# Ring structures in coarse $K$-theory

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# Abstract

The $K$-theory of the stable Higson corona of a coarse space carries a canonical ring structure. The present thesis covers two aspects of this ring:

Chapter 1: The $K$-theory of the stable Higson corona is the domain of an unreduced version of the coarse co-assembly map of Emerson and Meyer. We show that the target also carries a ring structure and co-assembly is a ring homomorphism, provided that the given coarse space is contractible in a coarse sense.

Chapter 2 (pursuing conjectures of John Roe): Applied to a foliated cone over a foliation, we show that the $K$-theory of the stable Higson corona can be considered as a new model for the $K$-theory of the leaf space, which is – in contrast to Connes' $K$-theory model – a ring. We show that Connes' $K$-theory model is a module over this ring and develop an interpretation of the module structure in terms of twisted longitudinally elliptic operators.

# Introduction

Let $X$ be a countably generated coarse space, e. g. a proper metric space. A well studied object in coarse geometry is the coarse assembly map

$$\mu : KX_*(X) \to K_*(C^*X)$$

from the coarse $K$-homology of $X$ into the $K$-theory of the Roe algebra $C^*X$ of $X$. It is subject of the coarse Baum-Connes conjecture, which has numerous implications to other famous conjectures like the Baum-Connes and the Novikov conjecture. We refer to [Roe96] for a detailed exposition.

The stable Higson corona $\mathfrak{c}(X)$ is the $C^*$-algebra of all continuous function of vanishing variation on $X$ with values in the $C^*$-algebra of compact operators $\mathfrak{K} := \mathfrak{K}(\ell^2)$ (where $\ell^2$ denotes our preferred infinite dimensional, separable Hilbert space, e. g. $\ell^2 = \ell^2(\mathbb{N})$ or $\ell^2 = \ell^2(\mathbb{Z})$) modulo the ideal $C_0(X, \mathfrak{K})$. It was introduced by Emerson and Meyer in [EM06] as an ingredient to the construction of a coarse co-assembly map

$$\mu^* : \tilde{K}_{1-*}(\mathfrak{c}(X)) \to KX^*(X)$$

which is dual to the coarse assembly map $\mu$. Here, $\tilde{K}_*(\mathfrak{c}X) = K_*(\mathfrak{c}X)/\mathbb{Z}$ is the reduced $K$-theory of the stable Higson corona. The co-assembly map, too, has applications to the Baum-Connes and the Novikov conjecture [EM07, EM08].

At first one would expect $K^*(C^*X)$, the $K$-homology of the Roe algebra $C^*X$, as the domain of $\mu^*$. However, it is not even clear how to define this group correctly, because $C^*X$ lacks separability. In contrast, $\tilde{K}_{1-*}(\mathfrak{c}(X))$ behaves in many ways as expected. For instance, we have functoriality under coarse maps and $\mu^*$ is an isomorphism for scalable spaces which are uniformly contractible and have bounded geometry.

The principle topic of this thesis is the canonical ring structure on the *unreduced* $K$-theory of the stable Higson corona: Multiplication of functions by means of an identification $\mathfrak{K} \otimes \mathfrak{K} \cong \mathfrak{K}$ yields a $*$-homomorphism

$$\mathfrak{c}(X) \otimes \mathfrak{c}(X) \to \mathfrak{c}(X)$$

which induces the ring multiplication

$$K_i(\mathfrak{c}(X)) \otimes K_j(\mathfrak{c}(X)) \to K_{i+j}(\mathfrak{c}(X)).$$

The two main Chapters of this thesis cover two different aspects of the ring structure:

### Chapter 1: Coarse co-assembly as a ring homomorphism

This chapter is concerned with an unreduced version of the coarse co-assembly map,

$$\mu^* : K_*(\mathfrak{c}(X)) \to KX^{1-*}(X \setminus \{pt\}),$$

which we derive from the co-assembly map $\mu^* : \tilde{K}_{1-*}(\mathfrak{c}(X)) \to KX^*(X)$ of Emerson and Meyer [EM06]. The target is by definition the $K$-theory of the Rips complex $\mathcal{P}$ of $X$ with one point $pt \in \mathcal{P}$ removed.

As the domain $K_*(\mathfrak{c}(X))$ of the unreduced coarse co-assembly map is a graded ring, it is a natural question to ask whether there is also a secondary ring structure

$$KX^i(X \setminus \{pt\}) \otimes KX^j(X \setminus \{pt\}) \to KX^{i+j-1}(X \setminus \{pt\})$$

on the target such that $\mu^*$ becomes a ring homomorphism. The usual primary ring structure in $K$-theory,

$$KX^i(X \setminus \{pt\}) \otimes KX^j(X \setminus \{pt\}) \to KX^{i+j}(X \setminus \{pt\}),$$

is obviously not suitable because of degree reasons.

The most enlightening example is the open cone $X = \mathcal{O}Y$ over a nice compact base space $Y$. In this case, the topological side can be identified with $K^{-*}(Y)$ and co-assembly becomes a ring isomorphism

$$K_*(\mathfrak{c}(X)) \cong K^{-*}(Y).$$

It turns out that an appropriate prerequisite for the existence of a secondary ring structure in the general case is the notion of coarse contractibility which we introduce in Definition 1.9.1. Open cones over compact spaces are coarsely contractible in this sense. So are foliated cones as defined in [Roe95] (see also Section 2.2), CAT(0) spaces and hyperbolic metric spaces, in particular Gromov's hyperbolic groups.

Given a coarse contraction, we obtain a suitable contraction $H : \mathcal{P} \times [0,1] \to \mathcal{P}$ onto the point $pt$ which we use to construct the proper continuous map

$$\begin{aligned}\Gamma:\ (0,1)\times(\mathcal{P}\setminus\{pt\}) &\to (\mathcal{P}\setminus\{pt\})\times(\mathcal{P}\setminus\{pt\})\\ (t,x) &\mapsto \begin{cases}(H(x,1-2t),x) & t\le 1/2\\ (x,H(x,2t-1)) & t\ge 1/2.\end{cases}\end{aligned}$$

Given this data, the secondary product on $KX^*(X\setminus\{pt\}) := K^*(\mathcal{P}\setminus\{pt\})$ is defined as $(-1)^i$ times the composition

$$\begin{aligned}K^i(\mathcal{P}\setminus\{pt\})\otimes K^j(\mathcal{P}\setminus\{pt\}) &\longrightarrow K^{i+j}(\,(\mathcal{P}\setminus\{pt\})\times(\mathcal{P}\setminus\{pt\})\,)\\ &\xrightarrow{\Gamma^*} K^{i+j}(\,(0,1)\times(\mathcal{P}\setminus\{pt\})\,)\\ &\xrightarrow{\cong} K^{i+j-1}(\mathcal{P}\setminus\{pt\}).\end{aligned}$$

Note that this secondary product arises like most secondary products by comparing two different reasons for the vanishing of the primary product. Here, the two reasons are the homotopies $\Gamma|_{(0,\frac{1}{2}]\times(\mathcal{P}\setminus\{pt\})}$ and $\Gamma|_{[\frac{1}{2},1)\times(\mathcal{P}\setminus\{pt\})}$ from the diagonal embedding to "infinity".

Our main result is the following.

**Theorem 1.9.8.** *Let $X$ be a coarsely contractible, countably generated coarse space. Then the unreduced coarse co-assembly map*

$$\mu^*: K_*(\mathfrak{c}(X)) \to KX^{1-*}(X\setminus\{pt\})$$

*is a ring homomorphism.*

There is also a more general coarse co-assembly map with coefficients in a $C^*$-algebra $D$. The stable Higson corona $\mathfrak{c}(X;D)$ with coefficients in $D$ is defined just like $\mathfrak{c}(X)$ with the only difference that we consider functions $X\to D\otimes\mathfrak{K}$. Furthermore, if we define

$$KX^*(X\setminus\{pt\};D) := K_{-*}(C_0(\mathcal{P}\setminus\{pt\})\otimes D)$$

then there are products

$$\begin{gathered}K_i(\mathfrak{c}(X;D))\otimes K_j(\mathfrak{c}(X;E)) \to K_{i+j}(\mathfrak{c}(X;D\otimes E))\\ KX^i(X\setminus\{pt\};D)\otimes KX^j(X\setminus\{pt\};E) \to KX^{i+j-1}(X\setminus\{pt\};D\otimes E)\end{gathered}$$

which are interior products with respect to the space but exterior products with respect to the coefficient algebra. Co-assembly also respects these products.

**Theorem 1.9.7.** *Let $X$ be a coarsely contractible, countably generated coarse space. Then the unreduced coarse co-assembly map with coefficients in $D$,*

$$\mu^* : K_*(\mathfrak{c}(X;D)) \to KX^{1-*}(X \setminus \{pt\}; D)\,,$$

*is multiplicative.*

In fact, using the stabilized version of the Higson corona is not necessary. Everything works just as well for the usual Higson corona or the unstabilized Higson corona with coefficients. However, the work of Emerson and Meyer indicates that it is the stabilized versions which one needs to consider in applications.

### Chapter 2: $K$-theory of leaf spaces of foliations

Roe suggested in [Roe95] to investigate foliated cones $\mathcal{O}(V,\mathcal{F})$ constructed from foliations $(V,\mathcal{F})$. Topologically they are cones over $V$, but they are equipped with a Riemannian metric which blows up only in the directions transverse to the foliation. From a coarse geometric point of view, one sees the leaves diverging and therefore coronas of these coarse spaces may be thought of as models for the leaf space.

Roe defined $K^*_{FJ}(V/\mathcal{F}) := K^{*+1}(C^*(\mathcal{O}(V,\mathcal{F}) \cup \mathbb{R}^+))$ as a new $K$-theory model for the leaf space of the foliation. The index $FJ$ stands for Farrell-Jones, as this construction was motivated by the foliated control theory of [FJ90]. Furthermore, Roe gave some conjectures which center around a hypothetical ring structure on these groups.

As mentioned earlier, the $K$-homology of the Roe algebra is not well behaved and one should therefore use the $K$-theory of the stable Higson corona instead. In view of our results in Section 1.5, we propose the following modification:

**Definition 2.3.1.** The "Farrell-Jones" model for the $K$-theory of the leaf space of a foliation $(V,\mathcal{F})$ is $K^*_{FJ}(V/\mathcal{F}) := K_{-*}(\mathfrak{c}(\mathcal{O}(V,\mathcal{F})))$. [...]

The ring structure on the "Farrell-Jones" $K$-theory groups, which Roe conjectured, is now simply our ring structure on the $K$-theory of the stable Higson corona.

One might expect further basic properties of $K$-theory of "spaces", and indeed the new model satisfies the following: The rings $K^*_{FJ}(V/\mathcal{F})$ are contravariantly functorial under smooth maps of leaf spaces (cf. Theorem 2.3.2) and if $(V,\mathcal{F})$ comes from a fibre bundle $V \to B$ with connected fibre, then there is a canonical ring isomorphism $K^*_{FJ}(V/\mathcal{F}) \cong K^*(B)$ (Example 2.3.4).

Usually, one considers the $K$-theory of Connes' foliation algebra,

$$K_C^*(V/\mathcal{F}) := K_{-*}(C_r^*(V,\mathcal{F})),$$

as the $K$-theory of the leaf space [Con82, Sections 5,6]. We use the reduced foliation algebra $C_r^*(V,\mathcal{F})$, because our proofs of the main theorems don't work for the full foliation algebra $C^*(V,\mathcal{F})$. These groups are the right receptacles of indices of longitudinally elliptic operators (see [Con82, Section 7] and Section 2.9) and there is a wrong way functoriality $f_! : K_C^*(V_1/\mathcal{F}_1) \to K_C^*(V_2/\mathcal{F}_2)$ under $K$-oriented smooth maps of leaf spaces $f : V_1/\mathcal{F}_1 \to V_2/\mathcal{F}_2$ [HS87].

Roe asked in [Roe95] for the relation between Connes' $K$-theory model and the new "Farrell-Jones" $K$-theory model. With our modified definition of the latter, Roe's conjectures work out fine:

**Corollary 2.8.6** (cf. [Roe95, Conjecture 0.2])**.** *[...] $K_C^*(V/\mathcal{F})$ is a module over $K_{FJ}^*(V/\mathcal{F})$.*

**Corollary 2.10.4** (cf. [Roe95, p. 204])**.** *Assume that $T\mathcal{F}$ is even dimensional and* spin$^c$ *and let $\not{D}$ be the corresponding Dirac operator. Then the map*

$$p_! \circ p^* : K_{FJ}^*(V/\mathcal{F}) \to K_C^*(V/\mathcal{F})$$

*is module multiplication with* $\mathrm{ind}(\not{D}) \in K_C^0(V/\mathcal{F})$.

Here, $p : V \to V/\mathcal{F}$ is the canonical smooth map of leaf spaces, its domain being $V/\mathcal{F}_0$ for the trivial 0-dimensional foliation $\mathcal{F}_0$ on $V$ (cf. Example 2.2.11).

Proving the same result for odd dimensional spin$^c$ foliations would require the index theory of selfadjoint longitudinally elliptic operators and its relation to the module structure, which is not discussed in this thesis.

The module structure can also be interpretated by indices of twisted longitudinally elliptic operators. If $D$ is a longitudinally elliptic operator on $(V,\mathcal{F})$ and $F \to V$ a smooth vector bundle, then the twisted operator $D_F$ again is longitudinally elliptic. In general, it is not possible to calculate $\mathrm{ind}(D_F)$ from $\mathrm{ind}(D)$ and $F$ alone. It is possible, however, if the bundle is a bundle over the leaf space in an asymptotic sense, as defined in Definition 2.4.7. This condition ensures that $[F] \in K^0(V)$ is the pullback of an element $x_F \in K_{FJ}^0(V/\mathcal{F})$.

**Corollary 2.10.3.** *If $D$ is a longitudinally elliptic operator, $F \to V$ a smooth vector bundle for which there is an element $x_F \in K_{FJ}^0(V/\mathcal{F})$ with $[F] = p^*(x_F)$, then the index of the twisted operator $D_F$ is*

$$\mathrm{ind}(D_F) = x_F \cdot \mathrm{ind}(D) \in K_C^0(V/\mathcal{F}).$$

An illustrative special case is provided by fibre bundles $p : V \to B$ with connected fibre: Here, a longitudinally elliptic differential operator $D$ is a family of operators parametrized by $B$ and the pullback $F = p^*F'$ of a vector bundle $F'$ over $B$ is asymptotically a bundle over the leaf space. Under the canonical isomorphism $K^*_C(V/\mathcal{F}) \cong K^*(B)$ [Con82, Section 5], the indices $\mathrm{ind}(D), \mathrm{ind}(D_F)$ correspond to the family indices of $D, D_F$, and under the isomorphism $K^*_{FJ}(V/\mathcal{F}) \cong K^*(B)$ mentioned above, $x_F$ corresponds to $[F'] \in K^0(B)$. In this case, the corollary specializes to the rather obvious statement

$$\mathrm{ind}(D_F) = [F'] \cdot \mathrm{ind}(D) \in K^0(B)\,.$$

### Final comments

In Chapter 3, we list some open questions and possible future research projects. In particular, Section 3.2 contains conjectures about how the ring structure could be applied to coarse index theory.

Information of how the material is organized can be found at the beginnings of the respective chapters.

A list of frequently used notations is found at the end of this thesis. In particular it should be said that the symbol $\otimes$ will always denote the maximal tensor product. The minimal tensor product is denoted by $\otimes_{\min}$.

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# Contents

# Chapter 1

# Coarse co-assembly as a ring homomorphism

This chapter is organized as follows. Section 1.1 recalls basic notions of coarse geometry and introduces Higson type corona $C^*$-algebras. In the subsequent three sections 1.2–1.4, we give an account of the basic facts about $\sigma$-coarse and $\sigma$-locally compact spaces as well as $\sigma$-$C^*$-algebras and their $K$-theory. This is necessary, because instead of working with the Rips complex $\mathcal{P}_{Z,R}$ of a discrete metric space $Z$ at a fixed scale $R > 0$, i. e. the simplicial complex consisting of one simplex with vertex set $S$ for every finite subset $S \subset Z$ of diameter at most $R$ (cf. [BH99, Definition 3.22]), we define the Rips complex just as in [EM06] as $\mathcal{P}_Z := \bigcup_{n\in\mathbb{N}} \mathcal{P}_{Z,n}$, i. e. as the direct limit of all finite scale Rips complexes, and this is only a $\sigma$-locally compact space. Properties of $K$-theory of $\sigma$-$C^*$-algebras concerning the products which cannot be found in the literature are proven in Appendices 1.A, 1.B.

In a nutshell, sections 1.2–1.4 say that these notions behave very much like their non-$\sigma$-counterparts. Thus, the impatient reader should get away with skimming over these sections.

In Section 1.5, we review the definition of the Rips complex and the concept of coarse co-assembly before deriving unreduced versions from the reduced ones. The $K$-theory product of the stable Higson corona is introduced in Section 1.6, where we also establish co-assembly as ring isomorphism in the case of open cones. The definition of the secondary ring structure on the target along with proofs of its most basic properties are found in Section 1.7. Multiplicativity of the co-assembly map is proven in Section 1.8. Finally, we introduce the notion of coarse contractibility as a prerequisite for $\sigma$-contractibility of the Rips complex in Section 1.9.

## 1.1 Coarse geometry and coronas

The purpose of this first section is to recall some basics of coarse geometry and Higson corona $C^*$-algebras. Less surprisingly, as far as coarse geometry is concerned, we shall stick quite closely to the very concise presentation of this topic in [EM06, Section 2]. More comprehensive references to coarse geometry are [Roe03] and [HR00, Chapter 6].

**Definition 1.1.1** ([EM06, Definition 2.1])**.** A *coarse structure* on a set $X$ is a collection $\mathcal{E}$ of subsets of $X \times X$, called controlled sets or entourages, which contains the diagonal and is closed under the formation of subsets, inverses, products and finite unions and contains all finite subsets, i. e.

1. $\Delta_X := \{(x,x) \mid x \in X\} \in \mathcal{E}$;
2. if $E \in \mathcal{E}$ and $E' \subset E$, then $E' \in \mathcal{E}$;
3. if $E \in \mathcal{E}$, then $E^{-1} := \{(y,x) \mid (x,y) \in E\} \in \mathcal{E}$;
4. if $E_1, E_2 \in \mathcal{E}$, then
$$E_1 \circ E_2 := \{(x,z) \mid \exists y \in X : (x,y) \in E_1 \text{ and } (y,z) \in E_2\} \in \mathcal{E};$$
5. if $E_1, E_2 \in \mathcal{E}$, then $E_1 \cup E_2 \in \mathcal{E}$;
6. if $E \subset X \times X$ is finite, then $E \in \mathcal{E}$.

A subset $B \subset X$ is called *bounded*, if $B \times B \in \mathcal{E}$. A *coarse space*[1] is a locally compact space $X$ equipped with a coarse structure $\mathcal{E}$ such that in addition

7. some neighborhood of the diagonal $\Delta \subset X \times X$ is an entourage;
8. every bounded subset of $X$ is relatively compact.

If $X$ is a coarse space and $A \subset X$, then the *subspace coarse structure* is the set of all subsets of $A \times A$ which are entourages of $X$. If $A$ is closed in $X$, then $A$ is a coarse space, too.

If $X, Y$ are coarse spaces with coarse structures $\mathcal{E}_X, \mathcal{E}_Y$, then the *product coarse structure* on $X \times Y$ consists of all subsets of $(X \times Y) \times (X \times Y)$ which are contained in

$$E_X \times E_Y := \{(x_1, y_1, x_2, y_2) \mid (x_1, x_2) \in E_X,\ (y_1, y_2) \in E_2\}$$

for some $E_X \in \mathcal{E}_X$, $E_Y \in \mathcal{E}_Y$. Again, $X \times Y$ is a coarse space.

A coarse space is called *countably generated* if there is an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of entourages such that any entourage is contained in $E_n$ for some $n \in \mathbb{N}$.

[1] In the terminology of [Roe03], this would be called a locally compact topological space equipped with a coarsely connected proper coarse structure.

**Example 1.1.2.** Let $(X,d)$ be a metric space in which every bounded set is relatively compact. The *metric coarse structure* on $X$ is the smallest coarse structure such that the sets

$$E_R := \{(x,y) \in X \times X \mid d(x,y) \leq R\}, \quad R \in \mathbb{N}$$

are entourages. Equipped with this coarse structure, $X$ becomes a countably generated coarse space.

**Definition 1.1.3.** A *coarse map* $\phi : X \to Y$ between coarse spaces is a Borel map such that $\phi \times \phi$ maps entourages to entourages and $\phi$ is proper in the sense that preimages of bounded sets are bounded. Two coarse maps $\phi, \psi : X \to Y$ are called *close* if $(\phi \times \psi)(\Delta_X) \subset Y \times Y$ is an entourage.

The *coarse category of coarse spaces* is the category of coarse spaces and closeness classes of coarse maps between them. A coarse map is called a *coarse equivalence* if it is an isomorphism in this category.

We are now in a position to introduce Higson type corona $C^*$-algebras which are the main coarse geometric players in this paper.

**Definition 1.1.4.** Let $X$ be a coarse space, $pt \in X$ a distinguished point, $(Y,d)$ a metric space and $D$ any $C^*$-algebra.

1. A Borel map $f : X \to Y$ is said to have *vanishing variation*, if for any entourage $E \subset X \times X$ the function

   $$\operatorname{Var}_E f :\ X \to [0,\infty), \quad x \mapsto \sup\{d(f(x),f(y)) \mid (x,y) \in E\}$$

   vanishes at infinity. [EM06, Definition 3.1]

2. For any coarse space $X$ and any $C^*$-algebra $D$, we let $\overline{\mathfrak{uc}}(X;D)$ be the $C^*$-algebra of bounded, continuous functions of vanishing variation $X \to D$. It is called the *unstable Higson compactification of $X$ with coefficients $D$*.

3. The *unstable Higson corona of $X$ with coefficients $D$* is the quotient $C^*$-algebra $\mathfrak{uc}(X;D) := \overline{\mathfrak{uc}}(X;D)/C_0(X;D)$.

4. The *pointed unstable Higson compactification of $X$ with coefficients $D$* is

   $$\overline{\mathfrak{uc}}_0(X;D) := \{f \in \overline{\mathfrak{uc}}(X;D) \mid f(pt) = 0\}.$$

   We have $\mathfrak{uc}(X;D) = \overline{\mathfrak{uc}}_0(X;D)/C_0(X \setminus \{pt\};D)$.

5. The stable counterparts of these function algebras are obtained by replacing $D$ with $D \otimes \mathfrak{K}$:

$$\begin{aligned}\bar{\mathfrak{c}}(X;D) &:= \overline{\mathfrak{uc}}(X;D\otimes\mathfrak{K}),\\ \bar{\mathfrak{c}}_0(X;D) &:= \overline{\mathfrak{uc}}_0(X;D\otimes\mathfrak{K}),\\ \mathfrak{c}(X;D) &:= \mathfrak{uc}(X;D\otimes\mathfrak{K}).\end{aligned}$$

   In particular, $\mathfrak{c}(X;D)$ is the *stable Higson corona of $X$ with coefficients $D$*. [EM06, Definition 3.2]

6. If $D = \mathbb{C}$, we usually omit $D$ from notation.

**Proposition 1.1.5.** *The assignments $X \mapsto \mathfrak{uc}(X,D)$, $X \mapsto \mathfrak{c}(X,D)$, $X \mapsto \mathfrak{c}(X)$, are contravariant functors from the coarse category of coarse spaces to the category of $C^*$-algebras.*

*The assignments $X \mapsto \overline{\mathfrak{uc}}(X,D)$, $X \mapsto \bar{\mathfrak{c}}(X,D)$, $X \mapsto \bar{\mathfrak{c}}(X)$, are contravariantly functorial with respect to continuous coarse maps.*

*The assignments $X \mapsto \overline{\mathfrak{uc}}_0(X,D)$, $X \mapsto \bar{\mathfrak{c}}_0(X,D)$, $X \mapsto \bar{\mathfrak{c}}_0(X)$, are contravariantly functorial with respect to pointed continuous coarse maps.*

*Furthermore, there is the obvious covariant functoriality in the coefficient algebra.*

*Proof.* See [EM06, Proposition 3.7] for the nontrivial parts of this proposition. □

## 1.2 $\sigma$-locally compact spaces and $\sigma$-coarse spaces

**Definition 1.2.1** ([EM06, Section 2])**.** A $\sigma$-locally compact space is an increasing sequence $(X_n)_{n\in\mathbb{N}}$ of subsets of a set $\mathcal{X}$ such that

- $\mathcal{X} = \bigcup_{n\in\mathbb{N}} X_n$,
- each $X_n$ is a locally compact Hausdorff space,
- $X_n$ carries the subspace topology of $X_m$ for all $n \le m$,
- $X_n$ is closed in $X_m$ for all $n \le m$.

By the usual abuse of notation, we will often call $\mathcal{X}$ instead of the sequence $(X_n)_{n\in\mathbb{N}}$ a $\sigma$-locally compact space. However, the sequence is an essential part of the definition.

Locally compact spaces are $\sigma$-locally compact spaces with all $X_n$ equal.

The set $\mathcal{X}$ can and will always be endowed with the final topology: A subset $\mathcal{A} \subset \mathcal{X}$ is open/closed if all the intersections $A_n = \mathcal{A} \cap X_n$ are open/closed. If $\mathcal{A} \subset \mathcal{X}$ is open or closed, then $\mathcal{X} \setminus \mathcal{A} = \bigcup_{n\in\mathbb{N}} X_n \setminus A_n$ is again a $\sigma$-locally compact space.

Note, that the following notion of morphism and cartesian product gives $\sigma$-locally compact spaces the structure of a monoidal category.

**Definition 1.2.2.** Let $\mathcal{X} = \bigcup_{n\in\mathbb{N}} X_n$ and $\mathcal{Y} = \bigcup_{n\in\mathbb{N}} Y_n$ be $\sigma$-locally compact spaces.

- A morphism $f : \mathcal{X} \to \mathcal{Y}$ is a map $f : \mathcal{U} \to \mathcal{Y}$ from an open subset $\mathcal{U} \subset \mathcal{X}$ such that for each $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $f(U_m) \subset Y_n$ and the restriction $f|_{U_m} : U_m \to Y_n$ is a proper continuous map.[2]

  We will also call such morphisms simply *proper continuous maps.*

- The cartesian product is defined by the sequence $(X_m \times Y_m)_{m\in\mathbb{N}}$. Passing forgetfully to the category of sets, we see that

$$\mathcal{X} \times \mathcal{Y} = \bigcup_{m\in\mathbb{N}} X_m \times Y_m$$

  and we apply the usual abuse of notation of denoting the product simply by $\mathcal{X} \times \mathcal{Y}$ instead of $(X_m \times Y_m)_{m\in\mathbb{N}}$.

It is worth noticing that continuity of the maps $f|_{U_m} : U_m \to Y_n$ is equivalent to continuity of $f : \mathcal{U} \to \mathcal{Y}$ in the final topologies.

A homotopy between two proper continuous maps $f, g : \mathcal{X} \to \mathcal{Y}$ is of course a proper continuous map $\mathcal{X} \times [0,1] \to \mathcal{Y}$ restricting to $f, g$ at $0, 1$, respectively.

Given a $\sigma$-locally compact space $\mathcal{X} = \bigcup_{n\in\mathbb{N}} X_n$, a $C^*$-algebra $D$ and a closed subset $\mathcal{A} \subset \mathcal{X}$, we can define the following function algebras:

$$\begin{aligned} C_0(\mathcal{X}; D) &= \{f : \mathcal{X} \to D :\ f|_{X_n} \in C_0(X_n; D)\, \forall n \in \mathbb{N}\} \\ C_b(\mathcal{X}, \mathcal{A}; D) &= \{f : \mathcal{X} \to D :\ f|_{X_n} \in C_b(X_n; D)\, \forall n \in \mathbb{N},\ f|_{\mathcal{A}} = 0\} \end{aligned}$$

In contrasts to the morphisms defined above, the functions in these function algebras are supposed to be defined on all of $\mathcal{X}$.

Note that they are $\sigma$-$C^*$-algebras in the sense of Definition 1.3.1 below, with $C^*$-seminorms given by $p_n(f) = \|f|_{X_n}\|_\infty$. Our definition of morphisms in the category of $\sigma$-locally compact spaces is such that a proper continuous map $f : \mathcal{X} \to \mathcal{Y}$ induces a $*$-homomorphism $f^* : C_0(\mathcal{Y}; D) \to C_0(\mathcal{X}; D)$. If

[2] These morphisms correspond to pointed continuous maps between one point compactifications.

additionally $f$ is defined on all of $\mathcal{X}$ and $\mathcal{A} \subset \mathcal{X}, \mathcal{B} \subset \mathcal{Y}$ are closed subsets such that $f(\mathcal{A}) \subset \mathcal{B}$, then $f$ induces a $*$-homomorphism $f^* : C_b(\mathcal{Y}, \mathcal{B}; D) \to C_b(\mathcal{X}, \mathcal{A}; D)$.

More properties of these function algebras are discussed in the next section.

We proceed with $\sigma$-coarse spaces.

**Definition 1.2.3.** A $\sigma$-coarse space is a $\sigma$-locally compact space in which each $X_n$ is a coarse space and $X_n$ has the subspace coarse structure of $X_m$ for all $n \leq m$ [EM06, Section 2]. A *coarse continuous map* is a map $f : \mathcal{X} \to \mathcal{Y}$ such that for each $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $f(X_m) \subset Y_n$ and the restriction $f|_{X_m} : X_m \to Y_n$ is a coarse continuous map.

Given a $\sigma$-coarse space $\mathcal{X} = \bigcup_{n \in \mathbb{N}} X_n$ and a $C^*$-algebra $D$, we define the $\sigma$-$C^*$-algebra

$$\overline{\mathfrak{uc}}(\mathcal{X}; D) := \{f : \mathcal{X} \to D : \; f|_{X_n} \in \overline{\mathfrak{uc}}(X_n; D) \, \forall n \in \mathbb{N}\}.$$

In the same manner we can generalize the other function algebras of Definition 1.1.4. In particular, if $pt \in X_0$ is a point then

$$\begin{aligned} \overline{\mathfrak{uc}}_0(\mathcal{X}; D) :&= \{f : \mathcal{X} \to D : \; f|_{X_n} \in \overline{\mathfrak{uc}}_0(X_n; D) \, \forall n \in \mathbb{N}\} \\ &= \{f \in \overline{\mathfrak{uc}}(\mathcal{X}; D) \; : \; f(pt) = 0\}, \\ \mathfrak{uc}(\mathcal{X}; D) :&= \overline{\mathfrak{uc}}(\mathcal{X}; D)/C_0(\mathcal{X}; D) = \overline{\mathfrak{uc}}_0(\mathcal{X}; D)/C_0(\mathcal{X} \setminus \{pt\}; D). \end{aligned}$$

A coarse continuous map $f : \mathcal{X} \to \mathcal{Y}$ mapping the basepoint in $X_0$ to the basepoint in $Y_0$ induces a $*$-homomorphism $f^* : \overline{\mathfrak{uc}}_0(\mathcal{Y}; D) \to \overline{\mathfrak{uc}}_0(\mathcal{X}; D)$.

## 1.3 $\sigma$-$C^*$-algebras

A good exposition of $\sigma$-$C^*$-algebras can be found in [Phi88]. In this section, we summarize all the properties that we shall need.

**Definition 1.3.1** (c.f. [Phi88])**.** A $\sigma$-$C^*$-algebra is a complex topological $*$-algebra such that its topology

- is Hausdorff,
- is generated by a countable family of $C^*$-seminorms $(p_\alpha)_{\alpha \in D}$, i.e. the $p_\alpha$ are submultiplicative and satisfy the $C^*$-identity

$$p_\alpha(a^* a) = p_\alpha(a)^2 \quad \forall a \in A,$$

- is complete with respect to the family of $C^*$-seminorms.

We can (and will) always assume without loss of generality that the index set is $D = \mathbb{N}$ and the $C^*$-seminorms are an increasing sequence: $p_n \leq p_m \ \forall n \leq m$

The quotients $A_n = A/\ker(p_n)$ are $C^*$-algebras and the continuous $*$-homomorphisms $A \to A_n$ and $A_m \to A_n$ $(n \leq m)$ are surjective. Furthermore, $A \cong \varprojlim_n A_n$ in the category of topological $*$-algebras.[3] In fact, we have the following equivalent characterization:

**Proposition 1.3.2** ([Phi88, Section 5]). *A topological $*$-algebra is a $\sigma$-$C^*$-algebra if and only if it is an inverse limit of an inverse system of $C^*$-algebras indexed over the natural numbers.*

As we have seen, the $*$-homomorphisms in this inverse system may be chosen to be surjections.

The $\sigma$-$C^*$-algebras defined in the previous section are the inverse limits of the inverse systems of $C^*$-algebras $C_0(X_n; D)$, $C_b(X_n, A_n; D)$, $\overline{\mathfrak{uc}}_0(X_n; D)$. In fact, we could have also chosen a $\sigma$-$C^*$-algebra $D$ as coefficient algebra by defining the function algebras as the inverse limits of $C_0(X_n; D_n)$, $C_b(X_n, A_n; D_n)$, $\overline{\mathfrak{uc}}_0(X_n; D_n)$. As before, they can also be defined as algebras of functions $\mathcal{X} \to D$.

We now collect some important features of $\sigma$-$C^*$-algebras.

**Proposition 1.3.3** ([Phi88, Theorem 5.2]). *Just as with $C^*$-algebras, $*$-homomorphisms between $\sigma$-$C^*$-algebras are automatically continuous.*

Just as in [Phi88], we will always call such continuous $*$-homomorphism simply *homomorphisms.*

*Ideals* in $\sigma$-$C^*$-algebras will always be closed two-sided selfadjoint ideals.

**Proposition 1.3.4** ([Phi88, Corollary 5.4]). *Let $A$ be a $\sigma$-$C^*$-algebra and let $I$ be an ideal in $A$. Then $A/I$ is a $\sigma$-$C^*$-algebra, and every homomorphism $\varphi : A \to B$ of $\sigma$-$C^*$-algebras such that $\varphi|_I = 0$ factors through $A/I$.*

If $\mathcal{X}$ is a $\sigma$-locally compact space, $\mathcal{A}$ a closed subset and $D$ a coefficient $C^*$-algebra, then $C_0(\mathcal{X} \setminus \mathcal{A}; D)$ is an ideal in $C_b(\mathcal{X}, \mathcal{A}; D)$. If $\mathcal{X}$ is a $\sigma$-coarse space and $D$ a coefficient $C^*$-algebra, then $C_0(\mathcal{X}; D)$ is an ideal in $\overline{\mathfrak{uc}}(\mathcal{X}; D)$, so in particular $\mathfrak{uc}(\mathcal{X}; D) = \overline{\mathfrak{uc}}(\mathcal{X}; D)/C_0(\mathcal{X}; D)$ is a $\sigma$-$C^*$-algebra. If $pt \in \mathcal{X}$ is a distinguished basepoint, then $C_0(\mathcal{X} \setminus \{pt\}; D)$ is an ideal in $\overline{\mathfrak{uc}}_0(\mathcal{X}; D)$.

[3] The algebra underlying this topological algebra is the inverse limit of the algebras underlying the $A_n$.

**Lemma 1.3.5** (cf. [EM06, Lemma 3.12 and preceding remarks])**.** *If all inclusions $X_n \subset X_m$ ($m \leq n$) are coarse equivalences, then $\mathfrak{uc}(\mathcal{X}; D)$ is a $C^*$-algebra which is canonically isomorphic to $\mathfrak{uc}(X_n; D)$ for all $n$. These isomorphisms are induced by the restriction maps $\overline{\mathfrak{uc}}(\mathcal{X}; D) \to \overline{\mathfrak{uc}}(X_n; D)$.*

A sequence

$$0 \to I \xrightarrow{\alpha} A \xrightarrow{\beta} B \to 0 \tag{1.1}$$

of $\sigma$-$C^*$-algebras and homomorphisms is called *exact* if it is algebraically exact, $\alpha$ is a homeomorphism onto its image, and $\beta$ defines a homeomorphism of $A/\ker(\beta) \to B$.

**Proposition 1.3.6.**

1. *For the sequence of $\sigma$-$C^*$-algebras and homomorphisms* (1.1) *to be exact, it is sufficient that it be algebraically exact. [Phi88, Corollary 5.5]*

2. *The sequence of $\sigma$-$C^*$-algebras and homomorphisms* (1.1) *is exact if and only if it is an inverse limit (with surjective maps) of exact sequences of $C^*$-algebras. [Phi88, Proposition 5.3(2)]*

If $\mathcal{X}$ is a $\sigma$-locally compact space and $\mathcal{A}$ a closed subset, then

$$0 \to C_0(\mathcal{X} \setminus \mathcal{A}) \to C_0(\mathcal{X}) \to C_0(\mathcal{A}) \to 0$$

is an exact sequence of $\sigma$-$C^*$-algebras. This assignment is natural under proper continuous maps between pairs of spaces $(\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$, i.e. proper continuous maps $\mathcal{X} \to \mathcal{Y}$ mapping the closed subset $\mathcal{A} \subset \mathcal{X}$ to the closed subset $\mathcal{B} \subset \mathcal{Y}$.

Just as for $C^*$-algebras, we can define the unitalization $\tilde{A}$ of a $\sigma$-$C^*$-algebra $A$ by extending multiplication and $C^*$-seminorms to $\tilde{A} = \mathbb{C} \oplus A$. Equivalently, if $A$ is written as inverse limit $\varprojlim_n A_n$ of $C^*$-algebras $A_n$, then $\tilde{A} = \varprojlim_n \tilde{A}_n$. We obtain the exact sequence

$$0 \to A \to \tilde{A} \to \mathbb{C} \to 0.$$

The maximal tensor product $\otimes$ of $C^*$-algebras can be extended to $\sigma$-$C^*$-algebras: Let $A, B$ be two $\sigma$-$C^*$-algebras and $p, q$ continuous $C^*$-seminorms on $A, B$, respectively. We denote by $p \otimes q$ the greatest $C^*$-cross-seminorm determined by $p$ and $q$, i. e. $p \otimes q$ is the greatest $C^*$-seminorm such that $(p \otimes q)(a \otimes b) = p(a)q(b)$ on elementary tensors $a \otimes b$.

**Definition 1.3.7** ([Phi88, Definition 3.1])**.** The maximal tensor product $A \otimes B$ of two $\sigma$-$C^*$-algebras $A, B$ is the completion of their algebraic tensor product $A \odot B$ with respect to the set of all $C^*$-cross-seminorms.

It is again a $\sigma$-$C^*$-algebra, with topology generated by the countable family of $C^*$-seminorms $(p_n \otimes q_n)_{n\in\mathbb{N}}$.

Of course, there is a universal property similar to the one for the maximal tensor product of $C^*$-algebras:

**Proposition 1.3.8** ([Phi88, Proposition 3.3])**.** *Let $A, B, C$ be $\sigma$-$C^*$-algebras and let $\varphi : A \to C$ and let $\psi : B \to C$ be two homomorphisms whose ranges commute. Then there is a unique homomorphism $\eta : A \otimes B \to C$ such that $\eta(a \otimes b) = \varphi(a)\psi(b)$ for all $a \in A, b \in B$.*

Directly from the definition of the maximal tensor product we deduce that if $f : A_1 \to A_2$, $g : B_1 \to B_2$ are continuous $*$-homomorphisms, then $f \otimes g : A_1 \odot B_1 \to A_2 \odot B_2$ extends to a continuous $*$-homomorphism $A_1 \otimes B_1 \to A_2 \otimes B_2$.

**Proposition 1.3.9** ([Phi88, Proposition 3.2])**.** *If $A = \varprojlim_n A_n$ and $B = \varprojlim_n B_n$, then $A \otimes B = \varprojlim_n A_n \otimes B_n$.*

This proposition allows the straightforward generalization of some known facts of $C^*$-algebras:

**Corollary 1.3.10.** *For any $\sigma$-locally compact space $\mathcal{X} = \bigcup_{n\in\mathbb{N}} X_n$ and coefficient $\sigma$-$C^*$-algebra $D$, the equation*

$$C_0(\mathcal{X}; D) = C_0(\mathcal{X}) \otimes D$$

*holds.*

*Proof.* $C_0(\mathcal{X}; D) = \varprojlim_{n\in\mathbb{N}} C_0(X_n; D_n) = \varprojlim_{n\in\mathbb{N}} C_0(X_n) \otimes D_n = C_0(\mathcal{X}) \otimes D$. □

**Corollary 1.3.11.** *Let $\mathcal{X} = \bigcup_{n\in\mathbb{N}} X_n$ and $\mathcal{Y} = \bigcup_{n\in\mathbb{N}} Y_n$ be $\sigma$-locally compact spaces. Then*

$$C_0(\mathcal{X} \times \mathcal{Y}) = C_0(\mathcal{X}) \otimes C_0(\mathcal{Y}).$$

*Proof.* $C_0(\mathcal{X} \times \mathcal{Y}) = \varprojlim_n C_0(X_n \times Y_n) = \varprojlim_n C_0(X_n) \otimes C_0(Y_n) = C_0(\mathcal{X}) \otimes C_0(\mathcal{Y})$. □

The property of maximal tensor products of $\sigma$-$C^*$-algebra, which is most important to us, is exactness:

**Theorem 1.3.12.** *The maximal tensor product of $\sigma$-$C^*$-algebras is exact.*

*Proof.* Given a short exact sequence

$$0 \to I \to A \to B \to 0$$

of $\sigma$-$C^*$-algebras, Proposition 1.3.6 allows us to write

$$I = \varprojlim I_n, \quad A = \varprojlim A_n, \quad B = \varprojlim B_n$$

where the directed systems fit into a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & I_{n+1} \arrow[r] \arrow[d] & A_{n+1} \arrow[r] \arrow[d] & B_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_n \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with exact rows and all vertical maps surjective.

The maximal tensor product of $C^*$-algebras is exact, so if $D = \varprojlim D_n$ is another $\sigma$-$C^*$-algebra, then

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & I_{n+1} \otimes D_{n+1} \arrow[r] \arrow[d] & A_{n+1} \otimes D_{n+1} \arrow[r] \arrow[d] & B_{n+1} \otimes D_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_n \otimes D_n \arrow[r] \arrow[d] & A_n \otimes D_n \arrow[r] \arrow[d] & B_n \otimes D_n \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

has exact rows and all vertical maps are surjective. Again by Proposition 1.3.6 the sequence

$$0 \to \varprojlim I_n \otimes D_n \to \varprojlim A_n \otimes D_n \to \varprojlim B_n \otimes D_n \to 0$$

is exact and the claim follows from Proposition 1.3.9. $\square$

**Corollary 1.3.13.** *If $I_{1,2} \subset A_{1,2}$ are ideals, then $I_1 \otimes I_2 \to A_1 \otimes A_2$ is a homeomorphism onto its image. Thus, $I_1 \otimes I_2$ is an ideal in $A_1 \otimes A_2$.*

**Corollary 1.3.14.** *There are canonical isomorphisms of $\sigma$-$C^*$-algebras*

$$\begin{aligned}(A_1 \otimes A_2)/(I_1 \otimes A_2) &\cong A_1/I_1 \otimes A_2,\\ (A_1 \otimes A_2)/(A_1 \otimes I_2) &\cong A_1 \otimes A_2/I_2,\\ (A_1 \otimes A_2)/(A_1 \otimes I_2 + I_1 \otimes A_2) &\cong A_1/I_1 \otimes A_2/I_2.\end{aligned}$$

*Proof.* The first two are direct consequences of exactness of the tensor product and the definition of short exact sequences of $\sigma$-$C^*$-algebras. For the third, note that algebraically we have

$$\begin{aligned}\frac{A_1 \otimes A_2}{A_1 \otimes I_2 + I_1 \otimes A_2} &\cong \frac{(A_1 \otimes A_2)/(I_1 \otimes A_2)}{(A_1 \otimes I_2 + I_1 \otimes A_2)/(I_1 \otimes A_2)}\\ &\cong \frac{(A_1 \otimes A_2)/(I_1 \otimes A_2)}{(A_1 \otimes I_2)/(I_1 \otimes A_2 \cap A_1 \otimes I_2)}\\ &= \frac{(A_1 \otimes A_2)/(I_1 \otimes A_2)}{(A_1 \otimes I_2)/(I_1 \otimes I_2)} \cong \frac{A_1/I_1 \otimes A_2}{A_1/I_1 \otimes I_2}\\ &\cong A_1/I_1 \otimes A_2/I_2.\end{aligned}$$

Continuity of this isomorphism and its inverse are automatic by Proposition 1.3.3. □

This corollary will come in handy for constructing homomorphisms

$$A_1/I_1 \otimes A_2 \to B, \quad A_1 \otimes A_2/I_2 \to B, \quad A_1/I_1 \otimes A_2/I_2 \to B$$

into another $\sigma$-$C^*$-algebra $B$.

The notion of homotopy is of course the canonical one:

**Definition 1.3.15.** A homotopy between two homomorphisms $f, g : A \to B$ is a homomorphism $A \to C[0,1] \otimes B$ such that evaluation at $0, 1$ yields $f$ and $g$.

A homotopy $H : [0,1] \times \mathcal{X} \to \mathcal{Y}$ between two proper continuous maps $f, g : \mathcal{X} \to \mathcal{Y}$ gives rise to a homotopy $H^* : C_0(\mathcal{Y}) \to C[0,1] \otimes C_0(\mathcal{X})$ between $f^*, g^*$.

## 1.4 $K$-theory of $\sigma$-$C^*$-algebras

The construction of the coarse co-assembly map in [EM06] requires $K$-theory for $\sigma$-$C^*$-algebras. This theory was developed in [Phi89] (see also [Phi91]). To make ourselves independent of a concrete picture of $K$-theory, we state the properties we need in the following five axioms. Their $C^*$-algebraic counterparts are well known.

**Axiom 1.4.1.** *$K$-theory of $\sigma$-$C^*$-algebras is a covariant homotopy functor from the category of $\sigma$-$C^*$-algebras into the category of $\mathbb{Z}/2$-graded abelian groups.*

**Axiom 1.4.2.** *Naturally associated to each exact sequence $0 \to I \to A \to B \to 0$ of $\sigma$-$C^*$-algebras is a six term exact sequence*

$$\begin{tikzcd} K_0(I) \arrow[r] & K_0(A) \arrow[r] & K_0(B) \arrow[d] \\ K_1(B) \arrow[u] & K_1(A) \arrow[l] & K_1(I). \arrow[l] \end{tikzcd}$$

**Axiom 1.4.3.** *Let $\tau : (0,1) \to (0,1)$, $t \mapsto 1 - t$. For any $\sigma$-$C^*$-algebra $A$, the induced homomorphisms*

$$K_*(C_0(0,1) \otimes A) \xrightarrow{(\tau^* \otimes \mathrm{id}_A)_*} K_*(C_0(0,1) \otimes A)$$

*is multiplication by $-1$.*

**Axiom 1.4.4.** *There is an associative and graded commutative exterior product*

$$K_i(A) \otimes K_j(B) \to K_{i+j}(A \otimes B)$$

*which is natural in both variables.*

**Axiom 1.4.5.** *The exterior product is compatible with boundary maps in the following sense: The diagram*

$$\begin{tikzcd} K_i(B) \otimes K_j(D) \arrow[r] \arrow[d] & K_{i-1}(I) \otimes K_j(D) \arrow[d] \\ K_{i+j}(B \otimes D) \arrow[r] & K_{i+j-1}(I \otimes D) \end{tikzcd}$$

*commutes and the diagram*

$$\begin{tikzcd} K_i(D)\otimes K_j(B) \arrow[r] \arrow[d] & K_i(D)\otimes K_{j-1}(I) \arrow[d] \\ K_{i+j}(D\otimes B) \arrow[r] & K_{i+j-1}(D\otimes I) \end{tikzcd}$$

*commutes up to a sign* $(-1)^i$ *whenever the upper horizontal arrows are connecting homomorphisms associated to a short exact sequence* $0\to I\to A\to B\to 0$ *of* $\sigma$-$C^*$*-algebras, the lower horizontal arrows are the connecting homomorphism associated to the short exact sequence obtained by tensoring the first with another* $\sigma$-$C^*$*-algebra* $D$ *and the vertical maps are exterior multiplication.*

Axioms 1.4.1–1.4.3 were proved in [Phi89, Phi91]. However, those versions of $K$-theory for $\sigma$-$C^*$-algebras are not very well adapted to the construction of products. We use the products of Cuntz' $kk$-theory [Cun97], which is much more general then $K$-theory for $\sigma$-$C^*$-algebras, to construct an exterior product satisfying Axioms 1.4.4, 1.4.5 in Appendix 1.B. Doing this forces us to work with Fréchet algebras which we shall review in Appendix 1.A.

The way we prove these axioms in the appendices is somewhat unsatisfactory as we have to mix up results from different theories. It would be nice to have a picture of $K$-theory for $\sigma$-$C^*$-algebras which is well adapted to both products and boundary maps and allows a direct approach to proving their compatibility as in Axiom 1.4.5.

## 1.5 Unreduced coarse co-assembly

This section gives an overview over the concept of coarse co-assembly. There is a variety of maps which deserve to be called coarse co-assembly maps:

1. Let $X$ be a countably generated, unbounded coarse space and $D$ a $C^*$-algebra. The coarse co-assembly map of Emerson and Meyer [EM06] (see Definition 1.5.2 below) is a map

$$\mu^*:\tilde K_{1-*}(\mathfrak{c}(X;D))\to KX^*(X;D)\,.$$

Its domain is the reduced $K$-theory of the stable Higson corona,

$$\tilde K_*(\mathfrak{c}(X,D)) := K_*(\mathfrak{c}(X,D))/\ \mathrm{im}[K_*(D\otimes\mathfrak{K}) \xrightarrow[\text{const. fu's}]{\text{incl. as}} K_*(\mathfrak{c}(X,D))]\,.$$

Its target $KX^*(X;D) := K_{-*}(C_0(\mathcal{P}_Z)\otimes D)$ is the $K$-theory of the Rips complex $\mathcal{P}_Z$ with coefficients in $D$. We recall the Rips complex construction below. In case $D=\mathbb{C}$, the coarse co-assembly map is dual to the coarse assembly map

$$\mu : KX_*(X) \to K_*(C^*(X))$$

($C^*(X)$ is the Roe algebra of $X$) in the sense that there are natural pairings

$$KX^*(X)\times KX_*(X)\to\mathbb{Z},\qquad \tilde{K}_{1-*}(\mathfrak{c}(X))\times K_*(C^*X)\to\mathbb{Z}$$

such that

$$\langle x,\mu(y)\rangle = \langle \mu^*(x), y\rangle \qquad \forall x\in \tilde{K}_{1-*}(\mathfrak{c}(X)), y\in KX_*(X).$$

2. If $X$ is a uniformly contractible metric space of bounded geometry, then $KX^*(X,D)\cong K_{-*}(C_0(X;D)) =: K^*(X;D)$ and the coarse co-assembly map of 1 corresponds to a map

$$\tilde{K}_{1-*}(\mathfrak{c}(X;D))\to K^*(X;D).$$

In fact, this map exists for any $\sigma$-coarse space $\mathcal{X}$ instead of $X$. We call it the uncoarsened version of the co-assembly map.

3. There are versions of 1 and 2 where the left hand side displays the unreduced $K$-theory of the stable Higson corona. We compensate this on the right hand side by removing a point of the Rips complex respectively the $\sigma$-coarse space. Thus, the unreduced co-assembly maps are

$$K_{1-*}(\mathfrak{c}(X;D))\to KX^*(X\setminus\{pt\};D),$$
$$K_{1-*}(\mathfrak{c}(X;D))\to K^*(X\setminus\{pt\};D)$$

(see Definition 1.5.6 for the first one). They can be obtained from the reduced versions of 1 and 2 by gluing a ray $\mathbb{R}^+=[0,\infty)$ to the space, which acts as a distinct base point at infinity.

4. The most general co-assembly map defined in Definition 1.5.4 below uses the unstable Higson corona $\mathfrak{uc}(X,D)$ instead of the stable Higson corona. For our purposes it is more convenient to do the calculations with the unstable version, as we do not have to keep track of the compact operators everywhere. Eventually, we will always return to the stable algebras by replacing $D$ by $D\otimes\mathfrak{K}$.

Details on the co-assembly maps of 1 and 2 can be found in [EM06]. However, the ring structures considered here work only in the unreduced cases. The reason for this is that the ring structure which we shall construct on $K_*(\mathfrak{c}(X))$ has a unit, namely the unit of $\mathbb{Z} \cong K_*(\mathfrak{K}) \subset K_*(\mathfrak{c}(X))$, and exactly this unit is identified with 0 in $\tilde{K}_*(\mathfrak{c}(X))$.

We will now take a closer look at the objects mentioned above starting with the Rips complex. We briefly recall its construction as presented in [EM06, Section 4].

Let $Z$ be a countably generated, discrete coarse space. Fix an increasing sequence $(E_n)$ of entourages generating the coarse structure. Let $\mathcal{P}_Z$ be the set of probability measures on $Z$ with finite support. This is a simplicial complex whose vertices are the Dirac measures on $Z$. It is given the corresponding topology. The locally finite subcomplexes

$$P_{Z,n} := \{\mu \in \mathcal{P}_Z \mid \operatorname{supp}\mu \times \operatorname{supp}\mu \subset E_n\}$$

are locally compact spaces and constitute a $\sigma$-locally compact space $\mathcal{P}_Z = \bigcup_{n\in\mathbb{N}} P_{Z,n}$. The $P_{Z,n}$ may be equipped with the coarse structure generated by the entourages

$$\{(\mu,\nu) \mid \operatorname{supp}\mu \times \operatorname{supp}\nu \subset E_m\}, \quad m \in \mathbb{N},$$

giving $\mathcal{P}_Z$ the structure of a $\sigma$-coarse space with all inclusions $Z \to P_{Z,n}$ being coarse equivalences.

**Definition 1.5.1** ([EM06, Definition 4.3])**.** Let $X$ be a countably generated coarse space and let $D$ be a $C^*$-algebra. Let $Z \subset X$ be a countably generated, discrete coarse subspace that is coarsely equivalent to $X$. The *coarse $K$-theory of $X$ with coefficients $D$* is defined as

$$KX^*(X; D) := K_{-*}(C_0(\mathcal{P}_Z; D)).$$

In this definition, $Z$ is equipped with the subspace coarse structure. It exists by [EM06, Lemma 2.4].

Furthermore, in the setting of the definition, $\mathfrak{c}(\mathcal{P}_Z; D)$ is a $C^*$-algebra isomorphic to $\mathfrak{c}(X; D)$ by Lemma 1.3.5.

**Definition 1.5.2** ([EM06, Definition 4.6])**.** Let $X$ be a countably generated, unbounded coarse space and let $D$ be a $C^*$-algebra. Choose $Z \subset X$ as in the previous definition. The *coarse co-assembly map*

$$\tilde{K}_{1-*}(\mathfrak{c}(X; D)) \to KX^*(X; D)$$

is induced by the connecting homomorphism of the short exact sequence

$$0 \to C_0(\mathcal{P}_Z; D \otimes \mathfrak{K}) \to \bar{\mathfrak{c}}(\mathcal{P}_Z; D) \to \mathfrak{c}(\mathcal{P}_Z; D) \cong \mathfrak{c}(X; D) \to 0.$$

We need an unreduced version of coarse co-assembly. It is obtained from the reduced version by implementing an idea of [Roe95]: Simply glue a ray $\mathbb{R}^+ = [0, \infty)$ to $X$ which acts as a distinct basepoint at infinity.

Given a countably generated coarse space $X$ with a distinguished point $pt$, we define

$$X^{\to} := X \cup_{pt \sim 0} \mathbb{R}^+.$$

If the coarse structure on $X$ is generated by the sequence of entourages $E_n \subset X \times X$, we can equip $X^{\to}$ with the coarse structure generated by the sequence of entourages

$$\begin{aligned} E_n^{\to} := E_n &\cup \{(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid |s-t| \le n\} \\ &\cup \{(x,t) \in X \times \mathbb{R}^+ \mid (x, pt) \in E_n \wedge t \le n\} \\ &\cup \{(t,x) \in \mathbb{R}^+ \times X \mid (x, pt) \in E_n \wedge t \le n\} \quad \subset X^{\to} \times X^{\to}. \end{aligned}$$

This coarse structure is obviously independent of the choice of the sequence $(E_n)$ and endows $X^{\to}$ with the structure of a coarse space.

**Lemma 1.5.3.** *Let $X, Z, D$ be as before. The coarse co-assembly map of $X^{\to}$ can be canonically identified with the connecting homomorphism*

$$K_{1-*}(\mathfrak{c}(X; D)) \to K_{-*}(C_0(\mathcal{P}_Z \setminus \{pt\}; D))$$

*associated to the short exact sequence of $\sigma$-$C^*$-algebras*

$$0 \to C_0(\mathcal{P}_Z \setminus \{pt\}; D \otimes \mathfrak{K}) \to \bar{\mathfrak{c}}_0(\mathcal{P}_Z; D) \to \mathfrak{c}(\mathcal{P}_Z; D) \cong \mathfrak{c}(X; D) \to 0.$$

*Proof.* We obviously have $\mathfrak{c}(X^{\to}; D) = \mathfrak{c}(X; D) \oplus \mathfrak{c}(\mathbb{R}^+; D)$, so

$$K_*(\mathfrak{c}(X^{\to}; D)) = K_*(\mathfrak{c}(X; D)) \oplus K_*(\mathfrak{c}(\mathbb{R}^+; D)).$$

Furthermore, the map $K_*(D \otimes \mathfrak{K}) \to K_*(\mathfrak{c}(\mathbb{R}^+; D))$ is an isomorphism by [EM06, Theorem 5.2]. It follows that the composition

$$K_*(\mathfrak{c}(X; D)) \xrightarrow{incl.} K_*(\mathfrak{c}(X^{\to}; D)) \to \tilde{K}_*(\mathfrak{c}(X^{\to}; D)). \tag{1.2}$$

is an isomorphism.

We choose $Z \cup_{pt\sim 0} \mathbb{N} \subset X^{\to}$ as preferred coarsely equivalent discrete subspace to calculate the coarse $K$-theory of $X^{\to}$. Consider the inclusion of simplicial complexes

$$\mathcal{P}_Z \cup_{pt\sim 0} \mathcal{P}_{\mathbb{N}} \subset \mathcal{P}_{Z\cup_{pt\sim 0}\mathbb{N}}. \tag{1.3}$$

The left hand side inherits the structure of a $\sigma$-coarse space by choosing the obvious filtration by the subcomplexes

$$P_{Z,n} \cup_{pt\sim 0} P_{\mathbb{N},n} \subset P_{Z\cup_{pt\sim 0}\mathbb{N},\, n}$$

and giving each of them the corresponding subspace coarse structure. These inclusions are coarse equivalences for all $n$, because both sides differ only by a finite number of simplices. Furthermore, all of these subcomplexes are coarsely equivalent to $X^{\to}$. We obtain canonical isomorphisms of $C^*$-algebras

$$\mathfrak{c}(\mathcal{P}_Z \cup_{pt\sim 0} \mathcal{P}_{\mathbb{N}}; D) \cong \mathfrak{c}(\mathcal{P}_{Z\cup_{pt\sim 0}\mathbb{N}}; D) \cong \mathfrak{c}(X^{\to}; D) \cong \mathfrak{c}(X; D) \oplus \mathfrak{c}(\mathbb{R}^+; D).$$

We claim that the inclusion (1.3) is also a homotopy equivalence. We shall construct a homotopy inverse as sketched in Figure 1.1.

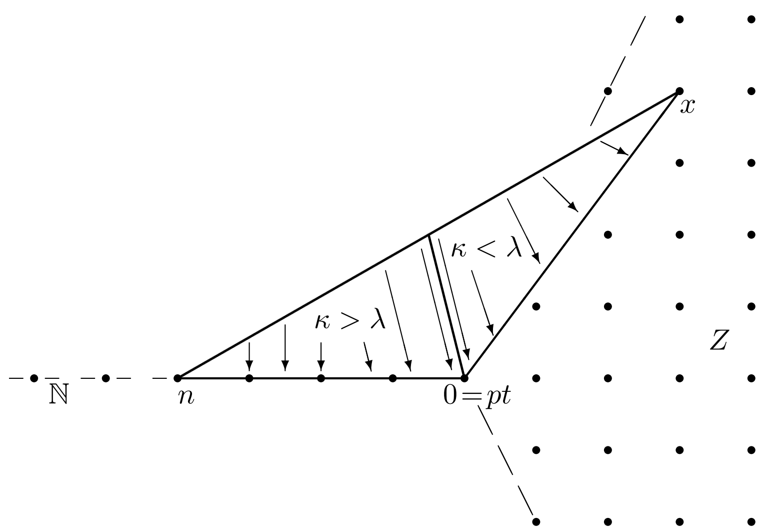


Figure 1.1: The simplex of $\mathcal{P}_{Z\cup_{pt\sim 0}\mathbb{N}}$ spanned by the points $n \in \mathbb{N}, x \in Z$ and $0 = pt \in \mathbb{N} \cap Z$ is mapped to the union of the simplex spanned by $n, 0$ and the simplex spanned by $pt, x$.

Let
$$\mu = \iota\delta_0 + \sum_{n\in\mathbb{N}\setminus\{0\}} \kappa_n\delta_n + \sum_{x\in Z\setminus\{pt\}} \lambda_x\delta_x \quad \in \mathcal{P}_{Z\cup_{pt\sim 0}\mathbb{N}},$$
where $\delta_x$ denotes the Dirac measure with support $x$. The sums are finite, $\iota, \kappa_n, \lambda_x \geq 0$ for all $n \in \mathbb{N}, x \in Z$ and $\iota + \sum \kappa_n + \sum \lambda_x = 1$. We map $\mu$ to
$$\sum_{n\in\mathbb{N}\setminus\{0\}} \left(\frac{2\kappa}{1-\iota} - 1\right)\kappa_n\delta_n + \left(1 - \frac{2\kappa^2}{1-\iota} + \kappa\right)\delta_0 \quad \in \mathcal{P}_{\mathbb{N}}$$
if $\kappa := \sum \kappa_n \geq \sum \lambda_x$ (note that $0 \leq \kappa_n \leq \kappa \leq 1-\iota$, so the definition can be extended continuously to the point $\mu = \delta_0$) and to
$$\sum_{x\in Z\setminus\{pt\}} \left(\frac{2\lambda}{1-\iota} - 1\right)\lambda_x\delta_x + \left(1 - \frac{2\lambda^2}{1-\iota} + \lambda\right)\delta_0 \quad \in \mathcal{P}_Z$$
if $\lambda := \sum \lambda_x \geq \sum \kappa_n$. It is easy to see that this construction is continuous, fixes the subcomplex $\mathcal{P}_Z \cup_{pt\sim 0} \mathcal{P}_{\mathbb{N}}$ and respects the filtration of these locally compact spaces. Thus, it defines a retraction
$$\mathcal{P}_{Z\cup_{pt\sim 0}\mathbb{N}} \to \mathcal{P}_Z \cup_{pt\sim 0} \mathcal{P}_{\mathbb{N}}$$
of $\sigma$-locally compact spaces. Similarly, we see that the composition
$$\mathcal{P}_{Z\cup_{pt\sim 0}\mathbb{N}} \to \mathcal{P}_Z \cup_{pt\sim 0} \mathcal{P}_{\mathbb{N}} \subset \mathcal{P}_{Z\cup_{pt\sim 0}\mathbb{N}}$$
is homotopic to the identity under linear homotopy.

Consider the short exact sequence of $\sigma$-$C^*$-algebras
$$0 \to C_0(\mathcal{P}_Z \setminus \{pt\}; D) \to C_0(\mathcal{P}_Z \cup_{pt\sim 0} \mathcal{P}_{\mathbb{N}}; D) \to C_0(\mathcal{P}_{\mathbb{N}}; D) \to 0.$$
By [EM06, Theorem 4.8] we have $K_*(C_0(\mathcal{P}_{\mathbb{N}}; D)) \cong K_*(C_0(\mathbb{R}^+; D)) = 0$, so the long exact sequence in $K$-theory proves
$$K_*(C_0(\mathcal{P}_Z \setminus \{pt\}; D)) \cong K_*(C_0(\mathcal{P}_Z \cup_{pt\sim 0} \mathcal{P}_{\mathbb{N}}; D)).$$
We obtain the second canonical isomorphism by exploiting homotopy invariance of $K$-theory:
$$\begin{aligned} K_*(C_0(\mathcal{P}_Z \setminus \{pt\}; D)) &\cong K_*(C_0(\mathcal{P}_Z \cup_{pt\sim 0} \mathcal{P}_{\mathbb{N}}; D)) \\ &\cong K_*(C_0(\mathcal{P}_{Z\cup_{pt\sim 0}\mathbb{N}}; D)) \\ &= KX^{-*}(X^{\to}; D) \end{aligned} \tag{1.4}$$

Thus, it remains to show that the canonical isomorphisms (1.2) and (1.4), coarse co-assembly and the connecting homomorphism mentioned in the statement of this lemma make the diagram

$$\begin{tikzcd}
K_{1-*}(\mathfrak{c}(X;D)) \arrow[r] \arrow[d, "\cong"'] & K_{-*}(C_0(\mathcal{P}_Z \setminus \{pt\};D)) \arrow[d, "\cong"] \\
\tilde{K}_{1-*}(\mathfrak{c}(X^{\rightarrow};D)) \arrow[r] & KX^*(X^{\rightarrow};D)
\end{tikzcd}$$

commute. This follows easily from naturality of the connecting homomorphism in $K$-theory applied to the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C_0(\mathcal{P}_Z \setminus \{pt\};D\otimes\mathfrak{K}) \arrow[r] \arrow[d] & \bar{\mathfrak{c}}_0(\mathcal{P}_Z;D) \arrow[r] \arrow[d] & \mathfrak{c}(\mathcal{P}_Z;D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_0(\mathcal{P}_Z \underset{pt\sim 0}{\cup} \mathcal{P}_{\mathbb{N}};D\otimes\mathfrak{K}) \arrow[r] & \bar{\mathfrak{c}}(\mathcal{P}_Z \underset{pt\sim 0}{\cup} \mathcal{P}_{\mathbb{N}};D) \arrow[r] & \mathfrak{c}(\mathcal{P}_Z \underset{pt\sim 0}{\cup} \mathcal{P}_{\mathbb{N}};D) \arrow[r] & 0 \\
0 \arrow[r] & C_0(\mathcal{P}_{Z\cup_{pt\sim 0}\mathbb{N}};D\otimes\mathfrak{K}) \arrow[r] \arrow[u, "\simeq"] & \bar{\mathfrak{c}}(\mathcal{P}_{Z\cup_{pt\sim 0}\mathbb{N}};D) \arrow[r] \arrow[u] & \mathfrak{c}(\mathcal{P}_{Z\cup_{pt\sim 0}\mathbb{N}};D) \arrow[r] \arrow[u, "\cong"'] & 0
\end{tikzcd}$$

and the obvious commutative diagram

$$\begin{tikzcd}
K_{1-*}(\mathfrak{c}(X;D)) \arrow[r, "\cong"] \arrow[dd] & K_{1-*}(\mathfrak{c}(\mathcal{P}_Z;D)) \arrow[d] \\
 & K_{1-*}(\mathfrak{c}(\mathcal{P}_Z \cup_{pt\sim 0} \mathcal{P}_{\mathbb{N}};D)) \\
K_{1-*}(\mathfrak{c}(X^{\rightarrow};D)) \arrow[r, "\cong"] & K_{1-*}(\mathfrak{c}(\mathcal{P}_{Z\cup_{pt\sim 0}\mathbb{N}};D)). \arrow[u, "\cong"']
\end{tikzcd}$$

□

This Lemma justifies the following definitions of unreduced co-assembly. It will be convenient to have also unstable and uncoarsened versions at hand. For any $\sigma$-coarse space $\mathcal{X} = \bigcup_{n\in\mathbb{N}} X_n$ and any $C^*$-algebra $D$ denote

$$K^*(\mathcal{X};D) := K_{-*}(C_0(\mathcal{X})\otimes D) \quad \text{and} \quad K^*(\mathcal{X}) := K_{-*}(C_0(\mathcal{X})).$$

**Definition 1.5.4.** Let $\mathcal{X} = \bigcup_{n\in\mathbb{N}} X_n$ be a $\sigma$-coarse space, $pt \in X_0$ and $D$ a $C^*$-algebra. The *unreduced, unstable and uncoarsened coarse co-assembly map with coefficients* $D$ is the connecting homomorphism

$$K_{1-*}(\mathfrak{uc}(\mathcal{X};D)) \to K^*(\mathcal{X}\setminus\{pt\};D)$$

associated to the short exact sequence

$$0 \to C_0(\mathcal{X} \setminus \{pt\}) \otimes D \to \overline{\mathfrak{uc}}_0(\mathcal{X}; D) \to \mathfrak{uc}(\mathcal{X}; D) \to 0.$$

**Definition 1.5.5.** Let $X$ be a coarse space, $pt \in X$ and $Z \subset X$ be a countably generated, discrete coarse subspace that is coarsely equivalent to $X$ with $pt \in Z$. We define

$$KX^*(X \setminus \{pt\}; D) := K^*(\mathcal{P}_Z \setminus \{pt\}; D) = K_{-*}(C_0(\mathcal{P}_Z \setminus \{pt\}) \otimes D).$$

It is known that $KX^*(X; D) = K_*(C_0(\mathcal{P}_Z) \otimes D)$ is independent of the choice of $Z \subset X$ [EM06, Corollary 4.2]. The exact sequence in $K$-theory associated to the short exact sequence

$$0 \to C_0(\mathcal{P}_Z \setminus \{pt\}) \otimes D \to C_0(\mathcal{P}_Z) \otimes D \to D \to 0$$

and the five lemma prove that $KX^*(X \setminus \{pt\}; D)$ is also independent of the choice of $Z$.

**Definition 1.5.6.** Let $X$ be a coarse space, $pt \in X$ and $D$ a $C^*$-algebra. The *unreduced, unstable coarse co-assembly map with coefficients* $D$ is obtained by appying the uncoarsened version to the Rips complex:

$$K_{1-*}(\mathfrak{uc}(X; D)) \cong K_{1-*}(\mathfrak{uc}(\mathcal{P}_Z; D)) \to KX^*(X \setminus \{pt\}; D)$$

Replacing $D$ by $D \otimes \mathfrak{K}$ or even by $\mathfrak{K}$, we obtain the *unreduced, stable coarse co-assembly map (with coefficients $D$ or without coefficients, respectively)*

$$K_{1-*}(\mathfrak{c}(X; D)) \to KX^*(X \setminus \{pt\}; D)\,,$$
$$K_{1-*}(\mathfrak{c}(X)) \to KX^*(X \setminus \{pt\})\,.$$

## 1.6 $K$-theory product for Higson coronas

Let $\mathcal{X} = \bigcup_{n\in\mathbb{N}} X_n$ be a $\sigma$-coarse space and $D, E$ coefficient $C^*$-algebras. The ranges of the two canonical homomorphisms

$$\overline{\mathfrak{uc}}_0(\mathcal{X}; D) \to \overline{\mathfrak{uc}}_0(\mathcal{X}; \tilde{D} \otimes \tilde{E}), \quad \overline{\mathfrak{uc}}_0(\mathcal{X}; E) \to \overline{\mathfrak{uc}}_0(\mathcal{X}; \tilde{D} \otimes \tilde{E})$$

commute, so they define a homomorphism

$$\overline{\mathfrak{uc}}_0(\mathcal{X}; D) \otimes \overline{\mathfrak{uc}}_0(\mathcal{X}; E) \to \overline{\mathfrak{uc}}_0(\mathcal{X}; \tilde{D} \otimes \tilde{E})$$

whose image is in fact contained in the ideal $\overline{\mathfrak{uc}}_0(\mathcal{X}; D \otimes E)$. If we pass to the quotient $\mathfrak{uc}(\mathcal{X}; D \otimes E)$, we obtain a homomorphism

$$\overline{\mathfrak{uc}}_0(\mathcal{X}; D) \otimes \overline{\mathfrak{uc}}_0(\mathcal{X}; E) \to \mathfrak{uc}(\mathcal{X}; D \otimes E)$$

which vanishes on the ideal

$$C_0(\mathcal{X} \setminus \{pt\}; D) \otimes \overline{\mathfrak{uc}}_0(\mathcal{X}; E) + \overline{\mathfrak{uc}}_0(\mathcal{X}; D) \otimes C_0(\mathcal{X} \setminus \{pt\}; E)$$

and thus yields a homomorphism

$$\nabla : \mathfrak{uc}(\mathcal{X}; D) \otimes \mathfrak{uc}(\mathcal{X}; E) \to \mathfrak{uc}(\mathcal{X}; D \otimes E)$$

by Corollary 1.3.14.

**Definition 1.6.1.** The product

$$K_i(\mathfrak{uc}(\mathcal{X}; D)) \otimes K_j(\mathfrak{uc}(\mathcal{X}; E)) \to K_{i+j}(\mathfrak{uc}(\mathcal{X}; D \otimes E))$$

is the composition of the exterior product in $K$-theory with $\nabla_*$. Replacing $D, E$ by $D \otimes \mathfrak{K}, E \otimes \mathfrak{K}$ or simply both by $\mathfrak{K}$ we obtain the products

$$K_i(\mathfrak{c}(\mathcal{X}; D)) \otimes K_j(\mathfrak{c}(\mathcal{X}; E)) \to K_{i+j}(\mathfrak{c}(\mathcal{X}; D \otimes E))$$
$$K_i(\mathfrak{c}(\mathcal{X})) \otimes K_j(\mathfrak{c}(\mathcal{X})) \to K_{i+j}(\mathfrak{c}(\mathcal{X}))$$

for the stable Higson corona. We denote all of them simply by "$\cdot$".

The identification $\mathfrak{K} \otimes \mathfrak{K} \cong \mathfrak{K}$ hidden in the definition is induced by adjoining with some Hilbert space isomorphism $\ell^2 \otimes \ell^2 \cong \ell^2$. Its homotopy class is independent of the chosen Hilbert space isomorphism.

With this in mind, the products given in the definition are obviously associative, graded commutative and independent of the choice of the identification $\mathfrak{K} \otimes \mathfrak{K} \cong \mathfrak{K}$.

We are now in a position to see a first instance of the multiplicativity of the co-assembly map. Let $Y$ be a compact metrizable space and embed it into the unit sphere of some real Hilbert space $H$. The open cone $\mathcal{O}Y$ of $Y$ is defined to be the union of all rays in $H$ starting at the origin and passing through points of $Y$. It is equipped with the subspace metric. The induced coarse structure of $\mathcal{O}Y$ is independent of the chosen embedding. Topologically, $\mathcal{O}Y \approx Y \times [0,\infty)/Y \times \{0\}$. Furthermore, let $\mathcal{C}Y := Y \times [0,\infty]/Y \times \{0\} \supset \mathcal{O}Y$ be the closed cone and $pt$ be the apex of these two cones.

If $Y$ is a "nice" compact space, e. g. a compact manifold or a finite simplicial complex, then $X := \mathcal{O}Y$ is a scalable and uniformly contractible metric space of bounded geometry. Therefore, the coarse co-assembly map of $X$ with coefficients in any $C^*$-algebra $D$ is an isomorphism

$$K_*(\mathfrak{c}(X,D)) \xrightarrow[\cong]{\mu^*} K_{*-1}(C_0(X \setminus \{pt\}) \otimes D)$$

by [EM06, Corollary 8.10].

Note that continuous functions on $\bar{X} := \mathcal{C}Y$ restrict to bounded continuous functions of vanishing variation on $X$. Thus, given a $C^*$-algebra $D$, there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C_0(X \setminus \{pt\}) \otimes D \arrow[r] \arrow[d] & C_0(\bar{X} \setminus \{pt\}) \otimes D \arrow[r] \arrow[d] & C(Y) \otimes D \arrow[r] \arrow[d, "p^*"'] & 0 \\
0 \arrow[r] & C_0(X \setminus \{pt\}) \otimes D \otimes \mathfrak{K} \arrow[r] & \bar{\mathfrak{c}}_0(X,D) \arrow[r] & \mathfrak{c}(X,D) \arrow[r] & 0
\end{tikzcd}$$

with vertical maps defined by choosing a rank one projection in $\mathfrak{K}$. The induced right vertical arrow $p^*$ is in fact given by pulling back functions along the projection $X \setminus \{pt\} \to Y$. We obtain a commutative diagram

$$\begin{tikzcd}
K_*(C(Y) \otimes D) \arrow[r, "\partial", "\cong"'] \arrow[d, "p^*"'] & K_{*-1}(C_0(X \setminus \{pt\})) \arrow[d, equal] \\
K_*(\mathfrak{c}(X,D)) \arrow[r, "\mu^*", "\cong"'] & K_{*-1}(C_0(X \setminus \{pt\})).
\end{tikzcd}$$

After identifying $K_{*-1}(C_0(X \setminus \{pt\}))$ with $K_*(C(Y) \otimes D)$ via the isomorphism $\partial$, we see that $p^*$ is inverse to the coarse co-assembly map $\tilde{\mu}^* := \partial^{-1} \circ \mu^*$.

It is now a triviality to check that $p^*$ and therefore also $\tilde{\mu}^*$ are multiplicative in the following sense:

**Proposition 1.6.2.** *Let $X := \mathcal{O}Y$ be the open cone over a compact metrizable space $Y$ and assume that the topology of $Y$ is "nice" enough such that $X$ is scalable and uniformly contractible of bounded geometry. If $D, E$ are any $C^*$-algebras, then the diagram*

$$\begin{tikzcd}
K_i(\mathfrak{c}(X,D)) \otimes K_j(\mathfrak{c}(X,E)) \arrow[r] \arrow[d, "\tilde{\mu}^* \otimes \tilde{\mu}^*"] & K_{i+j}(\mathfrak{c}(X, D \otimes E)) \arrow[d, "\tilde{\mu}^*"] \\
K^{-i}(Y;D) \otimes K^{-j}(Y;E) \arrow[r] & K^{-(i+j)}(Y; D \otimes E)
\end{tikzcd}$$

*commutes. In particular, the co-assembly map $K_*(\mathfrak{c}(X)) \to K^{-*}(Y)$ is a ring isomorphism.*

For more general $\sigma$-coarse spaces $\mathcal{X}$ we have to construct a secondary product directly on the groups $K^*(\mathcal{X} \setminus \{pt\}; D)$. This is done in the next section. The key property of cones which we will have to continue to assume is contractibility to the point $pt$.

## 1.7 Secondary product for $\sigma$-contractible spaces

Let $\mathcal{X} = \bigcup_{n\in\mathbb{N}} X_n$ be a $\sigma$-locally compact space and $D$ a $\sigma$-$C^*$-algebra. Recall the definitions

$$K^*(\mathcal{X}; D) := K_{-*}(C_0(\mathcal{X}) \otimes D) \quad \text{and} \quad K^*(\mathcal{X}) := K_{-*}(C_0(\mathcal{X})).$$

These groups are contravariantly functorial under proper continuous maps between $\sigma$-locally compact spaces and covariantly functorial under $*$-homomorphisms between coefficient $\sigma$-$C^*$-algebras.

The exterior product in $K$-theory of $\sigma$-$C^*$-algebras specializes to an exterior product

$$K^i(\mathcal{X}; D) \otimes K^j(\mathcal{Y}; E) \to K^{i+j}(\mathcal{X} \times \mathcal{Y}; D \otimes E).$$

In this section, we will construct the secondary product

$$K^i(\mathcal{X} \setminus \{pt\}; D) \otimes K^j(\mathcal{X} \setminus \{pt\}; E) \to K^{i+j-1}(\mathcal{X} \setminus \{pt\}; D \otimes E)$$

under the assumption that $\mathcal{X}$ is equipped with a $\sigma$-contraction onto the point $pt$ as defined below and prove the most basic properties like associativity.

**Definition 1.7.1.** Let $\mathcal{X} = \bigcup_{n\in\mathbb{N}} X_n$ be a $\sigma$-locally compact space. We call a map

$$H : \mathcal{X} \times [0,1] \to \mathcal{X}$$

a *$\sigma$-contraction* to the point $pt$ iff

- $H|_{\mathcal{X}\times\{0\}} = \mathrm{id}_{\mathcal{X}}$ and $H(\mathcal{X} \times \{1\} \cup \{pt\} \times [0,1]) = \{pt\}$,
- for each $n$ there is $m \geq n$ such that $H(X_n \times [0,1]) \subset X_m$ and the restriction

  $$H_{X_n\times[0,1]} : X_n \times [0,1] \to X_m$$

  is continuous.

For a locally compact space $X$ (i. e. $X_n = X\,\forall n$), a $\sigma$-contraction is nothing but a contraction which does not move the point $pt$.

Given such a $\sigma$-contraction on $\mathcal{X}$, we can define the map

$$\begin{aligned}\Gamma:\ [0,1]\times\mathcal{X} &\to \mathcal{X}\times\mathcal{X}\\ (t,x) &\mapsto \begin{cases}(H(x,1-2t),x) & t\le 1/2\\ (x,H(x,2t-1)) & t\ge 1/2\end{cases}\end{aligned}$$

We claim that it restricts to a a proper continuous map (a morphism, cf. Definition 1.2.2)

$$\Gamma:\ (0,1)\times(\mathcal{X}\setminus\{pt\})\to(\mathcal{X}\setminus\{pt\})\times(\mathcal{X}\setminus\{pt\}).$$

To be precise: the map defining this morphism is defined on the open subset $\mathcal{U}=\Gamma^{-1}((\mathcal{X}\setminus\{pt\})\times(\mathcal{X}\setminus\{pt\}))$.

To see this, first note that $\Gamma(\,[0,1]\times\{pt\}\cup\{0,1\}\times\mathcal{X}\,)\subset\mathcal{X}\times\{pt\}\cup\{pt\}\times\mathcal{X}$ and therefore indeed $\mathcal{U}\subset(0,1)\times(\mathcal{X}\setminus\{pt\})$. Second, for each $m\in\mathbb{N}$ we can choose $n\in\mathbb{N}$ as in the definition of the $\sigma$-contraction $H$. It remains to show that the continuous maps

$$\Gamma|_{U_m}: U_m\to(X_n\setminus\{pt\})\times(X_n\setminus\{pt\})$$

are proper. Given a compact subset $K\subset(X_n\setminus\{pt\})\times(X_n\setminus\{pt\})$, we denote its images under the canonical projections by $K_1,K_2\subset X_n\setminus\{pt\}$. Then $\Gamma^{-1}(K)$ is contained in the compact set $[0,\frac{1}{2}]\times K_2\cup[\frac{1}{2},1]\times K_1$ and is therefore itself compact. This proves properness.

**Definition 1.7.2.** Let $\mathcal{X}=\bigcup_{n\in\mathbb{N}}X_n$ be a $\sigma$-locally compact space together with a $\sigma$-contraction $H$ and let $D,E$ be coefficient $C^*$-algebras. The *secondary product* is $(-1)^i$ times the composition

$$\begin{aligned}&K^i(\mathcal{X}\setminus\{pt\};D)\otimes K^j(\mathcal{X}\setminus\{pt\};E)\to\\ &\qquad\longrightarrow K^{i+j}(\,(\mathcal{X}\setminus\{pt\})\times(\mathcal{X}\setminus\{pt\});\,D\otimes E)\\ &\qquad\xrightarrow{\Gamma^*} K^{i+j}(\,(0,1)\times(\mathcal{X}\setminus\{pt\});\,D\otimes E)\\ &\qquad\xrightarrow{\cong} K^{i+j-1}(\mathcal{X}\setminus\{pt\};D\otimes E),\end{aligned}$$

where the last isomorphism is the inverse of the connecting homomorphism associated to the short exact sequence

$$0\to C_0(0,1)\to C_0[0,1)\to\mathbb{C}\to 0$$

tensored with $C_0(\mathcal{X}\setminus\{pt\})\otimes D\otimes E$.

We shall use the infix notation $x \otimes y \mapsto x * y$ for this secondary multiplication. The remaining part of this section is devoted to proving basic properties of the secondary product.

**Proposition 1.7.3.** *The secondary product is independent of the choice of the $\sigma$-contraction to the given point pt and graded commutative.*

*Proof.* If $\tilde{H}$ is another $\sigma$-contraction to the same point $pt$ with associated proper continuous map $\tilde{\Gamma}$, then both $\Gamma$ and $\tilde{\Gamma}$ are homotopic to the proper continuous map

$$(t, x) \mapsto \begin{cases} (H(\tilde{H}(x, 1-2t), 1-2t),\ x) & t \leq 1/2 \\ (x,\ H(\tilde{H}(x, 2t-1), 2t-1)) & t \geq 1/2 \end{cases}$$

in the obvious way and thus induce the same map in $K$-theory.

Exchanging the factors in the exterior product gives an additional sign $(-1)^{ij}$, changing the orientation of the interval $(0,1)$ gives another $-1$ and instead of multiplying with $(-1)^i$ we have to multiply with $(-1)^j$. In total we obtain a change in sign by $(-1)^{(i+1)(j+1)}$. Note that this is the desired prefactor, as the correct degree of $K^i(\mathcal{X} \setminus \{pt\}; D)$ is in fact $i-1$, not $i$. $\square$

**Theorem 1.7.4.** *The secondary product is associative.*

*Proof.* For space reasons, we prove this for all coefficient algebras set to $\mathbb{C}$. The proof for the general case is obtained by simply dropping in the coefficient $C^*$-algebras $D, E, F$ at the appropriate places.

Consider the diagram in Figure 1.2. The upper vertical and left horizontal maps are exterior multiplication. All the $\delta_i$ are connecting homomorphisms associated to inclusions of closed subsets:

$\delta_1$ is associated to the inclusion $\mathcal{X} \setminus \{pt\} \to [0,1) \times (\mathcal{X} \setminus \{pt\}),\ x \mapsto (0, x)$.

$\delta_2$ is associated to the inclusion

$$(\mathcal{X} \setminus \{pt\}) \times (\mathcal{X} \setminus \{pt\}) \to [0,1) \times (\mathcal{X} \setminus \{pt\}) \times (\mathcal{X} \setminus \{pt\}),$$
$$(x, y) \mapsto (0, x, y).$$

$\delta_3$ is associated to the inclusion

$$(0,1) \times (\mathcal{X} \setminus \{pt\}) \to [0,1) \times (0,1) \times (\mathcal{X} \setminus \{pt\})$$
$$(s, x) \mapsto (0, s, x).$$

$$
\begin{tikzcd}
\begin{gathered} K^i(\mathcal{X}\setminus\{pt\})\otimes \\ \otimes K^j(\mathcal{X}\setminus\{pt\}) \\ \otimes K^k(\mathcal{X}\setminus\{pt\}) \end{gathered} \arrow[r] \arrow[d] &
\begin{gathered} K^{i+j}(\,(\mathcal{X}\setminus\{pt\})\times \\ \times(\mathcal{X}\setminus\{pt\})\,) \\ \otimes K^k(\mathcal{X}\setminus\{pt\}) \end{gathered} \arrow[r, "\Gamma^*\otimes\mathrm{id}"] \arrow[d] &
\begin{gathered} K^{i+j}(\,(0,1)\times \\ \times(\mathcal{X}\setminus\{pt\})\,) \\ \otimes K^k(\mathcal{X}\setminus\{pt\}) \end{gathered} \arrow[d] &
\begin{gathered} K^{i+j-1}(\mathcal{X}\setminus\{pt\}) \\ \otimes K^k(\mathcal{X}\setminus\{pt\}) \end{gathered} \arrow[l, "\delta_1\otimes\mathrm{id}"', "\cong"] \arrow[d] \\
\begin{gathered} K^i(\mathcal{X}\setminus\{pt\})\otimes \\ K^{j+k}(\,(\mathcal{X}\setminus\{pt\})\times \\ \times(\mathcal{X}\setminus\{pt\})\,) \end{gathered} \arrow[r] \arrow[d, "\mathrm{id}\otimes\Gamma^*"] &
\begin{gathered} K^{i+j+k}(\,(\mathcal{X}\setminus\{pt\}) \\ \times(\mathcal{X}\setminus\{pt\}) \\ \times(\mathcal{X}\setminus\{pt\})\,) \end{gathered} \arrow[r, "(\Gamma\times\mathrm{id})^*"] \arrow[d, "(\mathrm{id}\times\Gamma)^*"'] &
\begin{gathered} K^{i+j+k}(\,(0,1) \\ \times(\mathcal{X}\setminus\{pt\}) \\ \times(\mathcal{X}\setminus\{pt\})\,) \end{gathered} \arrow[d, "(\mathrm{id}\times\Gamma)^*"] &
\begin{gathered} K^{i+j+k-1}((\mathcal{X}\setminus\{pt\}) \\ \times(\mathcal{X}\setminus\{pt\})\,) \end{gathered} \arrow[l, "\delta_2"', "\cong"] \arrow[d, "\Gamma^*"] \\
\begin{gathered} K^i(\mathcal{X}\setminus\{pt\})\otimes \\ K^{j+k}(\,(0,1)\times \\ \times(\mathcal{X}\setminus\{pt\})\,) \end{gathered} \arrow[r] &
\begin{gathered} K^{i+j+k}(\,(\mathcal{X}\setminus\{pt\}) \\ \times(0,1) \\ \times(\mathcal{X}\setminus\{pt\})\,) \end{gathered} \arrow[r, "\tilde{\Gamma}^*"'] &
\begin{gathered} K^{i+j+k}(\,(0,1)^2 \\ \times(\mathcal{X}\setminus\{pt\})\,) \end{gathered} &
\begin{gathered} K^{i+j+k-1}((0,1) \\ \times(\mathcal{X}\setminus\{pt\})\,) \end{gathered} \arrow[l, "\cong"', "\delta_3"] \\
\begin{gathered} K^i(\mathcal{X}\setminus\{pt\})\otimes \\ K^{j+k-1}(\mathcal{X}\setminus\{pt\}) \end{gathered} \arrow[r] \arrow[u, "\cong", "\mathrm{id}\otimes\delta_1"'] &
\begin{gathered} K^{i+j+k-1}((\mathcal{X}\setminus\{pt\}) \\ \times(\mathcal{X}\setminus\{pt\})\,) \end{gathered} \arrow[r, "\Gamma^*"] \arrow[u, "\cong", "\delta_5"'] &
\begin{gathered} K^{i+j+k-1}((0,1) \\ \times(\mathcal{X}\setminus\{pt\})\,) \end{gathered} \arrow[u, "\cong", "\delta_4"'] \arrow[ur, equal] &
K^{i+j+k-2}(\mathcal{X}\setminus\{pt\}) \arrow[l, "\cong"', "\delta_1"] \arrow[u, "\cong", "\delta_1"']
\end{tikzcd}
$$

Figure 1.2: Proving Theorem 1.7.4

$\delta_4$ is associated to the inclusion

$$\begin{aligned}(0,1)\times(\mathcal{X}\setminus\{pt\}) &\to (0,1]\times(0,1)\times(\mathcal{X}\setminus\{pt\})\\ (s,x) &\mapsto (1,s,x).\end{aligned}$$

$\delta_5$ is associated to the inclusion

$$\begin{aligned}(\mathcal{X}\setminus\{pt\})\times(\mathcal{X}\setminus\{pt\}) &\to (\mathcal{X}\setminus\{pt\})\times[0,1)\times(\mathcal{X}\setminus\{pt\})\\ (x,y) &\mapsto (x,0,y).\end{aligned}$$

Note that all of them are isomorphisms, because $C_0[0,1)$ and $C_0(0,1]$ are contractible. Finally,

$$\begin{aligned}\tilde{\Gamma}:\ (0,1)^2\times\mathcal{X}\setminus\{pt\} &\to \mathcal{X}\setminus\{pt\}\times(0,1)\times\mathcal{X}\setminus\{pt\}\\ (s,t,x) &\mapsto \begin{cases}(\,H(x,1-2t),\,1-s,\,x\,) & t\le 1/2\\ (\,x,\,1-s,\,H(x,2t-1)\,) & t\ge 1/2.\end{cases}\end{aligned}$$

The upper left square commutes by associativity of the exterior product. The middle squares on the upper and left side commute by naturality of the exterior product. The the lower left square commutes up to a sign $(-1)^i$ and the upper right square commutes by Axiom 1.4.5. In the lower right square, $\delta_4$ is the negative of $\delta_3$ by Axiom 1.4.3.

The middle squares on the bottom and right side commute by naturality of the connecting homomorphism under the proper continuous maps of pairs of spaces

$$\begin{aligned}&\big([0,1)\times(0,1)\times(\mathcal{X}\setminus\{pt\}),\,\{0\}\times(0,1)\times(\mathcal{X}\setminus\{pt\})\big)\\ &\qquad\to\big([0,1)\times(\mathcal{X}\setminus\{pt\})\times(\mathcal{X}\setminus\{pt\}),\,\{0\}\times(\mathcal{X}\setminus\{pt\})\times(\mathcal{X}\setminus\{pt\})\big)\\ &(s,t,x)\mapsto(s,\Gamma(t,x))\end{aligned}$$

and

$$\begin{aligned}&\big((0,1]\times(0,1)\times(\mathcal{X}\setminus\{pt\}),\,\{1\}\times(0,1)\times(\mathcal{X}\setminus\{pt\})\big)\\ &\qquad\to\big((\mathcal{X}\setminus\{pt\})\times[0,1)\times(\mathcal{X}\setminus\{pt\}),\,(\mathcal{X}\setminus\{pt\})\times\{0\}\times(\mathcal{X}\setminus\{pt\})\big)\\ &(s,t,x)\mapsto\begin{cases}(\,H(x,1-2t),\,1-s,\,x\,) & t\le 1/2\\ (\,x,\,1-s,\,H(x,2t-1)\,) & t\ge 1/2.\end{cases}\end{aligned}$$

It remains to prove commutativity of the middle square. The proper continuous maps $(0,1)^2\times\mathcal{X}\setminus\{pt\}\to(\mathcal{X}\setminus\{pt\})^3$ inducing the two compositions are

$$(\Gamma\times\mathrm{id})\circ(\mathrm{id}\times\Gamma)(s,t,x)=\begin{cases}(\Gamma(s,H(x,1-2t)),\ x) & t\le 1/2\\ (\Gamma(s,x),\ H(x,2t-1)) & t\ge 1/2\end{cases}$$
$$=\begin{cases}(\,H(H(x,1-2t),1-2s),\ H(x,1-2t),\ x\,) & s\le 1/2, t\le 1/2\\ (\,H(x,1-2s),\ x,\ H(x,2t-1)\,) & s\le 1/2, t\ge 1/2\\ (\,H(x,1-2t),\ H(H(x,1-2t),2s-1),\ x\,) & s\ge 1/2, t\le 1/2\\ (\,x,\ H(x,2s-1),\ H(x,2t-1)\,) & s\ge 1/2, t\ge 1/2\end{cases}$$
$$(\mathrm{id}\times\Gamma)\circ\tilde{\Gamma}(s,t,x)=\begin{cases}(\,H(x,1-2t),\ \Gamma(1-s,\ x)\,) & t\le 1/2\\ (\,x,\ \Gamma(1-s,\ H(x,2t-1))\,) & t\ge 1/2\end{cases}$$
$$=\begin{cases}(\,H(x,1-2t),\ x,\ H(x,1-2s\,) & s\le 1/2, t\le 1/2\\ (\,x,\ H(x,2t-1),\ H(H(x,2t-1),1-2s)\,) & s\le 1/2, t\ge 1/2\\ (\,H(x,1-2t),\ H(x,2s-1),x\,) & s\ge 1/2, t\le 1/2\\ (\,x,\ H(H(x,2t-1),2s-1),\ H(x,2t-1)\,) & s\ge 1/2, t\ge 1/2.\end{cases}$$

These two maps are homotopic, as can be seen by performing the following four consecutive homotopies:

1. Homotop within $t\le 1/2$ (always leaving the rest constant):

$$(r,s,t,x)\mapsto\begin{cases}(H(H(x,1-2t),1-2s),H(x,(1-r)(1-2t)),x) & s\le 1/2\\ (H(x,1-2t),H(H(x,(1-r)(1-2t)),2s-1),x) & s\ge 1/2\end{cases}$$

2. Homotop within $s\le 1/2$:

$$(r,s,t,x)\mapsto\begin{cases}(H(H(x,1-2t),1-2s),x,H(x,r(1-2s))) & t\le 1/2\\ (H(x,1-2s),x,H(H(x,2t-1),r(1-2s))) & t\ge 1/2\end{cases}$$

3. Homotop again within $s\le 1/2$:

$$(r,s,t,x)\mapsto\begin{cases}(H(H(x,1-2t),(1-r)(1-2s)),x,H(x,1-2s)) & t\le 1/2\\ (H(x,(1-r)(1-2s)),x,H(H(x,2t-1),1-2s)) & t\ge 1/2\end{cases}$$

4. Homotop within $t\ge 1/2$:

$$(r,s,t,x)\mapsto\begin{cases}(x,H(x,r(2t-1)),H(H(x,2t-1),1-2s)) & s\le 1/2\\ (x,H(H(x,r(2t-1)),2s-1),H(x,2t-1)) & s\ge 1/2\end{cases}$$

Checking continuity is straightforward. To see properness, note that for any $(r,s,t,x)$ there is always one component equal to $x$, and if $s$ or $t$ is set equal to 0 or 1, then one component is equal to $pt$.

Composing the maps on the left and bottom side yields

$$x \otimes y \otimes z \mapsto (-1)^j x \otimes (y * z) \mapsto (-1)^{i+j} x * (y * z)$$

while the composition along top and right side is

$$x \otimes y \otimes z \mapsto (-1)^i (x * y) \otimes z \mapsto (-1)^{i+(i+j-1)} (x * y) * z\,.$$

As we have seen above, the diagram commutes up to a total sign of $(-1)^{i-1}$. This implies $x * (y * z) = (x * y) * z$. ☐

## 1.8 Multiplicativity of the coarse co-assembly map

**Theorem 1.8.1.** *Let $\mathcal{X} = \bigcup_{n\in\mathbb{N}} X_n$ be a $\sigma$-coarse space together with a $\sigma$-contraction $H$ and let $D, E$ be coefficient $\sigma$-$C^*$-algebras. Then the products we have constructed and coarse co-assembly fit into a commutative diagram.*

$$\begin{array}{ccc}
K_i(\mathfrak{uc}(\mathcal{X};D)) \otimes K_j(\mathfrak{uc}(\mathcal{X};E)) & \longrightarrow & K^{1-i}(\mathcal{X}\setminus\{pt\};D) \otimes K^{1-j}(\mathcal{X}\setminus\{pt\};E) \\
\downarrow & & \downarrow \\
K_{i+j}(\mathfrak{uc}(\mathcal{X};D\otimes E)) & \longrightarrow & K^{1-i-j}(\mathcal{X}\setminus\{pt\};D\otimes E)
\end{array}$$

*Proof.* We will use the following abbreviations: Let

$$I = C_0(\mathcal{X}\setminus\{pt\}), \quad I_D = C_0(\mathcal{X}\setminus\{pt\};D) = I \otimes D, \quad A_D = \overline{\mathfrak{uc}}_0(\mathcal{X};D),$$

so that

$$A_D/I_D = \mathfrak{uc}(\mathcal{X};D),$$

and $I_E, A_E, I_{D\otimes E}, A_{D\otimes E}$ are defined analogously. Furthermore, let

$$\begin{aligned}
A_1 &= C_b(\mathcal{X}, pt;\ D\otimes E),\\
A_2 &= C_b(\,[0,1]\times\mathcal{X},\ [0,1]\times\{pt\}\cup\{0,1\}\times\mathcal{X};\ D\otimes E),\\
A_3 &= C_b(\,[0,1]\times\mathcal{X},\ [0,1]\times\{pt\}\cup\{0\}\times\mathcal{X};\ D\otimes E),\\
I_2 &= C_0(\,(0,1)\times(\mathcal{X}\setminus\{pt\});\ D\otimes E)\\
&= C_0(0,1)\otimes I\otimes D\otimes E \quad = C_0(0,1)\otimes I_{D\otimes E},\\
I_3 &= C_0(\,(0,1]\times(\mathcal{X}\setminus\{pt\});\ D\otimes E)\\
&= C_0(0,1]\otimes I\otimes D\otimes E \quad = C_0(0,1]\otimes I_{D\otimes E},\\
I_4 &= C_0(\,[0,1)\times(\mathcal{X}\setminus\{pt\});\ D\otimes E)\\
&= C_0[0,1)\otimes I\otimes D\otimes E \quad = C_0[0,1)\otimes I_{D\otimes E}.
\end{aligned}$$

We will make use of the naturality of the connecting homomorphism in $K$-theory with respect to the following commutative diagrams with exact rows:

1. The ranges of the homomorphisms

$$\begin{aligned} I_D &\to C_b(\,[0,1]\times\mathcal{X},\,[0,1]\times\{pt\};\,\tilde{D}\otimes\tilde{E}) \\ f &\mapsto \left[(t,x)\mapsto \begin{cases} f(H(x,1-2t))\otimes 1 & t\le 1/2 \\ f(x)\otimes 1 & t\ge 1/2 \end{cases}\right], \\ A_E &\to C_b(\,[0,1]\times\mathcal{X},\,[0,1]\times\{pt\};\,\tilde{D}\otimes\tilde{E}) \\ g &\mapsto \left[(t,x)\mapsto \begin{cases} 1\otimes g(x) & t\le 1/2 \\ 1\otimes g(H(x,2t-1)) & t\ge 1/2 \end{cases}\right], \end{aligned}$$

   commute, so they define a homomorphism

$$\alpha:\, I_D\otimes A_E\to C_b(\,[0,1]\times\mathcal{X},\,[0,1]\times\{pt\};\,\tilde{D}\otimes\tilde{E}).$$

   Its image is contained in $A_2\subset C_b(\,[0,1]\times\mathcal{X},\,[0,1]\times\{pt\};\,\tilde{D}\otimes\tilde{E})$ and it maps $I_D\otimes I_E$ to $I_2=C_0(0,1)\otimes I_{D\otimes E}$. Note that the restriction of $\alpha$ to $I_D\otimes I_E$ is in fact $\Gamma^*$. Thus we obtain the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & I_D\otimes I_E \arrow[r] \arrow[d, "\Gamma^*"] & I_D\otimes A_E \arrow[r] \arrow[d, "\alpha"] & I_D\otimes A_E/I_E \arrow[r] \arrow[d, "\bar{\alpha}"] & 0 \\ 0 \arrow[r] & I_2 \arrow[r] & A_2 \arrow[r] & A_2/I_2 \arrow[r] & 0 \end{tikzcd}$$

   with $\bar{\alpha}$ being the quotient of $\alpha$.

2. The quotient homomorphism $\bar{\alpha}$ from the previous diagram also appears in the diagram

$$\begin{tikzcd} 0 \arrow[r] & I_D\otimes A_E/I_E \arrow[r] \arrow[d, "\bar{\alpha}"] & A_D\otimes A_E/I_E \arrow[r] \arrow[d, "\beta"] & A_D/I_D\otimes A_E/I_E \arrow[r] \arrow[d, "\bar{\beta}"] & 0 \\ 0 \arrow[r] & A_2/I_2 \arrow[r] & A_3/I_3 \arrow[r] & A_1/I_{D\otimes E} \arrow[r] & 0. \end{tikzcd}$$

   The lower row consists of the canonical maps: quotients of the inclusion $A_2\subset A_3$ and evaluation at one $A_3\to A_1$. It is easily seen to be an exact sequence.

The middle vertical homomorphism $\beta$ is defined the following way: The homomorphisms

$$\begin{aligned} A_D &\to C_b(\,[0,1]\times\mathcal{X},\,[0,1]\times\{pt\};\,\tilde{D}\otimes\tilde{E}) \\ f &\mapsto \left[(t,x)\mapsto\begin{cases} f(H(x,1-2t))\otimes 1 & t\le 1/2 \\ f(x)\otimes 1 & t\ge 1/2\end{cases}\right], \\ A_E &\to C_b(\,[0,1]\times\mathcal{X},\,[0,1]\times\{pt\};\,\tilde{D}\otimes\tilde{E}) \\ g &\mapsto [(t,x)\mapsto 1\otimes g(x)] \end{aligned}$$

induce a homomorphism $A_D\otimes A_E\to A_3$ into the subalgebra $A_3\subset C_b(\,[0,1]\times\mathcal{X},\,[0,1]\times\{pt\};\,\tilde{D}\otimes\tilde{E})$. The composition $A_D\otimes A_E\to A_3\to A_3/I_3$ vanishes on $A_D\otimes I_E$ yielding the homomorphism $\beta$.

We quickly check commutativity of the left square in the diagram by considering the images of elementary tensors $f\otimes\bar{g}\in I_D\otimes A_E/I_E$. The lower left path takes it to the residue class of

$$(t,x)\mapsto\begin{cases} f(H(x,1-2t))\otimes g(x) & t\le 1/2 \\ f(x)\otimes g(H(x,2t-1)) & t\ge 1/2.\end{cases}$$

Its difference to the representative obtained by following the upper right path is

$$(t,x)\mapsto\begin{cases} 0 & t\le 1/2 \\ f(x)\otimes[g(H(x,2t-1))-g(x)] & t\ge 1/2\,,\end{cases}$$

which is contained in $I_3$, because $f$ vanishes at infinity. Thus, the two compositions agree on elementary tensors and consequently on all elements.

By exactness of both rows and commutativity of the left square, $\beta$ passes to the quotient and we obtain

$$\bar{\beta}:\,A_D/I_D\otimes A_E/I_E\to A_1/I_{D\otimes E}.$$

3. The homomorphism

$$\gamma:\,I_4\to A_2,\;f\mapsto\left[(t,x)\mapsto\begin{cases} f(0,H(x,1-2t)) & t\le 1/2 \\ f(2t-1,x) & t\ge 1/2\end{cases}\right]$$

restricts to a homomorphism

$$\gamma':\,I_2\to I_2,\;f\mapsto\left[(t,x)\mapsto\begin{cases} 0 & t\le 1/2 \\ f(2t-1,x) & t\ge 1/2\end{cases}\right]$$

which is obviously homotopic to the identity. They induce the third diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & I_2 \arrow[r] & A_2 \arrow[r] & A_2/I_2 \arrow[r] & 0 \\
0 \arrow[r] & I_2 \arrow[r] \arrow[u, "\gamma'"] & I_4 \arrow[r] \arrow[u, "\gamma"] & I_{D\otimes E} \arrow[r] \arrow[u, "\bar\gamma"] & 0
\end{tikzcd}$$

The induced homomorphism $\bar\gamma$ is easily seen to map $f \in I_{D\otimes E}$ to the residue class of

$$(t,x) \mapsto \begin{cases} f(H(x,1-2t)) & t \le 1/2 \\ 2(1-t)\cdot f(x) & t \ge 1/2. \end{cases} \tag{1.5}$$

Note furthermore, that the connecting homomorphism

$$K_*(I_{D\otimes E}) \xrightarrow{\cong} K_{*+1}(I_2) = K_{*+1}(C_0(0,1)\otimes I_{D\otimes E})$$

is our preferred identification of these two groups.

4. The fourth is

$$\begin{tikzcd}
0 \arrow[r] & A_2/I_2 \arrow[r] & A_3/I_3 \arrow[r] & A_1/I_{D\otimes E} \arrow[r] & 0 \\
0 \arrow[r] & I_{D\otimes E} \arrow[r] \arrow[u, "\bar\gamma"] & A_{D\otimes E} \arrow[r] \arrow[u, "\delta"] & A_{D\otimes E}/I_{D\otimes E} \arrow[r] \arrow[u, "\bar\delta"] & 0
\end{tikzcd}$$

with $\bar\gamma$ as above and $\delta$ mapping $f \in A_{D\otimes E}$ to the residue class of

$$(t,x) \mapsto \begin{cases} f(H(x,1-2t)) & t \le 1/2 \\ f(x) & t \ge 1/2. \end{cases}$$

Note that for $f \in I_{D\otimes E}$ the difference between this function and the one given by (1.5) lies in $I_3$. Thus, the left square commutes and the quotient homomorphism $\bar\delta$ is simply inclusion.

We now obtain the diagram

$$\begin{tikzcd}
K_i\left(\frac{A_D}{I_D}\right)\otimes K_j\left(\frac{A_E}{I_E}\right) \arrow[r] \arrow[d] & K_{i-1}(I_D)\otimes K_j\left(\frac{A_E}{I_E}\right) \arrow[r] \arrow[d] & K_{i-1}(I_D)\otimes K_{j-1}(I_E) \arrow[d] \\
K_{i+j}\left(\frac{A_D}{I_D}\otimes\frac{A_E}{I_E}\right) \arrow[r] \arrow[d, "\bar{\beta}_*"] \arrow[dd, bend right=60, "\nabla_*"'] & K_{i+j-1}\left(I_D\otimes\frac{A_E}{I_E}\right) \arrow[r] \arrow[d, "\bar{\alpha}_*"] & K_{i+j-2}(I_D\otimes I_E) \arrow[d, "\Gamma^*"] \\
K_{i+j}\left(\frac{A_1}{I_{D\otimes E}}\right) \arrow[r] & K_{i+j-1}\left(\frac{A_2}{I_2}\right) \arrow[r] & K_{i+j-2}(I_2) \arrow[d, equal, "\gamma'_*=\mathrm{id}"] \\
K_{i+j}\left(\frac{A_{D\otimes E}}{I_{D\otimes E}}\right) \arrow[r] \arrow[u, "\bar{\delta}_*"] & K_{i+j-1}(I_{D\otimes E}) \arrow[r, "\cong"] \arrow[u, "\bar{\gamma}_*"] & K_{i+j-2}(I_2).
\end{tikzcd}$$

All horizontal maps are connecting homomorphisms and the upper vertical maps are exterior products.

The upper left square commutes and the upper right commutes up to a sign $(-1)^{i-1}$ by, Axiom 1.4.5. The lower four square commute by naturality of the connecting homomorphisms (Axiom 1.4.2) under the morphisms of exact sequences just constructed. The bulge on the left already commutes on the level of homomorphisms: $\bar{\beta}=\bar{\delta}\circ\nabla$

The outer left path from the upper left corner to the middle of the bottom side is multiplication on $K_*(\mathfrak{uc}(\mathcal{X};\_))$ followed by co-assembly and the outer right path is co-assembly applied twice followed by the secondary product on $K^*(\mathcal{X};\_)$ multiplied by $(-1)^{i-1}$. Exactly this factor appears inside the diagram, so the claim follows. $\square$

## 1.9 Coarse contractibility

So far we studied multiplicativity of uncoarsened co-assembly maps. In this section, we finally transfer the results to the coarsened versions. The required $\sigma$-contractibility of the Rips complex is provided by the following notion of coarse contractibility.

**Definition 1.9.1.** Let $X$ be a coarse space. We call a Borel map

$$H : X\times\mathbb{N}\to X$$

a *coarse contraction* to a point $pt\in X$ (and $X$ *coarsely contractible*) if

- $H \times H$ maps entourages of the product coarse structure on $X \times \mathbb{N}$ to entourages of $X$,
- the restriction of $H$ to $X \times \{0\}$ is the identity on $X$,
- for any bounded subset $B \subset X$ there is $N_B \in \mathbb{N}$ such that $H(x,n) = pt$ for all $(x,n) \in B \times \{n : n \geq N_B\}$,
- $H(pt,n) = pt$ for all $n \in \mathbb{N}$.

This map lacks properness and is thus no coarse map. However, the induced map

$$\{(x,n) \in X \times \mathbb{N} \,:\, n \leq N_{\{x\}}\} \to X \times X, \quad (x,n) \mapsto (H(x,n), x)$$

is a coarse map. Note the similarity to our earlier definition of the map $\Gamma$.

Coarse contractibility is obviously invariant under coarse equivalences such as passing from a countably generated coarse space $X$ to a coarsely equivalent discrete coarse subspace $Z \subset X$.

Now here are some examples of coarse contractibility:

**Example 1.9.2.** Open cones $\mathcal{O}Y \approx Y \times [0,\infty)/Y \times \{0\}$ are coarsely contractible for any compact metrizable space $Y$. The coarse contraction $H$ is defined by $H((y,t),n) := (y, \max\{t-n, 0\})$. The same argument shows that the foliated cones of [Roe95] are coarsely contractible.

**Example 1.9.3.** Locally compact, complete CAT(0) spaces $X$ are coarsely contractible by moving a point $y \in X$ with constant speed along the unique geodesic from $y$ to some fixed base point $p \in X$.

We only assume locally compactness and completeness in this and the following example, because our definition of coarse spaces involved a compatible locally compact topology on the underlying set. This premise becomes unnecessary if we define coarse contractibility more generally for sets equipped with a coarse structure.

**Example 1.9.4.** In this example we show that locally compact, complete, hyperbolic metric spaces (cf. [BH99, Gro87]) are coarsely contractible. So in particular, Gromov's hyperbolic groups equipped with an arbitrary word metric are coarsely contractible.

Of course, this is not surprising, because it is a well known fact that for any hyperbolic metric space $X$ with coarsely equivalent discrete subspace $Z$ the simplicial complexes $\mathcal{P}_{Z,n}$ are contractible for all large enough $n$ [BH99, Proposition 3.23].

We shall first recall the definition of hyperbolic metric spaces as presented in [HG04, Section 4.2]. Let $X$ be a metric space. A *geodesic segment* in $X$ is a curve $\gamma : [a,b] \to X$ such that

$$d(\gamma(s),\gamma(t)) = |s-t| \quad \forall s,t \in [a,b]\,.$$

A *geodesic metric space* is a metric space in which each two points are connected by a geodesic segment. A *geodesic triangle* in $X$ consists of three points of $X$ and for each two of these points a geodesic segment connecting them. A geodesic triangle is called $D$-slim, $D \geq 0$, if each point on each edge has distance at most $D$ from one of the other two edges.

**Definition 1.9.5.** A geodesic metric space $X$ is called *$D$-hyperbolic* if every geodesic triangle in $X$ is $D$-slim. It is called *hyperbolic* if it is $D$-hyperbolic for some $D \geq 0$.

Now let $X$ be a $D$-hyperbolic space and $pt \in X$. For each $x \in X$ we choose a geodesic segment $\gamma_x : [0,l_x] \to X$ with $\gamma(0) = x$ and $\gamma(l_x) = pt$, so $l_x = d(x,pt)$. We extend each $\gamma_x$ to $[0,\infty)$ by $\gamma(t) = pt$ for $t > l_x$ and define

$$H : X \times \mathbb{N} \to X, \quad (x,n) \mapsto \gamma_x(n)\,.$$

We claim that $H$ is a coarse contraction to the point $pt$.[4] Clearly $H|_{X\times\{0\}} = \mathrm{id}_X$, $H(pt,n) = pt$ for all $n \in \mathbb{N}$ and if $B \subset X$ is bounded and $x \in B$, then $H(x,n) = pt$ for all $n \geq N_B := \sup_{x\in B} d(pt,x)$.

It remains to show that entourages are mapped to entourages. Let $R > 0$. We claim that for all $x,y \in X$ with $d(x,y) \leq R$ and $n \in \mathbb{N}$ the inequality $d(H(x,n),H(y,n)) \leq 4D+R$ holds. Denote the geodesic triangle consisting of the points $pt,x,y$, the geodesics $\gamma_x,\gamma_y$ and another geodesic $\overline{xy}$ between $x,y$ by $\triangle$. We may assume w. l. o. g. that $l_x \geq l_y$. The triangle inequality in $\triangle$ implies $\delta := l_x - l_y \leq d(x,y) \leq R$.

Assume that there is $n \in \mathbb{N}$ such that

$$M := d(\gamma_x(n),\gamma_y(n)) = d(H(x,n),H(y,n)) > 4D + R\,.$$

Define $z := \gamma_x(n)$, $w := \gamma_y(n)$. Clearly $n < l_y$, because otherwise $d(z,w) = d(z,pt) = l_x - \min(n,l_x) \leq l_x - l_y = \delta \leq R$. Furthermore, the inequality

$$2n + R \geq d(z,x) + d(x,y) + d(y,w) \geq M > 4D + R$$

[4]We do not care whether this map is actually a Borel map, because we may always choose a discrete, coarsely equivalent subspace $Z \subset X$, restrict $H$ to $Z \times \mathbb{N}$ and finally re-extend to $X \times \mathbb{N}$ in a Borel fashion to obtain a Borel coarse contraction.

implies $n > 2D$. Because $\triangle$ is $D$-slim and

$$\operatorname{dist}(z, \overline{xy}) \geq d(z,x) - d(x,y) \geq n - D > D$$

there must be a point $v = \gamma_y(t)$ on the geodesic $\gamma_y$ such that $d(z,v) \leq D$.

**Case 1: $t \leq n$.** This means that $v$ lies between $w$ and $y$ on $\gamma_y$. The two triangle inequalities

$$\begin{aligned} M = d(z,w) &\leq d(z,v) + d(v,w) \leq D + (n-t)\,, \\ l_y - t = d(pt, v) &\leq d(pt, z) + d(z,v) \leq (l_x - n) + D \end{aligned}$$

imply $M \leq 2D + l_x - l_y \leq 2D + R$, contradicting our assumption.

**Case 2: $t \geq n$.** This time $v$ lies between $w$ and $pt$ on $\gamma_y$ and we consider the triangle inequalities

$$\begin{aligned} M = d(z,w) &\leq d(z,v) + d(v,w) \leq D + (t-n)\,, \\ l_x - n = d(pt, z) &\leq d(pt, v) + d(z,v) \leq (l_y - t) + D \end{aligned}$$

which imply $M \leq l_y - l_x + 2D \leq 2D$. Again, there is a contradiction.

For fixed $x$ we clearly have $d(H(x,n), H(x,m)) \leq |n-m|$ for all $n, m \in \mathbb{N}$. Thus, for arbitrary $(x,m), (y,m) \in X \times \mathbb{N}$ of distance at most $R$ the equality

$$d(H(x,m), H(y,m)) \leq 4D + 2R$$

holds. Therefore $H$ is a coarse contraction to the point $pt$.

**Theorem 1.9.6.** *Let $Z$ be a countably generated, discrete coarse space which is coarsely contractible. Then the Rips complex $\mathcal{P}_Z$ is $\sigma$-contractible.*

*Proof.* Let $H : Z \times \mathbb{N} \to Z$ be a coarse contraction of $Z$ to a point $pt \in Z$. For every $n \in \mathbb{N}$ we obtain a continuous map

$$H_n : \mathcal{P}_Z \to \mathcal{P}_Z$$

by pushing forward probability measures along the maps $Z \to Z$, $z \mapsto H(z,n)$.

A $\sigma$-contraction $H_* : \mathcal{P}_Z \times [0,1] \to \mathcal{P}_Z$ of the Rips complex to the same point $pt$ is then given on the strips $\mathcal{P}_Z \times [1-2^{-n}, 1-2^{-n-1}]$ by interpolating linearly between

$$H_*(\mu, 1-2^{-n}) = H_n \quad \text{and} \quad H_*(\mu, 1-2^{-n-1}) = H_{n+1}$$

and defining $H_*|_{\mathcal{P}_Z\times\{1\}} := pt$.

Obviously, $H_*$ restricts to the identity on $\mathcal{P}_Z\times\{0\}$ and $H_*(\mathcal{P}_Z\times\{1\}\cup\{pt\}\times[0,1]) = \{pt\}$.

Continuity of $H_*$ at $\mathcal{P}_Z\times\{1\}$ follows from continuity of the restrictions $H_*|_{\Delta\times(1-\delta_\Delta,1]}$ for every simplex $\Delta$ of $\mathcal{P}_Z$ and some $\delta_\Delta > 0$. But this is trivial, because the third condition in the definition of coarse contractibility ensures that these restrictions are constantly equal to $pt$ for $\delta_\Delta > 0$ small enough.

It remains to show that for each $n$ there is $m \geq n$ such that $H_*(\mathcal{P}_n\times[0,1]) \subset \mathcal{P}_m$. Denote the generating entourages of the coarse structure on $Z$ as usually by $E_n$. The first property of coarse contractibility ensures that the entourage

$$\{((x,k),\,(y,l)) \in (Z\times\mathbb{N})^2 \mid (x,y)\in E_n, |k-l|\leq 1\}$$

of the product coarse structure on $Z\times\mathbb{N}$ is mapped to some entourage $E_m$ of $Z$ by $H\times H$. If $\mu\in\mathcal{P}_n$, i.e. $\mu$ is a probability measure on $Z$ with $\operatorname{supp}\mu\times\operatorname{supp}\mu\subset E_n$, then for all $s\in[0,1]$ and $j\in\mathbb{N}$ the probability measure $\tilde\mu := (1-s)\cdot H_j(\mu) + s\cdot H_{j+1}(\mu)$ satisfies

$$\operatorname{supp}\tilde\mu \subset \operatorname{supp} H_j(\mu)\cup\operatorname{supp} H_{j+1}(\mu) = H(\operatorname{supp}\mu\times\{j\}\cup\operatorname{supp}\mu\times\{j+1\})$$

and therefore we have $\operatorname{supp}\tilde\mu\times\operatorname{supp}\tilde\mu\subset E_m$, i.e. $\tilde\mu\in P_m$. This proves $H_*(\mathcal{P}_n\times[0,1])\subset\mathcal{P}_m$. □

We shall now summarize our results to obtain the main result of this chapter:

**Theorem 1.9.7.** *Let $X$ be a coarsely contractible, countably generated coarse space. Then the unreduced coarse co-assembly map with coefficients in $D$,*

$$\mu^* : K_*(\mathfrak{c}(X;D)) \to KX^{1-*}(X\setminus\{pt\};D)\,,$$

*is multiplicative.*

Of course, the secondary product on $KX^{1-*}(X\setminus\{pt\};D)$ is obtained by applying Definition 1.7.2 to the $\sigma$-contraction of the Rips complex obtained as in Theorem 1.9.6 from the given coarse contraction. With these ingredients, the theorem follows trivially from Theorem 1.8.1.

For $D=\mathbb{C}$ we obtain a ring homomorphism:

**Theorem 1.9.8.** *Let $X$ be a coarsely contractible, countably generated coarse space. Then the unreduced coarse co-assembly map*

$$\mu^* : K_*(\mathfrak{c}(X)) \to KX^{1-*}(X\setminus\{pt\})$$

*is a ring homomorphism.*

## 1.A Fréchet algebras

$\sigma$-$C^*$-algebras are a special case of Fréchet algebras:

**Definition 1.A.1** (c.f. [Phi91, Section 1] and the reference therein)**.** A locally multiplicatively convex Fréchet algebra (over the complex numbers) is a complex topological algebra such that its topology

- is Hausdorff,
- is generated by a countable family of submultiplicative semi-norms $(p_n)_{n\in\mathbb{N}}$,
- is complete with respect to the family of semi-norms.

In the following, even without explicit mention, all Fréchet algebras appearing are assumed to be locally multiplicatively convex. Furthermore, we will always assume that there are constants $c_n$, such that $p_n \leq c_n p_{n+1}$ for all $n$.

Important examples of Fréchet algebras are the following algebras of smooth functions [Cun97, Section 1.1]:

Let $C^\infty[a,b]$ denote the algebra of smooth functions $f : [a,b] \to \mathbb{C}$ whose derivatives of all orders vanish at the endpoints. Furthermore, let $C_0^\infty(a,b]$, $C_0^\infty[b,a)$, $C_0^\infty(a,b)$ denote the subalgebras of $C^\infty[a,b]$ consisting of those functions which vanish at $a$ respectively $b$ respectively $a$ and $b$. All of them are Fréchet algebras with topology generated by the submultiplicative norms

$$p_n(f) = \|f\| + \|f'\| + \frac{1}{2}\|f''\| + \cdots + \frac{1}{n!}\|f^{(n)}\|.$$

They are the ingredients of a short exact sequence[5] of Fréchet algebras

$$0 \to C_0^\infty(0,1) \to C_0^\infty[0,1) \to \mathbb{C} \to 0 \tag{1.6}$$

which splits by a continuous linear map $\mathbb{C} \to C_0^\infty[0,1)$.

We will make use of the external product in $kk_*$-theory. It is formulated in terms of the projective tensor product of Fréchet algebras:

**Definition 1.A.2** (cf. [Trè67, Section 43],[Phi91, Theorem 2.3.(2e)] and [Cun97, Section 1])**.** Let $A, B$ be two Fréchet algebras whose topologies are generated by the seminorms $(p_n), (q_n)$. Their projective tensor product

[5] Again, a sequence $0 \to I \xrightarrow{\alpha} A \xrightarrow{\beta} B \to 0$ of Fréchet-algebras is called *exact* if it is algebraically exact, $\alpha$ is a homeomorphism onto its image, and $\beta$ defines a homeomorphism $A/\ker(\beta) \to B$.

$A \otimes_\pi B$ is the completion of the algebraic tensor product with respect to the seminorms

$$p \otimes_\pi q(z) = \inf \left\{ \sum_{i=1}^{m} p(a_i) q(b_i) : z = \sum_{i=1}^{m} a_i \otimes b_i,\ a_i \in A, b_i \in B \right\},$$

where $p$ is a continuous seminorm on $A$ and $q$ is a continuous seminorm on $B$.

The projective tensor product of Fréchet algebras is again a Fréchet algebra, its topology being generated by the seminorms $p_n \otimes_\pi q_n$.

Its universal property is of course the following:

**Lemma 1.A.3.** *Let $A, B, C$ be Fréchet algebras and let $f : A \times B \to C$ be a bilinear map such that for all continuous seminorms $r$ on $C$ there are continuous seminorms $p$ on $A$ and $q$ on $B$ and $K > 0$ satisfying $r(f(a,b)) \leq p(a)q(b)$ for all $a \in A$ and $b \in B$. Then there is a unique continuous linear map $g : A \otimes_\pi B \to C$ such that $g(a \otimes b) = f(a,b)$ for all $a \in A, b \in B$.*

*If $g$ restricted to $A \odot B$ is an algebra homomorphism, then so is $g$.*

The projective tensor product with nuclear Fréchet algebras is exact [Phi91, Theorem 2.3.(3b)]. So is the projective tensor product of any Fréchet algebra with short exact sequences of Fréchet algebras which split by a continuous linear map [Cun97, Remark following Definition 3.8]. This is, because the projective tensor product is in fact a tensor product of Fréchet spaces. In particular, the projective tensor product of (1.6) with another Fréchet algebra $A$,

$$0 \to C_0^\infty(0,1) \otimes_\pi A \to C_0^\infty[0,1) \otimes_\pi A \to A \to 0, \tag{1.7}$$

is exact.

For $\sigma$-$C^*$-algebras $A, B$, the universal property of Lemma 1.A.3 yields a continuous homomorphism

$$A \otimes_\pi B \to A \otimes B,$$

which will enables us to pass to the maximal tensor product of $\sigma$-$C^*$-algebras whenever an exact tensor product is needed.

## 1.B Cuntz's bivariant theory

The bivariant groups $kk_*(A,B)$ $(* = 0,1)$ of [Cun97] are defined for all locally multiplicatively convex topological algebras $A, B$, but we are only interested in the special case of Fréchet algebras here. $kk$-theory has properties analogous to those of $KK$-theory (cf. [Kas80b]) and $E$-theory (cf. Section 2.7). We mention some of the properties, which are all proved in [Cun97]:

1. There is a composition product
$$\cdot : kk_i(A,B) \otimes kk_j(B,C) \to kk_{i+j}(A,C)$$
and a graded commutative exterior product
$$\otimes : kk_i(A_1,B_1) \otimes kk_j(A_2,B_2) \to kk_{i+j}(A_1 \otimes_\pi A_2, B_1 \otimes_\pi B_2)$$
such that the $\mathbb{Z}_2$-graded groups $kk_*(A,B)$ become the morphism sets of an additive monoidal category **kk** whose objects are the Fréchet algebras. Some of the compatibility properties, but not all, are mentioned and proved in [Cun97]. The proof of the remaining properties is straightforward with the methods introduced therein.

2. There is a monoidal functor from the category of Fréchet algebras into **kk**: Continuous homomorphisms $\alpha : A \to B$ induce elements $kk(\alpha) \in kk_0(A,B)$. For $\alpha : A \to B$ and $\beta : B \to C$ we have $kk(\beta \circ \alpha) = kk(\alpha) \cdot kk(\beta)$. The groups $kk_*$ are functorial in the second and contravariantly functorial in the first variable with respect to continuous homomorphisms. This functoriality is given by composition product with the elements induced by the homomorphisms.

3. The covariant functor $kk_*(\mathbb{C}, \_)$ (defined on the category of Fréchet algebras) is naturally isomorphic to the $K$-theory functor $K_*$ of Phillips.

4. For each $A$ there is an invertible element $\mathfrak{b}_A \in kk_1(A, C_0^\infty(0,1) \otimes_\pi A)$. In fact, it is enough to know $\mathfrak{b} = \mathfrak{b}_\mathbb{C} \in K_1(C_0^\infty(0,1))$, because $\mathfrak{b}_A = \mathfrak{b} \otimes kk(\mathrm{id}_A)$. The composition product
$$\_ \cdot \mathfrak{b}_B : kk_i(A,B) \to kk_{i+1}(A, C_0^\infty(0,1) \otimes_\pi B)$$
is an isomorphism, which coincides up to a sign with the exterior product
$$\mathfrak{b} \otimes \_ : kk_i(A,B) \to kk_{i+1}(A, C_0^\infty(0,1) \otimes_\pi B).$$
We will always use the latter as the Bott periodicity identification $kk_i(A,B) \cong kk_{i+1}(A, C_0^\infty(0,1) \otimes_\pi B)$, which becomes
$$\mathfrak{b} \otimes \_ : K_i(A) \cong K_{i+1}(C_0^\infty(0,1) \otimes_\pi A)$$
in the special case of $K$-theory of Fréchet algebras.

5. There are the usual six-term exact sequences

$$\begin{tikzcd} kk_0(D,I) \arrow[r] & kk_0(D,A) \arrow[r] & kk_0(D,B) \arrow[d] \\ kk_1(D,B) \arrow[u] & kk_1(D,A) \arrow[l] & kk_1(D,I) \arrow[l] \end{tikzcd} \tag{1.8}$$

(and analogously in the first variable) for all short exact sequences

$$0 \to I \to A \xrightarrow{q} B \to 0 \tag{1.9}$$

of Fréchet algebras which split by a continuous linear map. We briefly recall the construction of the connecting homomorphisms. Even if the sequence does not split linearly, there is an exact sequence

$$\begin{aligned} \ldots \to kk_*(D, C_0^\infty(0,1) \otimes_\pi A) \to kk_*(D, C_0^\infty(0,1) \otimes_\pi B) \to \\ \to kk_*(D, C_q^\infty) \to kk_*(D,A) \to kk_*(D,B), \end{aligned}$$

where $C_q^\infty = \{(x,f) \in A \oplus C_0^\infty[0,1) \otimes_\pi B : q(x) = f(0)\}$ (the evaluation $f(0)$ is given by the map in (1.7)) is the smooth mapping cone.

We can allways replace $kk_*(D, C_0^\infty(0,1) \otimes_\pi B)$ by $kk_{*-1}(D,B)$ using Bott periodicity. The exact sequence

$$0 \to I \to C_q^\infty \to C_0^\infty[0,1) \otimes_\pi B \to 0 \tag{1.10}$$

splits, if (1.9) splits, and $C_0^\infty[0,1) \otimes_\pi B$ is diffeotopically contractible. By diffeotopy invariance of $kk_*$-theory, the exact sequence associated to (1.10) shows that

$$kk_*(D,I) \to kk_*(D, C_q^\infty)$$

is an isomorphism in this case. We now obtain (1.8) from these pieces and the boundary map is therefore given by the composition

$$kk_{*-1}(D,B) \xrightarrow{\mathfrak{b}\otimes -} kk_*(D, C_0^\infty(0,1) \otimes_\pi B) \to kk_*(D, C_q^\infty) \xleftarrow{\cong} kk_*(D,I).$$

However, if we work with $K$-theory of Fréchet algebras, i.e. $D = \mathbb{C}$, then there is always a six term exact sequence, even if (1.9) does not split. Thus, $K_*(I) \to K_*(C_q^\infty)$ is always an isomorphism by diffeotopy invariance of $K$-theory. So we see that the boundary map of the exact sequence

$$\begin{tikzcd} K_0(I) \arrow[r] & K_0(A) \arrow[r] & K_0(B) \arrow[d] \\ K_1(B) \arrow[u] & K_1(A) \arrow[l] & K_1(I) \arrow[l] \end{tikzcd}$$

can be chosen to be

$$K_{*-1}(B) \xrightarrow{\mathfrak{b}\otimes\_} K_*(C_0^\infty(0,1) \otimes_\pi B) \to K_*(C_q^\infty) \xleftarrow{\cong} K_*(I).$$

Now that we have reviewed some basic properties of $K$-theory of Fréchet algebras, we specialize to $\sigma$-$C^*$-algebras and prove that the remaining Axioms given in Section 1.4 are satisfied. We start with Axiom 1.4.4.

**Definition 1.B.1.** The exterior product in $K$-theory of $\sigma$-$C^*$-algebras with respect to the maximal tensor product is defined to be the composition

$$K_i(A) \otimes K_j(B) \to K_{i+j}(A \otimes_\pi B) \to K_{i+j}(A \otimes B).$$

Associativity follows from commutativity of

$$\begin{tikzcd} A \otimes_\pi B \otimes_\pi C \arrow[r] \arrow[d] & A \otimes_\pi (B \otimes C) \arrow[d] \\ (A \otimes B) \otimes_\pi C \arrow[r] & A \otimes B \otimes C. \end{tikzcd}$$

The product is obviously graded commutative and natural in both $A$ and $B$, because it has these properties with respect to the projective tensor product.

Given a short exact sequence of $\sigma$-$C^*$-algebras

$$0 \to I \to A \xrightarrow{q} B \to 0,$$

we can also define the continuous mapping cone

$$C_q = \{(x, f) \in A \oplus C_0[0,1) \otimes B : q(x) = f(0)\}.$$

It contains the ideals $C_0(0,1) \otimes B$ and $I$. As $C_q/I = C_0[0,1)$ is contractible, the inclusion $I \subset C_q$ induces an isomorphism $K_*(I) \cong K_*(C_q)$, too.

The canonical continuous homomorphisms $\iota : C_0^\infty(0,1) \to C_0(0,1)$, $C_0^\infty(0,1) \otimes_\pi B \to C_0(0,1) \otimes_\pi B \to C_0(0,1) \otimes B$ and $C_q^\infty \to C_q$ induce a commutative diagram

$$\begin{tikzcd}
K_{*-1}(B) \arrow[r, "\mathfrak{b}\otimes_"] \arrow[d, equal] & K_*(C_0^\infty(0,1)\otimes_\pi B) \arrow[r] \arrow[d] & K_*(C_q^\infty) \arrow[dd] & K_*(I) \arrow[l, "\cong"'] \arrow[dd, equal] \\
K_{*-1}(B) \arrow[r, "\iota_*(\mathfrak{b})\otimes_"] \arrow[d, equal] & K_*(C_0(0,1)\otimes_\pi B) \arrow[d] & & \\
K_{*-1}(B) \arrow[r, "\iota_*(\mathfrak{b})\otimes_"] & K_*(C_0(0,1)\otimes B) \arrow[r] & K_*(C_q) & K_*(I) \arrow[l, "\cong"']
\end{tikzcd}$$

If we use the letter $\mathfrak{b}$ for $\iota_*(\mathfrak{b})$, too, we obtain the $\sigma$-$C^*$-algebra description of the connecting homomorphism:

**Lemma 1.B.2.** *The connecting homomorphism in $K$-theory of $\sigma$-$C^*$-algebras associated to the short exact sequence*

$$0 \to I \to A \xrightarrow{q} B \to 0$$

*is the composition*

$$K_{*-1}(B) \xrightarrow{\mathfrak{b}\otimes\_} K_*(C_0(0,1)\otimes B) \to K_*(C_q) \xleftarrow{\cong} K_*(I).$$

Proving compatibility of boundary maps and exterior products as mentioned in Axiom 1.4.5 is now straightforward: Consider the diagram

$$\begin{tikzcd}
K_1(C_0(0,1))\otimes K_i(B)\otimes K_j(D) \arrow[rr] \arrow[d] & & K_1(C_0(0,1))\otimes K_{i+j}(B\otimes D) \arrow[d] \\
K_{i+1}(C_0(0,1)\otimes B)\otimes K_j(D) \arrow[rr] \arrow[d] & & K_{i+j+1}(C_0(0,1)\otimes B\otimes D) \arrow[d] \arrow[dl] \\
K_{i+1}(C_q)\otimes K_j(D) \arrow[r] & K_{i+1}(C_q\otimes D) \arrow[r] & K_{i+j+1}(C_{q\otimes \mathrm{id}_D}) \\
K_{i+1}(I)\otimes K_j(D) \arrow[rr] \arrow[u, "\cong"] & & K_{i+j+1}(I\otimes D). \arrow[u, "\cong"'] \arrow[ul]
\end{tikzcd}$$

The upper square commutes by associativity of the external product, the other two quadrilaterals commute by functoriality of the external product under the canonical inclusions $C_0(0,1)\otimes B \subset C_q$ and $I \subset C_q$ and the triangles commute already on the level of homomorphisms.

This proves commutativity of

$$\begin{tikzcd} K_i(B) \otimes K_j(D) \arrow[r] \arrow[d] & K_{i+1}(I) \otimes K_j(D) \arrow[d] \\ K_{i+j}(B \otimes D) \arrow[r] & K_{i+j+1}(I \otimes D). \end{tikzcd}$$

Commutativity of

$$\begin{tikzcd} K_i(D) \otimes K_j(B) \arrow[r] \arrow[d] & K_i(D) \otimes K_{j+1}(I) \arrow[d] \\ K_{i+j}(D \otimes B) \arrow[r] & K_{i+j+1}(D \otimes I) \end{tikzcd}$$

up to a sign $(-1)^i$ follows from the commutativity of the first by graded commutativity of the external product.

# Chapter 2

# $K$-theory of leaf spaces of foliations

This Chapter is organized as follows: In Section 2.1, we recall the definition of the holonomy groupoid and prove important properties of its length function, which is one of our main tools. Section 2.2 comprises the definition of foliated cones and a detailed proof of functoriality. The new "Farrell-Jones" $K$-theory model is introduced in 2.3, followed by elementary examples which serve as a basic test for the theory. Less trivial examples are then presented in Section 2.4.

We recall the definition of Connes' foliation algebra $C_r^*(V,\mathcal{F})$ and Hilbert modules over $C_r^*(V,\mathcal{F})$ coming from vector bundles in Sections 2.5 and 2.6, respectively. Section 2.7 is a short introduction to the asymptotic category and $E$-theory, which will be the main ingredient to the module structure. The module structure is finally introduced in Section 2.8 and the applications to index theory are discussed in Sections 2.9, 2.10.

## 2.1 The holonomy groupoid and its length function

Let $(V,\mathcal{F})$ be a foliation of a compact manifold $V$. We start by recalling the definition of the *holonomy groupoid* (also called the *graph*) of $(V,\mathcal{F})$, which lies in the heart of almost all further constructions. For details we refer to [Win83].

Let $c$ be a leafwise path (e. g. $c$ is piecewise smooth and $\dot{c}(t) \in T_{c(t)}\mathcal{F}$ for all $t$) between $x_0, x_1 \in V$. Choose foliation charts

$$\phi_i : U_i \to V_i \times W_i \subset \mathbb{R}^{\dim \mathcal{F}} \times \mathbb{R}^{\operatorname{codim} \mathcal{F}}$$

around $x_i$, $i = 0, 1$. By following paths which stay close to $c$, we obtain a well defined germ of local diffeomorphisms $W_0 \to W_1$ at $\phi_0(x_0)$. This germ is called the holonomy of $c$ (with respect to the chosen coordinate charts).

Two leafwise paths $c_1, c_2$ between $x_0, x_1$ are said to have the same holonomy if their holonomy with respect to some foliation charts around $x_0, x_1$ agree. This notion is independent of the choice of the charts.

**Definition 2.1.1** ([Win83])**.** The *holonomy groupoid* (or *graph*) $G \underset{r}{\overset{s}{\rightrightarrows}} V$ of $(V, \mathcal{F})$ is the smooth (possibly non-Hausdorff) groupoid of dimension $\dim V + \dim \mathcal{F}$ consisting of holonomy classes of leafwise paths together with the obvious source and range maps.

The manifold structure on $G$ is obtained as follows: For a leafwise path $c$, choose $\phi_0, \phi_1$ as above. We may assume that the chart domains are small enough such that $V_0, V_1, W_0, W_1$ are homeomorphic to open balls and $W_0 \cong W_1$ is a diffeomorphism representing the holonomy of $c$. Then there is an obvious chart from a set of leafwise paths which stay close to $c$ onto the open subset $V_0 \times W_0 \times V_1 \subset \mathbb{R}^{\dim V + \dim \mathcal{F}}$.

For each $x \in V$, the sets $G_x := s^{-1}(\{x\}), G^x := r^{-1}(\{x\})$ are smooth Hausdorff submanifold of $G$ of dimension $\dim \mathcal{F}$. For arbitrary subsets $A, B \subset V$ we define

$$G_A := s^{-1}(A), \quad G^B := r^{-1}(B), \quad G_A^B := s^{-1}(A) \cap r^{-1}(B).$$

From now on, we shall assume that $V$ is equipped with a Riemannian metric $g_V$. We will need a grasp on longitudinal distances. This is provided by the length function on the holonomy groupoid.

**Definition 2.1.2.** The length function of $G$ assigns to each holonomy class the infimum of the lengths of its representatives:

$$L : G \to [0, \infty), \quad \gamma \mapsto \inf_{c \in \gamma} L(c)$$

**Lemma 2.1.3.** *The length function is upper semi-continuous.*

*Proof.* Let $a > 0$. Assume $\gamma \in L^{-1}([0, a))$ is represented by a piecewise smooth path $c$ of length $L(c) < a$. From the construction of the manifold structure on $G$ it is evident that there is an open neighbourhood $U$ of $\gamma$ in (some coordinate chart of) $G$ whose elements can be represented by a family of piecewise smooth paths $\{c_\rho\}_{\rho\in U}$ in such a way that $U \to [0,\infty)$, $\rho \mapsto L(c_\rho)$ is continuous and $c_\gamma = c$. Thus, $U_0 := \{\rho \in U : L(c_\rho) < a\}$ is an open neighbourhood of $\gamma$. For any $\rho \in U_0$ we have $L(\rho) \leq L(c_\rho) < a$, so $U_0 \subset L^{-1}([0,a))$.

This shows that $L^{-1}([0,a)) \subset G$ is open and therefore $L : G \to [0,\infty)$ is upper semi-continuous. □

**Corollary 2.1.4.** *The length function is bounded on every compact subset of $G$.*

## 2.2 Foliated cones

Let $g_N$ be the transverse component of the chosen metric $g_V$ with respect to the orthogonal decomposition $TV = T\mathcal{F} \oplus N\mathcal{F}$.

**Definition 2.2.1** ([Roe95]). The foliated cone $\mathcal{O}(V,\mathcal{F})$ is the manifold $[0,\infty) \times V$ equipped with the Riemannian metric

$$g := dt^2 + g_V + t^2 g_N$$

which blows up only in the transverse direction.

Note that this is actually not a cone, as we did not crush $\{0\} \times V$ to a point. This minor difference is not detected by coarse geometry and thus the Higson coronas $\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),E)$ are unchanged. In many cases, our convention is more convenient to work with. We will see some examples of this in Section 2.4.

In this chapter, we consider the open cone $\mathcal{O}(V)$ as a special case of foliated cones by choosing the trivial 0-dimensional foliation on $V$. This means that we do not crush $\{0\} \times V$ to a point in the open cone $\mathcal{O}(V)$ either, which is different to our conventions of Chapter 1. Furthermore, the metric on $\mathcal{O}(V)$ is given by $g := dt^2 + (1+t^2)g_V$ instead of $dt^2 + t^2 g_V$. Again, these minor differences are not detected by coarse geometry.

We are interested in the algebras $\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),E)$. To get a grasp on the vanishing variation condition we use the length function.

**Lemma 2.2.2.** *Let $K \subset G$ be compact. Then the set*

$$E_K := \{(t, s(\gamma), t, r(\gamma)) \in \mathcal{O}(V,F) \times \mathcal{O}(V,F) : t \in [0,\infty),\ \gamma \in K\}$$

*is contained in*

$$E_R := \{(x,y) \in \mathcal{O}(V,F) \times \mathcal{O}(V,F) : \operatorname{dist}(x,y) \leq R\}$$

*for some* $R > 0$.

*Proof.* Let $R$ be a little bit bigger then the upper bound for $L|_K$ given by Corollary 2.1.4. The points $(t, s(\gamma))$ and $(t, r(\gamma))$ in $\mathcal{O}(V,F)$ are connected by the path $c_t : s \mapsto (t, c(s))$, where $c$ is a representative of $\gamma$ of length $L(c) < R$. As the metric on $\mathcal{O}(V,F)$ blows up only in transverse direction, we have

$$d\big((t,s(\gamma)),\, (t,r(\gamma))\big) \leq L(c_t) = L(c) < R$$

and the claim follows. □

**Corollary 2.2.3.** *Let* $g \in \overline{\mathfrak{uc}}(\mathcal{O}(V,F), E)$ *and denote its restriction to* $\{t\} \times V \subset \mathcal{O}(V,F)$ *by* $g_t \in C(V) \otimes E$. *Then the norm of* $s^*g_t - r^*g_t \in C(G,E)$ *restricted to any compact set* $K$ *tends to zero as* $t$ *goes to infinity.*

*Proof.* This follows directly from the previous Corollary and vanishing variation of $g$. □

The foliated cone construction is (almost) functorial under smooth maps of leaf spaces. This notion is defined as follows:

**Definition 2.2.4** ([HS87, Section I])**.** Let $(V_1, \mathcal{F}_1), (V_2, \mathcal{F}_2)$ be two foliations and denote their graphs by $G_1, G_2$. A *smooth map* $f : V_1/\mathcal{F}_1 \to V_2/\mathcal{F}_2$ *between the leaf spaces* consists of

- an open cover $\{\Omega_\alpha\}_{\alpha \in I}$ of $V_1$ and
- a collection of smooth maps $f_{\alpha\beta} : (G_1)_{\Omega_\beta}^{\Omega_\alpha} \to G_2$

such that

$$\forall \gamma \in (G_1)_{\Omega_\beta}^{\Omega_\alpha} : f_{\beta\alpha}(\gamma^{-1}) = f_{\alpha\beta}(\gamma)^{-1}$$

and for all $\gamma_1 \in (G_1)_{\Omega_{\alpha_2}}^{\Omega_{\alpha_1}}, \gamma_2 \in (G_1)_{\Omega_{\alpha_3}}^{\Omega_{\alpha_2}}$ with $s(\gamma_1) = r(\gamma_2)$ we have

$$s(f_{\alpha_1\alpha_2}(\gamma_1)) = r(f_{\alpha_2\alpha_3}(\gamma_2))$$

and

$$f_{\alpha_1\alpha_2}(\gamma_1) f_{\alpha_2\alpha_3}(\gamma_2) = f_{\alpha_1\alpha_3}(\gamma_1\gamma_2).$$

The unit map $u : V_1 \to G_1$ restricts to $u|_\alpha : \Omega_\alpha \to (G_1)_{\Omega_\alpha}^{\Omega_\alpha}$. The second property shows that the image of the composition $f_{\alpha\alpha} \circ u|_\alpha$ lies in the unit space $V_2$ of $G_2$. We thus obtain a family of smooth maps

$$f_\alpha := f_{\alpha\alpha} \circ u|_\alpha : \Omega_\alpha \to V_2.$$

The second property in the definition also implies

$$\forall \gamma \in (G_1)_{\Omega_\beta}^{\Omega_\alpha} : \quad r(f_{\alpha\beta}(\gamma)) = f_\alpha(r(\gamma)), \quad s(f_{\alpha\beta}(\gamma)) = f_\beta(s(\gamma)). \tag{2.1}$$

In particular, any representative of $f_{\alpha\beta}(u(x)) \in G_2$ for $x \in \Omega_\alpha \cap \Omega_\beta$ is a leafwise path in $V_2$ between $f_\alpha(x)$ and $f_\beta(x)$.

Furthermore, (2.1) implies that if $c$ is a path in $\Omega_\alpha$ which is contained in a single leaf then $f_\alpha \circ c$ is also contained in a single leaf. Thus, $df_\alpha$ maps $T_x\mathcal{F}_1$ to $T_{f_\alpha(x)}\mathcal{F}_2$.

Before we start making some choices, let us fix Riemannian metrics $g_1, g_2$ on $V_1, V_2$ which we shall use for the construction of the foliated cones $\mathcal{O}(V_1, \mathcal{F}_1), \mathcal{O}(V_2, \mathcal{F}_2)$.

Because of compactness of $V_1$, we can find a finite open cover $(\Omega_i')_{i=1,\dots,m}$ such that the closure of each $\Omega_i'$ is compact and contained in some $\Omega_{\alpha(i)}$. Denote the restriction of $f_{\alpha(i)}$ to $\overline{\Omega_i'}$ by $f_i$ and the restriction of $f_{\alpha(i)\alpha(j)}$ to $(G_1)_{\overline{\Omega_j'}}^{\overline{\Omega_i'}}$ by $f_{ij}$. Furthermore, choose a Borel map $i : V_1 \to \{1, \dots, m\}, x \mapsto i_x$ such that $\forall x \in V_1 : x \in \Omega_{i_x}'$.

**Proposition 2.2.5.** *With notations as above, the map*

$$f_* : \mathcal{O}(V_1, \mathcal{F}_1) \to \mathcal{O}(V_2, \mathcal{F}_2),\ (t, x) \mapsto (t, f_{i_x}(x))$$

*is a coarse map. The coarse equivalence class of this map is independent of the choices made.*

*Proof.* The following observation is central to the proof.

**Lemma 2.2.6.** *There is a constant $L$ such that for all $1 \le i, j \le m$ and $x \in \Omega_i' \cap \Omega_j'$ the points $f_i(x), f_j(x)$ are joint by a leafwise path of length at most $L$. In particular, the points*

$$(t, f_i(x)), (t, f_j(x)) \in \mathcal{O}(V_2, \mathcal{F}_2)$$

*have distance at most $L$ for all $t \ge 0$.*

*Proof.* The length function on the holonomy groupoid $G_2$ is bounded on the compact subset

$$\bigcup_{1\leq i,j\leq m} f_{ij}(u(\overline{\Omega'_i}\cap\overline{\Omega'_j}))$$

by Corollary 2.1.4. As we observed above, the points $f_i(x), f_j(x)$ are joined by any representative of some element in this compact set. □

Because of this Lemma, the choice of the Borel map $i : V_1 \to \{1,\dots,m\}$ affects the map $f_*$ only by a distance of at most $2L$ and therefore the coarse equivalence class remains unchanged. Independence of the finite and compact refinement $(\Omega'_i)_{i=1,\dots,m'}$ is clear, as this family may be enlarged by a finite number of open sets while leaving the Borel map $i$ and therefore also $f_*$ unchanged.

It remains to show that $f_*$ is a coarse map. Properness is clear, but the expansion condition has to be verified. We start with a little calculation.

**Lemma 2.2.7.** *There is a constant $K$ such that whenever $i \in \{1,\dots,m\}$ and $c : [0,1] \to [0,\infty)\times\Omega'_i$ is a smooth path then the length estimate*

$$L((id\times f_i)\circ c) \leq K\cdot L(c)$$

*holds. Here, the length of the curve $c$ is measured by equipping $[0,\infty)\times\Omega'_i$ with the restricted metric as a subset of $\mathcal{O}(V_1,\mathcal{F}_1)$.*

*Proof.* Let

$$K' := \max_{i=1,\dots,m}\sup_{x\in\overline{\Omega'_i}}\|df_i(x)\|\,.$$

Consider a tangent vector $\xi\in T_{(t,x)}\mathcal{O}(V_1,\mathcal{F}_1)$ at a point $(t,x)\in[0,\infty)\times\Omega'_i$. We write $\xi = \lambda\frac{\partial}{\partial t}+\xi_{\|}+\xi_{\perp}$ according to the orthogonal decomposition

$$T_{(t,x)}\mathcal{O}(V_1,\mathcal{F}_1) = \mathbb{R}\frac{\partial}{\partial t}\oplus T_x\mathcal{F}_1\oplus N_x\mathcal{F}_1.$$

Its norm is given by

$$\|\xi\|^2 = \lambda^2+\|\xi\|^2+t^2\|\xi_\perp\|^2 = \lambda^2+\|\xi_\|\|^2+(1+t^2)\|\xi_\perp\|^2.$$

Now recall that $df_i(\xi_\parallel)$ is tangent to $\mathcal{F}_2$. Thus, we can calculate

$$\begin{aligned}
\|df_i(\xi)\|^2 &= \|\lambda\frac{\partial}{\partial t} + df_i(\xi_\parallel) + df_i(\xi_\perp)\|^2 \\
&= \lambda^2 + \|df_i(\xi_\parallel) + df_i(\xi_\perp)_\parallel\|^2 + (1+t^2)\|df_i(\xi_\perp)_\perp\|^2 \\
&\leq \lambda^2 + (\|df_i(\xi_\parallel)\| + \|df_i(\xi_\perp)\|)^2 + (1+t^2)\|df_i(\xi_\perp)\|^2 \\
&\leq \lambda^2 + K'^2(\|\xi_\parallel\| + \|\xi_\perp\|)^2 + K'^2(1+t^2)\|\xi_\perp\|^2 \\
&\leq \lambda^2 + 2K'^2\|\xi_\parallel\|^2 + K'^2(3+t^2)\|\xi_\perp\|^2
\end{aligned}$$

and now we see that $\|df_i(\xi)\| \leq K\|\xi\|$ for

$$K = \sqrt{\max(1, 3K'^2)}.$$

The claim follows. □

Let $R > 0$. We have to find $S > 0$ such that whenever $z, z' \in \mathcal{O}(V_1, \mathcal{F}_1)$ are points of distance less than $R$ then the distance between $f_*(z), f_*(z')$ is less then $S$. We can estimate the distance between $f_*(z), f_*(z')$ by a sequence of paths as in Lemma 2.2.7 and jumps between points $(t, f_i(x)), (t, f_j(x))$ of length $\leq L$ as in Lemma 2.2.6. The only thing we need more is an upper bound for the number of jumps needed.

**Lemma 2.2.8.** *Let $R > 0$. There is $k \in \mathbb{N}$ such that whenever $c : [0,1] \to V_1$ is a path of length $\leq R$, then there are $0 = s_0 \leq \cdots \leq s_k = 1$ such that for each $l = 0, \ldots, k-1$ the image of $c|_{[s_l, s_{l+1}]}$ is contained in some $\Omega'_{i(l)}$.*

*Proof.* Let $\varepsilon > 0$ be such that every $\varepsilon$-ball in $V_1$ is contained in some $\Omega'_i$. Then the claim follows easily for some fixed $k > R/2\varepsilon$. □

Now if $z, z' \in \mathcal{O}(V_1, \mathcal{F}_1)$ are less than distance $R$ apart, then they are joined by a path

$$(t, c) : [0,1] \to \mathcal{O}(V_1, \mathcal{F}_1)$$

of length less than $R$. It follows that $c : [0,1] \to V_1$ has length less than $R$ (measured in $g_1$), too, and we can apply Lemma 2.2.8. With $s_l, i(l)$ as in the lemma, we can now go from

$$f_*(z) = (t(s_0), f_{i_{c(s_0)}}(c(s_0))) \quad \text{to} \quad f_*(z') = (t(s_k), f_{i_{c(s_k)}}(c(s_k)))$$

by first jumping to the point $(t(s_0), f_{i(0)}(c(s_0)))$, then following the path

$$(t|_{[s_0,s_1]},\ f_{i(0)} \circ c|_{[s_0,s_1]})$$

to the point $(t(s_1), f_{i(0)}(c(s_1)))$, then jumping again to $(t(s_1), f_{i(1)}(c(s_1)))$ and so on. We reach the endpoint after $k+1$ jumps of length at most $L$ and $k$ smooth paths in between, whose total length is at most $K \cdot R$.

Thus, the claim follows for $S := (k+1)L + KR$. □

An easy corollary, which one could also prove directly, is the following.

**Corollary 2.2.9.** *The coarse structure of the foliated cone $\mathcal{O}(V, \mathcal{F})$ is independent of the chosen metric on $V$.*

*Proof.* Applying the Proposition to the smooth map of leaf spaces $V/\mathcal{F} \to V/\mathcal{F}$ given by the one element open cover $\{V\}$ of $V$ and the map $\mathrm{id} : G = G_V^V \to G$ shows that the identity between the cones on the left and right hand side, which are allowed to be constructed with different metrics on $V$, is always a coarse map. □

The whole construction is obviously functorial. To summarize:

**Theorem 2.2.10.** *The foliated cone construction is a functor from the category of foliations and smooth maps of leaf spaces between them to the category of metrizable coarse spaces and coarse equivalence classes of coarse Borel maps between them.*

We conclude this chapter with two important special cases of functoriality.

**Example 2.2.11.** Given a foliation $(V, \mathcal{F})$, there is a canonical smooth map of leaf spaces

$$p : V \to V/\mathcal{F},$$

the left hand side being $V/\mathcal{F}_0$ for the trivial 0-dimensional foliation $\mathcal{F}_0$ on $V$. The graph of $(V, \mathcal{F}_0)$ is simply $V$ while we denote the graph of $(V, \mathcal{F})$ by $G$. The map $p$ consists of the one element open cover $\{V\}$ of $V$ and unit map $V = V_V^V \to G$.

The induced coarse map $\mathcal{O}V \to \mathcal{O}(V, \mathcal{F})$ is just the identity on the underlying topological space $[0, \infty) \times V$. It is obviously 1-Lipschitz.

**Example 2.2.12.** Assume that the foliation $(V, F)$ comes from a submersion $p : V \to B$. This submersion factors through a smooth map of leaf spaces $\tilde{p} : V/\mathcal{F} \to B$ which consists of the one element covering $\{V\}$ of $V$ and the map

$$p \circ s = p \circ r : G = G_V^V \to B$$

where $G = V \times_B V = \{(x,y)|p(x) = p(y)\}$ is the holonomy groupoid of $(V, \mathcal{F})$ and $B$ is the holonomy groupoid of the trivial 0-dimensional foliation on $B$. The induced coarse map we obtain is simply

$$\tilde{p}_* = \mathrm{id}_{[0,\infty)} \times p : \mathcal{O}(V, \mathcal{F}) \to \mathcal{O}B.$$

Now assume that $p$ is surjective and all fibers of $p$ are connected. In this case the fibers are uniformly bounded when measured by the length of smooth leafwise paths. As leafwise measured distances do not blow up in the foliated cone, we see that $\tilde{p}_*$ is a coarse equivalence.

In fact, an inverse coarse map can be chosen to be $f_*$ for a smooth map of leaf spaces $f : B \to V/\mathcal{F}$ defined as follows. Let $\{\Omega_\alpha\}$ be an open cover of $B$ such that for each $\alpha$ there is a smooth section $s_\alpha : \Omega_\alpha \to p^{-1}(\Omega_\alpha)$ of $p$ over $\Omega_\alpha$. Then the smooth map of leaf spaces $f : B \to V/\mathcal{F}$ given by

$$f_{\alpha\beta} = (s_\alpha, s_\beta) : \Omega_\alpha \cap \Omega_\beta \to V \times_B V$$

induces a coarse inverse to $\tilde{p}$.

If we had established a suitable equivalence relation on smooth maps between leaf spaces, then we would have obtained an isomorphism $V/\mathcal{F} \cong B$ already on the level of leaf spaces.

## 2.3 The new $K$-theory model

In [Roe95], Roe suggested considering $K^*_{FJ}(V/\mathcal{F}) := K^{*+1}(C^*(\mathcal{O}(V, \mathcal{F})^{\to}))$ as a new $K$-theory model for the leaf space of a foliation $(V, \mathcal{F})$. The index $FJ$ stands for Farrell-Jones, as this construction was motivated by the foliated control theory of [FJ90].

As the $K$-homology of the Roe algebra is not well behaved, we propose the following alternative definition, which leads to better behaved groups.

**Definition 2.3.1.** The "Farrell-Jones" model for the $K$-theory of the leaf space of a foliation $(V, \mathcal{F})$ is

$$K^{-*}_{FJ}(V/\mathcal{F}) := K_*(\mathfrak{c}(\mathcal{O}(V, \mathcal{F}))).$$

We also define the $K$-theory with coefficients in a $C^*$-algebra $D$ by

$$K^{-*}_{FJ}(V/\mathcal{F}, D) := K_*(\mathfrak{c}(\mathcal{O}(V, \mathcal{F}), D)).$$

It will be more convenient to perform proofs using the even more general groups $K_*(\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D))$. However, we will not give them any special name.

We spend the rest of this chapter showing properties and examples of these groups which indicate that they are a good model for $K$-theory of leaf spaces.

One might expect that $K$-theory models of "spaces" have some ring structure and contravariant functoriality. These properties are provided by Definition 1.6.1 and Theorem 2.2.10.

**Theorem 2.3.2.** *The groups $K^*_{FJ}(V/\mathcal{F})$ constitute a contravariant functor from the category of foliations and smooth maps between leaf spaces into the category of $\mathbb{Z}_2$-graded graded commutative rings.*

*The obvious analogous statements hold for the groups $K^*_{FJ}(V/\mathcal{F},D)$ and $K_*(\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D))$.*

The following two examples are a basic test for the Farrell-Jones $K$-theory model.

**Example 2.3.3.** Let $Y$ be a compact connected manifold. Equipped with the trivial 0-dimensional foliation, the foliated cone is equal to $\mathcal{O}Y$. From Proposition 1.6.2, we know that the inclusion $C(Y)\otimes\mathfrak{K}\subset\mathfrak{c}(\mathcal{O}Y)$ induces a ring isomorphism

$$K^*(Y)\xrightarrow{\cong}K^*_{FJ}(Y).$$

**Example 2.3.4.** Assume that the foliation $(V,\mathcal{F})$ comes from a surjective submersion $p:V\to B$ with all fibers connected. There is a canonical ring isomorphism

$$K^*(B)\cong K^*_{FJ}(B)\xrightarrow[\cong]{\tilde{p}^*}K^*_{FJ}(V/\mathcal{F})$$

induced by the smooth map of leaf spaces $\tilde{p}:V/\mathcal{F}\to B$ of Example 2.2.12. Thus, this composition is induced by the inclusion

$$C(B)\otimes\mathfrak{K}\hookrightarrow\mathfrak{c}(\mathcal{O}(V,\mathcal{F})),\quad g\mapsto\overline{(t,x)\mapsto p^*g(x)}.$$

Analogously there are multiplicative homomorphisms

$$K_{-*}(C(B)\otimes D)\cong K^*_{FJ}(B,D)\cong K^*_{FJ}(V/\mathcal{F},D)$$

for any coefficient $C^*$-algebra $D$.

## 2.4 Examples

In this section, we give some more examples of nontrivial elements in the ring $K^*_{FJ}(V/\mathcal{F})$. Before getting started, here are some general ideas of how to construct such elements:

In the definition of the foliated cone $\mathcal{O}(V,\mathcal{F})$, we refrained from crushing $\{0\}\times V$ to a point. Therefore, $C_0(\mathcal{O}(V,\mathcal{F}))\cong C_0([0,\infty)\times V)$ is contractible and the short exact sequence

$$0\to C_0([0,\infty)\times V)\otimes E\to \overline{\mathfrak{uc}}(\mathcal{O}(V,\mathcal{F}),E)\to \mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),E)\to 0$$

implies:

**Lemma 2.4.1.** *There are canonical isomorphisms*

$$K_*(\overline{\mathfrak{uc}}(\mathcal{O}(V,\mathcal{F}),E))\cong K_*(\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),E)).$$

This allows us to construct elements in $K_*(\bar{\mathfrak{c}}(\mathcal{O}(V,\mathcal{F})))$ instead of elements in $K_*(\mathfrak{c}(\mathcal{O}(V,\mathcal{F})))$, namely as vector bundles over $\mathcal{O}(V,\mathcal{F})$ of "vanishing variation", i. e. as projections in $\bar{\mathfrak{c}}(\mathcal{O}(V,\mathcal{F}))$.

One way to do this is by constructing a map $\phi:\mathcal{O}(V,\mathcal{F})\to X$ of vanishing variation into some compact metric space $X$ and using it to pull back vector bundles over $X$. More precisely, there is an induced $*$-homomorphism $\phi^*:C(X)\otimes\mathfrak{K}\to\bar{\mathfrak{c}}(\mathcal{O}(V,\mathcal{F}))$ and subsequently a homormorphism $\phi^*:K^*(X)\to K_*(\bar{\mathfrak{c}}(\mathcal{O}(V,\mathcal{F})))$. Note that $\phi^*$ is in fact a ring homomorphism, because multiplicativity follows from the commutative diagram

$$\begin{tikzcd}
\mathcal{O}(V,\mathcal{F}) \arrow[r, "\phi"] \arrow[d, "\Delta"'] & X \arrow[d, "\Delta"] \\
\mathcal{O}(V,\mathcal{F})\times\mathcal{O}(V,\mathcal{F}) \arrow[r, "\phi\times\phi"'] & X\times X.
\end{tikzcd}$$

We shall also need a method of distinguishing elements of $K^*_{FJ}(V/\mathcal{F})$. In some situations, an effective way to do this is to use the homomorphism

$$p^*:K^*_{FJ}(V/\mathcal{F})\to K^*(V)$$

induced by the canonical smooth map $p:V\to V/\mathcal{F}$ of Example 2.2.11. After identifying $K^*_{FJ}(V/\mathcal{F})$ with $K_{-*}(\bar{\mathfrak{c}}(\mathcal{O}(V,\mathcal{F})))$, one easily sees that $p^*$ is induced by the restriction $*$-homomorphism

$$\bar{\mathfrak{c}}(\mathcal{O}(V,\mathcal{F}))\to C(V)\otimes\mathfrak{K},\quad f\mapsto f|_{\{0\}\times V}\,. \tag{2.2}$$

Note that we would not have this simple formula if we had sticked to the stable Higson corona instead of the stable Higson compactification.

**Example 2.4.2.** Consider the one dimensional foliation of the two torus sketched in figure 2.1. The slices $T^2 \times \{t\} \subset \mathcal{O}(T^2, \mathcal{F})$ of the foliated cone

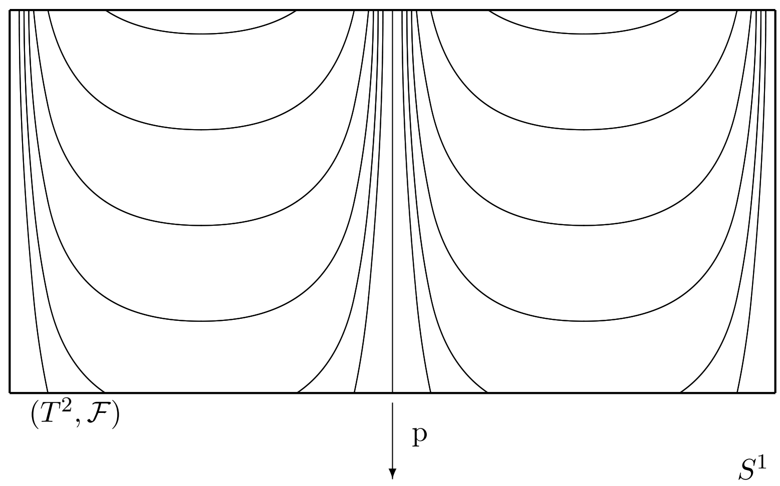


Figure 2.1: A one dimensional foliation on the two torus.

become larger and larger in the horizontal direction as $t \to \infty$. Given a unitary over $C(S^1)$, we can pull it back to $T^2$ via the projection $p$ onto the horizontal $S^1$. Subsequently, the variation of the unitary may be pushed into small neighborhoods of the compact leafs, where the metric blows up horizontally. By doing this, we obtain a unitary in $\bar{\mathfrak{c}}(\widetilde{\mathcal{O}(V, \mathcal{F})})$ and thus an element of $K_1(\mathfrak{c}(\mathcal{O}(V, \mathcal{F})))$ which is kind of a pullback of an element of $K^1(S^1)$ by the projection $p$.

The precise calculations involved are quite elaborate. We will perform them for a more general setup in Example 2.4.3.

**Example 2.4.3.** Let $\mathcal{F}$ be a one dimensional foliation on some compact manifold $V$ and $L$ a leaf of this foliation which is diffeomorphic to $S^1 = \mathbb{R}/\mathbb{Z}$. Assume that the normal bundle of $L$ is trivial, such that there is a tubular neighborhood of $L$ diffeomorphic to $D^n \times S^1$ in which $L$ corresponds to $\{0\} \times S^1$. Assume further that within this neighborhood the foliation is given by the trajectories of a unit vector field of the form

$$v(x, s) = \left(\lambda(x)x, \sqrt{1 - \lambda(x)^2\|x\|^2}\right) \in \mathbb{R}^n \times \mathbb{R} \cong T_{(x,s)}(D^n \times S^1)$$

for some continuous function $\lambda : D^n \to \mathbb{R}$. In particular, the vector field is $S^1$-invariant and Lipschitz continuous with Lipschitz constant $L := \|\lambda\|_\infty$.

The objectives of this example are to construct a ring homomorphism $K^*(S^n) \cong \mathbb{Z}[X]/(X^2) \to K^*_{FJ}(V/\mathcal{F})$ and to show that it is injective for $V = S^n \times S^1$. Thus, it is an example with nontrivial ring structure which is quite different from example 2.3.4.

We are free to choose any riemannian metric $g$ on $V$ to construct the foliated cone. Therefore, we may assume without loss of generality that it is the canonical one on the tubular neighborhood $D^n \times S^1$.

The first step is to construct a continuous map of vanishing variation $\Phi : \mathcal{O}(V, \mathcal{F}) \to S^n$ as follows. Consider the map

$$\phi : D^n \times S^1 \times [0, \infty) \to \mathbb{R}^n, \quad (x, s, t) \mapsto \begin{cases} \frac{\log(Lt\|x\|+1)}{\log(Lt+1)} \frac{x}{\|x\|} & t > 0 \\ x & t = 0. \end{cases}$$

It is smooth and maps $S^{n-1} \times S^1 \times [0, \infty)$ to $S^{n-1}$. Furthermore, let $\exp : \mathbb{R}^n \to S^n$ be the exponential map at the north pole $e$ of the sphere. It maps $\pi \cdot S^{n-1}$ to the south pole $-e$ and is Lipschitz continuous with constant 1.

**Lemma 2.4.4.** *The continuous map*

$$\Phi : \mathcal{O}(V, \mathcal{F}) \to S^n$$
$$x \mapsto \begin{cases} \exp(\pi \cdot \phi(x)) & x \in D^n \times S^1 \times [0, \infty) \\ -e & \textit{else} \end{cases}$$

*has vanishing variation.*

*Proof.* Note that it is enough to show that $\phi$ has vanishing variation with respect to the restricted metric on $D^n \times S^1 \times [0, \infty)$. To this end, let $w = (\xi, \mu) \in \mathbb{R}^n \times \mathbb{R} \cong T_{(x,s)}(D^n \times S^1)$. Denote by $w_\| = \langle w, v(x, s)\rangle \cdot v(x, s)$ its component tangential to the leafs. Furthermore, we decompose the $\mathbb{R}^n$-component $\xi = \xi_\perp + \xi_\|$ of $w$ into a component $\xi_\| := \frac{\langle \xi, x\rangle}{\|x\|^2} x$ parallel to $x$ and a component $\xi_\perp$ perpendicular to it. In the norm corresponding to the Riemannian metric $g_t := g + t^2 g_N$, we have

$$\begin{aligned}
\|w\|_t^2 &= \|w\|^2 + t^2\|w - w_\|\|^2 \\
&= (1 + t^2)\|w\|^2 - 2t^2\langle w, w_\|\rangle + t^2\|w_\|\|^2 \\
&= (1 + t^2)\|w\|^2 - t^2\langle w, v(x, s)\rangle^2 \\
&= (1 + t^2)(\|\xi\|^2 + \mu^2) - t^2 \left(\lambda(x)\langle \xi, x\rangle + \mu\sqrt{1 - \lambda(x)^2\|x\|^2}\right)^2 .
\end{aligned}$$

By minimizing this quadratic expression in $\mu$ and applying the inequality $\lambda(x)^2 \leq L^2$, we obtain the inequality

$$\|w\|_t^2 \geq (1+t^2)\|\xi_\perp\|^2 + \frac{1+t^2}{1+t^2L^2\|x\|^2}\|\xi_\|\|^2.$$

Thus, the norm of the tangential vector $w + \eta\frac{\partial}{\partial t} = (\xi, \mu, \eta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \cong T_{(x,s,t)}\mathcal{O}(V, \mathcal{F})$ is bounded from below by

$$\left\| w + \eta\frac{\partial}{\partial t} \right\|^2 \geq (1+t^2)\|\xi_\perp\|^2 + \frac{1+t^2}{1+t^2L^2\|x\|^2}\|\xi_\|\|^2 + \eta^2.$$

On the other hand, we define $f(r,t) := \frac{\log(rt+1)}{\log(t+1)}$ and calculate

$$D\phi\left(w + \eta\frac{\partial}{\partial t}\right) = f(\|x\|, t)\frac{\xi_\perp}{\|x\|} + \frac{\partial f}{\partial r}(\|x\|, t)\xi_\| + \eta\frac{\partial f}{\partial t}(\|x\|, t)\frac{x}{\|x\|}.$$

Thus,

$$\begin{aligned}\left\| D\phi\left(w + \eta\frac{\partial}{\partial t}\right) \right\|^2 &= \frac{f(\|x\|, t)^2}{\|x\|^2}\|\xi_\perp\|^2 + \left(\frac{\partial f}{\partial r}(\|x\|, t)\|\xi_\|\| \pm \frac{\partial f}{\partial t}(\|x\|, t)\eta\right)^2 \\ &\leq \left(\frac{f(\|x\|, t)^2}{(1+t^2)\|x\|^2} + 2\left(\frac{\partial f}{\partial r}(\|x\|, t)\right)^2 \frac{1+t^2L^2\|x\|^2}{1+t^2}\right. \\ &\qquad \left. +2\left(\frac{\partial f}{\partial t}(\|x\|, t)\right)^2\right) \left\| w + \eta\frac{\partial}{\partial t} \right\|^2.\end{aligned}$$

Vanishing variation of $\phi$ therefore follows from the fact that the three expressions

$$\begin{aligned}\frac{f(r,t)^2}{(1+t^2)r^2} &= \frac{\log(rt+1)^2}{\log(t+1)^2(1+t^2)r^2} \\ \left(\frac{\partial f}{\partial r}(r,t)\right)^2 \frac{1+t^2L^2r^2}{1+t^2} &= \frac{1}{\log(t+1)} \cdot \frac{t^2}{1+t^2} \cdot \frac{1+t^2L^2r^2}{(tr+1)^2} \\ \frac{\partial f}{\partial t}(r,t) &= \frac{r}{(rt+1)\log(t+1)} - \frac{\log(rt+1)}{(t+1)\log(t+1)^2}\end{aligned}$$

converge to 0 uniformly in $r \in (0, 1]$ for $t \to \infty$, as is readily verified. □

According to our remarks at the beginning of this section we obtain:

**Corollary 2.4.5.** *The map* $\Phi$ *induces a ring homomorphism*

$$\Phi^* : K^*(S^n) \to K^*_{FJ}(V/\mathcal{F}).$$

Furthermore, the composition $p^* \circ \Phi^* : K^*(S^n) \to K^*(V)$ is induced by the continuous map

$$\psi : V \to S^n, \quad x \mapsto \begin{cases} \exp(\pi \cdot y) & \text{if } x = (y, s) \in D^n \times S^1 \\ -e & \text{else.} \end{cases}$$

If we specialize to the case $V = S^n \times S^1$ where $D^n \times S^1 \subset V$ is assumed to come from an inclusion $D^n \subset S^n$, then the map $\psi$ is homotopic to the canonical projection $S^n \times S^1 \to S^n$. Thus, $p^* \circ \Phi^* = \psi^* : K^*(S^n) \to K^*(S^n \times S^1)$ is injective. In particular, the ring homomorphism

$$\Phi^* : K^*(S^n) \cong \mathbb{Z}[X]/(X^2) \hookrightarrow K^*_{FJ}(V/\mathcal{F})$$

is injective. We have thus detected some nontrivial ring structure inside of $K^*_{FJ}(V/\mathcal{F})$.

**Example 2.4.6.** The relation of the module structure to index theory will be discussed in Section 2.10, where Corollary 2.10.3 gives a formula for indices of longitudinally elliptic operators twisted by vector bundles $F \to V$ whose classes lie in the image of $p^* : K^0_{FJ}(V/\mathcal{F}) \to K^0(V)$. The following definition provides examples of such bundles.

**Definition 2.4.7.** Let $F \to V$ be a smooth vector bundle. We say that it is *asymptotically a bundle over the leaf space* if there is a smooth family of projections $(P_t)_{t\geq 0} \in M_n(C^\infty(V))$ such that $F \cong \mathrm{im}(P_0)$ and the norms

$$\|dP_t|_{T\mathcal{F}}\| = \sup_{0 \neq X \in T\mathcal{F}} \frac{\|dP_t(X)\|}{\|X\|}$$

converge to zero for $t \to \infty$.

Of course, the norm of $dP_t(X)$ is calculated in the $C^*$-algebra $M_n(C(V))$.

By reparametrising the $t$-parameter, we can always achieve that $\|\frac{\partial P_t}{\partial t}\|$ and $\frac{1}{1+t}\|dP_t|_{N\mathcal{F}}\|$ converge to zero for $t \to \infty$, too. Choose a monotonously decreasing function $K : [0, \infty) \to [0, \infty)$ converging to zero at infinity such that

$$\forall t : \ K(t) \geq \max\left( \|dP_t|_{T\mathcal{F}}\|, \ \frac{1}{1+t}\|dP_t|_{N\mathcal{F}}\|, \ \left\|\frac{\partial P_t}{\partial t}\right\| \right).$$

The $P_t$ compose to give a projection $P \in M_n(C_b(\mathcal{O}(V,\mathcal{F})))$. If $\gamma : [0,1] \to \mathcal{O}(V,\mathcal{F})$ is a smooth path with $t$-component bigger then some fixed $T$, then we decompose $\gamma' = v_L + v_N + \lambda\frac{\partial}{\partial t}$ into longitudinal, normal and $\partial/\partial t$-component and calculate

$$\begin{aligned}
\|P(\gamma(1)) - P(\gamma(0))\| &\leq \int_0^1 \|(P\circ\gamma)'(\tau)\| d\tau \\
&\leq \int_0^1 \left\| dP(v_L(\tau)) + dP(v_N(\tau)) + \lambda(\tau)\frac{\partial P_t}{\partial t}\right\| d\tau \\
&\leq K(T)\cdot \int_0^1 \left(\|v_L(\tau)\| + (1+t(\tau))\cdot\|v_N(\tau)\| + |\lambda(\tau)|\right) d\tau \\
&\leq 3K(T)\cdot\int_0^1 \|\gamma'(\tau)\| d\tau = 3K(T)\cdot L(\gamma)
\end{aligned}$$

This calculation shows that $P$ has vanishing variation and thus

$$P \in \overline{\mathfrak{uc}}(\mathcal{O}(V,\mathcal{F}); M_n(\mathbb{C})) \subset \bar{\mathfrak{c}}(\mathcal{O}(V,\mathcal{F})).$$

Let $x_F$ be its class in $K^0_{FJ}(V/\mathcal{F})$. Formula (2.2) now immediately implies $[F] = p^*(x_F)$. The element $x_F$ will become important in Corollary 2.10.3.

## 2.5 Connes' foliation algebra

We briefly recall the construction of Connes' foliation algebra $C^*_r(V,\mathcal{F})$. General references for this section are [Con82, Sections 5,6], [Con94, Section 2.8] or [Kor09, Section 5].

Instead of working with half densities as in [Con82, Con94], we fix once and for all a smooth, positive leafwise 1-density $\alpha \in C^\infty(V, |T\mathcal{F}|)$. It pulls back to smooth densities $r^*\alpha$ on $G_x$ and $s^*\alpha$ on $G^x$ for all $x \in V$ and we will always use these densities for integration. In particular, if $\gamma \in G$ with $x = s(\gamma), y = r(\gamma)$ and $f, g$ are functions on $G^y, G_x$, respectively, then we shall write

$$\begin{aligned}
\int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2) :&= \int_{\gamma_1\in G^y} f(\gamma_1)g(\gamma_1^{-1}\gamma)s^*\alpha(\gamma_1) \\
&= \int_{\gamma_2\in G_x} f(\gamma\gamma_2^{-1})g(\gamma_2)r^*\alpha(\gamma_2)\,.
\end{aligned}$$

In case $G$ is Hausdorff, the leafwise convolution product

$$(f*g)(\gamma) = \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2)$$

and the involution

$$f^*(\gamma) = \overline{f(\gamma^{-1})}$$

turn the vector space $C_c^\infty(G)$ of smooth complex valued functions with compact support on $G$ into a complex $*$-algebra.

If, however, the manifold structure on $G$ is non-Hausdorff, then $C_c^\infty(G)$ is by definition the vector space of complex functions on $G$ which are finite sums of smooth functions with compact support in some coordinate patch of $G$. In this case, the convolution product of two functions in $C_c^\infty(G)$ is again in $C_c^\infty(G)$, so $C_c^\infty(G)$ is a complex $*$-algebra in the non-Hausdorff case, too. This technicality does not interfere with our arguments at all, because we can always assume without loss of generality that our functions are compactly supported in coordinate patches.

For each $x \in V$, the Hilbert space $L^2(G_x)$ is defined by means of the density $r^*\alpha$ on $G_x$. There is a representation $\pi_x : C_c^\infty(G) \to \mathfrak{B}(L^2(G_x))$ given by

$$(\pi_x(f)\xi)(\gamma) = \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)\xi(\gamma_2)$$

for all $f \in C_c^\infty(G)$, $\xi \in L^2(G_x)$ and $\gamma \in G_x$.

**Definition 2.5.1.** The reduced foliation algebra $C_r^*(V,\mathcal{F})$ is defined as the completion of $C_c^\infty(G)$ in the pre-$C^*$-norm given by $\|f\|_r = \sup_{x\in V} \|\pi_x(f)\|$.

*Remark* 2.5.2. All the constructions above work equally well and give the same results if we use continuous instead of smooth functions everywhere. Note, however, that in this context the definition of continuous functions on a non-Hausdorff $G$ has to be adapted analogously.

**Definition 2.5.3** ([Con82])**.** Connes' $K$-theory model for the leaf space of the foliation $(V,\mathcal{F})$ is

$$K_C^*(V/\mathcal{F}) := K_{-*}(C_r^*(V,\mathcal{F})).$$

The reduced foliation algebra $C_r^*(V,\mathcal{F})$ can be understood as a sub-$C^*$-algebra of $\mathfrak{B}\left(\bigoplus_{x\in V} L^2(G_x)\right)$. We denote the canonical faithful representation by

$$\pi = \bigoplus_{x\in V} \pi_x : C_r^*(V,\mathcal{F}) \to \mathfrak{B}\left(\bigoplus_{x\in V} L^2(G_x)\right)$$

There is also a canonical faithful representation

$$\tau = \bigoplus_{x\in V} \tau_x : C(V) \to \mathfrak{B}\left(\bigoplus_{x\in V} L^2(G_x)\right)$$

given by $\tau_x(g)\xi := r^*g \cdot \xi$. For $f \in C_c(G)$ and $g \in C(V)$, the pointwise products $r^*g \cdot f$, $s^*g \cdot f$ lie in $C_c(G)$ and

$$\tau(g)\pi(f) = \pi(r^*g \cdot f), \quad \pi(f)\tau(g) = \pi(s^*g \cdot f). \tag{2.3}$$

**Lemma 2.5.4.** *$C(V)$ is canonically a sub-$C^*$-algebra of the multiplier algebra $\mathcal{M}(C_r^*(V,\mathcal{F}))$ of $C_r^*(V,\mathcal{F})$. For $f \in C_r^*(V,\mathcal{F})$ and $g \in C(V)$ we have*

$$\tau(g)\pi(f) = \pi(gf), \quad \pi(f)\tau(g) = \pi(fg). \tag{2.4}$$

*Furthermore, if $g \in C_c(G)$ then*

$$gf = r^*g \cdot f, \quad fg = s^*g \cdot f. \tag{2.5}$$

*Proof.* Formula (2.3) implies that the image of $\tau$ lies in the largest sub-$C^*$-algebra

$$D \subset \mathfrak{B}\left(\bigoplus_{x\in V} L^2(G_x)\right)$$

which contains the image of $\pi$ as an (essential) ideal. Thus, there is a canonical isometric $*$-homomorphism

$$C(V) \to D \to \mathcal{M}(C_r^*(V,\mathcal{F})).$$

Equations (2.4) and (2.5) are clear by definition. □

## 2.6 Hilbert modules associated to vector bundles

Smooth $\mathbb{Z}_2$-graded hermitian vector bundles $E \to V$ give rise to $\mathbb{Z}_2$-graded Hilbert modules over $C_r^*(V,\mathcal{F})$ which are particularly important in index theory. We review this construction as presented in [Kor09, Section 5.3]. For an introduction into the theory of Hilbert modules we refer to [Lan95].

As this is the first section featuring $\mathbb{Z}_2$-gradings, we take the opportunity to fix some notation: If $\mathcal{E}$ is an ungraded Hilbert module over an ungraded $C^*$-algebra $A$, then we denote $\mathcal{E}^{M,N} = \mathcal{E}^M \oplus \mathcal{E}^N$, where by definition the first summand is the even graded and the second summand is the odd gradded part. In particular this applies to the cases $A = \mathbb{C}$, where $\mathcal{E}$ is a Hilbert space, and $\mathcal{E} = A$ being a $C^*$-algebra. For any $\mathbb{Z}_2$-graded Hilbert module $\mathcal{E}$ we denote by $\mathfrak{B}(\mathcal{E})$ and $\mathfrak{K}(\mathcal{E})$ the $C^*$-algebras of adjointable respectively compact operators on $\mathcal{E}$ equipped with the canonical $\mathbb{Z}_2$-grading. In the special case $\mathcal{E} = A^{M,N}$ we obtain $M_{M,N}(A) := \mathfrak{K}(A^{M,N})$, which is the $C^*$-algebra of $(M+N)\times(M+N)$ matrices over $A$ where the diagonal $M\times M$-

and $N \times N$-blocks constitute the even part and the off-diagonal blocks are the odd part.

The symbol $\widehat{\otimes}$ will always denote graded tensor products. More specific, we use it in the context of $\mathbb{Z}_2$-graded $C^*$-algebras for the *maximal* graded tensor product. The minimal graded tensor product will be denoted by $\widehat{\otimes}_{\min}$.

Finally, all types of morphisms between $\mathbb{Z}_2$-graded objects are always assumed to be grading preserving, even without explicit mention.

Back to foliations: to define the Hilbert module $\mathcal{E}$ associated to $E$, let $\mathcal{E}^\infty := C_c^\infty(G, r^*E)$ be the vector space of smooth, compactly supported sections of the bundle $r^*E \to G$. Again, if $G$ is non-Hausdorff, we define it by summing up smooth sections compactly supported in coordinate patches of $G$. There is a right module structure of $\mathcal{E}^\infty$ over $C_c^\infty(G)$ by letting $f \in C_c^\infty(G)$ act on $\xi \in \mathcal{E}^\infty$ by the formula

$$(\xi * f)(\gamma) := \int_{\gamma_1\gamma_2=\gamma} \xi(\gamma_1) f(\gamma_2) \quad \forall \gamma \in G\,.$$

A $C_c^\infty(G) \subset C_r^*(V, \mathcal{F})$-valued inner product $\langle\,\_\,,\,\_\,\rangle_{\mathcal{E}^\infty}$ on $\mathcal{E}^\infty$ is defined by

$$\langle \xi, \zeta\rangle_{\mathcal{E}^\infty}(\gamma) := \int_{\gamma_1\gamma_2=\gamma} \langle \xi(\gamma_1^{-1}), \zeta(\gamma_2)\rangle_{E_{r(\gamma_2)}}\,.$$

This inner product is positive and defines a norm $\|\xi\|_r := \|\langle \xi, \xi\rangle_{\mathcal{E}^\infty}\|_r^{1/2}$ on $\mathcal{E}^\infty$.

**Definition 2.6.1.** The $\mathbb{Z}_2$-graded Hilbert module $\mathcal{E}$ associated to the hermitian vector bundle $E \to V$ is defined as the completion of $\mathcal{E}^\infty$ in the norm $\|\_\|_r$. The module multiplication of $C_r^*(V, \mathcal{F})$ on $\mathcal{E}$ and the $C_r^*(V, \mathcal{F})$-valued inner product $\langle\,\_\,,\,\_\,\rangle_{\mathcal{E}}$ on $\mathcal{E}$ are defined by extending module multiplication of $C_c^\infty(G)$ on $\mathcal{E}^\infty$ and $C_c^\infty(G)$-valued inner product on $\mathcal{E}^\infty$ continuously.

Again, one can perform these constructions using continuous instead of smooth sections.

If $E = \mathbb{C}^{M,N} \times V \to V$ is a trivial, $\mathbb{Z}_2$-graded bundle, then the associated $\mathbb{Z}_2$-graded Hilbert-$C_r^*(V, \mathcal{F})$-module is $\mathcal{E} = (C_r^*(V, \mathcal{F}))^{M,N}$. Its $\mathbb{Z}_2$-graded $C^*$-algebra of compact and adjointable operators are $\mathfrak{K}(\mathcal{E}) = M_{M,N}(C_r^*(V, \mathcal{F}))$ and $\mathfrak{B}(\mathcal{E}) = M_{M,N}(\mathcal{M}(C_r^*(V, \mathcal{F})))$, respectively.

An arbitrary $\mathbb{Z}_2$-graded vector bundle $E \to V$ may be embedded (grading preservingly) into a trivial bundle $\mathbb{C}^{M,N} \times V \to V$, such that $E$ is the image of a projection $p \in M_M(C(V)) \oplus M_N(C(V)) \subset M_{M,N}(C(V))$. Thus, $p$ may be seen as a projection in

$$\mathfrak{B}((C_r^*(V,\mathcal{F}))^{M,N}) = M_{M,N}(\mathcal{M}(C_r^*(V,\mathcal{F})))$$

which we also denote by $p$. It is easy to see, that the $\mathbb{Z}_2$-graded Hilbert-$C_r^*(V,\mathcal{F})$-module $\mathcal{E}$ associated to $E$ is canonically isomorphic to the image of this projection, $\mathcal{E} \cong \operatorname{im}(p) \subset (C_r^*(V,\mathcal{F}))^{M,N}$. Consequently,

$$\mathfrak{K}(\mathcal{E}) = pM_{M,N}(C_r^*(V,\mathcal{F}))p \subset M_{M,N}(C_r^*(V,\mathcal{F})). \tag{2.6}$$

We will need the following faithful representation of $\mathfrak{K}(\mathcal{E})$:

**Lemma 2.6.2.** *There is a canonical isometric inclusion*

$$\pi : \mathfrak{K}(\mathcal{E}) \overset{\subset}{\to} \mathfrak{B}\left(\bigoplus_{x\in V} L^2(G_x, r^*E)\right)$$

*with the following property: if $T \in \mathfrak{K}(\mathcal{E})$ is given on $C_c(G, r^*E)$ by convolution with $a \in C_c(G, r^*E \otimes s^*E^*)$, then $\pi(T)$ acts on each summand $L^2(G_x, r^*E)$ also by convolution with $a$.*

*Proof.* Simply compose (2.6) componentwise with the representation $\pi$ of $C_r^*(V,\mathcal{F})$. Using Lemma 2.5.4, it is straightforward to verify that the image of this composition is in fact contained in

$$\mathfrak{B}\left(\bigoplus_{x\in V} L^2(G_x, r^*E)\right) \subset \mathfrak{B}\left(\bigoplus_{x\in V} L^2(G_x, \mathbb{C}^{M,N})\right).$$

The claimed property of this representation of $\mathfrak{K}(\mathcal{E})$ follows directly from the analogous property of the canonical representation of $C_r^*(V,\mathcal{F})$. □

There is also a canonical isometric inclusion

$$\tau : C(V) \overset{\subset}{\to} \mathfrak{B}\left(\bigoplus_{x\in V} L^2(G_x, r^*E)\right)$$

where $g \in C(V)$ acts on each $L^2(G_x, r^*E)$ by pointwise multiplication with $r^*g$. Completely analogous to Lemma 2.5.4 we have:

**Lemma 2.6.3.** *$C(V)$ is canonically a sub-$C^*$-algebra of the multiplier algebra $\mathcal{M}(\mathfrak{K}(\mathcal{E})) = \mathfrak{B}(\mathcal{E})$ of $\mathfrak{K}(\mathcal{E})$. For $T \in \mathfrak{K}(\mathcal{E})$ and $g \in C(V)$ we have*

$$\tau(g)\pi(T) = \pi(gT), \quad \pi(T)\tau(g) = \pi(Tg).$$

*Furthermore, if $T \in \mathfrak{K}(\mathcal{E})$ is given by convolution with $a \in C_c(G, r^*E \otimes s^*E^*)$ and $g \in C(V)$, then $r^*g \cdot a, s^*g \cdot a \in C_c(G, r^*E \otimes s^*E^*)$, too, and $gT$ is convolution with $r^*g \cdot a$ whereas $Tg$ is convolution with $s^*g \cdot a$.*

## 2.7 The asymptotic category and $E$-theory

This section is a brief summary of the basic definitions and properties of the asymptotic category and $E$-theory. We use the picture of $E$-theory presented in [HG04]. A more detailed exposition of $E$-theory, which is based on a slightly different definition, is found in [GHT00].

**Definition 2.7.1** ([HG04, Definition 2.2],[GHT00, Definition 1.1])**.** Let $B$ be a $\mathbb{Z}_2$-graded $C^*$-algebra. The *asymptotic $C^*$-algebra of $B$* is

$$\mathfrak{A}(B) := C_b([1,\infty), B)/C_0([1,\infty), B).$$

$\mathfrak{A}$ is a functor from the category of $\mathbb{Z}_2$-graded $C^*$-algebras into itself.

An *asymptotic morphisms* is a graded $*$-homomorphisms $A \to \mathfrak{A}(B)$.

**Definition 2.7.2** ([HG04, Definition 2.3],[GHT00, Definition 2.2])**.** Let $A, B$ be $\mathbb{Z}_2$-graded $C^*$-algebras. The *asymptotic functors* $\mathfrak{A}^0, \mathfrak{A}^1, \dots$ are defined by $\mathfrak{A}^0(B) = B$ and

$$\mathfrak{A}^n(B) = \mathfrak{A}(\mathfrak{A}^{n-1}(B)).$$

Two $*$-homomorphisms $\phi^0, \phi^1 : A \to \mathfrak{A}^n(B)$ are $n$-homotopic if there exists a $*$-homomorphism $\Phi : A \to \mathfrak{A}^n(B[0,1])$, called $n$-homotopy between $\phi^0, \phi^1$, from which the $*$-homomorphisms $\phi^0, \phi^1$ are recovered as the compositions

$$A \xrightarrow{\Phi} \mathfrak{A}^n(B[0,1]) \xrightarrow{\text{evaluation at } 0,1} \mathfrak{A}^n(B).$$

**Lemma 2.7.3** ([GHT00, Proposition 2.3])**.** *The relation of $n$-homotopy is an equivalence relation on the set of $*$-homomorphisms from $A$ to $\mathfrak{A}^n(B)$.*

**Definition 2.7.4** ([HG04, Definition 2.4],[GHT00, Definition 2.6])**.** Let $A, B$ be $\mathbb{Z}_2$-graded $C^*$-algebras. Denote by $[\![A, B]\!]_n$ the set of $n$-homotopy classes of $*$-homomorphisms from $A$ to $\mathfrak{A}^n(B)$.

There are two natural transformations $\mathfrak{A}^n \to \mathfrak{A}^{n+1}$: The first is defined by including $\mathfrak{A}^n(B)$ into $\mathfrak{A}^{n+1}(B) = \mathfrak{A}(\mathfrak{A}^n(B))$ as constant functions. The second is defined by applying the functor $\mathfrak{A}^n$ to the inclusion of $B$ into $\mathfrak{A}B$ as constant functions. Both of them define maps $[\![A, B]\!]_n \to [\![A, B]\!]_{n+1}$.

**Lemma 2.7.5** ([GHT00, Proposition 2.8])**.** *The above natural transformations define the same map $[\![A, B]\!]_n \to [\![A, B]\!]_{n+1}$.*

These maps organize the sets $[\![A, B]\!]_n$ into a directed system

$$[\![A, B]\!]_0 \to [\![A, B]\!]_1 \to [\![A, B]\!]_2 \to \dots$$

**Definition 2.7.6** ([HG04, Definition 2.5],[GHT00, Definition 2.7]). Let $A, B$ be $\mathbb{Z}_2$-graded $C^*$-algebras. Denote by $[\![A, B]\!]_\infty$ the direct limit of the above directed system. We denote the class of a $*$-homomorphism $\phi : A \to \mathfrak{A}^n(B)$ by $[\![\phi]\!]$.

**Proposition 2.7.7** ([GHT00, Proposition 2.12]). *Let $\phi : A \to \mathfrak{A}^n(B)$ and $\psi : B \to \mathfrak{A}^m(C)$ be $*$-homomorphisms. The class $[\![\psi]\!] \circ [\![\phi]\!] \in [\![A, C]\!]_\infty$ of the composite $*$-homomorphism*

$$A \xrightarrow{\phi} \mathfrak{A}^n(B) \xrightarrow{\mathfrak{A}^n(\psi)} \mathfrak{A}^{n+m}(C)$$

*depends only on the classes $[\![\phi]\!] \in [\![A, B]\!]_\infty$, $[\![\psi]\!] \in [\![B, C]\!]_\infty$ of $\phi, \psi$. The composition law*

$$[\![A, B]\!]_\infty \times [\![B, C]\!]_\infty \to [\![A, C]\!]_\infty, \quad ([\![\phi]\!], [\![\psi]\!]) \mapsto [\![\psi]\!] \circ [\![\phi]\!]$$

*so defined is associative.*

For example, if $n = m = 1$ and $\phi, \psi$ lift to continuous maps

$$\begin{aligned} \tilde{\phi} &: A \to C_b([1, \infty), B), & a &\mapsto [t \mapsto \tilde{\phi}_t(a)], \\ \tilde{\psi} &: B \to C_b([1, \infty), C), & b &\mapsto [s \mapsto \tilde{\psi}_s(b)], \end{aligned}$$

respectively, then $[\![\psi]\!] \circ [\![\phi]\!]$ is represented by

$$A \to \mathfrak{A}^2(C), \quad a \mapsto \overline{t \mapsto \overline{s \mapsto \tilde{\psi}_s(\tilde{\phi}_t(a))}},$$

where the overline denotes equivalence classes. We will make use of this formula a few times later on.

According to the proposition, we obtain a category:

**Definition 2.7.8** ([HG04, Definition 2.6],[GHT00, Definition 2.13]). The *asymptotic category* is the category whose objects are $\mathbb{Z}_2$-graded $C^*$-algebras, whose morphisms are elements of the sets $[\![A, B]\!]_\infty$, and whose composition law is defined in Proposition 2.7.7.

The identity morphism $1_A \in [\![A, A]\!]_\infty$ is represented by the identity $\mathrm{id}_A : A \to A = \mathfrak{A}^0(A)$.

For arbitrary $\mathbb{Z}_2$-graded $C^*$-algebras $B, D$, there are canonical asymptotic morphisms

$$\begin{aligned} \mathfrak{A}(B) \mathbin{\widehat{\otimes}} D &\to \mathfrak{A}(B \mathbin{\widehat{\otimes}} D) & \bar{g} \mathbin{\widehat{\otimes}} d &\mapsto \overline{t \mapsto g(t) \mathbin{\widehat{\otimes}} d} \\ D \mathbin{\widehat{\otimes}} \mathfrak{A}(B) &\to \mathfrak{A}(D \mathbin{\widehat{\otimes}} B) & d \mathbin{\widehat{\otimes}} \bar{g} &\mapsto \overline{t \mapsto d \mathbin{\widehat{\otimes}} g(t)} \end{aligned}$$

and inductively also canonical $*$-homomorphisms $\mathfrak{A}^n(B) \mathbin{\widehat{\otimes}} D \to \mathfrak{A}^n(B \mathbin{\widehat{\otimes}} D)$, $D \mathbin{\widehat{\otimes}} \mathfrak{A}^n(B) \to \mathfrak{A}^n(D \mathbin{\widehat{\otimes}} B)$. This is a consequence of [GHT00, Lemmas 4.1, 4.2 & Chapter 3].

**Proposition 2.7.9** ([GHT00, Theorem 4.6])**.** *The asymptotic category is a monoidal category with respect to the* maximal graded tensor product $\widehat{\otimes}$ *of $C^*$-algebras and a tensor product on the morphism sets,*

$$\widehat{\otimes} : [\![A_1, B_1]\!]_\infty \times [\![A_2, B_2]\!]_\infty \to [\![A_1 \mathbin{\widehat{\otimes}} A_2, B_1 \mathbin{\widehat{\otimes}} B_2]\!]_\infty,$$

*with the following property: If $[\![\phi]\!] \in [\![A_1, B_1]\!]_\infty$ and $[\![\psi]\!] \in [\![A_2, B_2]\!]_\infty$ are represented by $\phi : A_1 \to \mathfrak{A}^m(B_1)$ and $\psi : A_2 \to \mathfrak{A}^n(B_2)$ , respectively, and $D$ is another $\mathbb{Z}_2$-graded $C^*$-algebra, then*

$$[\![\phi]\!] \mathbin{\widehat{\otimes}} 1_D \in [\![A_1 \mathbin{\widehat{\otimes}} D, B_1 \mathbin{\widehat{\otimes}} D]\!]_\infty,$$
$$1_D \mathbin{\widehat{\otimes}} [\![\psi]\!] \in [\![D \mathbin{\widehat{\otimes}} A_2, D \mathbin{\widehat{\otimes}} B_2]\!]_\infty$$

*are represented by the compositions*

$$A_1 \mathbin{\widehat{\otimes}} D \xrightarrow{\phi \mathbin{\widehat{\otimes}} \mathrm{id}_D} \mathfrak{A}^m(B_1) \mathbin{\widehat{\otimes}} D \to \mathfrak{A}^m(B_1 \mathbin{\widehat{\otimes}} D),$$
$$D \mathbin{\widehat{\otimes}} A_2 \xrightarrow{\mathrm{id}_D \mathbin{\widehat{\otimes}} \psi} D \mathbin{\widehat{\otimes}} \mathfrak{A}^n(B_2) \to \mathfrak{A}^n(D \mathbin{\widehat{\otimes}} B_2),$$

*respectively.*

The general form of the tensor product is of course

$$[\![\phi]\!] \mathbin{\widehat{\otimes}} [\![\psi]\!] = ([\![\phi]\!] \mathbin{\widehat{\otimes}} 1_{B_2}) \circ (1_{A_1} \mathbin{\widehat{\otimes}} [\![\psi]\!]) = (1_{B_1} \mathbin{\widehat{\otimes}} [\![\psi]\!]) \circ ([\![\phi]\!] \mathbin{\widehat{\otimes}} 1_{A_2}).$$

There is an obvious monoidal functor from the category of $\mathbb{Z}_2$-graded $C^*$-algebras into the asymptotic category which is the identity on the objects and maps a $*$-homomorphism $A \to B$ to its class in $[\![A, B]\!]_\infty$ by considering it as a $*$-homomorphism $A \to \mathfrak{A}^0(B)$.

The definition of $E$-theory involves the following two $\mathbb{Z}_2$-graded $C^*$-algebas. The first is $\widehat{\mathfrak{K}} = \mathfrak{B}(\widehat{\ell^2}) = M_{1,1}(\mathfrak{K})$ – the $\mathbb{Z}_2$-graded $C^*$-algebra of compact operators on the $\mathbb{Z}_2$-graded Hilbert space $\widehat{\ell^2} = \ell^2 \oplus \ell^2$ with even and odd part equal to the standard separable, infinite dimensional Hilbert space $\ell^2$.

The role of $\widehat{\mathfrak{K}}$ is stabilization: Given two separable, $\mathbb{Z}_2$-graded Hilbert spaces $H_1, H_2$, any isometry $V : H_1 \mathbin{\widehat{\otimes}} \widehat{\ell^2} \to H_2 \mathbin{\widehat{\otimes}} \widehat{\ell^2}$ defines an injective $*$-homomorphisms

$$\mathrm{Ad}_V : \mathfrak{K}(H_1) \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \to \mathfrak{K}(H_2) \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}}, \quad T \mapsto VTV^*.$$

The homotopy class of $\mathrm{Ad}_V$ is independent of the choice of $V$ and therefore defines a canonical isomorphism between $\mathfrak{K}(H_1) \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}}$ and $\mathfrak{K}(H_2) \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}}$ in the asymptotic category. In particular, $\widehat{\mathfrak{K}} \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}}$, $\mathfrak{K} \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}}$ and $M_{M,N}(\mathbb{C}) \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}}$ are all canonically isomorphic to $\widehat{\mathfrak{K}}$.

The second $\mathbb{Z}_2$-graded $C^*$-algebra is $C_0(\mathbb{R})$, but with non-trivial grading given by the direct sum decomposition into even and odd functions. This $\mathbb{Z}_2$-graded $C^*$-algebra is denoted by $\mathcal{S}$.

Recall from [HG04, Section 1.3] that $\mathcal{S}$ is also a co-algebra with co-unit $\eta : \mathcal{S} \to \mathbb{C}, f \mapsto f(0)$ and a co-multiplication $\Delta : \mathcal{S} \to \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S}$. The definition of $\Delta$ is not relevant to us, as we shall explain below. It is enough to know the axioms of a co-algebra, i. e. that

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S} \arrow[d, "\mathrm{id} \mathbin{\widehat{\otimes}} \Delta"] \\ \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S} \arrow[r, "\Delta \mathbin{\widehat{\otimes}} \mathrm{id}"'] & \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, "\Delta"] & \mathcal{S} \\ \mathcal{S} & \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S} \arrow[l, "\eta \mathbin{\widehat{\otimes}} \mathrm{id}"] \arrow[u, "\eta \mathbin{\widehat{\otimes}} \mathrm{id}"'] \end{tikzcd} \tag{2.7}$$

commute.

**Definition 2.7.10.** Let $A, B$ be $\mathbb{Z}_2$-graded $C^*$-algebras. The *E-theory* of $A, B$ is

$$E(A, B) = [\![\mathcal{S} \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}}, B \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}}]\!]_\infty.$$

It is a group with addition given by direct sum of $*$-homomorphisms

$$\mathcal{S} \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \to \mathfrak{A}^n(B \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}})$$

(via an inclusion $\widehat{\mathfrak{K}} \oplus \widehat{\mathfrak{K}} \hookrightarrow \widehat{\mathfrak{K}}$, which is canonical up to homotopy) and the zero element represented by the zero $*$-homomorphism.

*Remark* 2.7.11. By [GHT00, Theorem 2.16], this definition is equivalent to [HG04, Definition 2.1] when $A, B$ are separable. For non-separable $C^*$-algebras, however, it is essential to use Definition 2.7.10, because otherwise the products defined below might not exist.

There is a composition product

$$E(A, B) \otimes E(B, C) \to E(A, C), \quad (\phi, \psi) \mapsto \psi \circ \phi,$$

where $\psi \circ \phi \in E(A, C)$ is defined to be the composition

$$\mathcal{S} \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \xrightarrow{\Delta \mathbin{\widehat{\otimes}} \mathrm{id}_{A \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}}}} \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S} \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \xrightarrow{\mathrm{id}_{\mathcal{S}} \mathbin{\widehat{\otimes}} \phi} \mathcal{S} \mathbin{\widehat{\otimes}} B \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \xrightarrow{\psi} C \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}}$$

of morphisms in the asymptotic category.

There is also an exterior product

$$E(A_1, B_1) \otimes E(A_2, B_2) \to E(A_1 \mathbin{\widehat{\otimes}} A_2, B_1 \mathbin{\widehat{\otimes}} B_2), \quad (\phi, \psi) \mapsto \phi \mathbin{\widehat{\otimes}} \psi,$$

where $\phi \mathbin{\widehat{\otimes}} \psi \in E(A_1 \mathbin{\widehat{\otimes}} A_2, B_1 \mathbin{\widehat{\otimes}} B_2)$ is defined to be the composition

$$\begin{aligned} \mathcal{S} \mathbin{\widehat{\otimes}} A_1 \mathbin{\widehat{\otimes}} A_2 \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} &\xrightarrow{\Delta \mathbin{\widehat{\otimes}} \mathrm{id}} \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S} \mathbin{\widehat{\otimes}} A_1 \mathbin{\widehat{\otimes}} A_2 \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \cong \mathcal{S} \mathbin{\widehat{\otimes}} A_1 \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \mathbin{\widehat{\otimes}} \mathcal{S} \mathbin{\widehat{\otimes}} A_2 \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \\ &\xrightarrow{\phi \mathbin{\widehat{\otimes}} \psi} B_1 \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \mathbin{\widehat{\otimes}} B_2 \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \cong B_1 \mathbin{\widehat{\otimes}} B_2 \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \end{aligned}$$

of morphisms in the asymptotic category.

**Theorem 2.7.12** ([HG04, Theorems 2.3, 2.4]). *With these composition and exterior products, the $E$-theory groups $E(A, B)$ are the morphism groups in an additive monoidal category* $\mathbf{E}$ *whose objects are the $\mathbb{Z}_2$-graded $C^*$-algebras.*

We conclude this section by mentioning some properies of $E$-theory needed for our computations. Our earlier observations imply:

**Theorem 2.7.13** (Stability). *For any separable $\mathbb{Z}_2$-graded Hilbert space $H$, the $\mathbb{Z}_2$-graded $C^*$-algebra $\mathfrak{K}(H)$ is canonically isomorphic in the category* $\mathbf{E}$ *to $\mathbb{C}$. In particular, this applies to $\widehat{\mathfrak{K}}$, $\mathfrak{K}$ and $M_{M,N}(\mathbb{C})$.*

**Theorem 2.7.14** ([HG04, Theorems 2.3, 2.4]). *There is a monoidal functor from the asymptotic category into* $\mathbf{E}$ *which is the identity on the objects and maps $\phi \in [\![A, B]\!]_\infty$ to the morphism*

$$\mathcal{S} \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \xrightarrow{[\![\eta]\!] \mathbin{\widehat{\otimes}} \phi \mathbin{\widehat{\otimes}} 1_{\widehat{\mathfrak{K}}}} B \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}}$$

*in the asymptotic category, which we denote by the same letter $\phi$.*

Thus, by taking the $E$-theory product with this $E$-theory element, we obtain homomorphisms

$$E(D, A) \xrightarrow{\phi\circ} E(D, B), \quad E(B, D) \xrightarrow{\circ\phi} E(A, D)$$

for any third $\mathbb{Z}_2$-graded $C^*$-algebra $D$. Consequently, the $E$-theory groups are contravariantly functorial in the first variable and covariantly functorial in the second variable with respect to morphisms in the asymptotic category and in particular with respect to $*$-homomorphisms.

These functorialities can be computed more easily than arbitrary composition products in $E$-theory: If $\psi \in E(D, A)$ and $\phi \in [\![A, B]\!]_\infty$, then $\phi \circ \psi \in E(D, B)$ is the composition

$$\mathcal{S} \mathbin{\widehat{\otimes}} D \mathbin{\widehat{\otimes}} \mathfrak{K} \xrightarrow{\psi} A \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \xrightarrow{\phi \mathbin{\widehat{\otimes}} 1_{\widehat{\mathfrak{K}}}} B \mathbin{\widehat{\otimes}} \mathfrak{K}$$

in the asymptotic category. This is, because the co-multiplication $\Delta : \mathcal{S} \to \mathcal{S} \mathbin{\widehat{\otimes}} \mathcal{S}$ in the definition of the composition product in $E$-theory cancels with the co-unit $\eta : \mathcal{S} \to \mathbb{C}$ appearing in the functor from the asymptotic category to $E$-theory by (2.7).

Similarly, if $\psi \in E(B, D)$ and $\phi \in [\![A, B]\!]_\infty$, then $\psi \circ \phi \in E(A, D)$ is the composition

$$\mathcal{S} \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \xrightarrow{1_{\mathcal{S}} \mathbin{\widehat{\otimes}} \phi \mathbin{\widehat{\otimes}} 1_{\widehat{\mathfrak{K}}}} \mathcal{S} \mathbin{\widehat{\otimes}} B \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \xrightarrow{\psi} B \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}},$$

and the exterior product of $\phi \in E(A_1, B_1)$ and $\psi \in [\![A_2, B_2]\!]_\infty$ is the composition

$$\mathcal{S} \mathbin{\widehat{\otimes}} A_1 \mathbin{\widehat{\otimes}} A_2 \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}} \xrightarrow{\phi \mathbin{\widehat{\otimes}} \psi} B_1 \mathbin{\widehat{\otimes}} B_2 \mathbin{\widehat{\otimes}} \widehat{\mathfrak{K}}.$$

In our applications in the following sections, we have to compute products only in cases where one of the factors comes from an asymptotic morphism and not from an $E$-theory class. This is the reason why our computations will not involve $\Delta$ and thus we don't have to know its definition.

Generalizing the functor from the asymptotic category to the $E$-theory category, elements of $E(A, B)$ are also obtained from any morphism in the asymptotic category of the form

$$A \mathbin{\widehat{\otimes}} \mathfrak{K}(H_1) \to B \mathbin{\widehat{\otimes}} \mathfrak{K}(H_2) \quad \text{or} \quad \mathcal{S} \mathbin{\widehat{\otimes}} A \mathbin{\widehat{\otimes}} \mathfrak{K}(H_1) \to B \mathbin{\widehat{\otimes}} \mathfrak{K}(H_2)$$

where $H_1, H_2$ are arbitrary separable, $\mathbb{Z}_2$-graded Hilbert spaces. The $E$-theory element is obtained by tensoring with $[\![\eta]\!] \mathbin{\widehat{\otimes}} \mathrm{id}_{\widehat{\mathfrak{K}}}$ respectively $\mathrm{id}_{\widehat{\mathfrak{K}}}$ and applying stability.

We shall also need invariance under Morita equivalence, which is the second part of the following theorem:

**Theorem 2.7.15.** *Let $\mathcal{E}$ be a countably generated, $\mathbb{Z}_2$-graded Hilbert module over a $\mathbb{Z}_2$-graded $C^*$-algebra $B$. Given an isometric, grading preserving inclusion $V : \mathcal{E} \subset B \mathbin{\widehat{\otimes}} H$, where $H$ is a separable, $\mathbb{Z}_2$-graded Hilbert space, we obtain an isometric $*$-homomorphism $\mathrm{Ad}_V : \mathfrak{K}(\mathcal{E}) \subset B \otimes \mathfrak{K}(H)$ which induces an element $\Theta_{\mathcal{E}} \in E(\mathfrak{K}(\mathcal{E}), B)$.*

1. *This element $\Theta_{\mathcal{E}}$ always exists and is independent of the choice of the inclusion.*

2. *If $\mathcal{E}$ is full, i. e. $\langle \mathcal{E}, \mathcal{E} \rangle = B$, and $B$ has a strictly positive element, then $\Theta_{\mathcal{E}}$ is invertible.*

*Proof.* The existence of an inclusion $\mathcal{E} \subset B \mathbin{\widehat{\otimes}} \widehat{\ell}^2$ is guaranteed by Kasparov's stabilization Theorem [Kas80a, Theorem 2]. A standard argument shows that all inclusions $\mathcal{E} \subset B \mathbin{\widehat{\otimes}} \widehat{\ell}^2$ – including $V \mathbin{\widehat{\otimes}} \mathrm{id}_{\widehat{\ell}^2}$ – are homotopic. This proves uniqueness.

The same argument also proves that we can homotop $V \mathbin{\widehat{\otimes}} \mathrm{id}_{\widehat{\ell}^2}$ to the isomorphism $\mathcal{E} \otimes \widehat{\ell}^2 \cong B \mathbin{\widehat{\otimes}} \widehat{\ell}^2$ which always exists under the additional assumptions of the second part by [MP84, Theorems 1.9]. Thus, $m_{\mathcal{E}}$ is represented by a $*$-isomorphism and is therefore invertible.

Note that the cited theorems are only formulated for the ungraded case, but $\mathbb{Z}_2$-gradings are readily implemented into their proofs. □

**Corollary 2.7.16.** *If $E \to V$ is a nowhere zero dimensional, then the inclusion*

$$\mathfrak{K}(\mathcal{E}) \subset M_{M,N}(C_r^*(V, \mathcal{F}))$$

*of Equation* (2.6) *induces an invertible element of $E(\mathfrak{K}(\mathcal{E}), C_r^*(V, \mathcal{F}))$.*

*Proof.* The foliation algebra $C_r^*(V, \mathcal{F})$ is separable and thus contains a strictly positive element by [AK69]. The Hilbert-$C_r^*(V, \mathcal{F})$-module $\mathcal{E}$ is full, because $E$ is nowhere zero dimensional. □

The additive monoidal category **KK**, whose objects are separable $\mathbb{Z}_2$-graded $C^*$-algebras, has similar properties as **E** ([Kas80b], see also [Bla98]). Recall that its morphism groups $KK(A, B)$ are defined for all $\mathbb{Z}_2$-graded $C^*$-algebras where $A$ is separable ($B$ need not be separable). The functor from the category of $\mathbb{Z}_2$-graded separable $C^*$-algebras and $*$-homomorphisms to **E** factors canonically through **KK**. The maps $KK(A, B) \to E(A, B)$ of this functor also exist when $B$ is not separable and are isomorphisms if $A$ is nuclear. In particular, the functor $E(\mathbb{C}, \_)$ from the category of $\mathbb{Z}_2$-graded $C^*$-algebras to abelian groups is canonically naturally isomorphic to $K$-theory.

Index theory is usually formulated in terms of $KK$-theory whereas we have to use $E$-theory. Therefore, we have to know these maps explicitly to transfer basic notions to $E$-theory.

In the unbounded picture of $KK$-theory of [BJ83] (see also [Bla98, Section 17.11]), elements of $KK(A, B)$ are represented by triples $(\mathcal{E}, \rho, D)$, where

- $\mathcal{E}$ is a countably generated, $\mathbb{Z}_2$-graded Hilbert-$B$-module,

- $\rho : A \to \mathfrak{B}(\mathcal{E})$ is a grading preserving representation of $A$ on $\mathcal{E}$,
- $D$ is an odd selfadjoint regular operator on $\mathcal{E}$

such that for all $a$ in a dense subset of $A$, the commutator $[\rho(a), D]$ is densely defined and extends to a bounded operator on $\mathcal{E}$ and $\rho(a)(D \pm i)^{-1} \in \mathfrak{K}(\mathcal{E})$.

**Proposition 2.7.17** (cf. [CH, Section 8])**.** *Under the canonical map*

$$KK(A,B) \to E(A,B),$$

*the element represented by the triple $(\mathcal{E}, \rho, D)$ is mapped to the class of the asymptotic morphism*

$$\mathcal{S} \mathbin{\widehat{\otimes}} A \to \mathfrak{A}(\mathfrak{K}(\mathcal{E})), \quad f \mathbin{\widehat{\otimes}} a \mapsto \overline{t \mapsto \rho(a) f(t^{-1}D)}$$

*in $E(A, \mathfrak{K}(\mathcal{E}))$ composed with the element $\Theta_{\mathcal{E}} \in E(\mathfrak{K}(\mathcal{E}), B)$ of Theorem 2.7.15.*

## 2.8 The module structure on Connes' $K$-theory model

We are now going to define the module structure. Recall the objects we have introduced so far: $(V, \mathcal{F})$ is a foliation and $\mathcal{O}(V, \mathcal{F})$ its foliated cone constructed with respect to some Riemannian metric on $V$. Furthermore, $E$ denotes a smooth hermitian vector bundle over $V$ and $\mathcal{E}$ the associated Hilbert module.

If $D$ is any coefficient $C^*$-algebra and $g \in \overline{\mathfrak{uc}}(\mathcal{O}(V, \mathcal{F}), D)$, then $g_t \in C(V) \mathbin{\widehat{\otimes}} D$ denotes the restriction of $g$ to $\{t\} \times V \subset \mathcal{O}(V, \mathcal{F})$. Furthermore, we may consider $g_t$ as an element of $\mathfrak{B}(\mathcal{E}) \mathbin{\widehat{\otimes}} D$ by Lemma 2.6.3.

The main ingredient of the module structure is the following asymptotic morphism.

**Theorem 2.8.1.** *For each coefficient $C^*$-algebra $D$, there is an asymptotic morphism*

$$\begin{aligned} \mathfrak{m}_D : \ \mathfrak{uc}(\mathcal{O}(V, \mathcal{F}), D) \mathbin{\widehat{\otimes}} \mathfrak{K}(\mathcal{E}) &\to \mathfrak{A}(\mathfrak{K}(\mathcal{E}) \mathbin{\widehat{\otimes}}_{\min} D) \\ \bar{g} \mathbin{\widehat{\otimes}} T &\mapsto \overline{t \mapsto g_t \cdot (T \mathbin{\widehat{\otimes}} 1_{\tilde{D}})}. \end{aligned}$$

*Proof.* There is an obvious inclusion

$$\alpha_{\max} : \mathfrak{K}(\mathcal{E}) \to \mathfrak{A}(\mathfrak{B}(\mathcal{E}) \mathbin{\widehat{\otimes}} \tilde{D})$$

as constant functions. Furthermore, the composition

$$\begin{aligned}\overline{\mathfrak{uc}}(\mathcal{O}(V,\mathcal{F}),D) &\subset C_b([0,\infty)\times V,D) = C_b([0,\infty),C(V)\mathbin{\widehat{\otimes}} D)\\ &\subset C_b([0,\infty),\mathfrak{B}(\mathcal{E})\mathbin{\widehat{\otimes}}\tilde{D}) \twoheadrightarrow \mathfrak{A}(\mathfrak{B}(\mathcal{E})\mathbin{\widehat{\otimes}}\tilde{D})\end{aligned}$$

obviously descends to give a $*$-homomorphism

$$\beta_{\max}\colon \mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D)\to\mathfrak{A}(\mathfrak{B}(\mathcal{E})\mathbin{\widehat{\otimes}}\tilde{D}).$$

Let $\alpha,\beta$ be obtained from $\alpha_{\max},\beta_{\max}$ by passing from the maximal tensor product to the minimal tensor product $\mathfrak{B}(\mathcal{E})\mathbin{\widehat{\otimes}}_{\min}\tilde{D}$.

In the following lemma, the vanishing variation of $g$ enters the game:

**Lemma 2.8.2.** *For all $T\in\mathfrak{K}(\mathcal{E})$ and $g\in\overline{\mathfrak{uc}}(\mathcal{O}(V,\mathcal{F}),D)$, the commutator $[\beta(\bar{g}),\alpha(T)]\in\mathfrak{A}(\mathfrak{B}(\mathcal{E})\mathbin{\widehat{\otimes}}_{\min}\tilde{D})$ vanishes.*

*Proof.* Let $\rho:\tilde{D}\to\mathfrak{B}(H')$ be a faithful representation. The minimal tensor product $\mathfrak{K}(\mathcal{E})\mathbin{\widehat{\otimes}}_{\min}\tilde{D}$ may be defined as the image of the $*$-homomorphism

$$\pi\mathbin{\widehat{\otimes}}\rho\colon \mathfrak{K}(\mathcal{E})\mathbin{\widehat{\otimes}}\tilde{D}\to\mathfrak{B}\left(\bigoplus_{x\in V}L^2(G_x,r^*E)\mathbin{\widehat{\otimes}} H'\right).$$

Lemma 2.6.3 implies

$$\begin{aligned}(\pi\mathbin{\widehat{\otimes}}\rho)(g_t\cdot(T\mathbin{\widehat{\otimes}}1_{\tilde{D}})) &= (\tau\mathbin{\widehat{\otimes}}\rho)(g_t)\cdot(\pi\mathbin{\widehat{\otimes}}\rho)(T\mathbin{\widehat{\otimes}}1_{\tilde{D}}),\\ (\pi\mathbin{\widehat{\otimes}}\rho)((T\mathbin{\widehat{\otimes}}1_{\tilde{D}})\cdot g_t) &= (\pi\mathbin{\widehat{\otimes}}\rho)(T\mathbin{\widehat{\otimes}}1_{\tilde{D}})\cdot(\tau\mathbin{\widehat{\otimes}}\rho)(g_t).\end{aligned}$$

By definition of $\tau$, the operator $(\tau\mathbin{\widehat{\otimes}}\rho)(g_t)$ acts on each $L^2(G_x,r^*E)\mathbin{\widehat{\otimes}} H'\cong L^2(G_x,r^*E\mathbin{\widehat{\otimes}} H')$ by multiplication with $r^*(\rho\circ g_t)\in C(G_x,\mathfrak{B}(H'))$.

We may assume without loss of generality that the operator $T\in\mathfrak{K}(\mathcal{E})$ acts on $C_c(G,r^*E)\subset\mathcal{E}$ by convolution with some $a\in C_c(G,r^*E\mathbin{\widehat{\otimes}} s^*E^*)$ supported in a compact subset $K$ of some coordinate chart of $G$. According to Corollary 2.2.3, the norms

$$\varepsilon_t := \|(s^*g_t - r^*g_t)|_K\|_{C(K,D)}$$

tend to zero as $t\to\infty$. For $\xi\in L^2(G_x,r^*E)\mathbin{\widehat{\otimes}} H'\cong L^2(G_x,r^*E\mathbin{\widehat{\otimes}} H')$ we may now calculate:

$$
\begin{aligned}
&\|(\pi \mathbin{\widehat{\otimes}} \rho)([g_t, T \mathbin{\widehat{\otimes}} 1_{\tilde{D}}])(\xi)\|^2_{L^2(G_x, r^*E) \mathbin{\widehat{\otimes}} H'} = \\
&\quad = \|[(\tau \mathbin{\widehat{\otimes}} \rho)(g_t),\ (\pi \mathbin{\widehat{\otimes}} \rho)(T \mathbin{\widehat{\otimes}} 1_{\tilde{D}})](\xi)\|^2_{L^2(G_x, r^*E) \mathbin{\widehat{\otimes}} H'} = \\
&\quad = \int_{\gamma \in G_x} \Bigg\| (\mathrm{id}_{E_{r(\gamma)}} \mathbin{\widehat{\otimes}} \rho(g_t(r(\gamma)))) \int_{\gamma_1\gamma_2 = \gamma} (a(\gamma_1) \mathbin{\widehat{\otimes}} \mathrm{id}_{H'}) \xi(\gamma_2) - \\
&\qquad\qquad - \int_{\gamma_1\gamma_2=\gamma} (a(\gamma_1) \mathbin{\widehat{\otimes}} \mathrm{id}_{H'})(\mathrm{id}_{E_{r(\gamma_2)}} \mathbin{\widehat{\otimes}} \rho(g_t(r(\gamma_2))))\xi(\gamma_2) \Bigg\|^2_{E_{r(\gamma)} \widehat{\otimes} H'} \\
&\quad = \int_{\gamma \in G_x} \left\| \int_{\gamma_1\gamma_2=\gamma} (a(\gamma_1) \mathbin{\widehat{\otimes}} \rho(r^*g_t - s^*g_t)(\gamma_1))\xi(\gamma_2) \right\|^2_{E_{r(\gamma)} \widehat{\otimes} H'} \\
&\quad \le \int_{\gamma \in G_x} \Bigg( \int_{\gamma_1\gamma_2=\gamma} \|(r^*g_t - s^*g_t)(\gamma_1)\|_D \cdot \|a(\gamma_1)\|_{(r^*E \mathbin{\widehat{\otimes}} s^*E^*)_{\gamma_1}} \cdot \\
&\qquad\qquad \cdot \|\xi(\gamma_2)\|_{E_{r(\gamma_2)} \mathbin{\widehat{\otimes}} H'} \Bigg)^2 \\
&\quad \le \varepsilon_t^2 \cdot \int_{\gamma \in G_x} \left( \int_{\gamma_1\gamma_2=\gamma} \|a(\gamma_1)\|_{(r^*E \mathbin{\widehat{\otimes}} s^*E^*)_{\gamma_1}} \cdot \|\xi(\gamma_2)\|_{E_{r(\gamma_2)} \mathbin{\widehat{\otimes}} H'} \right)^2 \\
&\quad = \varepsilon_t^2 \cdot \left\| \pi_x(\shortmid a \shortmid)^2 \left( \|\xi\|_{r^*E \mathbin{\widehat{\otimes}} H'} \right) \right\|^2 \le \varepsilon_t^2 \cdot \|\pi_x(\shortmid a \shortmid)\|^2 \cdot \|\xi\|^2_{L^2(G_x, r^*E) \mathbin{\widehat{\otimes}} H'}
\end{aligned}
$$

Here, $\shortmid a \shortmid \in C_c(G)$ denotes the point-wise norm of $a \in C_c(G, r^*E \mathbin{\widehat{\otimes}} s^*E^*)$. It is a function in $C_c(G)$, because we have assumed that $a$ is supported in some coordinate chart and is therefore continuous in the usual sense. Thus, the inequality

$$\|(\pi_x \mathbin{\widehat{\otimes}} \rho)([g_t, T \mathbin{\widehat{\otimes}} 1_{\tilde{D}}])\| \le \varepsilon_t \cdot \|\pi_x(\shortmid a \shortmid)\|$$

holds for all $x \in V$. Taking the supremum over all $x \in V$, we see that the norm of the commutator

$$[g_t, T \mathbin{\widehat{\otimes}} 1_{\tilde{D}}] \in \mathfrak{K}(\mathcal{E}) \mathbin{\widehat{\otimes}}_{\min} D \subset \mathfrak{B}\left( \bigoplus_x L^2(G_x, r^*E) \mathbin{\widehat{\otimes}} H' \right)$$

is bounded by $\varepsilon_t$ times the norm of $\shortmid a \shortmid \in C^*_r(V, \mathcal{F})$ and thus tends to zero for $t \to \infty$. This implies $[\beta(g), \alpha(f)] = 0$ in $\mathfrak{A}(\mathfrak{B}(\mathcal{E}) \mathbin{\widehat{\otimes}}_{\min} \tilde{D})$. □

**Question 2.8.3.** *This proof uses an explicit calculation for the minimal tensor product. Nevertheless, one can ask whether the analogous statement still holds for the commutator of $\alpha_{\max}, \beta_{\max}$.*

According to this lemma, $\alpha$ and $\beta$ combine to a $*$-homomorphism

$$\mathfrak{m}_D : \mathfrak{uc}(\mathcal{O}(V,F),D) \widehat{\otimes} \mathfrak{K}(\mathcal{E}) \to \mathfrak{A}(\mathfrak{B}(\mathcal{E}) \widehat{\otimes}_{\min} \tilde{D}).$$

Its image is contained in $\mathfrak{A}(\mathfrak{K}(\mathcal{E}) \widehat{\otimes}_{\min} D)$ and indeed it maps elementary tensors $\bar{g} \widehat{\otimes} T$ as claimed. $\square$

From now on we will always assume that the coefficient algebras are nuclear to avoid mixing up minimal and maximal tensor products.

**Theorem 2.8.4.** *Let $D_1, D_2$ be nuclear $C^*$-algebras. Then, in the asymptotic category,*

$$([\![\mathfrak{m}_{D_1}]\!] \widehat{\otimes} 1_{D_2}) \circ (1_{\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_1)} \widehat{\otimes} [\![\mathfrak{m}_{D_2}]\!]) = [\![\mathfrak{m}_{D_1 \widehat{\otimes} D_2}]\!] \circ ([\![\nabla]\!] \widehat{\otimes} 1_{\mathfrak{K}(\mathcal{E})}).$$

Recall that the $*$-homomorphism

$$\nabla : \mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_1) \widehat{\otimes} \mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_2) \to \mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_1 \widehat{\otimes} D_2)$$

is defined by multiplication of functions.

*Proof.* The left hand side is represented by the 2-homotopy class of

$$\begin{aligned} \mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_1) \widehat{\otimes} \mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_2) \widehat{\otimes} \mathfrak{K}(\mathcal{E}) &\to \mathfrak{A}^2(\mathfrak{K}(\mathcal{E}) \widehat{\otimes} D_1 \widehat{\otimes} D_2) \\ \bar{f} \widehat{\otimes} \bar{g} \widehat{\otimes} T &\mapsto \overline{t \mapsto \overline{s \mapsto (f_s \widehat{\otimes} g_t) \cdot (T \widehat{\otimes} 1_{\tilde{D}_1} \widehat{\otimes} 1_{\tilde{D}_2})}}, \end{aligned}$$

while the right hand side is represented by the 1-homotopy class of

$$\begin{aligned} \mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_1) \widehat{\otimes} \mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_2) \widehat{\otimes} \mathfrak{K}(\mathcal{E}) &\to \mathfrak{A}(\mathfrak{K}(\mathcal{E}) \widehat{\otimes} D_1 \widehat{\otimes} D_2) \\ \bar{f} \widehat{\otimes} \bar{g} \widehat{\otimes} T &\mapsto \overline{s \mapsto (f_s \widehat{\otimes} g_s) \cdot (T \widehat{\otimes} 1_{\tilde{D}_1} \widehat{\otimes} 1_{\tilde{D}_2})}. \end{aligned}$$

In these formulas, we have, of course, interpreted $f_s \widehat{\otimes} g_t$ and $f_s \widehat{\otimes} g_s$ as elements of $C(V) \widehat{\otimes} D_1 \widehat{\otimes} D_2 \subset \mathfrak{B}(\mathcal{E}) \widehat{\otimes} D_1 \widehat{\otimes} D_2$.

Similar to the definition of $\mathfrak{m}$, we construct a 2-homotopy

$$\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_1) \widehat{\otimes} \mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_2) \widehat{\otimes} \mathfrak{K}(\mathcal{E}) \to \mathfrak{A}^2((\mathfrak{K}(\mathcal{E}) \widehat{\otimes} D_1 \widehat{\otimes} D_2)[0,1]).$$

There are three $*$-homomorphisms:

$$\begin{aligned} \tilde{\alpha} : \mathfrak{K}(\mathcal{E}) &\to \mathfrak{A}^2((\mathfrak{B}(\mathcal{E}) \widehat{\otimes} \tilde{D}_1 \widehat{\otimes} \tilde{D}_2)[0,1]) \\ T &\mapsto \overline{t \mapsto \overline{s \mapsto [r \mapsto T \widehat{\otimes} 1_{\tilde{D}_1} \widehat{\otimes} 1_{\tilde{D}_2}]}} \\ \tilde{\beta} : \overline{\mathfrak{uc}}(\mathcal{O}(V,\mathcal{F}),D_2) &\to \mathfrak{A}^2((\mathfrak{B}(\mathcal{E}) \widehat{\otimes} \tilde{D}_1 \widehat{\otimes} \tilde{D}_2)[0,1]) \\ g &\mapsto \overline{t \mapsto \overline{s \mapsto [r \mapsto g_{rs+(1-r)t} \widehat{\otimes} 1_{\tilde{D}_1}]}} \\ \tilde{\gamma} : \overline{\mathfrak{uc}}(\mathcal{O}(V,\mathcal{F}),D_1) &\to \mathfrak{A}^2((\mathfrak{B}(\mathcal{E}) \widehat{\otimes} \tilde{D}_1 \widehat{\otimes} \tilde{D}_2)[0,1]) \\ f &\mapsto \overline{t \mapsto \overline{s \mapsto [r \mapsto f_s \widehat{\otimes} 1_{\tilde{D}_2}]}} \end{aligned}$$

There is only one non-trivial property to be verified here, namely the continuity of

$$\begin{aligned}\phi:\ [1,\infty) &\to \mathfrak{A}((\mathfrak{B}(\mathcal{E})\mathbin{\widehat{\otimes}}\tilde{D}_1\mathbin{\widehat{\otimes}}\tilde{D}_2)[0,1])\\ t &\mapsto \overline{s\mapsto [r\mapsto g_{rs+(1-r)t}\mathbin{\widehat{\otimes}}1_{\tilde{D}_1}]}\end{aligned}$$

in the definition of $\tilde{\beta}$. This is a consequence of the following Lemma.

**Lemma 2.8.5.** *Let $X$ and $Y$ be metric spaces, $X$ complete. If $g \in C_b(X,Y)$ has vanishing variation, then it is uniformly continuous.*

*Proof.* Let $\varepsilon > 0$. Because of vanishing variation there is a compact subset $K \subset X$ such that

$$(x \notin K \vee y \notin K) \wedge d(x,y) < 1 \quad \Rightarrow \quad d(g(x), g(y)) < \varepsilon.$$

On the compact set $K$ however, uniform continuity is automatic. Combining these features, the claim follows. □

Using this lemma and the distance estimate

$$d(\,(rs+(1-r)t, x)\,,\,(rs+(1-r)t', x)\,) \le |t-t'|$$

in $\mathcal{O}(V,\mathcal{F})$, we conclude

$$\begin{aligned}\|\phi(t)-\phi(t')\| &= \left\|\overline{s\mapsto [r\mapsto (g_{rs+(1-r)t}-g_{rs+(1-r)t'})\mathbin{\widehat{\otimes}}1_{\tilde{D}_2}]}\right\|\\ &\le \sup_{s\in[0,\infty),r\in[0,1],x\in V}\left\|g(rs+(1-r)t,x)-g(rs+(1-r)t',x)\right\|\\ &\xrightarrow{t'\to t} 0.\end{aligned}$$

Now, $\tilde{\beta}$ and $\tilde{\gamma}$ factor through $\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}), D_2)$ and $\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}), D_1)$, respectively, and their images commute with each other. Furthermore, their images commute with the image of $\tilde{\alpha}$: The vanishing of $[\tilde{\gamma}(f), \tilde{\alpha}(T)]$ is completely analogous to Lemma 2.8.2. For the vanishing of $[\tilde{\beta}(g), \tilde{\alpha}(T)]$ we assume that $T \in \mathfrak{K}(\mathcal{E})$ acts on $C_c(G, r^*E)$ by convolution with an $a \in C_c(G, r^*E\mathbin{\widehat{\otimes}}s^*E^*)$ compactly supported in some coordinate chart of $G$ and calculate

$$\begin{aligned}\|[\tilde{\beta}(g), \tilde{\alpha}(T)]\| &= \limsup_{t\to\infty}\limsup_{s\to\infty}\sup_{r\in[0,1]}\|\,[g_{rs+(1-r)t}, T\mathbin{\widehat{\otimes}}1_{\tilde{D}_2}]\,\|\\ &\le \limsup_{t\to\infty}\limsup_{s\to\infty}\sup_{r\in[0,1]}\varepsilon_{rs+(1-r)t}\cdot\|\,|a|\,\|_r = 0,\end{aligned}$$

where $\varepsilon_t$, $|a| \in C_c(G)$ and the inequality are analogous to the ones in the proof of Lemma 2.8.2.

Thus, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ combine to a 2-homotopy

$$\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_1)\mathbin{\widehat{\otimes}}\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_2)\mathbin{\widehat{\otimes}}\mathfrak{K}(\mathcal{E}) \to \mathfrak{A}^2((\mathfrak{B}(\mathcal{E})\mathbin{\widehat{\otimes}}\tilde{D}_1\mathbin{\widehat{\otimes}}\tilde{D}_2)[0,1]).$$

The image is obviously contained in $\mathfrak{A}^2((\mathfrak{K}(\mathcal{E})\mathbin{\widehat{\otimes}}D_1\mathbin{\widehat{\otimes}}D_2)[0,1])$ so it is actually a 2-homotopy

$$\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_1)\mathbin{\widehat{\otimes}}\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_2)\mathbin{\widehat{\otimes}}\mathfrak{K}(\mathcal{E}) \to \mathfrak{A}^2((\mathfrak{K}(\mathcal{E})\mathbin{\widehat{\otimes}}D_1\mathbin{\widehat{\otimes}}D_2)[0,1]).$$

Evaluation at 0 yields the representative of

$$([\![\mathfrak{m}_{D_1}]\!]\mathbin{\widehat{\otimes}}1_{D_2})\circ(1_{\mathfrak{uc}(\mathcal{O}(V,\mathcal{F}),D_1)}\mathbin{\widehat{\otimes}}[\![\mathfrak{m}_{D_2}]\!])$$

while evaluation at 1 is equivalent to the representative of

$$[\![\mathfrak{m}_{D_1\mathbin{\widehat{\otimes}}D_2}]\!]\circ([\nabla]\mathbin{\widehat{\otimes}}1_{\mathfrak{K}(\mathcal{E})}).$$

□

One of our main results is now an easy corollary:

**Corollary 2.8.6** (cf. [Roe95, Conjecture 0.2])**.** *$K_*(\mathfrak{K}(\mathcal{E}))$ is a module over $K^{-*}_{FJ}(V/\mathcal{F})$. The multiplication is*

$$K^{-i}_{FJ}(V/\mathcal{F})\otimes K_j(\mathfrak{K}(\mathcal{E}))\to K_{i+j}(\mathfrak{c}(\mathcal{O}(V,\mathcal{F}))\mathbin{\widehat{\otimes}}\mathfrak{K}(\mathcal{E}))\xrightarrow{[\![\mathfrak{m}_{\mathfrak{K}}]\!]\circ}K_{i+j}(\mathfrak{K}(\mathcal{E})).$$

*In particular, $K^*_C(V/\mathcal{F})$ is a module over $K^*_{FJ}(V/\mathcal{F})$.*

*Proof.* Associativity of the module multiplication is a direct consequence of Theorem 2.8.4 applied with $D_1 = D_2 = \mathfrak{K}$. If $E = \mathbb{C}\times V$ is the trivial one dimensional bundle, then $\mathfrak{K}(\mathcal{E}) = C^*_r(V,\mathcal{F})$. This implies the special case mentioned, as $K^*_C(V/\mathcal{F}) = K_{-*}(C^*_r(V,\mathcal{F}))$ by definition. □

**Lemma 2.8.7.** *The Morita equivalence isomorphism*

$$K_*(\mathfrak{K}(\mathcal{E}))\cong K_*(C^*_r(V,\mathcal{F})) = K^{-*}_C(V/\mathcal{F})$$

*provided by Corollary 2.7.16 is a module isomorphism.*

*Proof.* The inclusion $\mathfrak{K}(\mathcal{E})\subset M_{M,N}(\mathbb{C})\mathbin{\widehat{\otimes}}C^*_r(V,\mathcal{F})$, which induces this isomorphism in $K$-theory, obviously commutes with the asymptotic morphisms $\mathfrak{m}_{\mathfrak{K}}$ associated to $\mathfrak{K}(\mathcal{E})$ and $M_{M,N}(\mathbb{C})\mathbin{\widehat{\otimes}}C^*_r(V,\mathcal{F})$. □

More general multiplications involving arbitrary coefficient $C^*$-algebras can be derived similarly from Theorem 2.8.4.

## 2.9 Longitudinal index theory

This section is a very short introduction to longitudinal index theory. For more details we refer primarily to [Kor09, Section 8.2], but of course also to [Con82, Con94, CS84].

**Definition 2.9.1** (cf. [Con94, Section 2.9])**.** Let $(V, \mathcal{F})$ be a foliated manifold, $E \to V$ a $\mathbb{Z}_2$-graded smooth vector bundle and $D : C^\infty(V, E) \to C^\infty(V, E)$ a first order symmetric differential operator of grading degree one. The operator $D$ is called *longitudinally elliptic* if it restricts to the leafs $L_x$ of the foliation and the restricted operators

$$D_{L_x} : C_c^\infty(L_x, E|_{L_x}) \to C_c^\infty(L_x, E|_{L_x})$$

are elliptic.

A special case are longitudinal Dirac type operators. Assume that $V$ is equipped with a Riemannian metric and $E \to V$ is a $\mathbb{Z}_2$-graded, smooth, hermitian vector bundle equipped with a Clifford action of $T\mathcal{F}$ and a compatible connection $\nabla$. The associated Dirac operator is defined locally by the usual formula

$$D = \sum_{i=1}^{\dim \mathcal{F}} e_i \nabla_{e_i},$$

where $e_1, \ldots, e_{\dim \mathcal{F}}$ is any local orthonormal frame of $T\mathcal{F}$.

If the bundle $T\mathcal{F}$ carries a spin$^c$-structure, we obtain the longitudinal spin$^c$-Dirac operator $\not{D}$ by choosing $E$ to be the corresponding spinor bundle. If the foliation is in addition even dimensional, then $E$ is $\mathbb{Z}_2$-graded and $\not{D}$ is a symmetric, grading degree one, longitudinally elliptic operator.

Let $\mathcal{E}$ be the Hilbert module associated to $E$ an let $D_{G_x} : C_c^\infty(G_x, r^*E) \to C_c^\infty(G_x, r^*E)$ be the lift of $D_{L_x}$ to the holonomy cover $G_x \to L_x$. The family $\{D_{G_x} \,|\, x \in M\}$ assembles to a differential operator

$$D_G : \, C_c^\infty(G, r^*E) \to C_c^\infty(G, r^*E).$$

Its closure, which we also denote by $D_G$, is an odd, unbounded, selfadjoint operator on the Hilbert module $\mathcal{E}$ constructed in Section 2.6. It is regular in the sense of [BJ83]. The proof of regularity relies on the existence of a parametrix and can be found in [Vas01, Proposition 3.4.9].

**Definition 2.9.2** ([Kor09, Section 8.2])**.** The $KK$-theory class of $D$ is the element

$$[D] \in KK(C(V), C_r^*(V, \mathcal{F}))$$

given in the unbounded picture of $KK$-theory by the triple $(\mathcal{E}, \phi, D_G)$ where $\phi : C(V) \to \mathfrak{B}(\mathcal{E})$ is the inclusion of Lemma 2.6.3. The index of $D$ is the element

$$\operatorname{ind}(D) \in K(C_r^*(V, \mathcal{F})) = K_C^0(V/\mathcal{F})$$

obtained from $[D]$ by crushing $V$ in the first variable to a point.

There are several advantages of having the $KK$-theory class $[D]$ at hand. One of them is the usual index pairing with vector bundles: Given a longitudinally elliptic operator $D$ and a smooth vector bundle $F \to V$, we can construct the twisted operator

$$D_F = D \otimes F : C_c^\infty(V, E \otimes F) \to C_c^\infty(V, E \otimes F),$$

which is again a longitudinally elliptic operator. A direct consequence of [Kuc97, Theorem 13] is the following generalization of the index pairing formula.

**Lemma 2.9.3.** $\operatorname{ind}(D_F) = [D] \circ [F] \in K_C^0(V/\mathcal{F})$.

Another appearence of the $KK$-theory class is the following. If $T\mathcal{F}$ is even dimensional and endowed with a spin$^c$ structure, then the map $p : V \to V/\mathcal{F}$ of Example 2.2.11 is $K$-oriented and induces a wrong way map $p! : K^*(V) \to K_C^*(V/\mathcal{F})$ given by composition product with an element $p! \in KK(C(V), C_r^*(V, \mathcal{F}))$ [CS84]. In this particular case, $p!$ is in fact the $KK$-theory class of the spin$^c$-Dirac operator $\not{D}$.

For our purposes, we have to pass from $KK$- to $E$-theory. Under the canonical isomorphism of Proposition 2.7.17, $[D]$ corresponds to the following $E$-theory classes:

**Definition 2.9.4.** The $E$-theory class

$$[\![D]\!] \in E(C(V), \mathfrak{K}(\mathcal{E})) \cong E(C(V), C_r^*(V, \mathcal{F}))$$

of a longitudinally elliptic Dirac type operator $D$ over $(V, \mathcal{F})$ is represented by the asymptotic morphism

$$\rho : \mathcal{S} \otimes C(V) \to \mathfrak{A}(\mathfrak{K}(\mathcal{E})), \quad f \otimes g \mapsto \overline{t \mapsto g \cdot f(t^{-1} D_G)}.$$

**Lemma 2.9.5.** *The index*

$$\operatorname{ind}(D) \in E(\mathbb{C}, \mathfrak{K}(\mathcal{E})) \cong E(\mathbb{C}, C_r^*(V, \mathcal{F})) \cong K_C^0(V/\mathcal{F})$$

*is represented by the $*$-homomorphism*

$$\mathcal{S} \to \mathfrak{K}(\mathcal{E}), \quad f \mapsto f(D_G). \tag{2.8}$$

*Proof.* By definition, $\mathrm{ind}(D)$ is represented by the asymptotic morphism

$$\mathcal{S} \to \mathfrak{A}(\mathfrak{K}(\mathcal{E})), \quad f \mapsto \overline{t \mapsto f(t^{-1}D_G)}\,. \tag{2.9}$$

A natural candidate for a 1-homotopy between (2.9) and (2.8) is

$$\mathcal{S} \to \mathfrak{A}(\mathfrak{K}(\mathcal{E})[0,1]), \quad f \mapsto \overline{t \mapsto [r \mapsto f((r+(1-r)t^{-1})D_G)]}\,. \tag{2.10}$$

We have to show that, for each $f \in \mathcal{S}$, the function

$$[1,\infty) \times [0,1] \to \mathfrak{K}(\mathcal{E}), \quad (t,r) \mapsto f((r+(1-r)t^{-1})D_G)$$

is continuous. This follows from continuity of

$$(0,1] \to \mathcal{S}, \quad \lambda \mapsto f(\lambda \cdot \_)$$

and the continuity of the functional calculus

$$\mathcal{S} \to \mathfrak{K}(\mathcal{E}), \quad f \mapsto f(D_G).$$

Furthermore, (2.10) is a $*$-homomorphism, because the functional calculus is. Thus, it fulfills all requirements on a 1-homotopy. □

## 2.10 Twisted operators and the module structure

In this section, we clarify the relation of our module structure to index theory. To this end, we make the following definition:

**Definition 2.10.1.** We denote by $[\![p^*]\!] \in E(\mathfrak{c}(\mathcal{O}(V,\mathcal{F})), C(V))$ the $E$-theory class of the asymptotic morphism

$$p^* : \mathfrak{c}(\mathcal{O}(V,\mathcal{F})) \to \mathfrak{A}(C(V) \otimes \mathfrak{K}), \quad \bar{g} \mapsto \overline{t \mapsto g_t}.$$

This notation comes from the fact that the composition product with $[\![p^*]\!]$ yields the homomorphism

$$p^* : K^*_{FJ}(V/\mathcal{F}) \to K^*_{FJ}(V) \cong K^*(V)$$

induced by the smooth map of leaf spaces $p : V \to V/\mathcal{F}$ defined in Example 2.2.11. To see this, recall that the isomorphism $K^*_{FJ}(V) \cong K^*(V)$ comes from the inclusion $C(V) \otimes \mathfrak{K} \subset \mathfrak{c}(\mathcal{O}V)$ as constant functions. An inverse to this isomorphism is induced by the asymptotic morphism

$$\mathfrak{c}(\mathcal{O}V) \to \mathfrak{A}(C(V) \otimes \mathfrak{K}), \quad \bar{g} \mapsto \overline{t \mapsto g_t},$$

because the composition

$$C(V) \otimes \mathfrak{K} \to \mathfrak{c}(\mathcal{O}V) \to \mathfrak{A}(C(V) \otimes \mathfrak{K})$$

is simply the inclusion as constant functions and therefore the identity morphism in the asymptotic category. The claim now follows from the fact that the homomorphism $p^* : K^*_{FJ}(V/\mathcal{F}) \to K^*_{FJ}(V)$ comes from the inclusion $\mathfrak{c}(\mathcal{O}(V,\mathcal{F})) \subset \mathfrak{c}(\mathcal{O}V)$.

Our main result relating the module structure to index theory is the following:

**Theorem 2.10.2.** *Let $D$ be a longitudinally elliptic operator over $(V,\mathcal{F})$ and $[\![p^*]\!] \in E(\mathfrak{c}(\mathcal{O}(V,\mathcal{F})), C(V))$ the element defined above. Then*

$$[\![D]\!] \circ [\![p^*]\!] = [\![\mathfrak{m}_{\mathfrak{K}}]\!] \circ (1_{\mathfrak{c}(\mathcal{O}(V,\mathcal{F}))} \otimes \operatorname{ind}(D)) \in E(\mathfrak{c}(\mathcal{O}(V,\mathcal{F})), \mathfrak{K}(\mathcal{E})).$$

Before proving this theorem, here are two consequences:

**Corollary 2.10.3.** *If $D$ is a longitudinally elliptic operator, $F \to V$ a smooth vector bundle for which there is an element $x_F \in K^0_{FJ}(V/\mathcal{F})$ with $[F] = p^*(x_F)$ (e. g. $F$ asymptotically a bundle over the leaf space as in Definition 2.4.7), then the index of the twisted operator $D_F$ is*

$$\operatorname{ind}(D_F) = x_F \cdot \operatorname{ind}(D) \in K^0_C(V/\mathcal{F}).$$

*Proof.* $\operatorname{ind}(D_F) = [D] \circ [F] = [D] \circ [\![p^*]\!] \circ x_F = [\![\mathfrak{m}_{\mathfrak{K}}]\!] \circ (x_F \otimes \operatorname{ind}(D)) = x_F \cdot \operatorname{ind}(D)$. □

**Corollary 2.10.4** (cf. [Roe95, p. 204])**.** *Assume that $T\mathcal{F}$ is even dimensional and* spin$^c$ *and let $D\!\!\!/$ be the corresponding Dirac operator. Then the map*

$$p_! \circ p^* : K^*_{FJ}(V/\mathcal{F}) \to K^*_C(V/\mathcal{F})$$

*is module multiplication with* $\operatorname{ind}(D\!\!\!/) \in K^0_C(V/\mathcal{F})$.

*Proof.* $p_! \circ p^*(x) = [\![D\!\!\!/]\!] \circ [\![p^*]\!] \circ x = [\![\mathfrak{m}_{\mathfrak{K}}]\!] \circ (x \otimes \operatorname{ind}(D\!\!\!/)) = x \cdot \operatorname{ind}(D\!\!\!/)$. □

For the proof of Theorem 2.10.2 it would be beneficial if one had defined the stable Higson corona in a more analytic way. Recall that Higson originally defined his corona $\eta X$ of a complete Riemannian manifold $X$ as the maximal ideal space of the $C^*$-algebra generated by the bounded smooth functions $X \to \mathbb{C}$ whose gradient vanishes at infinity (cf. [Roe93, Section 5.1]). It is unknown to the author under which conditions this definition is equivalent to Roe's definition (ibid.). In other words: given a complete Riemannian manifold and a bounded continuous function of vanishing variation, can this function be approximated by smooth functions whose gradients vanish at infinity? For the present situation, it is sufficient to have the following partial result for the stable Higson corona of foliated cones:

**Lemma 2.10.5.** *Every element of $\mathfrak{c}(\mathcal{O}(V,\mathcal{F}))$ has a representative $g \in \bar{\mathfrak{c}}(\mathcal{O}(V,\mathcal{F}))$ such that $g_t \in C(V) \otimes \mathfrak{K}$ is differentiable in leafwise direction for all $t$ and the leafwise derivatives $X.g_t \in C(V) \otimes \mathfrak{K}$ vanish in the limit $t \to \infty$ for every leafwise vector field $X \in C(V, T\mathcal{F})$.*

*Proof.* Let $\{\phi_i\}_{i=1,\dots,k}$ be an atlas of foliation charts $\phi_i : U_i \xrightarrow{\approx} \mathbb{R}^{\dim \mathcal{F}} \times \mathbb{R}^{\operatorname{codim} \mathcal{F}}$ and $\{\chi_i\}_{i=1,\dots,k}$ a subordinate smooth partition of unity. Choose a smooth function $\delta : \mathbb{R}^{\dim \mathcal{F}} \to [0,\infty)$ supported in the compact unit ball $\overline{B}_1(0)$ such that $\int \delta = 1$.

Given any $h \in \bar{\mathfrak{c}}(\mathcal{O}(V,\mathcal{F}))$, we define the functions

$$g_i, h_i : [0,\infty) \times \mathbb{R}^{\dim \mathcal{F}} \times \mathbb{R}^{\operatorname{codim} \mathcal{F}} \to \mathfrak{K}$$

for $i = 1, \dots, k$ by the formulas

$$\begin{aligned} h_i(t,x,z) &:= h(t, \phi_i^{-1}(x,z)), \\ g_i(t,x,z) &:= \int_{\mathbb{R}^{\dim \mathcal{F}}} \delta(x-y) h_i(t,y,z)\,dy \end{aligned}$$

and $g : \mathcal{O}(V,\mathcal{F}) \to \mathfrak{K}$ by

$$g(t,p) := \sum_{i=1}^{k} \chi_i(p) g_i(t, \phi_i(p)).$$

This function $g$ is clearly continuous and we claim that it is a representative of $\overline{h} \in \mathfrak{c}(\mathcal{O}(V,\mathcal{F}))$ with the desired properties.

To this end, let $K_i := \operatorname{supp}(\chi_i)$ and note that there is $R > 0$ with the following property: Whenever $i = 1, \dots, k$ and $(x,z),(y,z) \in \phi_i(K_i) + \overline{B}_1(0)$, then the points $\phi_i^{-1}(x,z), \phi_i^{-1}(y,z) \in V$ are joined by a leafwise path of length at most $R$. In particular, for all $t \geq 0$ the distance between the two points

$$(t, \phi_i^{-1}(x,z)),\ (t, \phi_i^{-1}(y,z)) \in \mathcal{O}(V,\mathcal{F})$$

is at most $R$ and therefore

$$\|h_i(t,x,z) - h_i(t,y,z)\| \leq \operatorname{Var}_R h(t, \phi_i^{-1}(x,z)).$$

This implies

$$\begin{aligned}\|g_i(t,x,z) - h_i(t,x,z)\| &= \left\| \int_{\overline{B}_1(x)} \delta(x-y)(h_i(t,y,z) - h_i(t,x,z))dy \right\| \\ &\leq \int_{\overline{B}_1(x)} \delta(x-y)\|h_i(t,y,z) - h_i(t,x,z)\|dy \\ &\leq \operatorname{Var}_R h(t, \phi_i^{-1}(x,z))\end{aligned}$$

for all $t \geq 0$ and $(x,z) \in \phi_i(K_i)$ and therefore $\|g(t,p) - h(t,p)\| \leq \operatorname{Var}_R h(t,p)$ for all $(t,p) \in \mathcal{O}(V,\mathcal{F})$. As $\operatorname{Var}_R h$ vanishes at infinity, $g$ must also have vanishing variation and $\overline{g} = \overline{h}$ in $\mathfrak{c}(\mathcal{O}(V,\mathcal{F}))$.

It remains to estimate the longitudinal derivatives: Let $X \in C(V, T\mathcal{F})$ be a tangential vector field. Given $p \in V$, we denote $(x_i, z_i) := \phi_i(p)$ and

$$X_i(x_i, z_i) := d\phi_i(X(p)) \in T\mathbb{R}^{\dim \mathcal{F}} \subset T\mathbb{R}^{\dim \mathcal{F}} \times \mathbb{R}^{\operatorname{codim} \mathcal{F}}$$

for those $i$ with $p \in U_i$. The derivative of $g_t$ along the tangential vector field $X$ is

$$X.g_t(p) = \sum_{i=1}^k X.\chi_i(p) g_i(t, x_i, z_i) + \sum_{i=1}^k \chi_i(p) \int_{\overline{B}_1(x_i)} X_i.\delta(x_i - y) h_i(t, y, z_i) dy.$$

As for the first summand, note that

$$\sum_{i=1}^k X.\chi_i(p) h_i(t, x_i, z_i) = \sum_{i=1}^k X.\chi_i(p) h(t,p) = (X.1)h(t,p) = 0$$

and therefore

$$\begin{aligned}\left\| \sum_{i=1}^k X.\chi_i(p) g_i(t, x_i, z_i) \right\| &= \left\| \sum_{i=1}^k X.\chi_i(p)(g_i(t,x_i,z_i) - h_i(t,x_i,z_i)) \right\| \\ &\leq \sum_{i=1}^k |X.\chi_i(p)| \operatorname{Var}_R h(t,p)\end{aligned}$$

which converges to 0 uniformly in $p \in V$ for $t \to \infty$.

For the second summand, we use that $\int \delta \equiv 1$ and therefore

$$\int_{\overline{B}_1(x_i)} X_i.\delta(x_i - y) dy = 0$$

to estimate

$$\begin{aligned}
&\left\| \int_{\overline{B}_1(x_i)} X_i.\delta(x_i - y) h_i(t,y,z_i) dy \right\| = \\
&\qquad = \left\| \int_{\overline{B}_1(x_i)} X_i.\delta(x_i - y)(h_i(t,y,z_i) - h_i(t,x_i,z_i)) dy \right\| \\
&\qquad \leq \int_{\overline{B}_1(x_i)} |X_i.\delta(x_i - y)| \operatorname{Var}_R h(t,p) dy
\end{aligned}$$

which also converges to 0 uniformly in $p \in V$ for $t \to \infty$, because $X_i.\delta$ is bounded on the compact set $K_i + \overline{B}_1(0)$. □

*Proof of Theorem 2.10.2.* The left hand side is represented by

$$\begin{aligned}
\mathcal{S} \mathbin{\widehat{\otimes}} \mathfrak{c}(\mathcal{O}(V,\mathcal{F})) &\to \mathfrak{A}^2(\mathfrak{K}(\mathcal{E}) \mathbin{\widehat{\otimes}} \mathfrak{K}) \\
f \mathbin{\widehat{\otimes}} g &\mapsto \overline{t \mapsto \overline{s \mapsto g_t \cdot (f(s^{-1}D_G) \mathbin{\widehat{\otimes}} 1_{\tilde{\mathfrak{K}}})}}\,.
\end{aligned}$$

The right hand side, on the other hand, is represented by

$$\begin{aligned}
\mathcal{S} \mathbin{\widehat{\otimes}} \mathfrak{c}(\mathcal{O}(V,\mathcal{F})) &\to \mathfrak{A}(\mathfrak{K}(\mathcal{E}) \mathbin{\widehat{\otimes}} \mathfrak{K}) \\
f \mathbin{\widehat{\otimes}} g &\mapsto \overline{t \mapsto g_t \cdot (f(D_G) \mathbin{\widehat{\otimes}} 1_{\tilde{\mathfrak{K}}})}\,.
\end{aligned}$$

To construct a 2-homotopy between them, let

$$\begin{aligned}
\alpha :\ \mathcal{S} &\to \mathfrak{A}^2((\mathfrak{B}(\mathcal{E}) \mathbin{\widehat{\otimes}} \tilde{\mathfrak{K}})[0,1]) \\
f &\mapsto \overline{t \mapsto \overline{s \mapsto [r \mapsto f((r + (1-r)s^{-1})D_G) \mathbin{\widehat{\otimes}} 1_{\tilde{\mathfrak{K}}}]}}
\end{aligned}$$

be the 2-homotopy obtained from the 1-homotopy in the proof of Lemma 2.9.5. Furthermore, by interpreting $g_t \in C(V) \otimes \mathfrak{K}$ as an element of $\mathfrak{B}(\mathcal{E}) \mathbin{\widehat{\otimes}} \mathfrak{K}$ for all $t$, we obtain an inclusion

$$\beta :\ \mathfrak{c}(\mathcal{O}(V,\mathcal{F})) \to \mathfrak{A}^2((\mathfrak{B}(\mathcal{E}) \mathbin{\widehat{\otimes}} \tilde{\mathfrak{K}})[0,1]), \quad \bar{g} \mapsto \overline{t \mapsto \overline{s \mapsto [r \mapsto g_t]}}.$$

We have to show that the commutators $[\alpha(f), \beta(\bar{g})]$ vanish for all $f \in \mathcal{S}$ and $\bar{g} \in \mathfrak{c}(\mathcal{O}(V,\mathcal{F}))$. We may assume $f(x) = (x \pm i)^{-1}$, as these functions generate $\mathcal{S}$, and that $g$ is as in Lemma 2.10.5. Then

$$\begin{aligned}
\|[\alpha(f),\beta(\bar g)]\| &= \limsup_{t\to\infty}\limsup_{s\to\infty}\sup_{r\in[0,1]}\|[f((r+(1-r)s^{-1})D_G)\mathbin{\widehat{\otimes}}1_{\tilde{\mathfrak{K}}},g_t]\| \\
&= \limsup_{t\to\infty}\sup_{\lambda\in(0,1]}\|[f(\lambda D_G)\mathbin{\widehat{\otimes}}1_{\tilde{\mathfrak{K}}},g_t]\| \\
&= \limsup_{t\to\infty}\sup_{\lambda\in(0,1]}\|\lambda\cdot((\lambda D_G\pm i)^{-1}\mathbin{\widehat{\otimes}}1_{\tilde{\mathfrak{K}}})\cdot[D_G\mathbin{\widehat{\otimes}}1_{\tilde{\mathfrak{K}}},g_t]\cdot \\
&\qquad\qquad\cdot((\lambda D_G\pm i)^{-1}\mathbin{\widehat{\otimes}}1_{\tilde{\mathfrak{K}}})\| \\
&\leq \limsup_{t\to\infty}\|[D_G\mathbin{\widehat{\otimes}}1_{\tilde{\mathfrak{K}}},g_t]\|.
\end{aligned}$$

If we write $D=\sum_i A_iX_i$ with bundle endomorphisms $A_i\in C(V,\mathrm{End}(E))$ and tangential vector fields $X_i\in C(V,T\mathcal{F})$, then

$$[D_G\mathbin{\widehat{\otimes}}1_{\tilde{\mathfrak{K}}},g_t]=\sum_i(A_i\mathbin{\widehat{\otimes}}1_{\tilde{\mathfrak{K}}})(1_{\mathrm{End}(E)}\mathbin{\widehat{\otimes}}X_i.g_t)\in C(V,\mathrm{End}(E))\mathbin{\widehat{\otimes}}\mathfrak{K}$$

and this vanishes for $t\to\infty$ by the choice of $g$.

Thus, $\alpha$ and $\beta$ combine to a 2-homotopy

$$\begin{aligned}
\mathcal{S}\mathbin{\widehat{\otimes}}\mathfrak{c}(\mathcal{O}(V,\mathcal{F}))&\to\mathfrak{A}^2(C[0,1]\mathbin{\widehat{\otimes}}\mathfrak{B}(\mathcal{E})\mathbin{\widehat{\otimes}}\tilde{\mathfrak{K}}) \\
f\mathbin{\widehat{\otimes}}g&\mapsto\overline{t\mapsto\overline{s\mapsto[r\mapsto g_t\cdot(f((r+(1-r)s^{-1})D_G)\mathbin{\widehat{\otimes}}1_{\tilde{\mathfrak{K}}})]}}
\end{aligned}$$

whose image lies in the sub-$C^*$-algebra $\mathfrak{A}^2(C[0,1]\mathbin{\widehat{\otimes}}\mathfrak{K}(\mathcal{E})\mathbin{\widehat{\otimes}}\mathfrak{K})$. Evaluating at $0,1$ yields representatives of left and right hand side of the equation. $\square$

# Chapter 3

# Open questions

The present thesis extensively covered two different applications of the given ring structure. Here in the final chapter, we mention some open questions and interesting problems arising.

First of all, one might ask whether the reduced $K$-theory of the stable Higson corona is really a perfect replacement for the $K$-homology of the Roe algebra. Section 3.1 gives an account of different aspects of this question.

Section 3.2 makes speculations about how the ring $K_*(\mathfrak{c}(M))$ might be of relevance in coarse index theory on a complete Riemannian manifold $M$. The given conjectures are motivated by the results of Chapter 2 and shed a completely new light on coarse index theory.

Finally, Section 3.3 is a short list of quite specific questions arising in this thesis and ideas for future research.

## 3.1 How to replace $K^*(C^*X)$?

In constructing a dual $\mu^*$ to the coarse assembly map $\mu : KX_*(X) \to K_*(C^*X)$, one would expect the $K$-homology of the Roe algebra, $K^*(C^*X)$, to be its domain. However, $K$-homology of non-separable and non-nuclear $C^*$-algebras is not well behaved. Therefore, one needs a replacement.

In general, a good replacement for $K^*(C^*X)$ should be a functor

$$X \mapsto \text{“}K^*(C^*X)\text{”}$$

from the coarse category of coarse spaces into the category of $\mathbb{Z}_2$-graded abelian groups with the following properties:

1. There is a natural pairing “$K^*(C^*X)$” $\times K_*(C^*X) \to \mathbb{Z}$;

2. there is a natural co-assembly map $\mu^* : \text{“}K^*(C^*X)\text{”} \to KX^*(X)$;

3. pairing and co-assembly are compatible with the pairing between $KX^*$ and $KX_*$ and the coarse assembly map $\mu : KX_*(X) \to K_*(C^*X)$ in the sense that

$$\langle x, \mu(y)\rangle = \langle \mu^*(x), y\rangle \qquad \forall x \in \text{“}K^*(C^*X)\text{”}, y \in KX_*(X);$$

4. the co-assembly map is an isomorphism for scalable, uniformly contractible spaces;

5. the co-assembly map is an isomorphism for groups which uniformly embed in Hilbert space.

The work of Emerson and Meyer shows that the $K$-theory of the stable Higson corona, $K_{1-*}(\mathfrak{c}(X))$, is a good replacement in this sense [EM06]. Furthermore, the existence of applications [EM07, EM08] underlines its significance.

However, it is not clear whether this model is best behaved for strange spaces like foliated cones. It might be that there are other good replacements which reveal more geometrically relevant information.

Any functor $F$ from the coarse category of coarse spaces into the category of $\mathbb{Z}_2$-graded abelian groups together with a natural transformation $\alpha : F \to K_{1-*}(\mathfrak{c}(\_))$ yields a new replacement satisfying the first three properties with co-assembly $\mu_F^* := \mu^* \circ \alpha$ and pairing $\langle \alpha(\_), \_\rangle$. The $K$-theory of the (unstabilized) Higson corona is an example. Nothing new can be expected for these replacements.

The real question is, how far to the *right* can we go in the diagram

$$\begin{tikzcd}
K_*(C^*X) \arrow[d, dotted, dash, "\langle\cdot,\cdot\rangle"'] \arrow[drr, dotted, dash, "\langle\cdot,\cdot\rangle"] & & & KX_*(X) \arrow[lll, "\mu"'] \arrow[d, dotted, dash, "\langle\cdot,\cdot\rangle"] \\
K_{1-*}(\mathfrak{c}(X)) \arrow[rr] & & \text{“}K^*(C^*X)\text{”} \arrow[r, "\mu^*"] & KX^*(X),
\end{tikzcd}$$

i. e. is there a good replacement $X \mapsto \text{“}K^*(C^*X)\text{”}$ such that all good replacements, in particular $X \mapsto K_{1-*}(\mathfrak{c}(X))$, factor through $X \mapsto \text{“}K^*(C^*X)\text{”}$ in the above described manner?

A positive answer to this question might yield an even better model for the $K$-theory of the leaf space of a foliation when applied to foliated cones.

Finally, it should be said that the construction of a ring structure on the groups “$K^*(C^*X^{\to})$” might become a quite delicate problem for other good replacements. For example, if one ignores the technical difficulties arising from the lack of separability and considers the $K$-homology of the Roe algebra itself, then the product would be a secondary product constructed by comparing two different reasons for the vanishing of a primary product (cf. [Roe95]). But already the definition of the primary product causes problems, because it should be a composition like

$$K^i(C^*X) \otimes K^j(C^*X) \to K^{i+j}(C^*X \otimes C^*X) \leftarrow K^{i+j}(C^*(X \times X)) \\ \xrightarrow{\Delta^*} K^{i+j}(C^*X)$$

and it is absolutely unclear how to bypass the wrong way map in the middle, which is induced by a strict inclusion $C^*X \otimes C^*X \subsetneq C^*(X \times X)$.

## 3.2 Coarse indices of twisted operators

We discussed an application of the ring structure on the $K$-theory of the stable Higson corona to longitudinal index theory on foliations in Chapter 2. This raises the question whether there is a more direct application to coarse index theory. In this section, we give conjectures in coarse index theory which very much resemble our results in foliation index theory. The author believes, that these conjectures can be easily proven by methods similar to those of Chapter 2.

To get started, recall the role of $KK$-theory classes in $C^*$-algebraic index theory: Given an elliptic operator $D$ over a compact manifold $M$, there is an associated $K$-homology class $[D] \in K_0(M) = KK(C(M), \mathbb{C})$ and, more generally, a longitudinal elliptic operator $D$ on a foliation $(M, \mathcal{F})$ has a $KK$-theory class $[D] \in KK(C(M), C^*_r(M, \mathcal{F}))$, as we have seen in Section 2.9.

Given a vector bundle $F \to M$, the index of the twisted operator $D_F$ is the element in $K_0(\mathbb{C}) \cong \mathbb{Z}$ respectively $K_0(C^*_r(M, \mathcal{F}))$ given by the composition product of $[D]$ with $[F] \in K^0(M) = K_0(C(M))$. A different method, yielding the index of $D$ itself, is to crush $M$ in the first variable to a point.

However, things look different in coarse index theory. A Dirac type operator $D$ over a complete, connected Riemannian manifold $M$ also has a $K$-homology class $[D] \in K_0(M) = KK(C_0(M), \mathbb{C})$ and the coarse index $\mathrm{ind}(D) \in K_0(C^*M)$ is obtained from the fundamental class $[D]$ under the coarse assembly map $\mu : K_0(M) \to K_0(C^*M)$. This map is a generalization of the map $K_0(M) \to K_0(pt)$ crushing $M$ to a point which only exists for compact $M$.

In the non-compact case, a vector bundle $F \to M$ is given by a projection in $C_b(M, \mathfrak{K})$ and thus defines a class $[F] \in K_0(C_b(M, \mathfrak{K}))$. Note that we really have to use the $C^*$-algebra $C_b(M, \mathfrak{K})$ and not $C_b(M)$ (or even $C_0(M)$), because $F$ might not be embeddable into a finite rank trivial bundle. This appearance of the compact operators underlines the importance of working with the stable Higson corona instead of the unstabilized Higson corona.

In calculating the index of the twisted operator $D_F$ by a Kasparov composition product, the fundamental class $[D] \in K_0(M) = KK(C_0(M), \mathbb{C})$ is obviously of no use. Instead, one should expect the following $E$-theory class:

**Conjecture 3.2.1.** *The elliptic operator $D$ defines a class*

$$[\![D]\!] \in E(C_b(M, \mathfrak{K}), C^*M)$$

*similar to the one of Definition 2.9.4.*

Here, the usage of $E$-theory instead of $KK$-theory is important, because $C_b(M, \mathfrak{K})$ is not separable.

Given this $E$-theory class, it should now be possible to calculate the index of twisted operators by the composition product

$$K(C_b(M, \mathfrak{K})) \otimes E(C_b(M, \mathfrak{K}), C^*M) \to K(C^*M), \quad [F] \otimes [\![D]\!] \mapsto \operatorname{ind}(D_F).$$

Having Corollary 2.10.3 in mind, one might now ask the following question: Is it possible to compute the index of $D_F$ from the index of $D$ and $F$, provided that $F$ satisfies some asymptotic triviality condition, e. g. $F$ being determined by a projection in $\bar{\mathfrak{c}}(M) \subset C_b(M, \mathfrak{K})$?

It is definitely not possible in the general case, because it fails for compact manifolds $M$: Here, any vector bundle satisfies the asymptotic triviality condition, because $\bar{\mathfrak{c}}(M) = C(M) \otimes \mathfrak{K}$, but $\operatorname{ind}(D)$ may vanish while $\operatorname{ind}(D_F)$ does not.

The following adaption of the coarse index seems necessary to get rid of such special cases: Instead of taking the index in $K(C^*M)$ we pass to its image in $K(C^*M/\mathfrak{K})$ by dividing out the canonically embedded ideal $\mathfrak{K} \subset C^*M$ of compact operators[1]. This has the same effect as gluing a ray onto $M$ as we did in Section 1.5, as is easily seen by establishing canonical canonical isomorphisms $K(C^*M/\mathfrak{K}) \cong K(C^*(M^{\to}))$.

How much information is lost by passing to $C^*M/\mathfrak{K}$? If $M$ is a non-compact complete manifold, then there is a coarsely embedded ray $[0, \infty) \subset$

[1]Recall that $C^*M$ is defined as a sub-$C^*$-algebra of some $\mathfrak{B}(H)$ and contains $\mathfrak{K}(H)$. See [HR00] for details.

$M$ and the map $K_*(\mathfrak{K}) \to K_*(C^*M)$ – which is induced by the inclusion of a point into $M$ – factors through $K_*(C^*[0,\infty)) = 0$ and therefore vanishes. Thus, the long exact sequence induced by $0 \to \mathfrak{K} \to C^*M \to C^*M/\mathfrak{K} \to 0$ splits into short exact sequences

$$0 \to K_i(C^*M) \to K_i(C^*M/\mathfrak{K}) \to K_{i-1}(\mathfrak{K}) \to 0$$

and we see that no information is lost in this case.

On the other hand, if $M$ is a compact manifold, then $K_i(C^*M/\mathfrak{K}) = 0$ and all information is lost.

Is this bad? Thomas Schick pointed out to the author that this loss of information also extinguishes some awkward special cases, like in the following theorem, which was first stated in [Roe96, Proposition 3.11 and following remark] without proof. Proofs of this theorem can be found in [Roe12, Pap11, HPS].

**Theorem 3.2.2.** *Let $M$ be a complete connected non-compact Riemannian spin manifold such that the scalar curvature is uniformly positive outside of a compact subset. Then the coarse index of the Dirac operator $\operatorname{ind}(\not{D}) \in K_*(C^*M)$ vanishes.*

If one uses the coarse index in $K_*(C^*M/\mathfrak{K})$ instead, then the theorem is trivially also true for compact $M$.

There is probably also an $E$-theoretic fundamental class for this new index.

**Conjecture 3.2.3.** *There is also an $E$-theory class*

$$[\![D]\!] \in E(C_b(M,\mathfrak{K})/C_0(M,\mathfrak{K}), C^*M/\mathfrak{K})$$

*such that the following diagram in the category **E** commutes:*

$$\begin{tikzcd} C_b(M,\mathfrak{K}) \arrow[r, "{[\![D]\!]}"] \arrow[d] & C^*M \arrow[d] \\ C_b(M,\mathfrak{K})/C_0(M,\mathfrak{K}) \arrow[r, "{[\![D]\!]}"'] & C^*M/\mathfrak{K} \end{tikzcd}$$

Coming back to the question about twisted operators, the obvious analogue of Corollaries 2.8.6 and 2.10.3 in coarse index theory would be the following:

**Conjecture 3.2.4.** *$K_*(C^*M/\mathfrak{K})$ is a module over the ring $K_*(\mathfrak{c}(M))$. If $M$ is a complete Riemannian manifold of bounded geometry, $D$ is an elliptic operator over $M$ and a vector bundle $F$ is determined by a projection $P$ in $\bar{\mathfrak{c}}(M)$, then* $\operatorname{ind}(D_F) \in K(C^*M/\mathfrak{K})$ *is obtained from* $\operatorname{ind}(D) \in K(C^*M/\mathfrak{K})$ *by module multiplication with the class of $P$ in $K(\mathfrak{c}(M))$.*

We assumed bounded geometry here, because the analog of Lemma 2.10.5 might not hold in full generality.

If these conjectures are true, then they reveal interesting new aspects of coarse index theory. Furthermore, they justify from the index theoretic point of view, why the $K$-theory of the stable Higson corona is a good replacement for the $K$-homology of the Roe algebra.

## 3.3 Further questions

The big open task is to find applications for the theory developed in this thesis.

The (reduced) co-assembly map of Emerson and Meyer has proven valuable in [EM07, EM08]. Analyzing their work should give hints of how to make use of the ring structures constructed in Chapter 1. In this context, it will definitely be necessary to construct the ring structures for equivariant co-assembly maps, too.

The ring and module structures in $K$-theory of leaf spaces and the results on indices of twisted longitudinally elliptic operators might find their applications to questions about leafwise positive scalar curvature. To this end, it might be useful to develop some of the following variations:

- In Section 3.1, we already discussed the possibility of using other replacements of the $K$-homology of the Roe algebra. They might reveal more structure of the leaf space, but for index theory one still needs to find a description of elements of "$K^*(C^*(\mathcal{O}(V,\mathcal{F})^{\rightarrow}))$" in terms of some kind of "vector bundles over the leaf space".

- Another idea would be to use cyclic homology instead of $K$-theory, as this has always been very promising in index theory.

- In Section 2.8 we had to use the reduced foliation $C^*$-algebra $C^*_r(V,\mathcal{F})$ and Higson coronas with nuclear coefficient $C^*$-algebras to make our proofs work. This was, because the proofs relied on an explicit calculation using the minimal tensor product. Generalizations to the full $C^*$-algebra $C^*(V, \mathcal{F})$ and the maximal tensor product might be obtained with different proofs.

Finally, we had already mention in Section 1.4 that it would be useful to have picture of $K$-theory for $\sigma$-$C^*$-algebras which is well adapted to both products and boundary maps and allows a direct proof of the axioms given. The author believes that this problem should be attacked by generalizing the spectral picture of $K$-theory of $C^*$-algebras presented in [HG04, Section 1.5] to $\sigma$-$C^*$-algebras.

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# Notations and conventions

| | |
|---|---|
| $\otimes$ | In contrast to established conventions, we use this symbol to denote the *maximal* tensor product of $C^*$-algebras or $\sigma$-$C^*$-algebras. For the latter, see Definition 1.3.7. |
| $\otimes_{\min}$ | Minimal tensor product of $C^*$-algebras. |
| $\widehat{\otimes}$, $\widehat{\otimes}_{\min}$ | Graded maximal resp. minimal tensor product of $\mathbb{Z}_2$-graded $C^*$-algebras. |
| $C_b(X, D)$, $C_b(X)$ | $C^*$-algebra of bounded continuous functions on the locally compact space $X$ with values in the $C^*$-algebra $D$ resp. $\mathbb{C}$. |
| $C_0(X, D)$, $C_0(X)$ | $C^*$-algebra of continuous functions vanishing at infinity on the locally compact space $X$ with values in the $C^*$-algebra $D$ resp. $\mathbb{C}$. |
| $C_b(\mathcal{X}, \mathcal{A}, D)$, $C_0(\mathcal{X}, D)$ | Generalization of the above function algebras to $\sigma$-locally compact spaces. See Section 1.2. Omitting $D$ from the notation means $D = \mathbb{C}$. |
| $\mathfrak{c}$, $\bar{\mathfrak{c}}$, $\mathfrak{uc}$, $\overline{\mathfrak{uc}}$, $\bar{\mathfrak{c}}_0$, $\overline{\mathfrak{uc}}_0$ | Higson corona and compactification $C^*$-algebras. See Definition 1.1.4 for the case of coarse spaces and Section 1.2 for the generalization to $\sigma$-coarse spaces. |
| $\ell^2$ | Standard separable infinite dimensional Hilbert space, e. g. $\ell^2(\mathbb{N})$ or $\ell^2(\mathbb{Z})$. |
| $\widehat{\ell^2}$ | Standard $\mathbb{Z}_2$-graded separable Hilbert space with even and odd graded parts equal to $\ell^2$. |
| $\mathcal{M}(A)$ | Multiplier algebra of $A$. |
| $\widetilde{A}$ | unitalization of $A$. |

| | |
|---|---|
| $\mathfrak{B}(H)$, $\mathfrak{B}(\mathcal{E})$ | ($\mathbb{Z}_2$-graded) $C^*$-algebra of bounded operators on the ($\mathbb{Z}_2$-graded) Hilbert space $H$ resp. ($\mathbb{Z}_2$-graded) $C^*$-algebra of adjointable operators on the ($\mathbb{Z}_2$-graded) Hilbert module $\mathcal{E}$. |
| $\mathfrak{K}(H)$, $\mathfrak{K}(\mathcal{E})$ | ($\mathbb{Z}_2$-graded) $C^*$-algebra of compact operators on the ($\mathbb{Z}_2$-graded) Hilbert space $H$ resp. ($\mathbb{Z}_2$-graded) Hilbert module $\mathcal{E}$. |
| $\mathfrak{K}$, $\widehat{\mathfrak{K}}$ | Standard ungraded resp. $\mathbb{Z}_2$-graded $C^*$-algebra of compact operators: $\mathfrak{K} := \mathfrak{K}(\ell^2)$, $\widehat{\mathfrak{K}} := \mathfrak{K}(\widehat{\ell^2})$. |
| $\mathcal{E}^{M,N}$ | The $\mathbb{Z}_2$-graded Hilbert module with even part $\mathcal{E}^M$ and odd part $\mathcal{E}^N$, where $\mathcal{E}$ is an ungraded Hilbert module. Important special cases are $\mathcal{E}$ a Hilbert space or a $C^*$-algebra. |
| $M_{M,N}(A)$ | $\mathbb{Z}_2$-graded $C^*$-algebra of $(M+N) \times (M+N)$ matrices over the ungraded $C^*$-algebra $A$. The even graded part consists of the $M \times M$ and $N \times N$ diagonal blocks. The off diagonal $M \times N$ and $N \times M$ blocks constitute the odd part. Note that $M_{M,N}(A) = \mathfrak{K}(A^{M,N})$. |