\rho(f)$ consists of only one vector) which are non-wandering.

Corollary 3 ([JKP13], Corollary 5)**.** *Suppose f is a non-wandering pseudo-rotation with rotation vector ρ and $\mathcal{M} \neq \mathbb{T}^2$ is a minimal set.*

(a) If ρ is totally irrational (i.e. its coordinates are rationally independent), then $\mathcal{M}$ is an extension of a Cantor set.

(b) If ρ is rational (i.e. $\rho \in \mathbb{Q}^2$), then $\mathcal{M}$ is either an extension of a Cantor set, or the periodic orbit of either a point or a homotopically trivial circloid.

1.1.6 Classification in closed hyperbolic surfaces.

The classification result in surfaces is given by two theorems. In the first one, Theorem 3, we find three complementary cases for the classification, and in the second one, Theorem 4, we give a more detailed description for the second case.

Theorem 3 ([PX], Theorem A)**.** *Let S be a closed hyperbolic surface and $\mathcal{M} \subset S$ be a minimal set of $f \in \mathrm{Homeo}(S)$. Then either:*

(1) *$\Pi_0(\mathcal{M}^c)$ consists of disks, with at least one unbounded and periodic. Moreover, any bounded disk in $\Pi_0(\mathcal{M}^c)$ is wandering;*

(2) *$\mathcal{E}(\mathcal{M}^c)$ is a family of locally simply connected sets, and $\mathcal{M} = \mathcal{C}_0 \cup \ldots \cup \mathcal{C}_{n-1}$ where $\{\mathcal{C}_i\}_{i=0}^{n-1}$ is a family of pairwise disjoint continua with $f(\mathcal{C}_i) = \mathcal{C}_{i+1 mod(n)}$ for all $i = 0, \ldots, n-1$. Furthermore, each element in $\{\mathcal{C}_i\}_{i=0}^{n-1}$ coincides with a connected component of ∂V for some $V \in \mathcal{E}(\mathcal{M}^c)$, and there are no periodic bounded discs in $\Pi_0(\mathcal{M}^c)$.*

(3) $\Pi_0(\mathcal{M}^c)$ *consists of exactly one fully essential component and bounded disks. Furthermore,* $\mathcal{M}$ *is either an extension of a periodic orbit or an extension of a minimal Cantor set.*

Notice than in case (2) fully essential domains are naturally excluded of being in $\Pi_0(\mathcal{M}^c)$. As before, we refer to a minimal set $\mathcal{M}$ as a type 1, 2 or 3 minimal set whenever it is in the case 1, 2 or 3 of the classification. Examples for each case in the classification are presented in Section 3.9.

If S is any closed surface (not necessarily hyperbolic), we say that a minimal set is of type 1 if $\Pi_0(\mathcal{M}^c)$ is a family of disks. This is consistent with the classification given in Section 1.1.5 when $S = \mathbb{T}^2$, and it is the situation considered by Matsumoto and Nakayama in [MN11] when S is the sphere.

Given a continuum and type 2 minimal set $\mathcal{C} \subset S$ of $f \in \text{Homeo}(S)$, we say that it is reducible to a type 1 minimal set in S' if there exists a closed surface S' with $g(S') < g(S)$, a homeomorphism $\hat{f} : S' \to S'$ and $\mathcal{C}'$ a type 1 minimal set for $\hat{f}$ such that $f|_{\mathcal{C}}$ is conjugated to $\hat{f}|_{\mathcal{C}'}$. Furthermore, we say that a reducible minimal set $\mathcal{C} \subset S$ is g-reducible, if $g = \min\{g(S') : \mathcal{C}$ is reducible to a minimal set in $S'\}$.

The second result, which completes the classification, is the following.

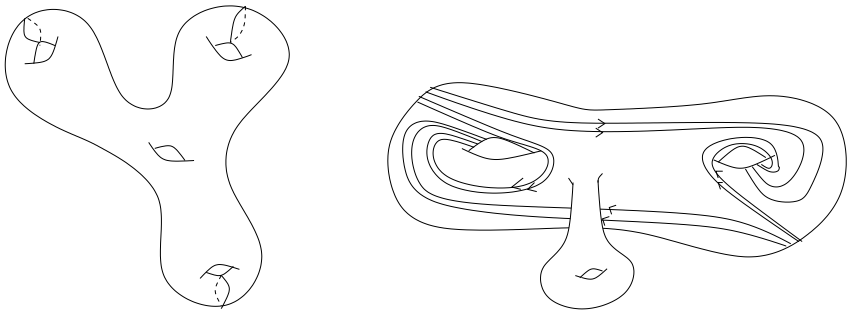
Theorem 4 ([PX], Theorem B)**.** *Let* S *be a closed hyperbolic surface and* $\mathcal{M} = \mathcal{C}_0 \cup ... \cup \mathcal{C}_{n-1}$ *a type 2 minimal set of* $f \in \text{Homeo}(S)$*. Then, for every* $k = 0, ..., n-1$ *there exists* $N > 0$ *such that* $\mathcal{C}_k$ *is reducible to a type 1 minimal set for* f^N*. Moreover,*

(2i) if $\mathcal{C}_0$ *is* 0*-reducible, then* $\mathcal{C}_k$ *is the boundary of a circloid for every* $k = 0, ..., n-1$ *and* $\#\mathcal{E}(\mathcal{C}_k^c) \leq 2$*. Furthermore, every disk in* $\Pi_0(\mathcal{M}^c)$ *is wandering.*

(2ii) if $\mathcal{C}_0$ *is* g*-reducible with* $g > 0$*, then* $\mathcal{C}_k$ *is* g*-reducible for* $k = 0, ..., n-1$*, and there are at least* $\#\mathcal{E}(\mathcal{C}_k^c)$ *unbounded fixed disks in* $\Pi_0(S' \backslash \mathcal{C}_k')$ *for* $\hat{f}$*.*

Notice that while for a minimal set $\mathcal{M}$ in the situation of the last theorem we know due to Theorem 3 that every bounded disk in $\Pi_0(\mathcal{M}^c)$ is wandering, this last theorem asserts that in the situation (2i) unbounded disks have to be wandering as well.

Basically, the last result means that there are only two ways for constructing type 2 minimal sets: one is to choose a finite family of pairwise disjoint compact essential annuli in S where the minimal set will be supported; and the other one is to consider a type 1 minimal set $\mathcal{C}'$ in some surface S' and then to extend the dynamics to a connected sum of S' and some other surfaces, by the complement of $\mathcal{C}'$. See the figures below, and Section 3.9 for a detailed description. More information about the conjugacy map between $f^N|_{\mathcal{C}_k}$ and $\hat{f}|_{\mathcal{C}_k'}$ can be obtained from its construction (see 3.6), in particular about the homotopy type of this map. We omit this here for the sake of brevity.



(a) (2i) (b) (2ii)

Figure 1.1.2: This figure illustrates the two possibilities for Type 2 minimal sets described in Theorem 4.

As we did in the case of the torus, we consider the classification for the non-wandering case, and in the different homotopy classes of homeomorphisms in light of the Nielsen-Thurston Theory. Based again on the result by A. Koropecki [Kor10], we have the following classification.

Theorem 5 ([PX], Introduction)**.** *Let S be a closed hyperbolic surface, and $\mathcal{M} \subset S$ a minimal set for a non-wandering homeomorphism $f \in \mathrm{Homeo}(S)$. Then either:*

(1^{nw}) *$\mathcal{M} = \mathcal{C}_0 \cup ... \cup \mathcal{C}_{n-1}$ where $\{\mathcal{C}_i\}_{i=0}^{n-1}$ is a family of pairwise disjoint essential circloids with $f(\mathcal{C}_i) = \mathcal{C}_{i+1 mod(n)}$ for all $i = 0, \ldots, n-1$. Furthermore, there are no disks in $\Pi_0(\mathcal{M}^c)$.*

(2^{nw}) *$\mathcal{M}$ is an extension of either a periodic orbit or a minimal Cantor set.*

Note the slight difference between the cases (1^{nw}) of last theorem and ($2i$) of Theorem 4. In the non-wandering case we have that the continua are circloids, while in the general case we only know that they are the boundary of a circloid.

Finally, we study the classification depending on the isotopy class of f. In light of Nielsen-Thurston Theory, we define the following set:

$$\mathcal{D} := \{f \in \mathrm{Homeo}(S) |\ f \sim g\ :\ g \text{ is either pA or reducible with pA components}\},$$

where "$\sim$" is the homotopy relation, and "pA" refers to pseudo-Anosov maps. We obtain thus the following:

Theorem 6 ([PX], Introduction)**.** *Let S be a closed hyperbolic surface. Then type 1 minimal sets can only occur for homeomorphisms in $\mathcal{D}^c$.*

In other words, type 1 minimal sets can only occur for homeomorphisms which are either periodic or reducible without pseudo-Anosov components. We only know examples of type 1 minimal sets for periodic homeomorphisms (indeed only for homotopic to the identity maps), and we think that only for this class these minimal sets can occur. So, we pose the following question:

Question 1.1.1. Do type 1 minimal sets exist for homeomorphisms in the homotopy class of reducible maps without pA components?

As a direct consequence of the last corollary we prove the following final result of this section.

Corollary 4 ([PX], Introduction)**.** *Let S be a closed hyperbolic surface and $f \in \mathrm{Homeo}(S)$ homotopic to a pseudo-Anosov homeomorphism. Then every minimal set of $\mathcal{M}$ is type 3.*

Now that we have the classification of minimal sets established for both cases, the torus and the closed surfaces of genus higher than one, we want to comment on their main difference. The reader may notice that while a (sub-) class in the case of the torus is given by those minimal sets which contain a wandering essential annulus in the complement, this kind of minimal sets cannot occur for surfaces of genus bigger than one.

For the homotopy class of the identity in hyperbolic surfaces there exists some developments of Rotation Theory (see [Les11] and references therein), but there is not such a concept as the rotation set. Nevertheless, although we do not consider this here, it seems to be possible to relate the Classification Theorem to Rotation Theory in this context as well.

1.2 Rotation Theory of $\mathbb{T}^2$.

In this section we summarize the state of knowledge of Rotation Theory on the two dimensional torus, and present our contributions.

1.2.1 State of the art.

Rotation Theory in the two-dimensional torus can be seen as the generalization of Poincaré's combinatorial theory for the dynamics of orientation-preserving circle homeomorphisms to the two dimensional context. In this theory, which is paradigmatic in several directions of Dynamical Systems, a complete description of the qualitative dynamics associated to an orientation-preserving homeomorphism f can be given via the so called rotation number $\rho(F)$. The latter which is defined by $\rho(F) := \lim_{n\to\infty}(F^n(x) - x)/n$ for some lift F of f and $x \in \mathbb{R}$. This limit always exists and it does not depend on x. If $\rho(F) = \frac{p}{q}$ with p, q co-prime positive integer numbers, then the limit set of the map f coincides with the set of periodic points of period q of f. Furthermore, given a lift x of a periodic point, we have $F^q(x) = x + p$. For the complementary case, that is $\rho(F) \in \mathbb{Q}^c$, one has that f is semiconjugate to the irrational rotation of angle $\rho(F)$. Further, the limit set of f is given by either the whole circle and is a minimal for f, or a minimal Cantor set whose gaps are wandering intervals. In case the whole circle is a minimal set for f, we have that the semiconjugacy is actually a conjugacy between f and the irrational rotation of angle $\rho(F)$.

In the context of the two-dimensional torus one considers the space of homeomorphisms homotopic to the identity, $\mathrm{Homeo}_0(\mathbb{T}^2)$. However, since it is impossible to ensure the independence of the asymptotic rotation vector with respect to x or even the existence as in the circle case, one has to consider the set of all possible limits. Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and $C \subset \mathbb{T}^2$ be a compact and f-invariant set, and $F : \mathbb{R}^2 \to \mathbb{R}^2$ a lift of f. The rotation set on C is defined by

$$\rho_C(F) := \left\{ \rho \in \mathbb{R}^2 \mid \exists x_i \in \pi^{-1}(C), n_i \nearrow +\infty : \lim_{i\to\infty} (F^{n_i}(x_i) - x_i)/n_i = \rho \right\}.$$

We refer to $\rho_{\mathbb{T}^2}(F)$ as the rotation set of f and denote this by $\rho(F)$. The following fundamental result is due to M. Misiurewicz and K. Zieman.

Theorem 1.2.1 ([MZ89], Theorem 3.4)**.** *$\rho(F)$ is a compact and convex set for any lift F.*

We also have the following properties for the rotation set:

(1) $\rho_C(F)$ is a compact set.

(2) If there exists a continuous map $h : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the identity such that $h \circ g = f \circ h$ for some $f, g \in \mathrm{Homeo}_0(\mathbb{T}^2)$, we say that g is semi-conjugate to f by h. In this case, we have $\rho_C(G) = \rho_{h(C)}(F)$ for any closed and g-invariant set C, and some lifts F, G of f, g respectively.

(3) Given a compact set $A \subset \mathbb{R}^2$, we denote by $\mathrm{conv}(A)$ the convex hull of A. Then we have that $\mathrm{conv}(\rho_C(F)) = \mathrm{conv}(\rho_{\Omega(f)\cap C}(F))$ where $\Omega(f)$ is the non-wandering set of f.

(4) For every $n \in \mathbb{Z}$, $\rho_C(F^n) = n \cdot \rho_C(F)$.

Properties 1,2 and 4 are not difficult to prove directly from the definition. For property 3 some ergodic-theoretical concepts related to the rotation set need to be introduced, see [MZ89] and [Dáv11].

After Misiurewicz's and Zieman's result, it can be said that Rotation Theory in the two torus has been developed by two different kind of results:

- those which establish dynamical consequences assuming different geometries for the rotation set **(A)**, and
- those trying to establish which convex sets of the plane can be realized as rotation sets **(B)**.

The state of the art for the first kind of results is more developed than for the second kind. In fact, many of the main questions are solved in the first case, while in the second there are basic problems with no answers yet. In what follows, we describe some of the most relevant results in the first class, and the state of the art for the second class.

Results in class (A).

As we said, results of this kind draw conclusions about the dynamics of the map from assumptions on the geometry of the rotation set. We have the following three principal cases for this geometry:

(1) the rotation set has non-empty interior;

(2) the rotation set is given by a line of positive length;

(3) the rotation set is given by a single point.

For the first case (1) there are two classical results which imply high dynamical complexity for maps verifying this assumption. First, a result by J. Franks which can be stated as follows.

Theorem 1.2.2 ([Fra89], Theorem 3.2)**.** *Let* $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ *and* $\frac{(p_1,p_2)}{q} \in \rho(\overset{\circ}{F}) \cap \mathbb{Q}^2$*. Then there exists a periodic point* $x \in \mathbb{T}^2$ *of period* q *such that* $F^q(\tilde{x}) = \tilde{x} + (p_1, p_2)$ *for every lift* $\tilde{x}$ *of* x*.*

This result shows the abundance of periodic orbits for maps with rotation sets having non-empty interior, which shows certain dynamical complexity. However, if one tries to estimate the growth of the cardinality of orbits of period q using this result, only a sub-exponential relation with respect to the period is obtained. So, we do not a priory obtain positive topological entropy. Nevertheless, J. Libre and R. MacKay proved the following.

Theorem 1.2.3 ([LM91], Theorem 1)**.** *Let* $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ *with* $\rho(\overset{\circ}{F}) \neq \emptyset$*. Then* f *has positive topological entropy.*

In [LM91] the authors do not provide a relation between the geometry of the rotation set and the entropy. This is done in the PhD thesis of J. Kwapisz [Kwa95a].

In [Jäg11], T. Jäger found an interesting dynamical structure for conservative maps whose rotation set has non-empty interior. In a recent article by A. Koropecki and F. Tal [KT13] the authors obtain a more detailed description of this phenomena. This structure is given by a *chaotic* region in which every neighborhood *sees* the rotation set, and its complement which is a union of periodic bounded discs. Another interesting result in case (1) is given in [MZ91], where it is proved that for any vector $v \in \rho(\overset{\circ}{F})$ there exists a compact and f-invariant set $C \subset \mathbb{T}^2$ such that $\rho_C(F) = \{v\}$. This in particular implies that for every point v in the interior of $\rho(F)$, there exits a minimal set $\mathcal{M}$ such that $\rho_{\mathcal{M}}(F) = \{v\}$.

For the second case (2) there were recently important advances. The first result in this case was given by P. LeCalvez.

Theorem 1.2.4 ([LC91]). *Suppose $\rho(F)$ is a segment with irrational slope for $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$. Then every rational point in $\rho(F)$ is realized by a periodic orbit.*

The proof in the quoted article was given for diffeomorphisms, but it can be generalized to homeomorphisms. Recently, in [Dáv12] P. Dávalos proved that if a rational point in a rotation set which is a segment is not realized, then there exists a periodic open annulus. After this, a slightly different version was proved in the conservative case under the assumption of having a rotation set given by a segment with infinitely many rational points, but avoiding the assumption about the unrealized rational point [GKT]. This last result is also announced by P. Dávalos for the non-conservative case.

The maps in case (3) are known as pseudo-rotations or pseudo-translations. For this case, there are two particular cases to which most of the attention has been paid: when $\rho(F)$ is given by a rational point (i.e. an element in $\mathbb{Q}^2$) and when $\rho(F)$ is given by a totally irrational vector (i.e. its coordinates are rationally independent).

For the first case it is known that there exists a periodic orbit realizing the rotation vector (see [Fra88]). In a recent result [KT] a classification for these pseudo-translations is given in terms of the deviations of orbits of the lift with respect to the respective rigid rotation.

In the second case one tries to see to what extent the situation in the circle can be generalized. There are examples which show that totally irrational pseudo-rotations are not necessarily semiconjugate to the respective translation [Jäg09a]. Nevertheless there exists a positive result in this sense due to T. Jäger under the assumptions of having bounded deviations with respect to the translation and being conservative [Jäg09b].

Those pseudo-translations which are semiconjugate to the respective rigid translation, but non-conjugate, are called Denjoy-type maps (inspired by the classical examples and results of A. Denjoy on the circle). In contrast with the one-dimensional case, the dynamical properties for this class of maps can be very rich, both from a topological and a measurable point of view (see [BCLR07, BCLR12]). Further, a plausible generalization for the rigidity result of Denjoy in the circle theory seems to be far from being obtained (see e.g. [McS95, PS12]).

State of the art of results in class (B).

As we said before, for this kind of problems there are still no answers to fundamental questions. For the cases (1), (2) and (3) stated in the last subsection, we have the following.

- In the first case it is not known whether every convex set in the plane with non-empty interior can be the rotation set of some homeomorphism. In

this direction, J. Kwapisz has shown that every polygon in the plane with rational vertices can be realized as a rotation set [Kwa92]. Also, Kwapisz constructs a diffeomorphism in $\mathrm{Homeo}_0(\mathbb{T}^2)$ such that its rotation set contains infinitely countably many extremal points [Kwa95b]. This last example is the unique one in which it is known that the rotation set has infinitely many vertices. Finally, a last result due to Béguin, Crovisier and Le Roux provides for every irrational vector $\rho \in \mathbb{R}^2$ an example with a rotation set containing ρ as an extremal point, and having non-empty interior [Bég06].

- For the second case, there is an open conjecture due to J. Franks and M. Misiurewicz [FM90], which states that the segments which can be realized as rotation sets either contain a rational point as an endpoint or they contain infinitely many rational points. There exist examples for all the possible cases which are allowed by this conjecture.

- For the third case, any single point in the plane can obviously be realized as a rotation set of some homeomorphism.

Concerning the Franks-Misuirewicz conjecture there are certain special families of maps where the conjecture is verified. A first family is given by all the skew-products over irrational rotations. In this case a theorem by M.Herman [Her83] proves the uniqueness of the rotation vector. A second family are those maps which are the time-one map of a flow (see [FM90]). Finally, a third family is given by those maps which preserve the leaves of a foliation of the torus by irrational lines (see [Kwa00]).

1.2.2 Presentation of the results.

Our contributions to this theory are related to the second class of results **(B)**. First, we show that generically with respect to the C^0 topology in $\mathrm{Homeo}_0(\mathbb{T}^2)$ the rotation set is given by a rational polygon (i.e. a convex set whose extremal points are finitely many points in $\mathbb{Q}^2$). Moreover, this fact is proved for an open and dense set of the space. We thus extend this result to the class of all axiom A diffeomorphisms (also known as Smale diffeomorphisms), which is a very relevant class for different topics in Dynamical Systems such as the study of the stability of maps and the study of generic dynamic. For these first two results, we deal with techniques coming from hyperbolic dynamics and classic results on axiom A diffeomorphisms. As a byproduct, the techniques we used allow us to prove the existence of minimal sets of certain maps whose rotation set is given by a non-trivial segment. These are the first examples of this type, and can be considered unexpected, since minimal sets generalize periodic orbits. Further, it is proved that in general the rotation set over a minimal set of a map is a connected set. Moreover, this last result can be related to

the Franks and Misiurewicz conjecture as this last in particular claims that a minimal torus homemorphism has necessarily a singleton rotation set. Then, our result proves that the recurrence is not enough for this, so in case of being the conjecture true the topology of $\mathbb{T}^2$ must be highly involved in the proof.

In a second group of results, we study the family of maps of extensions of irrational rotations of the circle in $\mathrm{Homeo}_0(\mathbb{T}^2)$, in light of the conjecture by Franks and Misiurewicz. If for some member in this family the rotation set is not a singleton, then it would be a counter-example for the conjecture. We first find a topological and dynamical characterization for the maps in this family, and then give a criterion concerning this characterization which guarantees the uniqueness of the rotation vector. The characterization places this family close to the skew-products over irrational rotations and the criterion generalizes the result by M. Herman [Her83]. As a byproduct, we get some results about the rotation theory for invariant *essential thin annular continua* in the annulus.

As we saw, the contributions to the Rotation Theory of the torus are given by two groups of results. We present each of them in the following two subsections.

1.2.3 Rational polygons as rotation sets of generic homeomorphisms and axiom A diffeomorphisms. Existence of minimal sets with non-singleton rotation sets.

The first result proves that generically the rotation set is given by a rational polygon. Here, a rational polygon is any convex set in the plane with finitely many extremal points which are rational. Note that this could be even segments and single points, to which we refer as degenerate rational polygons. We consider the C^0-topology in $\mathrm{Homeo}_0(\mathbb{T}^2)$ and obtain the following result.

Theorem 7 ([Pas13], Theorem 1)**.** *There exists an open and dense set $\mathcal{D} \subset \mathrm{Homeo}_0(\mathbb{T}^2)$ such that for every $f \in \mathcal{D}$ the rotation set $\rho(F)$ is a rational polygon (possibly degenerate).*

This result can be seen as a generalization of the corresponding result in the theory of the circle, which asserts that for an open and dense set in $\mathrm{Homeo}_+(S^1)$ the rotation number of any element belonging to this set is rational. Further, our result generalizes S. Zanata's result [AZ04], in which it is proved that if for some homeomorphism f the rotation set has an irrational extremal point, then the rotation set can be modified by arbitrary small perturbations of f.

The idea for proving this result is to see that the rotation set of certain *axiom A diffeomorphisms* (see Section 4.4 for the definition) are rational polygons, for which we were inspired by the articles [Kwa92] and [Zie95], and then to make use of the *genericity* and *robustness* of these diffeomorphism, to construct the set $\mathcal{D}$. After having obtained this first result, we extend it to all

Axiom A diffeomorphism in $\mathrm{Homeo}_0(\mathbb{T}^2)$ by using techniques from hyperbolic dynamics.

Theorem 8 ([Pas13], Theorem 2)**.** *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be an axiom A diffeomorphism. Then $\rho(F)$ is a rational polygon.*

Finally, we consider rotation sets on minimal sets of elements in $\mathrm{Homeo}_0(\mathbb{T}^2)$. As the definition of minimal sets can be seen as a natural generalization of periodic orbits, one could guess that the rotation set on a minimal set is given by a unique point, as in the case of periodic orbits. We see here that this is not true by using the techniques developed for the first two results. We observe in general that the rotation sets on minimal sets are continua, and then we prove the following result.

Theorem 9 ([Pas13], Theorem 3)**.** *There exists an open set $\mathcal{D}' \subset \mathrm{Homeo}_0(\mathbb{T}^2)$, such that any $f \in \mathcal{D}'$ has a minimal set $\mathcal{M}$ with $\rho_{\mathcal{M}}(F)$ given by a non-trivial segment. Furthermore, $\mathcal{D}'$ is dense in the set of homeomorphisms whose rotation set has non-empty interior.*

1.2.4 On extensions of irrational circle rotations on $\mathbb{T}^2$.

This group of results is devoted to the study of homeomorphisms in $\mathrm{Homeo}_0(\mathbb{T}^2)$ which are semiconjugate to an irrational rotation of the circle. This class of maps is interesting in light of the Franks-Misiurewicz conjecture. One can observe that in order for this conjecture to be true, elements in this family should have a unique rotation vector. Indeed, it is not difficult to see that the rotation set for this kind of maps is always contained in a line containing no rational points.

Denote by $\mathrm{Homeo}_0(\mathbb{T}^d)$ the set of homeomorphisms of the d-dimensional torus $\mathbb{T}^d$ that are homotopic to the identity and let $\mathbb{A}^d = \mathbb{S}^{d-1} \times \mathbb{R}$. There is a natural way for extending the definitions of annular continuum and circloids to $\mathbb{A}^d, d \geq 2$. We refer to Section 2.2 for these definitions in higher dimensions. Note that in any family of pairwise disjoint essential continua there exists a natural circular order on $\mathbb{T}^d$. We say a wandering essential continuum has irrational combinatorics (with respect to $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$) if its orbit is ordered in $\mathbb{T}^d$ in the same way as the orbit of an irrational rotation on $\mathbb{T}^1$ (see Section 5.3 for more details).

The first result gives a topological and dynamical characterization of the family of elements in $\mathrm{Homeo}_0(\mathbb{T}^d)$ which are semiconjugate to irrational circle rotations.

Theorem 10 ([JP], Theorem 1)**.** *Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$. Then the following statements are equivalent:*

(i) f is semiconjugate to an irrational rotation R of the circle;

(ii) there exists a wandering essential circloid with irrational combinatorics;

(iii) there exists a wandering essential continuum with irrational combinatorics;

(iv) there exists a semiconjugacy h from f to R such that for all $\xi \in \mathbb{T}^1$ the fiber $h^{-1}\{\xi\}$ is an essential annular continuum.

We note that the implication "(iii) $\Rightarrow$(i)" for the two-dimensional case is contained in [Kor07], and the proof easily extends to higher dimensions.

Related to this theorem is the question about weather the semiconjugacy is unique or not. It is easy to see that if we compose a semiconjugacy with a rigid rotation of the circle, then we obtain a new semiconjugacy, and hence we have no uniqueness. However, one could ask if the semiconjugacy is unique up to composing with rigid rotations. In Section 5.4 we give a simple example in order to see that this is not true, but we prove that such a uniqueness is true in each *externally transitive* invariant set of the considered map. This bring us by means of a result due to Potrie [Pot12], to the following.

Corollary 5 ([JP], Corollary 4.4)**.** *Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ be a totally irrational pseudo-rotation which is seminconjugate to the respective rigid translation of $\mathbb{T}^2$. Then the semiconjugacy is unique up to composing with rigid translations of $\mathbb{T}^2$.*

Back to Theorem 10, in our context the most important fact will be the equivalence "(i) $\Leftrightarrow$ (iv)", which says that the semiconjugacy can always be chosen such that its fibres are essential annular continua. This places the considered systems close to skew products over irrational rotations, with the only difference that the topological structure of the fibres can be more complicated. For this reason, one may hope to generalize Herman's result to this larger class of systems, thus proving the existence of a unique rotation vector. To that end, however, we here have to make an additional assumption on topological regularity of the fibres of the semiconjugacy.

An essential annular continuum $A \subseteq \mathbb{T}^2$ admits essential simple closed curves in its complement. The homotopy type of such curves is unique, and we define it to be the homotopy type of A. We say A is *horizontal* if its homotopy type is $(1, 0)$. Given a horizontal annular continuum A, we denote by $\widehat{A}$ a connected component of $\pi^{-1}(A)$, where $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is the canonical projection. Let $T : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x + 1, y)$. Then we say A is *compactly generated* if there exists a compact connected set $G_0 \subseteq \widehat{A}$ such that $A = \bigcup_{n\in\mathbb{Z}} T^n(G_0)$. An essential annular continuum with arbitrary homotopy type is said to be compactly generated if there exists a homeomorphism of $\mathbb{T}^2$ which maps it to a compactly generated horizontal one.

Theorem 11 ([JP], Theorem 2)**.** *Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is semiconjugate to an irrational rotation of the circle. Further, assume that the semiconjugacy h is chosen such that its fibres $h^{-1}(\xi)$ are all essential annular continua and there exists a set $\Omega \subseteq \mathbb{T}^1$ of positive Lebesgue measure such that $h^{-1}\{\xi\}$ is compactly generated for all $\xi \in \Omega$. Then f has a unique rotation vector.*

We say an annular continuum is *thin* if it has empty interior. Note that in the situation of Theorem 11, all but at most countably many of the fibres are thin in this sense.

As a byproduct of our methods, we also obtain the uniqueness of the rotation vector for invariant compactly generated thin annular continua.

Theorem 12 ([JP], Theorem 3)**.** *Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and A is a thin annular continuum that is compactly generated and f-invariant. Then $f_{|A}$ has a unique rotation vector, that is, there exists a vector $\rho \in \mathbb{R}^2$ such that $\lim_{n\to\infty} F^n(z) - z/n = \rho$ for all $z \in \mathbb{R}^2$ with $\pi(z) \in A$. Moreover, the convergence is uniform in z.*

For decomposable[3] circloids the above statement was proved before by Le Calvez [Cal], using Caratheodory's prime ends, which is a classical approach to study the rotation theory of continua [Car13a, Car13b, KLCN]. It should be noted that there exist important examples of invariant thin annular continua which are not compactly generated. One example is the Birkhoff attractor [LC88], which does not have a unique rotation vector and therefore cannot have a compact generator due to the above statement. Another well-known example is the pseudo-circle, which was constructed by Bing in [Bin51] and latter shown to occur as a minimal set of smooth surface diffeomorphisms [Han82, Her86].

We close by collecting some observations on the topology and dynamics of invariant thin annular continua. It is known that any thin annular continuum A contains a unique circloid C_A (see Lemma 5.2.2). If $\hat{A} = \pi^{-1}(A), C_{\hat{A}} = \pi^{-1}(C_A)$, we define the set of *spikes* of A as $\mathcal{S}_A := \{[\hat{A} \setminus C_{\hat{A}}]_x |\ x \in \hat{A}\}$, where $[Y]_x$ denotes the connected component of x in the topological space Y. Further, we associate to every spike $S \in \mathcal{S}_A$ the number $h(S) := \sup\{x_2 - x_1 \mid (x_1, y_1), (x_2, y_2) \in S\}$, and we say a spike S is infinite if $h(S) = \infty$. Thus, we obtain the following topological result.

Proposition 1 ([JP], Theorem 6.1)**.** *Let $A \subset \mathbb{A}$ be an essential thin annular continuum. Then the following holds.*

(1) If A has no generator either:

- *(1a) C_A has no generator, or*
- *(1b) C_A has generator and A contains an infinite spike.*

(2) If A has generator, then C_A has generator.

We remark here that two important lemmas used for the proof of this result are also crucial in the proof of Theorem 11.

[3] A continuum is called *decomposable* if it can be written as the union of two strict subcontinua.

After this topological result, we consider relations between each of the classes (1a),(1b) and (2), and the Rotation Theory of invariant continua. Given $f \in \text{Homeo}(\mathbb{A})$ which leaves invariant an essential annular continuum A, the rotation set on A for a lift F of f is given by

$$\rho_A(F) \;=\; \left\{ \rho \in \mathbb{R} \;\middle|\; \exists z_i \in \hat{A},\; n_i \nearrow \infty : \lim_{i\to\infty} \pi_1\left(F^{n_i}(z_i) - z_i\right)/n_i = \rho \right\}.$$

It is not difficult to see that this set is always a compact interval. For the class (1a), as we quoted before, there exists examples in which $\rho_A(F)$ is a non-trivial segment (see e.g. [LC88]). Nevertheless, it is not know if this is possible for the pseudo-circle, which has major importance amongst these kind of continua. For the class (2) Theorem 12 implies the uniqueness of the rotation number. For the class (1b), by reproducing some examples due to Walker [Wal91], we prove the following result.

Proposition 2 ([JP], Proposition 6.3)**.** *Given any segment $I \subset \mathbb{R}$, there exists a map $f \in \text{Homeo}(A)$ which leaves invariant an essential thin annular continuum $A \subset \mathbb{A}$ such that C_A has compact generator, A has an infinite spike, and $\rho_A(f) = I$.*

The structure of the thesis is straightforward: we devote one chapter to each group of results presented in subsections 1.1.5, 1.1.6, 1.2.3 and 1.2.4. Along these chapters several definitions and concepts will be repeated in order to have certain independence between them, and for the sake of clarity.

Chapter 2

Classification of minimal sets of torus homeomorphisms.

2.1 Statements of the results.

We first restate the results that we want to prove in this chapter. These are the theorems 1,2 and the corollaries 1,2,3, which were presented in Subsection 1.1.5. Our first Theorem is the general classification.

Theorem 2.1.1 (Classification Theorem in the Torus)**.** *Suppose* $f \in \mathrm{Homeo}(\mathbb{T}^2)$ *and* $\mathcal{M} \neq \mathbb{T}^2$ *is a minimal set. Then the elements of* $\Pi_0(\mathcal{M}^c)$ *consist of either:*

(1) *topological disks.*

(2) *at least one essential annulus and topological disks, where either:*

   (i) *the essential annuli in* $\mathcal{M}^c$ *are periodic and* $\mathcal{M}$ *is the orbit of the boundary of an essential periodic circloid, or*

   (ii) *every connected component in* $\mathcal{M}^c$ *is wandering and* f *is semi-conjugate to a one-dimensional irrational rotation,*

(3) *exactly one doubly essential component and a number of bounded topological disks, where either:*

   (i) $\mathcal{M}$ *is an extension of a periodic orbit, or*

   (ii) $\mathcal{M}$ *is an extension of a Cantor set.*

Moreover, in case (1) bounded periodic disks cannot occur and in case (2) only essential annuli can be periodic.

After the classification is established, we consider it for the non-wandering case.

Theorem 2.1.2 (Classification Theorem, non-wandering version)**.** *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is non-wandering and $\mathcal{M} \neq \mathbb{T}^2$ is a minimal set. Then one of the following holds:*

(1^{nw}) *$\mathcal{M}$ is a periodic orbit;*

(2^{nw}) *$\mathcal{M}$ is the orbit of a periodic circloid;*

(3^{nw}) *$\mathcal{M}$ is the extension of a Cantor set, with all connected components non-separating.*

Finally we consider the classification in different homotopy classes. We obtain for the Anosov case the following.

Corollary 2.1.3. *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is homotopic to an Anosov homeomorphism. Then any minimal set of f is of type 3.*

For the homotopy class of the identity, the relation between the given classification and the Rotation Theory is considered. We obtain the following results.

Corollary 2.1.4. *Suppose the rotation set of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ has non-empty interior. Then any minimal set is of type 3.*

Corollary 2.1.5. *Suppose f is a non-wandering pseudo-rotation with rotation vector ρ and $\mathcal{M} \neq \mathbb{T}^2$ is a minimal set.*

(a) If ρ is totally irrational (its coordinates are rationally independent), then $\mathcal{M}$ is an extension of a Cantor set.

(b) If ρ is rational, then $\mathcal{M}$ is either an extension of a Cantor set, or the periodic orbit of either a point or a homotopically trivial circloid.

The chapter is organized as follows. In Section 2.2, we collect several preliminary topological results which will be used in the later sections. Section 2.3 then contains the proof of the main classification. In Section 2.4, we consider the classification for the non-wandering case, and we discuss the classification in the different homotopy classes.

2.2 Topological background

In this section we collect several topological facts which are later mixed with dynamical arguments to obtain our main results. In particular, we describe a procedure to *'fill in'* subsets of the torus which is similar to a standard construction in the plane, but requires take care of some subtleties of surface topology. Even though some of these constructions may be considered folklore, we therefore spell out the details.

2.2.1 Notation

To an essential set $U \subset \mathbb{T}^2$ we can associate a homotopy type given by a vector $(p,q) \in \mathbb{Z}^2$ with $\gcd(p,q) = 1$, where (p,q) is the generator of $\mathrm{Im}(i_*)$. In this case, we call U a (p,q)-essential set. It is verified that an essential set U is (p,q)-essential if and only if every connected component $\widetilde{U}$ of $\pi^{-1}(U)$ satisfies $\widetilde{U} + (p,q) = \widetilde{U}$.

Given a set $X \subset \mathbb{T}^2$ we denote as $\Pi_0(X)$ to the set of connected components of $X \subseteq \mathbb{T}^2$. We call a connected set $A \subset \mathbb{T}^2$ *bounded*, if all connected components of $\pi^{-1}(A)$ are bounded. Note that this does not imply a uniform bound on the size of the connected components of $\pi^{-1}(A)$ (see Remark 1). We use $\overline{\mathbb{R}^2} = \mathbb{R}^2 \cup \{\infty\}$ to denote the Riemann sphere.

Given a homotopically trivial simple loop $\gamma \subseteq \mathbb{T}^2$, define the embedded Jordan disk $B(\gamma) := \pi(B(\gamma_0)) \subset \mathbb{T}^2$, where $\gamma_0 \subset \mathbb{R}^2$ is a lift of γ and $B(\gamma_0)$ the Jordan disk bounded by γ_0.

We denote the closure of any set X in any topological space by $\mathrm{cl}[X]$.

2.2.2 Fill-in of planar sets

Given any connected set $A \subseteq \mathbb{R}^2$, let $U_\infty(A)$ be the connected component of $\overline{\mathbb{R}^2} \setminus A$ which contains the point ∞. The standard way to fill in the set A is to define

$$\mathrm{Fill}_{\mathbb{R}^2}(A) \;=\; \mathbb{R}^2 \setminus U_\infty(A)\ .$$

Note that when γ is a loop in $\mathbb{R}^2$, then $\mathrm{Fill}_{\mathbb{R}^2}(\gamma)$ is just the closure of the Jordan domain of γ, which will be denoted by $B(\gamma)$. Equivalent definitions of $\mathrm{Fill}_{\mathbb{R}^2}(A)$ are the following. First, if $\{A_\alpha\}_{\alpha \in I}$ is the set of bounded connected components of $\mathbb{R}^2 \setminus A$, then

$$\mathrm{Fill}_{\mathbb{R}^2}(A) := A \cup \bigcup_{\alpha \in I} A_\alpha$$

Since the union of a connected set with a connected component of its complement is connected[1], this allows to see in particular that $\mathrm{Fill}_{\mathbb{R}^2}(A)$ is always connected. Secondly, we say that a set $A \subseteq \mathbb{R}^2$ is *filled-in* if $\overline{\mathbb{R}^2} \setminus A$ is connected, then $\mathrm{Fill}_{\mathbb{R}^2}(A)$ is just the smallest filled-in set that contains A. For domains, a third equivalent characterization is given by the first part of the next statement.

Lemma 2.2.1. *Let $A_0 \subseteq \mathbb{R}^2$ be open and connected. Then*

$$\mathrm{Fill}_{\mathbb{R}^2}(A_0) = \{z \in \mathbb{T}^2 \mid \exists \text{ a loop } \gamma \subseteq A_0 : z \in B(\gamma)\}.$$

Further, for all $v \in \mathbb{R}^2$ we have that $A_0 \cap (A_0 + v) = \emptyset$ implies $\mathrm{Fill}_{\mathbb{R}^2}(A_0) \cap (\mathrm{Fill}_{\mathbb{R}^2}(A_0) + v) = \emptyset$.

[1]This is true in any σ-compact connected Hausdorff space.

Proof. Let

$$\widetilde{A}_0 \;=\; \{z \in \mathbb{R}^2 \mid \exists \text{ a loop } \gamma \subseteq A_0 : z \in B(\gamma)\}\ .$$

Then $\widetilde{A}_0$ is simply-connected and therefore a topological disk by the Riemann Mapping Theorem. In particular, it is filled-in. Now, suppose B is another filled-in set that contains A_0, but does not contain $\widetilde{A}_0$. Then there is a loop in A_0 such that its bounded component contains a point that is not in B. However, this point cannot belong to the unbounded component of $\mathbb{R}^2 \setminus A$, which is a contradiction. It follows that any filled-in set that contains A_0 also contains $\widetilde{A}_0$, and therefore $\widetilde{A}_0 = \mathrm{Fill}_{\mathbb{R}^2}(A_0)$. The second statement is a consequence of the first statement. □

2.2.3 Fill-in of domains in the torus

Our aim is now to define a similar Fill-operation for connected subsets of the torus. This does not work for arbitrary connected subsets of the torus (see the remarks at the end of this section), but we show it does apply to subsets of $\mathbb{T}^2$ which are either domains or bounded continua. At the end of this section we collect several basic results that will be used later in the proof of our main results.

Lemma 2.2.2. *If a domain $A \subseteq \mathbb{T}^2$ is locally homotopically trivial, then any connected component of $\pi^{-1}(A) \subset \mathbb{R}^2$ is simply connected.*

Proof. Consider a connected component $\tilde{A}$ of $\pi^{-1}(A)$. Then, we have for the map $\pi_{\tilde{A}} := \pi_{|_{\tilde{A}}} : \tilde{A} \to A$ that $(\tilde{A}, \pi_{\tilde{A}})$ is a cover of A. Hence we have that the induced homotopy map $\pi_{\tilde{A}}* : \pi_1(\tilde{A}, x_0) \to \pi_1(A, \pi(x_0))$ is injective (see [Hat02], Proposition 1.31). Now, given a loop $\tilde{\gamma}$ in $\tilde{A}$ we have that $\pi(\tilde{\gamma})$ is a trivial loop in $\mathbb{T}^2$. Thus, as A is homotopically trivial we have that γ is homotopic to a constant in A. Then, the injectivity of the map $\pi_{\tilde{A}}*$ implies that $\tilde{\gamma}$ is homotopic to a constant in $\tilde{A}$[2].

□

Locally homotopically trivial domains in the torus in the above sense can be classified as follows.

Lemma 2.2.3 ([Kwa11], Lemma 7)**.** *Let $A \subset \mathbb{T}^2$ be open, trivial (respectively essential) and locally homotopically trivial. Then A is a disk (respectively essential annulus) in $\mathbb{T}^2$.*

Given a domain $A \subseteq \mathbb{T}^2$ and a connected component A_0 of its lift, define

$$\mathrm{Fill}(A) \;:=\; \pi(\mathrm{Fill}_{\mathbb{R}^2}(A_0)).$$

Note that this definition does not depend on the choice of connected component of $\pi^{-1}(A)$. Let us collect basic properties of the Fill-operation for domains.

[2]This proof was suggested by Andreas Knauf, and simplifies the proof given in the article [JKP13]

Proposition 2.2.4 (Fill-in of torus domains)**.** *Suppose $A \subseteq \mathbb{T}^2$ is a domain. Then the following hold.*

(*a*) $\mathrm{Fill}(A) = \{z \in \mathbb{T}^2 \mid \exists$ *a trivial loop* $\gamma \subseteq A : z \in B(\gamma)\}$.

(*b*) *A is trivial and bounded iff* $\mathrm{Fill}(A)$ *is a bounded disk.*

(*c*) *A is trivial and unbounded iff* $\mathrm{Fill}(A)$ *is an unbounded disk.*

(*d*) *A is (p, q)-essential iff* $\mathrm{Fill}(A)$ *is a (p, q)-annulus.*

(*e*) *A is doubly essential iff* $\mathrm{Fill}(A) = \mathbb{T}^2$.

(*f*) $\partial\mathrm{Fill}(A) \subseteq \partial A$.

(*g*) *If* $f \in \mathrm{Homeo}(\mathbb{T}^2)$, *then* $f(\mathrm{Fill}(A)) = \mathrm{Fill}(f(A))$.

Proof. The property (*a*) is a direct consequence of Lemma 2.2.1 combined with Lemma 2.2.2 applied to a connected component A_0 of $\pi^{-1}(A)$. As a consequence, we obtain that $\mathrm{Fill}(A)$ is simply-connected in $\mathbb{T}^2$, such that according to Lemma 2.2.3 it is either a disk, an essential annulus or a doubly essential set. The properties (*b*)–(*e*) therefore follow from the second part of Lemma 2.2.1.

To prove (*f*), assume for a contradiction that $z \in \partial\mathrm{Fill}(A)$ but $z \notin \partial A$. Then, since $A \subseteq \mathrm{Fill}(A)$ is open, we have $z \notin \mathrm{cl}[A]$. Let z_0 be a lift of z and denote by C the connected component of z_0 in $\mathbb{R}^2 \setminus \pi^{-1}(\mathrm{cl}[A])$. Then either C is contained in some integer translate of $\mathrm{Fill}(A_0)$, but then $z \subseteq \pi(C) \subseteq \mathrm{int}(\mathrm{Fill}(A))$, or C is disjoint from $\pi^{-1}(\mathrm{Fill}(A))$, but then $z \in \pi(C) \subseteq \mathrm{int}(\mathbb{T}^2 \setminus \mathrm{Fill}(A))$. Hence, in both cases we arrive at a contradiction.

Finally, in order to show (*g*) let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of $f \in \mathrm{Homeo}(\mathbb{T}^2)$. We then have

$$f(\mathrm{Fill}(A)) = f{\circ}\pi(\mathrm{Fill}_{\mathbb{R}^2}(A_0)) = \pi{\circ}F(\mathrm{Fill}_{\mathbb{R}^2}(A_0)) \overset{(a)}{=} \pi(\mathrm{Fill}_{\mathbb{R}^2}(F(A_0))) = \mathrm{Fill}(A).$$

This finishes the proof. □

2.2.4 Fill-in of continua in the torus

We now proceed to construct the fill-in of bounded continua in the torus. As the following remark shows, some subtleties have to be addressed and the construction does not work for general subsets of $\mathbb{T}^2$.

Remark 1. *(1) If $A \subset \mathbb{T}^2$ is a continuum, but not bounded, it is not clear how to define a fill-in. For example, consider the disjoint union of two essential loops $\gamma_1, \gamma_2 \subseteq \mathbb{T}^2$ with an infinite embedded line $\gamma \subseteq \mathbb{T}^2$ that accumulates on γ_1 in one direction and on γ_2 in the other direction. Then the complement of any lift of $\gamma_1 \cup \gamma_2 \cup \gamma$ to $\mathbb{R}^2$ will have four connected components, and* $\mathrm{Fill}(\gamma_1 \cup \gamma_2 \cup \gamma)$ *will depend on the particular choice and position of these components in the plane.*

(2) A second problem comes from the fact that even if a connected subset of $A \subseteq \mathbb{T}^2$ is bounded in the sense of Section 2.2.1, there is not necessarily a

uniform bound on the diameter of the connected components of $\pi^{-1}(A)$*. Indeed, consider an irrational foliation of the torus given by the orbits of a Kronecker flow. For each* $n \in \mathbb{N}$ *let* A_n *be a segment of length* n *in one of the leaves of this foliation, chosen such that no two segments are in the same leave. Then for any* $\varepsilon > 0$ *there exists* $n \in \mathbb{N}$ *such that* A_n *is* ε*-dense. This implies that* $A = \bigcup_{n\in\mathbb{N}} A_n$ *is connected: Suppose* $\{U, V\}$ *is an open disjoint cover of* A*. Then for* $\varepsilon > 0$ *sufficiently small both* U *and* V *contain a disk of radius* ε*, and consequently* A_n *intersects both* U *and* V *when* n *is large. Since* $A_n \subseteq A \subseteq U \cup V$*, this contradicts the connectedness of* A_n*. The connected components of* $\pi^{-1}(A)$ *are exactly the connected components of the lifts of the segments* A_n*. Hence,* A *is bounded according to the above definition, but the connected components of* $\pi^{-1}(A)$ *are not uniformly bounded in diameter.*

In order to avoid problems, we restrict to bounded continua and first show that their complement is always doubly essential. This implies immediately that for any bounded continuum $A \subseteq \mathbb{T}^2$ there is only one connected component of $\pi^{-1}(A)$ up to translation by integer vectors. As a consequence, we will be able to define the fill-in in the same way as for domains.

In what follows, for a given family of pairwise disjoint sets $\{X_n\}_{n\in\mathbb{N}}$ either in the plane or the torus, we denote its union by $\biguplus_{n\in\mathbb{N}} X_n$.

Lemma 2.2.5. *Let* $\{D_n\}_{n\in\mathbb{N}}$ *be a family of pairwise disjoint bounded open disks in the plane. Then,* $A = \mathbb{R}^2 \setminus \biguplus_{n\in\mathbb{N}} D_n$ *is a connected set.*

Proof. In order to prove the connectedness of A, we show that given any two points $z_0, z_1 \in A$ there exists a connected subset of A containing z_0 and z_1. Denote the straight line segment from z_0 to z_1 by S. We assume without loss of generality that $S = [0, 1] \times \{0\}$ and equip it with the canonical order on the unit interval. Further, define

$$N = \{n \in \mathbb{N} \mid D_n \cap S \neq \emptyset\} \text{ and } C = (S \cap A) \cup \bigcup_{n\in N} \partial D_n.$$

We claim that C is connected. In order to see this, suppose for a contradiction that $U, V \subseteq \mathbb{R}^2$ are disjoint open sets which both intersect C and whose union covers C. Suppose $z_0 \in U$ and let $z_1' \in C \cap V$. Further, let $z_- = \sup\{z \in S \cap A \cap U \mid z \leq z_1'\}$. By compactness, $z_- \in S \cap A \cap V^c = S \cap A \cap U$. Consequently, z_- is the left endpoint of an interval $I = S \cap D_n \subseteq S \setminus A$ for some $n \in \mathbb{N}$, and the right endpoint z_+ of this interval belongs to $S \cap A \cap V$. However, this means that U and V both intersect ∂D_n, contradicting the connectedness of ∂D_n. □

Using this together with Proposition 2.2.4, we can now show that compact and bounded subsets of the torus have doubly essential complement.

Lemma 2.2.6. *If $A \subseteq \mathbb{T}^2$ is compact and bounded, then A^c is doubly essential. Consequently, if A is connected, all connected components of $\pi^{-1}(A)$ project injectively onto A and coincide up to translation by an integer vector.*

Proof. Suppose that A^c is not doubly essential. Then due to Proposition 2.2.4, for every $U \in \Pi_0(A^c)$ the set $\mathrm{Fill}(U)$ is either a bounded or unbounded topological disk or an essential annulus. We distinguish between three corresponding cases and show that in each of them $\pi^{-1}(A)$ contains an unbounded connected component, contradicting the boundedness of A.

First assume that there exists $U \in \Pi_0(A^c)$ such that $\mathrm{Fill}(U)$ is an essential annulus. Then $\partial\mathrm{Fill}(U)$ consists of one or two connected components, which are both contained in A but at the same time lift to unbounded components in $\mathbb{R}^2$. Secondly, suppose there exists $U \in \Pi_0(A^c)$ such that $D = \mathrm{Fill}(U)$ is an unbounded disk. Fix a connected component $D_0 \subseteq \pi^{-1}(D)$ and a point $z_0 \in \partial D_0 \subseteq \pi^{-1}(A)$. Then for any $N \geq 0$ we can choose a sequence $\eta_n \subseteq D_0$ of arcs of diameter N converging in Hausdorff topology and such that $\eta := \lim^{\mathcal{H}}_{n\to\infty} \eta_n$ contains z_0 and is contained in ∂D_0. For example, we can identify D_0 with $\mathbb{D}$ by the Riemann Mapping Theorem and choose the η_n suitable segments in the circles of radius $1 - 1/n$. Then η is connected (as the Hausdorff limit of connected sets) and of diameter N. Since N was arbitrary, this shows that the connected component of z_0 in $\partial D_0 \subseteq \pi^{-1}(A)$ is unbounded. Finally, assume that all connected components of A^c are bounded. Then

$$\widetilde{A}_0 = \mathbb{R}^2 \setminus \bigcup_{U \in \Pi_0(\mathbb{R}^2 \setminus \pi^{-1}(A))} \mathrm{Fill}_{\mathbb{R}^2}(U)$$

is the complement of a family of bounded disks and therefore connected and unbounded by Lemma 2.2.5.

Thus, as claimed the complement of A contains a doubly essential component. This implies that A is contained in a bounded topological disk D. Every connected component D_0 of $\pi^{-1}(D)$ therefore contains a subset A_0 that projects injectively onto A. Since $\pi : D_0 \to D$ is a homeomorphism we obtain that A_0 is connected. □

Given a bounded continuum $A \subseteq \mathbb{T}^2$, we now choose an arbitrary connected component A_0 of $\pi^{-1}(A)$ and let

$$\mathrm{Fill}(A) \;:=\; \pi(\mathrm{Fill}_{\mathbb{R}^2}(A_0)) \;.$$

Since all connected components of $\pi^{-1}(A)$ coincide up to integer translations, this definition does not depend on the choice of A_0. Furthermore, we have the following.

Lemma 2.2.7. *Suppose A is a bounded continuum. Then*

$$\mathrm{Fill}(A) = A \cup \biguplus_{n \in \mathcal{N}} D_n(A),$$

where the $D_n(A)$ are bounded disks and

$$\partial D_n(A) = A \cap \mathrm{cl}[D_n(A)]$$

for every $n \in \mathcal{N} \subseteq \mathbb{N}$.

Proof. Let A_0 be a connected component of $\pi^{-1}(A)$ and $\{D_n(A_0)\}_{n\in\mathcal{N}}$ be the bounded connected components of $\mathbb{R}^2 \setminus A_0$. As A is connected, $D_n(A_0)$ is a disk for every $n \in \mathcal{N}$. Further, we have

$$\partial D_n(A_0) = A_0 \cap \mathrm{cl}[D_n(A_0)]$$

and since A_0 is bounded we have that

$$D_n(A_0) \cap (D_n(A_0) + v) = \emptyset$$

for every $v \in \mathbb{Z}^2 \setminus \{0\}$. Consequently $\pi(D_n(A_0)) = D_n(A)$ is a bounded disk for every $n \in \mathcal{N}$. We obtain that

$$\mathrm{Fill}(A) = \pi(\mathrm{Fill}(A_0)) = \pi(A_0 \cup \biguplus_{n\in\mathcal{N}} D_n(A_0)) = A \cup \biguplus_{n\in\mathcal{N}} D_n(A)$$

and

$$\partial D_n(A) = \partial\pi(D_n(A_0)) = \pi(\partial D_n(A_0)) = \pi(A_0 \cap \mathrm{cl}[D_n(A_0)]) = A \cap \mathrm{cl}[D_n(A)].$$

This finishes the proof. □

2.3 Classification of minimal sets

In this section, we prove the main classification given in the introduction.

2.3.1 Proof of the Classification Theorem

We start the proof with the following trichotomy.

Proposition 2.3.1. *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and $\mathcal{M} \neq \mathbb{T}^2$ is a minimal set. Then one of the following holds:*

(1) *$\mathcal{M}^c$ is a disjoint union of topological disks.*

(2) *$\mathcal{M}^c$ is a disjoint union of one or more essential annuli and topological disks.*

(3) *$\mathcal{M}^c$ is a disjoint union of one double essential component and bounded topological disks.*

Proof. Let $\mathcal{T}(\mathcal{M}^c) \subset \Pi_0(\mathcal{M}^c)$ be the trivial connected components and $\mathcal{E}(\mathcal{M}^c)$ the essential connected components of $\mathcal{M}^c$. Consider

$$\mathcal{M}' := \mathbb{T}^2 \setminus \bigcup_{\Sigma \in \mathcal{T}(\mathcal{M}^c)} \mathrm{Fill}(\Sigma),$$

which is compact and f-invariant by Proposition 2.2.4(g). We claim that $\mathcal{M}' \cap \mathcal{M} \neq \emptyset$. In order to see this, let

$$\mathcal{F} = \{\mathrm{Fill}(\Sigma) \mid \Sigma \in \mathcal{T}(\mathcal{M}^c)\}$$

and

$$\mathcal{D} = \left\{ \bigcup_{n \in \mathbb{N}} F_n \mid F_n \text{ is an increasing sequence in } \mathcal{F} \right\}.$$

First, note that all elements in $\mathcal{F}$ are topological disks with boundary contained in $\mathcal{M}$. Further, $\mathcal{F}$ is partially ordered by inclusion and two elements of $\mathcal{F}$ are either disjoint or one is contained in the other. Consequently, the same is true for $\mathcal{D}$. Furthermore, let $\{F_n\}_{n\in\mathbb{N}}$ be an increasing sequence in $\mathcal{F}$. Then $D = \bigcup_{n\in\mathbb{N}} F_n \in \mathcal{D}$ is a disk, and

$$\lim\nolimits^{\mathcal{H}}_{n\to\infty} \partial F_n = \partial D \subseteq \mathcal{M}.$$

Now, suppose $\{D_n\}_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{D}$. Then $\widehat{D} = \bigcup_{n\in\mathbb{N}} D_n$ is again an element of $\mathcal{D}$. Hence, if we define $\mathcal{D}_{\max}$ as the set of maximal elements of $\mathcal{D}$, then due to the Lemma of Zorn every element of $\mathcal{D}$ is contained in an element of $\mathcal{D}_{\max}$. Thus, we have

$$\mathcal{M}' \;=\; \mathbb{T}^2 \setminus \bigcup_{D\in\mathcal{D}} D \;=\; \mathbb{T}^2 \setminus \biguplus_{D\in\mathcal{D}_{max}} D\,.$$

Since $\partial D \subset \mathcal{M}$ for every $D \in \mathcal{D}_{max}$, this implies in particular that $\mathcal{M}' \cap \mathcal{M} \neq \emptyset$. By minimality of $\mathcal{M}$ we therefore have $\mathcal{M} \subset \mathcal{M}'$, so that $\mathrm{Fill}(\Sigma) = \Sigma$ for every $\Sigma \subset \mathcal{T}(\mathcal{M}^c)$. In other words, every trivial component of $\mathcal{M}^c$ is a disk.

There exists an integer vector $(p, q) \in \mathbb{Z}^2 \setminus \{0\}$ such that every element in $\mathcal{E}(\mathcal{M}^c)$ is (p, q)-essential. Moreover, the above argument adapted to this case shows that for every $\Sigma \in \mathcal{E}(\mathcal{M}^c)$ we have $\mathrm{Fill}(\Sigma) = \Sigma$, and hence every $\Sigma \in \mathcal{E}(\mathcal{M}^c)$ is a (p, q)-annulus.

To conclude the proof, it now suffices to remark that unbounded disks or essential components cannot coexist with a doubly essential component, and that any doubly essential component is necessarily unique. □

In order to show the non-existence of periodic bounded disks in $\Pi_0(\mathcal{M}^c)$, we start with a preliminary lemma.

Lemma 2.3.2. *Let $\mathcal{M} \subset \mathbb{T}^2$ be a connected minimal set of a homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$. Further, assume that there exists a periodic bounded disk $D_0 \in \Pi_0(\mathcal{M}^c)$ of period p. Then, $\mathcal{M} = \partial D_0 = ... = \partial D_{p-1}$, where $D_k = f^k(D_0)$ for $k = 1, ..., p-1$.*

Proof. Since $\partial D_0 \cup ... \cup \partial D_{p-1}$ is an f-invariant set contained in $\mathcal{M}$, we have that

$$\mathcal{M} = \partial D_0 \cup ... \cup \partial D_{p-1}.$$

Given $x \in \mathcal{M}$, let $r(x) \in \{1, ..., p\}$ be the number of disks in $\{D_0, ..., D_{p-1}\}$ for which $x \in \text{cl}[D_k]$ $(k = 0, ..., p-1)$. Now, for any $k_0 = 0, \ldots, p-1$ the set $r^{-1}(\{k : k \geq k_0\}) \subseteq \mathcal{M}$ is closed and invariant, and therefore by minimality of $\mathcal{M}$ either empty or equal to $\mathcal{M}$. This implies that r is constant, say $r = m \in \{1, \ldots, p\}$.

Now, for every $x \in \mathcal{M}$ define $I_x = \bigcap_{i=1}^{m} \partial D_{k_i}$, where $D_{k_1}, ..., D_{k_m}$ are the disks for which $x \in \text{cl}[D_{k_i}]$. For every $x \in \mathcal{M}$ the set I_x is closed, and the collection of these sets $\mathcal{Y} = \{I_x : x \in \mathcal{M}\}$ is a finite family. Suppose that it is given by $I_{x_1}, ..., I_{x_N}$. Then, we have that $\mathcal{M} = \bigcup_{i=1}^{N} I_{x_i}$. Further, $I_{x_i} \cap I_{x_j} = \emptyset$ if $i \neq j$, since $z \in I_{x_i} \cap I_{x_j}$ would imply $r(z) > m$. Therefore, by connectedness of $\mathcal{M}$ we have

$$I_{x_1} = ... = I_{x_N} = \mathcal{M},$$

which implies that

$$\mathcal{M} = \partial D_0 = ... = \partial D_{p-1}.$$

This finishes the proof. □

Lemma 2.3.3. *Let $\mathcal{M} \subset \mathbb{T}^2$ be a connected minimal set of a homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$, and assume that $\Pi_0(\mathcal{M}^c)$ does not contain any doubly essential component. Then every bounded disk $D_0 \in \Pi_0(\mathcal{M}^c)$ is non periodic.*

Proof. Let us suppose for a contradiction that there exists a periodic bounded disk $D_0 \in \Pi_0(\mathcal{M}^c)$. Since $\mathcal{M}$ is connected, Lemma 2.3.2 shows that $\mathcal{M} = \partial D_0$. Let $\widetilde{D}_0$ be a connected component of $\pi^{-1}(D_0)$. Then $\pi : \widetilde{D}_0 \to D_0$ is a homeomorphism and $\pi : \partial\widetilde{D}_0 \to \partial D_0$ is onto. We now split the proof into two cases, each leading to a contradiction. First suppose

$$\Big(\text{cl}[\widetilde{D}_0] + v\Big) \cap \text{cl}[\widetilde{D}_0] = \emptyset \text{ for every } v \in \mathbb{Z}^2 \setminus \{0\}.$$

Then $\text{cl}[D_0]$ is bounded and compact, such that Lemma 2.2.6 implies the existence of a doubly essential component in D_0^c. Secondly, assume that

$$\Big(\text{cl}[\widetilde{D}_0] + v\Big) \cap \text{cl}[\widetilde{D}_0] \neq \emptyset \text{ for some } v \in \mathbb{Z}^2 \setminus \{0\}.$$

In this case, define $r : \mathcal{M} \to \mathbb{Z}$ as $r(x) = \#\{v \in \mathbb{Z}^2 \setminus \{0\} : \widetilde{x} + v \in \partial\widetilde{D}_0\}$ where $\widetilde{x} \in \partial\widetilde{D}_0 \cap \pi^{-1}(x)$. It is verified that $r(x)$ does not depend on $\widetilde{x}$. Since $\widetilde{D}_0$ is

bounded, r is finite. Further, we have that $r^{-1}(\{k : k \geq k_0\})$ is a closed and f-invariant subset of $\mathcal{M}$ for every $k_0 \in \mathbb{Z}$. This implies by minimality of $\mathcal{M}$ that $r^{-1}(\{k : k \geq k_0\})$ is either empty or equal to $\mathcal{M}$, so $r(x)$ does not depend on $x \in \mathcal{M}$. Therefore, $r(x) = m$ for some positive integer m. Define

$$\mathcal{Y} = \left\{(v_1, ..., v_m) \in (\mathbb{Z}^2)^m : \exists z \in \partial\widetilde{D}_0 \text{ such that } z + v_1, ..., z + v_m \in \partial\widetilde{D}_0\right\}.$$

Since $\widetilde{D}_0$ is bounded, the set $\mathcal{Y}$ has to be finite, say $\mathcal{Y} = \{\xi_1, ..., \xi_N\}$ with $\xi_i = (v_1^i, \ldots, v_m^i)$. For $k = 1, ..., N$ define the sets

$$A_k = \{z \in \partial\widetilde{D}_0 : z + v_1^k, ..., z + v_m^k \in \partial\widetilde{D}_0\}.$$

It is readily verified that $A_k \subset \partial\widetilde{D}_0$ is closed and that $\partial\widetilde{D}_0 = \bigcup_{k=1}^N A_k$.

When ξ_i is just a permutation of the vector ξ_j, then obviously $A_i = A_j$. Otherwise, we must have $A_i \cap A_j = \emptyset$, since in this case the value $r(z)$ would be strictly greater than m for any $z \in A_i \cap A_j$, which is not possible. Therefore the sets A_i are either equal or pairwise disjoint. As $\partial\widetilde{D}_0$ is connected all sets A_i have to coincide, and this implies $\partial\widetilde{D}_0 = A_1$. However, this means that $z + nv_j^1 \in \partial\widetilde{D}_0$ for every $n \in \mathbb{N}$ and $j = 1, \ldots, m$, contradicting the boundedness of $\widetilde{D}_0$. □

Proposition 2.3.4. *If $\mathcal{M}$ in Proposition 2.3.1 is of type 1 or 2, then $\Pi_0(\mathcal{M}^c)$ does not contain any bounded periodic disk.*

Proof. Suppose for a contradiction that $\mathcal{M}$ is of type 1 or 2 and $D_0 \in \Pi_0(\mathcal{M}^c)$ is a bounded periodic disk of period p. Let $D_k = f^k(D_0)$ as before. Then $\bigcup_{k=0}^{p-1} \partial D_k \subseteq \mathcal{M}$ is compact and invariant, so that by minimality $\bigcup_{k=0}^{p-1} \partial D_k = \mathcal{M}$.

Let Λ_0 be the connected component of $\mathcal{M}$ which contains ∂D_0. Then Λ_0 is q-periodic for some $q \leq p$ and minimal for f^q. By Lemma 2.3.3, Λ_0^c contains a doubly essential component, and so does $f^i(\Lambda_0)^c$ for $i = 0, \ldots, q-1$. However, due to Lemma 2.2.6 this implies that

$$\mathcal{M}^c = \left(\bigcup_{i=0}^{q-1} f^i(\Lambda_0)\right)^c$$

contains a doubly essential component, contradicting our assumption. □

In the next section we will see that also unbounded disk are wandering for type 2 minimal sets.

2.3.2 Minimal sets of type 2

In this section, we give a more detailed description of minimal sets of type 2. Our aim is the following addendum to Proposition 2.3.1.

Addendum 2.3.5. *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and $\mathcal{M}$ is a minimal set of type 2. Then one of the following holds.*

(i) *The essential annuli in $\Pi_0(\mathcal{M}^c)$ are periodic and $\mathcal{M}$ is the orbit of the boundary of an essential periodic circloid. Further any disk in $\Pi_0(\mathcal{M}^c)$ is wandering.*

(ii) *f is semiconjugate to a one-dimensional irrational rotation, and every element in $\Pi_0(\mathcal{M}^c)$ is wandering.*

We start with some purely topological facts concerning circloids. We call a set $A \subseteq \mathbb{A}$ *essential*, if $\mathbb{A} \setminus A$ does not contain a connected component which is unbounded above and below. If A is bounded above, we denote by $\mathcal{U}^+(A)$ the connected component of $\mathbb{A} \setminus \mathrm{cl}(A)$ which is unbounded above. Similarly, we define $\mathcal{U}^-(A)$ when A is bounded below. Further, we write $\mathcal{U}^{-+}(A)$ instead of $\mathcal{U}^+(\mathcal{U}^-(A))$, and use analogous notation for longer concatenations of these operations. This leads to a simple procedure to produce circloids.

Lemma 2.3.6 ([Jäg09b])**.** *Suppose $A \subseteq \mathbb{A}$ is essential and bounded above. Then*

$$\mathcal{C}^+(A) \;=\; \mathbb{A} \setminus (\mathcal{U}^{+-}(A) \cup \mathcal{U}^{+-+}(A))$$

is a circloid. Further $\mathcal{U}^{+-+-}(A) = \mathcal{U}^{+-}(A)$.

We call $\mathcal{C}^+(A)$ the *upper frontier* of A, and similarly one can define a *lower frontier* $\mathcal{C}^-(A)$.

Lemma 2.3.7. *Under the assumptions of Lemma 2.3.6, we have that*

$$\partial\mathcal{C}^+(A) \;\subseteq\; \partial A.$$

In particular, any essential continuum in $\mathbb{A}$ contains the boundary of an essential circloid.

Proof. In general, when $S \subseteq \mathbb{A}$ is essential and bounded above we have $\partial\mathcal{U}^+(S) \subseteq \partial S$, and the analogous statement holds if S is bounded below. Applying this several times, we obtain

$$\partial\mathcal{U}^{+-+}(A) \;\subseteq\; \partial\mathcal{U}^{+-}(A) \;\subseteq\; \partial\mathcal{U}^{+}(A) \;\subseteq\; \partial A.$$

Furthermore, $E^+ = \partial\mathcal{U}^{+-+}(A)$ is an essential continuum which is disjoint from $\mathcal{U}^{+-}(A)$ and $\mathcal{U}^{+-+}(A)$ and therefore contained in $\mathcal{C}^+(A)$. Consequently $C^+ = \mathbb{A} \setminus (\mathcal{U}^-(E^+) \cup \mathcal{U}^+(E^+))$ is an annular continuum contained in $\mathcal{C}^+(A)$, and by minimality of the latter we obtain $C^+ = \mathcal{C}^+(A)$. However, this means that

$$\begin{aligned}\partial\mathcal{C}^+(A) \;&=\; \partial C^+ \;=\; \partial(\mathbb{A} \setminus \partial C^+)\\ &=\; \partial(\mathcal{U}^-(E^+) \cup \mathcal{U}^+(E^+))) \;\subseteq\; \partial E^+ \;=\; \partial\mathcal{U}^{+-+}(A) \;\subseteq\; \partial A,\end{aligned}$$

as required. □

Given two essential continua E_1 and E_2, we write $E_1 \prec E_2$ if $E_1 \subseteq \mathcal{U}^-(E_2)$. We say that a sequence of essential continua $\{E_n\}_{n\in\mathbb{N}} \subset \mathbb{A}$ is bounded if there exist two essential continua $E, F \subset \mathbb{A}$ such that $E \prec E_n \prec F$ for every $n \in \mathbb{N}$.

Lemma 2.3.8. *Suppose $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathbb{A}$ is a bounded sequence of essential continua with $E_n \prec E_{n+1}$ for all $n \in \mathbb{N}$. Then the E_n converge in Hausdorff limit to the essential continuum $\partial\mathcal{U}^-$, where $\mathcal{U}^- = \bigcup_{n\in\mathbb{N}} \mathcal{U}^-(E_n)$.*

Proof. We have to show that

$$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ :\quad \partial\mathcal{U}^- \subseteq B_\varepsilon(E_n) \text{ and } E_n \subseteq B_\varepsilon(\partial\mathcal{U}^-)\ .$$

Note that for all $n \in \mathbb{N}$ the set E_n is contained in $\mathcal{U}^-(E_k)$ for all $k > n$. Conversely, $\partial\mathcal{U}^- \subseteq \mathcal{U}^+(E_n)$ for all $n \in \mathbb{N}$, since the bounded connected components of $\mathbb{A} \setminus E_n$ are all contained in $\mathcal{U}^-(E_{n+1}) \subseteq \mathcal{U}^-$ and can therefore not intersect $\partial\mathcal{U}^-$.

Now, first assume that there exist infinitely many $n \in \mathbb{N}$ with $\partial\mathcal{U}^- \nsubseteq B_\varepsilon(E_n)$. Choose a sequence $n_i \nearrow \infty$ and $z_i \in \partial\mathcal{U}^-$ with $z_i \notin B_\varepsilon(E_{n_i})$. Note that this implies $z_i \notin B_\varepsilon(E_{n_j})$ for all $j \leq i$, since the straight arc from z_i to the nearest point in E_{n_j} first has to pass through E_{n_i}. By compactness, we may assume that the limit $z = \lim_{i\to\infty} z_i \in \partial\mathcal{U}^-$ exists. Then $B_\varepsilon(z) \cap E_n = \emptyset$ for all $n \in \mathbb{N}$. Obviously $B_\varepsilon(z)$ cannot be contained in $\mathcal{U}^-(E_n)$ for any $n \in \mathbb{N}$. Further, $B_\varepsilon(z)$ can also not be contained in a bounded component of $\mathbb{A} \setminus E_n$, since it would then be contained in $\mathcal{U}^-(E_{n+1})$. Consequently $B_\varepsilon(z) \subseteq \mathcal{U}^+(E_n)$ for all $n \in \mathbb{N}$. However, this means that $\mathcal{U}^-$ does not intersect $B_\varepsilon(z)$, contradicting $z \in \partial\mathcal{U}^-$.

Conversely, suppose $E_n \nsubseteq B_\varepsilon(\partial\mathcal{U}^-)$ for infinitely many $n \in \mathbb{N}$. Choose $n_i \nearrow \infty$ and $z_i \in E_n \setminus B_\varepsilon(\partial\mathcal{U}^-)$ so that the limit $z = \lim_{i\to\infty} z_i$ exists. Then on the one hand we have $z \notin B_{\varepsilon/2}(\partial\mathcal{U}^-)$, but on the other hand z is a limit point of points $z_i \in E_n \subseteq \mathcal{U}^-(E_{n+1}) \subseteq \mathcal{U}^-$, a contradiction. This shows that (2.3.2) holds and thus $\lim^{\mathcal{H}}_{n\to\infty} E_n = \partial\mathcal{U}^-$ as claimed. □

We now turn to minimal sets of type 2, starting with a simple observation.

Lemma 2.3.9. *Suppose $\mathcal{M}$ is a minimal set of $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and $\mathcal{M}^c$ contains an essential annulus $\mathcal{A}$ of homotopy type (p, q). Then (p, q) is an eigenvector of the induced action f_* on homotopy.*

In fact, the assertion of the lemma is true for any annulus which is either invariant or disjoint from its image.

Proof of Lemma 2.3.9. When $\mathcal{A}$ is invariant, then the fact that its homotopy vector is preserved is obvious. When $f(\mathcal{A})$ and $\mathcal{A}$ are disjoint, this follows from the fact that essential annuli of different homotopy types have to intersect. □

Now, choose $A \in \mathrm{SL}(2,\mathbb{Z})$ such that $(p,q) = A \cdot (1,0)^t$ and let f_A be the torus homeomorphism induced by A. Then $(1,0)$ is an eigenvector of the action on homotopy of

$$\widehat{f} \;=\; f_A^{-1} \circ f \circ f_A \;,$$

and this implies that there exists a lift $\widetilde{f} : \mathbb{A} \to \mathbb{A}$ which projects to $\widehat{f}$ under the canonical projection $\pi_{\mathbb{A}} : \mathbb{A} \to \mathbb{T}^2$. We either have $\widetilde{f}(z + (0,1)) = \widetilde{f}(z) + (0,1)$ or $\widetilde{f}(z + (0,1)) = \widetilde{f}(z) - (0,1)$. We call $\widetilde{f}$ *order-preserving* in the first case and *order-reversing* in the second. When $\widetilde{f}$ is order-preserving, we define the *rotation interval of f orthogonal to (p,q)* by

$$\rho_{(p,q)}(f) \;=\; \frac{1}{\|(p,q)\|_2} \cdot \left\{ \lim_{i\to\infty} \left(\pi_2 \circ \widetilde{f}^{n_i}(z_i) - \pi_2(z_i) \right) / n_i \;\middle|\; n_i \nearrow \infty,\; z_i \in \mathbb{A} \right\}.$$

Of course, due to the freedom in the choice of the lift $\widetilde{f}$ the interval $\rho_{(p,q)}$ is only well-defined up to translation by integer multiples of $\|(p,q)\|_2$, and we will implicitly understand it in this sense. Note that when f is homotopic to the identity, then $\rho_{(1,0)}(\widetilde{f})$ is just the projection of $\rho(F)$ to the second coordinate. In general, it is the projection of $\rho(F)$ to the line $(-q,p)\cdot\mathbb{Z}$.

In the order-reversing case, we apply the above definition to f^2 and let $\rho_{(p,q)}(f) = \rho_{(p,q)}(f^2)/2$. In this case, we have

Lemma 2.3.10. *If $\widetilde{f}$ is order-reversing, then $\rho_{(p,q)}(f)$ contains 0.*

Proof. For every $n \in 2\mathbb{Z}+1$ the map $\widetilde{f}^n$ reverses orientation, so that $D(k) = \pi_2 \circ \widetilde{f}^n(0,k) - k$ goes to $\pm\infty$ as k goes to $\mp\infty$. Consequently, for sufficiently large k the numbers $D(k)$ and $D(-k)$ have opposite sign. Therefore, by the Intermediate Value Theorem any arc joining $(0,k)$ to $(0,-k)$ contains a point with $\pi_2 \circ \widetilde{\pi}_2 \circ f^n(z) - \pi_2(z) = 0$. □

In the same way, it is shown that $\rho_{(p,q)}(f)$ is connected and, in the order-reversing case, symmetric around 0. In the situation we consider, the rotation interval is degenerate.

Lemma 2.3.11. *Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and suppose there exists an annulus $\mathcal{A} \subseteq \mathbb{T}^2$ of homotopy type (p,q) which is either periodic or wandering. Then $\rho_{(p,q)}(f)$ contains a single number. If $\mathcal{A}$ is periodic, this number is rational.*

In particular, suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ has a periodic or wandering annulus. Then the rotation set $\rho(F)$ is contained in a rational line.

This is a direct corollary to [Kor07, Lemma 1.4], and we omit the simple proof. From now on, we identify $\rho_{(p,q)}(f)$ with the unique real number ρ it contains and call it the *rotation number of f orthogonal to (p,q)*. The following lemma deals with the case where this rotation number is irrational. We omit the proof, which can be found in [Kor07]. The author uses an additional minimality assumption, but this is actually not needed. Alternatively, the result also follows from a minor modification of [Jäg09b, Proof of Theorem C].

Lemma 2.3.12. *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ has a wandering annulus $\mathcal{A}$ of homotopy type (p,q) and $\rho_{(p,q)}(f)$ is irrational. Then f is semiconjugate to the corresponding irrational rotation on $\mathbb{T}^1$.*

In order to treat the rational case, we first need some more information concerning circloids.

Lemma 2.3.13. *Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and suppose there exists a wandering circloid $C \subseteq \mathbb{T}^2$ of homotopy type (p,q). Further, assume $\rho_{(p,q)}(f) = 0$ and let $C_0 \subseteq \mathbb{A}$ be a lift of $f_A(C)$, where $A \in SL(2,\mathbb{Z})$ is chosen as in (2.3.2). Then $C_0 - (0,1) \prec \widetilde{f}^{2n}(C_0) \prec C_0 + (0,1)$ for all $n \in \mathbb{N}$. In particular, the orbit of C_0 under $\widetilde{f}$ is bounded.*

Proof. Since C is wandering, $\widetilde{f}^{2n}(C)$ is disjoint from $C_0 + (0,1)\cdot\mathbb{Z}$ for all $n \neq 0$. Suppose for a contradiction that $\widetilde{f}^{2n}(C_0)$ does not lie between $C_0 - (0,1)$ and $C_0 + (0,1)$ for some $n \in \mathbb{N}$, for example $\widetilde{f}^{n}(C_0) \prec C_0 + (0,1)$. Then, by induction $\widetilde{f}^{i2n} \prec C_0 + (0,i)$. This, however, implies that the rotation number is strictly positive, contradicting the assumptions. □

Lemma 2.3.14. *Let $\widetilde{f} \in \mathrm{Homeo}(\mathbb{A})$ and suppose E is an essential continuum which is disjoint from its image and has a bounded orbit. If $\widetilde{f}$ is order-preserving, then $\lim^{\mathcal{H}}_{n\to\infty} \widetilde{f}^{n}(E)$ exists and contains an invariant circloid. If $\widetilde{f}$ is order-reversing, then $\lim^{\mathcal{H}}_{n\to\infty} \widetilde{f}^{2n}(E)$ exists and contains a circloid which is either invariant or two-periodic.*

Proof. It suffices to treat the order-preserving case, since we only have to consider $\widetilde{f}^2$ when $\widetilde{f}$ reverses order. We either have $E \prec \widetilde{f}(E)$ or $E \succ \widetilde{f}(E)$. We treat the first case, the other one is similar.

If we let $E_n := \widetilde{f}^{n}(E)$, then this is an increasing sequence with respect to $\prec$ and converges in Hausdorff distance to the essential continuum ∂U^- given by Lemma 2.3.8. Since $\partial U^- = \lim^{\mathcal{H}}_{n\to\infty} \widetilde{f}^{n}(E)$ is contained in $\mathcal{M}$, the statement follows from Lemma 2.3.7. □

We now turn to the proof of the fact that only essential annuli can be periodic connected components of the complement of a type two minimal set. We start with the following well-known fact.

Lemma 2.3.15. *Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and suppose $\mathcal{A} \subset \mathbb{T}^2$ is an f-invariant essential annulus. Then, for every essential simple loop γ in $\mathcal{A}$ and any neighbourhood $V \subseteq \mathcal{A}$ of γ there exists $g \in \mathrm{Homeo}(\mathbb{T}^2)$ such that:*

(i) g is homotopic to f;

(ii) $g(\gamma) = \gamma$;

(iii) $f(x) = g(x)$ for every $x \in V^c$.

To a homeomorphism g as in the lemma, we can naturally associate a homeomorphism $\overline{g} : \mathbb{S}^2 \to \mathbb{S}^2$, where $\mathbb{S}^2$ is the two-dimensional sphere by cutting the torus along γ to obtain an open annulus $\mathbb{T}^2 \setminus \gamma$ and then compactifying this annulus by adding two points N and S. Then $g|_{\gamma^c}$ is conjugate to $\overline{g}|_{\{N,S\}^c}$ by a semiconjugacy $h : \mathbb{T}^2 \setminus \gamma \to \mathbb{S}^2 \setminus \{N, S\}$ and $\overline{g}(\{N, S\}) = \{N, S\}$. Furthermore, h maps the two components of $\mathrm{cl}[\mathcal{A}] \setminus \gamma$ to two different components U_1 and U_2 in S^2 with $\mathrm{cl}[U_1] = U_1 \cup \{N\}$ and $\mathrm{cl}[U_2] = U_2 \cup \{S\}$. The advantage that this transformation to a sphere homeomorphism has, is that it allows to apply the following theorem by Matsumoto and Nakayama [MN11].

Theorem 2.3.16 ([MN11])**.** *Let $\overline{g} : S^2 \to S^2$ be a homeomorphism and $C \subset S^2$ be a non singleton compact and connected set. Further assume that C is a minimal set of $\overline{g}$. Then, there are exactly two periodic connected components A_1 and A_2 in C^c.*

In our context, it is obvious from the invariance of $\mathcal{A}$ that A_1 and A_2 are the images of the two components of $\mathcal{A} \setminus \gamma$ under h. We obtain the following.

Proposition 2.3.17. *Suppose that $\mathcal{M}$ is a type 2 minimal set for $f \in \mathrm{Homeo}(\mathbb{T}^2)$. Further suppose there exists an essential annulus $\mathcal{A}$ in $\Pi_0(\mathcal{M}^c)$ which is periodic. Then any disk in $\Pi_0(\mathcal{M}^c)$ is wandering.*

Proof. We denote by $C_1, C_2 \subset \mathbb{T}^2$ the two connected components of $\partial\mathcal{A}$, allowing $C_1 = C_2$ in case $\partial\mathcal{A}$ is connected. Then due to the fact that $\mathcal{A}$ is periodic there exists $n \in \mathbb{N}$ such that C_1 and C_2 are minimal sets of f^n and $\mathcal{M} = (C_1 \cup C_2) \cup ... \cup f^n(C_1 \cup C_2)$. Hence, given a disk $V \in \Pi_0(\mathcal{M}^c)$, due to ∂V be a connected set contained in $\mathcal{M}$ there exists $n_0 \in \mathbb{N}$ such that $V_1 := f^{n_0}(\partial V) \subset C_1 \cup C_2$. We assume without loss of generality $\partial V \subseteq C_1$. This means that V_1 is a connected component of C_1^c. However, if we consider $g \in \mathrm{Homeo}(\mathbb{T}^2)$ given by Lemma 2.3.15 applied to f^n and some essential loop $\gamma \subseteq \mathcal{A}$, we have that C_1 is a minimal set for g.

Suppose for a contradiction that V is periodic by f. Then V_1 is a periodic connected component of C_1^c for g. However, as V_1 is a disk it cannot coincide with one of the two periodic components A_1 and A_2 that g admits. This contradicts Theorem 2.3.16. □

We are ready now to give the proof of the Addendum 2.3.5.

Proof of Addendum 2.3.5. Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and $\mathcal{M}$ is a minimal set of type 2. Let $\mathcal{A}$ be an essential annulus of homotopy type (p, q) in $\Pi_0(\mathcal{M}^c)$. Then $\mathcal{A}$ is either wandering or periodic, and in each case $\rho_{(p,q)}(\widetilde{f})$ contains a unique number $\rho \in \mathbb{R}$ by Lemma 2.3.11. If ρ is irrational, then $\mathcal{A}$ is wandering by Lemma 2.3.11, and Lemma 2.3.12 provides the existence of a semiconjugacy to an irrational rotation of $\mathbb{T}^1$. Furthermore, due to the existence of such a semi-conjugacy any element in $\Pi_0(\mathcal{M}^c)$ is wandering. Thus, we are in case (ii) of the addendum.

Now, assume ρ is rational. Passing to an iterate f^k and choosing the right lift $\widetilde{f}$ of $\widehat{f^k}$ in (2.3.2), we may assume without loss of generality that $\rho_{(p,q)}(f^k) = 0$. Let $\mathcal{A}_0$ be a lift of $f_A(\mathcal{A})$, where $A \in \mathrm{SL}(2,\mathbb{Z})$ is chosen as in (2.3.2), and let $\widetilde{f}$ be the lift of f^k used to compute the rotation interval of f^k. Then either by invariance or by Lemma 2.3.13, the orbit of $C_0 = \mathcal{C}^+(\mathcal{A}_0)$ under $\widetilde{f}$ is bounded. Hence, by Lemma 2.3.14, $\lim^{\mathcal{H}}_{n\to\infty} \widetilde{f}^{2n}(C_0)$ contains an $\widetilde{f}^2$-invariant circloid $\widetilde{C}$. Since $\widetilde{C}$ is disjoint from $\mathcal{A}_0 + (0,1)\cdot\mathbb{Z}$, it projects to a circloid C on $\mathbb{T}^2$ which is $2k$-periodic under f. Furthermore, C is contained in the Hausdorff limit of $f^{2kn}(\partial A)$ and thus in $\mathcal{M}$. By minimality, we obtain $\mathcal{M} = \bigcup_{n=1}^{2k} f^n(C)$. Moreover, in this case Proposition 2.3.17 implies that any disk in $\Pi_0(\mathcal{M}^c)$ has to be wandering, which means that we are in case (i) of the addendum. □

Remark 2. *The above proof shows that case (i) of the addendum corresponds exactly to a rational rotation number orthogonal to the homotopy vector of the essential annuli, whereas case (ii) corresponds to an irrational rotation number.*

2.3.3 Minimal sets of type 3

The following addendum to Proposition 2.3.1 concerning the structure of minimal sets of type 3 completes the proof of Theorem 2.1.1.

Addendum 2.3.18. *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and $\mathcal{M}$ is a minimal set of type 3. Then $\mathcal{M}$ is an extension of either a periodic orbit or a Cantor set.*

Again, we first recall some purely topological facts. We say $U \subseteq \mathbb{T}^2$ is *non-separating* if $\mathbb{T}^2 \setminus U$ is connected. We call a partition into continua $\mathcal{U} = \{U_i\}_{i\in I}$ of $\mathbb{T}^2$ an *upper semi-continuous decomposition* if it satisfies

(i) $\biguplus_{i\in I} U_i = \mathbb{T}^2$ and $U_i \cap U_j = \emptyset$ if $i \neq j$,

(ii) U_i is a compact, bounded and non-separating set for every $i \in I$;

(iii) if $\{U_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ has Hausdorff limit $\mathcal{C}$, then there exists $U_0 \in \mathcal{U}$ so that $\mathcal{C} \subset U_0$ (upper semi-continuity property).

Further, we say that a map $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ is a Moore projection for the decomposition $\mathcal{U}$ if it satisfies

(i) Φ is continuous and surjective;

(i) Φ is homotopic to the identity;

(ii) $\Phi^{-1}(x) \in \mathcal{U}$ for all $x \in \mathbb{T}^2$.

Now, the following is a classical decomposition theorem by R. Moore (see e.g.[Why42]).

Moore's Theorem. *For any upper semi-continuous decomposition of* $\mathbb{T}^2$ *there exists a Moore projection.*

For our purposes, we have to ensure that under suitable conditions this projection produces a totally disconnected set.

Lemma 2.3.19. *Let* $A \subset \mathbb{T}^2$ *be closed and denote its connected components by* $\{A_i\}_{i\in I}$*. Suppose that the decomposition* $\mathcal{U} = \{A_i\}_{i\in I} \cup \{\{x\} : x \notin \bigcup_{i\in\mathbb{N}} A_i\}$ *is upper semi-continuous. Then for any Moore projection* Φ *associated to* $\mathcal{U}$*, the image* $\Phi(A)$ *is totally disconnected.*

Proof. Suppose for a contradiction that $\Phi(A)$ is not totally disconnected, then there exists a connected component $C \subseteq \Phi(A)$ which has more than one element. Since connected components of A project to single points, the set $\Phi^{-1}(C) \subseteq A$ cannot be connected and therefore decomposes into two disjoint relatively closed subsets C_1 and C_2. As a connected component of a compact set, C is compact, and the same is true for its pre-image $\Phi^{-1}(C)$. Hence, both C_1 and C_2 are compact.

For any $x \in \mathbb{T}^2$, the continuum $\Phi^{-1}(x)$ of A is completely contained in either C_1 or C_2, and therefore the images $\Phi(C_1)$ and $\Phi(C_2)$ are disjoint. However, this means that C decomposes into two disjoint compact sets, contradicting its connectedness. □

Finally, the following statement will be useful to verify the upper semi-continuity of decompositions.

Lemma 2.3.20. *Let* A_n*,* $n \in \mathbb{N}$ *be a family of compact, connected and bounded sets in* $\mathbb{T}^2$*. If the sets* $\mathrm{Fill}(A_n)$ *are pairwise disjoint and* $A_n \to_{\mathcal{H}} \mathcal{A}$*, then* $\mathrm{Fill}(A_n) \to_{\mathcal{H}} \mathcal{A}$*.*

Proof. For any fixed $\varepsilon > 0$, the fact that $A_n \to_{\mathcal{H}} \mathcal{A}$ implies that, for any fixed $\varepsilon > 0$, $\mathcal{A} \subseteq B_\varepsilon(A_n) \subseteq B_\varepsilon(\mathrm{Fill}(A_n))$ for sufficiently large n. Therefore, it suffices to show that conversely $\mathrm{Fill}(A_n) \subseteq B_\varepsilon(\mathcal{A})$ for sufficiently large n.

Suppose for a contradiction that for some $\varepsilon > 0$ there is a sequence of integers $n_k \nearrow \infty$ such that for each $k \in \mathbb{N}$ there exists some $x_k \in \mathrm{Fill}(A_{n_k}) \setminus B_{2\varepsilon}(\mathcal{A})$. Since $A_n \to_{\mathcal{H}} \mathcal{A}$, for k large enough, we have that $A_{n_k} \subset B_\varepsilon(\mathcal{A})$. By Lemma 2.2.7, the point x_k is contained in some disk D_k with $\partial D_k \subseteq A_{n_k}$, and for large k we have $\partial D_k \subseteq B_\varepsilon(\mathcal{A})$. Let x_0 be an accumulation point of $\{x_k\}_{k\in\mathbb{N}}$. Then $x_0 \notin B_\varepsilon(A)$, and further x_0 cannot belong to any disk D_k since these are pairwise disjoint, which follows from the assumption that the sets $\mathrm{Fill}(A_n)$ are pairwise disjoint combined with Lemma 2.2.7. Now, let $k \in \mathbb{N}$ such that $D_k \cap B_\varepsilon(x_0) \neq \emptyset$ and $\partial D_k \subset B_\varepsilon(\mathcal{A}) \subseteq B_\varepsilon(x_0)^c$. Then the closest point of $\mathrm{cl}[D_k]$ to x_0 is contained in $B_\varepsilon(x_0)$ and can therefore not belong to the boundary of D_k, a contradiction. □

We now finish the proof

Proof of Addendum 2.3.18. Let $\{\Lambda_i\}_{i\in I} := \Pi_0(\mathcal{M})$. Since $\mathcal{M}$ is of type 3, every element in $\{\Lambda_i\}_{i\in I}$ is bounded. Now, $\mathcal{U} = \{\mathrm{Fill}(\Lambda_i)\}_{i\in I} \cup \{\{x\} : x \notin \bigcup_{i\in I} \mathrm{Fill}(\Lambda_i)\}$ is a family of bounded continua in $\mathbb{T}^2$. We claim that it is an upper semi-continuous decomposition of $\mathbb{T}^2$.

First let us see that $\mathcal{U}$ is a partition. Suppose for a contradiction that $\mathrm{Fill}(\Lambda_i) \cap \mathrm{Fill}(\Lambda_j) \neq \emptyset$ for some $i \neq j \in I$. Then since $\Lambda_i \cap \Lambda_j = \emptyset$, Λ_i has to be contained in a bounded connected component D of Λ_j^c or vice versa. By Proposition 2.3.1, since the compact boundary of the unique doubly essential component of $\mathcal{M}^c$ is left invariant and it is contained in the minimal set $\mathcal{M}$, by minimality, $\mathcal{M}$ equals the boundary of the unique doubly essential component. If $\Lambda_i \subset D$, with D a bounded complementary domain of Λ_j^c, then an open neighborhood of a point of Λ_i does not intersect the doubly essential component, even though $\Lambda_j \subset \mathcal{M}$, a contradiction.

To proceed, the elements of $\mathcal{U}$ are non-separating. Hence, it remains to check the upper semi-continuity. For this the only non-trivial case is when the sequence of elements in $\mathcal{U}$ is given by elements in $\{\mathrm{Fill}(\Lambda_i)\}_{i\in I}$. Take a countable subsequence $\{\mathrm{Fill}(\Lambda_n)\}_{n\in\mathbb{N}} \subset \{\mathrm{Fill}(\Lambda_i)\}_{i\in I}$ such that $\mathrm{Fill}(\Lambda_n) \to_{\mathcal{H}} \mathcal{X}$. Passing to a subsequence if necessary, we may assume that the Λ_n converge to a continuum $\mathcal{Y} \subseteq \mathcal{M}$. We therefore have $\mathcal{Y} \subseteq \Lambda_i$ for some $i \in I$, and by Lemma 2.3.20 we have that

$$\mathrm{Fill}(\Lambda_n) \to_{\mathcal{H}} \mathcal{Y} \subseteq \Lambda_i \subseteq \mathrm{Fill}(\Lambda_i).$$

Hence, $\mathcal{U}$ is upper semi-continuous.

Now, let $\Phi\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a Moore projection for $\mathcal{U}$. Given a point $x \in \mathbb{T}^2$, since $\mathcal{U}$ is preserved by f (i.e for every element $U_i \in \mathcal{U}$ we have $f(U_i) \in \mathcal{U}$) the set $\Phi \circ f \circ \Phi^{-1}(x)$ contains only one point $y_x \in \mathbb{T}^2$ for every $x \in \mathbb{T}^2$. Define $\widetilde{f} : \mathbb{T}^2 \to \mathbb{T}^2$ by $f(x) = y_x$. We claim that $\widetilde{f}$ is a homeomorphism. Let us prove first the continuity. For this we take $x \in \mathbb{T}^2$ and fix $\varepsilon > 0$. Then, there exists a neighborhood V of $\Phi^{-1}(y_x)$ such that $\Phi(V) \subset B_\varepsilon(y_x)$. Moreover, there exists a neighborhood U of $f^{-1}(\Phi^{-1}(y_x))$ such that $f(U) \subset V$. On the other hand, since $\Phi^{-1}(x) = f^{-1}(\Phi^{-1}(y_x))$ we have that there exist $\delta > 0$ such that $\Phi^{-1}(B_\delta(x)) \subset U$. Therefore, given a point $z \in B_\delta(x)$ the set $\Phi \circ f \circ \Phi^{-1}(z)$ is contained in $B_\varepsilon(y_x)$. Hence, $\widetilde{f}$ is continuous. If we define $\widetilde{f}^{-1} : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\widetilde{f}^{-1}(x)$ is the unique point in $\Phi \circ f^{-1} \circ \Phi^{-1}(x)$, we have that $\widetilde{f}^{-1}$ is exactly the inverse function of $\widetilde{f}$. Moreover, by an analogous argument as above we have that $\widetilde{f}^{-1}$ is continuous. Therefore $\widetilde{f}$ is a homeomorphism.

By definition of $\widetilde{f}$, $\Phi \circ f(x) = \widetilde{f} \circ \Phi(x)$ holds for every $x \in \mathbb{T}^2$. This implies in particular that $\widetilde{\mathcal{M}} = \Phi(\mathcal{M})$ is minimal for $\widetilde{f}$. Thus, we have that $(f, \mathcal{M})$ is an extension of $(\widetilde{f}, \widetilde{\mathcal{M}})$. Moreover, Proposition 2.3.19 implies that $\Phi(\mathcal{M})$ is totally disconnected and therefore either a periodic orbit or a Cantor set. □

2.4 Special Cases and Applications

In this section, we consider several special cases of the classification. First we study this for the non-wandering case, and we thus restrict the classification to different homotopy classes. In the Anosov case we get strong restrictions, and in the homotopy class of the identity we find some relations between the classification and the Rotation Theory.

2.4.1 Non-wandering torus homeomorphisms

We start with three statements on periodic circloids of non-wandering torus homeomorphisms (see 1.1.3).

Lemma 2.4.1 ([Jäg09b], Corollary 3.6)**.** *Suppose C is a periodic circloid of a non-wandering torus homeomorphism. If C does not contain periodic points, then C has empty interior.*

We call a straight line $L \subseteq \mathbb{R}^2$ *rational*, if it contains infinitely many rational points. Note that in particular, this implies that the slope of L is rational.

Lemma 2.4.2 ([Jäg09b], Proposition 3.9)**.** *Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ has a periodic circloid. Then the rotation set of f is contained in a rational line.*

Given $f \in \mathrm{Homeo}(\mathbb{T}^2)$, we say an f-invariant continuum $C \subseteq \mathbb{T}^2$ is aperiodic if it does not contain a periodic point. In [Kor10], Koropecki identified annular continua as the only possible aperiodic invariant proper subcontinua of non-wandering torus homeomorphisms.

Theorem 2.4.3 ([Kor10], Theorem 1.1)**.** *Let S be a compact orientable surface and suppose $f \in \mathrm{Homeo}(S)$ is non-wandering. Then every aperiodic invariant subcontinuum of $\mathbb{T}^2$ is an annular continuum.*

The following consequence will be crucial in the proof of Theorem 2.1.2 below.

Lemma 2.4.4. *Suppose $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is non-wandering. Then every periodic unbounded disk D contains a periodic point in its boundary.*

Proof. Let p be the period of D. As the boundary of an invariant open disk, ∂D is an f^p-invariant continuum. Suppose for a contradiction that ∂D does not contain a periodic point. Then it is an annular continuum A by Theorem 2.4.3. However, the complement of an annular continuum in $\mathbb{T}^2$ is either an open annulus $\mathcal{A}$ or the union of a punctured torus $\mathcal{T}$ and a bounded disk $\mathcal{D}$. If any of these sets contains the unbounded disk D, then by connectedness D must have further boundary points in the respective set, a contradiction. □

Proof of Theorem 2.1.2. Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ non-wandering and $\mathcal{M} \neq \mathbb{T}^2$ a minimal set for f. First, assume $\mathcal{M}$ is of type 1. Then $\Pi_0(\mathcal{M}^c)$ cannot contain bounded disks since these would have to be wandering by the Classification Theorem. Likewise, $\Pi_0(M^c)$ cannot contain an unbounded disk, since by

Lemma 2.4.4 this unbounded disk has to contain periodic points in the boundary, and the boundary belongs to $\mathcal{M}$. Hence $\mathcal{M} = \mathbb{T}^2$. Secondly, suppose $\mathcal{M}$ is of type 2. Obviously, the essential annuli cannot be wandering, therefore $\mathcal{M}$ is equal to the orbit of the boundary of a periodic essential circloid C. However, by Lemma 2.4.1 the interior of C is empty, and it thus follows that $\partial C = C$.

Finally, suppose $\mathcal{M}$ is of type 3. If $\mathcal{M}$ is a Cantor extension, then for any connected component $\Lambda \in \Pi_0(\mathcal{M})$ the set $\mathrm{Fill}(\Lambda)$ is a wandering set. Therefore $\mathrm{int}(\mathrm{Fill}(\Lambda)) = \emptyset$, which means that Λ is non-separating. If $\mathcal{M}$ is a periodic orbit extension, but not a periodic orbit, then every connected component of $\mathcal{M}$ is an aperiodic invariant continuum for some iterate of f. By Lemma 2.4.4 it is an annular continuum, and by minimality this annular continuum must coincide with its frontiers. By Lemma 2.3.6 the frontiers are circloids. □

2.4.2 Homeomorphisms homotopic to an Anosov

To prove Corollary 2.1.3, recall classical results on Anosov diffeomorphisms.

(1) Manning [Man74] showed that any Anosov diffeomorphism is topologically conjugate to an algebraic Anosov, i.e. an Anosov induced by a hyperbolic element of $\mathrm{SL}(2, \mathbb{R})$.

(2) Bowen [Bow70a] showed that a minimal set of an algebraic Anosov diffeomorphism is either a periodic orbit or a Cantor set.

(3) Walters [Wal70] provides the existence of a semiconjugacy, homotopic to the identity, between a homeomorphism homotopic to an Anosov and the underlying Anosov.

Proof of Corollary 2.1.3. If $\mathcal{M}$ would be of type 1 or 2, then by Lemma 2.2.6, there exists at least one unbounded connected component of $\mathcal{M}$, which we denote Λ. Let $h : \mathbb{T}^2 \to \mathbb{T}^2$ be the semiconjugacy between f and f_A, given by [Wal70]. Then $\mathcal{M}' := h(\mathcal{M})$ is a minimal set of f_A and is totally disconnected due to [Bow70a, Man74]. On the other hand, since h is continuous and homotopic to the identity, and Λ is unbounded, then $h(\Lambda)$ must be an unbounded continuum in $\mathcal{M}'$, a contradiction. Since $\mathcal{M}$ can not be the whole torus either, $\mathcal{M}$ has to be of type 3. □

2.4.3 Relations with the rotation set

We now give a proof of the relation of the structure of minimal sets to the rotation set for homeomorphisms homotopic to the identity. We start with the proof of Corollary 2.1.4, which states that if the rotation set of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ has non-empty interior, then any minimal set is of type 3.

Proof of Corollary 2.1.4. Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ have lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ and assume that the rotation set $\rho(F)$ has non-empty interior. Suppose for a contradiction that $\mathcal{M}$ is a minimal set of f that is not of type 3, that is, there

exists no doubly essential component in its complement. By Lemma 2.3.11 the existence of an essential component in $\mathcal{M}^c$ is excluded, so that all connected components are disks. Now, [MZ91, Theorem A] states that for every vector $\rho \in \operatorname{int}(\rho(F))$ there exists a minimal set $\mathcal{M}_\rho$ with rotation vector ρ, that is,

$$\lim_{n\to\infty} \frac{F^n(z) - z}{n} = \rho \text{ for all } z \in \pi^{-1}(\mathcal{M}_\rho).$$

Fix $\mathcal{M}_\rho$ for some totally irrational vector $\rho \in \mathbb{R}^2$. Then, since all points in $\mathcal{M}_\rho$ are recurrent, $\mathcal{M}_\rho$ has to be contained in the orbit of a periodic disk $D \subseteq \mathcal{M}^c$. This implies that there exists a set $\mathcal{M}'_\rho \subseteq D$ which is minimal for f^p, where p is the period of D.

Choose a connected component D_0 of $\pi^{-1}(D)$, and a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of f that leaves D_0 invariant. Fix $z \in \mathcal{M}'_\rho$ with lift $z_0 \in \mathbb{R}^2$ and $\delta > 0$ such that $B_\delta(z) \subseteq D$. Then irrationality of ρ together with the recurrence of z implies that there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ of integers such that $\lim_{k\to\infty} f^{n_k p}(z) = z$, whereas $F^{n_k p}(z_0)$ is unbounded. Consequently, for sufficiently large $k \in \mathbb{N}$ we have that $F^{n_k p}(z_0) \subseteq B_\delta(z_0) + v$ for some $v \in \mathbb{Z}^2 \setminus \{0\}$. However, this means that D_0 contains both z_0 and $z_0 + v$, contradicting the fact that D is homotopically trivial in $\mathbb{T}^2$. □

Finally, we turn to the proof of Corollary 2.1.5, which states that if f is a non-wandering pseudo-rotation with rotation vector ρ and $\mathcal{M} \neq \mathbb{T}^2$ is a minimal set, then

(a) if ρ is totally irrational, then $\mathcal{M}$ is an extension of a Cantor set, and

(b) if ρ is rational, then $\mathcal{M}$ is either an extension of a Cantor set, or the periodic orbit of a point or a homotopically trivial circloid.

Given a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of $f \in \operatorname{Homeo}_0(\mathbb{T}^2)$, the function $\varphi(z) = F(z) - z$ is doubly periodic and can therefore be interpreted as a function on the torus.

Theorem 2.4.5 ([Mat10], Theorem 2)**.** *Suppose $f \in \operatorname{Homeo}_0(\mathbb{T}^2)$ is non-wandering and A is an essential annular continuum. Further, suppose there exists an f-invariant probability measure μ with support in A and $\rho = \int \varphi \, d\mu \in \mathbb{Q}^2$. Then there exists a periodic point in A with rotation vector ρ.*

Proof of Corollary 2.1.5. Suppose $f \in \operatorname{Homeo}_0(\mathbb{T}^2)$ is a non-wandering pseudo-rotation with rotation vector ρ and $\mathcal{M} \neq \mathbb{T}^2$ a minimal set for f.

(a) Let ρ be totally irrational. We have to rule out cases 1^{nw} and 3^{nw} in Theorem 2.1.2. First, by Lemma 2.4.2, $\mathcal{M}$ cannot be a union of periodic essential circloids, since these force the rotation set to be included in a rational line. Similarly, $\mathcal{M}$ cannot be a periodic orbit or a periodic orbit extension, since this implies the existence of a rational rotation vector. Note here that the factor map in the definition of a periodic orbit extension preserves rotation vectors.

(b) Let ρ be rational. In this case, we have to show that only cases 2^{nw} and 3^{nw} in Theorem 2.1.2 can occur. However, as a rational pseudo-rotation f has at least one periodic orbit [Fra89]. Thus $\mathcal{M} \neq \mathbb{T}^2$. Further, due to Theorem 2.4.5 any periodic essential circloid has to contain a periodic point. This rules out case 1^{nw}.

□

2.5 Examples.

In this finial section we give a list of examples fullfiling the different cases of the classification.

2.5.1 Type 1 minimal sets.

For type one minimal sets we have two main examples:

- a first one constructed by McSweegen in [McS95], which consist of a map $f \in \text{Homeo}(\mathbb{T}^2)$ with a wandering bounded disk D, such that $\mathcal{M} := (\bigcup f^n(D))^c$ is a minimal set. In this work, the author constructs the examples with differentiability C^r for every $r < 3$.

- The second example can be constructed by means of the suspension of a Denjoy example g on the circle, which is given by a flow ϕ_g in $\mathbb{T}^2$. These flow leaves invariant a disk D which is given by the suspension of the wandering interval of g, and is an unbounded disk. Now, the set $\mathcal{M} = D^c$ is a minimal set for the flow ϕ_g, thus we can consider a time t such that if f is the time-t map of ϕ_g, then $\mathcal{M}$ is a minimal set of f (see e.g. [IO07]).

For the first example $\Pi_0(\mathcal{M}^c)$ is a collection of bounded disks, and for the second one $\Pi_0(\mathcal{M}^c)$ is given by a unique unbounded disk. An example in which both bounded and unbounded disks can be found in $\Pi_0(\mathcal{M}^c)$ can be constructed by means of blowing up an orbit $\mathcal{O}(x, f)$ of the second example, to an orbit of a wandering disk $\mathcal{O}(D', f)$, where the minimal set is now given by $\mathcal{M} := (D \cup \mathcal{O}(D', f))^c$ (see [Kwa11]).

2.5.2 Type 2 minimal sets.

For type 2 minimal sets again we can consider two main examples:

- for the first example we consider the torus modeled by the product of two circles, that is $T^2 = \mathbb{T}^1 \times \mathbb{T}^1$, and a map $f \in \text{Homeo}(T^2)$ defined as $f = R_{\frac{p}{q}} \times R_\rho$ where $R_{\frac{p}{q}}$ is a rotation or rational angle $\frac{p}{q}$ and R_ρ is an

irrational rotation. This yields that any set $\mathcal{M} = C \cup f(C) \cup \cdots \cup f^{q-1}(C)$, where $C = \{\theta\} \times \mathbb{T}^1$ for some $\theta \in \mathbb{T}^1$, is a type two minimal set of f. Moreover, the complement of $\mathcal{M}$ is given by the orbit of a periodic essential annulus.

- For the second example we consider again T^2, but this time the map f is given by $f = g \times R_\rho$ where g is a Denjoy example on the circle. This implies that the set $A = I \times \mathbb{T}^1 \subset T^2$ is a wandering annulus, where I is the wandering interval of g. Furthermore, we have that $\mathcal{M} = \left(\bigcup_{n\in\mathbb{N}} f^n(A)\right)^c$ is a minimal set of f.

The first two examples do not combine annuli and disks in $\Pi_0(\mathcal{M}^c)$. In order to do this one can again blow up some orbit in $\mathcal{M}$ to an orbit of a wandering disk. Nevertheless, this can not be done in the first example but for the second. In order to do this for the first kind of examples one have to consider $\mathcal{M} = \mathcal{C} \cup \cdots \cup f^{q-1}(\mathcal{C})$, where $\mathcal{C}$ is now a more complicated continuum but not the circle (see e.g. [MN11]).

A final comment concerning type two minimal sets is that we do not know whether there exists an example with both annuli and unbounded disks in $\Pi_0(\mathcal{M}^c)$. So we pose this as a question:

Question 2.5.1. Does there exist a type 2 minimal set $\mathcal{M}$ of $f \in \text{Homeo}(\mathbb{T}^2)$ such that $\Pi_0(\mathcal{M}^c)$ contains both annuli and unbounded disks?

2.5.3 Type 3 minimal sets.

For type three minimal sets, the trivial examples are either periodic orbits, or minimal Cantor sets. A further simple example is given by a map $f \in \text{Homeo}(\mathbb{T}^2)$ which has a periodic disk D of period p, such that $\{\partial f^i(D)\}_{i=0}^{p-1}$ is a family of pairwise disjoint trivial loops, and $\mathcal{M} = \partial D \cup \cdots \cup \partial f^{p-1}(D)$ is a minimal set. In this situation we have that $\mathcal{M}$ is a type 3 minimal set, such that the map Φ which defines the extension can be considered in a way that $\Phi(\mathcal{M}) = \{x_0, \ldots, x_{p-1}\}$ where $\Phi^{-1}(x_0) = f^i(D \cup \partial D)$ for $i = 0, \ldots, p-1$.

The construction of non-trivial examples of minimal sets which are an extension of a Cantor minimal set is difficult and can be find in [Kwa11]. Notice that in this case only countably many orbits can be blow up to orbits of wandering disks.

Chapter 3

Classification of minimal sets of surfaces homeomorphisms.

3.1 Statements of the results.

The results we want to prove in this chapter are Theorems 3,4,5 and the corollaries 6 and 4, which were presented in Subsection 1.1.6.

The classification result is given by the two first theorems 3,4. The first one is the following result.

Theorem 3.1.1. *Let S be a closed hyperbolic surface and $\mathcal{M} \subset S$ a minimal set of $f \in \mathrm{Homeo}(S)$. Then either:*

(1) *the elements in $\Pi_0(\mathcal{M}^c)$ are all disks, with at least one unbounded and periodic. Moreover, any bounded disk in $\Pi_0(\mathcal{M}^c)$ is wandering;*

(2) *$\mathcal{E}(\mathcal{M}^c)$ is a finite family of locally simply connected sets, and $\mathcal{M} = \mathcal{C}_0 \cup ... \cup \mathcal{C}_{n-1}$ where $\{\mathcal{C}_i\}_{i=0}^{n-1}$ is a family of pairwise disjoint continua with $f(\mathcal{C}_i) = \mathcal{C}_{i+1 mod(n)}$ for all $i = 0, \dots, n-1$. Furthermore each element in $\{\mathcal{C}_i\}_{i=0}^{n-1}$ coincides with a connected component of ∂V for some $V \in \mathcal{E}(\mathcal{M}^c)$, and there are no periodic bounded discs in $\Pi_0(\mathcal{M}^c)$.*

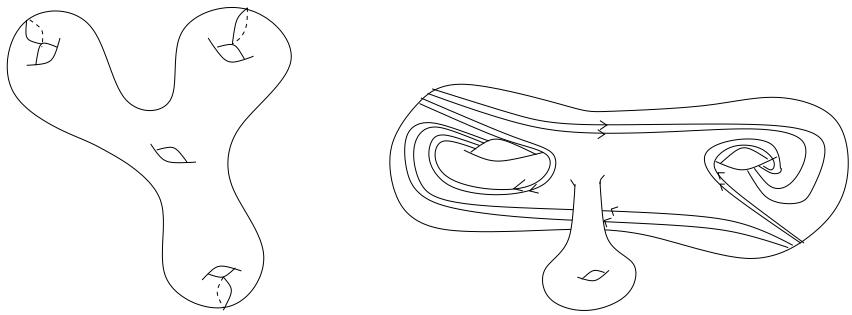
(3) *$\mathcal{M}$ is either an extension of a periodic orbit or an extension of a minimal Cantor set.*

The second result which completes the classification is given by the following theorem.

Theorem 3.1.2. *Let S be a closed hyperbolic surface and $\mathcal{M} = \mathcal{C}_0 \cup ... \cup \mathcal{C}_{n-1}$ a type 2 minimal set of $f \in \mathrm{Homeo}(S)$. Then, for every $k = 0, ..., n-1$ there exists $N > 0$ such that $\mathcal{C}_k$ is reducible to a type 1 minimal set for f^N. Moreover,*

(2i) *if $\mathcal{C}_0$ is 0-reducible, then $\mathcal{C}_k$ is the boundary of a circloid for every $k = 0, ..., n-1$ and $\#\mathcal{E}(\mathcal{C}_k^c) \leq 2$. Furthermore every disk in $\Pi_0(\mathcal{M}^c)$ is wandering.*

(2ii) *if $\mathcal{C}_0$ is g-reducible with $g > 0$, then $\mathcal{C}_k$ is g-reducible for $k = 0, ..., n-1$, and there are at least $\#\mathcal{E}(\mathcal{C}_k^c)$ unbounded fixed disks in $\Pi_0(S'\backslash\mathcal{C}_k')$ for $\hat{f}$.*



(a) (2i) (b) (2ii)

Figure 3.1.1: Type 2 minimal sets.

For the non-wandering case the classification is given by the following result.

Theorem 3.1.3. *Let S be a closed hyperbolic surface, and $\mathcal{M} \subset S$ a minimal set for a non-wandering homeomorphism $f \in \text{Homeo}(S)$. Then either:*

(1^{nw}) *$\mathcal{M} = \mathcal{C}_0 \cup ... \cup \mathcal{C}_{n-1}$ where $\{\mathcal{C}_i\}_{i=0}^{n-1}$ is a family of pairwise disjoint essential circloids with $f(\mathcal{C}_i) = \mathcal{C}_{i+1 mod(n)}$ for all $i = 0, \ldots, n-1$. Furthermore, there are no disks in $\Pi_0(\mathcal{M}^c)$.*

(2^{nw}) *$\mathcal{M}$ is an extension of either a periodic orbit or a minimal Cantor set.*

The classification result is considered in the different homotopy classes, in light of the Nielsen-Thurston Theory. The following theorem and corollary are obtained (see section 1.1.6 for the preliminaries definitions).

Theorem 3.1.4. *Let S be a closed hyperbolic surface. Then type 1 minimal sets can only occur for homeomorphisms in $\mathcal{D}^c$.*

Corollary 3.1.5. *Let S be a closed hyperbolic surface and $f \in \text{Homeo}(S)$ homotopic to a pseudo-Anosov homeomorphism. Then every minimal set of $\mathcal{M}$ is type 3.*

For the classification result (mainly the first theorem) we purpose the following mental pictures to have in mind:

- Type 3 minimal set: a fixed point;

- Type 2 minimal set: a non trivial simple loop $\mathcal{C}$ in S which supports an irrational rotation (more interesting examples are described in Section 3.9);

- Type 1 minimal set : consider a full geodesic lamination Λ in S of genus g, as constructed in [CB88] such that $S\backslash\Lambda$ is the union of a collection of disks that lift to the universal covering space to ideal $2g$-gones[1] with alternating orientations on its sides (see Figure 3.1.2). This last condition allows to complete the lamination to a flow ϕ in S with a fixed point in each disk. Moreover, the lamination is a minimal set for the flow, and one can show that there exists $t \in \mathbb{R}$ such that Λ is minimal for the time t of this flow ([IO07]).

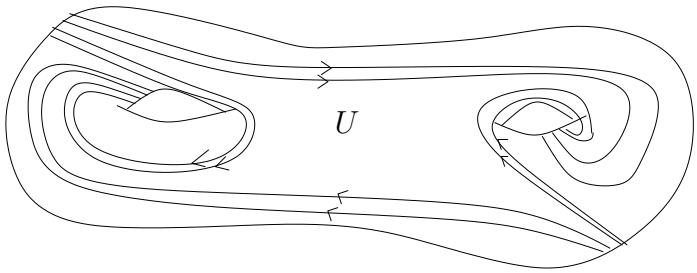


Figure 3.1.2: Type 1 minimal set.

The chapter is organized as follows. We start with a preliminary classification which gives us the three different classes of Theorem 3.1.1. In the next section 3.3 we study the existence of periodic disks in $\Pi_0(\mathcal{M}^c)$ for minimal sets in any of the classes given by the preliminary classification. Then in Sections 3.4 and 3.5, a more detailed description of type 2 and 3 minimal sets is given (in the second case we just need to follow what is done for the respective case in the torus). In Section 3.6 we give the proof of Theorems 3.1.1 and 3.1.2, in Section 3.7 we give the classification in the non-wandering case, and in Section 3.8 we relate the classification to Nielsen-Thurston Theory. Finally, in Section 3.9 we provide a list of examples for the different cases of the classification.

3.2 A preliminary classification.

Let $S = \mathbb{D}/G$ be a compact hyperbolic surface. In this section we make the following preliminary classification of minimal sets:

Theorem 3.2.1. *One and only one of the following holds:*

1. *Every $U \in \Pi_0(\mathcal{M}^c)$ is a disk;*

[1] That is, a $2g$-gone bounded by geodesics.

2. *There exists an essential component* $U \in \Pi_0(\mathcal{M}^c)$*, and* $i_* : \pi_1(V, x_0) \to \pi_1(S, x_0)$ *is injective for all* $V \in \Pi_0(\mathcal{M}^c)$*;*

3. *There exists a fully essential component* $V \in \Pi_0(\mathcal{M}^c)$ *and every other* $U \in \Pi_0(\mathcal{M}^c)$ *is a bounded disk.*

Figure 3.1.2 illustrates the first item, and Figure 3.1 the second one.

We begin with purely topological facts. For a subset U in a surface S we denote the closure of U by $\mathrm{cl}[U]$.

Lemma 3.2.2. *If* V *is a fully essential open and connected subset of* S*,* $U \subset S$ *is a connected set, and* $V \cap U = \emptyset$ *then* U *is bounded.*

Proof. If U is not bounded, and $\tilde{U}$ is a connected component of $p^{-1}(U)$, there exists $x \in \mathbb{S}^1 \cap \mathrm{cl}[\tilde{U}]$ where $\mathbb{S}^1 = \partial D$. Take $x_0 \in \tilde{U}$ and let α be the segment of geodesic joining x_0 and x, where x_0 is considered in the interior of a fundamental geodesic polygon P. As α is unbounded, there exists a geodesic γ containing one of the sides of P and an infinite sequence $(g_n)_{n\in N}$ of covering transformations such that $\alpha \cap g_n(\tilde{\gamma}) \neq \emptyset$. Let W_n be the connected component of $\mathbb{D}\backslash g_n(\gamma)$ containing x in its closure. Then, by construction we have that $(\mathrm{cl}[W_n])_{n\in\mathbb{N}}$ is a basis of neighborhoods of x in $\mathrm{cl}[\mathbb{D}]$. Furthermore, $\mathrm{cl}[g_n(\gamma)]$ separates x_0 from x for all $n \in \mathbb{N}$. As V is fully essential, one can find $\beta \subset V$ homotopic to γ. Moreover, for any $n \in \mathbb{N}$, there exists a lift β_n of β asymptotic to $g_n(\gamma)$. So, there exists $\tilde{\beta}$ which lifts β such that $\mathrm{cl}[\tilde{\beta}]$ separates x from $x_0 \in \mathrm{cl}[\mathbb{D}]$. This implies that $\tilde{\beta} \cap \tilde{U} \neq \emptyset$ which contradicts the fact $V \cap U = \emptyset$. □

Lemma 3.2.3. *If* $X \subset S$ *is contained in a finite union of pairwise disjoint closed disks* $D_1, \ldots, D_n$*, then* $i_* : \pi_1(S\backslash X, x_0) \to \pi_1(S, x_0)$ *is surjective for any* $x_0 \in S\backslash \cup_{i=1}^n D_i$*.*

Proof. Let $X \subset \cup_{i=1}^n D_i$, with the D_i's closed and pairwise disjoint disks. Then, $p^{-1}(\cup_{i=1}^n D_i)$ is a union of pairwise disjoint closed disks containing $p^{-1}(X)$. Indeed, any two lifts $\tilde{D}_i$ and $\tilde{D}_j$ of D_i and D_j with $i \neq j$ cannot intersect as $D_i \cap D_j = \emptyset$. Moreover, for any non trivial $T \in G$ and $\tilde{D}_i$ a lift of D_i, $\tilde{D}_i \cap T(\tilde{D}_i) = \emptyset$ because D_i is a disk and cannot contain a non trivial loop. In particular, $\tilde{S}\backslash p^{-1}(\cup_{i=1}^n D_i)$ is arc connected. If $i_* : \pi_1(S\backslash X, x_0) \to \pi_1(S, x_0)$ is not surjective, there exists $T \in G$ such that for any lift $\tilde{x}$ of x_0, $\tilde{x}$ and $T(\tilde{x})$ cannot be joined by a path in $\tilde{S}\backslash p^{-1}(X)$. As $\tilde{S}\backslash p^{-1}(\cup_{i=1}^n D_i)$ is arc connected and both $\tilde{x}$ and $T(\tilde{x})$ belong to $\tilde{S}\backslash p^{-1}(\cup_{i=1}^n D_i)$, this is a contradiction.

□

A set $F \subset \mathbb{R}^2$ is non-separating if F^c has a unique connected component. Given a compact and connected set C in $\mathbb{R}^2$ the filled-in set of C is defined

as the minimal (with respect to the inclusion) non-separating continuum containing C. We denoted it by $\mathrm{Fill}_{\mathbb{R}^2}(C)$. This kind of sets and a list of their properties were already introduced in Subsection 2.2.4. If now we have a continuum C in a surface S such that C is contained in a topological disk D, i.e. C is contractible, then by identifying D with $\mathbb{R}^2$ via a homeomorphism, we can define the filled-in set of C in D. We denote this as $\mathrm{Fill}_D(C)$. When a contractible set F coincides with the filled-in set, we say that F is non-separating. It is not difficult to see by means of a standard argument with the Riemann Mapping Theorem, that these sets can be expressed as a decreasing union of closed topological disks.

The classical decomposition theorem by Moore stated in 2.3.3, implies that given a set X in any surface S so that $\Pi_0(X)$ is a collection of contractible non-separating sets, we can consider a family of pairwise disjoint closed disks $D_1, \ldots, D_n$ so that $X \subset \bigcup_{i=1}^n \mathrm{int}(D_i)$. Making use of this fact, we can prove the following lemma.

Lemma 3.2.4. *If there exists $U \in \Pi_0(\mathcal{M}^c)$ such that the map $i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$ is not injective, then there exists a finite number of pairwise disjoint closed disks in S, $D_1, \ldots, D_n$ such that $\mathcal{M} \subset \mathrm{int}(\cup_{i=1}^n D_i)$.*

Proof. Take $U \in \Pi_0(\mathcal{M}^c)$ such that the map $i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$ is not injective. Then, there exists a simple closed curve $\gamma \subset U$ such that γ bounds an open disk $D \subset S$, but $D \cap \mathcal{M} \neq \emptyset$. By minimality, for any $\mathcal{C} \in \Pi_0(\mathcal{M})$ there exists $n \geq 0$ such that $f^n(\mathcal{C}) \subset D$. Indeed, $f^n(\mathcal{C}) \cap D \neq \emptyset$ for some $n \geq 0$, and $f^n(\mathcal{C}) \cap \partial D = \emptyset$ as $\partial D = \gamma \subset U \subset \mathcal{M}^c$, and $f^n(\mathcal{C})$ is connected. For $\mathcal{C} \in \Pi_0(\mathcal{M})$, let $k_{\mathcal{C}} = \min\{n \geq 0 \mid f^n(\mathcal{C}) \subset \mathcal{D}\}$.

Then, we can consider for any $\mathcal{C} \in \Pi_0(\mathcal{M})$ the contractible and non-separating set $\mathrm{Fill}(\mathcal{C}) = f^{-k_{\mathcal{C}}}(\mathrm{Fill}_D(f^{k_{\mathcal{C}}}(\mathcal{C})))$. For the family of sets

$\mathcal{G} = \{\mathrm{Fill}(\mathcal{C})\}_{\mathcal{C} \in \Pi_0(\mathcal{M})}$ we have that either any two elements are disjoint or one is contained into the other. Any other situation would imply $\mathcal{C}' \cap \mathcal{C} \neq \emptyset$ for some $\mathcal{C} \neq \mathcal{C}' \in \Pi_0(\mathcal{M})$ (this fact is shown for instance in 2.3.1).

Let $X = \bigcup_{F \in \mathcal{G}} F$. Then, $\mathcal{M} \subset X$ and the connected components of X are all contractible and non-separating. Therefore, as we state right before this lemma, there exits a number of pairwise disjoint disks $D_1, \ldots, D_n$ so that $\mathcal{M} \subset X \subset \mathrm{int}(\cup_{i=1}^n D_i)$.

□

Corollary 3.2.5. *Suppose that there exists $U \in \Pi_0(\mathcal{M}^c)$ such that the map $i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$ is not injective. Then there exists a fully essential component $V \in \Pi_0(\mathcal{M}^c)$.*

Proof. We know that $\mathcal{M} \subset \mathrm{int}(\cup_{i=1}^n D_i)$, where the D_i's are pairwise disjoint closed disks in S. Then, $V' = S \setminus \cup_{i=1}^n D_i$ is open and connected, $V' \subset \mathcal{M}^c$, and $i_* : \pi_1(V', x_0) \to \pi_1(S, x_0)$ is surjective (see Lemma 3.2.3). We take V to be the connected component of $\mathcal{M}^c$ containing V'.

□

Lemma 3.2.6. *If $U \in \Pi_0(\mathcal{M}^c)$ is bounded, then U is a topological disk.*

Proof. As U is bounded, if $\tilde{U}$ is a connected component of $p^{-1}(U)$, then $\partial\tilde{U}$ has a unique connected component $\mathcal{C}$ that separates $\mathbb{S}^1$ from $\tilde{U}$. Let D be the bounded connected component of $\mathbb{D}\backslash\mathcal{C}$. We claim that for any lift $\tilde{f}$ of f and any $k \in \mathbb{Z}$, either $\tilde{f}^k(D) = D$, or $\tilde{f}^k(D) \cap D = \emptyset$. It is clear that one can not have $\tilde{f}^k(D) \subsetneq D$ or $D \subsetneq \tilde{f}^k(D)$ as this would imply the existence of a wandering point in $\mathcal{M}$. As D and $f^k(D)$ are topological disks, $\tilde{f}^k(D) \cap D \neq \emptyset$ but $\tilde{f}^k(D) \neq D$ is impossible because this implies that $\mathcal{C}$ cannot be a connected component of $\partial\tilde{U}$, proving the claim. If U is not a disk, then $\mathcal{M} \cap p(D) \neq \emptyset$, and there exists a lift $\tilde{f}$ of f and $k \in \mathbb{Z}$ such that $\tilde{f}^k(D) \cap D \neq \emptyset$. Therefore, $\tilde{f}^k(D) = D$, which implies $\tilde{f}^k(\mathcal{C}) = \mathcal{C}$. If $\mathcal{K}$ is the connected component of $\mathcal{M}$ containing $p(\mathcal{C})$, then the (periodic) orbit of $\mathcal{K}$ is a closed an invariant set, which is different from $\mathcal{M}$, as $\mathcal{M} \cap p(D) \neq \emptyset$. This contradiction finishes the proof.

□

Lemma 3.2.7. *If $U \in \Pi_0(\mathcal{M}^c)$ is not fully essential, then $i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$ is injective.*

Proof. If U is essential, then there are no fully essential components in $\Pi_0(\mathcal{M}^c)$ (see Lemma 3.2.2). So, $i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$ is injective, this being the content of Corollary 3.2.5. If U is trivial and there are no fully essential components in $\Pi_0(\mathcal{M}^c)$, then
$i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$ is injective again by Corollary 3.2.5. If there is a fully essential component $V \in \Pi_0(\mathcal{M}^c)$, then U is bounded (see Lemma 3.2.2) and so by Lemma 3.2.6 $i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$ is injective. □

The proof of Theorem 3.2.1 follows.

Proof. If (1) holds, then evidently neither (2) nor (3) hold. If (2) holds, then evidently (1) does not hold, and (3) cannot hold due to lemma 3.2.2. If (3) holds, then evidently (1) does not hold, and (2) cannot hold because of lemma 3.2.2. So, only one of them can hold.

Suppose that (3) does not hold. So, by Corollary 3.2.5, for every $U \in \Pi_0(\mathcal{M}^c)$ the map $i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$ is injective. Then necessarily (1) or (2) hold and we are done. □

Corollary 3.2.8. *If $U \in \Pi_0(\mathcal{M}^c)$ is essential, then U is homeomorphic to a finitely punctured closed surface.*

Proof. By a theorem of Richards (see [Ric91] Theorem 3), U is homeomorphic to $\Sigma\backslash X$, where Σ is a closed surface and X is a closed and totally disconnected set. We just have to prove that X has finitely many connected components.

If this was not the case, there would be infinitely many homotopically non trivial, pairwise non freely homotopic and pairwise disjoint loops $(\gamma_n)_{n\in\mathbb{N}} \subset U$. As $i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$ is injective and S is a closed surface, this is a contradiction. □

Corollary 3.2.9. *If $U \in \Pi_0(\mathcal{M}^c)$ is essential, then ∂U has finitely many connected components.*

Proof. By the previous Corollary, U is homeomorphic to a finitely punctured closed surface. It follows that the number of punctures is greater or equal than $\#\Pi_0(\partial U)$. □

3.3 Periodic disks in $\Pi_0(\mathcal{M}^c)$

We will say that a minimal set $\mathcal{M}$ is of type 1, 2 or 3 if it corresponds to cases (1), (2) or (3) of Theorem 3.2.1.

We devote this section to proving that there cannot be bounded periodic disks in $\Pi_0(\mathcal{M}^c)$ for a minimal set of type (1) or (2). As an outcome of this fact we obtain that for type (1) minimal sets there is always a periodic unbounded disk in $\Pi_0(\mathcal{M}^c)$.

We remark that the following four lemmas are just an adaptation of the ideas in Sub-section 2.3.1 to higher genus.

Lemma 3.3.1. *If $\mathcal{M}$ is connected and $U \in \Pi_0(\mathcal{M}^c)$ is a periodic disk of period p, then $\mathcal{M} = \partial U = \ldots = \partial f^{p-1}(U)$.*

Proof. The set $\partial U \cup \ldots \cup \partial f^{p-1}(U)$ is closed, invariant and contained in $\mathcal{M}$. So, $\mathcal{M} = \partial U \cup \ldots \cup \partial f^{p-1}(U)$. Let $r(x) = \#\{0 \leq k < p \mid x \in \partial f^k(U)\}$. Then, for any $k_0 \in \{1, \ldots, p\}$ the set $r^{-1}(\{k \mid k \geq k_0\})$ is closed and invariant, and therefore empty or equal to $\mathcal{M}$. This implies that there exists $m \geq 1$ such that $r(x) = m$ for all $x \in \mathcal{M}$.

For every $x \in \mathcal{M}$ define $I_x = \cap_{i=1}^m \partial f^{k_i}(U)$, where $k_1, \ldots, k_m \in \{0, ..., p-1\}$ are such that $x \in \partial f^{k_i}(U)$. Then, I_x is closed and $\mathcal{M} = I_{x_1} \cup \ldots \cup I_{x_n}$ with $I_{x_i} \neq I_{x_j}$ if $i \neq j$. Furthermore, $I_{x_i} \cap I_{x_j} = \emptyset$ if $i \neq j$, otherwise there would exist $z \in \mathcal{M}$ with $r(z) > m$. As $\mathcal{M}$ is connected, we have $\mathcal{M} = I_{x_1} = \ldots = I_{x_n}$, which implies $\mathcal{M} = \partial U = \ldots = \partial f^{p-1}(U)$. □

Lemma 3.3.2. *Suppose that $\mathcal{M}$ is connected and there exists $U \in \Pi_0(\mathcal{M}^c)$ such that U is bounded and periodic. Let $\tilde{U}$ be a connected component of $p^{-1}(U)$. Then, for each $x \in \mathcal{M} = \partial U$ and $\tilde{x} \in \partial\tilde{U}$, $r(x) = \#\{T \in G \mid T(\tilde{x}) \in \partial\tilde{U}\}$ is a well defined constant function in $\mathcal{M}$.*

Proof. As U is bounded, for each $x \in \mathcal{M} = \partial U$ and $\tilde{x} \in \partial\tilde{U}$ the set

$$\{T \in G \mid T(\tilde{x}) \in \partial\tilde{U}\}$$

is finite. Moreover,

$$r(x) = \#\{T \in G \mid T(\tilde{x}) \in \partial\tilde{U}\}$$

is independent of $\tilde{x} \in \partial\tilde{U} \cap p^{-1}(x)$. So, we have a well defined function $r : \mathcal{M} \to \mathbb{N}$. We claim that for any $n_0 \in \mathbb{N}$, the set $r^{-1}(\{n < n_0\})$ is open. Indeed, take $x \in r^{-1}(\{n < n_0\})$ and $\tilde{x} \in \partial\tilde{U} \cap p^{-1}(x)$. Consider a neighborhood V of $\tilde{x}$ such that the sets $T(V), T \in G$, are pairwise disjoint and $p(V)$ is a neighborhood of x. There exists $n < n_0$ and $T_1, \ldots, T_n \in G$ such that for any $T \in G, T \notin \{T_1, \ldots, T_n\}$, $T(\tilde{x}) \notin \partial\tilde{U}$. Taking V smaller if necessary, we can suppose that $T(V) \cap \partial\tilde{U} = \emptyset$ for all $T \in G, T \notin \{T_1, \ldots, T_n\}$. So, for any $z \in V$, $T(z) \in \partial\tilde{U}$ if and only if $T \in \{T_1, \ldots, T_n\}$. In particular, $p(z) \in r^{-1}(\{n < n_0\})$. So, $p(V) \subset r^{-1}(\{n < n_0\})$ showing that $r^{-1}(\{n < n_0\})$ is open. Equivalently, the sets $X_{n_0} = r^{-1}\{n \geq n_0\}$ are closed for all $n_0 \in \mathbb{N}$. To see that these sets are also f- invariant it is enough to be able to lift f to a homeomorphism $\tilde{f} : \tilde{S} \to \tilde{S}$ such that $\partial\tilde{U}$ is $\tilde{f}$-invariant. As $\mathcal{M} = \partial U = \partial f(U)$, if $\tilde{g}$ is any lift of f, then $\tilde{g}(\partial\tilde{U}) = \partial\tilde{g}(U) = T(\partial\tilde{U})$ for some $T \in G$. Then, $\tilde{f} = T^{-1}g$ is a lift of f fixing $\partial\tilde{U}$. By minimality, one obtains that $r(x)$ is independent of $x \in \mathcal{M}$. □

Lemma 3.3.3. *Suppose that $\mathcal{M}$ is connected and that there exists $U \in \Pi_0(\mathcal{M}^c)$ such that U is bounded and periodic. Then $\mathcal{T}_x = \{T \in G \mid T(x) \in \partial\tilde{U}\}$ is independent of $x \in \partial\tilde{U}$.*

Proof. The previous lemma implies that there exists $m \geq 0$ such that for any $x \in \partial\tilde{U}$, $\#\mathcal{T}_x = m$. As U is bounded, the family $\{\mathcal{T}_x \mid x \in \partial U\}$ is finite. For all $x \in \partial U$ define $C_x = \cap_{i=1}^{m} T_{x_i}(\partial U)$, where $x \in T_{x_i}(\partial U)$ for all $i = 1, \ldots, m$. Then, each C_x is closed and the family $\mathcal{F} = \{C_x \mid x \in \partial U\}$ is finite, $\mathcal{F} = \{C_{x_1}, \ldots, C_{x_n}\}$. Furthermore, $C_{x_i} \cap C_{x_j} = \emptyset$ if $i \neq j$: otherwise, there would exist $x \in \partial\tilde{U}$ with $\#\mathcal{T}_x > m$. As $\partial\tilde{U}$ is connected, we have $\partial\tilde{U} = C_{x_1} = \ldots = C_{x_n}$ and we are done.

□

Lemma 3.3.4. *If $\mathcal{M}$ is a connected type 1 or 2 minimal set and $U \in \Pi_0(\mathcal{M}^c)$ is bounded, then U is wandering.*

Proof. Otherwise, Lemma 3.3.1 implies that $\mathcal{M} = \partial U$. As U is bounded, Lemma 3.3.3 implies that for any lift $\tilde{U}$ of U and any $T \in G$ we have $T\left(\mathrm{cl}[\tilde{U}]\right) \cap \mathrm{cl}[\tilde{U}] = \emptyset$. So, $\bigcup_{T \in G} \mathrm{cl}[T(\tilde{U})]$ can be covered by a pairwise disjoint family of closed disks, and there is a fully essential component in $\Pi_0(\mathcal{M}^c)$. □

Lemma 3.3.5. *If $\mathcal{M}$ is of type 1 or 2, then there is no bounded periodic disk in $\Pi_0(\mathcal{M}^c)$.*

Proof. If $U \in \Pi_0(\mathcal{M}^c)$ is a bounded periodic disk, then $\mathcal{M} = \partial U \cup f(\partial U) \ldots \cup f^{p-1}(\partial U)$. Let C be the connected component of $\mathcal{M}$ containing ∂U. Then, C is periodic and minimal for f^p. So, by lemma 3.3.4, C^c contains a fully essential component. Therefore $f^i(C)$ contains a fully essential component for $i = 1, \ldots, q$, and so does $\mathcal{M} = \cup_{i=1}^n f^i(C)$. This contradicts the fact that $\mathcal{M}$ is of type 1 or 2. □

We finish by proving:

Lemma 3.3.6. *If $\mathcal{M}$ is a type 1 minimal set, then there is at least one unbounded periodic disk in $\mathcal{M}^c$.*

Proof. Any homeomorphism of a surface of genus $g > 1$ has a periodic point p ([Ful53]). As bounded disks in $\mathcal{M}^c$ are wandering, $p \in \mathcal{M}$ or there exists one unbounded periodic disk in $\mathcal{M}^c$. If $p \in \mathcal{M}$, then $\mathcal{M}$ is the orbit of p, which contradicts that $\mathcal{M}$ is a type 1 minimal set.

□

3.4 Type 2 minimal sets.

In this section we first prove the following:

Proposition 3.4.1. *Let $\mathcal{M}$ be a type 2 minimal set of $f \in \mathrm{Homeo}(S)$. Then, $\mathcal{E}(\mathcal{M}^c)$ is a finite family of locally simply connected sets, and $\mathcal{M} = \mathcal{C}_0 \cup \ldots \cup \mathcal{C}_{n-1}$ where $\{\mathcal{C}_i\}_{i=0}^{n-1}$ is a family of pairwise disjoint continua with $f(\mathcal{C}_i) = \mathcal{C}_{i+1 mod(n)}$ for all $i = 0, \ldots, n-1$. Furthermore each element in $\{\mathcal{C}_i\}_{i=0}^{n-1}$ coincides with a connected component of ∂V for some $V \in \mathcal{E}(\mathcal{M}^c)$, and there are no periodic bounded discs in $\Pi_0(\mathcal{M}^c)$.*

The idea of the proof is simple. First we prove that an essential component in $\Pi_0(\mathcal{M}^c)$ must be periodic. As such a component $U \in \Pi_0(\mathcal{M}^c)$ exists because we are considering type 2 minimal sets, one has $\mathcal{M} = \partial U \cup \partial f(U) \cup \ldots \cup \partial f^{p-1}(U)$, where p is the period of U. We finish by taking $\mathcal{C}_0$ to be a connected component of $\mathcal{M}$ contained in ∂U.

After proving this, we construct for any continuum $\mathcal{C} \subset S$ which is a type two minimal set for some homeomorphism $f \in \mathrm{Homeo}(S)$, a surface S^* containing $\mathcal{C}$ with $\partial S^* = \bigcup_{i=1}^n \beta_i$ where $\{\beta_i\}_{i=1}^n$ is a family of pairwise disjoint essential loops, such that the following theorem holds. We denote $S^*/_\partial$ to the quotient space of S^* under the relation $x \sim y$ iff $x, y \in \beta_i$ for some $i = 1, \ldots n$.

Theorem 3.4.2. *There exist $n_0 \in \mathbb{N}$ and a homeomorphism g homotopic to f^{n_0}, with $g(\beta) = \beta$ for all $\beta \in \partial S^*$, such that if $S' = S^*/_\partial$ and $q : S^* \to S'$ is the quotient map, then:*

1. *S' is a surface with $g(S') < g(S)$ and $\mathcal{C}' := q(\mathcal{C})$ is a minimal set for the induced map $\hat{g} \in$ Homeo(S'). Furthermore q conjugates $f^{n_0}|_{\mathcal{C}}$ with $\hat{g}|_{\mathcal{C}'}$;*

2. *$S'\backslash\mathcal{C}'$ is a union of disks and there are at least $\#\mathcal{E}(S\backslash\mathcal{C}^c)$ unbounded disks containing fixed points of $\hat{g}$.*

These two results are crucial for the proofs of Theorem 3.1.1 and Theorem 3.1.2.

3.4.1 Essential components are periodic

Throughout this section $\mathcal{M}$ will stand for a type 2 minimal set. We begin by the following.

Lemma 3.4.3. *If $U \in \Pi_0(\mathcal{M}^c)$ is essential and is not an annulus, then U is periodic.*

Proof. Let g be the genus of U. If $g \neq 0$, then clearly U is periodic, as S is a compact surface. If $g = 0$ as $i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$ is injective (see Lemma 3.2.7) U is homeomorphic to the n-punctured sphere, $n \geq 3$, since we are assuming that U is not an annulus. Let $U_i = f^i(U), i \geq 0$. For all $i \geq 0$, U_i is an n-punctured sphere with $n \geq 3$ whose fundamental group injects in $\pi_1(S, x_0)$. As S is compact, necessarily $(U_i)_{i\geq 0}$ has to be a finite family. □

Let $\mathcal{A}$ be the family of all essential open annuli in S. To each element $U \in \mathcal{A}$ we can associate $U_\infty \subset \mathbb{S}^1$ as follows. Take a simple loop $\alpha \subset U$ generating $\pi_1(U, x_U)$, and pick a lift $\tilde{\alpha} \subset \mathbb{D}$ of α. Let $a, b \in \mathbb{S}^1$ such that $\mathrm{cl}[\tilde{\alpha}] \cap \partial\mathbb{D} = \{a, b\}$. Let $U_\infty = \{(T(a), T(b)) \mid T \in G\}$. The set U_∞ is independent of the choice of α and $\tilde{\alpha}$. So, for each $A \in \mathcal{A}$ there is a well defined subset $U_\infty \subset \mathbb{S}^1$.

Notice that given two annuli $U, V \in \mathcal{A}$ we have $U_\infty = V_\infty$ if and only if there exist two essential loops $\alpha \subset U$, $\beta \subset V$, and two lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β respectively, such that $\mathrm{cl}[\tilde{\alpha}] \cap \mathbb{S}^1 = \mathrm{cl}[\tilde{\beta}] \cap \mathbb{S}^1$.

Lemma 3.4.4. *Let $U \in \mathcal{A} \cap \Pi_0(\mathcal{M}^c)$, and define*

$$\mathcal{F} = \{V \in \mathcal{A} \cap \Pi_0(\mathcal{M}^c) \mid U_\infty = V_\infty\}.$$

Then, there exists $A \in \mathcal{A}$ such that

1. *$A_\infty = U_\infty$;*
2. *for all $V \in \mathcal{F}, V \subset A$;*
3. *$\partial A \subset \mathcal{M}$;*
4. *for all $x \in \partial A$ and all neighborhood N of x, there exists $V \in \mathcal{F}$ such that $\partial V \cap N \neq \emptyset$.*

Proof. Let $\gamma \subset U$ be a generator of $\pi_1(U, x_U)$, and $\tilde{\gamma} \subset \mathbb{D}$ a lift of γ. Let $a, b \in \mathbb{S}^1$ such that $\mathrm{cl}[\tilde{\gamma}] \cap \mathbb{S}^1 = \{a, b\}$, and $\tilde{U}$ the connected component of $p^{-1}(U)$ containing $\tilde{\gamma}$.

For all $V \in \mathcal{F}$, pick a loop $\alpha(V) \subset V$ generating $\pi_1(V, x_V)$ and let $\tilde{\alpha(V)}$ be the lift of $\alpha(V)$ such that $\mathrm{cl}[\tilde{\alpha(V)}] \cap \mathbb{S}^1 = \{a, b\}$. We define $\tilde{V}$ to be the connected component of $p^{-1}(V)$ containing $\tilde{\alpha(V)}$.

Further let $\mathcal{D}$ be the family of topological disks in $\mathbb{D}$ which are bounded by $\mathrm{cl}[\tilde{\alpha}]$ and $\mathrm{cl}[\tilde{\beta}]$, where $\tilde{\alpha} \subset \tilde{V_0}$ and $\tilde{\beta} \subset \tilde{V_1}$ lift two essential loops $\alpha \subset V_1 \in \mathcal{F}$ and $\beta \subset V_2 \in \mathcal{F}$ respectively. Notice that in particular $\mathrm{cl}[\tilde{\alpha}] \cap \mathbb{S}^1 = \mathrm{cl}[\tilde{\beta}] \cap \mathbb{S}^1 = \{a, b\}$.

We define:

$$\tilde{A} := \bigcup_{D \in \mathcal{D}} D$$

The construction of $\tilde{A}$ implies for any compact set $K \subset \tilde{A}$, that $K \subset D$ for some $D \in \mathcal{D}$. Then we have that $\tilde{A}$ is open, connected and simply connected.

We claim that $A := p(\tilde{A})$ is an annulus. If A is not an annulus there exists $T \in G$ with $T(\{a, b\}) \neq \{a, b\}$ and $T(\tilde{A}) \cap \tilde{A} \neq \emptyset$ (as A has to contain at least two essential loops in different homotopy classes). this implies that we have $V_0, V_1 \in \mathcal{F}$ with $T(\tilde{V_1}) \cap \tilde{V_0} \neq \emptyset$ and $\tilde{\xi} \subset T(\tilde{V_1})$, which lifts an essential loop $\xi \subset V_1$, such that $\mathrm{cl}[\tilde{\xi}] \cap \mathbb{S}^1 \neq \{a, b\}$. This implies since $\tilde{V_0}$ is a connected component of the complement of $p^{-1}(\mathcal{M}^c)$ that $\tilde{\xi} \subset \tilde{V_0}$ which is impossible by definition of $\tilde{V_0}$.

Therefore we have that A is an annulus verifying the points 1 and 2 of the Theorem. The points 3 and 4 are direct consequences of the construction.

□

Lemma 3.4.5. *Let $U \in \mathcal{A} \cap \Pi_0(\mathcal{M}^c)$, and $\mathcal{F}$ as in the previous lemma. Then, $\mathcal{F} = \{U\}$.*

Proof. Suppose for a contradiction that $\mathcal{F} \neq \{U\}$, and let $A \in \mathcal{A}$ be given by the previous lemma. By item 3 in this lemma, $\partial A \subset \mathcal{M}$. Moreover, $\mathcal{M} \cap A \neq \emptyset$ because $\mathcal{F} \neq \{U\}$. Therefore there exists $x \in \partial A$ and $n > 0$ such that $f^n(x) \in A$.

We first claim that $f^n(A)_\infty = A_\infty$. By item 4 in the previous lemma there exists $U_0 \in \mathcal{F}$ and $z \in \partial U_0$ such that $f^n(z) \in A$. Then, $f^n(U_0) \cap A \neq \emptyset$, $f^n(U_0) \cap \mathcal{M} = \emptyset$ and so $f^n(U_0) \subset A$. This implies that $A_\infty = f^n(U_0)_\infty = f^n(A_\infty)$.

Now, we claim that $f^n(\partial A) = \partial A$. If $f^n(\partial A) \neq \partial A$ then due to the minimality of $\mathcal{M}$, the claim above, and item 4 in previous lemma, we have necessarily that $\partial A \setminus \mathrm{cl}[f^n(A)] \neq \emptyset$ (otherwise ∂A would contain wandering points). Then, due to the point 4 of the last lemma, there exists an annulus $V_0 \in \mathcal{F}$ such that $A_0 \subset A \setminus f^n(A)$.

Let us fix $\{a, b\} = \mathbb{S}^1 \cap \tilde{\alpha}$ where $\tilde{\alpha}$ is the lift of an essential loop in V_0 and a lift g of f^n such that g leaves invariant $\{a, b\}$. Further let $\tilde{A}$ be the lift of A which contains $\tilde{\alpha}$ and $\tilde{V}_0 \subset \tilde{A}$ lift of V_0. Then, $\tilde{V}_0 \subset \tilde{A} \backslash \tilde{g}(\tilde{A})$ which implies $\tilde{g}^{-1}(\tilde{V}_0) \cap \tilde{A} = \emptyset$. But $f^{-n}(V_0) \in \mathcal{F}$, so $g^{-1}(\tilde{V}_0) \subset \tilde{A}$. This contradiction proves the claim.

Therefore $f^n(\partial A) = \partial A$, which contradicts the fact that $x \in \partial A$ and $f^n(x) \in A$.

□

Corollary 3.4.6. *The number of essential annuli in* $\Pi_0(\mathcal{M}^c)$ *is finite.*

Proof. Let $\mathcal{A} \subset \Pi_0(\mathcal{M}^c)$ be the family of all the essential annuli. Hence to each element $U \in \mathcal{A}$ we can associate a simple and closed geodesic γ_U. Furthermore, due to the previous lemma, the set $\mathcal{G} = \{\gamma_U \mid U \in \mathcal{A}\}$ is a pairwise disjoint family. Therefore, since the cardinal of any family of pairwise disjoint simple and closed geodesics is upper bounded by $3g - 3$ (see [Hug05]), we have that $\mathcal{G}$ is finite. Thus $\mathcal{A}$ is finite and its cardinal is upper bounded by $3g - 3$.

□

The last corollary and the same arguments as in Lemma 3.4.3 lead to the following Corollary.

Corollary 3.4.7. *The number of essential components in* $\Pi_0(\mathcal{M}^c)$ *is finite.*

Proof. It is obvious that the number of components with genus in $\Pi_0(\mathcal{M}^c)$ is finite. Furthermore, as $i_* : \pi_1(C, x_0) \to \pi_1(S, x_0)$ is injective for $C \in \Pi_0(\mathcal{M}^c)$, the number of genus zero components which are not annuli in $\Pi_0(\mathcal{M}^c)$ is also finite (see proof of Lemma 3.4.3). We are now done by Corollary 3.4.6. □

Corollary 3.4.8. *Every essential component* U *in* $\Pi_0(\mathcal{M}^c)$ *is periodic. In particular,* $\mathcal{M} = \partial U \cup \partial f(U) \cup \ldots \cup \partial f^p(U)$.

We finish this section with the proof of Proposition 3.4.1

Proof. The definition of type 2 minimal set (see the preliminary classification) and the corollary 3.4.7 imply that $\mathcal{E}(\mathcal{M})^c$ is a non-empty finite family of simply connected domains. In particular, the previous corollary implies for $U \in \mathcal{E}(\mathcal{M}^c)$ that

$$\mathcal{M} = \partial U \cup \partial f(U) \cup \ldots \cup \partial f^p(U).$$

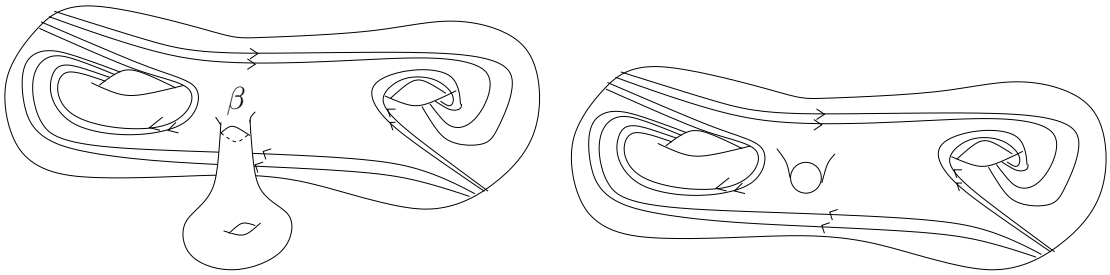
Corollary 3.2.9 implies that $\mathcal{F} = \{C \subset S \mid C \in \Pi_0(\partial f^i(U))$ for some $i = 0, ..., p\}$ is a finite family. Hence $\mathcal{M}$ has finitely many connected components $\{\mathcal{C}_i\}_{i=0}^{n-1}$ which due to the minimality of $\mathcal{M}$ are permuted by f. Furthermore, for every $i = 0, \ldots, n-1$ we have by construction that $\mathcal{C}_i$ is a union of elements in $\mathcal{F}$, which implies that $\mathcal{C}_i \supset [\partial V]_x$ for some $V \in \mathcal{E}(\mathcal{M}^c)$, where $[\partial V]_x$ is some connected component of ∂V. Moreover, the minimality of $\mathcal{M}$ implies that we

can consider $m_i \in \mathbb{N}$ with $f^{m_i}([\partial V]_x) = [\partial V]_x$, and $f^{m_i}(\mathcal{C}_i) = \mathcal{C}_i$, hence we have by minimality that $\mathcal{C}_i = [\partial V]_x$.

□

3.4.2 Surgery

Given a continuum $K \subset S$ by means of the arguments as in the proof of Lemma 3.4.3 one can easily observe that $\#\mathcal{E}(S \setminus K)$ is finite. If $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ is a finite family of loops in $S \backslash K$, we define $S^*_{\mathcal{B}}$ to be the closure of the connected component of $S \backslash \cup_{i=1}^n \beta_i$ containing K.



(a) (b)

Figure 3.4.1: $\mathcal{B}$ and $S^*_{\mathcal{B}}$

Lemma 3.4.9. *Let $K \subset S$ be a continuum. Further assume that if $U \in \mathcal{E}(S \backslash K)$, then $i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$ is injective.*

Then, there exists a finite family of simple, essential and pairwise disjoint loops in $S \backslash K$, $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ such that:

1. *the elements in $\Pi_0(S^*_{\mathcal{B}} \backslash K)$ are either annuli with one boundary component contained in $\cup_{i=1}^n \beta_i$ or disks with a disjoint closure of $\bigcup_{i=1}^n \beta_i$;*
2. *If $g : U \to U$ is a homeomorphism, where $U \in \mathcal{E}(S \backslash K)$, then $\{[\beta] \mid \beta \in \mathcal{B} \cap U\} = \{[g(\beta)] \mid \beta \in \mathcal{B} \cap U\}$[2], where $[\gamma] \in \pi_1(S)/_{\alpha \sim \alpha^{-1}}$.*

Proof. If $U \in \mathcal{E}(S \backslash K)$, the fact that $i_* : \pi_1(U, x_0) \to \pi_1(S, x_0)$ is injective implies due to Corollary 3.2.8 that U is homeomorphic to a surface of finite type, that is, U is homeomorphic to a k_U-punctured surface, $k_U \in \mathbb{N}, k_U \geq 1$. For every $i = 1, \ldots, k_U$ pick a simple loop β_i^U surrounding each puncture in such a way that the β_i^U's are pairwise disjoint, and define $\mathcal{B}_U = \{\beta_1^U, \ldots, \beta_{k_U}^U\}$. By the choice of the β_i^U's, for any $i = 1, \ldots, k_U$, the set $U \setminus \bigcup_{i=1}^{k_U} \beta_i^U$ has only one connected component which is an annulus A_i^U with one boundary component given by β_i^U and the other contained in K. Define

$$\mathcal{B} = \bigcup_{U \in \mathcal{E}(S \backslash K)} \mathcal{B}_U.$$

[2]Here a usuall abuse of notation is done in order to get the sets defined.

If $V \in \mathcal{E}(S^*_{\mathcal{B}} \backslash K)$, then $V = A^U_i$ for some $U \in \Pi_0(S \backslash K)$ and $i = 1, \ldots, k_U$. On the other hand if $V \in \Pi_0(S^*_{\mathcal{B}} \setminus K) \setminus \mathcal{E}(S^*_{\mathcal{B}} \backslash K)$ we have that it is homotopically trivial and, due to K be connected, simply connected. This gives us 1.

Furthermore, by construction, for any $U \in \mathcal{E}(S \backslash K)$ the set $\{[\beta] \mid \beta \in \mathcal{B} \cap U\}$ is determined by the topology of U, which gives 2.

□

Notice that in the last lemma we have by construction for every loop in ∂S^*_β, that they are accumulated by points in S^*_β only by one side, with respect some fixed orientation. In particular, this implies that $S' := S^*_\beta / \partial$ is a closed surface. Furthermore, due to ∂S^*_β being given by essential loops of S we have that $g(S') < g(S)$.

If $\mathcal{C} \subset S$ is a continuum and type 2 minimal set for $f \in \text{Homeo}(S)$, then $\mathcal{C}$ verifies the hypothesis of Lemma 3.4.9 (see Proposition 3.4.1).

Lemma 3.4.10. *Let $\mathcal{C}$ be a continuum and type 2 minimal set for $f \in \text{Homeo}(S)$, and let $\mathcal{B}$ be given by Lemma 3.4.9. Then, there exists $n_0 \in \mathbb{N}$ and $g : S \to S$ such that $g = f^{n_0}$ in $\mathcal{C}$ and $g(\beta) = \beta$ for all $\beta \in \mathcal{B}$.*

Proof. By Corollary 3.4.8, essential components in $\Pi_0(\mathcal{C}^c)$ are periodic. So, there exists $n \in \mathbb{N}$ such that $f^n(U) = U$ for all essential component $U \in \Pi_0(\mathcal{C}^c)$. Furthermore, by Lemma 3.4.9, Item 2 $\{[\beta] \mid \beta \in \mathcal{B} \cap U\} = \{[f^n(\beta)] \mid \beta \in \mathcal{B} \cap U\}$. As $\mathcal{B}$ is a finite family, we take a power m of f^n fixing every $[\beta]$, $\beta \in \mathcal{B}$. Let $n_0 = n + m$. It is a well known fact that one can modify f^{n_0} with an isotopy supported in $\mathcal{C}^c$ to obtain $g : S \to S$ such that $g(\beta) = \beta$ for all $\beta \in \mathcal{B}$. □

Let $\mathcal{C} \subset S$ be a continuum and a type 2 minimal set of $f \in \text{Homeo}(S)$. Further, consider the family of pairwise essential loops $\mathcal{B}$ given by Lemma 3.4.9 and $S^* = S^*_{\mathcal{B}}$. Let n_0 and g be given by Lemma 3.4.10. We now give a proof for Theorem 3.4.2.

Proof. The quotient map is a homeomorphism if we restrict to $\overset{\circ}{S}{}^*$, and therefore conjugates $g|_{\mathcal{C}}$ and $\hat{g}|_{\mathcal{C}'}$, which gives us 1. By Lemma 3.4.9, item 2, $S^* \backslash \mathcal{C}$ is a union of disks or annuli with one boundary component contained in $\mathcal{B}$. So, $S' \backslash \mathcal{C}'$ is a union of disks. Furthermore, for each $C \in \mathcal{E}(\mathcal{C}^c)$ there is at least one $\beta \in \mathcal{B} \cap C$, whose projection gives a fixed point for $\hat{g}$, thus we obtain 2.

□

3.5 Type 3 minimal sets.

As usual, we consider a closed hyperbolic surface S. Recall that a minimal set $\mathcal{M} \subset S$ of $f \in \text{Homeo}(S)$ is an extension of a minimal Cantor set (of a

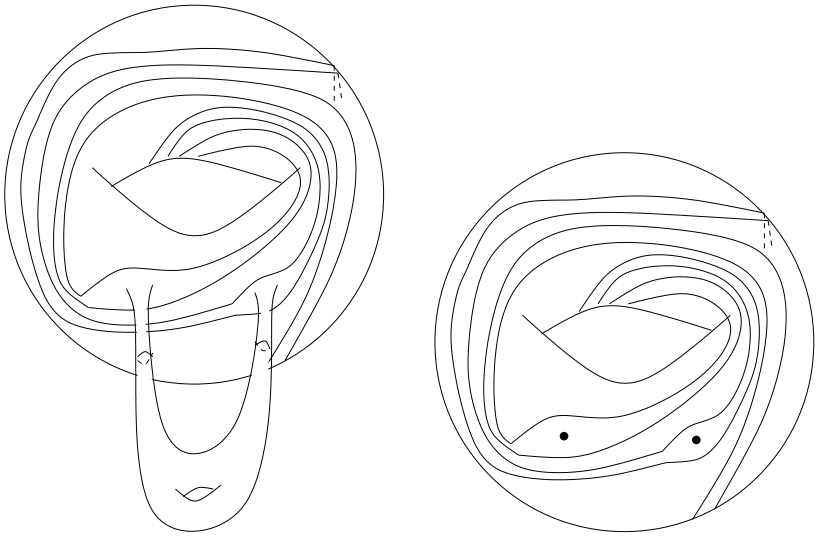


Figure 3.4.2: Proof of Theorem 3.4.2

periodic orbit) if there exists a map $h : S \to S$ homotopic to the identity such that:

(i) $h \circ f = \hat{f} \circ h$ for some $\hat{f} \in \mathrm{Homeo}(S)$;

(ii) $\mathcal{M}' = h(\mathcal{M})$ is a minimal Cantor set (is a periodic orbit) of $\hat{f}$, and $h^{-1}(y)$ contains a unique connected component of $\mathcal{M}$ for every $y \in \mathcal{M}'$.

In this section we prove the following

Proposition 3.5.1. *If $\mathcal{M}$ is a type 3 minimal set, then $\mathcal{M}$ is an extension of a periodic orbit or a minimal cantor set.*

The analogue result for the torus is proved in Section 2.3.3. As much of the work is already done there, rather than repeating the proofs we give precise references.

The topological tool needed for the proof is a classical decomposition theorem by Moore [Why42]. It concerns upper semi-continuous partitions of surfaces into continua.

A partition $\mathcal{G}$ of S into closed sets is upper semi-continuous if for each $g \in \mathcal{G}$ and each open set U containing g, there exists an open set $V \subset U$ containing g and such that every $g' \in \mathcal{G}$ which meets V lies in U.

In our context (S is a compact metric space), $\mathcal{G}$ is upper semi-continuous if and only if for every $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{G}$ with a Hausdorff limit X, there exists $g \in \mathcal{G}$ such that $X \subset g$.

We say that a set $X \subset S$ is *filled contractible* if each open neighborhood of X contains a disk D having X in its interior, such that $D \setminus X$ has a unique connected component.

Theorem 3.5.2. [Why42] *If $\mathcal{G}$ is an upper semi-continuous partition of S such that every $g \in \mathcal{G}$ is a filled contractible continuum, then $S/\mathcal{G}$ is homeomorphic to S.*

If a closed set $X \subset S$ is contained in a bounded disk $D \subset S$, then $\mathrm{Fill}(X)$ is defined as to be the union of X with the bounded components of its complement (where ∂D is identify with ∞). So, $\mathrm{Fill}(X)$ is a filled contractible set.

As $\mathcal{M}$ is a type 3 minimal set, then every $\mathcal{C} \in \Pi_0(\mathcal{M})$ is contained in a bounded disk (see Lemma 3.2.2). So, $\mathrm{Fill}(\mathcal{C})$ is defined for all $\mathcal{C} \in \Pi_0(\mathcal{M})$.

Lemma 3.5.3. *Let $\mathcal{G} = \{\mathrm{Fill}(\mathcal{C}) : \mathcal{C} \in \Pi_0(\mathcal{M})\} \cup \{x \in S : x \notin \cup_{\mathcal{C} \in \Pi_0(\mathcal{M}}\mathrm{Fill}(\mathcal{C})\}$. Then, $\mathcal{G}$ is an upper semi-continuous partition of S in filled contractible sets.*

Proof. We first prove that $\mathcal{G}$ is a partition. It is enough to prove that $\mathrm{Fill}(\mathcal{C}) \cap \mathrm{Fill}(\mathcal{C}') = \emptyset$ if $\mathcal{C}$ and $\mathcal{C}'$ are different elements in $\Pi_0(\mathcal{M})$. But $\mathrm{Fill}(\mathcal{C}) \cap \mathrm{Fill}(\mathcal{C}') \neq \emptyset$ implies that $\mathrm{Fill}(\mathcal{C}) \subset \mathrm{Fill}(\mathcal{C}')$ (or the other way around). However, this is impossible because of Lemma 3.2.6.

Let $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{G}$ with a Hausdorff limit X. As every set in $\mathcal{G}$ is a continuum, then X is a continuum. Furthermore, by definition of $\mathrm{Fill}(\mathcal{C})$ we have for every point $x \in X$ a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in g_n \cap \mathcal{M}$ for every $n \in \mathbb{N}$ and $(y_n)_{n\in\mathbb{N}}$ converges to x (see 2.3.3 for a detailed argument). Besides, as $\mathcal{M}$ is closed, $X \subset \mathcal{M}$. So, there exists $g \in \mathcal{G}$ such that $X \subset \mathcal{G}$.

□

The proof of Proposition 3.5.1 follows exactly as in 2.3.3 (see Lemma 2.3.19, and the proof of Addendum 2.3.18).

3.6 Proof of Theorem 3.1.1 and Theorem 3.1.2.

In this section we prove the main results of the chapter, which are a classification of minimal set for surface homeomorphisms given by Theorem 3.1.1, and the topological and dynamical characterization for type 2 minimal sets given in Theorem 3.1.2 (see the introduction).

Theorem 3.6.1. *Let S be a closed hyperbolic surface and $\mathcal{M} \subset S$ minimal set of $f \in \mathrm{Homeo}(S)$. Then either:*

(1) *the elements in $\Pi_0(\mathcal{M}^c)$ are all disks, with at least one unbounded and periodic. Moreover, any bounded disk in $\Pi_0(\mathcal{M}^c)$ is wandering;*

(2) *$\mathcal{E}(\mathcal{M}^c)$ is a finite family of locally simply connected sets, and $\mathcal{M} = \mathcal{C}_0 \cup ... \cup \mathcal{C}_{n-1}$ where $\{\mathcal{C}_i\}_{i=0}^{n-1}$ is a family of pairwise disjoint continua with $f(\mathcal{C}_i) = \mathcal{C}_{i+1 mod(n)}$ for all $i = 0, \ldots, n-1$. Furthermore each element in $\{\mathcal{C}_i\}_{i=0}^{n-1}$ coincides with a connected component of ∂V for some $V \in \mathcal{E}(\mathcal{M}^c)$, and there are not periodic bounded discs in $\Pi_0(\mathcal{M}^c)$.*

(3) *$\mathcal{M}$ is either an extension of a periodic orbit or an extension of a minimal cantor set.*

Proof. Theorem 3.2.1 gives us three complementary cases:

1. Every $U \in \Pi_0(\mathcal{M}^c)$ is a disk;

2. There exists an essential component $U \in \Pi_0(\mathcal{M}^c)$, and $i_* : \pi_1(V, x_0) \to \pi_1(S, x_0)$ is injective for all $V \in \Pi_0(\mathcal{M}^c)$;

3. There exists a fully essential component $V \in \Pi_0(\mathcal{M}^c)$ and every other component $U \in \Pi_0(\mathcal{M}^c)$ is a bounded disk.

In the first case, Lemma 3.3.6 gives us an unbounded periodic disk in $\mathcal{M}^c$, and Lemma 3.3.5 gives us that any bounded disk in $\Pi_0(\mathcal{M}^c)$ is wandering. In the second case, Proposition 3.4.1 gives us the desired result. Finally, in the third case we apply directly Theorem 3.5.1.

□

From now on we refer to a minimal set $\mathcal{M}$ as a type 1,2 or 3 minimal set, whenever it is in the case 1,2 or 3 of the last theorem. The following result can be found in subsection 2.3.2.

Lemma 3.6.2. *If $\mathcal{C} \subset S$ is an essential continuum contained in a open annulus which is a minimal set for $f \in \text{Homeo}(S)$, then $\mathcal{C}$ is the boundary of a circloid.*

Before stating the main theorem of this section, we need a result which is contained in S. Matsumoto's and H. Nakayama's work [MN11], concerning continua which are minimal set of homeomorphisms in the sphere. This result was already quoted in Chapter 2. We denote by S^2 the Riemann sphere.

Theorem 3.6.3. *Let $\mathcal{C} \subset S^2$ be a continuum and minimal set for a homeomorphism $f \in Homeo(S^2)$. Then, there are at most two components in $\Pi_0(S^2 \setminus \mathcal{C})$ which are periodic by f.*

We recall some preliminaries before proving Theorem 3.1.2. Given a continuum and type 2 minimal set $\mathcal{C} \subset S$ of $f \in \text{Homeo}(S)$, we say that it is reducible to a type 1 minimal set in S', if there exists a closed surface S' with $g(S') < g(S)$, a homeomorphism $\hat{f} : S' \to S'$ and $\mathcal{C}'_k$ a type 1 minimal set for $\hat{f}$ such that $f|_{\mathcal{C}_k}$ is conjugated to $\hat{f}|_{\mathcal{C}'_k}$. Furthermore we say that a reducible minimal set $\mathcal{C} \subset S$ is g-reducible, where $g = \min\{g(S') : \mathcal{C} \text{ is reducible to a minimal set in } S'\}$.

The second result, which completes the classification, is the following:

Theorem 3.6.4. *Let S be a closed hyperbolic surface and $\mathcal{M} = \mathcal{C}_0 \cup ... \cup \mathcal{C}_{n-1}$ a type 2 minimal set of $f \in \text{Homeo}(S)$. Then, there exists $N > 0$ such that $\mathcal{C}_k$ is reducible to a type 1 minimal set for f^N. Moreover,*

(2i) if it is 0-reducible, then $\mathcal{C}_k$ is the boundary of a circloid for every $k = 0, ..., n-1$ and $\#\mathcal{E}(\mathcal{C}_k^c) \leq 2$. Further, every disk in $\Pi_0(\mathcal{M}^c)$ is wandering.

(2ii) if it is g-reducible with $g > 0$, then every bounded disk in $\mathcal{M}^c$ is wandering, and there are at least $\#\mathcal{E}(\mathcal{C}_k^c)$ unbounded fixed disks in $\Pi_0(S' \backslash \mathcal{C}_k')$.

Proof. For a fixed $k \in \{0, ..., n-1\}$ we consider $S^* \supset \mathcal{C}_k$, $g \in \text{Homeo}(S)$ and $n_0 \in \mathbb{N}$ associated to $\mathcal{C}_k$ and f^n by Theorem 3.4.2. Let $N = n + n_0$. Then due to Theorem 3.4.2 we have that $\mathcal{C}_k$ is reducible to a type 1 minimal set for f^N.

Furthermore, we have the two following complementary cases:

(2i) $\mathcal{C}_k$ is 0-reducible: in this case we have that $\mathcal{C}'$ is a minimal set of $\hat{g}$ in S', and S' can be identify with the sphere S^2 . Furthermore $\mathcal{C}'$ is a continuum contained in $q(\overset{\circ}{S^*})$ and necessarily there are at least two different essential loops β_1, β_2 in ∂S^* (otherwise $g(S') \geq 1$). Hence Theorem 3.4.2 implies that there are at least two fixed disks $U_0, U_1 \in \Pi_0(\mathcal{C}'^c)$ of $\hat{g}$ separated by $\mathcal{C}'$. Moreover due to Theorem 3.6.3 these two are the unique loops in ∂S^*.

Consider now a neighborhood $V \subset S^2 \setminus \{q(\beta_1), q(\beta_2)\}$ of $\mathcal{C}'$, which is an essential annulus in the open annulus $S^2 \setminus \{q(\beta_1), q(\beta_2)\}$. Thus due to $q : \overset{\circ}{S^*} \to S^2 \setminus \{q(\beta_1), q(\beta_2)\}$ be a homeomorphism, we have that $W = q^{-1}(V)$ is an essential annulus containing $\mathcal{C}$. This, together with Lemma 3.6.2 implies that $\mathcal{C}_k$ is the boundary of a circloid. Hence the same holds for every $k = 0, ..., n-1$, so we are in case (2i) of the Theorem.

(2ii) $\mathcal{C}_k$ is g-reducible with $g > 0$: in this case the point (2ii) of the Theorem is a direct consequence of Theorem 3.4.2 and Lemma 3.3.5.

□

3.7 Classification Theorem in the non-wandering case.

We now restrict our classification to the class of non-wandering homeomorphisms. We denote by $\text{Homeo}_{nw}(S)$ the class of all such homeomorphisms (see Subsection 1.1.3).

As was done in the toral case, we make use of the result by A. Koropecki [Kor10] which describes the topology of continua in surfaces which are f-invariant and aperiodic (i.e it does not contain periodic orbits) for some non-wandering homeomorphism f.

Theorem 3.7.1. *Suppose that $\mathcal{C} \subset S$ is an f-invariant and aperiodic continuum for $f \in \text{Homeo}_{nw}(S)$. Then $\mathcal{C}$ is an intersection of a decreasing sequence of annuli.*

As a consequence, our classification theorem in the non-wandering setting comes down to:

Corollary 3.7.2. *Let S be a closed hyperbolic surface, and $\mathcal{M} \subset S$ a minimal set for a non-wandering homeomorphism $f \in \mathrm{Homeo}(S)$. Then either:*

(1^{nw}) *$\mathcal{M} = \mathcal{C}_0 \cup ... \cup \mathcal{C}_{n-1}$ where $\{\mathcal{C}_i\}_{i=0}^{n-1}$ is a family of pairwise disjoint essential circloids with $f(\mathcal{C}_i) = \mathcal{C}_{i+1}$ mod(n) for all $i = 0, \ldots, n-1$. Furthermore, there are no disks in $\Pi_0(\mathcal{M}^c)$.*

(2^{nw}) *$\mathcal{M}$ is an extension of either a periodic orbit or a minimal cantor set.*

Proof. We first want to see that type 1 and type 2ii minimals sets can not occur for $f \in \mathrm{Homeo}_{nw}(S)$. If S' is an oriented surface and $\mathcal{M} \subset S'$ is a type 1 minimal set for some homeomorphism $g \in \mathrm{Homeo}(S')$, then $\mathcal{M}$ is an aperiodic continuum which is not a decreasing intersection of annuli. Hence, due to the result above, type 1 minimal sets can not occur for $g \in \mathrm{Homeo}_{nw}(S')$. This implies that type 1 minimal sets can not occur for $f \in \mathrm{Homeo}_{nw}(S)$.

Suppose for a contradiction that $\mathcal{M} = \mathcal{C}_0 \cup ... \cup \mathcal{C}_{n-1}$ is a minimal set of $f \in \mathrm{Homeo}_{nw}(S)$ in the case 2ii of Theorem 3.1.2. The map $\hat{g}$ associated by Theorem 3.1.2 to $\mathcal{C}_0$, can be considered by construction non-wandering (we do not give a proof, but this fact is well known). On the other hand $\mathcal{C}_0'$ is a type 1 minimal set for $\hat{g}$, which is absurd.

So, now we consider $\mathcal{M}$ as in 2i of Theorem 3.1.2. We know that $\mathcal{M} = \partial\mathcal{C}_0 \cup ... \cup \partial\mathcal{C}_{n-1}$ where $\{\mathcal{C}_i\}_{i=0}^{N-1}$ is a family of pairwise disjoint circloids. Further, by definition of circloid we know that the existence of an interior point x for some $\mathcal{C}_i$ would imply the existence of a disks in $\Pi_0(\mathcal{M}^c)$, which has to be wandering due to Theorem 3.1.2. Hence due to f be non-wandering we have that $\partial\mathcal{C}_i = \mathcal{C}_i$ for every $i = 0, ..., N-1$, which finishes the proof. □

We point out again the slight difference between the cases (1^{nw}) of last theorem and ($2i$) of Theorem 3.1.2. In the non-wandering case we have that the continua are circloids, while in the general case we only know that they are the boundary of a circloid.

3.8 Classification and Nielsen-Thurston theory.

In this section we study the given classification of minimal sets in light of Nielsen-Thurston theory (see [Thu98]). We will see that type 1 minimal sets can only occur in the homotopy class of either periodic homeomorphisms or reducible maps without pseudo-Anosov components. This will trivially imply for elements in the homotopy class of pseudo-Anosov homeomorphisms, that

only type 3 minimal sets can occur (which is the analogous of the result for the classification in the torus). Let us first introduce some notions, which are related to the dynamics of pseudo-Anosov components (see [Boy94]).

Consider a metric d in a space $\mathcal{X}$ and $f : \mathcal{X} \to \mathcal{X}$ bijective. We say that f is expansive with respect d if there exists $\alpha > 0$ such that $\sup\{d(f^n(x), f^n(y)) \mid n \in \mathbb{Z}\} > \alpha$ for any $x, y \in \mathcal{X}$. Given now $f : \mathcal{X} \to \mathcal{X}$ and $g : \mathcal{X} \to \mathcal{X}$ bijective functions we say that the pair (x, f) and (y, g) globally shadows with respect to d, if there exists $K > 0$ such that $d(f^n(x), g^n(y)) < K$ for every $n \in \mathbb{Z}$. Is easy to see that this defines an equivalence relation between these kind of pairs.

Consider a hyperbolic surface S and two homeomorphisms $f, \phi \in \text{Homeo}(S)$ which are homotopic. Further fix $\tilde{f}, \tilde{\phi}$ equivariantly homotopic lifts of f, ϕ. We say for $x, y \in S$ that $(x, \tilde{f})$ and $(y, \tilde{\phi})$ globally shadows if $(\tilde{x}, \tilde{f})$ and $(\tilde{y}, \tilde{\phi})$ globally shadows with respect the hyperbolic metric, where $p(\tilde{x}) = x$ and $p(\tilde{y}) = y$.

We denote by $\text{Homeo}_+(S)$ the set of orientation preserving homeomorphisms in S and by ρ the hyperbolic metric of S. In light of the Nielsen-Thurston theory we define:

$$\mathcal{A} = \{f \in \text{Homeo}_+(S) |\ f \sim g\ :\ g \text{ is either pA or reducible with pA components}\},$$

The following result can be found in [Boy99] (Theorem 3.2).

Theorem 3.8.1. *Suppose S hyperbolic surface and $f \in \mathcal{A}$. Then there exists $\phi \in Homeo_+(S)$ and a non-empty ϕ-invariant and compact surface $S' \subset S$ (possibly with boundary and non-connected) such that:*

(1) the boundary of S' is either empty or union of essential curves. Further it is empty if and only if f is homotopic to a pseudo-Anosov map;

(2) ϕ is homotopic to f and $\phi|_{int(S')}$ is expansive;

(3) There exist a compact f-invariant set $\mathcal{Y} \subset S$, a continuous homotopic to the projection map $\alpha : \mathcal{Y} \to S/_{\partial S'}$ (where $S/_{\partial S'} = S$ if $\partial S' = \emptyset$), and two equivariantly homotopic lifts $\tilde{f}, \tilde{\phi}$ of f, ϕ verifying:

(3a) $Im(\alpha) = S'/_{\partial S'}$ and $\alpha \circ f|_{\mathcal{Y}} = \phi_q \circ \alpha$, where $\phi_q : S'/_{\partial S'} \to S'/_{\partial S'}$ is the induced map by ϕ (equals to ϕ if $\partial S' = \emptyset$);

(3b) if $x, x' \in \mathcal{Y}$ where $(x, \tilde{f})$ and $(x', \tilde{f})$ globally shadows, then $\alpha(x) = \alpha(x')$.

In order to prove the next fundamental lemma, we need to introduce a result due to R. Mañé (see [Mañ79]), which in our context can be stated as follows.

Theorem 3.8.2. *If $f \in \text{Homeo}(S)$ is expansive, then every minimal set of f is either a periodic orbit or a Cantor set.*

Lemma 3.8.3. *Consider $\alpha : \mathcal{Y} \to S/_{\partial S'}$ given by Theorem 3.8.1 and assume that $\mathcal{M} \subset S$ is a type 1 minimal set of f. Then $\mathcal{Y} \subset \mathcal{M}^c$.*

Proof. Suppose for a contradiction that $\mathcal{Y} \cap \mathcal{M} \neq \emptyset$. Thus because of the minimality of $\mathcal{M}$ we have that $\mathcal{M} \subset \mathcal{Y}$. This implies, since α semiconjugates $f|_{\mathcal{Y}}$ with ϕ_q, that $\mathcal{M}' = \alpha(\mathcal{M})$ is a minimal set of ϕ_q. We now split the proof in the following two complementary cases:

> $\partial S' = \emptyset$: in this case we have that ϕ is expansive in the surface S, and hence due to Theorem 3.8.2 $\mathcal{M}'^c$ is given by a doubly essential component. This is impossible since $\Pi_0(\mathcal{M}^c)$ is a family of disks, and the map α is homotopic to the inclusion (the identity restricted to $\mathcal{M}$).
>
> $\partial S' \neq \emptyset$: due to $\mathcal{M}$ being a type 1 minimal set of f and point (1) in Theorem 3.8.1, we have necessarily that $\mathcal{M} \cap \partial S' \neq \emptyset$. Thus if $q : S' \to S'/_{\partial S'}$ is the quotient map we have that $\mathcal{M}' \supset \{P_0\}$, where $\{P_0\} = q(\partial S')$. Therefore since P_0 is fixed by ϕ_q we have that $\mathcal{M}' = \{P_0\}$. Now, by Theorem 3.8.1 we have that $S/_{\partial S'} \setminus \{P_0\}$ is a union of open surfaces with non-trivial fundamental group, while since $\mathcal{M}$ is a type 1 minimal set $q(\mathcal{M})^c$ is a union of disks. This is impossible due to α be homotopic to q.

□

We now state the non-existence of type 1 minimal sets for elements in $\mathcal{A}$.

Theorem 3.8.4. *If $\mathcal{M} \subset S$ is a type 1 minimal set of $f \in \mathrm{Homeo}(S)$, then f is not in $\mathcal{A}$.*

Proof. We suppose for a contradiction that $f \in \mathcal{A}$. Let us consider S', ϕ and $\alpha : \mathcal{Y} \to S/_{\partial S'}$ given by Theorem 3.8.1. Due to Lemma 3.8.3 we have that $\mathcal{Y} \subset \mathcal{M}^c$. Then there exists a family of pairwise disjoint closed disks $\{D_i\}_{i=0}^{N-1}$ and of pairwise disjoint elements in $\Pi_0(\mathcal{M}^c)$, $\{U_i\}_{i=0}^{N-1}$, such that:

(i) $D_i \subset U_i$ for every $i = 0, ..., N-1$;

(ii) $\mathcal{Y} \subset \bigcup_{i=0}^{n-1} D_i$;

(iii) for every $i \in \{0, ..., N-1\}$ there exits $j \in \{0, ..., N-1\}$ with $f(\mathcal{Y}_i) = \mathcal{Y}_j$, where $\mathcal{Y}_k = \mathcal{Y} \cap D_k$ for every $k = 1, ..., N-1$.

Let us now consider the lifts $\tilde{f}, \tilde{\phi}$ of f, ϕ given by Theorem 3.8.1. We claim that if $x, y \in \mathcal{Y} \cap \mathcal{Y}_i$ for some $i = 0, ..., N-1$ then $(x, \tilde{f})$ and $(y, \tilde{f})$ globally shadows.

In fact for such x, y consider two lifts $\tilde{x}, \tilde{y} \in \tilde{D}_i$ where $\tilde{D}_i$ is a connected component of $p^{-1}(D_i)$, and let $R = \max\{\mathrm{diam}(D_i) \mid i = 0, ..., N-1\}$. Then we have that $\sup\{d(\tilde{f}^n(\tilde{x}), \tilde{f}(\tilde{y})) \mid n \in \mathbb{Z}\} \leq R$. Otherwise by definition of

R and the point (iii) we have for some $k \in \mathbb{Z}$ that $\tilde{f}^k(\tilde{D}_i) \cap \tilde{D}_j \neq \emptyset$ and $\tilde{f}^k(\tilde{D}_i) \cap \sigma(\tilde{D}_j) \neq \emptyset$, where $\tilde{D}_j$ is a connected component of $p^{-1}(D_j)$ and $\sigma \in \Gamma \setminus \{e\}$. This implies that $p(\tilde{D}_j \cup \tilde{f}^k(\tilde{D}_i) \cup \sigma(\tilde{D}_j))$ is contained in a essential element of $\Pi_0(\mathcal{M}^c)$ which is impossible since $\mathcal{M}$ is a type 1 minimal set.

Thus, by the point (3b) in Theorem 3.8.1 we have that $\mathrm{Im}(\alpha)$ has at most N elements, which is absurd since $\mathrm{Im}(\alpha) = S'/_{\partial S'}$.

□

Corollary 3.8.5. *Let S be a closed and oriented hyperbolic surface and $f \in$ Homeo(S) homotopic to a pseudo-Anosov map. Then every minimal set of $\mathcal{M}$ is type 3.*

Before giving the proof, we point out that for a pseudo-Anosov homeomorphism $\phi \in \mathrm{Homeo}(S)$, due to the expansiveness of such a system, one can apply the result [Mañ79] and obtain that every minimal set is either a periodic orbit or a minimal cantor set. However, in general, elements in their homotopy classes are not expansive.

Proof. Theorem 3.8.4 asserts that type 1 minimal set can not occur. On the other hand, we have seen in section 3.4 that the existence of a type 2 minimal set of f would imply that $f^n_{\#}([\beta]) = [\beta]$ for some $n \in \mathbb{N}$ and $\beta \subset S$ essential loop, which can not happen since f is in the homotopy class of a pseudo-Anosov homeomorphism. Then, every minimal set of f has to be type 3.

□

3.9 Examples.

In this final section we state examples for each non-trivial case of the Theorem 3.1.1 and Theorem 3.1.2.

3.9.1 Type 1 minimal sets.

We say that a flow $\phi : S \times \mathbb{R} \to S$ is irreducible if:

(i) ϕ has a unique minimal set $\mathcal{M} \subset S$;

(ii) $\mathcal{M}$ intersect any essential loop in S.

Irreducible flows can be constructed in any closed hyperbolic surface, see for instance [ABZ96]. On the other hand given a flow ϕ in a surface S with a minimal set $\mathcal{M} \subset S$, we can consider $t \in \mathbb{R}$ such that $\mathcal{M}$ is a minimal set of the time-t map f_t of the flow (this is true for flows in general contexts, see [IO07]).

Thus we can consider for any surface S an irreducible flow ϕ, its minimal set $\mathcal{M}$ given by definition, and a time t-map f with $\mathcal{M}$ as minimal set. Then due to property (ii), $\mathcal{M}$ is a type 1 minimal set of f in the surface S.

Although we give a reference for the existence of irreducible flows in any closed hyperbolic surface, for the sake of completion, we give a hint below as how to construct such examples using geodesic laminations [CB88].

One can always construct an oriented geodesic lamination Λ in the surface S of genus $g > 1$ such that $S\backslash\Lambda$ is the union of two disks that lift to the universal covering space to ideal $2g$- gones with alternating orientations on its sides. Indeed, a standard way of constructing pseudo-Anosov maps is by lifting an Anosov diffeomorphism on $\mathbb{T}^2$ to a branched covering $\pi : S \to \mathbb{T}^2$. If π has exactly two ramification points where π is $g - 1$, then the stable (or unstable) lamination of the corresponding pseudo-Anosov diffeomorphism has this property.

This lamination can be completed to a flow in S with a fixed point of index $1 - g$ in each disk. Moreover, the lamination is a minimal set for the flow, and hence there exists $t \in \mathbb{R}$ such that Λ is minimal for the time t of this flow ([IO07]).

Notice that by construction the map f is homotopic to the identity, which agrees with Theorem 3.8.4.

3.9.2 Type 2 minimal sets.

Examples in this category can be constructed defining a homeomorphism of an essential annulus $A \subset S$ and then extending it to a homeomorphism f of the whole surface S. A minimal set of the annulus homeomorphism will of course be a minimal set of f. The simplest example in this spirit is to take an irrational rotation in A, obtaining simple closed curves as minimal sets. Of course, much wilder minimal sets can be obtained with this method. The following Theorem appears in [Cro06] but gathers together previous ideas of Handel [Han82], Herman [Her86] and Fayad-Katok [FK04].

Theorem 3.9.1. *There exists a homeomorphism of the annulus $\mathbb{S}^1 \times \mathbb{R}$ which has a minimal invariant set Λ which is connected, not locally connected at any point and has positive Lebesgue measure.*

If one constructs examples in this way, one obtains type 2 connected minimal sets $\mathcal{M}$. Indeed, there is an essential component $U \in \Pi_0(\mathcal{M}^c)$ which is a surface of genus $g - 1$. One may of course generalize this example by allowing the minimal set to have more than one connected component, and therefore different homotopy types for the essential components in $\Pi_0(\mathcal{M}^c)$. Take a symmetry $\varphi : S \to S$ taking an essential closed annulus $A \subset S$ to an essential closed annulus $A' \subset S$, $A \cap A' = \emptyset$. Let $f \in \mathrm{Homeo}(S)$ supported in a neighborhood U of A, leaving A invariant and such that the restriction of f to A has a minimal set $\mathcal{C}$. Let $g = \varphi \circ f \circ \varphi^{-1}(x)$. Then, g is supported

in $\varphi(U)$, leaves A' invariant, and $\mathcal{C}' = \varphi(\mathcal{C})$ is a minimal set for $g|_{A'}$. Define $h \in \mathrm{Homeo}(S)$ by $h = f \circ g$. Now, $\mathcal{C} \cup \mathcal{C}'$ is a minimal set for the map $\varphi \circ h$.

This kind of constructions give us examples of type (2i) minimal sets. The construction of type (2ii) minimal sets is not difficult, since we have constructed type 1 minimal sets. Indeed, one just has to consider connected sum of surfaces, by the complement of a type 1 minimal set in a base surface S, and then extend the dynamics.

3.9.3 Type 3 minimal sets.

Non-trivial examples of this type of minimal sets are those which are not periodic orbits nor Cantor sets. These examples can be easily constructed in such a way that the minimal set has finitely many connected components which are not a singleton but periodic. In [Kwa11] non-trivial examples with uncountable many connected components, some of them being not a singleton, are constructed.

Chapter 4

Rotation sets of generic homeomorphisms and axiom A diffeomorphisms.

4.1 Statements of the results.

We devote this chapter to the proofs of Theorems 7,8 and 9, which we first restate. For the preliminaries definitions see Subsection 1.2.3.

Theorem 4.1.1. *There exists an open and dense set $\mathcal{D} \subset \mathrm{Homeo}_0(\mathbb{T}^2)$ such that for every $f \in \mathcal{D}$ the rotation set $\rho(F)$ is a rational polygon (possibly degenerate).*

This result is then extended to the family of axiom A diffeomorphisms (or Smale's diffeomorphisms).

Theorem 4.1.2. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be an axiom A diffeomorphism. Then $\rho(F)$ is a rational polygon.*

Finally, we observe in general that the rotation sets on minimal sets are continua, and then we prove the following result.

Theorem 4.1.3. *There exists an open set $\mathcal{D}' \subset \mathrm{Homeo}_0(\mathbb{T}^2)$, such that any $f \in \mathcal{D}'$ has a minimal set $\mathcal{M}$ with $\rho_{\mathcal{M}}(F)$ given by a non-trivial segment. Furthermore, $\mathcal{D}'$ is dense in the set of homeomorphisms whose rotation set has non-empty interior.*

The organization of the chapter is given as follows. In Section 4.2 we study the relation between symbolic dynamics and the displacement of orbits in the universal cover for maps in $\mathrm{Homeo}_0(\mathbb{T}^2)$. In Section 4.3 we construct the set $\mathcal{D}$ of Theorem 4.1.1, in Section 4.4 we prove Theorem 4.1.2, and finally in Section 4.5 we prove Theorem 4.1.3.

4.2 Symbolic representation and rotation set.

Let us consider a closed and invariant set $C \subset \mathbb{T}^2$ of $f \in \text{Homeo}_0(\mathbb{T}^2)$, and a family $\mathcal{S} = \{S_0, ..., S_M\}$ of pairwise disjoint sets contained in $\mathbb{T}^2$ such that:

- S_i is homeomorphic to the closed unit square for every $i = 0, ..., M$;
- $C \subset \bigcup_{i=0}^{M} S_i$.

We say that $\mathcal{S}$ is a finite partition of C by rectangles. In the previous chapters we call the elements of $\mathcal{S}$ disks, but along this chapter we call them rectangles since it is the appopiate geometric form when Markov partitions are considered. For the sake of simplicity we refer to $\mathcal{S}$ as a partition. As usual, for every partition $\mathcal{S}$ we can associate the space of sequences of symbols $\Omega_{\mathcal{S}} \subset \{0, ..., M\}^{\mathbb{Z}}$ defined as

$$\Omega_{\mathcal{S}} = \left\{\xi \in \{0, ..., M\}^{\mathbb{Z}} \mid \exists\, x \in C \text{ such that } f^n(x) \in S_{\xi(n)} \forall n \in \mathbb{Z}\right\}.$$

The space $\Omega_{\mathcal{S}}$ is closed and invariant under the shift $\sigma : \{0, ..., M\}^{\mathbb{Z}} \to \{0, ..., M\}^{\mathbb{Z}}$, defined by $\sigma(\xi)(n) := \xi(n+1)$ for every $n \in \mathbb{Z}$. Furthermore, there is a natural onto map $h : C \to \Omega_{\mathcal{S}}$ given by the relation $f^n(x) \in S_{h(x)(n)}$ for every $n \in \mathbb{Z}$, which semi-conjugates $f \mid_C$ with $\sigma \mid_{\Omega_{\mathcal{S}}}$ (that is $h \circ f = \sigma \circ h$ holds in C).

4.2.1 Symbolic representation and displacement.

We say that an open, connected and bounded set $D \subset \mathbb{R}^2$ is a trivial domain if $\pi_{|D} : D \to \pi(D)$ is a homeomorphism. If C and $\mathcal{S} = \{S_1, ..., S_M\}$ are as above, we say that $\mathcal{S}$ is a rotational partition if there exists a trivial domain D such that:

(1) $\pi(D) \supset \bigcup_{i=0}^{M} S_i$;

(2) if we consider $\tilde{S}_i := \pi^{-1}(S_i) \cap D$ for $i = 0, ..., M$ and a lift F of f, then for each $i = 1, ..., M$ there is a unique vector $s_i \in \mathbb{Z}^2$ such that $F(\tilde{S}_i) \subset D + s_i$.

Notice that a rotational partition has the following properties: (1) $\tilde{S}_i := \pi^{-1}(S_i)$ is a rectangle for every $i \in \{0, ..., M\}$, and (2) for every $i, j \in \{0, ..., M\}$ the image of the rectangle $\tilde{S}_i$ by the lift F meets at most one integer translation of the rectangle $\tilde{S}_j$.

Given a surface S, $A \subset S$ and $x \in A$, we denote by $[A]_x$ the connected component of x in A. Let us consider a rotational partition $\mathcal{S} = \{S_0, ..., S_M\}$ of

a closed and f-invariant set C, and a fixed $n \in \mathbb{N}$. Then for every $i \in \{0, ..., M\}$ we define the following families of subsets of $\mathbb{T}^2$:

$$\mathcal{I}_i = \left\{ \pi([K]_x) \;\middle|\; K = F^{-n}(\tilde{S}_i) \cap \pi^{-1}\left(\bigcup_{j=0}^{M} S_j\right), x \in \pi^{-1}(C) \cap K \right\}$$

and

$$f_*^n(\mathcal{S}) = \bigcup_{i=0}^{N} \mathcal{I}_i.$$

For a partition $\mathcal{S}$ we define the positive number $d_{\mathcal{S}} := \max\{\text{diam}(S_i) \mid S_i \in \mathcal{S}\}$. The following two are simple observations.

Proposition 4.2.1. *For every closed f-invariant set C which admits a partition $\mathcal{R}$, there exists $\varepsilon_0 > 0$ such that if $\mathcal{S}$ is a partition of C with $d_{\mathcal{S}} < \varepsilon_0$, then it is a rotational partition.*

Proof. Due to C being contained in a union of pairwise disjoint rectangles we can consider two essential loops $\gamma_1, \gamma_2 \subset C^c$, with $D = (\gamma_1 \cup \gamma_2)^c$ being a topological disk in $\mathbb{T}^2$. Then, by the invariance of C there exists a positive number $\varepsilon_0 < d(C, \partial D)$, such that for every connected set $X \subset B_{\varepsilon_0}(C)$ we have $f(X) \cap \partial D = \emptyset$.

Let $\mathcal{S}$ be a partition with $d_{\mathcal{S}} < \varepsilon_0$. Thus if $\tilde{D}$ is a connected component of $\pi^{-1}(D)$ we have for every $S \in \mathcal{S}$ that $\tilde{S} := \pi^{-1}(S) \cap \tilde{D}$ is given by a unique rectangle. Furthermore, we have that $F(\tilde{S})$ is contained in a unique integer translation of $\tilde{D}$, otherwise $F(\tilde{S})$ would intersect $\partial\pi^{-1}(D)$ and hence $f(S) \cap \partial D \neq \emptyset$ which is absurd by definition of ε_0. Therefore $\mathcal{S}$ is a rotational partition.

□

Notice that in general a partition $\mathcal{S}$ as in the proposition above may not exists.

Proposition 4.2.2. *If $f_*^n(\mathcal{S})$ is a partition of C, then it is a rotational partition of C for f^n.*

Proof. Let D be a trivial domain associated to the rotational partition $\mathcal{S}$. By definition each element of $f_*^n(\mathcal{S})$ is contained in some element of $\mathcal{S}$. Hence $f_*^n(\mathcal{S})$ verifies the condition (1) of the definition of rotational partition, with respect the domain D.

Let K be an element of $f_*^n(\mathcal{S})$. Then by definition $F^n(K) \subset \tilde{S}$, where $\tilde{S}$ is a lift of some $S \in \mathcal{S}$. Thus, there exists a unique translation of the domain D containing $F^n(K)$. Hence $f_*^n(\mathcal{S})$ is a rotational partition of C for f^n.

□

Given a partition $\mathcal{S}$ we consider $\Omega_{\mathcal{S}}$ and the semi-conjugacy h.

Proposition 4.2.3. *Let $\mathcal{S}$ be a rotational partition of C. Then, for every $x \in \pi^{-1}(C)$ and $n \in \mathbb{N}$ we have $F^n(x) \in B_{2\cdot d_{\mathcal{S}}}(x+\sum_{j=0}^{n-1} s_{i_j})$, where $i_j = h(\pi(x))(j)$ for $j = 0, ..., n-1$.*

Proof. Let $R = 2 \cdot d_{\mathcal{S}}$. For a fixed $n > 0$ and $x \in \pi^{-1}(C)$ we have that $f^j(\pi(x)) \in S_{i_j}$ where $i_j = h(\pi(x))(j)$ for every $j = 0, ..., n$. Therefore for every $j = 0, ..., M$ we have $F^j(x) \in \tilde{S}_{i_j} + v_j$, for some $v_j \in \mathbb{Z}^2$ ($\tilde{S}_i$ as in the definition of the rotational partition). Moreover, for every $j = 1, ..., M$ we have $v_j = v_{j-1} + s_{i_{j-1}}$ and $v_0 = u$ with $u \in \mathbb{Z}^2$ such that $x \in \tilde{S}_{i_0} + u$. This implies by replacement that $F^n(x) \in \tilde{S}_{i_n} + s_{i_{n-1}} + ... + s_{i_0} + u$, hence $F^n(x) \in B_R(x + \sum_{j=0}^{n-1} s_{i_j})$ where $i_j = h(\pi(x))(j)$ for $j = 0, ..., n-1$.

□

Now we introduce some preliminary notation before stating an important corollary. We say that $\alpha : \{0, 1, ..., l_\alpha - 1\} \to \{0, ..., M\}$ is a finite word if for some $\xi \in \Omega_{\mathcal{S}}$ we have $\alpha(i) = \xi(i) \forall i = 0, ..., l_\alpha - 1$. We define the set of finite words $\Omega_{\mathcal{S}}^f \subset \Omega_{\mathcal{S}}$ as For $\alpha \in \Omega_{\mathcal{S}}^f$ we denote the length of α by l_α. Further, we define a function $\psi : \Omega_{\mathcal{S}}^f \to \mathbb{R}^2$ by $\psi(\alpha) = \sum_{j=0}^{l_\alpha - 1} s_{\alpha(j)}$.

Corollary 4.2.4. *Let $\mathcal{S}$ be a rotational partition of C, then*

$$\rho_C(F) = \left\{ v \in \mathbb{R}^2 \mid \exists\ (\alpha_i)_{i\in\mathbb{N}} \subset \Omega_{\mathcal{S}}^f \text{ with } l\alpha_i \nearrow +\infty \text{ and } v = \lim_i \tfrac{\psi(\alpha_i)}{l_{\alpha_i}} \right\}$$

4.2.2 Finite Markov partitions and the rotation set.

The results presented in this section and the theorem 4.2.7 can be proved by simple adaptations of the main result in [Zie95]. Nevertheless, we present the proofs here for the sake of convenience of the reader.

Let $\mathcal{P}(X)$ denote the set of all the subsets of a given set X. We call a function $\tau : \{0, ..., M\} \to \mathcal{P}(\{0, ..., M\})$ with $\emptyset \notin Im(\tau)$ transition law. Given a transition law we define $\Omega_\tau \subset \{0, ..., M\}^{\mathbb{Z}}$ as the set of sequences ξ such that $\xi(n+1) \in \tau(\xi(n))$ for every $n \in \mathbb{Z}$. Let us consider a partition $\mathcal{S} = \{S_0, ..., S_M\}$. We say that $\mathcal{S}$ is a (finite) Markov partition if $\Omega_{\mathcal{S}} = \Omega_\tau$ for some transition law τ. In this subsection we want to prove that the convex hull of $\rho_C(F)$ is a rational polygon whenever C admits a rotational Markov partition.

If we consider the space of finite words $\Omega_{\mathcal{S}}^f$ for a Markov partition and two elements $\alpha_1, \alpha_2 \in \Omega_{\mathcal{S}}^f$ with lengths l_1, l_2 respectively such that $\alpha_2(0) \in \tau(\alpha_1(l_1))$, the function $\alpha_1\alpha_2 : \{0, ..., l_1+l_2-1\} \to \{0, ..., M\}$ given by $\alpha_1\alpha_2(i) = \alpha_1(i)$ if $i = 0, ..., l_1 - 1$, and $\alpha_1\alpha_2(i) = \alpha_2(i - (l_1))$ if $i = l_1, ..., l_1 + l_2 - 1$ is an element of $\Omega_{\mathcal{S}}^f$. We call $\alpha_1\alpha_2$ the concatenation of α_1 and α_2.

In particular, if $\nu \in \Omega_{\mathcal{S}}^f$ verifies that $\nu(0) \in \tau(\nu(l_\nu - 1))$ then one can consider for each $n \in \mathbb{N}$ the word $\nu^n \in \Omega_{\mathcal{S}}^f$ given by $\nu^n = \underbrace{\nu...\nu}_{n-times}$. We call this kind of elements periodic words and denote by $\Omega_{\mathcal{S}}^{Per} \subset \Omega_{\mathcal{S}}^f$ the set of periodic words. Then is easy to see that for any $\nu \in \Omega_{\mathcal{S}}^{Per}$ the vector $\frac{\psi(\nu)}{l_\nu}$ belongs to

$\rho_C(F)$. We also add an extra word θ to $\Omega_{\mathcal{S}}^f$ that we call the trivial word which verifies $\theta\alpha = \alpha\theta = \alpha$ for every $\alpha \in \Omega_{\mathcal{S}}^f$.

In what follows we denote by cl$[A]$ the closure of $A \subset \mathbb{R}^2$, by conv(A) the convex hull of A and let

$$\rho_C^{Per}(F) = \left\{ v \in \mathbb{R}^2 \mid \exists\ (\alpha_i)_{i\in\mathbb{N}} \subset \Omega_{\mathcal{S}}^{Per} \text{ with } v = \lim_i \frac{\psi(\alpha_i)}{l_{\alpha_i}} \right\}.$$

Lemma 4.2.5. *Suppose that $\mathcal{S}$ is a rotational Markov partition of C. Then $\rho_C(F) \subset cl[conv(\rho_C^{Per}(F))]$.*

Proof. Let $v \in \rho_C(F)$. Then by Corollary 4.2.4 we have $v = \lim_i \frac{\psi(\alpha_i)}{l_{\alpha_i}}$ with $\alpha_i \in \Omega_{\mathcal{S}}^f$ for every $i \in \mathbb{N}$ and l_{α_i} strictly increasing to $+\infty$. For a fixed $\varepsilon > 0$, we want to show the existence of an element of conv$(\rho_C^{Per}(F))$ which is ε-close to v. We consider $k_\varepsilon \in \mathbb{N}$ which verifies that for every $i > k_\varepsilon$ we have $l_{\alpha_i} > M$ and $\left\| \frac{\psi(\alpha_i)}{l_{\alpha_i}} - \frac{\psi(\alpha_i)-\psi(\beta)}{l_{\alpha_i}-l_\beta} \right\| < \frac{\varepsilon}{2}$ for every $\beta \in \Omega_{\mathcal{S}}^f$ with $l_\beta \leq M$.

Let $i > k_\varepsilon$ such that $\left\| v - \frac{\psi(\alpha_i)}{l_{\alpha_i}} \right\| < \frac{\varepsilon}{2}$. Since $l_{\alpha_i} > M$ there exist at least two different elements $j_1, j_2 \in \{0, ..., l_{\alpha_i}\}$ such that $\alpha_i(j_1) = \alpha_i(j_2)$. This implies that $\alpha_i = \gamma_1\nu_0\gamma_2$ for some $\gamma_1, \gamma_2 \in \Omega_{\mathcal{S}}^f$ with at least one different of θ and $\nu_0 \in \Omega_{\mathcal{S}}^{Per}$. Moreover since ν_0 is a periodic word $\alpha_0 = \gamma_1\gamma_2 \in \Omega_{\mathcal{S}}^f$ is a well defined non trivial word and $\psi(\alpha_i) = \psi(\alpha_0) + \psi(\nu_0)$ with $l_{\nu_0} + l_{\alpha_0} = l_{\alpha_i}$. Now if $l_{\alpha_0} > M$ we can do the same as we did with α_i and produce $\nu_1 \in \Omega_{\mathcal{S}}^{Per}$, $\alpha_1 \in \Omega_{\mathcal{S}}^f$ such that $\psi(\alpha_i) = \psi(\alpha_1) + \psi(\nu_1) + \psi(\nu_0)$ with $l_{\nu_0} + l_{\nu_1} + l_{\alpha_1} = l_{\alpha_i}$.

If we apply this argument recursively we obtain $\nu_0, ..., \nu_r \in \Omega_{\mathcal{S}}^{Per}$ and $\alpha_f \in \Omega_{\mathcal{S}}^f$ with $l_{\alpha_f} \leq M$ such that $\psi(\alpha_i) = \sum_{k=0}^r \psi(\nu_k) + \psi(\alpha_f)$ and $\sum_{k=0}^r l_{\nu_k} + l_{\alpha_f} = l_{\alpha_i}$. Hence by definition of k_ε we have that $\left\| \frac{\psi(\alpha_i)}{l_{\alpha_i}} - \frac{\sum_{k=0}^r \psi(\nu_k)}{\sum_{k=0}^r l_{\nu_k}} \right\| < \frac{\varepsilon}{2}$ which implies that $\left\| v - \frac{\sum_{k=0}^r \psi(\nu_k)}{\sum_{k=0}^r l_{\nu_k}} \right\| < \varepsilon$. Since $w = \frac{\sum_{k=0}^r \psi(\nu_k)}{\sum_{k=0}^r l_{\nu_k}}$ belongs to conv$(\rho_C^{Per}(F))$, this finishes the proof.

□

As we have said the goal of this subsection is prove that conv$(\rho_C(F))$ is a rational polygon whenever C admits a rotational Markov partition $\mathcal{S}$. Due to Lemma 4.2.5 it is sufficient to prove that conv$(\rho_C^{Per}(F))$ is a rational polygon. The next observation is rather simple, so we do not give the proof.

Proposition 4.2.6. *Suppose that there exists a constant $K > 0$ such that for every $w \in \mathbb{R}^2$ the function $< w, . >: \rho_C^{Per}(F) \to \mathbb{R}$ is maximized by $\frac{\psi(\nu)}{l_\nu}$ with $\nu \in \Omega_{\mathcal{S}}^{Per}, l_\nu \leq K$. Then, conv$(\rho_C^{Per}(F))$ is a rational polygon with vertices in the finite set $\left\{ \frac{\psi(\nu)}{l_\nu} \mid \nu \in \Omega_{\mathcal{S}}^{Per}, l_\nu \leq K \right\}$.*

Theorem 4.2.7. *Suppose that $\mathcal{S} = \{S_0, ..., S_M\}$ is a rotational Markov partition for C. Then conv$(\rho_C^{Per}(F))$ is a rational polygon with vertices in*
$\left\{ \frac{\psi(\nu)}{l_\nu} \mid \nu \in \Omega_{\mathcal{S}}^{Per}, l_\nu \leq M \right\}$.

Proof. By definition $\rho_C^{Per}(F)$ is contained in the compact set $\rho_C(F)$. Hence, for a fixed $w \in \mathbb{R}^2$ there exists

$$L = \sup\{\langle w, v\rangle \mid v \in \rho_C^{Per}(F)\} = \sup\left\{\frac{\langle w, \psi(\nu)\rangle}{l_\nu} \mid \nu \in \Omega_{\mathcal{S}}^{Per}\right\}.$$

Our goal is to prove that $L = \sup\left\{\frac{\langle w,\psi(\nu)\rangle}{l_\nu} \mid \nu \in \Omega_{\mathcal{S}}^{Per}, l_\nu \leq M\right\}$ which due to the last proposition, implies the desired result. Hence for every $\varepsilon > 0$ we want to find an element $\nu_\varepsilon \in \Omega_{\mathcal{S}}^{Per}$ with $l_{\nu_\varepsilon} \leq M$ such that $L - \frac{\langle w,\psi(\nu_\varepsilon)\rangle}{l_{\nu_\varepsilon}} < \varepsilon$.

For a fixed $\varepsilon > 0$ we consider $\nu_0 \in \Omega_{\mathcal{S}}^{Per}$ such that $L - \left\langle w, \frac{\psi(\nu_0)}{l_{\nu_0}}\right\rangle < \varepsilon$. If $l_{\nu_0} \leq M$ we are done. If $l_{\nu_0} > M$ there exists $\nu_0' \in \Omega_{\mathcal{S}}^{Per}$ and $\alpha_1, \alpha_2 \in \Omega_{\mathcal{S}}^{f}$ with at least one of the two different from θ such that $\nu_0 = \alpha_1\nu_0'\alpha_2$, $\alpha_1\alpha_2 \in \Omega_{\mathcal{S}}^{Per}$ and $l_{\alpha_1} + l_{\alpha_2} + l_{\nu_0'} = l_{\nu_0}$. Therefore $\frac{\langle w,\psi(\nu_0)\rangle}{l_{\nu_0}} = \frac{\langle w,\psi(\alpha_1\alpha_2)\rangle + \langle w,\psi(\nu_0')\rangle}{l_{\alpha_1}+l_{\alpha_2}+l_{\nu_0'}}$ which implies that either $\frac{\langle w,\psi(\nu_0)\rangle}{l_{\nu_0}} \leq \frac{\langle w,\psi(\nu_0')\rangle}{l_{\nu_0'}}$ or $\frac{\langle w,\psi(\nu_0)\rangle}{l_{\nu_0}} \leq \frac{\langle w,\psi(\alpha_1\alpha_2)\rangle}{l_{\alpha_1\alpha_2}}$. Thus there exist $\nu_1 \in \Omega_{\mathcal{S}}^{Per}$ such that $\frac{\langle w,\psi(\nu_0)\rangle}{l_{\nu_0}} \leq \frac{\langle w,\psi(\nu_1)\rangle}{l_{\nu_1}}$ and $l_{\nu_1} < l_{\nu_0}$.

Applying the above argument recursively, we eventually find a periodic word $\nu_\varepsilon \in \Omega_{\mathcal{S}}^{Per}$ such that $l_{\nu_\varepsilon} \leq M$ and $L - \frac{\langle w,\psi(\nu_\varepsilon)\rangle}{l_{\nu_\varepsilon}} < \varepsilon$.

□

Corollary 4.2.8. *Suppose that $\mathcal{S}$ is a rotational Markov partition of C. Then $conv(\rho_C(F))$ is a rational polygon with vertices in $\left\{\frac{\psi(\nu)}{l_\nu} \mid \nu \in \Omega_{\mathcal{S}}^{Per}, l_\nu \leq M\right\}$.*

4.3 Construction of the set $\mathcal{D}$.

Let M be a compact manifold and $f : M \to M$ be a diffeomorphism. We say that a compact set $\Lambda \subset M$ is a hyperbolic set if there exist $C > 0$, $\lambda \in (0,1)$ and a decomposition of the tangent bundle over Λ given by $T_xM = E_x^s \oplus E_x^u$ (stable and unstable vector bundles) such that:

(1) $D_xf(E_x^s) = E_{f(x)}^s, D_xf^{-1}(E_x^u) = E_{f^{-1}(x)}^u$ for every $x \in \Lambda$.

(2) $\|D_xf^n|_{E_x^s}\| < C \cdot \lambda^n$ and $\|D_xf^{-n}|_{E_x^u}\| < C \cdot \lambda^n$ for every $x \in \Lambda, n \in \mathbb{N}$.

Further, we say that Λ is a basic piece of f if it is a compact invariant hyperbolic set, transitive ($\Lambda = cl[\{f^n(x) \mid n \in \mathbb{Z}\}]$ for some $x \in \Lambda$) and locally maximal (there exist a neighborhood U of Λ such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$).

Let $f : M \to M$ be a homeomorphism and $x \in M$, Then:

- for $\varepsilon > 0$ the local stable set of size ε is defined as

$$W_\varepsilon^s(x, f) = \{y \in M \mid d(f^n(y), f^n(x)) < \varepsilon \text{ for every } n \in \mathbb{N}\},$$

and the local unstable set of size ε as

$$W^u_\varepsilon(x,f) = \{y \in M \mid d(f^{-n}(y), f^{-n}(x)) < \varepsilon \text{ for every } n \in \mathbb{N}\};$$

- the stable set is defined as

$$W^s(x,f) = \{y \in M \mid \lim_{n\to+\infty} d(f^n(y), f^n(x)) = 0\},$$

and the unstable set as

$$W^u(x,f) = \{y \in M \mid \lim_{n\to+\infty} d(f^{-n}(y), f^{-n}(x)) = 0\}.$$

For a hyperbolic set Λ of some diffeomorphism f we have the so called Stable Manifold Theorem.

Theorem 4.3.1 (Stable Manifold Theorem, [KH97], Section 6)**.** *Let M be a compact manifold and $f : M \to M$ a C^r diffeomorphism with hyperbolic set Λ. Then there exists $\varepsilon > 0$ such that for every point $x \in \Lambda$ we have:*

(1) $W^s_\varepsilon(x,f)$ is a C^r embedded manifold such that $T_x W^s_\varepsilon(x,f) = E^s_x$ and $W^s_\varepsilon(x,f)$ varies continuously with x (as C^r sub-manifold);

(2) $W^s_\varepsilon(x,f) \subset W^s(x,f)$;

(3) $W^s(x,f) = \bigcup_{n\in\mathbb{N}} f^{-n}(W^s_\varepsilon(x,f))$ is a C^r immersed manifold.

The analogous result holds for the unstable set.

A first remark concerning this theorem is that for a basic piece the stable and unstable bundles have constant dimension, and they vary continuously (this can be proved directly from the definition of basic piece). This implies that we can define the index of a basic piece as the dimension of its unstable vector bundle. When a basic piece has index 1 we say it is a saddle piece. Further properties for basic sets will be presented in the next section. Next we want to define axiom A diffeomorphisms. We denote by $\mathrm{Per}(f)$ the set of periodic points of f and recall that the non-wandering set of f is defined as:

$$\Omega(f) := \{x \in M \mid \forall\ U \text{ neighborhood of } x\ f^n(U) \cap U \neq \phi \text{ for some } n \in \mathbb{N}\}$$

Let M be a compact manifold. We say that a diffeomorphism $f : M \to M$ is axiom A if

- $\Omega(f) = \mathrm{cl}[Per(f)]$;
- $\Omega(f)$ is hyperbolic.

The dynamical properties of these systems are well-understood. Further, under additional hypothesis these systems are *robust* in different senses and play a crucial role in the theory of generic dynamics (see references below). We will make use of these facts for constructing the set $\mathcal{D}$.

Theorem 4.3.2 (Spectral decomposition theorem, [New72]). *Let $f : M \to M$ be an axiom A diffeomorphism. Then, there exist a finite number of pairwise disjoint basic pieces $\Lambda_1, ..., \Lambda_n \subset M$ for f such that $\Omega(f) = \Lambda_1 \cup ... \cup \Lambda_n$. Further, for each $i = 1, ..., n$ there exists a finite number of compact and pairwise disjoint sets $\Lambda_{i,0}, ..., \Lambda_{i,k_i-1}$ such that:*

- *for every $j = 0, ..., k_i - 1$ we have $f(\Lambda_{i,j}) = \Lambda_{i,j+1 \ mod(k_i)}$;*
- *for every $j = 0, ..., k_i - 1$ and $x \in \Lambda_{i,j}$ the sets $W^u(x, f), W^s(x, f)$ are dense in $\Lambda_{i,j}$;*
- *for every $j = 0, ..., k_i - 1$ we have that $f^{k_i} \mid_{\Lambda_{i,j}}$ is topologically mixing.*

We say that an axiom A diffeomorphism satisfies the *transversality condition* if for any two points $x, y \in \Omega(f)$, we have that $W^u(x, f)$ intersects $W^s(y, f)$ transversally.

The following theorem shows the semi-stability of axiom A diffeomorphisms with the transversality condition (for any compact manifold) and is due to Z. Nitecki.

Theorem 4.3.3 ([Nit71]). *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be an axiom A diffeomorphism with the transversality condition. Then, there exists a neighborhood $\mathcal{U}(f) \subset \mathrm{Homeo}_0(\mathbb{T}^2)$ of f such that any g of $\mathcal{U}(f)$ is semi-conjugate to f by a semiconjugacy h homotopic to the identity.*

We mention here that the fact of the map h to be homotopic to the identity is not explicitly stated in Z. Nietecki's article, but is a well known fact that this kind of maps can be constructed by the *Shadowing Lemma* such that they are close to the identity, and therefore homotopic to the identity. The above theorem, together with the properties of the rotation set given in Subsection 1.2, implies the following corollary:

Corollary 4.3.4. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ be an axiom A diffeomorphism that satisfies the transversality condition. Then there exists an open neighborhood $\mathcal{U}(f) \subset \mathrm{Homeo}_0(\mathbb{T}^2)$ of f such that for every $g \in \mathcal{U}(f)$ we have $\rho(G) = \rho(F)$ for some lifts G, F of g, f respectively.*

We say that a diffeomorphism $f : M \to M$ is a fitted axiom A diffeomorphism if it is axiom A, has the transversality condition and $\Omega(f)$ is totally disconnected. For a given manifold M we denote by $\mathcal{F}$ the family of fitted axiom A diffeomorphisms.

Theorem 4.3.5 ([SS75]). *Let M be a compact manifold. Then $\mathcal{F}$ is dense in $Homeo(M)$ with the C^0 topology.*

A nice exposition of this result in surfaces can be found in [Fra80]. We now turn to the existence of Markov partitions for totally disconnected saddle basic pieces of $f \in \text{Homeo}_0(\mathbb{T}^2)$, with some extra properties needed for further applications in the following sections. The main result that we are going to provide can be found for instance in [Bég99], where Markov partitions are defined in a stronger way, verifying at once all the properties that we request. We should say that the general idea of associating Markov partitions to basic pieces in any dimension, is due to R. Bowen [Bow70b]. Next we introduce some notations and remarks which will be used not in this section, but in the last section of this article.

We say that a hyperbolic set Λ has local product structure if there exists $\delta_0 > 0$ such that given $\varepsilon_0 > 0$ as in the (Un)Stable Manifold Theorem and any two points $x, y \in \Lambda$ with $d(x, y) < \delta_0$ we have that $W^u_{\varepsilon_0}(x, f) \cap W^s_{\varepsilon_0}(y, f) \subset \Lambda$. It turns out that for a hyperbolic set Λ, the existence of a local product structure is equivalent to be locally maximal. Hence every basic piece has local product structure.

Let $\mathcal{S}$ be a partition of a totally disconnected saddle basic piece Λ. We say that $S \in \mathcal{S}$ is an *us*-box if

- $\text{diam}(S) < \delta_0$, where δ_0 is given by the local product structure;
- there exists a homeomorphism $h : [0, 1]^2 \to S$ such that for every $x \in S \cap \Lambda$ we have that

$$W^u_{\varepsilon_0}(x, f) \cap S = h([0, 1] \times \{b_x\}),\ W^s_{\varepsilon_0}(x, f) \cap S = h(\{a_x\} \times [0, 1])$$

  where $(a_x, b_x) = h^{-1}(x)$, and

$$h(\{(0, 0), (0, 1), (1, 0), (1, 1)\}) \subset \Lambda.$$

We say that $\mathcal{S}$ is a *us*-partition if every element in $\mathcal{S}$ is a *us*-box. When $\mathcal{S}$ is a also a Markov partition we say that $\mathcal{S}$ is a *us*-Markov partition. Let S be a *us*-box of a partition of Λ. We say that $H \subset S$ is a horizontal rectangle if $H = h([0, 1] \times [c, d])$. Analogously we say that $V \subset S$ is a vertical rectangle if $V = h([a, b] \times [0, 1])$.

Given an *us*-partition $\mathcal{S} = \{S_0, ..., S_N\}$ of Λ, we say that it has the intersection property if for every $i, j \in \{0, ..., N\}$, $x \in \Lambda \cap S_i$, $y \in \Lambda \cap S_j$ and $n \in \mathbb{N}$ with $f^n(x) = y$ we have

- $V = [f^{-n}(S_j) \cap S_i]_x$ is a vertical rectangle;
- $H = [f^n(S_i) \cap S_j]_y$ is a horizontal rectangle.

The following is an easy observation.

Proposition 4.3.6. *Let $\mathcal{S}$ be an us-partition with the intersection property and $i, j \in \{0, ..., N\}$. Further assume that there exists $x \in S_i$ and $y \in S_j$ such that $f^n(x) = y$ for some $n \in \mathbb{N}$. Then the vertical rectangle $V = [f^{-n}(S_j) \cap S_i]_x$ and the horizontal rectangle $H = [f^n(S_i) \cap S_j]_y$ verify $f^n(V) = H$.*

We are now ready to state the theorem about Markov partitions associated to totally disconnected saddle basic pieces with the desired properties.

Theorem 4.3.7 ([Bég99])**.** *Let Λ be a totally disconnected saddle basic piece of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$. Then for every $\varepsilon > 0$ there exists a partition $\mathcal{S}$ of Λ such that:*

(1) $\mathcal{S}$ is a us-Markov partition for f with the intersection property, and $d_{\mathcal{S}} < \varepsilon$;

(2) for every $n \in \mathbb{N}$, $f^n_*(\mathcal{S})$ is a Markov partition of Λ for f^n.

The point (2) in the last Theorem is not explicit in the reference as a property for the partitions, but it is a direct consequence of their construction (and in general a well-known fact). Now by making use of the propositions 4.2.1 and 4.2.2 we have the following crucial consequence:

Corollary 4.3.8. *Let Λ be a totally disconnected saddle basic piece of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$. Then there exists a partition $\mathcal{S}$ such that:*

(1) $\mathcal{S}$ is a rotational us-Markov partition for f with the intersection property, with $d_{\mathcal{S}}$ arbitrarily small;

(2) for every $n \in \mathbb{N}$, $f^n_*(\mathcal{S})$ is a rotational Markov partition of Λ for f^n.

The following is an useful observation for computing the rotation set of an axiom A diffeomorphism.

Proposition 4.3.9. *Let $\Lambda_1, ..., \Lambda_n$ be basic pieces of f such that $conv(\rho_{\Lambda_i}(F))$ are rational polygons for $i = 1, ..., n$. Then, if $\Lambda = \Lambda_1 \cup ... \cup \Lambda_n$ we have that $conv(\rho_\Lambda(F))$ is a rational polygon. Furthermore, if $\Lambda = \Omega(f)$ then $\rho(F)$ is a rational polygon.*

This observation is easily proved by making use of the properties of the rotation set given in Subsection 1.2. Now we are able to construct the set $\mathcal{D}$ in the following theorem.

Theorem 4.3.10. *There exists an open and dense set $\mathcal{D} \subset \mathrm{Homeo}_0(\mathbb{T}^2)$ such that for any $f \in \mathcal{D}$, the rotation set $\rho(F)$ is a rational polygon.*

Proof. Let us consider

$$\mathcal{F}_0 = \{f \in \mathrm{Homeo}_0(\mathbb{T}^2) \mid f \text{ is a fitted axiom A diffeomorphism}\}$$

.

Given an element $f \in \mathcal{F}$, we have that any basic piece Λ of f is either a periodic orbit of index 0 or 2 or a totally disconnected saddle basic piece. If Λ is a totally disconnected saddle basic piece then Corollary 4.3.8 implies that we can associate a rotational Markov partition to Λ. Therefore, Corollary 4.2.8 implies that $\text{conv}(\rho_\Lambda(F))$ is a rational polygon for each basic piece Λ.

Hence by the last proposition, $\rho(F)$ is a rational polygon for every $f \in \mathcal{F}_0$.

Now, if for every $f \in \mathcal{F}_0$ we consider the neighborhood $\mathcal{U}(f)$ given by the corollary 4.3.4, then we have that $\mathcal{D} = \bigcup_{f\in\mathcal{F}_0} \mathcal{U}(f)$ is an open and dense set in $\text{Homeo}_0(\mathbb{T}^2)$, and for every $g \in \mathcal{D}$ the set $\rho(G)$ is a rational polygon for any lift G of g.

□

4.4 Extension to any axiom A diffeomorphism.

The aim of this section is prove that the rotation set of any axiom A diffeomorphism in $\text{Homeo}_0(\mathbb{T}^2)$ is a rational polygon. From the last section, we know that this is true for any fitted axiom A diffeomorphism in $\text{Homeo}_0(\mathbb{T}^2)$. Thus, one first may ask about the existence or not of axiom A diffeomorphisms which are not fitted in $\text{Homeo}_0(\mathbb{T}^2)$. However, is not difficult to construct, by using the Plykin attractor, an axiom A diffeomorphism $f \in \text{Homeo}_0(\mathbb{T}^2)$ with a basic piece Λ that is non-totally disconnected (see [KH97], Section 17). Further, one can manage the construction in a way that Λ is not contained in a fordward invariant topological disk, so we don't have $\rho_\Lambda(F) = \{v\}$ with v rational.

In what follows we introduce some preliminary notions and results, which are folklore of hyperbolic dynamics. As reference for these results see for instance [KH97], Section 18. We have that every hyperbolic sets is expansive, that is: given a hyperbolic set Λ there exists a positive constant γ (constant of expansiveness) such that for any two points $x, y \in \Lambda$ there exists $n \in \mathbb{Z}$ for which $d(f^n(x), f^n(y)) > \gamma$.

Let M be a Riemannian manifold, $f : M \to M$ a diffeomorphism and $\{x_n\}_{n\in\mathbb{Z}} \subset M$. We say that $\{x_n\}_{n\in\mathbb{Z}}$ is a δ-pseudo orbit if $d(f(x_n), x_{n+1}) < \delta$ for every $n \in \mathbb{Z}$. It is said to be ε-shadowed by the orbit of $y \in M$ if $d(f^n(y), x_n) < \varepsilon$ for every $n \in \mathbb{Z}$.

Theorem 4.4.1 (Shadowing Lemma)**.** *Let M be a Riemannian manifold, $f : M \to M$ a diffeomorphism, $\Lambda \subset M$ a basic piece of f and $\gamma > 0$ a constant of expansiveness in Λ. Then, there exists a neighborhood $U(\Lambda)$ of Λ such that for every positive number $\varepsilon < \frac{\gamma}{2}$, there exists $\delta > 0$ for which every δ-pseudo orbit is ε-shadowed by a unique orbit in Λ.*

We say that a basic piece Λ is an attractor if there exists an open set U such that $f(cl[U]) \subset U$ and $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(U)$, and it is a repeller if it is an attractor for f^{-1}. We say that it is a non-trivial attractor (repeller) when it is not a periodic orbit.

Proposition 4.4.2. *If Λ is a non-trivial attractor for a diffeomorphism f, then Λ has index one and $W^u(x,f) \subset \Lambda$ for every $x \in \Lambda$. The analogous result holds for repellers.*

Proposition 4.4.3. *Let Λ be a non-trivial attractor (repeller) for $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$. Then $\Lambda \neq \mathbb{T}^2$.*

In fact, if $\Lambda = \mathbb{T}^2$ it is known that f is conjugate to an Anosov diffeomorphism, which is not possible since f is homotopic to the identity.

Proposition 4.4.4. *Let $\Lambda \neq \mathbb{T}^2$ be a non-trivial attractor with a fixed point P_0 of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$. Then, there exists $r' > 0$ such that for every positive number $r < r'$ we have $[B_r(P_0) \cap \Lambda]_{P_0} = W^u_r(P_0, f)$. An analogous result holds for repellers.*

Otherwise, due to the local product structure one can observe that a stable arc is contained in Λ. This, due to Proposition 4.4.2 and the local product structure, would imply that Λ is an open set. Hence Λ is closed and open so $\Lambda = \mathbb{T}^2$ which contradicts the last proposition. In general, this result holds for every point in Λ, even if they are not periodic. However, we just need the result with the given generality.

Proposition 4.4.5. *Let Λ be a non-totally disconnected basic piece of a diffeomorphism f in $\mathrm{Homeo}_0(\mathbb{T}^2)$. Then, we have that either it is a non-trivial attractor or a non-trivial repeller. Furthermore, if Λ contains a fixed point then it is a connected set.*

This is a well-known statement in hyperbolic dynamics and can be proved by similar arguments as the remark above.

The following Lemma refers to the topological stability of hyperbolic fixed points of saddle type. Given two continuous maps f, g defined in either $\mathbb{R}^2$ or $\mathbb{T}^2$, we denote the C^0 distance between them by $d_0(f,g)$.

Lemma 4.4.6. *Let P_0 be a hyperbolic fixed point of saddle type for f. Then there exists $\delta_0 > 0$ such that given two positives numbers $r \in (0, \delta_0)$ and $\eta > 0$ there exists $\xi > 0$ with the following property: if $d_0(\tilde{f}, f) < \xi$ then there is a compact and connected set K that verifies*

- *$d_{\mathcal{H}}(K, W^u_r(P_0, f)) < \eta$, and for some $n \in \mathbb{N}$ we have that $K = \tilde{f}^n(K_0)$ where $K_0 \subset \bigcap_{n\in\mathbb{N}} \tilde{f}^n(B_\eta(P_0))$;*
- *there exists a $\tilde{f}$-invariant set $K' \subset B_\eta(P_0)$ such that $K' \cap K \neq \emptyset$.*

(see figure 4.4.1).

Proof. Let $\phi : \mathbb{R}^2 \to U$ be the Hartman-Grobman coordinates with $\phi([-1,1] \times \{0\}) = W^u_{\delta_0}(P_0, f)$ and $\phi(\{0\} \times [-1,1]) = W^s_{\delta_0}(P_0, f)$. For positive numbers $r < \delta_0$ and η let us fix a box $B = [a,b] \times [c,d]$, such that:

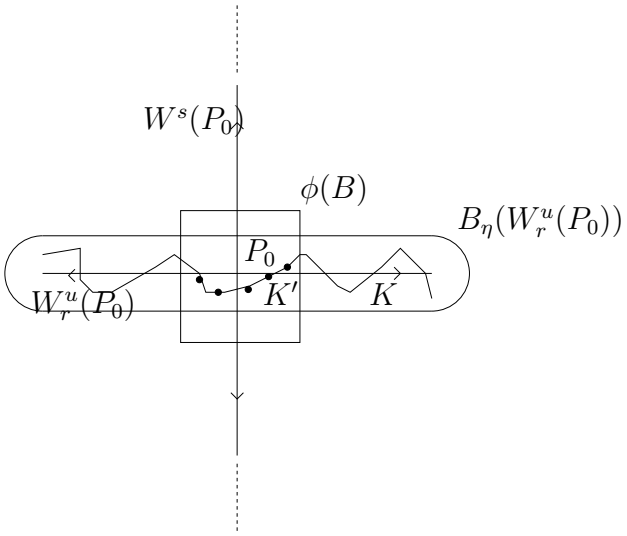


Figure 4.4.1: Lemma 4.4.6. The set $\phi(B)$ is constructed in the proof.

(i) $0 \in \text{int}(B)$, $\phi(B) \subset B_\eta(P_0) \cap U$;

(ii) $f(\phi(B)) \subset U$;

(iii) there exists $n \in \mathbb{N}$ such that for any compact and connected set $L \subset B$ which intersect the sets $A_1 = \{a\} \times [c, d]$ and $A_3 = \{b\} \times [c, d]$, we have $d_{\mathcal{H}}(\phi((Df)^n_{P_0}(L)), W^u_r(P_0, f)) < \eta$.

Is not difficult to see that this box can be constructed. In what follows we say that a compact and connected set $L \subset B$ joins A_1 and A_3 if it intersects A_1 and A_3. Further, we use the analogous definitions for the sets $A_2 = [a, b] \times \{c\}$ and $A_4 = [a, b] \times \{d\}$.

Now, we consider $\xi_1 > 0$ such that if $d_0(\tilde{f}, f) < \xi_1$ then $\tilde{g} = \phi^{-1} \circ \tilde{f} \circ \phi$ verifies:

(a) for every compact and connected set $C \subset B$ which joins A_1 and A_3, the set $\tilde{g}(C) \cap B$ contains at least one connected component which joins A_1 and A_3;

(b) for every compact and connected set $C \subset B$ which joins A_2 and A_4, the set $\tilde{g}^{-1}(C) \cap B$ contains at least one connected component which joins A_2 and A_4;

Therefore property (a) allows us to do the following inductive definition:

$K_0 = B$,

$\mathcal{F}_0 = \{C \mid\ C$ connected component of $\tilde{g}(K_0) \cap B$ which joins A_1 and $A_3\}$;

$K_1 = \bigcup_{C \in \mathcal{F}_0} C$,

$\mathcal{F}_1 = \{C \mid\ C$ connected component of $\tilde{g}(K_1) \cap B$ which joins A_1 and $A_3\}$;

and in general,

$K_n = \bigcup_{C \in \mathcal{F}_{n-1}} C$,

$\mathcal{F}_n = \{C \mid\ C$ connected component of $\tilde{g}(K_n) \cap B$ which joins A_1 and $A_3\}$.

We remark here that due to consider C^0-perturbations of f, we can not ensure that K_n has a unique element for every $n \in \mathbb{N}$. Now, we define $K_I = \bigcap_{n\in\mathbb{N}} K_n$ which is a non-empty compact set, since it is a decreasing intersection of non-empty compact set (notice that the families $\mathcal{F}_n$ are finite). This implies that $B^+ = \bigcap_{n\in\mathbb{N}} \tilde{g}^n(B)$ is not empty since $K_I \subset B^+$. Moreover if we pick a point $x \in K_I \cap B^+$, we have that $[K_I]_x$ joins A_1 and A_3.

In an analogous way by using the property (b), we can construct a non-empty compact set K_{II} with $K_{II} \subset B^- := \bigcap_{n\in\mathbb{N}} \tilde{g}^{-n}(B)$. Furthermore if we pick now $y \in K_{II}$, we have that $[K_{II}]_y$ joins A_2 and A_4. This implies that the set $\tilde{K}' := B^+ \cap B^- =: \tilde{K}'$ is a non-empty compact and $\tilde{g}$-invariant set. Let $z \in \tilde{K}'$ and $\tilde{K}_0 := [K_I]_z \subset B^+$.

Due to the property (iii) in the definition of B, we can suppose the existence of $\xi < \xi_1$ such that if $d_0(\tilde{f}, f) < \xi$ then for $K := \tilde{f}^n(\phi(\tilde{K}_0))$ we have $d_{\mathcal{H}}(K, W^u_r(P_0, f)) < \eta$. Furthermore, if we set $\phi(\tilde{K}_0) =: K_0$ we have by definition that $K = \tilde{f}^n(K_0)$, and by construction that $K_0 \subset \bigcap_{n\in\mathbb{N}} \tilde{f}^n(B_\eta(P_0))$. Finally, if we take $K' := \phi(\tilde{K}')$ we have that $K' \subset B_\eta(P_0)$ and $K \cap K' \neq \emptyset$ since $\tilde{f}^n(\phi(z)) \in K \cap K'$.

□

The following theorem is the key result for our purpose. It relates, in a dynamical way, totally disconnected basic pieces with connected ones.

Theorem 4.4.7. *Let Λ be a non-trivial attractor of a diffeomorphism f in* $\mathrm{Homeo}_0(\mathbb{T}^2)$*, P_0 a fixed point in Λ, $U(\Lambda)$ as in Theorem 4.4.1 and $U \subset U(\Lambda)$ an open and connected set such that $f(cl[U]) \subset U$ and $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$. Then, there exists $\varepsilon_0 > 0$ such that if $d_0(\tilde{f}, f) < \varepsilon_0$ we have:*

(1) $\tilde{f}(cl[U]) \subset U$ and $\tilde{\Lambda} = \bigcap_{n\in\mathbb{N}} \tilde{f}^n(U)$ is a non-empty compact and connected set;

(2) there exists a continuous and onto map $h : \tilde{\Lambda} \to \Lambda$ such that $d_0(h, Id) < \frac{1}{4}$ and $h \circ \tilde{f} = f \circ h$;

(3) h lifts to a continuous and onto map $H : \pi^{-1}(\tilde{\Lambda}) \to \pi^{-1}(\Lambda)$ such that for any two lifts $\tilde{F}, F$ of $\tilde{f}, f$ we have $H \circ \tilde{F} = F \circ H + v$ for some $v \in \mathbb{Z}^2$, and $d_0(H, id) < \frac{1}{4}$.

Proof. It is easy to see that there exists $\varepsilon' > 0$ such that for all $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ with $d_0(\tilde{f}, f) < \varepsilon'$ statement (1) holds. In the following we want to achieve point (2) in the theorem in two steps. We first find $\varepsilon_1 > 0$ for which the existence and continuity of the map h is guaranteed for any $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ with $d_0(\tilde{f}, f) < \varepsilon_1$ (i). The second step is find a positive number $\varepsilon_2 \leq \varepsilon_1$, such that if $d_0(\tilde{f}, f) < \varepsilon_2$ then the map h is onto (ii).

i) Let $\alpha > 0$ be such that $\alpha < \min\{\frac{\gamma}{2}, \frac{1}{4}\}$, where γ is the constant of expansiveness of f in Λ, and consider $\delta_\alpha > 0$ associated to α by Theorem 4.4.1. Consider now a positive number $\varepsilon_1 < \min\{\delta_\alpha, \varepsilon'\}$. Then for every $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ with $d_0(\tilde{f}, f) < \varepsilon_1$ we have that $\tilde{\Lambda}$ is a non-empty invariant set of $\tilde{f}$ contained in U and every orbit of $\tilde{f}$ in $\tilde{\Lambda}$ is a δ_α-pseudo orbit of f. Hence, by Theorem 4.4.1 to any point $x \in \tilde{\Lambda}$ we can associate a unique point $y_x \in B_\alpha(x) \cap \Lambda$ such that its orbit by f α-shadows the orbit of x by $\tilde{f}$. Thus, we define $h : \tilde{\Lambda} \to \Lambda$ by $h(x) = y_x$.

   Now we want to see that h is a continuous function. Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\tilde{\Lambda}$ converging to x_0, and $\{y_n = h(x_n)\}_{n\in\mathbb{N}}$. Suppose for a contradiction that $\{y_n\}_{n\in\mathbb{N}}$ does not converge to $h(x_0) = y_0$. Then we can consider a sequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converging to x_0 such that $\{y_{n_k}\}_{k\in\mathbb{N}}$ converges to $z_0 \neq y_0$. It is not difficult to see that this implies $d(f^n(z_0), \tilde{f}^n(x_0)) \leq \frac{\gamma}{2}$ for very $n \in \mathbb{Z}$. On the other hand by definition of h $d(f^n(y_0), \tilde{f}^n(x_0)) \leq \frac{\gamma}{2}$ for very $n \in \mathbb{Z}$. Therefore $d(f^n(z_0), f^n(y_0)) \leq \gamma$ for very $n \in \mathbb{Z}$ which is absurd since γ is the constant of expansiveness of f in Λ. So we have for every $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ with $d_0(\tilde{f}, f) < \varepsilon_1$ a continuous map $h : \tilde{\Lambda} \to \Lambda$ such that $d_0(h, id) < \frac{1}{4}$ and $h \circ \tilde{f} = f \circ h$. Moreover, we have that $d_0(h, id) \leq d_0(\tilde{f}, f)$.

ii) Now we want to find a positive number $\varepsilon_2 \leq \varepsilon_1$, such that for every $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ with $d_0(\tilde{f}, f) < \varepsilon_2$ the map h is onto. Consider $\delta_0 > 0$ given by Lemma 4.4.6 for P_0. Define a positive number $r_0 < \min\{\delta_0, \frac{\gamma}{2}, r'\}$, where r' is given by Remark 4.4.4, such that $B_{r_0}(P_0) \subset U$. Let $W = W^u_{r_0}(P_0, f)$, $r_1 = \frac{r_0}{2}$, and $W' = W^u_{r_1}(P_0, f)$. Given a parametrization with some orientation for W we can make use of standard notation for defining arcs $[c, d]$ in W. In particular W itself is an arc $[a, b]$ such that $P_0 \in (a, b)$ and W' is an arc $[a', b']$ with $a < a' < P_0 < b' < b$ (see figure 4.4.2).

   Now, let us consider $\eta > 0$ such that

$$\min\{d(a, a'), d(b', b), d(a', [P_0, b]), d([a, P_0], b'), d(W', \partial B_{r_0}(P_0))\} > 2\eta$$

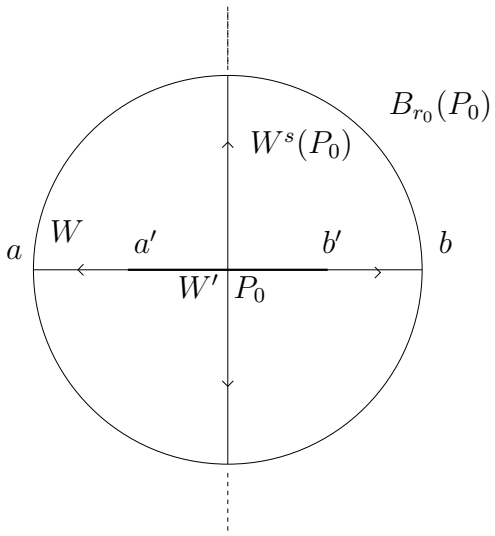


Figure 4.4.2:

and take $\xi' > 0$ associated to r_1 and η by Lemma 4.4.6. Thus, if we consider $\xi = \min\{\xi', \varepsilon_1\}$ by Lemma 4.4.6 we have for any $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ with $d_0(\tilde{f}, f) < \xi$, the existence of a non-empty compact and connected set K such that:

- K is contained in $\tilde{\Lambda}$, since $K = \tilde{f}^n(K_0)$ with $K_0 \subset \bigcap_{n\in\mathbb{N}} \tilde{f}^n(B_\eta(P_0))$;
- K intersects a compact and $\tilde{f}$-invariant set K' with $K' \subset B_\eta(P_0)$.

(see figure 4.4.3).

Due to the fact that $r_0 < \frac{\gamma}{2}$ we have necessarily that $h(z) = P_0$ for every $z \in K'$, thus $h(K)$ contains P_0. Let $\varepsilon_2 = \min\{\xi, \eta\}$. Then for every $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ with $d_0(\tilde{f}, f) < \varepsilon_2$ we have that $h(K) \subset B_{2\eta}(W') \subset B_{r_0}(P_0)$ is a compact and connected set containing P_0. Thus, by Remark 4.4.4 $h(K) \subset W$. Furthermore if we choose $x' \in K \cap B_\eta(a')$ and $y' \in K \cap B_\eta(b')$, then for any $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ with $d_0(\tilde{f}, f) < \varepsilon_2$, we have that $d(h(x'), a') < 2\eta$ and $d(h(y'), b') < 2\eta$. Hence by definition of η we have that $c := h(x') \in [a, P_0)$ and $d := h(y') \in (P_0, b]$. By connectedness of K and continuity of h we have that $[c, d] \subset h(K)$.

Due to the properties of the semi-conjugacy, the set $D = \bigcup_{n\in\mathbb{N}} f^n([c, d])$ is contained in $h(\tilde{\Lambda})$. On the other hand $D = W^u(P_0, f)$, which is dense in Λ (see Theorem 4.3.2). Thus $h(\tilde{\Lambda}) = \Lambda$ which means that h is onto for every $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ with $d_0(\tilde{f}, f) < \varepsilon_2$.

We have already shown the existence of $\varepsilon_0 = \min\{\varepsilon', \varepsilon_1, \varepsilon_2\}$ such that for any $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ with $d_0(\tilde{f}, f) < \varepsilon_0$ (1) and (2) in the theorem are verified.

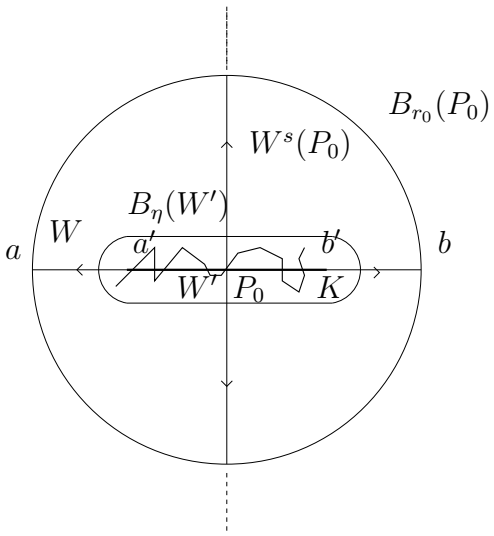


Figure 4.4.3:

Finally, we want to see that if $d_0(\tilde{f}, f) < \varepsilon_0$ then the map h lift to a map H with the properties given in the point (3).

For this, consider a map $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ with $d_0(\tilde{f}, f) < \varepsilon_0$ and for $\tilde{x} \in \pi^{-1}(\tilde{\Lambda})$ the point $h(\pi(\tilde{x})) = y \in \Lambda$. Then since $d_0(h, Id) < \frac{1}{4}$, there is a unique point in $B_{\frac{1}{4}}(\tilde{x}) \cap \pi^{-1}(y)$ that we call $\tilde{y}_x$. We define $H : \pi^{-1}(\tilde{\Lambda}) \to \pi^{-1}(\Lambda)$ by $H(\tilde{x}) = \tilde{y}_x$. It is not difficult to see that this function is continuous, onto, and $d_0(H, Id) < \frac{1}{4}$. Furthermore if we consider $x \in \pi^{-1}(\tilde{\Lambda})$ and $v \in \mathbb{Z}^2$, then $H(x + v) = H(x) + v$. So H lifts h.

Further, for fixed lifts $\tilde{F}, F$ of $\tilde{f}, f$ and any point $x \in \pi^{-1}(\tilde{\Lambda})$ we have by the definition of H that $H \circ \tilde{F}(x) = F \circ H(x) + v(x)$ with $v(x) \in \mathbb{Z}^2$. Therefore we have that $v(x) = H \circ \tilde{F}(x) - F \circ H(x)$ defines a continuous and integer-valued function over the set $\Lambda \subset \mathbb{T}^2$, which is compact and connected. Hence there exists $v \in \mathbb{Z}^2$ such that $v(x) = v$ for every $x \in \pi^{-1}(\tilde{\Lambda})$. This completes the proof.

□

In the following we make use of the above theorem to compute the rotation set on basic pieces which are either a non-trivial attractor or a non-trivial repeller.

Theorem 4.4.8. *Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ be a diffeomorphism, Λ be a non-trivial attractor of f and P_0 a fixed point in Λ. Then $conv(\rho_\Lambda(F))$ is a rational polygon. An analogous result holds for repellers.*

Proof. Let U be an open and connected set such that $f(cl[U]) \subset U$ and $\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda$. Let $\varepsilon_0 > 0$ be given by the Theorem 4.4.7. Due to Theorem

4.3.5 we can consider a fitted axiom A diffeomorphism $\tilde{f} \in \text{Homeo}_0(\mathbb{T}^2)$ that $d_0(\tilde{f}, f) < \varepsilon_0$. Therefore we have:

- $\tilde{f}(cl[U]) \subset U$ and $\tilde{\Lambda} = \bigcap_{n\in\mathbb{Z}} \tilde{f}(U)$ is a non-empty compact and connected set;
- there exists a continuous and onto $H : \pi^{-1}(\tilde{\Lambda}) \to \pi^{-1}(\Lambda)$ such that for any two lifts $\tilde{F}, F$ of $\tilde{f}, f$ we have $H \circ \tilde{F} = F \circ H + v$ for some $v \in \mathbb{Z}^2$, and $d_0(H, id) < \frac{1}{4}$. Further $H(x)+v = H(x+v)$ for every $x \in \pi^{-1}(\tilde{\Lambda}), v \in \mathbb{Z}^2$.

This implies on one hand that for every $x \in \pi^{-1}(\tilde{\Lambda})$ if $y = H(x)$ then $H \circ \tilde{F}^n(x) = F^n(y)+v$. Therefore, for every $x \in \pi^{-1}(\tilde{\Lambda})$ there exists $y \in \pi^{-1}(\Lambda)$ such that $d(F^n(y), \tilde{F}^n(x)) < \frac{1}{4} + \|v\|$ for every $n \in \mathbb{Z}$. On the other hand for every $y \in \pi^{-1}(\Lambda)$ one can consider $x \in H^{-1}(y)$ and have that $H \circ \tilde{F}^n(x) = F^n(y) + v$, so we have for every $y \in \pi^{-1}(\Lambda)$ a point $x \in \pi^{-1}(\tilde{\Lambda})$ such that $d(F^n(y), \tilde{F}^n(x)) < \frac{1}{4} + \|v\|$ for every $n \in \mathbb{Z}$.

Is not difficult to see that the last observation implies $\rho_\Lambda(F) = \rho_{\tilde{\Lambda}}(\tilde{F})$. Due to the fact that $\tilde{f}(cl[U]) \subset U$, and that $\tilde{f}$ is a fitted axiom A diffeomorphism, we have that $\tilde{\Lambda} \cap \Omega(\tilde{f}) = \Lambda_1 \cup ... \cup \Lambda_m$ with Λ_i totally disconnected basic pieces of $\tilde{f}$ for $i = 1, ..., m$. This implies by Remark 4.3.9 that $\text{conv}(\rho_{\tilde{\Lambda}}(\tilde{F}))$ is a rational polygon, so it is $\text{conv}(\rho_\Lambda(F))$.

□

Corollary 4.4.9. *Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ be an axiom A diffeomorphism. Then $\rho(F)$ is a rational polygon.*

Proof. Due to Theorem 4.3.2 there exists $n \in \mathbb{N}$ for which f^n is an axiom A diffeomorphism with basic pieces $\Lambda_1, ..., \Lambda_m$, such that for each $i = 1, ..., m$ the piece Λ_i contains a fixed point P_i of f^m. Whenever Λ_i is a totally disconnected basic piece of f we have seen in section 4.3 that $\text{conv}(\rho_{\Lambda_i}(F))$ is a rational polygon. On the other hand when it is not totally disconnected due to Remark 4.4.5 it is either a fixed point or a non-trivial attractor or a non-trivial repeller. Thus, by Theorem 4.4.8 we have that $\text{conv}(\rho_{\Lambda_i}(F^m))$ is a rational polygon in any case. Therefore Proposition 4.3.9 implies that $\rho(F^m)$ is a rational polygon, and since $\rho(F) = \frac{1}{m}\rho(F^m)$ we have that $\rho(F)$ is a rational polygon.

□

4.5 Minimal sets with rotation set given by a non trivial continuum.

In this section we observe the existence of minimal sets $\mathcal{M} \subset \mathbb{T}^2$ for some $f \in \text{Homeo}_0(\mathbb{T}^2)$, such that $\rho_{\mathcal{M}}(F)$ is a non trivial segment. We also observe that this phenomena is abundant in $\text{Homeo}_0(\mathbb{T}^2)$ (it occurs on an open set of

the space). Let us start by proving that the rotation sets associated to minimal sets are always continua (compact and connected sets).

Given F lift of $f \in \text{Homeo}_0(\mathbb{T}^2)$ we define for any $n \in \mathbb{N}$ the function $\varphi_n : \mathbb{T}^2 \to \mathbb{R}^2$ by $\varphi_n(x) := \frac{F^n(\tilde{x})-\tilde{x}}{n}$, where $\tilde{x}$ is some lift of the point x. As an easy observation we have the following.

Proposition 4.5.1. *Let $f \in \text{Homeo}_0(\mathbb{T}^2)$. Then there exists $K > 0$ such that $\| \varphi_{n+1}(x) - \varphi_n(x) \| < \frac{K}{n}$ for every $x \in \mathbb{R}^2$, F lift of f and $n \in \mathbb{N}$.*

Theorem 4.5.2. *Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ and assume that $\mathcal{M}$ is a minimal set of f and F is a lift of f. Then $\rho_{\mathcal{M}}(F)$ is a continuum.*

Proof. Suppose for a contradiction that $\rho_{\mathcal{M}}(F) = A_1 \cup A_2$, where A_1, A_2 are non-empty disjoint closed sets. Let $\delta = \frac{1}{3}d(A_1, A_2)$. Definition of the rotation set implies that there exists $n_0 \in \mathbb{N}$ such that $\varphi_n(\mathcal{M}) \subset B_\delta(A_1) \cup B_\delta(A_2)$ for every $n \geq n_0$. Due to Proposition 4.5.1 we can assume at the same time that $\|\varphi_{n+1}(x) - \varphi_n(x)\| < \delta$ for all $n \geq n_0$ and $x \in \mathcal{M}$. As a consequence, $\varphi_{n_0}(x) \in B_\delta(A_i)$ implies $\varphi_n(x) \in B_\delta(A_i)$ for every $n \geq n_0$, $x \in \mathcal{M}$ and $i = 1, 2$. This directly yields that $\mathcal{M}$ decomposes into two disjoint sets:

$$C_i = \{x \in \mathcal{M} \mid \rho_x(F) \subset A_i\} = \{x \in \mathcal{M} \mid \varphi_{n_0}(x) \in \text{cl}[B_\delta(A_i)]\}; i = 1, 2.$$

From the first characterization of C_i we see that they are f-invariant, whereas the second characterization implies that they are closed. Further none of them can be empty, since this would mean that $\mathcal{M} = C_i$ for some C_i, and therefore $\varphi_n(\mathcal{M}) \subset \text{cl}[B_\delta(A_i)]$ for all $n > n_0$, which implies that $\rho_{\mathcal{M}}(F) = A_i$ in contradiction to the definition of A_1 and A_2.

Hence, we have decomposed $\mathcal{M}$ in two non-empty closed and f-invariant sets, contradicting minimality.

□

Let us now consider a space of sequences of finite symbols $\Omega = \{0, ..., N\}^{\mathbb{Z}}$. Let $\xi' : \{0, ..., n_1\} \to \{0, ..., N\}$ be a finite word and $\xi : \{0, ..., n_2\} \to \{0, ..., N\}$ either be a finite or infinite ($n_2 = \infty$) word, with $n_2 \geq n_1$. We say that ξ' is a sub-word of ξ if there exist $t \in \mathbb{N}, 0 \leq t \leq n_2 - n_1$ such that $\xi(i + t) = \xi'(i)$ for every $i = 0, ..., n_1$. We say that $\xi \in \Omega$ is an almost periodic word if for every sub-word ξ' of ξ there exist $T_{\xi'} \in \mathbb{N}$, such that for every sub-word α of ξ with length $T_{\xi'}$, we have that ξ' is a sub-word of α.

Proposition 4.5.3. *Let $\sigma : \Omega \to \Omega$ be the shift, that is $\sigma(\xi)(i) = \xi(i + 1)$. Then for any almost periodic sequence $\xi \in \Omega$ the set $\mathcal{M} = cl[\{\sigma^n(\xi) : n \in \mathbb{Z}\}]$ is a minimal set.*

In the following lemma some special almost periodic sequences are constructed. Given $\xi \in \Omega$ we define $S_n(\xi) = \frac{\sum_{i=0}^{n-1} \xi(i)}{n}$.

Lemma 4.5.4 (Communicated by Tobias Jäger)**.** *For every $\delta > 0$ there exists an almost periodic sequence $\xi \in \{0,1\}^{\mathbb{Z}}$, such that* $\liminf S_n(\xi) < \delta$ *and* $\limsup S_n(\xi) > 1 - \delta$.

Proof. Let us fix $\delta > 0$ and consider $t \in \mathbb{N}$ such that $2^{-t} < \delta$. We define recursively the sequence $\{a_n\}_{n\in\mathbb{N}}$ by $a_{n+1} = 2^{t+n}a_n$ and $a_0 = 1$. Further we consider the following family of subsets of the positive integers:

- $A_n = [0, ..., a_n]$, $B_0 = A_0$;
- $B_n = (A_n + a_{n+1}\mathbb{N}) \setminus \bigcup_{i=0}^{n-1} B_i$ for every $n > 0$. Note that $\biguplus_{n=0}^{\infty} B_n = \mathbb{N}$.

We consider now $\xi \in \Omega$ defined by

$$\xi(i) = \begin{cases} 0 :| \ i \ |\in B_n, \ n \text{ even} \\ \\ 1 :| \ i \ |\in B_n, \ n \text{ odd} \end{cases}$$

Given a finite sub-word ξ' of ξ we have by construction a sub-word w of ξ, which is the restriction of ξ to some A_{n_0}, such that ξ' is a sub-word of w. Furthermore again by construction, we have that every sub-word of ξ with length $a_{n_0} + 2^{t+n_0}a_{n_0}$ contains w as a sub-word. This implies the almost periodicity for ξ.

Now, if we consider $n \in \mathbb{N} \setminus \{0\}$ then:

- if n is even we have by construction that $S_{a_n} \leq \frac{1}{a_n} \cdot \sum_{i=0}^{n-1} \#A_i \cdot \frac{a_n}{a_{i+1}} = \sum_{i=0}^{n-1} \frac{a_i}{a_{i+1}}$, which tends to $2^{-t} = \delta$ when n goes to infinity.
- it n is odd we have that $S_{a_n} \geq \frac{1}{a_n} \cdot \left(a_n - \sum_{i=0}^{n-1} \#A_i \cdot \frac{a_n}{a_{i+1}}\right) = 1 - \sum_{i=0}^{n-1} \frac{a_i}{a_{i+1}}$, which tends to $1 - \delta$ when n goes to infinity.

So, $\liminf S_n(\xi) < \delta$ and $\limsup S_n(\xi) > 1 - \delta$.

□

An important corollary of the last lemma is the following:

Corollary 4.5.5. *Let C be an invariant set of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ which admits a rotational Markov partition $\mathcal{S} = \{S_0, S_1\}$. Further, let F be a lift of f and assume that $\tau(0) = \{0,1\}$, $\tau(1) = \{0,1\}$ and $s_0 \neq s_1$. Then there exists a minimal set $\mathcal{M}$ of f such that $\rho_{\mathcal{M}}(F)$ is a non-trivial segment.*

Proof. Let us first assume that $s_0 = (0,0)$ and $s_1 = (1,0)$. Consider $\delta \in (0, \frac{1}{2})$, and $\xi \in \Omega_{\mathcal{S}}$ given by Lemma 4.5.4. By Proposition 4.5.3 we have that $\mathcal{M}' = cl[\{\sigma^n(\xi) : n \in \mathbb{Z}\}]$ is a minimal set for the shift $\sigma : \Omega_S \to \Omega_{\mathcal{S}}$.

Let $\mathcal{M}$ be a minimal set of f contained in $h_{\mathcal{S}}^{-1}(\mathcal{M}')$. Since $\mathcal{M}'$ is a minimal set of σ, we have that $h_{\mathcal{S}}(\mathcal{M}) = \mathcal{M}'$, and in particular $\xi \in h_{\mathcal{S}}(\mathcal{M})$. Then, by

defining
$\xi_n : \{0, ..., n-1\} \to \{0, 1\}$ as $\xi_n(i) = \xi(i)$ for every $i \in \mathbb{Z}$, we have that $\xi_n \in \Omega_{\mathcal{S}}^f$ and $\frac{\psi(\xi_n)}{n} = S_n$ for every $n \in \mathbb{N}$. Thus, by Corollary 4.2.4 $\liminf S_n(\xi)$ and $\limsup S_n(\xi)$ are two different points in $\rho_{\mathcal{M}}(F)$. Further, by construction we know that $\rho_{\mathcal{M}}(F) \subset \text{conv}(\{(0,0),(1,0)\})$. Then by Theorem 4.5.2 we have that $\rho_{\mathcal{M}}(F)$ is a non trivial segment.

Now, let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a affine linear isomorphism with $L(\mathbb{Z}^2) \supset \mathbb{Z}^2$ and denote by A_L the induced torus automorphism. Then if C is a closed f-invariant set and $g := A_L \circ f \circ A_L^{-1}$, we have $\rho_{A_L^{-1}(C)}(g) = L(\rho_C(f))$ (see [Kwa92]). Hence by considering L taking s_0 to $(0,0)$ and s_1 to $(1,0)$ and applying the argument above to g, we get the result in full generality.

□

In what follows, we want to see that the existence of minimal sets with non-trivial segments as rotation sets is abundant in the C^0 topology. In fact, we will prove first that for any axiom A diffeomorphism whose rotaion set has non-empty interior this kind of minimal sets exists, and then we extend its existence to an open set in $\text{Homeo}_0(\mathbb{T}^2)$. Given any basic piece $\Lambda \subset \mathbb{T}^2$ we write $\tilde{\Lambda} = \pi^{-1}(\Lambda)$.

Lemma 4.5.6. *Let Λ be a totally disconnected saddle basic piece of $f \in \text{Homeo}_0(\mathbb{T}^2)$ with a fixed point $P_0 \in \Lambda$. Further consider a rotational Markov partition $\mathcal{S} = \{S_0, ..., S_N\}$ of Λ such that $P_0 \in S_0$ and assume that conv$(\rho_\Lambda(F))$ is not a singleton. Then, for a lift F of f and $\tilde{P}_0 \in \pi^{-1}(P_0)$ with $F(\tilde{P}_0) = \tilde{P}_0$, we have that $W^u(\tilde{P}_0, F) \cap \tilde{\Lambda}$ has non-empty intersection with at least two connected components of $\pi^{-1}(S_0)$. Moreover, there exits a heteroclinic orbit of F joining $\tilde{P}_0$ with $\tilde{P}_0 + v$ for some $v \in \mathbb{Z}^2 \setminus \{0\}$.*

Proof. To prove this it is sufficient to do it for some particular $\tilde{P}_0$ and F with $F(\tilde{P}_0) = \tilde{P}_0$. Let D be a trivial domain such that $\tilde{P}_0 \in D$, $\pi(D) \supset \bigcup_{i=0}^N S_i$ and let $\tilde{S}_i = D \cap \pi^{-1}(S_i)$. We affirm first that there exists $i_0 \in \{0, ..., N\}$ such that $W^u(\tilde{P}_0, F) \cap \tilde{\Lambda} \cap \tilde{S}_{i_0} + v_0 \neq \emptyset$ and $W^u(\tilde{P}_0, F) \cap \tilde{\Lambda} \cap \tilde{S}_{i_0} + v_1 \neq \emptyset$ for some $v_0, v_1 \in \mathbb{Z}^2$ with $v_0 \neq v_1$. Otherwise we have necessarily that $cl[W^u(\tilde{P}_0, F)] \cap \tilde{\Lambda} =: \tilde{\Lambda}_0$ is a bounded F-invariant set with $\pi(\tilde{\Lambda}_0) = \Lambda$. Hence $\rho_\Lambda(F) = \{0\}$ which is absurd.

Now, we can consider two points $\tilde{x}_0, \tilde{x}_1 \in \tilde{S}_0 \cap \tilde{\Lambda}$ such that $F^n(\tilde{x}_0) \in \tilde{S}_{i_0} + v_0$ and $F^n(\tilde{x}_1) \in \tilde{S}_{i_0} + v_1$. For $x_0 = \pi(\tilde{x}_0)$ and $x_1 = \pi(\tilde{x}_1)$ we consider the finite words
$\alpha_0 : \{0, ..., n\} \to \{0, ..., N\}$ and $\alpha_1 : \{0, ..., n\} \to \{0, ..., N\}$ such that $\alpha_0(i) = h_{\mathcal{S}}(x_0)(i)$ and $\alpha_1(i) = h_{\mathcal{S}}(x_1)(i)$ for every $i \in \{0, ..., n\}$. Further since Λ is transitive we can consider a finite word $\beta : \{0, ..., m\} \to \{0, ..., N\}$ such that $\beta(0) = i_0$ and $\beta(m) = 0$.

As we have seen in the section 4.2.2 we can take now $\xi_0, \xi_1 \in \Omega_{\mathcal{S}}$ defined as:

$$\xi_0(i) = \begin{cases} 0 : i \leq 0, i \geq n+m \\ \\ \alpha_0\beta(i) : 0 \leq i \leq n+m \end{cases}, \xi_1(i) = \begin{cases} 0 : i \leq 0, i \geq n+m \\ \\ \alpha_1\beta(i) : 0 \leq i \leq n+m \end{cases}$$

Let $y_0 \in h_{\mathcal{S}}^{-1}(\xi_0)$, $y_1 \in h_{\mathcal{S}}^{-1}(\xi_1)$. Then we have that $f^{-n}(y_0), f^{-n}(y_1) \in W^u(P_0, f) \cap S_0$ for every $n \in \mathbb{N}$, since by construction $h(y_0), h(y_1) \in W^u(\theta, \sigma)$, where $\theta(i) = 0$ for all $i \in \mathbb{Z}$ and $W^u(\theta, \sigma)$ is the unstable set of θ. This implies that $\tilde{y}_0 := \pi^{-1}(y_0) \cap \tilde{S}_0$ and $\tilde{y}_1 := \pi^{-1}(y_1) \cap \tilde{S}_0$ are contained in $W^u(\tilde{P}_0, F)$.

Due to the fact that $\mathcal{S}$ is a rotational Markov partition, we have on the one hand that $F^i(\tilde{y}_0)$ and $F^i(\tilde{x}_0)$ belong to the same connected component of $\pi^{-1}(\bigcup_{i=0}^N S_i)$ for every $i = 0, ..., n$, and the same holds for $F^i(\tilde{y}_1)$ and $F^i(\tilde{x}_1)$. Otherwise we necessarily have that $F(S_{i*})$ intersects $S_{j*} + u_0$ and $S_{j*} + u_1$, for some $i_*, j_* \in \{0, ..., N\}$ and $u_0 \neq u_1 \in \mathbb{Z}^2$.

On the other hand, by the same reason, for every $j \in \{0, ..., m\}$ we have the following:

$$\text{if } F^{n+j}(\tilde{y}_0) \in \tilde{S}_{i_j} + w_j,\ w_j \in \mathbb{Z}^2, \text{ then } F^{n+j}(\tilde{y}_1) \in \tilde{S}_{i_j} + w_j + v, \text{ where } v = v_1 - v_0.$$

Hence $F^{n+m}(\tilde{y}_0) \in \tilde{S}_0 + w_m$ and $F^{n+m}(\tilde{y}_1) \in \tilde{S}_0 + w_m + v$ with $v \neq 0$, which implies the first part of the statement. Furthermore, we have by construction that $\tilde{y}_0, \tilde{y}_1$ are heteroclininc orbits joining $\tilde{P}_0$ with $\tilde{P}_0 + w_m$ and $\tilde{P}_0 + w_m + v$ respectively, with $v \in \mathbb{Z}^2 \setminus \{0\}$. This implies the final part of the statement. □

Theorem 4.5.7. *Let Λ be a totally disconnected saddle basic piece of $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ with a fixed point $P_0 \in \Lambda$. Further assume that conv$(\rho_\Lambda(F))$ is not a singleton for some F lift of f. Then Λ contains a minimal set $\mathcal{M}$ such that $\rho_{\mathcal{M}}(F)$ is a non-trivial segment.*

Proof. Let us first fix $\tilde{P}_0 \in \pi^{-1}(P_0)$ and a lift F of f with $F(\tilde{P}_0) = \tilde{P}_0$. We consider a rotational *us*-Markov partition $\mathcal{S} = \{S_0, ..., S_N\}$ given by Corollary 4.3.8 with $P_0 \in S_0$. Further we consider a trivial domain D such that $\pi(D) \supset \bigcup_{i=0}^N S_i$ and $\tilde{S}_i = D \cap \pi^{-1}(S_i)$ for all $i = 0, \ldots, N$. Thus we know by Lemma 4.5.6 that there exists $n \in \mathbb{N}$ and $\tilde{x} \in \tilde{S}_0 \cap \tilde{\Lambda}$ such that $F^n(\tilde{x}) \in \tilde{S}_0 + v \cap \tilde{\Lambda}$ with $v \in \mathbb{Z}^2 \setminus \{0\}$. Set $\pi(\tilde{x}) = x$.

We now define $g = f^n$. Due to the choice of $\mathcal{S}$ we know that $g_*(\mathcal{S})$ is a rotational
us-Markov partition of Λ for g. Furthermore the sets $S_0' := [f^{-n}(S_0) \cap S_0]_{P_0} = \pi([F^{-n}(\tilde{S}_0) \cap \tilde{S}_0]_{\tilde{P}_0})$ and $S_1' := [f^{-n}(S_0) \cap S_0]_x = \pi([F^{-n}(\tilde{S}_0 + v) \cap \tilde{S}_0]_{\tilde{x}})$ are two elements in $g_*(\mathcal{S})$, which are vertical rectangles of S_0, and the sets $T_0' = f^n(S_0')$, $T_1' = f^n(S_1')$ are horizontal rectangles in S_0.

Due to the local product structure of Λ and the fact that $\mathcal{S}$ is a us-partition, we have that
$S_0' \cap T_0' \cap \Lambda \neq \emptyset$, $S_0' \cap T_1' \cap \Lambda \neq \emptyset$, $S_1' \cap T_0' \cap \Lambda \neq \emptyset$ and $S_1' \cap T_1' \cap \Lambda \neq \emptyset$. Hence we have that $\tau_g(0) \supset \{0,1\}$ and $\tau_g(1) \supset \{0,1\}$. Further by definition we have $s_0' = 0$ and $s_1' = v$ for F^n.

Let us now consider $C = h^{-1}_{g_*(\mathcal{S})}(\{\xi \in \Omega_{g_*(\mathcal{S})} : \xi(i) \in \{0,1\} \text{ for every } i \in \mathbb{Z}\})$. The above argument implies that C is a a non-empty g-invariant closed set. Furthermore by construction $\mathcal{S}' = \{S_0', S_1'\}$ is a rotational Markov partition of C for $g = f^n$ with $s_0' = 0$ and $s_1' = v$ for F^n. Therefore due to Corollary 4.5.5 there exists a minimal set $\mathcal{M}'$ of g such that $\rho_{\mathcal{M}'}(F^n)$ is a non-trivial segment. Hence, defining $\mathcal{M} = \mathcal{M}' \cup ... \cup f^{n-1}(\mathcal{M}')$ we have that $\mathcal{M}$ is a minimal set of f contained in Λ with $\rho_{\mathcal{M}}(F) = \frac{1}{n}\rho_{\mathcal{M}'}(F^n)$.

□

We have thus obtained a criterion to ensure the existence of minimal sets with non-trivial segments as rotation sets. Using this result we can now show that this phenomena is abundant in $\text{Homeo}_0(\mathbb{T}^2)$.

Corollary 4.5.8. *Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ be a fitted axiom A diffeomorphism with int$(\rho(F))$ non-empty. Then, there exists a minimal set $\mathcal{M}$ of f such that $\rho_{\mathcal{M}}(F)$ is a non-trivial segment.*

Proof. By the same argument as in the final part of the previous proof, is sufficient to show the existence of a minimal set with the rotation set given by a non-trivial segment for some power of f. Let us consider $n \in \mathbb{N}$ such that each basic piece of $g := f^n$ has a fixed point of g. Due to a result by Franks [Fra95], we have for any vector $v \in \mathbb{Q}^2 \cap \text{int}(\rho(G))$ a periodic point $x \in \mathbb{T}^2$ such that $\lim_n \frac{G^n(\tilde{x}) - \tilde{x}}{n} = v$, where $G = F^n$. Therefore, if $\Lambda_1, ..., \Lambda_m$ are the basic pieces of g, there exists $i_0 \in \{1, ..., m\}$ such that Λ_{i_0} contains two periodic points x_1, x_2 with $\lim_n \frac{G^n(\tilde{x}_1) - \tilde{x}_1}{n} = v_1$ and $\lim_n \frac{G^n(\tilde{x}_2) - \tilde{x}_2}{n} = v_2$ for two different vectors v_1 and v_2. Thus $\text{conv}(\rho_\Lambda(G))$ is not a singleton.

Furthermore Λ has to be a saddle, otherwise it is a periodic orbit and hence has a unique rotation vector. Then due to f be a fitted axiom A diffeomorphism we have that Λ is a totally disconnected saddle basic piece, which by definition of g contains some fixed point. Thus, by Theorem 4.5.7 we obtain the existence of a minimal set of g with a non-trivial segment as rotation set.

□

The same techniques used in the proof of Theorem 4.4.8, one can achieve the generalization of the last corollary to any axiom A diffeomorphism in $\text{Homeo}_0(\mathbb{T}^2)$.

Corollary 4.5.9. *Let $f \in \text{Homeo}_0(\mathbb{T}^2)$ be a axiom A diffeomorphism with int$(\rho(F)) \neq \emptyset$. Then, there exists $\mathcal{M}$ minimal set of f such that $\rho_{\mathcal{M}}(F)$ is a non-trivial segment.*

Let $\mathcal{A} \subset \text{Homeo}_0(\mathbb{T}^2)$ be defined as

$$\mathcal{A} = \{f \in \text{Homeo}_0(\mathbb{T}^2) \mid \text{int}(\rho(F)) \neq \phi \text{ for some } F \text{ lift of } f\}.$$

As shown in [MZ91] this set is open in $\text{Homeo}_0(\mathbb{T}^2)$.

Corollary 4.5.10. *There exists an open and dense set $\mathcal{D}' \subset \mathcal{A}$, such that any $f \in \mathcal{D}'$ has a minimal set $\mathcal{M}$ with $\rho_{\mathcal{M}}(F)$ given by a non trivial segment.*

Proof. Let $\mathcal{D}$ be the set constructed in Section 4.3.10, and $\mathcal{D}' := \mathcal{D} \cap \mathcal{A}$. Thus by construction of $\mathcal{D}$ for any element $f \in \mathcal{D}'$ there exist a fitted axiom A diffeomorphism $g \in \mathcal{D}'$ and a homotopic to the identity map $h : \mathbb{T}^2 \to \mathbb{T}^2$ such that h semi-conjugates f with g ($h \circ f = g \circ h$).

Due to the corollary 4.5.8 we have the existence of a minimal set $\mathcal{M}'$ of g such that $\rho_{\mathcal{M}'}(G)$ is a non trivial segment for some lift G of g. Let us define the f-invariant set $\mathcal{M}'' = h^{-1}(\mathcal{M}')$, and consider a minimal set $\mathcal{M} \subset \mathcal{M}''$ of f. Hence, since $\mathcal{M}'$ is a minimal set of g we have necessarily that $h(\mathcal{M}) = \mathcal{M}'$. Thus by the properties of the rotation set we have that $\rho_{\mathcal{M}}(F)$ is a non-trivial segment.

□

Finally, inspired by the results given in [MZ91] and [LM91] we pose the following question:

Question 4.5.11. Given $f \in \mathcal{A}$ and a continuum $C \subset \text{int}(\rho(F))$, does there exists a minimal set $\mathcal{M}$ of f such that $\rho_{\mathcal{M}}(F) = C$?

Chapter 5

Toral homeomorphisms semiconjugate to irrational rotations.

5.1 Statements of the results.

In this chapter we prove the theorems 10, 11, 12 and the propositions 1, 2, which were presented in Subsection 1.2.4. We establish in what follows, the statements for these results.

Theorem 5.1.1. *Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$. Then the following statements are equivalent.*

(i) f is semiconjugate to an irrational rotation R of the circle;

(ii) there exists a wandering essential circloid with irrational combinatorics;

(iii) there exists a wandering essential continuum with irrational combinatorics;

(iv) there exists a semiconjugacy h from f to R such that for all $\xi \in \mathbb{T}^1$ the fiber $h^{-1}\{\xi\}$ is an essential annular continuum.

The second result gives a sufficient condition in order to toral extensions of irrational rotations, have a singleton rotation set. The condition is related with the characterization above.

Theorem 5.1.2. *Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is semiconjugate to an irrational rotation of the circle. Further, assume that the semiconjugacy h is chosen such that its fibres $h^{-1}(\xi)$ are all essential annular continua and there exists a set $\Omega \subseteq \mathbb{T}^1$ of positive Lebesgue measure such that $h^{-1}\{\xi\}$ is compactly generated for all $\xi \in \Omega$. Then f has a unique rotation vector.*

As by product of the last theorem we obtain the following result, related to rotation theory for invariant thin essential annular continua.

Theorem 5.1.3. *Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and A is a thin annular continuum that is compactly generated and f-invariant. Then $f_{|A}$ has a unique rotation vector, that is, there exists a vector $\rho \in \mathbb{R}^2$ such that $\lim_{n\to\infty} F^n(z) - z/n = \rho$ for all $z \in \mathbb{R}^2$ with $\pi(z) \in A$. Moreover, the convergence is uniform in z.*

Related to Theorem 5.1.1 arise the question about whether the semiconjugacy is "unique" or not. In this direction we obtain the following result.

Proposition 5.1.4. *Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ be a totally irrational pseudo-rotation which is seminconjugate to the respective rigid translation of $\mathbb{T}^2$. Then the semiconjugacy is unique up to composing with rigid translations of $\mathbb{T}^2$.*

The following result describes different kinds of thin essential annular continua, which try to place the family of those with compact generator.

Proposition 5.1.5. *Let $A \subset \mathbb{A}$ be an essential thin annular continuum. Then the following holds.*

(1) If A has no generator either:

(1a) C_A has no generator, or

(1b) C_A has generator and A contains an infinite spike.

(2) If A has generator, then C_A has generator.

Finally we give a finial results which complements Theorem 5.1.3, in order to relate Rotation Theory to our topological observation.

Proposition 5.1.6. *Given any segment $I \subset \mathbb{R}$, there exists a map $f \in \mathrm{Homeo}(A)$ which leaves invariant an essential thin annular continuum $A \subset \mathbb{A}$ such that C_A has compact generator, A has an infinite spike, and $\rho_A(f) = I$.*

In the next section we give the preliminaries definition and results. In Section 5.3 we give the proof of Theorem 5.1.1, in Section 5.5 the proof of the theorems 5.1.3 and 5.1.2, and in Section 5.4 the proof of 5.1.4. Finally, in the last section we prove the propositions 5.1.5 and 5.1.6.

5.2 Notation and preliminaries

Some of the following notations and results were already used in the study of dynamics on the two-dimensional torus in Chapter 2. For convenience, we stick to the same terminology also in higher dimensions. We let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ and denote by $\mathbb{A}^d = \mathbb{T}^{d-1} \times \mathbb{R}$ the d-dimensional annulus. If $d = 2$, we simply write $\mathbb{A}$ instead of $\mathbb{A}^2$. We will often compactify $\mathbb{A}$ by adding two points $-\infty$ and $+\infty$, thus making it a sphere. As long as no ambiguities can arise, we will always denote canonical quotient maps like $\mathbb{R} \to \mathbb{T}^1$, $\mathbb{R}^d \to \mathbb{T}^d$, $\mathbb{R}^d \to \mathbb{A}^d$ by π. Likewise, on any product space π_i denotes the projection to the i-th

coordinate. We call a subset $A \subseteq \mathbb{A}^d$ or $A \subseteq \mathbb{R}^d$ bounded from above (from below) if $\pi_d(A)$ is bounded from above (from below).

We say a continuum (that is, a compact and connected set) $E \subseteq \mathbb{A}^d$ is *essential* if $\mathbb{A}^d \setminus E$ contains two unbounded connected components. In this case, one of these components will be unbounded above and bounded below, and we denote it by $\mathcal{U}^+(A)$. The second unbounded component will be bounded above and unbounded below, and we denote it by $\mathcal{U}^-(A)$. A is called an *essential annular continuum* if $\mathbb{A}^d \setminus A = \mathcal{U}^+(A) \cup \mathcal{U}^-(A)$. Note that in dimension two, one can show by using the Riemann Mapping Theorem that both unbounded components are homeomorphic to $\mathbb{A}$ and A is the intersection of a decreasing sequence of topological annuli. This is not true anymore in higher dimensions, but at least we have the following.

Lemma 5.2.1. *If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of essential annular continua, then $A = \bigcap_{n\in\mathbb{N}} A_n$ is an essential annular continuum as well.*

Proof. As a decreasing intersection of essential continua, A is an essential continuum. Further, we have that $\mathbb{T}^d \setminus A = \bigcup_{n\in\mathbb{N}} \mathbb{T}^d \setminus A_n$ is the union of the two sets

$$U^+ = \bigcup_{n\in\mathbb{N}} \mathcal{U}^+(A_n) \quad \text{and} \quad U^- = \bigcup_{n\in\mathbb{N}} \mathcal{U}^-(A_n) \ .$$

As the union of an increasing sequence of open connected sets is connected, both these sets are connected. Hence, $\mathbb{T}^d \setminus A$ consists of exactly two connected components $\mathcal{U}^+(A) = U^+$ and $\mathcal{U}^-(A) = U^-$, both of which are unbounded. □

Given $S \subseteq \mathbb{R}^d$, we say S is *horizontal* if $\pi_d(S)$ is bounded and $\mathbb{R}^d \setminus S$ contains two different connected components $\mathcal{U}^+(S)$ and $\mathcal{U}^-(S)$ whose image under π_d is unbounded. Note that in this case one of the two components, which we always denote by $\mathcal{U}^+(S)$, is bounded below whereas the other component, denoted by $\mathcal{U}^-(S)$, is bounded above. Similarly, given any subset $B \subseteq \mathbb{A}^d$ bounded above (below) we denote by $\mathcal{U}^+(B)$ ($\mathcal{U}^-(B)$) the unique connected component of $\mathbb{A}^d \setminus B$ which is unbounded above (below). The same notation is used on $\mathbb{R}^d$. A horizontal connected closed set S is called a *horizontal strip*, if $\mathbb{R}^d \setminus S = \mathcal{U}^+(S) \cup \mathcal{U}^-(S)$. Note that thus the lift of an essential annular continuum $A \subseteq \mathbb{A}^d$ to $\mathbb{R}^d$ is a horizontal strip.

In any d-dimensional manifold M, we say A is an *annular continuum* if it is contained in a topological annulus $\mathcal{A} \simeq \mathbb{A}^d$ and it is an essential annular continuum in the above sense when viewed as a subset of $\mathcal{A}$. In this situation, we say A is *essential* if essential loops in $\mathcal{A}$ are also essential in M. We call $C \subseteq \mathbb{A}^d$ an *essential circloid* if it is an essential annular continuum and does not contain any other essential annular continuum as a strict subset. Circloids in general manifolds are then defined in the same way as annular continua. Finally, we call a horizontal strip S *minimal* if it is a minimal element of the set of horizontal strips with the partial ordering by inclusion. Note that thus

an annular essential continuum in $\mathbb{A}^d$ is a circloid if and only if its lift to $\mathbb{R}^d$ is a minimal strip. Finally, we call a closed set *thin*, if it has empty interior.

Lemma 5.2.2 ([Jäg09b], Lemma 3.4)**.** *Every thin annular continuum A contains a unique circloid C_A, which is given by*

$$C_A \;=\; cl[\mathcal{U}^+(A)] \cap cl[\mathcal{U}^-(A)] \;. \tag{5.2.1}$$

The same statement applies to thin horizontal strips.

The proof in [Jäg09b] is given for essential annular continua and for $d = 2$, but it literally goes through in higher dimensions and for strips. The same is true for the following result, which describes an explicit construction to obtain essential circloids from arbitrary essential continua. Given an essential set $A \subseteq \mathbb{R}^d$ which is bounded above, we write $\mathcal{U}^{+-}(A)$ instead of $\mathcal{U}^-(\mathrm{cl}[\mathcal{U}^+(A)])$ and use analogous notation for other concatenations of these procedures.

Lemma 5.2.3 ([Jäg09b], Lemma 3.2)**.** *If $A \subseteq \mathbb{A}^d$ is an essential continuum, then*

$$\mathcal{C}^+(A) = \mathbb{T}^d \backslash \big(\mathcal{U}^{+-}(A) \cup \mathcal{U}^{+-+}(A)\big) \quad \textit{and} \quad \mathcal{C}^-(A) = \mathbb{T}^d \backslash \big(\mathcal{U}^{-+}(A) \cup \mathcal{U}^{-+-}(A)\big)$$

are circloids. Further, we have $\partial \mathcal{C}^{\pm}(A) \subseteq A$.

It is not true in general that in the above situation $\mathcal{C}^{\pm}(A) \subseteq A$. One can construct an example in which the the essential continuum A has empty interior but $\mathcal{C}^{\pm}(A)$ contains a disk.

Given two horizontal essential continua $A_1, A_2 \subseteq \mathbb{T}^d$, we say $\hat{A}_i \subseteq \mathbb{A}^d$ is a lift of A_i if it is a connected component of $\pi^{-1}(A_i)$. We write $\hat{A}_1 \preccurlyeq \hat{A}_2$ if $\hat{A}_2 \subseteq \mathcal{U}^+(\hat{A}_1)$. Given two lifts $\hat{A}_1 \preccurlyeq \hat{A}_2$ we say they are *adjacent*, if if the compact region $\mathbb{A}^d \setminus \Big(\mathcal{U}^-(\hat{A}_1) \cup \mathcal{U}^+(\hat{A}_2)\Big)$ does not contain any integer translates of $\hat{A}_1$ and $\hat{A}_2$. In this situation, we let $\Big[\hat{A}_1, \hat{A}_2\Big] = \mathbb{A}^d \setminus \Big(\mathcal{U}^-(\hat{A}_1) \cup \mathcal{U}^+(\hat{A}_2)\Big)$ and $\Big(\hat{A}_1, \hat{A}_2\Big) = \mathcal{U}^+(\hat{A}_1) \cap \mathcal{U}^-(\hat{A}_2)$. Then we let $[A_1, A_2] = \pi\left(\Big[\hat{A}_1, \hat{A}_2\Big]\right)$ and $(A_1, A_2) = \pi\left(\Big(\hat{A}_1, \hat{A}_2\Big)\right)$. With these notions, we define a circular order on pairwise disjoint essential continua $A_1, A_2, A_3 \subseteq \mathbb{T}^d$ by

$$A_1 \preccurlyeq A_2 \preccurlyeq A_3 \quad \Leftrightarrow \quad A_2 \in [A_1, A_3] \;.$$

The strict relation $\prec$ is defined by replacing closed *intervals* with open ones. Using these notions, we now say a sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint essential continua in $\mathbb{T}^d$ has *irrational combinatorics* if there exists $\rho \in \mathbb{R} \setminus \mathbb{Q}$ such that for arbitrary $y_0 \in \mathbb{T}^1$ the sequence $y_n = y_0 + n\rho \bmod 1$ satisfies

$$A_k \prec A_m \prec A_n \quad \Leftrightarrow \quad y_k < y_m < y_n$$

for all $k, m, n \in \mathbb{Z}$.

We finish the topological preliminaries with the following proposition.

Proposition 5.2.4. *Let A, B be compact subsets of $\mathbb{A}^n$ such that:*

- $A \cap B = \emptyset$
- $\mathbb{A}^n \setminus A$ *has exactly one unbounded component.*
- $\mathbb{A}^n \setminus B$ *has exactly one unbounded component.*

Then $\mathbb{A}^n \setminus A \cup B$ has one exactly unbounded component.

Proof. Let $\Pi_{0,b}(A^c), \Pi_{0,b}(B^c)$ be the sets of bounded connected components of A^c, B^c respectively. We define

$$\mathrm{Fill}(A) = A \cup \left[\bigcup_{U \in \Pi_{0,b}(A^c)} U\right] , \mathrm{Fill}(B) = B \cup \left[\bigcup_{U \in \Pi_{0,b}(B^c)} U\right]$$

Hence $A \subset \mathrm{Fill}(A)$, $B \subset \mathrm{Fill}(B)$ and $\mathrm{Fill}(A)^c, \mathrm{Fill}(B)^c$ have each only one connected component which is unbounded. Furthermore, we may choose sets $C, D \subseteq \mathbb{A}^n$ such that $\mathrm{Fill}(A) \cup \mathrm{Fill}(B) = C \cup D$ and we have that

- C, D are compact;
- $C \cap D = \emptyset$;
- C^c, D^c has each only one connected component.

Notice that one of the two sets C or D could be empty.

Now, it suffices to show that $(C \cup D)^c$ has only one connected component. For this we consider the usual compactification of the annulus by the sphere $\mathbb{S}^n$, and the unreduced Mayer-Vietoris long sequence (see [Hat02]) for $(\mathbb{S}^n, C^c, D^c)$ starting from $H_1(\mathbb{S}^n)$:

$$H_1(\mathbb{S}^n) \xrightarrow{\partial_*} H_0(C^c \cap D^c) \xrightarrow{\theta_*} H_0(C^c) \oplus H_0(D^c) \xrightarrow{\xi_*} H_0(\mathbb{S}^n) \to 0$$

Then, we obtain that

$$0 \xrightarrow{\partial_*} H_0(C^c \cap D^c) \xrightarrow{\theta_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\xi_*} \mathbb{Z} \to 0$$

Thus we have that $H_0(C^c \cap D^c)$ is isomorphic to $\ker(\xi_*)$, so $H_0(C^c \cap D^c) \cong \mathbb{Z}$. This implies that there is only one connected component in $C^c \cap D^c$. □

Finally, we will frequently use the following Uniform Ergodic Theorem (e.g. [KH97, SS00]).

Theorem 5.2.5. *Suppose X is a compact metric space and $f : X \to X$ and $\varphi : X \to \mathbb{R}$ are continuous. Further, assume that there exists $\rho \in \mathbb{R}$ such that*

$$\int_X \varphi \, d\mu \;=\; \rho$$

for all f-invariant ergodic probability measures μ on X. Then

$$\lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=0}^{n-1} \varphi \circ f^i(x)\right) \;=\; \rho \quad \textit{for all } x \in X \ .$$

Furthermore, the convergence is uniform on X.

5.3 Semiconjugacy to an irrational rotation

We now turn to the proof of Theorem 5.1.1. The implications (ii)⇒(iii) and (iv)⇒(i) in Theorem 5.1.1 are obvious. Hence, in order to prove all the equivalences, it suffices to prove (iii)⇒(ii), (i)⇒(iii) and (ii)⇒(iv). We do so in three separate lemmas and start by treating the easiest of the three implications, which is (iii)⇒(ii).

Lemma 5.3.1. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ and suppose E is a wandering essential continuum. Then $\mathcal{C}^+(E)$ is a wandering essential circloid and the circular ordering of the orbits of E and $\mathcal{C}^+(E)$ are the same.*

Proof. Suppose $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ and E is a wandering essential continuum with irrational combinatorics. Let $E_n = f^n(E)$ and $C_n = \mathcal{C}^+(E_n) = f^n(\mathcal{C}^+(E))$. Assume for a contradiction that the C_n are not pairwise disjoint, that is, $C_i \cap C_j \neq \emptyset$ for some integers $i \neq j$. Since $\partial C_n \subseteq E_n$ for all $n \in \mathbb{Z}$, C_i must intersect the interior of C_j or vice versa. Assuming the first case, by connectedness C_i has to be contained in a single connected component of $\mathbb{T}^d \setminus \partial C_j$, since otherwise it would have to intersect ∂C_j. As C_i is not contained in $\mathbb{T}^d \setminus C_j$, it has to be contained in the interior of C_j. However, this contradicts the minimality of circloids with respect to inclusion. Hence C_0 is wandering.

The fact that the circular ordering is preserved when going from $(E_n)_{n\in\mathbb{N}}$ to $(C_n)_{n\in\mathbb{N}}$ is now obvious. □

The next lemma shows (ii)⇒(iv).

Lemma 5.3.2. *Let $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ and suppose C is a wandering essential circloid with irrational combinatorics of type ρ. Then there exists a semiconjugacy $h : \mathbb{T}^d \to \mathbb{T}^1$ to R_ρ such that the fibres $h^{-1}\{\xi\}$ are all essential annular continua.*

Proof. We let $C_n := f^n(C)$ and denote the connected components of the lifts of these circloid by $\hat{C}_{n,m}$, where the indices are chosen such that for all integers n, m we have

- $\pi(\hat{C}_{n,m}) = C_n$;
- $F(\hat{C}_{n,m}) = \hat{C}_{n+1,m}$;
- $T(\hat{C}_{n,m}) = \hat{C}_{n,m+1}$.

We claim that

$$H(z) \;=\; \sup\Big\{n\rho+m \mid z\in\mathcal{U}^+(\hat{C}_{n,m})\Big\}$$

is a lift of a semiconjugacy h with the required properties. Note that due to the irrational combinatorics we have $n\rho+m\le\tilde{n}\rho+\tilde{m}$ if and only if $\hat{C}_{n,m}\preccurlyeq\hat{C}_{\tilde{n},\tilde{m}}$, such that in particular $H(z)$ is well-defined and finite for all $z\in\mathbb{A}^d$. Further, for any $z\in\mathbb{A}^d$ we have

$$\begin{aligned} H\circ F(z) &= \sup\Big\{n\rho+m \mid F(z)\in\mathcal{U}^+(\hat{C}_{n,m})\Big\} \\ &= \sup\Big\{n\rho+m \mid z\in\mathcal{U}^+(\hat{C}_{n-1,m})\Big\} \;=\; H(z)+\rho\ . \end{aligned}$$

In a similar way one can see that $H\circ T(z)=H(z)+1$, such that H projects to a map $h:\mathbb{T}^d\to\mathbb{T}^1$ which satisfies $h\circ f=R_\rho\circ h$.

In order to check the continuity of H, suppose $U\subseteq\mathbb{R}$ is an open interval and let $z\in H^{-1}(U)$. Choose $r=n\rho+m<H(z)<\tilde{n}\rho+\tilde{m}=s$ with $r,s\in U$. Then $z\in\mathcal{U}^+(C_{n,m})\cap\mathcal{U}^-(C_{\tilde{m},\tilde{n}})=:V$. From the definition of H we see that $H(V)\subseteq[r,s]\subseteq U$, and thus $H^{-1}(U)$ contains an open neighbourhood of z. Since U and $z\in H^{-1}(U)$ were arbitrary, H is continuous. The fact that h is onto follows easily from the minimality of R_ρ, so that h is indeed a semiconjugacy from f to R_ρ.

It remains to prove the fact that the fibres $h^{-1}\{\xi\}$ are annular continua. In order to do so, note that for $\xi\in\mathbb{T}^1$

$$\begin{aligned} H^{-1}\{\xi\} &= \bigcap_{n\rho+m<\xi}\mathcal{U}^+(\hat{C}_{n,m})\;\cap\bigcap_{\tilde{n}+\rho\tilde{m}>\xi}\mathcal{U}^-(\hat{C}_{\tilde{n},\tilde{m}}) \\ &= \bigcap_{n\rho+m<\xi}\mathbb{A}^d\setminus\mathcal{U}^-(\hat{C}_{n,m})\;\cap\bigcap_{\tilde{n}\rho+\tilde{m}>\xi}\mathbb{A}^d\setminus\mathcal{U}^+(\hat{C}_{\tilde{n},\tilde{m}})\ . \end{aligned} \tag{5.3.1}$$

Note here that for all n,m,n',m' with $n\rho+m<n'\rho+m'$ we have

$$\mathcal{U}^+(\hat{C}_{n',m'})\;\subseteq\;\mathbb{A}^d\setminus\mathcal{U}^-(\hat{C}_{n',m'})\;\subseteq\;\mathcal{U}^+(\hat{C}_{n,m})$$

and similar inclusions hold in the opposite direction. This explains

the second equality in (5.3.1). Choosing sequences $n_i,m_i,\tilde{n}_i,\tilde{m}_i$ with $n_i\rho+m_i\nearrow\xi$ and $\tilde{n}_i\rho+\tilde{m}_i\searrow\xi$, we can rewrite (5.3.1) as

$$H^{-1}\{\xi\}\;=\;\bigcap_{i\in\mathbb{N}}\mathbb{A}^d\setminus\Big(\mathcal{U}^-(\hat{C}_{n_i,m_i})\cup\mathcal{U}^+(\hat{C}_{\tilde{n}_i,\tilde{m}_i})\Big)\ .$$

Since the sets of the intersection are all essential annular continua, so is $H^{-1}\{\xi\}$ by Lemma 5.2.1. □

It remains to prove the implication (i)⇒(iii).

Lemma 5.3.3. *Suppose $h : \mathbb{T}^d \to \mathbb{T}^1$ is a semiconjugacy from $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$ to an irrational rotation R_ρ. Then every fibre $h^{-1}\{\xi\}$ contains a wandering essential continuum with irrational combinatorics.*

Proof. We first show that the action $h^* : \pi_1(\mathbb{T}^d) \to \pi_1(\mathbb{T}^1)$ of h on the fundamental groups is non-trivial. Suppose for a contradiction that $h^* = 0$. Then any lift $H : \mathbb{R}^d \to \mathbb{R}$ of h is bounded since in this case $\sup_{z\in\mathbb{R}^d} \|H(z)\| = \sup_{z\in[0,1]^d} \|H(z)\|$. However, this contradicts the unboundedness of

$$H \circ F^n(z) \;=\; R_\rho^n \circ H(z)\ .$$

Consequently, h^* is non-trivial, and by composing h with a linear torus automorphism we may assume that h^* is just the projection to the last coordinate. We then obtain a lift $\hat{h} : \mathbb{A}^d \to \mathbb{R}$ which satisfies $\hat{h}(z) \to \pm\infty$ if $z \to \pm\infty$.

As a consequence, the Intermediate Value Theorem implies that every properly embedded line $\Gamma = \{\gamma(t) \mid t \in \mathbb{R}\}$ intersects all level sets $\widehat{E}_x = \hat{h}^{-1}\{x\}$. Hence, all $\widehat{E}_x$ are essential.

If $\widehat{E}_x$ is not connected, we consider the family of all compact and essential subsets of $\widehat{E}_x$ and choose and element $\widehat{\mathcal{E}}$ which is minimal with respect to the inclusion. Note that such minimal elements exist by the Lemma of Zorn. By Proposition 5.2.4 $\widehat{\mathcal{E}}$ is connected. Further, $\mathcal{E} = \pi(\widehat{\mathcal{E}})$ is wandering since $\widehat{\mathcal{E}} \subseteq h^{-1}\{x\}$. Hence, $\mathcal{E}$ is the wandering essential continuum we are looking for. The fact that $\mathcal{E}$ has irrational combinatorics can be seen from the semiconjugacy equation. □

5.4 On the uniqueness of the semiconjugacy.

In the light of the preceding section, it is an obvious question to ask to what extent a semiconjugacy between $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and an irrational rotation R_ρ of the circle is unique. It is easy to check that for every rigid rotation $R : \mathbb{T}^1 \to \mathbb{T}^1$ the map $R \circ h$ is a semiconjugacy between f and R_ρ as well. Hence there is non-uniqueness of the semiconjugacy in general. Nevertheless, one could ask whether there is uniqueness up to post-composition with rotations. In brief, we will speak of *uniqueness modulo rotations*.

Consider $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ given by $f(x, y) = (x + \rho_1, y)$ with $\rho_1 \in \mathbb{Q}^c$. For any continuous function $\alpha : \mathbb{T}^1 \to \mathbb{T}^1$, we have that $h_\alpha(x, y) = x + \alpha(y)$ is a semiconjugacy from f to R_{ρ_1}. Thus we do not have uniqueness of the semiconjugacy even modulo rotations. However, it is not difficult to see that all the possibles semiconjugacies between f and R_{ρ_1} are given by h_α for some continuous function α. This implies in particular that on every minimal set $Y_r = \{(x, y) \in \mathbb{T}^2 |\ y = r\}$, $r \in \mathbb{T}^1$, given any two semiconjugacies h_1 and h_2 we have that $h_{1|Y_r} = (R \circ h_2)_{|Y_r}$ for some rigid rotation R. This, as we will see, is a general fact.

We say that an f-invariant set Ω is *externally transitive* if for every $x, y \in \Omega$ and neighbourhoods U_x, U_y of x and y, respectively, there exists $n \in \mathbb{N}$ such that $f^n(U_x) \cap U_y \neq \emptyset$. Notice that $f^n(U_x)$ and U_y do not need to intersect in Ω as in the usual definition of topological transitivity. In the above example the sets Y_r are transitive, hence externally transitive.

Given $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ semiconjugate to a rigid rotation R_ρ and an f-invariant set $\Omega \subseteq \mathbb{T}^2$, we say the semiconjugacy is *unique modulo rotations on* Ω if for all semiconjugacies h_1, h_2 from f to R_ρ we have $h_{1|\Omega} = (R \circ h_2)_{|\Omega}$ for some rigid rotation R.

Proposition 5.4.1. *Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ be semiconjugate to a rigid rotation of $\mathbb{T}^1$. Further, assume that $\Omega \subset \mathbb{T}^2$ is an externally transitive set of f. Then the semiconjugacy is unique modulo rotations on Ω.*

Proof. Let h_1, h_2 be two semiconjugacies between f and R_ρ. By post-composing with a rigid rotation, we may assume that $h_1(x) = h_2(x)$ for some $x \in \Omega$. Suppose for a contradiction that $h_1(y) \neq h_2(y)$ for some $y \in \Omega$.

Let $\varepsilon = \frac{1}{2} \cdot d(h_1(y), h_2(y))$ and $\delta > 0$ such that $d(h_1(x'), h_2(x')) < \varepsilon$ if $x' \in B_\delta(x)$ and $d(h_1(y'), h_2(y')) > \varepsilon$ if $y' \in B_\delta(y)$. Due to Ω being externally transitive, there exists $z \in B_\delta(x)$ and $n \in \mathbb{N}$ such that $f^n(z) \in B_\delta(y)$. However, at the same time we have that $\varepsilon < d(h_1(f^n(z)), h_2(f^n(z))) = d(R_\rho^n(h_1(z)), R_\rho^n(h_2(z))) = d(h_1(z), h_2(z)) < \varepsilon$, which is absurd. □

As a consequence, we obtain the uniqueness of the semiconjugacy modulo rotations whenever the non-wandering set of f is externally transitive. The reason is the following simple observation.

Lemma 5.4.2. *If $h_1(x) = h_2(x)$ for two semiconjugacies between $f \in \mathrm{Homeo}(\mathbb{T}^2)$ and a rigid rotation of $\mathbb{T}^1$, then $h_1(y) = h_2(y)$ for all y with $x \in cl[\mathcal{O}(y, f)]$.*

Proof. Suppose for a contradiction that $x \in \mathrm{cl}[\mathcal{O}(y, f)]$ but $h_1(y) \neq h_2(y)$. Let $\varepsilon = d(h_1(y), h_2(y))/2$ and $\delta > 0$ such that if $x' \in B_\delta(x)$ then $h_1(x'), h_2(x') \in B_\varepsilon(h_1(x))$. Further, let $n \in \mathbb{N}$ be such that $z := f^n(y) \in B_\delta(x)$. Then on one hand $h_1(z), h_2(z) \in B_\varepsilon(h_1(x))$, and on the other hand $d(h_1(z), h_2(z)) = d(h_1(y), h_2(y)) = 2\varepsilon$, which is absurd. □

Given $f \in \mathrm{Homeo}(\mathbb{T}^2)$ we denote its non-wandering set by $\Omega(f)$. Since any orbit accumulates in the non-wandering set, the combination of Proposition 5.4.1 and Lemma 5.4.2 immediately yields

Corollary 5.4.3. *Suppose that $f \in \mathrm{Homeo}(\mathbb{T}^2)$ is semiconjugate to a rigid rotation of $\mathbb{T}^1$. Further assume that $\Omega(f)$ is externally transitive. Then the semiconjugacy is unique modulo rotations.*

For irrational pseudorotations of the torus,[1] external transitivity of the

[1] That is, torus homeomorphisms homotopic to the identity with unique and totally irrational rotation vector.

non-wandering set was proved by R. Potrie in [Pot12]. Hence, applying Corollary 5.4.3 in both coordinates yields

Corollary 5.4.4. *Let $f \in \mathrm{Homeo}(\mathbb{T}^2)$ be an irrational pseudo-rotation which is seminconjugate to the respective rigid translation of $\mathbb{T}^2$. Then the semiconjugacy is unique up to composing with rigid translations of $\mathbb{T}^2$.*

Finally, one may ask the following.

Question 5.4.5. Does every semiconjugacy between $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and a rigid rotation on $\mathbb{T}^1$ have essential annular continua as fibres?

We note that in the example $f(x,y) = (x + \rho_1, y)$ discussed above this is true, since the fibres of the semiconjugacy h_α are the essential circles $\{(x - \alpha(y), y) \mid y \in \mathbb{T}^1\}$, $x \in \mathbb{T}^1$. By Theorem 5.1.1 it is also true whenever the semiconjugacy is unique modulo rotations, since there always exists one semiconjugacy with this property and the topological structure of the fibres is certainly preserved by post-composition with rotations.

5.5 Fibred rotation number for foliations of circloids

The aim of this section is to prove Theorem 5.1.2. In order to do so, we need some further preliminary results. Given two open connected subsets U, V of a manifold M, we say that $K \subseteq M \setminus (U \cup V)$ *separates* U and V if U and V are contained in different connected components of $M \setminus K$.

Lemma 5.5.1. *Suppose $S \subseteq \mathbb{R}^d$ is a thin horizontal strip and $K \subseteq S$ is a connected closed set that separates $\mathcal{U}^+(S)$ and $\mathcal{U}^-(S)$. Then $C_S \subseteq K$.*

Proof. Suppose $C_S \nsubseteq K$ and let $z \in C_S \setminus K$. Then $B_\varepsilon(z) \subseteq \mathbb{R}^d \setminus K$. However, as $B_\varepsilon(z)$ intersects both $\mathcal{U}^+(S)$ and $\mathcal{U}^-(S)$ by Lemma 5.2.2, this means that $\mathcal{U}^+(S) \cup B_\varepsilon(z) \cup \mathcal{U}^-(S)$ is contained in a single connected component of $\mathbb{R}^d \setminus K$, contradicting the fact that K separates $\mathcal{U}^+(S)$ and $\mathcal{U}^-(S)$. □

Given an essential thin annular continuum $A \subseteq \mathbb{A}$, we denote its lift to $\mathbb{R}^2$ by $\widehat{A} = \pi^{-1}(A)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto (x+1, y)$. Then we say A has a *compact generator*, if there exists a compact connected set $G_0 \subseteq \widehat{A}$ such that $\bigcup_{n\in\mathbb{Z}} G_n = \widehat{A}$, where $G_n = T^n(G_0)$.

Lemma 5.5.2. *If $A \subseteq \mathbb{A}$ is a thin annular continuum with generator G_0, then $G_n \cap G_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$.*

Proof. It suffices to prove that $G_0 \cap G_1 \neq \emptyset$. As $C_{\widehat{A}} = \bigcup_{n\in\mathbb{Z}} G_n \cap C_{\widehat{A}}$ and $C_{\widehat{A}}$ is connected, the compact sets $G_n \cap C_{\widehat{A}}$ cannot be pairwise disjoint. Hence, for some $k \in \mathbb{N}$ we have $G_0 \cap G_k \cap C_{\widehat{A}} \neq \emptyset$ and we can therefore choose a point $z_0 \in G_0 \cap C_{\widehat{A}}$ with $z_k = T^k(z_0) \in G_0 \cap G_k$. If $k = 1$, then $z_1 = T(z_0) \in G_0 \cap G_1$, and we are finished in this case.

Hence, suppose for a contradiction that $k > 1$ and $z_1 = T(z_0) \notin G_0$. Then $B_\varepsilon(z_1) \cap G_0 = \emptyset$ for some $\varepsilon > 0$. Since $B_\varepsilon(z_1)$ intersects both $\mathcal{U}^+(\widehat{A})$ and $\mathcal{U}^-(\widehat{A})$ by Lemma 5.2.2, this means that we can choose a proper curve $\Gamma \subseteq \mathcal{U}^+(\widehat{A}) \cup B_\varepsilon(z_1) \cup \mathcal{U}^-(\widehat{A})$ passing through z_1 such that $\pi_1(\Gamma)$ is bounded and $\pi_2(\Gamma) = \mathbb{R}$. In addition, by choosing Γ as the lift of a proper curve $\gamma \subseteq \mathcal{U}^+(A) \cup B_\varepsilon(\pi(z_1)) \cup \mathcal{U}^-(A) \subseteq \mathbb{A}$ joining $-\infty$ and $+\infty$, we can assume that $T(\Gamma) \cap \Gamma = \emptyset$.

Γ separates $\mathbb{R}^2$ into two open connected components D^- unbounded to the left and D^+ unbounded to the right. As $T(\Gamma)$ is disjoint from Γ we have $T(\mathrm{cl}[D^+]) \subseteq D^+$. This implies that $z_0 \in D^-$, otherwise we would have $z_1 = T(z_0) \in T(\mathrm{cl}[D^+]) \subseteq D^+$, contradicting $z_1 \in \Gamma$. At the same time, we have $z_k = T^{k-1}(z_1) \in T^{k-1}(\mathrm{cl}[D^+]) \subseteq D^+$. However, this means that z_0 and z_k are in different connected components of $\mathbb{R}^2 \setminus \Gamma$, contradicting the connectedness of G_0. □

Given any bounded set $B \subseteq \mathbb{R}^2$, we let

$$\nu_B \;=\; \max\{n \in \mathbb{N} \mid \exists z \in B : T^n(z) \in B\} \;.$$

Lemma 5.5.3. *Suppose $A, A' \subseteq \mathbb{A}$ are thin essential annular continua with compact generators G_0, G_0'. Further, assume $f \in \mathrm{Homeo}_0(\mathbb{A})$ maps A to A'. Then for any lift F of f the set $F(G_0)$ intersects at most $\nu_{G_0} + \nu_{G_0'} + 1$ integer translates of G_0'.*

Proof. Suppose $F(G_0)$ intersects G_n' and G_m' for some $m > n$. Then due to Lemma 5.5.2, the set

$$\bigcup_{k \le n} G_k' \cup F(G_0) \cup \bigcup_{k \ge m} G_k' \;\subseteq\; \widehat{A'}$$

is connected and therefore separates $\mathcal{U}^+(\widehat{A'})$ and $\mathcal{U}^-(\widehat{A'})$. Hence, by Lemma 5.5.1 it contains $C_{\widehat{A'}} = \widehat{C_A'}$. Let $z_0 \in G_0' \cap C_{\widehat{A'}}$ and assume without loss of generality that $z_j \notin G_0'$ for all $j \ge 1$. Then $z_n \in G_n'$ and $z_j \notin \bigcup_{k \le n} G_k'$ for all $j > n$. Furthermore, since $z_m \in G_m'$ we have that $z_j \notin \bigcup_{k \ge m} G_k'$ for all $j < m - \nu_{G_0'}$. Thus, we must have

$$\{z_{n+1}, \ldots, z_{m-\nu_{G_0'}-1}\} \;\subseteq\; F(G_0) \;.$$

However, since $F(G_0)$ contains at most ν_{G_0} integer translates of z_0, this implies $m - n \le \nu_{G_0} + \nu_{G_0'} + 1$. □

As a first consequence, this yields the following.

Corollary 5.5.4. *Let $f \in \mathrm{Homeo}_0(\mathbb{A})$ with lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ and suppose A is a thin essential annular continuum which is compactly generated. Then $f_{|A}$ has a unique rotation number, that is,*

$$\rho_A(F) \;=\; \lim_{n \to \infty} \pi_1 \circ (F^n(z) - z)/n$$

exists for all $z \in \pi^{-1}(A)$ and is independent of z. Moreover, the convergence is uniform in z.

Proof. As $\rho(F,z) = \lim_{n\to\infty} \pi_1 \circ (F^n(z) - z)/n = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ f^i(\pi(z))$ is an ergodic sum with observable $\varphi(z) = \pi_2(F(z) - z)$, we have that $\rho(f,z) = \int_A \varphi \, d\mu =: \rho(\mu)$ μ-a.s. for every f-invariant probability measure supported on A. Note here that φ is well-defined as a function $\mathbb{A} \to \mathbb{R}$. Assume for a contradiction that the rotation number is not unique on A. Then Theorem 5.2.5 implies the existence of two f-invariant ergodic measures μ_1, μ_2 supported on A with $\rho(\mu_1) \neq \rho(\mu_2)$. Consequently, we can choose $z_1, z_2 \in A$ with $\rho(F, z_1) = \rho(\mu_1) \neq \rho(F, z_2) = \rho(\mu_2)$. However, at the same time we may choose lifts $\hat{z}_1, \hat{z}_2 \in G_A$ of z_1, z_2, where G_A is a compact generator of A. Then Lemma 5.5.3 implies that $F^n(\hat{z}_1)$ and $F^n(\hat{z}_2)$ are contained in the union of $2\nu_{G_A} + 1$ adjacent copies of G_A. Consequently, we have that $d(F^n(\hat{z}_1), F^n(\hat{z}_2)) \leq \operatorname{diam}(G_A) + 2\nu_{G_A} + 1$ for all $n \in \mathbb{N}$, a contradiction. The uniform convergence follows from the same argument. □

The following result is a complement of the corollary above.

Corollary 5.5.5. *Let $f \in \operatorname{Homeo}_0(\mathbb{A})$ with lift $F : \mathbb{R}^2 \to \mathbb{R}^2$. Further, suppose A is a f-invariant thin essential annular continuum which is compactly generated and $\rho_A(F) = \{0\}$. Then F has a fixed point in $\pi^{-1}(A)$.*

Proof. Let G_0 be a compact generator of A, and for every $k \in \mathbb{Z}$ the set $G_k := T^k(G_0)$. We first claim that $F^n(G_0) \cap G_0 \neq \emptyset$ for all $n \in \mathbb{Z}$. Otherwise we have $n_0 \in \mathbb{N}$ with either $F^{n_0}(G_0) \subset \bigcup_{k\geq 1} G_k$ or $F^{n_0}(G_0) \subset \bigcup_{k\leq -1} G_k$. Let us consider the first situation, for the complementary one a symmetric argument gives the contradiction. Then, for every $j \in \mathbb{N}$ we have that $F^{jn_0}(G_0) \subset \bigcup_{k\geq j} G_k$. This, however, implies that the rotation number of A is strictly positive, a contradiction.

Consider now $C := \operatorname{cl}[\big(\bigcup_{k\in\mathbb{Z}} F^k(G_0)\big)]$. Thus Lemma 5.5.3 implies that C is a compact and invariant set. Moreover, as A is thin, C is a non-separating continuum. Then, the Cartwright and Littlewood Theorem [CL51] implies the existence of a fixed point of F in C.

□

Remark 5.5.6. We note that as a special case, Corollary 5.5.4 applies to decomposable essential thin circloids. In order to see this, recall that a continuum C is called *decomposable* if it can be written as the union of two non-empty continua C_1 and C_2. If C is a thin circloid, then due to the minimality of circloids C_1 and C_2 have to be non-essential. Hence, connected components $\widehat{C}_i$ of $\pi^{-1}(C_i) \subseteq \mathbb{R}^2$, $i = 1, 2$, are bounded. If these lifts are chosen such that their intersection is non-empty, then $G = \widehat{C}_1 \cup \widehat{C}_2$ is a compact generator of C.

For this special case, Corollary 5.5.4 was already proved in [Cal] by using Caratheodory's Prime End Theory. Examples of (hereditarily) non-decomposable

circloids were constructed by Bing [Bin48] and may occur as minimal sets of smooth surface diffeomorphisms [Han82, Her86].

As Lemma 5.5.3 works for any combination of two compactly generated thin annular continua, we can prove Theorem 5.1.2 in a similar way as the above Corollary 5.5.4. However, what we need as a technical prerequisite is the measurable dependence of the size of the generators of fibres $h^{-1}(\xi)$ under the assumptions of the theorem. We obtain this in several steps. We place ourselves in the situation of Theorem 5.1.2 and assume again without loss of generality that the action $h^* : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^1)$ on the fundamental group is the projection to the second coordinate. This implies that the annular continua $\mathcal{A}_\xi = h^{-1}\{\xi\}$ are all of homotopy type $(1, 0)$. We denote by $\hat{f}$ the lift of f to $\mathbb{A}$ and by F the lift to $\mathbb{R}^2$. Further, we denote by $\hat{h} : \mathbb{A} \to \mathbb{R}$ the lift of h to $\mathbb{A}$ and by $H : \mathbb{R}^2 \to \mathbb{R}$ the lift to $\mathbb{R}^2$.

Let $\Omega_0 = \{\xi \in \mathbb{T}^1 \mid \mathcal{A}_\xi \text{ is thin}\}$, $\Omega = \pi^{-1}(\Omega_0) \subset \mathbb{R}$ and $A_\xi = \hat{h}^{-1}\{\xi\}$ ($\xi \in \mathbb{R}$). Then all A_ξ are essential annular continua in $\mathbb{A}$, and A_ξ is thin if and only if $\xi \in \Omega$. Further, define $A^+_\xi = \partial\mathcal{U}^+(A_\xi)$ and $A^-_\xi = \partial\mathcal{U}^-(A_\xi)$. Then for all $\xi \in \Omega$ we have $A_\xi = A^+_\xi \cup A^-_\xi$ and, by Lemma (5.2.1), $A^+_\xi \cap A^-_\xi = C_{A_\xi} =: C_\xi$.

Lemma 5.5.7. *If A_ξ is thin, then $\lim^{\mathcal{H}}_{\xi' \nearrow \xi} A^-_{\xi'} = \lim^{\mathcal{H}}_{\xi' \nearrow \xi} A_{\xi'} = A^-_\xi$ and $\lim^{\mathcal{H}}_{\xi' \searrow \xi} A^+_{\xi'} = \lim^{\mathcal{H}}_{\xi' \searrow \xi} A_{\xi'} = A^+_\xi$.*

Proof. We prove $\lim^{\mathcal{H}}_{\xi' \nearrow \xi} A^-_{\xi'} = \lim^{\mathcal{H}}_{\xi' \nearrow \xi} A_{\xi'} = A^-_\xi$, the opposite case follows by symmetry. Since $A^-_{\xi_n} \subseteq A_{\xi_n}$, it suffices to show that for all $\varepsilon > 0$ there exists $\delta > 0$ such for all $\xi' \in (\xi - \delta, \xi)$ we have

$$A_{\xi'} \subseteq B_\varepsilon(A^-_\xi) \quad \text{and} \quad A^-_\xi \subseteq B_\varepsilon(A^-_{\xi'}) \ . \tag{5.5.1}$$

We start by showing the first inclusion. Fix $\varepsilon > 0$. Assume for a contradiction that there exists a sequence $\xi_n \nearrow \xi$ such that $A_{\xi_n} \subsetneq B_\varepsilon(A^-_\xi)$ for all $n \in \mathbb{N}$. Let $z_n \in A_{\xi_n} \setminus B_\varepsilon(A^-_\xi)$ and $z = \lim_{n\to\infty} z_n$. Then $z \notin B_\varepsilon(A^-_\xi)$ and thus, since all the z_n are below A_ξ, we have $z \notin A_\xi$. However, at the same time $h(z) = \lim_{n\to\infty} h(z_n) = \lim_{n\to\infty} \xi_n = \xi$, a contradiction.

Conversely, in order to show the second inclusion in (5.5.1), assume for a contradiction that there exists a sequence $\xi_n \nearrow \xi$ such that $A^-_\xi \subsetneq B_\varepsilon(A^-_{\xi_n})$ for all $n \in \mathbb{N}$. Let $K_n = A^-_\xi \setminus B_\varepsilon(A^-_{\xi_n})$. Then $(K_n)_{n\in\mathbb{N}}$ is a decreasing sequence of non-empty compact sets, such that $K = \bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$. Note here that $A^-_\xi \cap B_\varepsilon(A^-_{\xi_n}) = A^-_\xi \cap B_\varepsilon(\mathcal{U}^-(A^-_{\xi_n}))$, which is increasing in n. Let $z \in K$. Then $B_\varepsilon(z) \cap A^-_{\xi_n} = \emptyset$ and thus $B_\varepsilon(z) \subseteq \mathcal{U}^+(A^-_{\xi_n})$ for all $n \in \mathbb{N}$. This implies $h(z') \geq \xi$ for all $z' \in B_\varepsilon(z)$, contradicting the fact that $B_\varepsilon(z)$ intersects $\mathcal{U}^-(A_\xi)$ and $h < \xi$ on $\mathcal{U}^-(A_\xi)$. □

Given a compactly generated thin annular continuum A, we let

$$\tau(A) \ = \ \inf\{\operatorname{diam}(G) \mid G \text{ is a compact generator of } A\} \ .$$

Lemma 5.5.8. *$\xi \mapsto \tau(A_\xi^-)$ is lower semi-continuous from the left on Ω, that is,*

$$\liminf_{\xi' \nearrow \xi} \tau(A_{\xi'}^-) \;\geq\; \tau(A_\xi^-) \quad \text{for all } \xi \in \Omega \;.$$

Similarly, $\xi \mapsto \tau(A_\xi^+)$ is lower semi-continuous from the right on Ω.

Proof. Let $\xi_n \nearrow \xi$ and assume without lose of generality that $\tau := \lim_{n\to\infty} \tau(A_{\xi_n}^-)$ exists and is finite. Choose generators G_{ξ_n} of A_{ξ_n} of diameter smaller than $\tau(A_{\xi_n}^-) + \frac{1}{n}$. Then, using Lemma 5.5.7, it is straightforward to verify that any limit point G of $(G_{\xi_n})_{n\in\mathbb{N}}$ in the Hausdorff metric is a compact generator of A_ξ^- of diameter smaller than τ. □

It is easy to check that real-valued functions which are lower semi-continuous from one side are also measurable. Consequently, since $\tau(A_\xi) \leq \eta(\xi) := \tau(A_\xi^-) + \tau(A_\xi^+)$, the function η provides a measurable majorant for the minimal diameter of the generators of A_ξ. Further, $A_\xi^\pm$ are compactly generated if and only if A_ξ is compactly generated, a fact which follows from the topological considerations on thin annular continua exposed in the next section, see Lemma 5.6.8(iv). Altogether, this yields

Corollary 5.5.9. *Assume that A_ξ has compact generator for almost every $\xi \in \mathbb{T}^1$. Then the map $\xi \mapsto \tau(A_\xi)$ has a measurable finite-valued majorant.*

We are now in the position to complete the

Proof of Theorem 5.1.2. Let $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ and suppose $h : \mathbb{T}^2 \to \mathbb{T}^1$ is a semiconjugacy to the irrational rotation R_ρ. We assume again that $h^* = \pi_2^*$, such that there exist a continuous lift $\hat{h} : \mathbb{A} \to \mathbb{R}$ of h and a lift $\hat{f} : \mathbb{A} \to \mathbb{A}$ of f which satisfy

$$\hat{h} \circ \hat{f} \;=\; R_\rho \circ \hat{h} \;.$$

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f. Assume for a contradiction that f has no unique rotation number. As in the proof of Corollary 5.5.4, this implies the existence of two f-invariant ergodic probability measures μ_1 and μ_2 with

$$\rho_1 \;=\; \int_{\mathbb{T}^2} \pi_2(F(z) - z)\; d\mu_1(z) \;\neq\; \int_{\mathbb{T}^2} \pi_2(F(z) - z)\; d\mu_2(z) \;=\rho_2 \;.$$

As $h^{-1}\{\xi\}$ is compactly generated for Lebesgue-a.e. $\xi \in \mathbb{T}^1$, Corollary 5.5.9 yields the existence of a finite-valued measurable majorant of $\xi \mapsto \tau(h^{-1}\{\xi\})$. Hence, we can find a constant $C > 0$ and a set $\Omega_C \subseteq \mathbb{T}^1$ of positive measure such that for all $\xi \in \Omega_C$ the annular continuum $h^{-1}\{\xi\}$ has a compact generator G_ξ with $\mathrm{diam}(G_\xi) \leq C$.

Both μ_1 and μ_2 must be mapped to the Lebesgue measure on $\mathbb{T}^1$ by h, since this is the only invariant probability measure of R_ρ. Hence, for almost every

$\xi \in \mathbb{T}^1$ there exist points $z_1, z_2 \in h^{-1}\{\xi\}$ which are generic with respect to μ_1 and μ_2, respectively. In particular, for any lift $\hat{z}_i \in \mathbb{R}^2$ of z_i we have that

$$\lim_{n\to\infty} \pi_2(F^n(\hat{z}_i) - \hat{z}_i)/n \;=\; \rho_i \quad (i = 1, 2)\ . \tag{5.5.2}$$

Without loss of generality, we may assume that $h^{-1}\{\xi\}$ has compact generator G_ξ and $R^n_\rho(\xi)$ visits Ω_C infinitely many times, say, $r^{n_i}_\rho(\xi) \in \Omega_C$ for a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ of integers. Given lifts $\hat{z}_1, \hat{z}_2 \in G_\xi$ of z_1, z_2, Lemma 5.5.3 implies that

$$\pi_2(F^{n_i}(\hat{z}_1)) - \pi_2(F^{n_i}(\hat{z}_2) \;\le\; \operatorname{diam}(G_{r^{n_i}_\rho(\xi)}) + \nu_{G_\xi} + \nu_{G_{r^{n_i}_\rho(\xi)}} + 1 \;\le\; \nu_{G_\xi} + 2C + 1$$

for all $i \in \mathbb{N}$. As $\rho_1 \neq \rho_2$, this contradicts (5.5.2). □

5.6 Comments on the Topology and Rotation Sets of thin annular continua.

Given $X \subset \mathbb{R}^2$ we denote by $[X]_y$ the connected component of y in X and define $h(X) = \sup\{x_1 - x_2 \mid (x_1, y_1), (x_2, y_2) \in X\}$. For an essential annular continuum $A \subset \mathbb{A}$, we define the set of *spikes* of A as

$$\mathcal{S}_A := \{[(\hat{A} \setminus C_{\hat{A}})]_x \mid x \in \hat{A} \setminus C_{\hat{A}}\}$$

and say that A has an infinite spike if there exists $S \in \mathcal{S}_A$ with $h(S) = \infty$. Further we let $H_{\mathcal{S}_A} := \sup\{h(S) \mid S \in \mathcal{S}_A\}$. The main result of this section is the following.

Theorem 5.6.1. *Let $A \subset \mathbb{A}$ be an essential thin annular continuum. Then the following holds.*

(1) If A is not compactly generated then either

- *(1a) C_A is not compactly generated, or*
- *(1b) C_A is compactly generated and A contains an infinite spike.*

(2) If A compactly generated, then so is C_A.

The proof is given in Section 5.6.1 below. As a example in the class of continua given in (1a), we can regard the *Birkhoff attractor*. This is an essential thin circloid which has a segment as a rotation set for some map that leaves it invariant (see e.g. [LC88]). Hence, due to Corollary 5.5.4 the Birkhoff attractor cannot have a compact generator. For the class given in (1b) we can consider the continuum given by $A = \pi\left(\mathbb{R} \times \{0\} \cup \left\{\left(x, \frac{1}{x}\right) \in \mathbb{R}^2 \mid x \geq 1\right\}\right)$, which contains the infinite spike $S = \left\{\left(x, \frac{1}{x}\right) \in \mathbb{R}^2 \mid x \geq 1\right\}$.

Note that Theorem 5.6.1 does not rule out the coexistence of an infinite spike and a compact generator. In fact, this may happen, and a way to construct such examples is the following. Let $I = [0,1] \times \{0\}$ and $J = \{0\} \times [0,1]$. We consider $K = J \cup I \cup T(J)$. Fix $x_0 \in J \setminus I$ and $x_1 = T(x_0)$ and and let $\gamma : [0, +\infty) \to \{(x,y) \in \mathbb{R}^2 | 0 < x < 1, y > 0\}$ be an injective curve that verifies

(i) $\gamma([n, +\infty)) \subset B_{\frac{1}{n}}(K)$ for every $n \in \mathbb{N}$;

(ii) $\lim_i \gamma(t_i) = x_0$, $\lim_j \gamma(t_j) = x_1$ for two strictly increasing sequences of positive integers $(t_i)_{i\in\mathbb{N}}, (t_j)_{j\in\mathbb{N}}$.

Now let $A = \pi(\tilde{A})$ where $\tilde{A} := \bigcup_{n\in\mathbb{Z}} T^n(K \cup \gamma)$. It is easy to see that A is a thin essential annular continuum. Furthermore the set $G = K \cup \gamma$ is compact and connected, and hence a generator of A. Finally the set $S := \tilde{A} \setminus (\mathbb{R} \times \{0\})$ is connected since $S = \bigcup_{n\in\mathbb{Z}} T^n((J \cup T(J) \setminus I) \cup \gamma)$. Hence A has compact generator G and at the same time contains the infinite spike S. What is not clear to us is whether similar examples can be produced with an infinite spike that is not T-invariant.

Question 5.6.2. Suppose A is a thin annular continuum which contains an infinite spike S with $T^n(S) \cap S = \emptyset$ for all $n \in \mathbb{N}$. Does this imply that A has no compact generator?

As seen in Corollary 5.5.4, an invariant compactly generated thin annular continuum has a unique rotation vector. As the Birkhoff attractor shows, this is not true if C_A has no generator. For the case of thin annular continua which are not compactly generated, but have a compactly generated circloid, the situation is similar.

Proposition 5.6.3. *Given any segment $I \subset \mathbb{R}$, there exists a map $f \in$ Homeo(A) which leaves invariant an essential thin annular continuum $A \subset \mathbb{A}$ such that C_A has compact generator, A has an infinite spike, and $\rho_A(f) = I$.*

The proof, which is similar to a construction by Walker [Wal91] of some examples in the context of prime end rotation, is given in Section 5.6.2.

5.6.1 Topology of thin annular continua: Proof of Theorem 5.6.1.

The proof of Theorem 5.6.1 mainly hinges on a number of purely plane-topological lemmas. In order to state and prove them, we need to introduce further notation. Given a Jordan curve $\gamma \subset \mathbb{R}^2$ we denote its Jordan domain by D_γ. Further, for two curves α, β in the plane with $\alpha(0) = \beta(1)$, we denote their concatenation by $\alpha \cdot \beta$. We say an arc is an injective curve $\alpha : [0,1] \to \mathbb{R}^2$ and define $\overset{\circ}{\alpha} := \alpha \setminus \{\alpha(0), \alpha(1)\}$. We will sometimes abuse terminology by identifying the function defining a curve and its image. The diameter of a

set $X \subset \mathbb{R}^2$ is denoted by $\operatorname{diam}(X)$. As before, given two sets $A, B \subset \mathbb{R}^2$ we denote the Hausdorff distance by $d_{\mathcal{H}}(A, B)$. Our first step is to prove that the existence of a generator for an essential thin annular continuum A implies the existence of a generator for the circloid C_A. We divide the proof into several lemmas.

Lemma 5.6.4. *Given a bounded set $X \subset \mathbb{R}^2$ and two positive numbers $\delta_1 < \delta_2$, there exists a neighborhood $U_{\delta_1\delta_2}(X)$ of X which verifies*

(i) $B_{\delta_1}(X) \subset U_{\delta_1\delta_2}(X) \subset B_{\delta_2}(X)$;

(ii) $U_{\delta_1\delta_2}(X) = \bigcup_{i=1}^{n} D_i$ *where* $\{cl[D_i]\}_{i=1}^{n}$ *is a family of pairwise disjoint topological closed disks with $\mathcal{C}^1$-boundaries ∂D_i.*

Proof. Let $\Delta : \mathbb{R}^2 \to \mathbb{R}$ be given by $\Delta(y) = d(y, \operatorname{cl}[X])$. Let $\varepsilon_1 = \min\left\{\frac{\delta_1}{10}, \frac{\delta_2 - \delta_1}{10}\right\}$, such that $a := \delta_1 + \varepsilon_1 < \delta_2 - \varepsilon_2 =: b$. Further, let $\Psi : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function such that $d_0(\Psi, \Delta) < \varepsilon_1$. Then $\Psi^{-1}((-\infty, a)) \supset B_{\delta_1}(\operatorname{cl}[X])$ and $\Psi^{-1}((-\infty, b)) \subset B_{\delta_2}(\operatorname{cl}[X])$.

Consider now a regular value [2] $r \in (a, b)$ of Ψ which exists due to Sard's Theorem [GP74, Chapter 1, Section 7]. Then we have that $\Psi^{-1}(r)$ is a union of pairwise disjoint C^1 Jordan curves $\{C_j\}_{j=1}^{m}$. Hence, $U_{\delta_1\delta_2}(X) := \Psi^{-1}((-\infty, r))$ has all the desired properties. Note here that $X \subseteq \Psi^{-1}((-\infty, r))$ and $\partial\Psi^{-1}((-\infty, r)) \subseteq \Psi^{-1}(\{r\})$. □

Lemma 5.6.5. *Let B_1, B_2 be two open sets in $\mathbb{R}^2$ with $d(B_1, B_2) = \varepsilon_0 > 0$ and fix $\varepsilon_1 > 0$. Then there exists $\xi = \xi(\varepsilon_0, \varepsilon_1) > 0$ such that given any pair of arcs α and β which verify*

(i) $\alpha(0) = \beta(1) \in B_1$, $\alpha(1) = \beta(0) \in B_2$ *and $\alpha \cdot \beta$ is a Jordan curve,*

(ii) $d(\alpha \setminus (cl[B_1 \cup B_2]), \beta \setminus (cl[B_1 \cup B_2])) \geq \varepsilon_1$,

there exists $z \in D_{\alpha\cdot\beta}$ for which $B_\xi(z) \subset D_{\alpha\cdot\beta} \setminus (cl[B_1 \cup B_2])$. (See Figure 5.6.1).

Proof. Let $\delta_1 = \frac{\varepsilon_0}{10}$ and $\delta_2 = \frac{\varepsilon_0}{5}$. We consider $U_1 := U_{\delta_1\delta_2}(B_1)$ and $U_2 := U_{\delta_1\delta_2}(B_2)$ given by the previous lemma. Due to the fact that continuous arcs can be approximated by C^1 arcs, we may assume that the arcs α, β are C^1. Furthermore, for similar reasons we may assume that the curves α, β are transverse to each Jordan curve in $\partial U_1 \cup \partial U_2$.

Due to U_1 being disjoint of U_2, we have a C^1 loop $\theta \subset \partial U_1$ that separates $\alpha(0)$ from $\alpha(1)$. We claim that there exists a subarc $J \subseteq \theta$ such that

(a) $J(0) \in \alpha \cap \theta$ and $J(1) \in \beta \cap \theta$;

(b) $\overset{\circ}{J} \subset D_{\alpha\cdot\beta}$.

[2] That is, a value r such that for all $y \in \Psi^{-1}(r)$ the derivative matrix $D\Psi(y)$ is surjective.

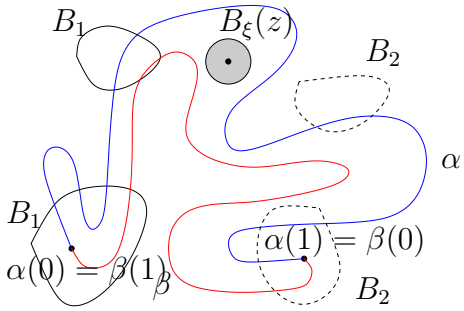


Figure 5.6.1:

In order to see this, fix a point $x_0 \in \theta$ outside of $\mathrm{cl}[D_{\alpha\cdot\beta}]$ and assume, by reparametrising if necessary, that $\theta(0) = x_0$. Then, if J with the above properties does not exist, this means that every time that θ goes into $D_{\alpha\cdot\beta}$ by passing through α, it goes out of $D_{\alpha\cdot\beta}$ through α as well. Consequently, the algebraic intersection number $[\alpha] \wedge [\theta]$ between the homotopy classes $[\alpha], [\theta]$ relative to the points $x_0, \alpha(0), \alpha(1)$, is zero. However, this is a contradiction since θ separates $\alpha(0)$ and $\alpha(1)$, so $[\alpha] \wedge [\theta] = 1$ (compare [GP74, Chapter 3]).

Therefore, we can consider an arc J with the above properties. Let $\xi := \min\{\frac{\varepsilon_0}{10}, \frac{\varepsilon_1}{2}\}$ and consider a bijective parametrisation $\gamma : [0,1] \to J$ with $\gamma(0) = z_0 \in \alpha$ and $\gamma(1) = z_1 \in \beta$ (see Figure 5.6.2).

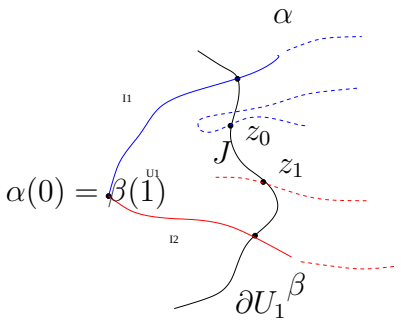


Figure 5.6.2:

Let $\lambda := \sup\{t \in [0,1] \mid d(\gamma(t), \alpha) < \xi\}$ which exists since 1 is an upper bound of $\{t \in [0,1] \mid d(\gamma(t), \alpha) < \xi\}$. Otherwise we would have that $d(\alpha, \beta) < \varepsilon_1$ in $(B_1 \cup B_2)^c$.

Then, we have for $z := \gamma(\lambda)$ that $B_\xi(z) \cap \alpha = \emptyset$ and $B_{\xi+\delta}(z) \cap \alpha \neq \emptyset$ for any positive number δ. Thus we have $B_\xi(z) \cap \beta = \emptyset$, otherwise $d(\alpha, \beta) < \varepsilon_1$ in $(\mathrm{cl}[B_1 \cup B_2])^c$ which is impossible. This implies that $B_\xi(z) \subset D_{\alpha\cdot\beta} \setminus (\mathrm{cl}[B_1 \cup B_2])$. □

We say that a compact and connected set $\mathcal{L} \subset \mathbb{R}^2$ separates $\mathbb{R}^2$ if $\mathbb{R}^2 \setminus \mathcal{L}$ has at least two connected components.

Lemma 5.6.6. *Suppose $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are two sequences of arcs in $\mathbb{R}^2$ with the following properties:*

(i) $\alpha_n, \beta_n \subset B_R(0)$ *and* $\alpha_n \cap \beta_n = \emptyset$ *for every* $n \in \mathbb{N}$ *and some* $R \in \mathbb{R}$;

(ii) $\lim^{\mathcal{H}}_{n\to\infty} \alpha_n = \mathcal{L}_1, \lim^{\mathcal{H}}_{n\to\infty} \beta_n = \mathcal{L}_2$;

(iii) $(\alpha_n(0))_{n\in\mathbb{N}}$ *and* $(\beta_n(0))_{n\in\mathbb{N}}$ *converge to* $x_0 \in \mathbb{R}^2$, $(\alpha_n(1))_{n\in\mathbb{N}}$ *and* $(\beta_n(1))_{n\in\mathbb{N}}$ *converges to* $x_1 \in \mathbb{R}^2$ *with* $x_0 \neq x_1$.

Then we have that either $[\mathcal{L}_1 \cap \mathcal{L}_2]_{x_0} = [\mathcal{L}_1 \cap \mathcal{L}_2]_{x_1}$ *or* $\partial(\mathcal{L}_1 \cup \mathcal{L}_2)$ *separates* $\mathbb{R}^2$.

Proof. Suppose $[\mathcal{L}_1 \cap \mathcal{L}_2]_{x_0} \neq [\mathcal{L}_1 \cap \mathcal{L}_2]_{x_1}$. We claim that for some $\delta > 0$ there exist two different connected components B_1', B_2' of $B_\delta(\mathcal{L}_1 \cap \mathcal{L}_2)$, such that $x_0 \in B_1, x_1 \in B_2$. Otherwise for every $\varepsilon > 0$ the points x_0 and x_1 would be in the same connected component of $B_\varepsilon(\mathcal{L}_1 \cap \mathcal{L}_2)$. However, this would imply that $K := \bigcap_{n\in\mathbb{N}} [B_{\frac{1}{n}}(\mathcal{L}_1 \cap \mathcal{L}_2)]_{x_0} \subset \mathcal{L}_1 \cap \mathcal{L}_2$ is a connected set containing x_0 and x_1, contradicting $[\mathcal{L}_1 \cap \mathcal{L}_2]_{x_0} \neq [\mathcal{L}_1 \cap \mathcal{L}_2]_{x_1}$.

We take now $B_1 = B_{\frac{\delta}{2}}(\mathcal{L}_1 \cap \mathcal{L}_2) \cap (B_2')^c$ and $B_2 = B_{\frac{\delta}{2}}(\mathcal{L}_1 \cap \mathcal{L}_2) \cap B_2'$. We have then:

- $d(B_1, B_2) = \varepsilon_0 > 0$;
- $\mathcal{L}_1 \cap \mathcal{L}_2 \subset B_1 \cup B_2$;
- $x_0 \in B_1$ and $x_1 \in B_2$.

For every $n \in \mathbb{N}$ we consider small arcs a_n, b_n, c_n and d_n such that a_n joins $\alpha_n(1)$ with x_1, b_n joins x_1 with $\beta_n(1)$, c_n joins $\beta_n(0)$ with x_0 and d_n joins x_0 with $\alpha_n(0)$. Further we may request that $\gamma_n := \alpha_n \cdot a_n \cdot b_n \cdot \beta_n^{-1} \cdot c_n \cdot d_n$ is a C^1 Jordan curve, and that $\lim^{\mathcal{H}}_{n\to\infty} a_n \cdot b_n = x_1$, $\lim^{\mathcal{H}}_{n\to\infty} c_n \cdot d_n = x_0$ (See Figure 5.6.3).

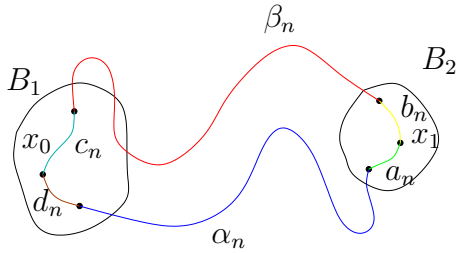


Figure 5.6.3:

Let $\omega_n := d_n \cdot \alpha_n \cdot a_n$ and $\theta_n := b_n \cdot \beta_n \cdot c_n$, $D_n := D_{\omega_n \cdot \theta_n} = D_{\gamma_n}$. Since $\mathcal{L}_1 \cap \mathcal{L}_2$ is contained in $B_1 \cup B_2$ and we have $\lim^{\mathcal{H}}_{n\to\infty} \omega_n = \mathcal{L}_1$ and $\lim^{\mathcal{H}}_{n\to\infty} \theta_n = \mathcal{L}_2$, there exists $\varepsilon_1 > 0$ such that for every $n \in \mathbb{N}$

$$d(\omega_n \setminus (\mathrm{cl}[B_1 \cup B_2]), \theta_n \setminus (\mathrm{cl}[B_1 \cup B_2])) \geq \varepsilon_1$$

Consequently, due to Lemma 5.6.5 there exists a positive number ξ and a sequence of points $(z_n)_{n\in\mathbb{N}}$ such that $B_\xi(z_n) \subset D_n \setminus (\mathrm{cl}[B_1 \cup B_2])$ for every $n \in \mathbb{N}$. Therefore, if we consider a subsequence $(D_{n_i})_{i\in\mathbb{N}}$ which converges to D_0, we have the existence of some $z_0 \in D_0$ such that $B_\xi(z_0) \subset D_0 \setminus (\mathrm{cl}[B_1 \cup B_2])$. By construction, every curve joining $z \in B_\xi(z_0)$ with ∞ intersects $\partial D_0 \subset \partial(\mathcal{L}_1 \cup \mathcal{L}_2)$. Hence $\partial(\mathcal{L}_1 \cup \mathcal{L}_2)$ separates $\mathbb{R}^2$ into at least two components. □

Lemma 5.6.7. *Let $D \subset \mathbb{R}^2$ be a smooth disk such that $\mathrm{int}(D) \cap \mathrm{int}(T(D)) \neq \emptyset$. Further consider the embedding of the real line $\Gamma_1 = \partial\mathcal{U}^-\left(\bigcup_{n\in\mathbb{Z}} T^n(cl[D])\right)$. Then, for every $x \in \Gamma_1$ and arc $\gamma_x \subseteq \Gamma_1$ with end points $x, T(x)$, we have $h(\gamma_x) < 4 \cdot diam(D) + 4$. The analogous statement holds for $\Gamma_2 = \partial\mathcal{U}^+\left(\bigcup_{n\in\mathbb{Z}} T^n(cl[D])\right)$.*

Proof. We first fix $x \in \Gamma_1$ with the property that $\pi_2(x) = \max \pi_2(\Gamma_1)$. Let $D_k := T^k(D)$. Suppose for a contradiction that $h(\gamma_x) \geq 2 \cdot \mathrm{diam}(D) + 2$. Let $z \in \gamma_x$ maximize the function $|\pi_1(.) - \pi_1(x)| : \gamma_x \to \mathbb{R}$. Then $|\pi_1(z) - \pi_1(x)| \geq \mathrm{diam}(D) + 1$. Further, due to the T-invariance of Γ_1 and the choice of x we have that γ_x verifies $\gamma_x \cap (\mathbb{R} \times \{\pi_2(x)\}) = \gamma_x \cap s_x$, where s_x is the interval joining x with $T(x)$.

Let $\Delta_x \subset \gamma_x$ be an arc containing z, such that $\Delta_x(0) = x_0, \Delta_x(1) = x_1 \in s_x$ and $\Delta_x(t) \notin s_x$ for all $t \in (0,1)$. Further, let I_x be the segment joining x_1 with x_0 and $\alpha_x := \Delta_x \cdot I_x$. Then $D_{\alpha_x} \subset \mathcal{U}^+(\Gamma_1)$ since $\overset{\circ}{I}_x \subset \mathcal{U}^+(\Gamma_1)$. This implies for D_k with $z \in \partial D_k$ that $D_k \subset D_{\alpha_x}$. Otherwise we would have $D_k \cap \overset{\circ}{I}_x \neq \emptyset$ which implies that $\mathrm{diam}(D_k) > \mathrm{diam}(D)$. On the other hand, we have by construction that $T(D_{\alpha_x}) \cap D_{\alpha_x} = \emptyset$. Hence $T(D_k) \cap D_k = \emptyset$, which is absurd.

Given now any point $y \in \Gamma_1$ we find $v = (n, 0)$ with $n \in \mathbb{Z}$ such that $y \in \gamma_{x+v}$. Thus $T(y) \in \gamma_{x+v+(1,0)}$, and γ_y is an arc contained in $\gamma_{x+v} \cup \gamma_{x+v+(1,0)}$. This implies that $h(\gamma_y) < 2 \cdot h(\gamma_x) < 4 \cdot \mathrm{diam}(D) + 4$. □

Lemma 5.6.8. *Let $A \subset \mathbb{A}$ be an essential thin annular continuum with compact generator. Then, there exist two sequences of arcs $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ which verify:*

(i) $\alpha_n \subset \mathcal{U}^-(\hat{A})$ and $\beta_n \subset \mathcal{U}^+(\hat{A})$ for every $n \in \mathbb{N}$;

(ii) $(\alpha_n(0))_{n\in\mathbb{N}}$, $(\beta_n(0))_{n\in\mathbb{N}}$ converges to $x_0 \in C_{\hat{A}}$ and $(\alpha_n(1))_{n\in\mathbb{N}}$, $(\beta_n(1))_{n\in\mathbb{N}}$ converges to $x_1 = T(x_0)$;

(iii) $\lim^{\mathcal{H}}_{n\to\infty} \alpha_n = \mathcal{L}_1$ and $\lim^{\mathcal{H}}_{n\to\infty} \beta_n = \mathcal{L}_2$ exist;

(iv) $\mathcal{L}_1$ and $\mathcal{L}_2$ are generators of $A^- = \partial\mathcal{U}^-(A)$ and $A^+ = \partial\mathcal{U}^+(A)$, respectively, and $G := \mathcal{L}_1 \cup \mathcal{L}_2$ is a generator of A.

Proof. Let G' be a generator of A. Due to the Riemann Mapping Theorem applied in $\mathbb{R}^2 \cup \{\infty\} \setminus G'$, we can consider a sequence of smooth Jordan curves $(\gamma_n)_{n\in\mathbb{N}}$ such that

(1) $\gamma_n \subset \mathbb{R}^2 \setminus G'$ for every $n \in \mathbb{N}$;

(2) $\lim^{\mathcal{H}}_{n\to\infty} \gamma_n = G'$;

(3) $D_{n+1} \subset D_n$, where $D_n = D_{\gamma_n}$ for every $n \in \mathbb{N}$.

Due to (2) we have $\operatorname{diam}(D_n) < K_0$ for some $K_0 \in \mathbb{R}$ and all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, $m \in \mathbb{Z}$ we define $D_{n,m} = T^m(D_n)$, $\mathcal{D}_n = \bigcup_{m\in\mathbb{Z}} D_{n,m}$, and $\Gamma^-_n = \partial\mathcal{U}^-(\mathcal{D}_n), \Gamma^+_n = \partial\mathcal{U}^+(\mathcal{D}_n)$. Notice that the curves Γ^-_n, Γ^+_n are embeddings of the real line of the same form as in the previous lemma. Let $x_0 \in G' \cap C_{\hat{A}}$.

Given $n \in \mathbb{N}$ we consider two arcs $\alpha'_n \subset \Gamma^-_n$ and $\beta'_n \subset \Gamma^+_n$ such that $\alpha'_n(1) = T(\alpha'_n(0))$, $\beta'_n(1) = T(\beta'_n(0))$ and $(\alpha'_n(0))_{n\in\mathbb{N}}, (\beta'_n(0))_{n\in\mathbb{N}}$ converge to x_0. Due to Lemma 5.6.7 we know that α'_n, β'_n are contained in a ball $B_R(x_0)$ where

$$R = 8(K_0 + 1) + \max\{d(\alpha'_n(0), \beta'_n(0)) \mid n \in \mathbb{N}\} .$$

Then by choosing subsequences of $(\alpha'_n)_{n\in\mathbb{N}}, (\beta'_n)_{n\in\mathbb{N}}$ we obtain two sequences $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ that verify statements (i) and (ii) of the theorem and the limits $\mathcal{L}_1$ and $\mathcal{L}_2$ exist. This implies in particular that $G = \mathcal{L}_1 \cup \mathcal{L}_2$ is a subcontinuum of $\hat{A}$. Moreover, as α_n, β_n are generators of $\Gamma^-_{k_n}, \Gamma^+_{k_n}$ for a suitable sequence $(k_n)_{n\in\mathbb{N}}$ and $\left(\lim^{\mathcal{H}}_{n\to\infty} \Gamma^-_{k_n} \cup \lim^{\mathcal{H}}_{n\to\infty} \Gamma^+_{k_n}\right) = \hat{A}$ since A is thin, we have that $\mathcal{L}_1$ is a generator of $\partial\mathcal{U}^-(A)$, $\mathcal{L}_2$ is a generator of $\partial\mathcal{U}^+(A)$ and G is a generator of A. □

As consequence of the above results, we obtain the following.

Corollary 5.6.9. *If an essential thin annular continuum A has generator, then the circloid C_A has generator.*

Proof. Let $G = \partial(\mathcal{L}_1 \cup \mathcal{L}_2)$ be the generator of A given by Lemma 5.6.8. Due to Lemma 5.6.6 we have $G' := [\mathcal{L}_1 \cap \mathcal{L}_2]_{x_0} = [\mathcal{L}_1 \cap \mathcal{L}_2]_{T(x_0)}$. Further, since $\mathcal{L}_1 \subseteq \partial\mathcal{U}^+(A)$ and $\mathcal{L}_2 \subseteq \partial\mathcal{U}^-(A)$ we have that $\mathcal{L}_1 \cap \mathcal{L}_2 \subseteq C_A$ by Lemma 5.2.2. This implies that $C := \pi\left(\bigcup_{n\in\mathbb{N}} T^n(G')\right)$ is an essential annular continuum contained C_A, and hence $C = C_A$ since C_A is a circloid. Thus G' is a generator for C_A. □

We now consider the family $\mathcal{A}$ of all the essential thin annular continua, and let $\mathcal{A}_1$ be the family of all those $A \in \mathcal{A}$ such that C_A has no generator and $\mathcal{A}_2 = \mathcal{A} \setminus \mathcal{A}_1$. Then due to the last corollary $\mathcal{A}_2$ contains those $A \in \mathcal{A}$ which admit compact generator. In order to finish the proof of Theorem 5.6.1, it remains to show that whenever $A \in \mathcal{A}_2$ has no generator then A contains an infinite spike. We proceed in two steps and start by showing that every finite spike has at least one 'base point' in C_A.

Lemma 5.6.10. *If A is a thin annular continuum and S is a spike of A with $h(S) < \infty$, then $cl[S] \cap C_{\hat{A}} \neq \emptyset$.*

Proof. Fix $x \in S$. For each $n \in \mathbb{N}$ we define $\mathcal{Y}_n$ as the family of all curves $\alpha : [0,1] \to \mathbb{R}^2$ which verify:

(i) $\alpha(0) \in B_{\frac{1}{n}}(x)$;

(ii) $\alpha(1) \in B_{\frac{1}{n}}(C_{\hat{A}})$;

(iii) $\alpha \subset B_{\frac{1}{n}}(\hat{A})$.

Let $\mathbb{R}^2 \cup \{\infty\}$ be the compactification of $\mathbb{R}^2$ by the sphere. Then, we can consider a sequence of curves $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in \mathcal{Y}_{k_n}$ such that:

- $k_n \nearrow \infty$;
- $\mathcal{L} = \lim^{\mathcal{H}}_{n\to\infty} \alpha_n$ exists in $\mathbb{R}^2 \cup \{\infty\}$ (the compactification).

Let $\mathcal{L}_x := [\mathcal{L} \setminus (C_{\hat{A}} \cup \{\infty\})]_x$. Then $\mathcal{L}_x \subset S$, and hence $\infty \notin \mathrm{cl}[\mathcal{L}_x]$ since $h(S) < \infty$, so $\mathrm{cl}[\mathcal{L}_x]$ is compact. By property (ii) of the curves α_n we have $\mathcal{L}_x \cap B_{\frac{1}{n}}(C_{\hat{A}}) \neq \emptyset$ for all $n \in \mathbb{N}$, and hence $\mathrm{cl}[\mathcal{L}_x] \cap C_{\hat{A}} \neq \emptyset$. □

Corollary 5.6.11. *If $A \in \mathcal{A}_2$ and $H_{\mathcal{S}_A} < \infty$, then A has generator.*

Proof. Let G' be a generator of C_A. For every spike S choose $n \in \mathbb{N}$ such that $S' := T^n(S)$ intersects G'. Note that this is possible due to Lemma 5.6.10. Since $H_{\mathcal{S}_A} < \infty$ we have that $G := \mathrm{cl}[\big(G' \cup \bigcup_{S\in\mathcal{S}_A} S'\big)]$ is a compact generator of A. □

Finally, we show that for every $A \in \mathcal{A}_2$ with $H_{\mathcal{S}_A} = \infty$, there exists an infinite spike contained in A.

Proposition 5.6.12. *Let $A \in \mathcal{A}_2$ with $H_{\mathcal{S}_A} = \infty$. Then, there exists an infinite spike $S \in \mathcal{S}_A$.*

Proof. We assume that the supremum $H_{\mathcal{S}_A}$ is obtained by spikes in $\mathcal{U}^-(C_{\hat{A}})$, the other case is symmetric. Suppose for a contradiction that $h(S) < \infty$ for every $S \in \mathcal{S}_A$.

Let $x_0 \in \hat{A} \setminus \mathcal{U}^+(C_{\hat{A}}) = \Big(\hat{A} \cap \mathcal{U}^-(C_{\hat{A}})\Big) \cup C_{\hat{A}}$ such that

$$\pi_2(x_0) = \min\left\{ \pi_2(x) \;\middle|\; x \in \bigcup_{S\in\mathcal{S}_A} S \cap \mathcal{U}^-(C_{\hat{A}}) \right\}.$$

By changing coordinates if necessary, we may assume that $x_0 \notin C_{\hat{A}}$.

Let $\gamma_{x_0}(t) = x_0 + t \cdot (1,0)$ and $S_0 \in \mathcal{S}_A$ such that $x_0 \in S_0$. Then due to Lemma 5.6.10 and the fact that C_A has generator, we can consider a generator L of C_A that verifies $L \cap \mathrm{cl}[S_0] \neq \emptyset$ and $L \cap \mathrm{cl}[T(S_0)] \neq \emptyset$ (see Figure 5.6.4).

Given now any spike $S \subset \mathcal{U}^-(C_{\hat{A}})$ different of S_0, due to the definition of x_0 we have:

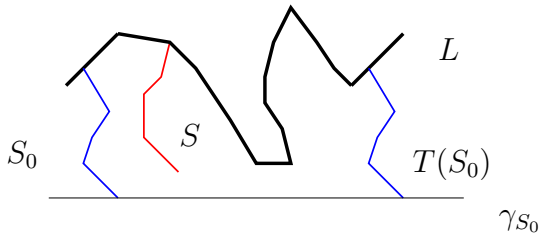


Figure 5.6.4:

$$S \subset \left(\mathrm{cl}[\mathcal{U}^+(\gamma_{x_0})] \cap \mathcal{U}^- \left(\bigcup_{n\in\mathbb{Z}} T^n(L) \right) \right) \setminus \bigcup_{n\in\mathbb{Z}} T^n(S_0).$$

Therefore we have that $h(S) < 2 \cdot h(S_0) + h(L)$. This contradicts $H_{\mathcal{S}_A} = \infty$. □

This last proposition finishes the proof of Theorem 5.6.1.

5.6.2 Rotation sets for thin annular continua.

Let $\mathcal{D} \subset \mathrm{Diffeo}_+(\mathbb{T}^1)$ be the set of orientation-preserving circle diffeomorphisms with a totally disconnected non-wandering set. Note that this means that $f \in \mathcal{D}$ has either rational rotation number and a totally disconnected set of periodic points, or f is a Denjoy example (with irrational rotation number).

The aim of this section is to construct a family of examples of homeomorphisms $f_{g,\alpha}$ of $\mathbb{A}$, parametrized by $g \in \mathcal{D}$ and $\alpha \in \mathbb{R}$ such that

- $f_{g,\alpha}$ leaves invariant some annular continuum $A_{g,\alpha} \in \mathcal{A}_2$ containing at least one infinite spike, and
- $\rho_{A_{g,\alpha}}(F) = \mathrm{conv}(\{\alpha, \rho(g)\})$.[3]

This will prove Proposition 5.6.3.

For any $t \in \mathbb{R}^+$, let $\mathcal{R}_t = \mathbb{R} \times \{t\}$ and define $i : (0, +\infty) \to \mathbb{R}$ by $i(x) = \frac{1}{x}$. Further, let $\mathcal{G} = \{L_p\}_{p\in\mathbb{R}}$ be the C^∞-foliation of $\mathbb{R} \times (0, +\infty]$ whose leaves are given by $L_p = \mathrm{gr}(i) + (p, 0)$ for every $p \in \mathbb{R}$, where $\mathrm{gr}(i) = \{(x, i(x)) | x > 0\}$. Notice that for $(x, y) \in \mathbb{R} \times (0, +\infty)$ the leave through (x, y) is given by $L_{p(x,y)}$ with $p(x, y) = x - \frac{1}{y}$.

Given $g \in \mathcal{D}$, we choose a lift $G : \mathbb{R} \to \mathbb{R}$. Then, we consider $F_1 : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}\times\mathbb{R}^+$ given by $F_1(x, y) = (x+v(x, y), y)$ where $v(x, y) = G(p(x, y)) - p(x, y)$.

[3]Where $\mathrm{conv}(X)$ denotes the convex hull of X, and $\rho(g)$ is the rotation number of g.

Notice that $F_1(L_{p(x,y)}) = L_{p(x,y)+v(x,y)}$ for every $(x,y) \in \mathbb{R} \times \mathbb{R}^+$. Hence, F_1 preserves the set

$$\mathcal{T} := \bigcup_{p \in \pi^{-1}(\Omega(g))} L_p \cap (\mathbb{R} \times [0,1]) \ .$$

Further, F_1 is a C^1 diffeomorphism since p is C^∞ and G is C^1.

Denote by $T_{(x,y)}L_p$ the tangent space of the curve L_p at the point (x,y). Let $X : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \times \mathbb{R}^+$ be the C^1 vector field given by

$$X(x,y) = \lambda(y)(\alpha - v(F_1^{-1}(x,y)), t(x,y)),$$

where:

- $t(x,y)$ is uniquely defined in order to have $X(x,y) \in T_{(x,y)}L_{p(x,y)}$;
- $\lambda : [0,+\infty) \to [0,+\infty)$ is a decreasing C^∞ function which equals 1 in $[0,\frac{1}{2}]$ and 0 in $[1,+\infty)$.

Note that as the leafs of $\mathcal{G}$ becomes horizontal when approach $\mathcal{R}_0$, the component $t(x,y)$ tends uniformly to zero when y goes to zero (see figure 5.6.5).

Let ϕ be the flow associated to X. Then, due to the last argument we have a positive number $y_0 < \frac{1}{2}$ so that $\phi(t,(x,y)) \in \mathbb{R} \times [0,\frac{1}{2}]$ for all $t \in [0,1]$, $x \in \mathbb{R}$, $y \in (0,y_0]$. We consider now the time one map H associated to ϕ. Then, H preserves each leaf of the foliation $\mathcal{G}$ and also the set $\mathcal{T}$. Moreover, for every $(x,y) \in \mathbb{R}^2$ with $y \in (0,y_0]$ we have

$$\pi_1(H(x,y)) = \pi_1(x,y) + \alpha - v(F_1^{-1}(x,y)).$$

Define now $F_{g,\alpha} : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$F_{g,\alpha}(x,y) = \begin{cases} H \circ F_1(x,y) & \text{if} \quad y \in \mathbb{R}^+ \\ (x,y) + (\alpha,0) & \text{if} \quad y \notin \mathbb{R}^+ \end{cases}$$

In order to see that $F_{g,\alpha}$ is a homeomorphism, the non-trivial issue to check is the continuity in points of the form $(x,0) \in \mathbb{R}^2$. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence converging to $(x,0)$. Further we assume that $\pi_2(w_n) \in [0,y_0]$ for all $n \in \mathbb{N}$. Then, we have that $F_{g,\alpha}(w_n) - w_n = (\alpha, d_n)$, where $d_n = \pi_2(\phi(1, F_1(w_n)))$. Hence, as $t(F_1(w_n))$ tends to zero when n goes to $+\infty$, we have that $\lim_n d_n = 0$ and therefore $\lim_n F_{g,\alpha}(w_n) = (x+\alpha, 0)$ (see figure 5.6.5).

Hence, we have that $F_{g,\alpha}$ is a T-invariant homeomorphism where T is the plane translation $x \mapsto x + (1,0)$, and thus it defines a homeomorphism $f_{g,\alpha} : \mathbb{A} \to \mathbb{A}$. Furthermore $f_{g,\alpha}$ leaves invariant the essential thin annular continuum given by $A_{g,\alpha} := \pi(\mathcal{R}_0 \cup \mathcal{T})$, which has infinite spikes and whose circloid $C_{A_{g,\alpha}} = \pi(\mathcal{R}_0) = \mathbb{T}^1 \times \{0\}$ is compactly generated.

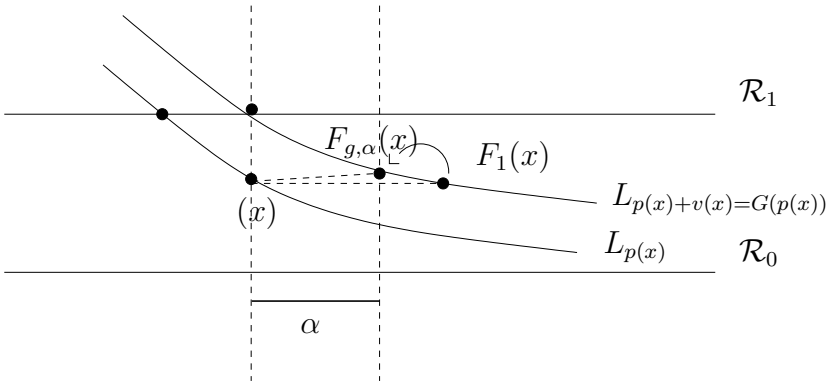


Figure 5.6.5: As the leaves of $\mathcal{G}$ becomes horizontal as x approaches the line $\mathcal{R}_0$, we have that $t(x)$ tends to zero as x approaches the line $\mathcal{R}_0$. Hence, so it does $\pi_2(F_{g,\alpha}(x) - (x))$.

Finally, it follows from the definition of $f_{g,\alpha}$ that points in $A \cap \pi(\mathcal{R}_1)$ have a unique rotation vector $(\rho(G), 0)$, and points in C_A have rotation vector $(\alpha, 0)$. Hence $\rho_A(F) \supset I = \text{conv}(\alpha, \rho(G))$. In order to prove that $\rho_A(F) = I$ we will assume that $\alpha < \rho(G)$. Otherwise we work with f^{-1}.

We claim that for every $x \in \mathcal{T}$ if there exists $\lim_n \frac{\pi_1(F^n_{g,\alpha}(x)-x)}{n}$ then we have $\lim_n \frac{\pi_1(F^n_{g,\alpha}(x)-x)}{n} = \rho(G)$.

As by construction $\pi_1(F^n_{g,\alpha}(x)) > \pi_1(F^n_{g,\alpha}(y))$ where y is the intersection point of $\mathcal{R}_1$ and $L_{p(x)}$. This implies that $\rho(G) = \lim_n \frac{\pi_1(F^n_{g,\alpha}(y)-y)}{n} \leq \lim_n \frac{\pi_1(F^n_{g,\alpha}(x)-x)}{n}$.

Assume now for a contradiction, that $\lim_n \frac{\pi_1(F^n_{g,\alpha}(x)-x)}{n} > \rho(G)$. Thus we necessarily have that $\lim \pi_2(F^n_{g,\alpha}(x)) = 0$, otherwise we find an increasing sequence $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$, so that being y a point as before

$$\rho(G) = \lim_i \frac{\pi_1(F^{n_i}_{g,\alpha}(y) - y)}{n_i} = \lim_i \frac{\pi_1(F^{n_i}_{g,\alpha}(x) - x)}{n_i}$$

which is imposible. Therefore, we have that there is $n_0 \in \mathbb{N}$ so that $F^n_{g,\alpha}(x) \in \mathbb{R} \times [0, y_0]$ for all $n > n_0$. Then $\pi_1(F^{n+1}_{g,\alpha}(x) - F^n_{g,\alpha}(x)) = \alpha$ for all $n > n_0$, which implies by a telescopic argument the absurd $\alpha = \lim_n \frac{\pi_1(F^n_{g,\alpha}(x)-x)}{n}$ (recall that we have assumed $\alpha < \rho(G)$).

Thus, we have proved our claim, and since the endpoints of the rotation interval are always realized by pointwise rotation vectors, we obtain $\rho_{A_{g,\alpha}}(F_{g,\alpha}) = [\alpha, \rho(G)]$.

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Fakultät Mathematik und Naturwissenschaften
Fachrichtung Mathematik

INSTITUT FÜR ANALYSIS

TECHNISCHE UNIVERSITÄT DRESDEN

T. Oertel-Jäger, FR Mathematik, TU Dresden, 01062 Dresden

Dr. Tobias Oertel-Jäger

e-Mail: Tobias.Oertel-Jaeger@tu-dresden.de

Telefon: +49 (351) 46335491

Promotionsbüro der Fakultät für Naturwissenschaften
FAU Erlangen-Nürnberg
Halbmondstrasse 6
91054 Erlangen

Dresden, 31. Juli 2013

Coauthor's statement concerning the PhD thesis of Alejandro Passeggi

To whom it may concern.

I have been the coauthor of two joint articles which form a part of the PhD thesis of Alejandro Passeggi. Concerning his contributions to these collaborations, the following can be said.

- **A classification of minimal sets of torus homeomorphisms.** By Ferry Kwakkel, Tobias Jäger and Alejandro Passeggi. Published in *Mathematische Zeitschrift* 274(1-2):405-426 (2013).
  This project originated from a visit of Ferry Kwakkel to our research group in Dresden. The aim was to extend a previous result of Ferry Kwakkel – namely a classification of minimal sets of so-called *irrational pseudo-rotations*[1] – to general torus homeomorphisms. Once we posed ourselves the initial problem, our joint discussion quickly led to an outline for the proof of the main result. This was facilitated by the fact that some of the arguments of the previous article of Ferry Kwakkel could be carried over to the more general setting with relatively small adjustments. In retrospective, it is difficult to determine exactly who contributed which idea on this level. In any case, it is well-justified to say that either of the three authors would have been able to make similar progress independently at this initial stage.

[1] A particular class of torus homeomorphisms, characterised by the existence of a unique and totally irrational rotation vector.

Postanschrift (Briefe)
Technische Universität Dresden
D – 01062 Dresden

Hausanschrift (Pakete etc.)
Technische Universität Dresden
Mommsenstr. 13
D – 01069 Dresden

Besucheradresse
Willersbau C 217
Zellescher Weg 12 - 14
D – 01069 Dresden

The rigorous implementation of the ideas and the solution of several technical problems was then mainly done by Alejandro Passeggi, with some iterations of discussions, proof-reading and corrections by Ferry Kwakkel and myself. Some additional problems which do not appear in the case of irrational pseudo-rotations were solved by Alejandro Passeggi alone. This concerns, amongst others, the exclusion of the existence of bounded periodic topological disks in one case of the classification and the treatment of torus homeomorphisms homotopic to Anosov maps.

- **On torus homeomorphisms semiconjugate to irrational rotations.** by Tobias Jäger and Alejandro Passeggi, Preprint 2013.
  This problem was posed by myself as a part of the PhD project of Alejandro Passeggi. The principal idea was to generalise a theorem of M. Herman on skew products over irrational rotations to torus homeomorphisms which are extensions of irrational rotations, but not necessarily skew products anymore. For this class, the aim was to prove the uniqueness of the rotation number in the system under certain topological regularity assumptions on the fibres of the semiconjugacy.
  The crucial concept, which eventually made it possible to implement the strategy in a rigorous way, was that of a *compact generator* of a thin annular continuum. This was Alejandro Passeggi's invention. The first part of the article was written jointly. The second part, which contains a classification of so-called thin annular continua, was mainly written by Alejandro Passeggi alone, with only occasional input in the form of discussions and proof-reading by myself. This second part eventually turned out to be crucial for the first part of the article as well, since the results were needed to solve some technical problems in the construction.

Summarising, one can say that Alejandro Passeggi had a major part in both of the mentioned collaborations. In particular, he contributed independent and original ideas and took a leading role in solving many problems on a technical level. My own role was mainly that of a supervisor.

With best regards,

Tobias Oertel-Jäger

Technische Universität Dresden
Fakultät Mathematik und Naturwissenschaften
Fachrichtung Mathematik
Institut für Analysis
01062 Dresden

July 16, 2013

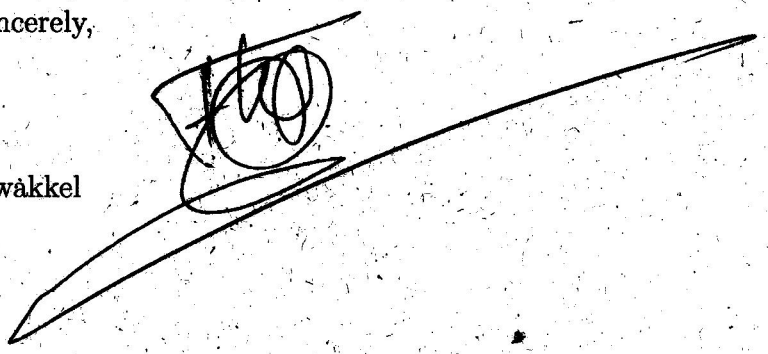
Dr. Tobias Jäger
TU Dresden
Institut für Analysis
Zellescher Weg 12-14
01219 Dresden, Germany

To whom it may concern,

in our joint paper "*A classication of minimal sets of torus homeomorphisms*", authored by Tobias Jäger, Ferry Kwakkel and Alejandro Passeggi, published in Mathematische Zeitschrift 274(1-2):405-426 (2013), an extended version of an earlier paper, published in Fundamenta Mathematica 211-1: 41-76 (2011), from the author's PhD thesis, was given.

This work builds forth upon the original paper, and it required both a rewriting of the results into the more general setting, as well as several technical results to carry out this full extension. Alejandro Passeggi, advised by Dr. Tobias Jäger, independently made good and original technical contributions to the joint paper that enabled this extension, and my main contribution to the paper was to simplify and correct parts of the paper to prepare it for publication, most of which was carried out during a research visit to the Emmy Noether group in Dresden led by Dr. Jäger.

Yours sincerely,



Ferry Kwakkel

COAUTHOR'S STATEMENT ON JOINT CONTRIBUTIONS IN THE PHD THESIS OF ALEJANDRO PASSEGGI

Minimal sets for surface homeomorphisms

In this paper we provide a classification of minimal sets of closed and oriented surfaces homeomorphisms throughout the study of the connected components in its complement. This classification arises as the extension of a classification first proposed by Ferry Kwakkel for irrational pseudo-rotations in the two torus, and later generalized by F. Kwakkel, T. Jäger and Alejandro Passeggi to any homeomorphism in the same space.

This project developed as a consequence of the enthusiasm regarding this topics that Alejandro Passeggi brougth to Uruguay in one of many visits. He wanted to generalize previous work he (and others) had done in the torus to higher genus. This very natural question left place to surprising results that enriched both of our knowledges in the subject. His work was impecable at all levels, starting with encouraging the project, fine mathematical work, writting and further correcting the manuscript.

The results we obtained were very satisfying, and new questions rised that we are still trying to answer at the time.

Summing up, Alejandro Passeggi did a great job in this collaboration, and I am looking forwards to working with him again.

MX
PhD Juliana Xavier
IM.E.R.L
Faultad de Ingeniería
Montevideo, Uruguay