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Research Article

ADVANCED COMPUTATIONAL TECHNIQUES IN ELECTROMAGNETICS

The error analysis and convergence evaluation of a computational technique for solving electromagnetic scattering problems

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Abstract
A computational technique for solving a specific class of electromagnetic scattering problems has been proposed in [1] based on integral equation modeling. We perform, in this article, an error analysis for that method to obtain some error bounds for it and estimate its convergence rate.

Keywords: Electromagnetic scattering; Computational technique; Error analysis; Convergence evaluation.

1 Introduction

One way for modeling of electromagnetic scattering problems is based on using Magnetic Field Integral Equation (MFIE). This equation, in view of mathematical form, is a Fredholm integral equation of the second kind. An interesting computational technique for solving the MFIE model has been proposed in [1] by using an appropriate set of basis functions. In this article, after a brief review on the method, we obtain some error bounds for it and then estimate the convergence rate analytically. The estimates will show that the error of the method will be rapidly convergent to zero if the degree of the quadrature rule used in implementation of the method is high enough.

2 A very quick review of the technique

As mentioned above, the computational technique presented in [1] has been proposed for solving MFIE which has the form of a Fredholm integral equation of the second kind. Therefore, let us consider the general form of such an integral equation as follows:

$$x(s) + \int_a^b k(s,t)x(t)\,\mathrm{d}t = f(s), \qquad s \in [a,b], \tag{2.1}$$

in which k and f are known functions and x is the unknown function to be determined. Also, $a,b \in \mathbb{R}$, $f \in \mathscr{L}^2\big([a,b]\big)$, and $k \in \mathscr{L}^2\big([a,b]\times[a,b]\big)$, where $\mathscr{L}^2$ is the space of square integrable functions.
For solving integral equation (2.1), the unknown function x is expanded in terms of Chebyshev cardinal functions as

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$x(s) \simeq x_N(s) = \sum_{j=0}^{N} a_j C_j(s)$; where x_N is the approximate solution of integral equation (2.1) which is obtained by the technique and C_j's are the Chebyshev cardinal functions defined by [2, 1]

$$C_j(t) = \frac{(-1)^{j+1}(1-t^2)\dot{T}_N(t)}{c_j N^2 (t-t_j)}, \qquad j = 0,1,\ldots,N, \tag{2.2}$$

with $c_0 = c_N = 2$, and $c_j = 1$, for $1 \leqslant j \leqslant N-1$. It is assumed, in (2.2), that

$$\dot{T}_N(t) = \frac{\mathrm{d}\, T_N(t)}{\mathrm{d}t}, \tag{2.3}$$

where

$$T_N(t) = \cos\left(N\cos^{-1}(t)\right). \tag{2.4}$$

Also, the coefficients in the $x(s)$ expansion are $a_j = x(t_j) = x_N(t_j)$, in which t_j's are the interpolation (grid) points defined as

$$t_j = \cos(\frac{j\pi}{N}). \tag{2.5}$$

Moreover, the integral operator in Eq. (2.1) is approximated by the Clenshaw-Curtis quadrature rule (for a much fuller treatment of this quadrature rule, see for example [3, 4, 5, 6]).
After several steps of various mathematical operations, the following equation is finally obtained [1]:

$$a_q + \sum_{p=0}^{N}{}'' w_p k(t_q,t_p) a_p = f(t_q), \qquad q = 0,1,\ldots,N, \tag{2.6}$$

where

$$w_p = \frac{4}{N} \sum_{\substack{n=0 \\ n \text{ even}}}^{N}{}'' \frac{1}{1-n^2} \cos\left(\frac{np\pi}{N}\right), \tag{2.7}$$

and the notation $\sum''$ means the first and last terms are to be halved before summing.
Eq. (2.6) is a linear system of $N+1$ algebraic equations for $N+1$ unknowns $a_0, a_1, \ldots, a_N$. Solution of this system gives the unknown coefficients. Hence, an approximate solution $x_N(s) = \sum_{j=0}^{N} a_j C_j(s)$ is obtained for integral equation (2.1).

3 Error analysis and convergence evaluation

Here, we perform an error analysis for the technique mentioned in the previous section to obtain some error bounds and evaluate its rate of convergence.
Let us consider the following integral equation:

$$x(s) + \int_a^b k(s,t)x(t)\,\mathrm{d}t = f(s), \qquad s \in [a,b]. \tag{3.8}$$

Eq. (3.8) can be rewritten in the following form:

$$(I + \mathscr{K})x = f, \tag{3.9}$$

where I is identity operator and operator $\mathscr{K}$ is defined as

$$\left(\mathscr{K}x\right)(s) = \int_a^b k(s,t)x(t)\,\mathrm{d}t. \tag{3.10}$$

Let us consider

$$\sum_{j=0}^{N} w_j k(s,t_j)x(t_j) = \left(\mathscr{K}x\right)(s) - E_t\left(k(s,t)x(t)\right), \tag{3.11}$$

in which E_t indicates the error functional for the quadrature rule operating on $k(s,t)x(t)$, viewed as a function of t for fixed s. Now, suppose x_N be the approximate solution of (3.8) obtained by the presented method. It satisfies the following equation:

$$x_N(s) + \sum_{j=0}^{N} w_j k(s,t_j) x_N(t_j) = f(s). \tag{3.12}$$

So, the error of the presented method can be defined as

$$e_N(s) = x(s) - x_N(s), \tag{3.13}$$

where $x(s)$ is the exact solution of (3.8). By combination of (3.11) and (3.12) we obtain

$$x_N(s) + \int_a^b k(s,t) x_N(t)\,\mathrm{d}t - E_t\big(k(s,t)x_N(t)\big) = f(s). \tag{3.14}$$

Subtracting (3.14) form (3.8) gives

$$e_N(s) + \int_a^b k(s,t) e_N(t)\,\mathrm{d}t = E_t\big(k(s,t)x_N(t)\big). \tag{3.15}$$

Thus, the error function $e_N(s)$ satisfies an integral equation with the same kernel as (3.8), but with different driving terms. Equation (3.15) can also be written in the form

$$\big(I + \mathscr{K}\big) e_N(s) = E_t\big(k(s,t)x_N(t)\big), \tag{3.16}$$

This equation has the solution

$$e_N(s) = \big(I + \mathscr{K}\big)^{-1} E_t\big(k(s,t)x_N(t)\big), \tag{3.17}$$

or

$$e_N(s) = \big(I - \mathscr{H}\big) E_t\big(k(s,t)x_N(t)\big), \tag{3.18}$$

where $\mathscr{H}$ is the resolvent operator (see [8, 7]). Hence, taking arbitrary norms we find

$$\|e_N\| \leqslant \big(1 + \|\mathscr{H}\|\big)\,\big\|E_t\big(k(s,t)x_N(t)\big)\big\|. \tag{3.19}$$

However, if $\|\mathscr{K}\| < 1$ we find, on taking norms in (3.15) and rearranging, the simpler bound

$$\|e_N\| \leqslant \frac{\|E_t\big(k(s,t)x_N(t)\big)\|}{1 - \|\mathscr{K}\|}. \tag{3.20}$$

The bounds of the error show that e_N is directly related to the error of the quadrature rule.
The simplest and most commonly used procedure for estimating the achieved accuracy is to choose a family of quadrature rules R_N and to compute an approximate solution for members of this family with increasing N, and hence decreasing error, until the results appear to have settled down to the required accuracy. It is usually straightforward to guarantee convergence; that is, to ensure that [3]

$$\lim_{N\to\infty} \| e_{R_N} \| = 0. \tag{3.21}$$

Sufficient conditions for this are given in [3, 9].
Now, we try to compute the rate at which convergence is achieved.

Lemma 3.1. *The error estimates for N–point quadrature rules of various types have the form*

$$\big|E_t f\big| = \left| \int_a^b f(t)\,\mathrm{d}t - \sum_{i=1}^{N} w_i f(t_i) \right| \leqslant C(f) N^{-p}, \tag{3.22}$$

where $C(f)$ is a constant dependent on the function $f(t)$ and the exponent p depends either on the degree of the rule (if $f(t)$ is smooth enough) or on the continuity properties of $f(t)$ (if the degree of the rule is high enough). We refer to p as the order of convergence.

Proof. See [3].
Now, considering Lemma 3.1 and using (3.19) we obtain

$$\|e_N\| \leqslant \big(1+ \| \mathscr{H} \|\big) \big\|C\big(k(s,t)x_N(t)\big)\big\|N^{-p}. \tag{3.23}$$

If C is uniformly bounded in s such that

$$\Big|C\big(k(s,t)x_N(t)\big)\Big| \leqslant C_0, \qquad a \leqslant s \leqslant b, \tag{3.24}$$

then, taking the maximum norm $\|\cdot\|_\infty$ on (3.23) gives

$$\|e_N\|_\infty \leqslant \big(1+ \| \mathscr{H} \|_\infty\big)C_0 N^{-p}, \tag{3.25}$$

where C_0 has a finite positive value. □

The estimates of convergence rate above show that the presented error will be rapidly convergent to zero if the degree of the quadrature rule is high enough.

4 Conclusion

An error analysis was performed for a computational technique for solving MFIE. Some error bounds were computed and the convergence rate was estimated. The evaluations showed that error of the technique will be rapidly convergent to zero if the degree of the quadrature rule used in implementation of the method is high enough.

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