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# 12 Tense or Temporal Logic

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## 1. Introduction

The logic of time is a subfield of logic of special philosophical interest. Its development in the twentieth century is connected with issues in metaphysics, the philosophy of language, philosophy of science, the philosophy of logic, and action theory. But tense logic, also called temporal logic, is interesting from a purely formal point of view as well. It played a decisive role in the development of modal logic, and it continues to play an important role in theoretical computer science and in many applications.

This introduction provides some motivational and historical background. Basic formal issues of propositional tense logic are treated in Section 2. Section 3 gives a glimpse of a number of related further topics.

### 1.1 Motivation and Terminology

Arthur Prior, who initiated the formal study of the logic of time in the 1950s, called his logic *tense logic* to reflect his underlying motivation: to advance the philosophy of time by establishing formal systems that treat the tenses of natural languages like English as sentence-modifying operators. While many other formal approaches to temporal issues have been proposed in the meantime, Priorean tense logic is still the natural starting point. Prior introduced four *tense operators* that are traditionally written as follows:

**P** it was the case that (Past)
**H** it has always been the case that
**F** it will be the case that (Future)
**G** it is always going to be the case that

Thus, a sentence in the past tense like 'John was happy' is analysed as a temporal modification of the sentence 'John is happy' via the operator 'it was the case that', and can be formally expressed as 'P Happy(John)'.

While the syntax of the tense operators is analogous to that of a simple one-place operator like negation, their semantical interpretation transcends the territory of standard extensional languages: tense operators are not truth functional (obviously one cannot compute the truth value of a temporally modified sentence from the *truth value* of the unmodified sentence alone), and tense logic is accordingly an intensional logic, or a modal logic in the wider sense.

Typical foundational questions of intensional logic acquire a specific twist when applied to tense logic. Once one considers tense to be part of the *content* of a proposition modified by a tense operator, one has to become clear on the meaning of an unmodified sentence. Basically there are two options: the tense operators could form something that is tensed out of material that is not yet tensed, or they could modify something that is already tensed. The plausibility of iterated tense operators ('it was the case that it was the case that it is raining') points towards option two: unmodified propositions should be considered to be in the present tense, not tenseless. Such a view was held by a number of ancient and medieval logicians starting with Aristotle (in fact, an additional motivation for the development of tense-logical systems was the hope that they could contribute to the exegesis and understanding of historical texts, especially on the much-discussed issue of future contingents). Furthermore, the view that unmodified propositions are already tensed accords well with the grammatical



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form of the sentences expressing unmodified propositions like 'John is happy' in English, which are in the present tense. However, this seems to make trouble for the traditional idea that a proposition has a truth value *simpliciter*. Frege for one took this as so basic that he argued that all common unmodified sentences are elliptical, expressing a proposition that in fact contains a time index ('John is happy on January 8, 2010, 1:07 pm GMT', with the 'is' interpreted as a *tenseless* copula). The idea that an ideally rational language should not make a difference between past, present, and future is indeed a motif found in a variety of philosophers ranging from Spinoza to the logical empiricists of the twentieth century. Tense logic, on the other hand, invites one to view *one and the same proposition* as true at one time, yet false at another. This was certainly Prior's view. Writers who wish to make a metaphysical point about this are perhaps more likely to use Prior's original term 'tense logic' than others, who may prefer the somewhat more neutral terms 'temporal logic' or 'the logic of time'.

## 1.2 Tense Logic and the History of Formal Modal Logic

As just mentioned, modern logicians did not commonly take modalities and tenses to be elements of the logical form of a proposition. Frege, continuing the Kantian tradition, completely banished modalities from that part of a sentence that was considered amenable to logical analysis. The late nineteenth century saw some work on the formalization of modalities; formal modal logic proper is mostly considered to begin with C. I. Lewis' early twentieth century investigations into the logic of conditionals leading, e.g., to the well-known systems S4 and S5.[1] These systems of modalities were characterized *syntactically*.

A *semantic* interpretation for these systems was not established until much later. The key step towards an adequate semantics, which occurred to a number of philosophers and mathematicians more or less simultaneously in the 1950s, was to allow for different states of affairs (often called 'possible worlds') in one single semantic model, and to impose a relational structure on them. Metaphorically this relation is often called an 'accessibility relation', but it just expresses the relation of relative possibility: given a set $W$ of worlds and the accessibility relation $R$ (set-theoretically, a subset of $W \times W$), if $wRw'$, then $w'$ is possible relative to $w$. Necessity is then taken to be truth in all accessible worlds, and accordingly, a sentence $\Box\phi$ ('it is necessary that $\phi$') is said to be true at $w$ in a model iff for all $w'$ for which $wRw'$, $\phi$ is true at $w'$. $\Box$ is called a *strong* modal operator since it corresponds to universal quantification over accessible worlds. Dually, the *weak* operator $\Diamond$ ('it is possible that') is defined via existential quantification: $\Diamond\phi$ is true at $w$ iff there is some $w'$ with $wRw'$ s.t. $\phi$ is true at $w'$.

In general, the accessibility relation $R$ is hard to make concrete sense of; this conceptual difficulty may account for the fact that formal semantics for modal languages were developed rather late. The relation $R$ is however easy to interpret



if the modalities involved are in fact tenses and the 'possible worlds' contain information about the state of things at different moments. In that case, the 'accessibility' relation is simply the relation $>$ of temporal succession: $P\phi$ is true at $m$ iff there is some moment $m'$ for which $m > m'$ and at which $\phi$ is true. It is no surprise, then, that some of the first ideas about a relational semantics for a modal language came from tense logic. Many interesting correspondences between modal formulae and conditions on relational structures – like the condition of transitivity and the validity of $FF\phi \rightarrow F\phi$, to be discussed in Section 2.1.2 below – can be intuitively well motivated by considering the structure of time, cf. e.g., [Prior, 1967]. In this way, tense logic facilitated the development of formal modal logic by providing a plausible reading of the relational semantics. On the other hand, tense logic is more complicated than the logic of necessity and possibility (so-called alethic modal logic) because it contains *two* interacting sets of modal operators, one for the future and one for the past, and because the interpretation of the future operator poses special problems (see Section 2.1.3 below). For a detailed overview of the development of the semantics of temporal and modal logic, cf. [Copeland, 2002] and [Goldblatt, 2005]. This history certainly contains a wealth of material for a case study of the phenomenon of 'multiple discovery'.

## 2. Basic Formal Tense Logic

Above we have already described informally the basic ingredients of the syntax and semantics of tense logic. In fact there are many variants. Here we give some basic information about *propositional* tense logic based on point structures. (There are also good motivations for building systems based on interval structures; cf., e.g., [van Benthem, 1991] for a balanced overview. Reichenbach introduced a somewhat different analysis of tenses; for references see Section 3.1. The predicate logic will be discussed very briefly in Section 3.3 below.) A more complete overview of propositional tense logic can be found, e.g., in [Burgess, 2002] and [Finger et al., 2002].

**Syntax:** Our common background for the following considerations is to assume for the syntax

- a countable set *Atoms* of atomic propositions $p, q, r, \ldots$;
- the usual propositional operators: negation ($\neg$) and conjunction ($\wedge$), and defined from these, disjunction ($\vee$), the conditional ($\rightarrow$) and the biconditional ($\leftrightarrow$); we will also use the (defined) propositional constants TRUE ($\top$) and FALSE ($\bot$);
- a choice of temporal operators: Prior's original one-place strong operators for past (H) and future (G) and, usually defined from these, their weak duals P and F (see Section 2.1), and/or the two-place

connectives for 'since' (S) and 'until' (U) introduced by Kamp (see Section 2.2).[2]

**Semantics:** On the semantic side, we will be working with

- a non-empty relational structure $\langle W, R\rangle$, also called a *frame* and often written as $\langle T, <\rangle$ to suggest 'time' and the earlier–later relation.

Note that the latter notation suggests, e.g., transitivity of $<$, but in the minimal setting with which we will start this is not yet required. In most cases, however, we will assume a structure which could intuitively be a *flow of time*, i.e., we will be working with

- a non-empty *strict partial ordering* of moments $\langle T, <\rangle$ as the underlying frame; being a partial ordering, $<$ is transitive (for all $m, m', m'' \in T$: if $m < m'$ and $m' < m''$, then $m < m''$) and irreflexive (for all $m \in T$, $m \not< m$), hence also asymmetric (for all $m$ and $m'$: if $m < m'$, then $m' \not< m$); the non-strict ordering $\leq$, defined in the obvious way ($m \leq m'$ iff $m < m'$ or $m = m'$), is antisymmetric (if $m \leq m'$ and $m' \leq m$, then $m = m'$);
- a *model* $\mathcal{M} = \langle T, <, V\rangle$ based on the frame $\langle T, <\rangle$, which is that frame together with a *valuation* $V$, which assigns to each $p \in Atoms$ the set $V(p) \subseteq T$ of moments at which that atom is true in the model.

In many cases the frame is assumed to have additional properties: most commonly, one only considers such partial orderings that are

- *backwards-linear* (also called 'left-linear' or 'future-branching', also simply 'branching'), i.e., if $m < n$ and $m' < n$, then ($m < m'$ or $m = m'$ or $m' < m$)

and very often one narrows this down further to orderings that are

- *linear* (or: trichotomous), i.e., for all $m, m' \in T$, we have one of the three following cases: $m < m'$ or $m = m'$ or $m' < m$.

The following discussion is split into a longer part on Priorean (P / F) tense logic (Section 2.1) and a much shorter part on tense logic with 'since' and 'until' (Section 2.2).

## 2.1 Priorean (Past, Future) Tense Logic

As mentioned, Prior's motivation was to capture the notion of tense as a sentence-modifying operator. He introduced the weak one-place modal operators P (past) and F (future) and their strong duals H (always in the past; $H\phi \Leftrightarrow \neg P\neg\phi$) and G (always in the future; $G\phi \Leftrightarrow \neg F\neg\phi$). Prior was of the opinion that the semantics of these operators could not be *given* via model-theoretic conditions, because the only way we could understand these conditions and the underlying models was via our natural language and its informal counterparts to these operators. He did however acknowledge the usefulness of a model-theoretic approach for getting clear about frame conditions (see Section 2.1.2 below), and it has become customary to lay out the semantics of tense operators in that fashion too. Prior's own 'internal' approach, which sticks to modal languages and increases their expressive power via special propositions (so-called nominals) that characterize individual moments, ultimately led to the now flourishing field of so-called hybrid logic; cf. [Prior, 1967, Chapter V], [Prior and Fine, 1977] and, for an overview, [Blackburn, 2000] and [Areces and ten Cate, 2007].

Given a model $\mathcal{M} = \langle T, <, V\rangle$ based on some frame $\langle T, <\rangle$ and a moment of evaluation $m \in T$, the truth conditions for a sentence $\phi$ are as follows (where $\mathcal{M}, m \models \phi$ is to be read '$\phi$ is true in model $\mathcal{M}$ at moment $m$'):[3]

- If $\phi = p \in Atoms$: $\mathcal{M}, m \models \phi$ iff $m \in V(p)$.
- If $\phi = \neg\psi$: $\mathcal{M}, m \models \phi$ iff $\mathcal{M}, m \not\models \psi$.
- If $\phi = \psi_1 \wedge \psi_2$: $\mathcal{M}, m \models \phi$ iff $\mathcal{M}, m \models \psi_1$ and $\mathcal{M}, m \models \psi_2$.
- If $\phi = H\psi$: $\mathcal{M}, m \models \phi$ iff for all $m' \in T$ s.t. $m' < m$, we have $\mathcal{M}, m' \models \psi$.
- If $\phi = G\psi$: $\mathcal{M}, m \models \phi$ iff for all $m' \in T$ s.t. $m < m'$, we have $\mathcal{M}, m' \models \psi$.

The more general notions of global truth, validity, and logical consequence are then defined in the usual way:

- $\phi$ is globally true in a model $\mathcal{M} = \langle T, <, V\rangle$ iff for all $m \in T$, $\mathcal{M}, m \models \phi$. We write $\mathcal{M} \models \phi$.
- $\phi$ is valid in a frame $F = \langle T, <\rangle$ iff $\phi$ is globally true in every model based on that frame. We write $F \models \phi$. Generalizing, $\phi$ is valid in a class of frames $\mathcal{F}$ iff it is valid in any frame from that class. We write $\mathcal{F} \models \phi$
- $\phi$ is a logical consequence of a set $\Psi$ of formulae in a model $\mathcal{M} = \langle T, <, V\rangle$ iff for all $m \in T$, if $\mathcal{M}, m \models \psi$ for all $\psi \in \Psi$, then $\mathcal{M}, m \models \phi$ as well ($\Psi \models_{\mathcal{M}} \phi$). Again, this is generalized to frames ($\Psi \models_F \phi$) and classes of frames ($\Psi \models_{\mathcal{F}} \phi$).

Once a tense-logical language with the above semantics is available, one can rephrase certain questions in the philosophy of time in a formal manner. For example, does time have a beginning? Certainly some frames have a first moment, while others don't. Can one express this fact tense-logically, as it were 'from within' the temporal structure? (It turns out that one can.) More generally, it is interesting to see how much the tense-logical language can reflect about the structure of the frames on which it is defined. One of the early questions in the

development of tense logic, mirrored by similar questions in alethic modal logic, was about useful axiomatic systems and the correspondence of the validity of tense-logical formulae and general properties of frames: such correspondences can lead the way to *complete* axiomatizations, in which the formulae derivable syntactically from the axioms are exactly the formulae valid in a suitably characterized class of models. Most interestingly, this will be a class of models singled out by simple conditions on the underlying frame, e.g., first-order conditions like transitivity, density, the existence of a first moment, or the like. This issue, which pertains to modal logic generally, will be discussed in Section 2.1.2. In tense logic, a further point of interest is the question of the adequacy of the definition of the F operator; for discussion see Section 2.1.3 below. We will also discuss the treatment of the indexical 'now' (Section 2.1.4) and metric aspects of tense logic (Section 2.1.5). But first, let us introduce the simplest tense logic, $K_t$.

### 2.1.1 The Minimal Tense Logic $K_t$

Above we have laid out the semantics of tense logic. What would be a proper axiomatic basis? Note that this style of approach turns the general historical development upside down: historically, syntactic systems of axioms and rules came much earlier than the semantics. Tense logic, however, has from the beginning been a rather semantically-driven field, in which intuitions about sensible temporal structures take precedence over the choice of particular axioms. Still one wants to know what that syntactic side looks like.

By the definition of the semantics given above, it is clear that all propositional tautologies are true in any model at any moment – thus, any useful axiomatic basis for a tense logic should contain a set of axioms for the propositional calculus. As in basic propositional logic, so in propositional tense logic, an axiomatization can be given either by specifying axiom schemata (which are not themselves formulae of the language, but characterize an infinite set of such formulae), or concrete axioms (which are formulae) together with a substitution rule (allowing for the uniform substitution of atomic propositions by arbitrary formulae). For simplicity's sake we choose to work with axiom schemata here, even though we will be lax in sometimes calling them 'axioms'.[4]

Over and above the propositional calculus, the semantics laid out above guarantees, even in the most general case in which there are no conditions on the relation $<$ (not even transitivity), the validity of the *K*-axiom (the modal distribution axiom) for each of the strong operators, like in alethic modal logic (cf. Chapter 11):

$$H(\phi \to \psi) \to (H\phi \to H\psi);$$

$$G(\phi \to \psi) \to (G\phi \to G\psi).$$



Furthermore, the semantics supports rules of temporal generalization (corresponding to the necessitation rule in alethic modal logic) for both strong operators, i.e.,:

- if $\phi$ is a theorem, then so is $H\phi$;
- if $\phi$ is a theorem, then so is $G\phi$.

These are, twice, the standard axioms and rules for a normal modal logic: tense logic as laid out above is a *normal bimodal logic*. The fact that two sets of modal operators are involved immediately raises the question whether these modalities interact in any specific way. After all, their semantics involves not two independent relations, but just one relation and its converse. As one can check easily, there are indeed two characteristic interaction axioms that follow from the semantics given above (again these are valid in any model based on any frame):[5]

$$\phi \to HF\phi$$

and

$$\phi \to GP\phi.$$

The minimal tense logic comprising just the mentioned axioms (a set of axioms for propositional calculus, the two *K* axioms for H and G, and the two interaction axioms) and rules (modus ponens and the two rules of temporal generalization) is called $K_t$, mimicking the name for the minimal normal monomodal logic, K (so-called after its single characteristic axiom, which in turn is named after one of the pioneers of the field, Saul Kripke). We write '$\vdash_{K_t} \phi$' for '$\phi$ is a theorem of $K_t$', and '$\Psi \vdash_{K_t} \phi$' for '$\phi$ can be proved from $\Psi$ in $K_t$'.

One can show that the logic $K_t$ is sound and complete with respect to the class of all models based on all frames, i.e., that all theorems (formulae derivable from the axioms) of the logic are validities (true at every moment in every model based on any frame) – the soundness direction – and that every validity is also in fact a theorem (weak completeness). Weak completeness corresponds to the property that every single $K_t$-consistent formula (i.e., every formula from which one cannot derive a contradiction in $K_t$) is satisfiable. $K_t$ is also *strongly complete* w.r.t. the class of all frames: if $\phi$ is a logical consequence of $\Psi$, then $\phi$ is also $K_t$-provable from $\Psi$. Strong completeness corresponds to the property that every $K_t$-consistent set of formulae is satisfiable.

Soundness can be proved formally, as usually, by induction on the length of a proof, i.e., by showing that the mentioned axioms are validities and that the mentioned inference rules preserve validity; we have motivated such a proof informally above. Completeness can be proved in a number of ways, e.g., via the canonical model technique (cf., e.g., [Blackburn et al., 2002, pp. 205f.]). The first completeness proof for $K_t$ is due to Lemmon; it is laid out, e.g., in

[Burgess, 2002].[6] Lemmon's proof in fact establishes that every $\mathbf{K}_t$-consistent set of formulae is satisfiable on an *irreflexive* frame; we will comment on this below.

Like many other useful modal logics, $\mathbf{K}_t$ also has the *finite model property*, from which it follows that $\mathbf{K}_t$ is *decidable*. The investigation of the computational complexity of the satisfiability problem for various tense logics is an important topic for theoretical computer science, with obvious repercussions for applications. To mention just one relevant result, the satisfiability problem for linear transitive flows of time is NP-complete. For further results, cf. [Gabbay et al., 1994, Chapter 15.8].

### 2.1.2 Frame Conditions

Having studied the basic semantics, one can ask which modal formulae can be used to express certain metaphysically interesting assumptions about a flow of time – a task that provided much motivation for the early tense logicians in the 1950s and 1960s. As Lemmon's completeness proof for $\mathbf{K}_t$ shows that any consistent set of $\mathbf{K}_t$-formulae is in fact satisfiable in a model based on an *irreflexive* frame, the class of all frames gives rise to the same tense logic as the class of irreflexive frames. Thus, tense-logical formulae cannot single out the irreflexive frames. This is one sign of the expressive limitations of tense logic that was noticed early on. Insofar as irreflexivity is metaphysically interesting (it certainly is in connection with avoiding circularity: circular time is reflexive), this is also a philosophically important negative result.[7] Many other metaphysically interesting conditions on frames can however be expressed tense-logically; we will discuss the most prominent cases. More generally, one can ask what the fundamental connections are between validity of formulae and conditions on a class of frames; we will comment briefly on this at the end of this section.

**Transitivity:** As one of the simplest cases, consider transitivity. It is easy to check that the formula

$$\mathsf{FF}\phi \rightarrow \mathsf{F}\phi \tag{4}$$

(or in the contrapositive, $\mathsf{G}\psi \rightarrow \mathsf{GG}\psi$) is valid in all transitive frames. In fact it defines the class of transitive frames: (4) is valid on a frame iff that frame is transitive. To prove the direction from right to left, let $F = \langle T, < \rangle$ be a transitive frame and $\mathcal{M} = \langle T, <, V \rangle$ a model based on $F$, and let $m \in T$. If $\mathcal{M}, m \models \mathsf{FF}\phi$, then by the semantic clause for F, there must be $m', m'' \in T$ s.t. $m < m'$, $m' < m''$, and $\mathcal{M}, m'' \models \phi$. By transitivity, we also have $m < m''$. (Note that while the notation suggests this anyway, this does not hold for all relations $<$; transitivity *is* needed at this step.) Applying the semantic clause for F once more, we get $\mathcal{M}, m \models \mathsf{F}\phi$, and thus (4). The direction from left to right can be proved contrapositively.



Thus, let $F = \langle T, < \rangle$ be a frame that is not transitive, i.e., in which there are $m, m', m'' \in T$ for which $m < m'$, $m' < m''$, but $m \not< m''$. Now, let $p$ be an atomic proposition, and let $V$ be a valuation for which $V(p) = \{m''\}$, i.e., according to $V$, $p$ is true only at $m''$ and false at all other moments. By the semantic clause for F, we have $\mathcal{M}, m \models \mathsf{FF}p$, but $\mathcal{M}, m \not\models \mathsf{F}p$; thus, for $\phi = p$, we have a counterexample to (4).

The fact that a tense-logical formula defines a certain class of frames does not yet amount to a completeness proof for the respective extension of $\mathbf{K}_t$. In the present case, however, a completeness proof for $\mathbf{K}_t$ can be extended fairly easily to show that the logic $\mathbf{K}_t\mathbf{4}$, i.e., the system of $\mathbf{K}_t$ enriched by the axiom (4),[8] is complete for the class of all transitive frames (and, by the remark about irreflexivity above, also for all partial orders). As the mirror image of (4),

$$\mathsf{PP}\phi \rightarrow \mathsf{P}\phi \tag{4'}$$

is also valid in all transitive frames, it must (by completeness) be provable from (4) given the other axioms. The proof amounts to about half a page (cf., e.g., [McArthur, 1976, Chapter 2] for this and other elementary results).

In a similar vein, there are characteristic axioms and corresponding completeness results for many other theoretically interesting and/or metaphysically significant frame conditions.

**Left Linearity (Past Non-Branching) / Right Linearity:** Frames that are left linear (for all $n, m, m'$, if $m < n$ and $m' < n$, then either $m < m'$ or $m = m'$ or $m' < m$) capture the widespread idea that the past is fixed while the future is open. Left linearity, or past non-branching, is defined by the axiom

$$\mathsf{FP}\phi \rightarrow (\mathsf{F}\phi \vee \phi \vee \mathsf{P}\phi). \tag{LLIN}$$

Many applications of tense logic in computer science as well as in philosophy presuppose left linearity, but allow for future branching (often just called 'branching'); e.g., left-linear frames form the basis for the Ockhamist tempo-modal language to be discussed below. We will give a rather philosophically tinted overview of branching time in Section 2.1.3.

Right linearity (future non-branching) is defined by the mirror image of (LLIN),

$$\mathsf{PF}\phi \rightarrow (\mathsf{P}\phi \vee \phi \vee \mathsf{F}\phi). \tag{RLIN}$$

**Linearity (Trichotomy):** Linearity amounts to backward and forward linearity.[9] Thus, linear frames are defined, e.g., by the above axioms taken together, i.e.,

$$LLIN \wedge RLIN. \tag{LIN}$$

It seems fair to say that linear tense logics are the most well-studied ones, and many specific applications have been developed on the basis of linear time. [Kamp, 1968] established a number of significant results early on.

**First/Last Moment:** The formula

$$G\bot \vee FG\bot \qquad \text{(LASTMO)}$$

defines frames with a last moment (some $m \in T$ for which there is no $m' \in T$ s.t. $m < m'$): at a moment that has no successors, in fact *any* formula starting with 'G' is vacuously true. Similarly, the mirror image

$$H\bot \vee PH\bot \qquad \text{(FIRSTMO)}$$

defines frames in which there is a first moment. These two conditions are obviously independent, i.e., one can hold in a frame without the other also holding.

**No First/Last Moment:** Frames in which there is no last moment (for all $m \in T$ there is some $m' \in T$ such that $m < m'$), also called *serial* (or, more precisely, right serial), are defined by the formula

$$G\phi \rightarrow F\phi; \qquad \text{(NOLAST)}$$

the mirror image

$$H\phi \rightarrow P\phi \qquad \text{(NOFIRST)}$$

defines frames without a first moment, also called left-serial. (The shorter formulae $F\top$ and $P\top$, respectively, also do the job.) In alethic modal logic, the axiom of this form, $\Box\phi \rightarrow \Diamond\phi$, is sometimes called $D$ (for 'deontic': given a deontic reading of the modalities, it expresses the desirable condition that what is obligatory is also permitted). Again, the two conditions for past and future are independent.

**Density:** A *dense* frame is one in which there is a moment between any two given ones: for all $m, m' \in T$ there is some $n \in T$ s.t. $m < n < m'$. The rational numbers are a well-known example of a dense ordering (in fact, any countable dense linear ordering without minima or maxima is isomorphic to the rationals). The axiom

$$F\phi \rightarrow FF\phi \qquad \text{(DENSE)}$$

defines dense frames. In fact $\mathbf{K}_t\mathbf{4}$ together with (LIN) and (DENSE) is sound and complete w.r.t. the class of dense linear orders. Again, like in the case of transitivity, this means that there must be a proof for the temporal mirror image of the axiom,

$$P\phi \rightarrow PP\phi, \qquad \text{(DENSE')}$$



which is also valid in all dense linear orders; as in the case of transitivity, the proof involves a little work.

**Discreteness:** In contradistinction to density, discreteness again splits into two cases, one claiming the existence of immediate successors (for any moment $m \in T$ that is not maximal, there is some $m' \in T$ for which $m < m'$ and there is no $n \in T$ s.t. $m < n < m'$) and the existence of immediate predecessors (the mirror image). The formula

$$(\phi \wedge H\phi) \rightarrow FH\phi \qquad \text{(RDISCR)}$$

defines right-discrete frames that have immediate successors, as is easy to check: if a moment doesn't have an immediate successor, like the number 0 in the reals, then it is possible to verify the antecedent (setting $\phi$ to be true up until and including the moment 0) and falsify the consequent (setting $\phi$ to be false at every moment after 0). The mirror image

$$(\phi \wedge G\phi) \rightarrow PG\phi \qquad \text{(LDISCR)}$$

defines left-discrete frames that have immediate predecessors.

**Characterizing Number Systems:** All of the common number systems are linear orders. Can they be captured tense-logically? From what was said above it is clear that we can characterize dense linear orders without endpoints, like the rationals ($\mathbb{Q}$). In fact the extension of the axiom system $\mathbf{K}_t$ by (LIN), (NOFIRST), (NOLAST), and (DENSE) is sound and strongly complete with respect to the class of dense linear orders without endpoints.[10] In that tense-logical system, a nice result by Hamblin shows that there are only 15 different combinations of the four tense operators, i.e., any sequence of tense operators, no matter its length, reduces to one of these 15 basic tenses [Prior, 1967, Chapter 3].[11]

Similarly to the case of the rationals, tense-logical characterizations exist for the natural numbers, the integers, and the reals. The relevant property of the reals, over and above linearity, density, and no endpoints, is Dedekind continuity, i.e., the absence of 'gaps' in the ordering. (Dedekind continuity is a second-order notion.) If a formula $\phi$ is first true and then false, then there has to be either a last moment at which it is true, or a first moment at which it is false. That property is characterized by the axiom

$$(F\phi \wedge FG\neg\phi) \rightarrow F(HF\phi \wedge G\neg\phi) \qquad \text{(DEDEKIND)}$$

Indeed, this formula is refutable in the rationals, which has gaps at irrational numbers like $\sqrt{2}$. Let $\phi$ be true for all $m \in \mathbb{Q}$ for which $m < \sqrt{2}$ and false for all $m$ with $m > \sqrt{2}$ (so a truth value has been defined for all $m \in \mathbb{Q}$). Then at 0, the

antecedent is true ($\phi$ will be true, e.g., at 1, and forever false from, e.g., 2 on), but the consequent is false, as one can check easily. In a Dedekind continuous ordering, on the other hand, this cannot happen; if the antecedent is true, then so is the consequent.

In order to characterize the natural numbers, one needs, over and above the presence of a first and the absence of a last moment, an axiom guaranteeing that between any two moments there is only a finite number of moments. This splits in two parts, the condition of future finite intervals

$$G(G\phi \rightarrow \phi) \rightarrow (FG\phi \rightarrow G\phi) \qquad \text{(FFIN)}$$

and its temporal mirror image,

$$H(H\phi \rightarrow \phi) \rightarrow (PH\phi \rightarrow H\phi). \qquad \text{(PFIN)}$$

In order to refute (FFIN), one needs a moment at which $G(G\phi \rightarrow \phi)$ and $FG\phi$ are true but $G\phi$ is false. The latter means that there has to be at least one future moment at which $\phi$ is false. Given the truth of $FG\phi$, on a frame with finite future intervals there has to be a *last* such moment; at that moment, $G\phi$ holds. But then by $G(G\phi \rightarrow \phi)$, $\phi$ has to be *true* at that moment, a contradiction. Thus (FFIN) holds on all frames with finite future intervals; and similarly for (PFIN).

Summing up, the following extensions of $\mathbf{K}_t$ characterize the common number systems in the sense that they provide sound and strongly complete axiomatizations of the respective frames (see, e.g., [Segerberg, 1971] and [Hodkinson and Reynolds, 2007]):

- The natural numbers ($\mathbb{N}$): (LIN), (FIRSTMO), (NOLAST), (PFIN), (FFIN).
- The integers ($\mathbb{Z}$): (LIN), (NOFIRST), (NOLAST), (PFIN), (FFIN).
- The rationals ($\mathbb{Q}$): (LIN), (NOFIRST), (NOLAST), (DENSE).
- The reals ($\mathbb{R}$): (LIN), (NOFIRST), (NOLAST), (DENSE), (DEDEKIND).

**Some Difficulties:** The discussion of the correspondence between temporal formulae and simple frame conditions so far may suggest that this is all smooth sailing. This would however be too optimistic. First, there is no guarantee of completeness – in fact, the first example of an incomplete modal logic, given by [Thomason, 1972], was tense-logical. Second, we have already discussed the fact that irreflexivity cannot be captured by a temporal formula – and in fact, Priorean tense logic is severely limited in its expressiveness. A systematic reason for this limitation can be seen from the interrelation between tense logic and first-order logic.



**General Remarks on Correspondence:** The metalogical properties of tense logic can be best understood by relating tense logic explicitly to first-order logic. Indeed the semantic clauses given above can be used to establish a direct correspondence via the so-called *standard translation*, which associates a monadic first-order formula $ST_x(\phi)$ with one free variable $x$ with each tense-logical formula $\phi$. Corresponding to every atomic proposition $p, q, r, \ldots$, the first-order language has a one-place predicate $P, Q, R, \ldots$:

- If $\phi = p$, $p$ atomic: $ST_x(\phi) = P(x)$.
- If $\phi = \neg\psi$: $ST_x(\phi) = \neg ST_x(\psi)$.
- If $\phi = \psi_1 \wedge \psi_2$: $ST_x(\phi) = ST_x(\psi_1) \wedge ST_x(\psi_2)$.
- If $\phi = H\psi$: $ST_x(\phi) = \forall y\,(y < x \rightarrow ST_y(\psi))$, where $y$ is a new variable.
- If $\phi = G\psi$: $ST_x(\phi) = \forall y\,(x < y \rightarrow ST_y(\psi))$, where $y$ is a new variable.

This translation establishes a link between the 'local' tense-logical perspective of evaluating formulae at moments, with formulae that are true or false *at* moments, and the 'global' first-order perspective on a relational structure (the model), with open formulae that are true or false *of* the elements of the structure. This shift of perspective allows one to bring to bear many tools of standard model theory and algebraic semantics.

The two most important observations resulting from this shift of perspective are:

- In the standard translation, one only really needs *two* different variables. Priorean tense logic is therefore contained in the two-variable fragment of first-order logic. This explains its expressive limitations, on the one hand, and its decidability, on the other.
- Completeness for a class of frames is a *second-order* notion, as it forces one to quantify over relational structures. This explains the incompleteness of many temporal logics.

This perspective is developed nicely in [Blackburn et al., 2002].

The technical considerations mentioned in this section all presuppose the adequacy of the model-theoretic semantics for tense logic laid out above. As mentioned, Prior had qualms about the model-theoretic presentation for metaphysical reasons. But he also raised important doubts about the semantics of the F operator, which led to a number of alternative semantics.

### 2.1.3 The Semantics of the Future Operator

If the future operator is to mimic the use of the future tense in natural language, or to provide a basis for discussions in the philosophy of time, the symmetrical

nature of the interaction axioms of $K_t$ may strike one as inappropriate. Consider first the axiom

$$\phi \rightarrow GP\phi.$$

The validity of this formula follows directly from the semantics of $K_t$: if $\phi$ is true at $m$, then either there is no $m' > m$, in which case the consequent is vacuously true, or one picks some $m' > m$, in which case $m$ itself provides the sought-for moment in the past of $m'$ at which $\phi$ is true. And metaphysically speaking, there seems to be nothing wrong with the following paraphrase according to the suggestions of Section 1.1: if $\phi$ is true now, it is always going to be the case that it was the case that $\phi$ is true. Intuitively, we think of the present as well as the past as fixed and unalterable, and the formula seems to express just such a sentiment. But what about the temporal mirror image,

$$\phi \rightarrow HF\phi?$$

The validity of this formula can be established in exactly the same way as for its mirror image: given that $\phi$ is true at $m$, either there is no $m' < m$ (so that the consequent $HF\phi$ is vacuously true), or one chooses some such $m' < m$, for which $m > m'$ then witnesses the truth of the consequent. But what about the corresponding paraphrase: if $\phi$ is true now, it has always been the case that it will be the case that $\phi$ is true? This seems to suggest the dubious view that whatever is so now, was going to be so anyway. And that is not in accord with our view of the world, in which we acknowledge indeterminism of various forms.[12] The point here is not that we have to assume indeterminism and therefore the formula is inadequate. Indeterminism is a strong metaphysical thesis, and it should be open to discussion. The point is just that the formula seems to make such a discussion impossible.

But *is* the formula really metaphysically inappropriate? Is the paraphrase appropriate at all? *Does* F express something close to the future tense – and not merely something like 'it may be the case that'? The point is important and subtle. Since the formula is simply a validity, there are two ways to go: either to adhere to the interpretation of F as 'it will be the case that' and to show that the formula is not in fact philosophically inappropriate (or at least, not inappropriate in certain important cases), or to interpret F differently and to give a different semantics for a more adequate future operator.

As to the first option, once one restricts attention to linear frames, the problems vanish: there is no branching, just a single linear chain of future moments. Both interaction axioms have exactly the same status, there is no remaining philosophical issue. Linear frames are important in many applications, and they are especially well studied in computer science. Are non-linear, branching frames ever needed? There is certainly room for debate. However, one fairly



obvious philosophical argument will not do: the fact that we experience the flow of time as linear, and that we give names (viz., dates and times) to moments in a linear fashion, proves nothing. It is simply a conceptual truth that we experience one moment at a time, not multiple incompatible moments at once.

In the absence of a conclusive argument for restricting attention to linear frames describing only a single possible future (thus, a deterministic flow of time), tense logic should be open towards other flows of time – and especially towards ones that incorporate the idea of an open future. The struggle for a suitable semantics for the future operator occupied a central place in the early development of tense logic. Prior, who was also interested in questions of freedom and predestination, rejected the basic semantics for $K_t$ given above. Indeed his first approach to the topic ([Prior, 1957]) resulted in a fairly complicated predicate logical system in which reference to nonexistent entities was a major issue. (The problem is still there to haunt temporal predicate logic.)

In the propositional case that we are dealing with here, there are two main options for alternative semantics, which are known as 'Peircean' and 'Ockhamist', after the nineteenth-century American philosopher C. S. Peirce and the medieval logician William of Ockham whom Prior ([Prior, 1967]) cited as sources of inspiration. Both alternative semantics for F are based on frames that are backwards linear (left linear), so that the past operator P, based on a linear past, poses no additional difficulties.

In a backwards linear frame $(T, <)$, also called *branching time*, one can single out the set *Hist* of maximal linear subsets (maximal chains) of $T$. These maximal linear subsets are called *histories*, the idea being that a maximal chain pictures a complete possible course of events: a possible history of the world. The set of all histories containing a moment $m$, which is a subset of *Hist*, is written $H_m$; this set may be viewed as the set of all possible futures of $m$ complemented by the unique past of $m$ and $m$ itself. Single histories are denoted $h, h'$ etc. Histories $h, h'$ are called *undivided* at a moment $m$ ($h \equiv_m h'$) iff they share a moment $m'$ that is properly later than that moment ($m' > m$). On plausible topological conditions on $(T, <)$, $\equiv_m$ is an equivalence relation inducing a partition of $H_m$ that may be thought of as the set of immediate future possibilities at $m$.

From backwards linearity it follows that the set of histories gets smaller as time progresses: if $m < m'$, then $H_m \supseteq H_{m'}$. This accords well with our intuitions about the flow of time excluding ever more of the once open possibilities; witness, e.g., the notion of a missed opportunity, or the wisdom in 'no use crying over spilt milk'. The Peircean and the Ockhamist semantics for F both make use of the notion of histories in branching time models, but they differ as to the role that histories play.[13]

**Peircean F:** The discussion of the controversial interaction axiom $\phi \rightarrow HF\phi$ also reveals another counterintuitive feature of the $K_t$ semantics for F: too many

future-tensed sentences turn out to be true. If $\phi$ will be true in one history from $H_m$ and $\neg\phi$ in another, then both $\mathsf{F}\phi$ and $\mathsf{F}\neg\phi$ count as true at $m$ – even if $\phi$ explicitly pertains to a specific date. This clashes with our idea that at a certain date, things will be one way or other, but not both ways. How then could both $\mathsf{F}\phi$ and $\mathsf{F}\neg\phi$ be true? The Peircean idea is to make many fewer future-tensed sentences true by considering a future-tensed sentence $\mathsf{F}_{\text{Peirce}}\phi$ to be true exactly if $\phi$ will be true *no matter what* – i.e., if it will be true in all (currently still) possible histories. Formally:[14]

- If $\phi = \mathsf{F}_{\text{Peirce}}\psi$: $\mathcal{M}, m \models \phi$ iff for all $h \in H_m$ there is some $m' \in h$ s.t. $m \prec m'$ and $\mathcal{M}, m' \models \psi$.

While this move resolves the problem of inconsistent truths about the future, it makes nonsense of many of our other ideas about the future. E.g., when I bet that it will rain tomorrow, I am not saying that there is no chance that it won't rain – I am just claiming that it is *actually* going to rain. Indeed, in betting, one of the main points seems to be that both parties agree that the outcome is contingent, not fixed one way or other – that's part of the fun! But if I say that it will rain tomorrow, and I acknowledge that it could fail to rain, then on the Peircean semantics I am contradicting myself: for a future-tensed sentence to be true, there cannot be a possible history on which it is false. It seems that too few future-tensed sentences are true on the Peircean reading.

**Ockhamist F:** The Ockhamist semantics was sketched by Prior ([Prior, 1967]) and developed fully by Thomason ([Thomason, 1970], [Thomason, 1984]). It amounts to a substantial revision of the standard semantics. The basic idea is to evaluate a formula not just at a moment, but at a moment/history pair $m/h$ (where $m \in h$).[15] As histories are linear by definition, the past and future tense operators are again symmetrical, like in $\mathbf{K}_t$:

- If $\phi = \mathsf{H}\psi$: $\mathcal{M}, m/h \models \phi$ iff for all $m' \in h$ s.t. $m' \prec m$, we have $\mathcal{M}, m' \models \psi$.
- If $\phi = \mathsf{G}\psi$: $\mathcal{M}, m/h \models \phi$ iff for all $m' \in h$ s.t. $m \prec m'$, we have $\mathcal{M}, m' \models \psi$.

In the clause for H, the requirement that $m' \in h$ is in fact superfluous since $T$ is backwards linear anyway – but in the clause for G, the requirement does some useful work. The derived semantic clause for F (remember that $\mathsf{F}\phi \Leftrightarrow \neg\mathsf{G}\neg\phi$),

- if $\phi = \mathsf{F}\psi$: $\mathcal{M}, m/h \models \phi$ iff there is some $m' \in h$ s.t. $m \prec m'$ and $\mathcal{M}, m' \models \psi$,

shows that the problem of inconsistent futures for the same date vanishes: histories, being linear, are consistent. Furthermore, it becomes possible to combine



tense and modality by defining the so-called *historical modalities* 'Poss' and 'Sett' (for 'possibly' and 'it is settled that'):

- If $\phi = \mathsf{Sett}\,\psi$: $\mathcal{M}, m/h \models \phi$ iff for all $h' \in H_m$, $\mathcal{M}, m/h' \models \psi$.
- If $\phi = \mathsf{Poss}\,\psi$: $\mathcal{M}, m/h \models \phi$ iff there is some $h' \in H_m$ s.t. $\mathcal{M}, m/h' \models \psi$.

As can be seen from these semantic clauses, Poss is the dual of Sett, and could as well be introduced as an abbreviation ($\mathsf{Poss}\,\phi :\Leftrightarrow \neg\mathsf{Sett}\neg\phi$).

On the Ockhamist semantics, $\phi \rightarrow \mathsf{HF}\phi$ is valid, but that doesn't cause any interpretational difficulties anymore, since a (linear) history has to be specified. The deterministic sentiment connected with that formula in basic tense logic is now expressed more clearly as $\phi \rightarrow \mathsf{HSett}\,\mathsf{F}\phi$, and that formula is not a validity.

Given the historical modalities, it is possible to define the Peircean F in the Ockhamist framework:

$$\mathsf{F}_{\text{Peirce}}\phi \quad :\Leftrightarrow \quad \mathsf{Sett}\,\mathsf{F}\phi.$$

So, if we do want to talk about inevitable future happenings in the Ockhamist framework, we can do that – but the framework is vastly more expressive. Does the Ockhamist solution therefore deliver *the* adequate semantics for the future tense? Opinions are divided. The main technical *and* philosophical issue is the question of the initialization of the $h$ parameter in a stand-alone sentence. In standard tense logic, there is one *mobile parameter*, $m$. This parameter is shifted by the tense operators. In a stand-alone sentence, uttered at a moment of context $m_C$, $m$ gets its initial value from that context: one starts evaluating with $m = m_C$. Now for a stand-alone sentence containing an Ockhamist future operator, we also need a history $h$ to evaluate its truth value – but where do we get that history from? One idea is that the context also furnishes a 'history of the context': *the* real future. But arguably that will lead to a collapse of branching time to linear time: if there is a unique real future, isn't the rest of the model just sugaring? (See [Belnap et al., 2001] for discussion, and [Belnap, 2007] for a detailed and systematic, general overview of parameters of truth.) Thomason ([Thomason, 1970]) suggests to use the technique of supervaluations to resolve the issue. In supervaluational semantics, uninitialized parameters are quantified over, and a truth value is assigned only if the verdict is uniform; otherwise, no truth value is assigned. Accordingly, if a stand-alone sentence is declared to be true, it is also settled true. In this respect, supervaluational semantics agrees with Peirceanism to a certain extent, but some of the unwelcome consequences of the semantics of $\mathsf{F}_{\text{Peirce}}$ are avoided – e.g., unlike in Peirceanism, $\mathsf{F}\phi \vee \mathsf{F}\neg\phi$ does turn out to be a validity (the disjunction is true on every history). Still there seems to be a difference between truth and settled truth that the supervaluational framework is unable to capture: the supervaluational framework is arguably unable to account for our assessment of bets. A better option

may therefore be to say that a future-tensed sentence can simply fail to have a truth value at its time of utterance, but later on (when more things have become settled) it will be true to say that at the time of utterance, the sentence was true (or false). That seems to accord better with the way we handle bets: once the outcome is settled, we assess the bet – not earlier. (See [Belnap, 2001] for technical details of such so-called double time references.) For some more comments on the formal analysis of tense in natural languages, see Section 3.1 below.

**Time and Modality:** We have seen that the Ockhamist framework affords a well-integrated notion of (historical) possibility and necessity based on branching time. For the record, there are also *purely temporal* interpretations of possibility and necessity, and these played an important role in the development of tense logic and in its application to historical sources. One idea is to call that necessary which is now and will always be:

$$\Box_D \phi :\Leftrightarrow (\phi \wedge G\phi).$$

This interpretation has been claimed to capture the notion of necessity in the writings of the Stoic philosopher Diodorus Cronus, and is accordingly called the Diodorean modality. Note that this modality also makes sense in linear time.

A similar notion of necessity that has been related to the works of Aristotle, the so-called Aristotelian modality, identifies necessity with truth at all times; in linear time:

$$\Box_A \phi :\Leftrightarrow (H\phi \wedge \phi \wedge G\phi).$$

Cf., e.g., [Rescher and Urquhart, 1971, Chapter 1] for some pointers to source texts.

### 2.1.4 The Indexical 'Now'

In English, words like 'I', 'you', 'here' and 'now' are special because their reference depends on the context of utterance. The semantical analysis of such indexical expressions went hand in hand with the development of tense logic, because the tenses themselves are indexical: past and future localize events relative to the present moment that 'now' refers to. Whether the sentence 'It was raining yesterday' is true or false depends on the weather on the day before the utterance is made, which may differ for different utterances of the same sentence. Can one get rid of this indexical context dependence? As mentioned, Frege suggested that an explicit date could always be substituted; Russell considered a token-reflexive analysis in which the content would refer back to the utterance act producing the sentence token. These approaches were however later seen to be inadequate vis-à-vis the phenomenon of so-called essential indexicality. Prior ([Prior, 1959]) gave the following example: when I say 'thank goodness that's



over', what I am thankful for is not that something is finished before a certain date (I knew *that* all along), nor that something is finished before a certain utterance of mine (what a weird thing to be thankful for). Rather, it seems to be the thing's being past, an indexical feature, that I am thankful for. (Cf., e.g., [Perry, 1977] for more on essential inexicality, and [Mellor, 1998] and [Oaklander and Smith, 1994] for a discussion of consequences for the philosophy of time.)

Given that indexicality won't go away, semantic analysis has to make room for it. The simple rule for 'now' is the following:

- If $\phi = \mathrm{Now}\ \psi$: $\mathcal{M}, m_C, m \models \phi$ iff $\mathcal{M}, m_C, m_C \models \psi$.

Thus, the 'Now' operator (sometimes also written 'J', from the German 'jetzt') simply shifts back the moment of evaluation $m$ to the moment of the context, $m_C$ (which has here been mentioned explicitly as a parameter of evaluation as well).

Does the presence of this operator enhance the expressive power of tense logic? Kamp ([Kamp, 1971]) gave a proof that 'Now' can in fact be eliminated from propositional tense logic – the relevant observation is that the operator is redundant at the beginning of a stand-alone sentence, and that there are transformations of propositional tense-logical formulae into a normal form that always puts 'Now' up front. The operator is however not eliminable in quantified tense logic, nor in the context of propositional attitudes (cf. [Burgess, 2002, pp. 32f.]).

The insights gained from the formal study of 'Now' have been generalized in various treatments of indexical semantics, sometimes known as 'two-dimensional semantics'; note that that term is used in a number of different ways.

### 2.1.5 Metric Operators and $At_t$

The P and F operators, Ockhamist or not, do not distinguish between the near and the far past, or future. We do, for obvious reasons – we want to be able say that it will rain tomorrow, or on New Year's Eve 2025; not just that it will rain sometime or other.

If time is discrete, as is presupposed in many computer science applications, then there is a natural way of defining a 'previous time' and a 'next time' operator (sometimes written 'Y' and 'X', for 'yesterday' and 'next'), and one can build up further operators from these. In general, however, there is no natural 'next time'. Still, metric tense operators have some popularity; they were extensively used by Prior ([Prior, 1967]). Such operators are often written $\mathsf{P}_x$ and $\mathsf{F}_x$, and there are suggestive combination axioms such as, e.g.,

$$\mathsf{F}_x\mathsf{F}_y\phi \quad \leftrightarrow \quad \mathsf{F}_{x+y}\phi,$$

sometimes with real numbers, sometimes with integers as index. In order to have a well-defined semantics for these operators, the underlying models need

to have an appropriate metric structure – e.g., all of the model's histories can be required to be isomorphic to the reals. Such structure can also be used to define '$At_t$' operators for indicating dates and times. In an Ockhamist framework, these operators will shift the moment of evaluation along the current history to the moment on that history that has the appropriate clock time. As one can see, it pays to distinguish the elements of the partial ordering $(T, <)$ from clock times: there can be many incompatible elements of $T$ at one and the same clock time, in different histories. Calling the elements of $T$ 'times' tends to blur this distinction, which is why we call the elements of $T$ 'moments'.

Formal details about '$At_t$' are given, e.g., in [Belnap et al., 2001, Chapter 8].

## 2.2 Temporal Logic with 'Since' and 'Until'

Hans Kamp not only contributed to the formal study of indexicals starting with 'now', but also introduced two more general operators into temporal logic: 'Since' (S) and 'Until' (U). Due to their greater expressive power these operators have found widespread application and are often taken as basic in advanced approaches to temporal logic. (Details of the semantics differ somewhat between the computer science and philosophy communities; we'll stick to the philosophy convention of [Kamp, 1968]. Cf. [Pnueli, 1977] for the origin of the other tradition.)

S and U are two-place operators; unlike Prior's connectives, they do not modify a single sentence, but combine two sentences in a temporally specific way, in analogy with conjunction. While one can also find them written in infix notation like conjunction, they are mostly written in prefix notation with explicit parentheses, and we adopt that convention. $S(\psi_1, \psi_2)$ means that there was a past moment at which $\psi_1$ was true, and that $\psi_2$ has been true since then; U is the temporal mirror image. Formally, the clauses are the following:

- If $\phi = S(\psi_1, \psi_2)$: $\mathcal{M}, m \models \phi$ iff there is some $m' \in T$ s.t. $m' < m$ for which $\mathcal{M}, m' \models \psi_1$, and for all $n$ for which $m' < n < m$, we have $\mathcal{M}, n \models \psi_2$.
- If $\phi = U(\psi_1, \psi_2)$: $\mathcal{M}, m \models \phi$ iff there is some $m' \in T$ s.t. $m < m'$ for which $\mathcal{M}, m' \models \psi_1$, and for all $n$ for which $m < n < m'$, we have $\mathcal{M}, n \models \psi_2$.

If the S and U connectives are present, the Priorean connectives can be defined easily:

- $P\phi :\Leftrightarrow S(\phi, \top)$;
- $F\phi :\Leftrightarrow U(\phi, \top)$.



Also the 'previous time' and 'next time' operators mentioned in Section 2.1.5 above are definable (obviously these definitions make good sense only if the underlying flow of time is discrete):

- $Y\phi :\Leftrightarrow S(\phi, \bot)$;
- $X\phi :\Leftrightarrow U(\phi, \bot)$.

Kamp introduced these connectives in his study of linear orders [Kamp, 1968]. It is customary to presuppose linear time (i.e., a model based on a linear frame) when discussing S and U. For the tense logic of Dedekind complete linear orders, Kamp showed that in fact any temporal operator that has a first-order truth definition can be defined on the basis of S and U. This important, early expressiveness result can be extended to linear orders generally, using an alternative set of connectives named after Jonathan Stavi (cf. [Gabbay et al., 1994, Chapter 6.3.9]).

The rich logic of S and U has been studied closely, especially with respect to computer science applications. For a detailed overview cf., e.g., [Finger et al., 2002].

In terms of viewing temporal logic as a fragment of first-order logic via the standard translation, it turns out that the tense logic with 'since' and 'until' is contained in the three-variable fragment. (A look at the semantic clauses already suggests as much.) As Priorean tense logic is already contained in the two-variable fragment, the greater expressive power of 'since' and 'until' (as well as of the Stavi connectives) can readily be accounted for. Since that greater expressive power is bought at the price of a rather modest increase in complexity, since-until-logics are used in many applications.

# 3. Some Further Topics

The overview of temporal logic given here is necessarily limited. We briefly look at a number of selected further topics.

## 3.1 Temporal Logic and Natural Languages

Is the tense-logical approach laid out here a useful approach to capturing the temporal phenomena of natural language? Certainly consideration of the English or Latin system of tenses has provided important motivation for the development of tense logic, but those logical systems also have an independent systematic interest – adequacy with respect to the linguistic facts is not necessarily crucial. Still it is interesting to ask how close the match really is between the formal systems and natural languages. (In what follows we stick to a discussion of English, which is sufficiently rich to point out the main issues.)

There are at least two ways in which one can understand the mentioned question. Is tense logic an adequate representation of the English verbal tenses? And if that is so, is tense logic also adequate vis-à-vis all the other temporal features of English?

The answer to the first question in fact depends on what one understands by 'tense'. Traditionally, the English simple past, past continuous, perfect, present tense, the 'will' future, etc., are all called tenses. Among linguists it has however become customary to limit the notion of tense to the specification of temporal location relative to the present. In this narrow sense, tense logic is indeed adequate as a formal means of expressing the English tenses – however, one could almost say that this holds by definition. Still even here there is room for debate. Reichenbach ([Reichenbach, 1947]), in an influential chapter on 'The tenses of verbs', proposed a scheme in which a fixed number of three moments – point of speech, point of reference, and point of event – are used to analyse all verbal tenses. In comparison, tense logic appears to be both too weak (it only allows for two moments to play a role in the analysis) and too strong (its recursive machinery allows tense operators to be iterated). Prior ([Prior, 1967, Ch. I.6]), on the other hand, while mentioning Reichenbach as one of the 'precursors of tense logic', criticizes the non-recursiveness of the Reichenbachian scheme.

Considering the wider sense of 'tense', it is clear that tense logic by itself is unable to express the difference between, e.g., the simple past ('I made tea') and the past continuous ('I was making tea'). Linguistically, this difference is considered to be not one of tense proper, but one of *aspect*: the simple past pictures a happening from the point of view of its completion (perfective aspect), while the past continuous pictures a happening as incomplete (imperfective aspect). Thus, an action that was never completed may be reported in the imperfective aspect ('I was making tea when the phone rang, I completely forgot about it and in fact never made tea'), but not in the perfective aspect.

Tense logic is not, and was not meant to be, a logic of aspect. For two rather different approaches to such a logic, see [Galton, 1984] and [van Lambalgen and Hamm, 2005]. At any rate, it is clear that there is more to the temporal structure of English – and most, if not all other languages – than what is captured by the formal framework of tense logic.

## 3.2 Tense Logic and Relativity Theory

Early on, tense logic was confronted with a scientific objection along the following lines: Tense logic, with its notion of past and future, is wedded to a Newtonian picture of time and space that has become obsolete through the advent of relativity theory. We have come to see that there isn't space and time separately, but just a space-time manifold that can be foliated (divided into space and time) in different ways, none of which is physically distinguished. Your rest frame is as good as mine. So, why bother developing a tense logic at all [Massey, 1969]?

One reaction to this worry is to develop a tense logic that is based on the Minkowskian partial ordering of causal connectability in special relativistic space-time. In that ordering, which is branching to the past as well as to the future, incomparability is not a sign of incompatibility like in branching time, but of space-like relatedness, or causal isolatedness. A number of interesting results in this area are surveyed by [Uckelman and Uckelman, 2007].

While this body of work manages to reconcile temporal logic and relativity in some fashion, one may wonder whether it isn't possible to establish a *purely temporal* tense logic before a relativistic background. This is still an area of debate, especially among those who want logic to have some metaphysical import. One option is to argue for a physically distinguished rest frame after all (e.g., based on some notion of cosmic time), which would single out a preferred foliation and thus a preferred time coordinate; another option is to work on the assumption of a *metaphysically* distinguished time coordinate. Rakić ([Rakić, 1997]) has shown that this move is logically unobjectionable; both options would allow one to stay within the standard Priorean paradigm. A further option is to extend tense logic to cover relativistic space-time in another way: somewhat parallel to the Ockhamist move of introducing a history as an additional parameter in the semantic clauses, one can include a reference frame as a further mobile parameter of truth. This move is unproblematic in so far as a given utterance context supplies a reference frame of the context, viz., the speaker's rest frame.

Even though Minkowski space-time is only partially ordered, the discussion so far parallels the case of linear time. A semantic parallel to branching time, *branching space-times*, has been developed by Belnap [Belnap, 1992]. The logical properties of branching space-times are the subject of ongoing research; cf., e.g., [Müller, tab] for pointers to the recent literature.

## 3.3 Temporal Predicate Logic

The extension of a propositional modal logic to a system of quantified modal logic is always an intricate matter. In fact, many choices need to be made – cf., e.g., [Garson, 2006] for a broad, philosophically oriented overview, and Chapter 11 for a succinct presentation of axiomatic and interpretational issues. If anything, the construction of a system of quantified *tense* logic is conceptually even more difficult than in the case of alethic modal logic. This may to a large part be due to the fact that we have fairly firm but not completely coherent intuitions about things and their persistence through time.

Many questions need to be answered. What are we talking about? What should the universe of discourse for a temporal predicate logic look like? Should it contain ordinary individuals like you and me, which, as we all know, come

to be and pass away, leading to different domains at different moments and the difficult question of what to do with names of mere have-beens and mere will-bes? Or should we try to capture (maybe as a first step) a temporal predicate logic of some eternal and perhaps ultimately simple objects? And if so, what would these be? Anyone who has tried to come to grips with atomistic philosophies like Wittgenstein's *Tractatus* knows that these are deep and troubling questions. The issue may look 'merely metaphysical', but it has obvious and far-reaching consequences for the logical formalism. Logical modelling and metaphysical argument tend to go hand in hand here, again substantiating the claim made at the beginning, viz., that tense logic is a philosophically especially rich subject.

A key question revolves around the so-called temporal Barcan formula,[16]

$$\forall x\, G\phi(x) \rightarrow G\forall x\, \phi(x),$$

or equivalently,

$$F\exists x\, \psi(x) \rightarrow \exists x\, F\psi(x).$$

While this formula helps one to simplify the semantics, it seems intuitively questionable: If it will be the case that there is someone who is flying to Mars, does this imply that there is someone for whom it will be the case that she is flying to Mars?

It seems safe to say that no consensus has been reached with respect to predicate tense logic, and a pragmatic attitude towards the construction of such systems may often be the most satisfactory choice.

## 3.4 Branching Time and the Logic of Agency

The tempo-modal framework of Ockhamist branching time has been used as a basis for logics of agency, again bringing formal and philosophical issues very closely together. The idea behind such logics is to view truth due to agency as a modus of a proposition, an idea that can be traced back to Anselm of Canterbury. In *stit* logics, developed by Belnap and collaborators, a branching time model is enriched with agent's choices at moments: Given a set $A$ of agents, for each agent $\alpha \in A$ and each moment $m \in T$ there is a partition $Choice^{\alpha}_{m}$ of $H_m$ that respects undividedness at $m$ ('no choice between undivided histories') and that forms the basis for the semantics of the *stit* ('seeing to it that') connective. $Choice^{\alpha}_{m}(h)$ indicates that element of the partition that contains $h$ (presupposing $h \in H_m$). The semantic clause then is as follows:

- If $\phi = [\alpha\ \mathrm{Stit}]\psi$: $\mathcal{M}, m/h \models \phi$ iff
  (1) for all $h' \in Choice^{\alpha}_{m}(h)$, $\mathcal{M}, m/h' \models \psi$; and
  (2) there is some $h^* \in H_m$ for which $\mathcal{M}, m/h^* \not\models \psi$.



Here (1) is a positive condition requiring *effectiveness* of the choice: the embedded sentence $\psi$ has to be true on all histories belonging to the choice under consideration (the choice has to secure the outcome, so to speak). The negative condition (2) ensures that, e.g., nobody sees to it that $2 + 2 = 4$: agency, on that view, presupposes contingency of the outcome. Horty ([Horty, 2001]) bases his development of a *stit* logic on a connective without condition (2); this move is popular in computer science applications. Cf., e.g., [Broersen, 2009]; cf. [Belnap et al., 2001] for a philosophy-centered overview.

## 4. Suggested Further Reading

The classic of the field is [Prior, 1967], which contains a wealth of background material and philosophical discussions. A more modern and more mathematically oriented treatment is given by [van Benthem, 1991]. [Belnap et al., 2001] has a strong focus on philosophical issues of branching time. The computer science oriented development of temporal logic is discussed in depth in [Gabbay et al., 1994, Gabbay et al., 2000].

## Notes

1. For the terminology of modal systems as well as for more on Lewis' motivation cf. Chapter 11.
2. In fact, Prior and other early tense logicians used Polish prefix notation, in which one-letter connectives are always put *before* their arguments, making parentheses and special symbols unnecessary. The connectives are written $N$ ('negation'), $C$ ('conditional'), $K$ (think 'konjunction'), $A$ ('alternation', or disjunction), and $E$ ('equivalence'). Tense operators are as stated; alethic modal operators are often symbolized as $M$ (for the weak modality, possibility, 'möglich') and $L$ (necessity). Handling that notation requires a little training. (As an exercise, you may wish to translate the formula $CACCKppqpNKpqq$ into standard infix notation, or transcribe some of the proofs in Prior's original writings.)
3. We give the official semantic rules for the *strong* operators here, treating the weak operators as defined. This way we avoid having to discuss a fine point about axiomatization later; cf. [Blackburn et al., 2002, p. 34].
4. As in the propositional calculus, the use of axiom schemata is thus just a small convenience. In a similar vein, many formal developments of propositional modal or tense logic do not hesitate to simply add the whole set of propositional tautologies to the axiom set. The important point to keep in mind is that such moves are just a convenience – in contradistinction to, e.g., the axiomatization of Peano arithmetic in first-order logic, where an axiom schema is needed essentially. First-order Peano arithmetic is *not* finitely axiomatizable; the logics we are dealing with here are finitely axiomatizable. (Thus in fact, a single axiom, the conjunction of the separate axioms, would be sufficient.)
5. In fact, if one views temporal logic as a bimodal logic based on two *independent* accessibility relations, these two axioms are strong enough to force the accessibility relations to be each other's converse.

6. A tableaux-based proof is given, e.g., in [Rescher and Urquhart, 1971, Chapter VI].
7. It is possible to tackle irreflexivity syntactically at the level of *rules*: The so-called Gabbay rule, from an unconditional proof of $\neg q \wedge H q \rightarrow \phi$ to infer $\phi$ (where $q$ is an atomic proposition not occurring in $\phi$), is valid for $\mathbf{K}_t$. While this adds no new theorems in $\mathbf{K}_t$, it forces irreflexivity for extensions of $\mathbf{K}_t$. Cf., e.g., [Gabbay et al., 1994, Chapter 3.2].
8. As in modal logic (cf. Chapter 11) it is customary to specify names of tense-logical systems by concatenating the names of the characteristic axioms.
9. This assertion has to be taken with a grain of salt. Certainly every linear frame is both backward and forward linear. But full trichotomy also requires that there be only *one* line – any two points have to be comparable – while both backward and forward linearity holds, e.g., if there are two unconnected lines. In general, modal formulae cannot 'see' disjoint unions. Cf. [Blackburn et al., 2002, Ch. 3] for a survey of this and similar results leading to the celebrated Goldblatt–Thomason theorem.
10. Due to the 'local' nature of the tense-logical modalities, global characteristics like the cardinality of such a flow are beyond the power of tense-logical characterization. To give a very simple example, no tense-logical characterization can distinguish between a single copy of the rationals and any number of disjoint copies (where that number could be any infinite cardinal). See also note 9.
11. From here it is a short step to speculations that the limited amount of explicit tenses in English may be due to some such collapsing property.
12. To cite Prior on the matter: 'Before God, or whoever it is that is responsible for these things, had decided to make parrots, "There are going to be parrots" wasn't true at all' [Prior, 1976, p. 128] .
13. The philosophy-driven discussion given in what follows is parallelled in computer science by a discussion of the relative merits of (Peircean) 'computational tree logic' CTL and its Ockhamist alternative, CTL*. We will keep the discussion at the philosophical level, referring the computer-science interested reader instead to, e.g., [Gabbay et al., 2000, Chapter 3]. Cf. also [Øhrstrøm and Hasle, 1995] for more historical background information.
14. A simplification is possible in discrete time: One can simply do away with the F operator, leave the semantics for G as in $\mathbf{K}_t$ above, and give a specific semantics to the 'next time' modality X (see Section 2.1.5): instead of the F-analogue existential quantification over immediate successors, one can read X as 'inevitably, at the next time step', employing universal quantification over immediate successors. This move avoids the second-order aspects of quantification over branches, and thus leads to a much more tractable logic. Cf. [Gabbay et al., 2000, p. 65]. At any rate, the interpretational problems mentioned below remain.
15. The question arises whether the valuation $V$ should remain, like in standard tense logic, a function from the atomic propositions to the powerset of $T$. Prior ([Prior, 1967, p. 123]) discusses the issue; Thomason ([Thomason, 2002, p. 214]) gives an argument to the effect that $V$ should map the atomic propositions to subsets of $T \times \mathit{Hist}$ instead: this way, all instances of the future necessitation rule could be preserved. Lacking a full axiomatic framework for Ockhamist tense logic, we leave this matter open.
16. Interesting questions can also be asked with respect to the formula's converse; again, see Chapter 11 for more information.