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Research Article

Numerical method for solving system of Fredhlom integral equations using Chebyshev cardinal functions

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Abstract

The focus of this paper is on the numerical solution of linear systems of Fredhlom integral equations of the second kind. For this purpose, the Chebyshev cardinal functions with Gauss-Lobatto points are used. By combination of properties of these functions and the effective Clenshaw-Curtis quadrature rule, an applicable numerical method for solving the mentioned systems is formulated. Some error bounds for the method are computed and its convergence rate is estimated. The method is numerically evaluated by solving some test problems caught from the literature by which the accuracy and computational efficiency of the method will be demonstrated.

Keywords: Chebyshev cardinal functions; Fredholm integral equations system; Numerical solution; Clenshaw-Curtis quadrature rule; Error analysis.

1 Introduction

In recent years, the cardinal functions have been finding an important role in numerical analysis. Especially, valuable efforts have been spent, by researchers, on introducing novel ideas for numerical solution of various functional equations by using the superior properties of these functions. A numerical technique is presented in [1] for solution of a parabolic partial differential equation which is derived by expanding the required approximate solution as the elements of the Chebyshev cardinal functions. [2] presents two numerical techniques for solving Riccati differential equation. These methods use the cubic B-spline scaling functions and Chebyshev cardinal functions. The Chebyshev cardinal functions are also used in [3] for solution of fourth-order integro-differential equations which are reduced to a set of algebraic equations using the operational matrix of derivative. A review on approximate cardinal preconditioning methods for solving partial differential equations by using radial basis functions has been performed in [4], in which, the authors have numerically compared the related preconditioners on some numerical examples of Poisson's, modified Helmholtz, and Helmholtz equations. [5] uses cardinal bases for implementation of some pseudospectral methods, then special cases of differential and integro-differential equations are solved by these bases. In [6], the cardinal interpolation of functions on the real line by splines is determined by certain formula free of solving large or infinite systems. The authors obtain an interpolation projection of the function which asymptotically maintains the

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optimal accuracy of the basic cardinal interpolation on the real line. An approach to identify multivariable Hammerstein systems is presented in [7]. By using cardinal cubic spline functions to model the static nonlinearities and with an appropriate transformation, the nonlinear models are parameterized such that the nonlinear identification problem is converted into a linear one. [8] proposes a pseudospectral method for generating optimal trajectories of linear and nonlinear constrained dynamic systems. The method consists of representing the solution of the optimal control problem by using cardinal functions. Further information regarding the cardinal functions may be found in [9–12].

A great deal of interest has been focused on the solution of linear Fredhlom integral equations systems. [13] proposes the Adomian decomposition method for solving such systems. In [14], numerical solution of system of linear Fredholm integral equations by means of the Sinc-collocation method is considered and the system is replaced by an explicit system of linear algebraic equations. Two other numerical techniques based on using rationalized Haar functions and block-pulse functions (BPFs) are respectively presented in [15] and [16]. A direct method to compute numerical solutions of the linear Volterra and Fredholm integral equations system is proposed in [17], where by using vector forms of triangular functions (TFs), solving of an integral equations system reduces to solve a system of algebraic equations.

This paper proposes a numerical method for solving system of Linear Fredholm integral equations of the second kind. For this purpose, the Chebyshev cardinal functions with Gauss-Lobatto points are used as a set of basis functions. By combination of properties of these functions and Clenshaw-Curtis quadrature rule, an effective numerical method will be formulated for solution of such systems. The main advantages of the presented method are enough accuracy (the numerical results will be compared with those of other methods), quick convergence, and relatively small size of calculations.

The organization of this paper is as follows. A review on the cardinal functions and their properties is provided in section 2 and specific Chebyshev cardinal functions are introduced. Section 3 gives a brief resume of Clenshaw-Curtis quadrature rule as an important tool for implementation of the proposed method. Section 4 presents the numerical method for solving system of Linear Fredholm integral equations of the second kind which is implemented by combination of the properties of the Chebyshev cardinal functions and the Clenshaw-Curtis quadrature rule. An error analysis regarding the proposed method will be done in section 5 where some error bounds are obtained and convergence rate is estimated. Some examples are caught from the literature and provided in section 6 to illustrate the computational efficiency of the method. Comments on the results is the subject of section 7 where, by referring to the obtained results in section 6, the method will be compared with other methods in view of accuracy. Also, the mean-absolute errors associated with the results obtained by the method will be given to confirm its quick convergence. Finally, conclusions will be in section 8.

2 Cardinal functions

Definition 2.1. *A cardinal function $C_j(t)$ for a specific interpolation function and for a set of interpolation points t_i is defined as [1–3, 8, 18, 19]*

$$C_j(t_i) = \delta_{i,j}, \qquad i,j = 1,2,\dots,N, \tag{2.1}$$

where N is the number of the interpolation points and $\delta_{i,j}$ is Kronecker delta defined as

$$\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{2.2}$$

That is to say, the cardinal functions are combination of the underlying basis (trigonometric functions, Chebyshev polynomials, or whatever) which are chosen so that the jth function is equal to one at the jth grid point and vanishes at all the other grid points.

In this paper a specific set of cardinal functions are considered based on the zeros of $(1-t^2)\dot{T}_N(t)$, where $\dot{T}_N(t) = \frac{\mathrm{d}T_N(t)}{\mathrm{d}t}$, such that $T_N(t) = \cos\left(N\cos^{-1}(t)\right)$, for $t \in [-1,1]$, is the Chebyshev polynomial of degree N. We choose these grid (interpolation) points as follows:

$$t_j = \cos\left(\frac{j\pi}{N}\right), \qquad j = 0,1,\dots,N, \tag{2.3}$$

therefore

$$t_N = -1 < t_{N-1} < \cdots < t_1 < t_0 = 1. \tag{2.4}$$

Definition 2.2. *Chebyshev cardinal function with Gauss-Lobatto grids of order N in $[-1,1]$ is defined as [5, 8, 18]*

$$C_j(t) = \frac{(-1)^{j+1}(1-t^2)\dot{T}_N(t)}{c_j N^2 (t-t_j)}, \qquad j = 0,1,\ldots,N, \tag{2.5}$$

with $c_0 = c_N = 2$, and $c_j = 1$, for $1 \leqslant j \leqslant N-1$.

It is easy to show that relation (2.1) is valid for the cardinal function defined by (2.5).

Definition 2.3. *A function $f(t)$ can be approximated in terms of cardinal functions by the following series of the form [5, 8, 18]:*

$$f(t) \simeq f_N(t) = \sum_{j=0}^{N} f(t_j)\, C_j(t), \tag{2.6}$$

such that

$$f_N(t_j) = f(t_j), \qquad j = 0,1,\ldots,N, \tag{2.7}$$

and $f_N(t)$ is a unique Nth-degree interpolating polynomial associated with the $N+1$ Chebyshev Gauss-Lobatto grids.

3 Clenshaw-Curtis numerical quadrature

A numerical quadrature (numerical integration) rule is the basis of many numerical methods for the solution of integral equations. In this section a brief resume of Clenshaw-Curtis quadrature which is important later on is given. For a much fuller treatment of the subject, see for example [20–23].

The Gauss-Chebyshev rules are of special interest as an easy-to-use sequence of Gauss rule. However, in practice the weight function $w(s) = 1$ occurs much more commonly than Chebyshev weight function $w(s) = (1-s^2)^{-\frac{1}{2}}$. It is always possible to write

$$\int_{-1}^{1} f(s)\,\mathrm{d}s = \int_{-1}^{1} \frac{\bar{f}(s)}{(1-s^2)^{\frac{1}{2}}}\,\mathrm{d}s, \tag{3.8}$$

where $\bar{f}(s) = (1-s^2)^{\frac{1}{2}} f(s)$.

However, if $f(s)$ is smooth near $s = \pm 1$, $\bar{f}(s)$ is not, and the direct use of a Gauss-Chebyshev rule on (3.8) will yield results which converge only slowly as the quadrature points or nodes increases. For avoiding this slow convergence, we can use a very effective Clenshaw-Curtis quadrature rule. Here, we present the Clenshaw-Curtis scheme as a "standard" integration rule as follows:

$$\begin{aligned} \int_{-1}^{1} f(s)\,\mathrm{d}s &= \sum_{\substack{n=0 \\ n \text{ even}}}^{N}{}'' \frac{4}{(1-n^2)N} \sum_{k=0}^{N}{}'' \cos\big(\frac{nk\pi}{N}\big) f\big(\cos\frac{k\pi}{N}\big) \\ &= \sum_{k=0}^{N}{}'' w_k f\big(\cos\frac{k\pi}{N}\big), \end{aligned} \tag{3.9}$$

where

$$w_k = \frac{4}{N} \sum_{\substack{n=0 \\ n \text{ even}}}^{N}{}'' \frac{1}{1-n^2} \cos\big(\frac{nk\pi}{N}\big), \qquad k = 0,1,\ldots,N, \tag{3.10}$$

and the notation $\sum''$ means the first and last terms are to be halved before summing.

Remark 3.1.

1. *The resulting formula can be shown to be exact if $f(s)$ is a polynomial of degree $2N-1$.*

2. *The apparent cost of implementing this rule is high; a direct sumation of (3.10) to compute w_k, $k = 0, 1, \ldots, N$, involves a total of $(N+1)^2$ multiplications and additions, compared with only $N+1$ to actually evaluating the sum (3.9). Equation (3.10) for the weights w_k can be viewed as the discrete cosine transformation of a vector* $\mathbf{v}$ *with entries*

$$v_n = \begin{cases} \frac{2}{1-n^2}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \tag{3.11}$$

The weights w_k can therefore be computed using Fast Fourier Transform (FFT) technique in $\mathcal{O}(N \ln N)$ operations; so the rule is reasonable in cost and very stable against rounding errors [20, 24].

4 Numerical solution of Fredholm integral equations system

Let us consider the following system of linear Fredholm integral equations:

$$\mathbf{U}(s)\,\mathbf{X}(s) = \mathbf{F}(s) + \int_a^b \mathbf{K}(s,t)\,\mathbf{X}(t)\,\mathrm{d}t, \qquad s \in [a,b], \tag{4.12}$$

where

$$\begin{aligned} &\mathbf{U}(s) = \left[u_{i,j}(s)\right], \qquad i,j = 1,2,\ldots,n, \\ &\mathbf{F}(s) = \left[f_1(s), f_2(s), \ldots, f_n(s)\right]^T, \\ &\mathbf{X}(s) = \left[x_1(s), x_2(s), \ldots, x_n(s)\right]^T, \\ &\mathbf{K}(s,t) = \left[\lambda_{i,j}\, k_{i,j}(s,t)\right], \qquad i,j = 1,2,\ldots,n, \end{aligned} \tag{4.13}$$

and superscript T indicates transposition. In (4.12), the parameters $\lambda_{i,j}$, the functions $f_i(s)$, $u_{i,j}(s)$, and $k_{i,j}(s,t)$, for $i,j = 1,2,\ldots,n$, are known, and $x_i(s)$, for $i = 1,2,\ldots,n$, are the unknown functions to be determined. Also, $k_{i,j}(s,t) \in \mathscr{L}^2\left([a,b]\times[a,b]\right)$, and $f_i(s), x_i(s), u_{i,j}(s) \in \mathscr{L}^2\left([a,b]\right)$, where $\mathscr{L}^2$ is the space of square integrable functions. Moreover, we assume that at least one component of any row of matrix $\mathbf{U}$ is non-zero on $[a,b]$.
Without loss of generality, it is supposed that $a = -1$ and $b = 1$, since any finite interval $[a,b]$ can be transformed to interval $[-1,1]$ by linear maps.
For convenience, let us consider the ith equation of (4.12) whom we can write as

$$\sum_{j=1}^{n} u_{i,j}(s)\,x_j(s) = f_i(s) + \sum_{j=1}^{n} \lambda_{i,j} \int_{-1}^{1} k_{i,j}(s,t)\,x_j(t)\,\mathrm{d}t. \tag{4.14}$$

Approximating the solution $x_j(s)$ by the Chebyshev cardinal functions from Eqs. (2.5) and (2.6) gives

$$x_j(s) \simeq x_{j,N}(s) = \sum_{k=0}^{N} a_{j,k} C_k(s), \tag{4.15}$$

where $a_{j,k} = x_j(t_k)$, and t_k's, for $k = 0,1,\ldots,N$, are defined by (2.3). Also, subscript N denotes an approximate solution $x_{j,N}(s)$ in terms of $N+1$ cardinal functions.
Substituting (4.15) into (4.14) yields

$$\sum_{j=1}^{n} u_{i,j}(s)\,x_{j,N}(s) - \sum_{j=1}^{n} \lambda_{i,j} \int_{-1}^{1} k_{i,j}(s,t)\,x_{j,N}(t)\,\mathrm{d}t \simeq f_i(s). \tag{4.16}$$

Now, approximating the integral operator in (4.16) by the Clenshaw-Curtis quadrature defined by (3.9) and (3.10) follows

$$\sum_{j=1}^{n} u_{i,j}(s)\,x_{j,N}(s) - \sum_{j=1}^{n} \lambda_{i,j} {\sum_{p=0}^{N}}'' w_p\, k_{i,j}(s,t_p)\,x_{j,N}(t_p) \simeq f_i(s). \tag{4.17}$$

From (2.7) and (4.15) we can write $x_{j,N}(t_p) = x_j(t_p) = a_{j,p}$. Therefore

$$\sum_{j=1}^{n} u_{i,j}(s)\,x_{j,N}(s) - \sum_{j=1}^{n}\sum_{p=0}^{N}{}'' \lambda_{i,j}\,w_p\,k_{i,j}(s,t_p)\,a_{j,p} \simeq f_i(s). \tag{4.18}$$

Substituting $s = t_q$, for $q = 0,1,\ldots,N$, defined by (2.3) into (4.18) follows

$$\sum_{j=1}^{n} u_{i,j}(t_q)\,x_{j,N}(t_q) - \sum_{j=1}^{n}\sum_{p=0}^{N}{}'' \lambda_{i,j}\,w_p\,k_{i,j}(t_q,t_p)\,a_{j,p} \simeq f_i(t_q), \qquad q = 0,1,\ldots,N. \tag{4.19}$$

Now, considering $x_{j,N}(t_q) = a_{j,q}$ and replacing "$\simeq$" sign with "$=$" sign gives

$$\sum_{j=1}^{n}\left[u_{i,j}(t_q)\,a_{j,q} - \lambda_{i,j}\sum_{p=0}^{N}{}'' w_p\,k_{i,j}(t_q,t_p)\,a_{j,p}\right] = f_i(t_q), \qquad q = 0,1,\ldots,N. \tag{4.20}$$

By considering a similar procedure for the other equations of system (4.12) we can finally obtain

$$\sum_{j=1}^{n}\left[u_{i,j}(t_q)\,a_{j,q} - \lambda_{i,j}\sum_{p=0}^{N}{}'' w_p\,k_{i,j}(t_q,t_p)\,a_{j,p}\right] = f_i(t_q), \qquad q = 0,1,\ldots,N, \quad i = 1,2,\ldots,n. \tag{4.21}$$

Now, (4.21) is a system of $n(N+1)$ algebraic equations for the $n(N+1)$ unknowns $a_{1,0}, a_{1,1}, \ldots, a_{1,N}, a_{2,0}, \ldots, a_{n,N}$. Hence, according to (4.15), an approximate solution $\mathbf{X}_N(s) = \left[x_{1,N}(s), x_{2,N}(s), \ldots, x_{n,N}(s)\right]^T$ is obtained for Fredholm integral equations system (4.12).

5 Error analysis and convergence evaluation

Without loss of generality, system (4.12) can be considered as follows

$$\mathbf{X}(s) = \mathbf{F}(s) + \int_a^b \mathbf{K}(s,t)\,\mathbf{X}(t)\,\mathrm{d}t, \qquad s \in [a,b] = [-1,1], \tag{5.22}$$

where $\mathbf{X}(s)$, $\mathbf{F}(s)$, $\mathbf{K}(s,t)$ are defined by (4.13). Equation (5.22) can be rewritten in the following form:

$$(\mathbf{I} - \mathscr{K})\,\mathbf{X} = \mathbf{F}, \tag{5.23}$$

where $\mathbf{I}$ is identity operator and operator $\mathscr{K}$ is defined as

$$(\mathscr{K}\,\mathbf{X})\,(s) = \int_a^b \mathbf{K}(s,t)\,\mathbf{X}(t)\,\mathrm{d}t. \tag{5.24}$$

Let us consider

$$\sum_{j=0}^{N}{}'' w_j\,\mathbf{K}(s,t_j)\,\mathbf{X}(t_j) = (\mathscr{K}\,\mathbf{X})\,(s) - \mathbf{E}_t\left(\mathbf{K}(s,t)\,\mathbf{X}(t)\right), \tag{5.25}$$

in which $\mathbf{E}_t$ indicates the error functional for the Clenshaw-Curtis quadrature rule operating on $\mathbf{K}(s,t)\,\mathbf{X}(t)$, viewed as a function of t for fixed s [20]. From (5.22) we can defined $\mathbf{E}_t\left(\mathbf{K}(s,t)\,\mathbf{X}(t)\right)$ as an n–vector of the form

$$\mathbf{E}_t\left(\mathbf{K}(s,t)\,\mathbf{X}(t)\right) = \begin{pmatrix} E_t^{1,1}\left(k_{1,1}(s,t)x_1(t)\right) + E_t^{1,2}\left(k_{1,2}(s,t)x_2(t)\right) + \cdots + E_t^{1,n}\left(k_{1,n}(s,t)x_n(t)\right) \\ E_t^{2,1}\left(k_{2,1}(s,t)x_1(t)\right) + E_t^{2,2}\left(k_{2,2}(s,t)x_2(t)\right) + \cdots + E_t^{2,n}\left(k_{2,n}(s,t)x_n(t)\right) \\ \vdots \\ E_t^{n,1}\left(k_{n,1}(s,t)x_1(t)\right) + E_t^{n,2}\left(k_{n,2}(s,t)x_2(t)\right) + \cdots + E_t^{n,n}\left(k_{n,n}(s,t)x_n(t)\right) \end{pmatrix}. \tag{5.26}$$

Suppose $\mathbf{X}_N$ be the approximate solution of (5.22) obtained by the presented method. It satisfies the following equation:

$$\mathbf{X}_N(s) = \mathbf{F}(s) + \sum_{j=0}^{N}{}'' w_j\, \mathbf{K}(s,t_j)\, \mathbf{X}_N(t_j). \tag{5.27}$$

If we consider $s = t_i = \cos\left(\frac{i\pi}{N}\right)$, for $i = 0,1,\ldots,N$, then (5.27) is equivalent to algebraic system (4.21).
Now, the error of the presented method can be defined as

$$\mathbf{e}_N(s) = \mathbf{X}(s) - \mathbf{X}_N(s), \tag{5.28}$$

where $\mathbf{e}_N(s)$ is an n–vector such that its ith component is the error corresponding to solving the ith integral equation of system (5.22). From (5.25) we obtain

$$\mathbf{X}_N(s) = \mathbf{F}(s) + \int_a^b \mathbf{K}(s,t)\, \mathbf{X}_N(t)\, \mathrm{d}t - \mathbf{E}_t\big(\mathbf{K}(s,t)\, \mathbf{X}_N(t)\big). \tag{5.29}$$

Subtracting (5.29) form (5.22) gives

$$\mathbf{e}_N(s) = \mathbf{E}_t\big(\mathbf{K}(s,t)\, \mathbf{X}_N(t)\big) + \int_a^b \mathbf{K}(s,t)\, \mathbf{e}_N(t)\, \mathrm{d}t. \tag{5.30}$$

Thus, the vector of the error functions $\mathbf{e}_N(s)$ satisfies a Fredholm integral equations system with the same kernels as (5.22), but with different driving terms. Equation (5.30) can also be written in the form

$$(\mathbf{I} - \mathscr{K})\, \mathbf{e}_N(s) = \mathbf{E}_t\big(\mathbf{K}(s,t)\, \mathbf{X}_N(t)\big). \tag{5.31}$$

This equation has the solution

$$\begin{aligned} \mathbf{e}_N(s) &= (\mathbf{I} - \mathscr{K})^{-1} \mathbf{E}_t\big(\mathbf{K}(s,t)\, \mathbf{X}_N(t)\big) \\ &= (\mathbf{I} + \mathscr{H})\, \mathbf{E}_t\big(\mathbf{K}(s,t)\, \mathbf{X}_N(t)\big), \end{aligned} \tag{5.32}$$

where $\mathscr{H}$ is the resolvent operator (see [25,26]). Hence, taking arbitrary norms we find

$$\big\|\, \mathbf{e}_N \,\big\| \leqslant (1 + \|\, \mathscr{H} \,\|)\, \big\|\, \mathbf{E}_t\big(\mathbf{K}(s,t)\, \mathbf{X}_N(t)\big) \,\big\|. \tag{5.33}$$

However, if $\|\, \mathscr{K} \,\| < 1$ we find, on taking norms in (5.30) and rearranging, the simpler bound

$$\big\|\, \mathbf{e}_N \,\big\| \leqslant \frac{\|\, \mathbf{E}_t\big(\mathbf{K}(s,t)\, \mathbf{X}_N(t)\big) \,\|}{(1 - \|\, \mathscr{K} \,\|)}. \tag{5.34}$$

The bounds of the error vector show that $\mathbf{e}_N$ is directly related to the error of the quadrature rule.
The simplest and most commonly used procedure for estimating the achieved accuracy is to choose a family of quadrature rules R_N and to compute an approximate solution for members of this family with increasing N, and hence decreasing error, until the results appear to have settled down to the required accuracy. It is usually straightforward to guarantee convergence; that is, to ensure that [20]

$$\lim_{N\to\infty} \big\|\, \mathbf{e}_{R_N} \,\big\| = 0. \tag{5.35}$$

Sufficient conditions for this are given in [20,27].
Now, we try to compute the rate at which convergence is achieved.

Lemma 5.1. *The error estimates for N–point quadrature rules of various types have the form*

$$\big|E_t f\big| = \left| \int_a^b f(t)\, \mathrm{d}t - \sum_{i=1}^{N} w_i f(t_i) \right| \leqslant C(f)\, N^{-p}, \tag{5.36}$$

where $C(f)$ is a constant dependent on the function $f(t)$ and the exponent p depends either on the degree of the rule (if $f(t)$ is smooth enough) or on the continuity properties of $f(t)$ (if the degree of the rule is high enough). We refer to p as the order of convergence.

Proof. [20].

Now, considering Lemma 5.1 and using (5.33) we obtain

$$\left\| \mathbf{e}_N \right\| \leqslant (1+ \| \mathscr{H} \|) \left\| \mathbf{C}\big(\mathbf{K}(s,t)\,\mathbf{X}_N(t)\big) \right\| N^{-p}, \tag{5.37}$$

where $\mathbf{C}\left(\mathbf{K}(s,t)\,\mathbf{X}_N(t)\right)$ is an n–vector of the form

$$\mathbf{C}\left(\mathbf{K}(s,t)\,\mathbf{X}(t)\right) = \begin{pmatrix} C_{1,1}\left(k_{1,1}(s,t)x_1(t)\right) + C_{1,2}\left(k_{1,2}(s,t)x_2(t)\right) + \cdots + C_{1,n}\left(k_{1,n}(s,t)x_n(t)\right) \\ C_{2,1}\left(k_{2,1}(s,t)x_1(t)\right) + C_{2,2}\left(k_{2,2}(s,t)x_2(t)\right) + \cdots + C_{2,n}\left(k_{2,n}(s,t)x_n(t)\right) \\ \vdots \\ C_{n,1}\left(k_{n,1}(s,t)x_1(t)\right) + C_{n,2}\left(k_{n,2}(s,t)x_2(t)\right) + \cdots + C_{n,n}\left(k_{n,n}(s,t)x_n(t)\right) \end{pmatrix}. \tag{5.38}$$

If $C_{i,j}$, for $i,j = 1,2,\ldots,n$, is uniformly bounded in s such that

$$\left| C_{i,j}\left(k_{i,j}(s,t)\,x_{j,N}(t)\right) \right| \leqslant M_{i,j}, \qquad a \leqslant s \leqslant b. \tag{5.39}$$

Then, taking the maximum norm $\|\cdot\|_\infty$ on (5.37) gives

$$\left\| \mathbf{e}_N \right\|_\infty \leqslant (1+ \| \mathscr{H} \|_\infty)\,\bar{M} N^{-p}, \tag{5.40}$$

where $\| \mathbf{C}\big(\mathbf{K}(s,t)\,\mathbf{X}_N(t)\big) \|_\infty \leqslant \bar{M}$. □

The estimates of convergence rate above show that the presented error will be rapidly convergent to zero if the degree of the quadrature rule is high enough and if for every fixed s, $\mathbf{K}(s,t)\,\mathbf{X}(t)$ is a vector of smooth functions with respect to t.

Referring to the fact that if $\mathbf{F}$ and $\mathbf{K}$ in (5.22) are continuous then so is $\mathbf{X}$ (see [25]), we pose the following theorem as the extension of Theorem 4.2.2 in [20].

Theorem 5.1. *Let in (5.22) $f_i(s) \in C^{(p)}[a,b]$, for $i = 1,2,\ldots,n$, and $k_{i,j}(s,t) \in C^{(p)}[a,b] \times C^{(p)}[a,b]$, where $C^{(p)}$ is space of continuous and differentiable functions of order p. Then, $x_i(s) \in C^{(p)}[a,b]$, for $i = 1,2,\ldots,n$.*

Proof. We have

$$x_i(s) = f_i(s) + \sum_{j=1}^{n} \lambda_{i,j} \int_a^b k_{i,j}(s,t)\,x_j(t)\,\mathrm{d}t, \qquad i = 1,2,\ldots,n, \tag{5.41}$$

whence

$$x_i'(s) = f_i'(s) + \sum_{j=1}^{n} \lambda_{i,j} \int_a^b \frac{\partial}{\partial s} k_{i,j}(s,t)\,x_j(t)\,\mathrm{d}t, \qquad i = 1,2,\ldots,n. \tag{5.42}$$

But, for $p \geqslant 1$, $\frac{\partial}{\partial s} k_{i,j}(s,t)$ is continuous by hypothesis, and the integrals $\int_a^b \frac{\partial}{\partial s} k_{i,j}(s,t)\,x_j(t)\,\mathrm{d}t$ are continuous functions of s, for $i,j = 1,2,\ldots,n$. So, $\sum_{j=1}^{n} \lambda_{i,j} \int_a^b \frac{\partial}{\partial s} k_{i,j}(s,t)\,x_j(t)\,\mathrm{d}t$ is a continuous function. Therefore, if $f_i'(s)$, for $i = 1,2,\ldots,n$, are continuous, then according to (5.42) $x_i'(s)$, for $i = 1,2,\ldots,n$, will be continuous. Similarly for any $q \leqslant p$

$$x_i^{(q)}(s) = f_i^{(q)}(s) + \sum_{j=1}^{n} \lambda_{i,j} \int_a^b \frac{\partial^q}{\partial s^q} k_{i,j}(s,t)\,x_j(t)\,\mathrm{d}t, \qquad i = 1,2,\ldots,n, \tag{5.43}$$

whence, proceeding inductively, it follows that $x_i(s) \in C^{(p)}[a,b]$, for $i = 1,2,\ldots,n$. □

6 Numerical results

Some examples are investigated by the proposed method in this section. Most of these examples are caught from various references, such that we are able to compare the numerical results obtained by the proposed method with both the exact solution and those presented in the related references. The numerical results are given for eleven points s in interval $[a,b]$. These points are set by dividing the interval to ten equal segments (according to the related literature).

Example 6.1. *For the following linear Fredholm integral equations system [13, 14, 17]:*

$$\begin{cases} x_1(s) = \frac{s}{18} + \frac{17}{36} + \int_0^1 \frac{s+t}{3}\left(x_1(t)+x_2(t)\right)\mathrm{d}t, \\ \\ x_2(s) = s^2 - \frac{19}{12}s + 1 + \int_0^1 st\left(x_1(t)+x_2(t)\right)\mathrm{d}t, \end{cases} \tag{6.44}$$

the presented method in this paper gives the exact solutions $x_1(s) = s+1$ and $x_2(s) = s^2+1$ for $N = 2$. Table 1 shows the numerical results obtained by the method and those given in [13, 17].

Table 1: Numerical results for Example 6.1

s	Exact solution	Presented method, $N = 2$	Direct method [17], $m = 32$	Decomposition method [13], $k = 11$
		Results for $x_1(s)$		
0.0	1.000000	1.000000	1.000088	0.988498
0.1	1.100000	1.100000	1.100104	1.086632
0.2	1.200000	1.200000	1.200119	1.184766
0.3	1.300000	1.300000	1.300134	1.282899
0.4	1.400000	1.400000	1.400150	1.381033
0.5	1.500000	1.500000	1.500165	1.479167
0.6	1.600000	1.600000	1.600180	1.577301
0.7	1.700000	1.700000	1.700196	1.675435
0.8	1.800000	1.800000	1.800211	1.773569
0.9	1.900000	1.900000	1.900226	1.871702
1.0	2.000000	2.000000	———	1.969836
		Results for $x_2(s)$		
0.0	1.000000	1.000000	1.000000	1.000000
0.1	1.010000	1.010000	1.010183	1.006549
0.2	1.040000	1.040000	1.040287	1.033099
0.3	1.090000	1.090000	1.090314	1.079648
0.4	1.160000	1.160000	1.160262	1.146198
0.5	1.250000	1.250000	1.250133	1.232747
0.6	1.360000	1.360000	1.360315	1.339296
0.7	1.490000	1.490000	1.490420	1.465846
0.8	1.640000	1.640000	1.640446	1.612695
0.9	1.810000	1.810000	1.810395	1.778945
1.0	2.000000	2.000000	———	1.965494

Example 6.2. *For the following linear Fredholm integral equations system [16]:*

$$\begin{cases} x_1(s) = \frac{11}{6}s + \frac{11}{15} - \int_0^1 (s+t)\,x_1(t)\,\mathrm{d}t - \int_0^1 (s+2t^2)\,x_2(t)\,\mathrm{d}t, \\ \\ x_2(s) = \frac{5}{4}s^2 + \frac{1}{4}s - \int_0^1 st^2 x_1(t)\,\mathrm{d}t - \int_0^1 s^2 t\,x_2(t)\,\mathrm{d}t, \end{cases} \tag{6.45}$$

the presented method gives the exact solutions $x_1(s) = s$ and $x_2(s) = s^2$ for $N = 4$. Table 2 shows the numerical results obtained by it and those given in [16].

Table 2: Numerical results for Example 6.2

s	Exact solution	Presented method, $N=4$	BPFs method [16], $m=32$
		Results for $x_1(s)$	
0.0	0.000000	0.000000	0.01421
0.1	0.100000	0.100000	0.09802
0.2	0.200000	0.200000	0.20345
0.3	0.300000	0.300000	0.29146
0.4	0.400000	0.400000	0.38790
0.5	0.500000	0.500000	0.48614
0.6	0.600000	0.600000	0.60641
0.7	0.700000	0.700000	0.70914
0.8	0.800000	0.800000	0.81314
0.9	0.900000	0.900000	0.91512
1.0	1.000000	1.000000	0.99315
		Results for $x_2(s)$	
0.0	0.000000	0.000000	0.04331
0.1	0.010000	0.010000	0.00971
0.2	0.040000	0.040000	0.04172
0.3	0.090000	0.090000	0.08866
0.4	0.160000	0.160000	0.16812
0.5	0.250000	0.250000	0.26409
0.6	0.360000	0.360000	0.38405
0.7	0.490000	0.490000	0.48991
0.8	0.640000	0.640000	0.65795
0.9	0.810000	0.810000	0.80551
1.0	1.000000	1.000000	0.99971

Example 6.3. *Consider the following Fredholm integral equations system [14–17]:*

$$\begin{cases} x_1(s) = 2e^s + \frac{e^{s+1}-1}{s+1} - \int_0^1 e^{s-t} x_1(t)\,\mathrm{d}t - \int_0^1 e^{(s+2)t} x_2(t)\,\mathrm{d}t, \\ x_2(s) = e^s + e^{-s} + \frac{e^{s+1}-1}{s+1} - \int_0^1 e^{st} x_1(t)\,\mathrm{d}t - \int_0^1 e^{s+t} x_2(t)\,\mathrm{d}t, \end{cases} \tag{6.46}$$

with the exact solutions $x_1(s) = e^s$ and $x_2(s) = e^{-s}$. The numerical results are shown in Table 3.

Table 3: Numerical results for Example 6.3

s	Exact solution	Presented method, $N=4$	Presented method, $N=6$	Direct method [17], $m=32$	Rationalized Haar method [15], $k=32$	BPFs method [16], $m=32$
			Results for $x_1(s)$			
0.0	1.000000	0.999978	1.000000	0.998849	1.01548	1.01047
0.1	1.105171	1.105132	1.105171	1.104019	1.11531	1.11641
0.2	1.221403	1.221401	1.221403	1.220211	1.22495	1.22496
0.3	1.349859	1.349882	1.349859	1.348574	1.34538	1.34547
0.4	1.491825	1.491837	1.491825	1.490378	1.47764	1.47776
0.5	1.648721	1.648696	1.648721	1.647027	1.6229	1.6230
0.6	1.822119	1.822055	1.822119	1.820420	1.83904	1.83910
0.7	2.013753	2.013678	2.013753	2.011984	2.01983	2.01982
0.8	2.225541	2.225497	2.225541	2.223613	2.2184	2.2190
0.9	2.459603	2.459608	2.459603	2.457404	2.43648	2.43651
1.0	2.718282	2.718276	2.718282	————	2.67601	2.67611
			Results for $x_2(s)$			
0.0	1.000000	1.000005	1.000000	1.000667	0.98456	0.98470
0.1	0.904837	0.904849	0.904837	0.905568	0.89646	0.89657
0.2	0.818731	0.818726	0.818731	0.819474	0.81625	0.81636
0.3	0.740818	0.740804	0.740818	0.741531	0.74322	0.74351
0.4	0.670320	0.670310	0.670320	0.670969	0.67673	0.67682
0.5	0.606531	0.606536	0.606531	0.607086	0.61619	0.61621
0.6	0.548812	0.548832	0.548812	0.549356	0.54382	0.54386
0.7	0.496585	0.496611	0.496585	0.497077	0.49518	0.49520
0.8	0.449329	0.449349	0.449329	0.449731	0.45091	0.45010
0.9	0.406570	0.406582	0.406570	0.406848	0.41060	0.41070
1.0	0.367879	0.367906	0.367879	————	0.37391	0.37401

Example 6.4. *This example includes a Fredholm integral equations system with variable coefficients with respect to s as follows:*

$$\begin{cases} s^2 x_1(s) + (s+1)x_2(s) = y_1(s) - \int_0^1 \sin(s-t)\,x_1(t)\,\mathrm{d}t - \int_0^1 \cos(s-t)\,x_2(t)\,\mathrm{d}t, \\ -s x_1(s) + (1-2s^2)x_2(s) = y_2(s) - \int_0^1 \sin(s+t)\,x_1(t)\,\mathrm{d}t - \int_0^1 \cos(s+t)\,x_2(t)\,\mathrm{d}t, \end{cases} \tag{6.47}$$

with the exact solutions $x_1(s) = s^3 + 4$ and $x_2(s) = 1 - s^2$ and suitable $y_1(s)$ and $y_2(s)$. Figure 1 shows the numerical results for this problem for $N = 6$.

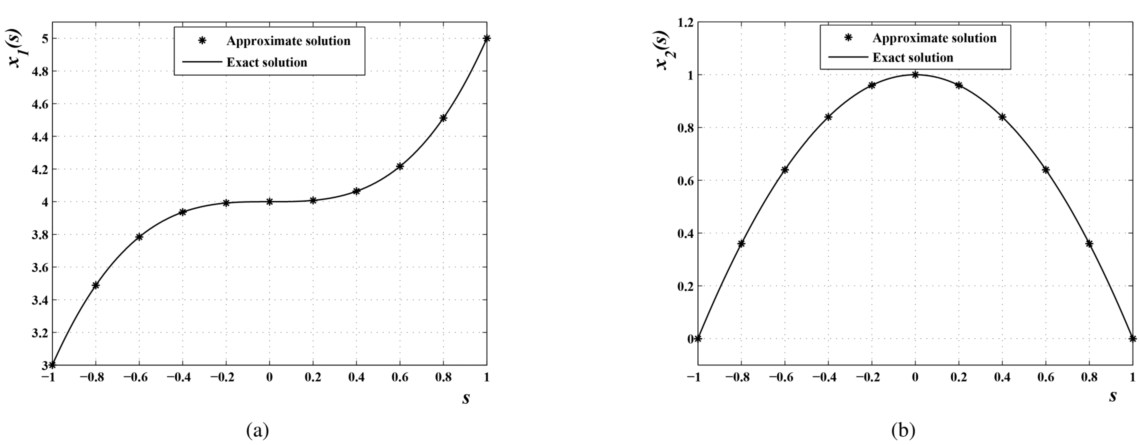

Figure 1: Numerical results for Example 6.4 obtained by the proposed method. (a) Results for $x_1(s)$. (b) Results for $x_2(s)$.

7 Comments on the results

Four test problems were illustrated above for evaluating the applicability and accuracy of the proposed method. Example 6.1 has been solved in [13], [14], and [17], too. [13] proposes the decomposition method and [17] introduces a direct method using the triangular functions to solve the problem. Also, a Sinc-collocation method is considered in [14] for numerical solution of Example 6.1. Our method gives the exact solution for this problem for a very small size of discretization ($N = 2$). The numerical results obtained by the decomposition and direct methods shown in Table 1 of this article and also the error values given in Table 2 of [14] confirm the superiority of the proposed method over the three mentioned methods in view of accuracy. On the other hand, the number of calculations in the decomposition method is higher.
For Example 6.2, [16] gives an approximate solution by using the block-pulse functions. The related results for $m = 32$ are shown in Table 2. Our method gives the exact solution for this example for $N = 4$, whence it follows that this method is much more accurate than the BPFs method.
[15] proposes a numerical method based on using rationalized Haar functions for linear Fredholm integral equations system. This method together with those presented in [14, 16, 17] obtain a numerical solution for Example6.3. The numerical results in Table 3 of this paper and also the error values given in Table 1 of [14] still confirm good accuracy of the method proposed in this paper.
For further evaluation of the computational efficiency of the method we give the mean-absolute errors associated with it within solving two of the examples . The mean-absolute error is calculated by considering the errors at n_0 points $s \in [a,b]$ and by using the following relation:

$$E_{j,N}^{(n_0)} = \frac{1}{n_0} \sum_{i=1}^{n_0} \left| x_j(s_i) - x_{j,N}(s_i) \right|, \tag{7.48}$$

where $E_{j,N}^{(n_0)}$ is the mean-absolute error, and $x_j(s)$ and $x_{j,N}(s)$ are the jth exact and approximate solutions, respectively. For Examples 6.3 and 6.4, these errors for $n_0 = 11$ points $s_i = a + \frac{b-a}{10} i$, $i = 0, 1, \ldots, 10$, and $N = 2, 4, \ldots, 12$ are illustrated in Table 4. Obviously, these results confirm very quick convergence of the proposed method meanwhile emphasis again on its excellent accuracy.

Table 4: Mean-absolute errors

N	Mean-absolute errors for Example 3		Mean-absolute errors for Example 4	
	Results for $x_1(s)$	Results for $x_2(s)$	Results for $x_1(s)$	Results for $x_2(s)$
2	$1.4E-2$	$6.0E-3$	$4.0E-1$	$1.9E-1$
4	$2.9E-5$	$1.4E-5$	$1.0E-1$	$4.1E-2$
6	$3.8E-8$	$1.7E-8$	$5.7E-5$	$3.1E-5$
8	$3.5E-11$	$1.4E-11$	$1.4E-7$	$7.3E-8$
10	$2.1E-14$	$9.2E-15$	$2.7E-10$	$1.3E-10$
12	$1.4E-14$	$5.0E-15$	$4.0E-13$	$1.7E-13$

8 Conclusion

An effective and accurate numerical approach for solving linear systems of Fredhlom integral equations of the second kind was proposed by using the Chebyshev cardinal functions with Gauss-Lobatto points and also the Clenshaw-Curtis quadrature rule. Moreover, two error bounds were computed for the method in terms of the error of the Clenshaw-Curtis quadrature rule and its convergence rate was estimated. Some test problems were solved by the presented method which showed that it is applicable and accurate in solving of the mentioned systems.

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