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Research Article

A numerical method for calculation of electrostatic charge distribution induced on conducting surfaces

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Abstract

The focus of this article is on calculation of electrostatic charge distribution induced on conducting surfaces. For this purpose, the integral equation concept is used for mathematical modeling of the problem. A special set of exponential basis functions is introduced and defined to be used in formulation of a numerical method for solving the integral equation to obtain the charge distribution. The method is numerically evaluated via calculation of charge density for some structures by which the computational efficiency of the method will be demonstrated.

Keywords: Numerical solution; Electrostatic charge distribution; Mathematical modeling; Integral equation; Conducting structure.

1 Introduction

Integral equation technique is a well-known approach for modeling of Electromagnetics problems [1–7] and many of such problems are modeled by integral equations of the first kind. These equations are in general ill-posed. That is, small changes to the problem's data can make very large changes to the answers obtained [8, 9]. Hence, obtaining the numerical solutions is difficult. In recent years, some numerical methods based on different basis functions have been illustrated. These methods often use a projection method and transform a first kind integral equation to a linear system of algebraic equations. This system usually has large condition number.

This article focuses on calculation of electrostatic charge distribution induced on conducting surfaces. For this purpose, the integral equation concept is used for mathematical modeling of the problem. For solving the integral equation, we define a special set of exponential basis functions to be used for formulation of a point-matching method. The main advantages of the proposed method are simplicity and enough accuracy.

The organization of this article is as follows. We survey in section 2 the integral equation modeling for calculation of electrostatic charge distribution induced on three conducting structures. In section 3 we firstly define a set of exponential basis functions and then present the formulation of the point-matching method based on these functions for calculation of charge density. The results obtained by the numerical method are given in section 4 to illustrate its computational efficiency. Finally, conclusions will be given in section 5.

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2 Integral equation modeling

Here, we survey the integral equation modeling for electrostatic charge density induced on three conducting surfaces; a finite-width strip, an incomplete cylindrical surface, and an incomplete spherical surface.

2.1 Finite-width strip

In Fig. 1 there is a conducting strip of width a containing electrostatic charge that is very long in $\pm z$ direction. It can be imagined that the strip consists of line charges stacked next to one another and parallel to z-axis. The potential ϕ at a filed point $\boldsymbol{\rho}$ produced by a line charge (electric filed source) of density q_L is given by [3]

$$\mathrm{d}\phi(\boldsymbol{\rho}) = \frac{q_L}{2\pi\varepsilon_0}\ln\left(\frac{1}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}\right), \tag{2.1}$$

where $\varepsilon_0 \simeq 8.854e-12$ (F/m) is free space permittivity, $\boldsymbol{\rho}$ is the position vector of observation (field) point, $\boldsymbol{\rho}'$ is the position vector of source point.

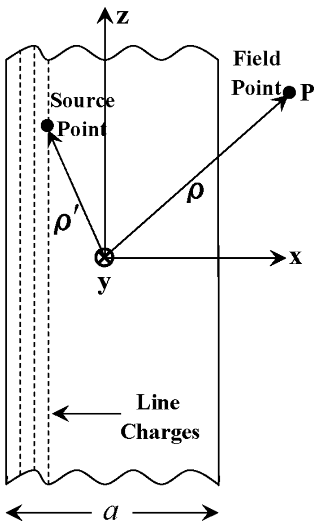


Figure 1: A conducting strip of width a containing electrostatic charge.

The surface charge density is constant along z at any x-axis point. It is possible to relate the surface charge on the strip to a line charge by multiplying the surface charge density by $\mathrm{d}x'$ as

$$q_L(x') = q_s(x')\,\mathrm{d}x'. \tag{2.2}$$

Replacing (2.2) into (2.1) and integrating both sides gtives

$$\phi(\boldsymbol{\rho}) = \int_{-a/2}^{a/2}\frac{q_s(x')}{2\pi\varepsilon_0}\ln\left(\frac{1}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}\right)\mathrm{d}x'. \tag{2.3}$$

Now, the observation point P can be moved to the surface of strip. So, the source and observation points are confined to x-axis. Hence, Eq. (2.3) is reduced to

$$\phi(x) = \int_{-a/2}^{a/2}\frac{q_s(x')}{2\pi\varepsilon_0}\ln\left(\frac{1}{|x-x'|}\right)\mathrm{d}x'. \tag{2.4}$$

Assume that the potential produced by the surface charge of strip is chosen 1 (V). Eq. (2.4) becomes

$$\int_{-a/2}^{a/2}\frac{q_s(x')}{2\pi\varepsilon_0}\ln\left(\frac{1}{|x-x'|}\right)\mathrm{d}x' = 1. \tag{2.5}$$

Equation (2.5) is a Fredholm integral equation of the first kind in terms of the unknown function q_s. Solution of this equation gives the surface charge density on the strip.
For solving Eq. (2.5) we will present a numerical method in section 3. However, we obtain an analytical solution here for a special case. Suppose that the width of the strip is set to $a = 2$. With this assumption, Eq. (2.5) becomes

$$\int_{-1}^{1} \frac{q_s(x')}{2\pi\varepsilon_0} \ln(\frac{1}{|x-x'|})\,\mathrm{d}x' = 1. \tag{2.6}$$

It is possible to choose to observe the potential at any point along x-axis that is desired. Choosing $x = 0$, Eq. (2.6) becomes

$$\int_{-1}^{1} \frac{q_s(x')}{2\pi\varepsilon_0} \ln(\frac{1}{|x'|})\,\mathrm{d}x' = 1. \tag{2.7}$$

This integral has the same form as a known definite integral [3]

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x'^2}} \ln(\frac{1}{|x'|})\,\mathrm{d}x' = \pi \ln 2. \tag{2.8}$$

Normalizing the right side and comparing with Eq. (2.7) we have

$$q_s(x') = \frac{2\varepsilon_0}{(\ln 2)\sqrt{1-x'^2}}. \tag{2.9}$$

Rearranging (2.9), the exact solution of the charge distribution as a function of x is obtained as follows:

$$q_s(x) = \frac{2\varepsilon_0}{(\ln 2)\sqrt{1-x^2}}. \tag{2.10}$$

2.2 Incomplete cylindrical surface

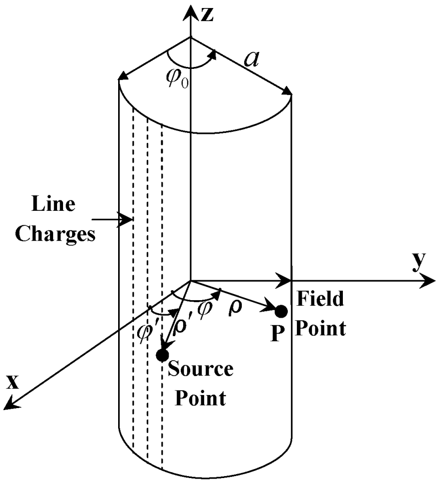


Figure 2: A conducting incomplete cylindrical surface of radius a containing electrostatic charge.

In Fig. 2 there is a conducting incomplete cylindrical surface of radius a containing electrostatic charge that is very long in $\pm z$ direction. According to the figure, it is clear that we can write

$$q_L(\varphi') = q_s(\varphi')(a\,\mathrm{d}\varphi'), \tag{2.11}$$

and

$$|\boldsymbol{\rho} - \boldsymbol{\rho}'| = a\sqrt{2(1-\cos(\varphi-\varphi'))}, \tag{2.12}$$

where $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are the position vectors of observation and source points, respectively. By replacing (2.11) and (2.12) in (2.1) and integrating both sides we have

$$\phi(\varphi) = \int_0^{\varphi_0} q_s(\varphi') \frac{a}{2\pi\varepsilon_0} \ln\left(\frac{1}{a\sqrt{2(1-\cos(\varphi-\varphi'))}}\right) \mathrm{d}\varphi'. \tag{2.13}$$

If the potential produced by the surface charge of cylindrical surface is chosen 1 (V), then Eq. (2.13) becomes

$$\int_0^{\varphi_0} q_s(\varphi') \frac{a}{2\pi\varepsilon_0} \ln\left(\frac{1}{a\sqrt{2(1-\cos(\varphi-\varphi'))}}\right) \mathrm{d}\varphi' = 1. \tag{2.14}$$

Equation (2.14) is a Fredholm integral equation of the first kind in terms of the unknown function q_s. Solution of this equation gives the surface charge density on the incomplete cylindrical surface.

2.3 Incomplete spherical surface

In Fig. 3, there is a conducting incomplete spherical surface of radius a containing electrostatic charge that produces a constant potential at its surface.
According to the symmetry of the problem (as shown in Fig. 3), it is clear that the charge distribution on the surface of the sphere is independent of variable φ of the spherical coordinates system. However, it depends on θ. Consider a differential surface element $\mathrm{d}s$ at an arbitrary source point on the surface of the sphere. For element $\mathrm{d}s$, the resulting electric potential at the field point P (that is moved to the surface of sphere) is

$$\mathrm{d}\phi = \frac{q_s(\theta')}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}s, \tag{2.15}$$

where $\mathbf{r}'$ and $\mathbf{r}$ are respectively the position vectors of source point and observation point (field point) on the surface of sphere.
According to Fig. 3 we have

$$|\mathbf{r}-\mathbf{r}'| = a\sqrt{2[1-(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'))]}, \tag{2.16}$$

and

$$\mathrm{d}s = a^2 \sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\varphi'. \tag{2.17}$$

Then the total potential is

$$\phi = \int_0^{2\pi}\int_0^{\theta_0} \frac{a}{4\pi\varepsilon_0} \frac{q_s(\theta')\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\varphi'}{\sqrt{2[1-(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi'))]}}. \tag{2.18}$$

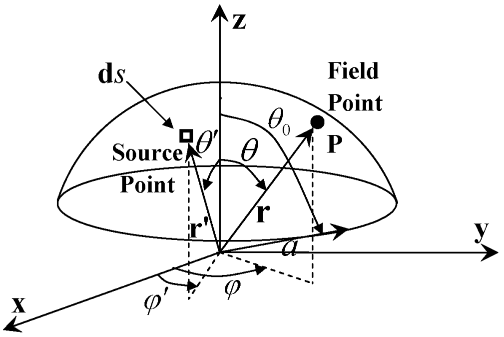


Figure 3: A conducting incomplete spherical surface of radius a containing electrostatic charge.

The potential is constant throughout the conducting surface, therefore the observation point (P) can be moved to $\varphi = 0$. So, Eq. (2.18) can be rewritten as

$$\phi(\theta) = \int_0^{2\pi}\int_0^{\theta_0} \frac{a}{4\pi\varepsilon_0} \frac{q_s(\theta')\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\varphi'}{\sqrt{2[1-(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\varphi')]}}, \tag{2.19}$$

where $\phi(\theta)$ has a constant value. Now, assume that the potential produced by the surface charge is chosen 1 (V). Then, Eq. (2.19) becomes

$$\int_0^{2\pi}\int_0^{\theta_0} \frac{a}{4\pi\varepsilon_0} \frac{q_s(\theta')\sin\theta'\,\mathrm{d}\theta'\,\mathrm{d}\varphi'}{\sqrt{2[1-(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\varphi')]}} = 1. \tag{2.20}$$

Equation (2.20) is a first kind Fredholm integral equation of the following form

$$\int_0^{\theta_0} k(\theta,\theta')x(\theta')\,\mathrm{d}\theta' = f(\theta), \tag{2.21}$$

in which

$$k(\theta,\theta') = \int_0^{2\pi} \frac{a}{4\pi\varepsilon_0} \frac{\sin\theta'\,\mathrm{d}\varphi'}{\sqrt{2[1-(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\varphi')]}}, \tag{2.22}$$

$x(\theta') = q_s(\theta')$ is the unknown function (surface charge density) to be determined, and $f(\theta) = 1$.

3 Numerical method for calculation of electrostatic charge distribution

A point-matching method is presented in this section for numerically solving the integral equation models arising in the charge distribution problems, because the analytical solution is often unknown. For formulation of the method, we firstly introduce a set of basis functions.

3.1 Basis functions

We define a special m-set of exponential basis functions for any positive integer m over real interval $[a,b)$ as

$$\mathscr{E}_i(t) = \begin{cases} \exp\left(-\left|t-a-ih-\frac{h}{2}\right|^n\right), & a+ih \leqslant t < a+(i+1)h, \\ 0, & \text{otherwise}, \end{cases} \tag{3.23}$$

where $h = \frac{b-a}{m}$, $i = 0,1,\ldots,m-1$, and n is a positive integer and may be considered as a regularization exponent. The above definition can obviously make a set of disjoint and orthogonal basis functions. For arbitrary i and j, such that $i = 0,1,\ldots,m-1$ and $j = 0,1,\ldots,m-1$, we have

$$\begin{aligned} \left\langle \mathscr{E}_i, \mathscr{E}_j \right\rangle &= \int_a^b \mathscr{E}_i(t)\mathscr{E}_j(t)\,\mathrm{d}t \\ &= \begin{cases} \int_{a+ih}^{a+(i+1)h} \exp\left(-2\left|t-a-ih-\frac{h}{2}\right|^n\right)\mathrm{d}t, & i=j, \\ 0, & i\neq j, \end{cases} \end{aligned} \tag{3.24}$$

in which $\langle .,. \rangle$ indicates the inner product.
The disjointness and orthogonality properties of $\mathscr{E}_i$'s can make them very efficient for approximation of functions. The expansion of a function f over $[a,b)$ with respect to $\mathscr{E}_i$, $i = 0, 1, \ldots, m-1$, may be compactly written as

$$f(t) \simeq \sum_{i=0}^{m-1} f_i \mathscr{E}_i(t), \tag{3.25}$$

where f_i's, the expansion coefficients, may be computed by

$$\begin{aligned} f_i &= \langle f, \mathscr{E}_i \rangle \\ &= \int_a^b f(t)\mathscr{E}_i(t)\,\mathrm{d}t. \end{aligned} \tag{3.26}$$

In the next subsection, we will formulate a numerical method based on the exponential basis functions for solving the integral equations mentioned in section 2.

3.2 Formulation of point-matching method based on the exponential basis functions

Let us consider first kind Fredholm integral equation of the form

$$\int_a^b k(s,t)x(t)\,\mathrm{d}t = f(s), \qquad a \leqslant s < b, \tag{3.27}$$

where the functions k and f are known but x is the unknown function to be determined. Also, $k \in \mathscr{L}^2\big([a,b) \times [a,b)\big)$ and $f \in \mathscr{L}^2\big([a,b)\big)$.
Approximating the unknown function x with respect to the exponential basis functions using (3.25) gives

$$x(t) \simeq \sum_{i=0}^{m-1} x_i \mathscr{E}_i(t), \tag{3.28}$$

where x_i's are defined as in (3.26). Substituting (3.28) into (3.27) results in

$$\int_a^b k(s,t) \left(\sum_{i=0}^{m-1} x_i \mathscr{E}_i(t) \right) \mathrm{d}t \simeq f(s), \tag{3.29}$$

or

$$\sum_{i=0}^{m-1} x_i \int_a^b k(s,t)\mathscr{E}_i(t)\,\mathrm{d}t \simeq f(s). \tag{3.30}$$

Now, choosing m appropriate points $s_j \in [a,b)$, $j = 0, 1, \ldots, m-1$, and replacing '$\simeq$' with '$=$', we obtain

$$\sum_{i=0}^{m-1} x_i \int_a^b k(s_j,t)\mathscr{E}_i(t)\,\mathrm{d}t = f(s_j), \qquad j = 0, 1, \ldots, m-1. \tag{3.31}$$

Eq. (3.31) is a linear system of m algebraic equations in terms of m unknown coefficients x_i. Solution of this systems gives x_i's, and then we obtain an approximate solution $x(s) \simeq \sum_{i=0}^{m-1} x_i \mathscr{E}_i(s)$ for (3.27).

4 Numerical results

Now, we apply the numerical method formulated in section 3 in solution of integral equations (2.5), (2.14), and (2.20) to determine the electrostatic surface charge distribution q_s induced on the conducting finite-width strip, incomplete cylindrical surface, and incomplete spherical surface. All the implementations have been performed by setting $m = 256$. The results for the strip problem are given in Fig. 4 for $a = 2$ (m) and $n = 4$, which are in good agreement with the exact solution. The results for the incomplete cylindrical surface for $a = 1$ (m), $\varphi_0 = \frac{3\pi}{2}$ (rad), and $n = 6$ are shown in Fig. 5.

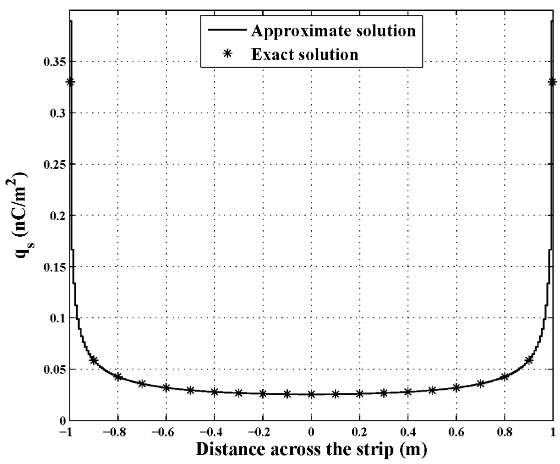


Figure 4: Electrostatic charge distribution induced on the conducting finite-width strip for $a = 2$ (m), obtained by the numerical method together with the exact results.

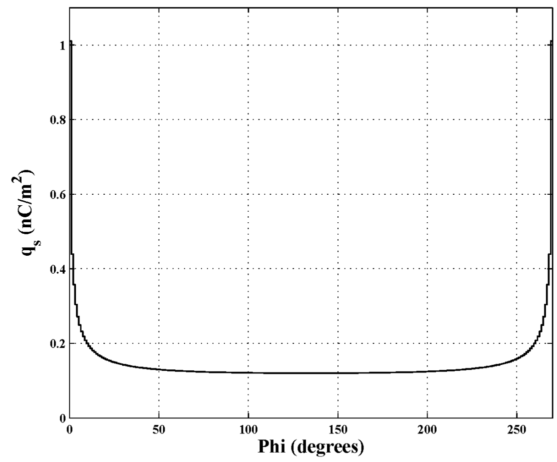
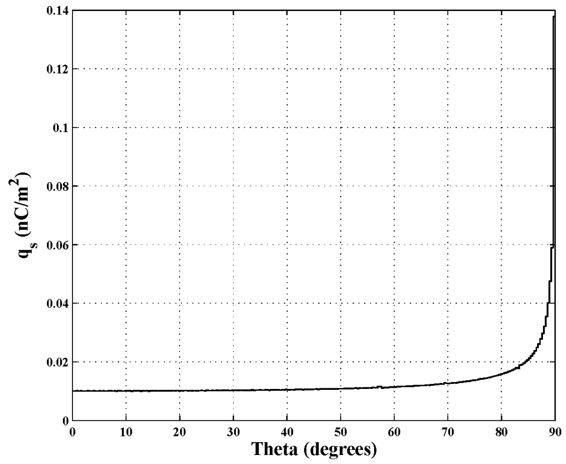
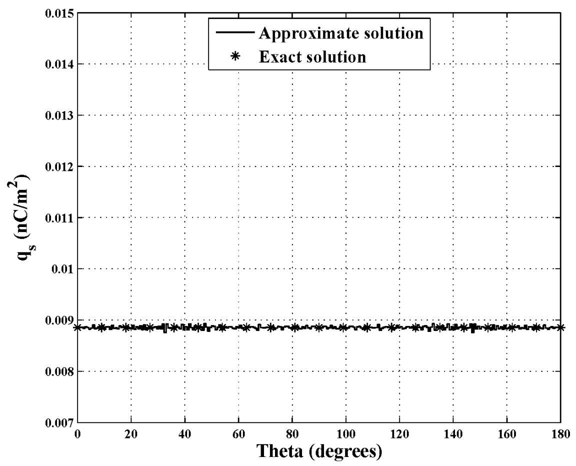


Figure 5: Electrostatic charge distribution induced on the conducting incomplete cylindrical surface for $a = 1$ (m) and $\varphi_0 = \frac{3\pi}{2}$ (rad), obtained by the numerical method.



(a)



(b)

Figure 6: (a) Electrostatic charge density induced on the conducting incomplete spherical surface for $a = 1$ (m) and $\theta_0 = \frac{\pi}{2}$ (rad), obtained by the numerical method. (b) Electrostatic charge density induced on the conducting spherical surface for $a = 1$ (m) and $\theta_0 = \pi$ (rad) (complete sphere), obtained by the numerical method, which is in agreement with the exact solution given by the Gauss's law.

The results for the incomplete spherical surface for $a = 1$ (m), $\theta_0 = \frac{\pi}{2}$ (rad), and $n = 12$ are given in Fig. 6(a). For $\theta_0 = \pi$ (rad) we have a complete sphere and the surface charge density q_s should have a constant value, according to the Gauss's law, throughout the surface of the sphere. To calculate the exact value of q_s in this case, we can write

$$\phi = \frac{Q}{4\pi\varepsilon_0 a}, \tag{4.32}$$

where Q is the total charge on the sphere. Considering $Q = q_s(4\pi a^2)$ and substituting $\phi = 1$ (V) and $a = 1$ (m) we obtain

$$\begin{aligned} q_s &= \varepsilon_0 \\ &\simeq 8.854e-12 \ \ (\mathrm{C/m^2}), \end{aligned} \tag{4.33}$$

for any arbitrary θ.
The results obtained by the numerical method for the case of $\theta_0 = \pi$ (rad), $a = 1$ (m), and $n = 12$ are shown in Fig. 6(b) which are in good agreement with the exact constant value of $q_s \simeq 8.854e-12$ (C/m^2) given by (4.33).

5 Conclusion

We analyzed in this article some problems of calculating the electrostatic charge distribution induced on conducting structures based on integral equation modeling. We defined a special set of exponential basis functions to present a numerical method for obtaining an approximate solution for the related integral equations because their analytical solution is often unknown. The numerical results obtained by the method confirmed its computational efficiency in solution of such problems.

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