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Solution Concepts for C–Convex, Assignment, and M2–Games

by

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Abstract

In this paper a new class of cooperative transferable utility games, called c–convex games, is introduced. The structure of the least core of c–convex games is shown to be similar to the structure of the core of convex games. Indeed, the extremal points of the least core are determined by certain (P,Q)–tight sequences of coalitions. Both, minima of two additive games and two–sided assignment games are c–concave. Moreover, it is proved that the modified least core of these particular c–concave games is contained in the classical least core of the dual game. The modified least core is a new solution concept, which takes into account both the "power" – i.e. the worth – and the "blocking power" of a coalition – i.e. the amount which the coalition cannot be prevented from by the complement coalition.

0. Introduction

This paper is organized as follows:

In Section 1 several definitions of cooperative game theory are recalled and some necessary notation is introduced. Moreover, two solution concepts are described, which are, in some sense, related to the prenucleolus and the least core. The modified nucleolus successively minimizes highest differences of excesses – the classical prenucleolus successively minimizes highest excesses, whereas the modified least core minimizes the highest difference of excesses – the classical least core minimizes the highest excess. The modified nucleolus is a singleton contained in the modified least core, which is a convex compact polyhedron as shown in [10]. Both modified solutions satisfy duality, i.e. coincide for the game and its dual. Next the class of complementary convex (c–convex) games is introduced. A game is c–convex w.r.t. (P,Q), if certain games (restrictions on the two parts P and Q of a partition of the player set) are convex and one "concavity" condition is satisfied. For the precise formulation Definition 1.4 (i) is referred to. Finally it is shown that weighted majority games are generically not c–convex and both, two–sided assignment games and minima of two additive games (M2–games), are c–concave, meaning their duals are c–convex.

Section 2 presents a characterization of all extreme points of the least core for c–convex games. Any extreme point generates a certain (P,Q)–tight sequence of coalitions and, conversely, each (P,Q)–tight sequence generates either a unique extreme point or a unique point being no member of the least core at all.

In Section 3 it is proved that the modified least core of both, assignment games and M2–games, is contained in the classical least core of the corresponding dual game. As a consequence both sides of an assignment game are treated equally by every preimputation of the modified least core.

1. Notation, Definitions, and Preliminary Results

A **cooperative game with transferable utility** – a **game** – is a pair (N,v), where N is a finite nonvoid set and

$$v : 2^N \to \mathbb{R},\ v(\emptyset) = 0$$

is a mapping. Here $2^N = \{S \mid S \subseteq N\}$ denotes the set of **coalitions** and v is the **coalitional function** of (N,v). Since the nature of the **player set** N is determined by the coalitional function, v is called game as well.

The **dual** game (N, v*) of v is given by

$$v^*(S) = v(N) - v(N \backslash S).$$

The set of **preimputations of** (N,v) is denoted

$$X(N,v) := X(v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\},$$

where $$x(S) := \sum_{i \in S} x_i \text{ for } S \subseteq N,\ x \in \mathbb{R}^N.$$

For $x \in \mathbb{R}^N$, $S \subseteq N$ the **excess of** S **at** x (w.r.t. v) is the real number

$$e(S,x,v) := v(S) - x(S).$$

Moreover, let

$$\mu_0(x,v) := \max \{e(S,x,v) \mid \emptyset \neq S \subsetneq N\}$$

denote the **maximal nontrivial** excess at x.

The **least core** of v is the set

$$\mathscr{LC}(v) = \{x \in X(v) \mid e(S,x,v) \leq \mu_0(y,v) \text{ for } y \in X(v),\ \emptyset \neq S \subsetneq N\}.$$

The least core of v is a nonvoid convex polytope containing the prenucleolus (see, e.g., [3]). Recall that the **prenucleolus** of v is defined to be

$$\mathscr{PN}(v) := \{x \in X(v) \mid \vartheta(x,v) \underset{lex}{\leq} \vartheta(y,v) \text{ for } y \in X(v)\},$$

where $\vartheta(x,v) = (e(S,x,v))_{S \subseteq N}$ is the vector of excesses in a nonincreasing order. The prenucleolus of v is a singleton (see [6]) and its unique element is abbreviated by $\nu(v)$. Maschler, Peleg, and Shapley ([3]) tried to give an intuitive meaning to the prenucleolus by regarding the excess of a coalition as a measure of dissatisfaction which should be minimized. Indeed, the prenucleolus can be reached by minimizing the highest excess, then minimizing the number of coalitions attaining highest excess, then minimizing the second highest excess, and so on.

Instead of considering the values of excesses as measures of dissatisfaction it is also natural to try to treat coalitions equally w.r.t. excesses as far as this is possible.
This leads to a procedure in which the values of excesses are replaced by the values of differences of excesses. A preimputation belongs to the **modified nucleolus** $\Psi(v)$ of a game v if it successively minimizes highest differences of excesses and numbers of pairs of coalitions attaining these differences. The modified least core arises from the modified nucleolus in the same way as the least core arises from the prenucleolus; by only proceeding along the first step of the minimizing procedure. The formal notation is contents of

Definition 1.1: Let (N,v) be a game and $x \in \mathbb{R}^N$. Let

$$\Theta(x,v) = (e(S,x,v) - e(T,x,v))_{(S,T) \in 2^N \times 2^N}$$

denote the vector of differences of excesses in a nonincreasing order. Then

$$\Psi(v) = \{x \in X(v) \mid \Theta(x,v) \underset{lex}{\leq} \Theta(y,v) \text{ for } y \in X(v)\}$$

is the **modified nucleolus** of v, whereas

$$\mathcal{MLC}(v) = \{x \in X(v) \mid e(S,x,v) - e(T,x,v) \leq \tilde{\mu}(y,v) \text{ for } y \in X(v),\ S,T \subseteq N\}$$

is the **modified least core of** v (here $\tilde{\mu}(y,v)$ denotes the maximal difference of excesses of v at y).

Let $\mu(x,v) = \max\{e(S,x,v) \mid S \subseteq N\}$ denote the maximal excess of v at x.

Remark 1.2:

(i) In Definition 1.1 $\Theta(x,v)$ can be replaced by the nonincreasing vector

$$(e(S,x,v) + e(T,x,v^*))_{(S,T) \in 2^N \times 2^N}$$

of sums of excesses w.r.t. v and v*. Hence the modified least core can be rewritten as

$$\mathcal{MLC}(v) = \{x \in X(v) \mid \mu(x,v) + \mu(x,v^*) \leq \mu(y,v) + \mu(y,v^*) \text{ for } y \in X(v)\}.$$

The modified least core of v consists of all preimputations minimizing the sum of maximal excesses w.r.t. v and v*.

(ii) In the definition of the least core only nontrivial coalitions S (i.e. $S \neq \emptyset, N$) play a role. Analogously the modified least core remains unchanged if only sums of excesses of **pairs** of nontrivial coalitions (S,T), i.e. $\{S,T\} \not\subseteq \{\emptyset,N\}$ are considered. This can easily be verified by observing that $\mu(x,v) = \mu(x,v^*)$ $(= 0)$ can only hold for some $x \in X(v)$, if v is **inessential (additive)**, i.e. if there exists a vector $m \in \mathbb{R}^N$ (take m = x in this case) such that v(S) = m(S).

(iii) Trivially the modified nucleolus is contained in the modified least core by definition. Moreover the modified nucleolus is a singleton (see [10]). The unique point $\psi(v)$ of $\Psi(v)$ is again called **modified nucleolus** (point).

A finite nonvoid set $X \subseteq \mathbb{R}^N$ is **weakly balanced (balanced)**, if X possesses a vector of **weakly balancing (balancing) coefficients** $(\delta_x)_{x \in X}$, i.e.

$$\sum_{x \in X} \delta_x x = 1_N \text{ and } \delta_x \geq 0 \ (\delta_x > 0) \text{ for } x \in X.$$

Here 1_S is the indicator function of S, considered as vector of $\mathbb{R}^N$. A nonvoid subset of D of coalitions or $\tilde{D}$ of pairs of coalitions is (weakly) **balanced** if

$$\{1_S \mid S \in D\} \quad \text{or} \quad \{1_S + 1_T \mid (S,T) \in \tilde{D}\} \text{ respectively}$$

is (weakly) balanced. For $x \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$ define

$$D(x,\alpha,v) = \{S \subseteq N \mid e(S,x,v) \geq \alpha\},$$
$$\tilde{D}(x,\alpha,v) = \{(S,T) \in 2^N \times 2^N \mid e(S,x,v) + e(T,x,v^*) \geq \alpha\}.$$

Lemma 1.3: Let (N,v) be a game, $\alpha \in \mathbb{R}$, and $x \in X(v)$.

(i) $x = \nu(v)$ iff each nonvoid $D(x,\alpha,v)$ is balanced.

(ii) $x \in \mathscr{LC}(v)$ iff $D(x,\mu_0(x,v),v)$ is weakly balanced or empty (i.e. $|N| = 1$).

(iii) $x = \Psi(v)$ iff each nonvoid $\tilde{D}(x,\alpha,v)$ is balanced.

(iv) $x \in \mathscr{MLC}(v)$ iff $\tilde{D}(x,\mu(x,v) + \mu(x,v^*),v)$ is weakly balanced.

For a proof of assertions (i) and (ii) Kohlberg ([1]) is referred to, whereas assertion (iii) and (iv) of Lemma 1.3 are proved in [10] .

We proceed by introducing some special classes of games.

Definition 1.4: Let (N,v) be a game and (P,Q) be a partition of N – i.e. P + Q = N (where A + B := A ∪ B iff A,B are disjoint sets).

(i) (N,v) is **complementary convex (c–convex) w.r.t.** (P,Q) iff

$$v(S) + v(T) \leq v((S \cap T)_P + (S \cup T)_Q) + v((S \cup T)_P + (S \cap T)_Q), \quad (1)$$

(where $S_R = S \cap R$ for S, $R \subseteq N$) for S, $T \subseteq N$.

(ii) (N,v) is an **assignment game** (w.r.t. (P,Q)) if there is a nonnegative $P \times Q$ matrix A such that

$$v(S) = \max \{ \sum_{i \in S_P} \sum_{j \in S_Q} a_{ij} x_{ij} \mid \sum_{i \in S_P} x_{ij} \leq 1 \geq \sum_{j \in S_Q} x_{ij}, x_{ij} \geq 0 \text{ for } i \in S_P, j \in S_Q \}$$

for $S \subseteq N$. (Note that x_{ij} can be chosen to be 0 or 1 by [8].)

(iii) (N,v) is a **minimum of two additive games** – an **M2–game**, if there are $m^1, m^2 \in \mathbb{R}^N$ such that

$$v(S) = \min \{m^1(S), m^2(S)\} \text{ for } S \subseteq N.$$

Remark 1.5:

(i) A game (N,v) is c–convex w.r.t. every partition (P,Q) of N, iff v is additive. Indeed, if v is additive, then inequality (1) is an equality. Conversely, assume that v is not additive, hence there are coalitions $S, T \subseteq N$ with

$$v(S) + v(T) \neq v(S \cap T) + v(S \cup T).$$

Two cases may occur. If $v(S) + v(T) < v(S \cap T) + v(S \cup T)$, then v is not c–convex w.r.t. $(S \setminus T, (N \setminus S) \cup T)$. In case the opposite inequality holds, $P = S \cup T$, $Q = N \setminus P$ shows the assertion.

(ii) Any **convex** game (N,v), i.e. a game v satisfying

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T) \text{ for } S, T \subseteq N,$$

is c–convex w.r.t. $(N,\emptyset)$.

(iii) Almost all weighted majority games are not c–convex at all. Here a **weighted majority game** (N,v) is a **simple** game (i.e. a coalition S is either winning – $v(S) = 1$ – or losing – $v(S) = 0$) possessing a **representation** $(\lambda;m)$:

$$\lambda > 0, m \in \mathbb{R}^N, m \geq 0, m(N) \geq \lambda,$$

$$v(S) = \begin{cases} 1 \text{ , if } m(S) \geq \lambda \\ 0 \text{ , otherwise} \end{cases} \text{ for } S \subseteq N.$$

A weighted majority game is **monotone**, i.e. $v(S) \leq v(T)$ if $S \subseteq T \subseteq N$, hence v is determined by its set of **minimal winning coalitions**

$$W_v^m = \{S \subseteq N \mid v(S) = 1 \text{ and } v(T) = 0 \text{ for } T \subsetneq S\}.$$

The set of **null players** is denoted $D(v) = N \setminus \bigcup_{S \in W_v^m} S$.

A weighted majority game (N,v) is c–convex, iff v is a "composition of at most one winning player i_0 and a unanimity game", i.e.

$$(v(\{i_0\}) = 1, N \setminus (D(v) \cup \{i_0\}) \in W_v^m) \text{ or } N \setminus D(v) \in W_v^m.$$

In this case v is c–convex w.r.t. any partition (P,Q) satisfying

$$\{i_0\} \subseteq P \subseteq \{i_0\} \cup D(v) \text{ or } P \subseteq D(v) \text{ in case } N \setminus D(v) \in W_v^m.$$

A proof of this assertion is given below.

(iv) If (N,v) is c–convex w.r.t. (P,Q), then the dual game (N,v*) is **c–concave w.r.t.** (P,Q), i.e. the opposite inequality of (1) holds true.

Proof of (iii):

The proof that each composition of at most one winning player and a unanimity game is c–convex in the desired sense is straightforward and therefore skipped. Conversely, assume that (N,v) is a weighted majority game with representation $(\lambda;m)$ which is c–convex w.r.t. (P,Q). If $P \supseteq N \setminus D(v)$, then $S \subseteq P$ for $S \in W_v^m$, hence $\{P \setminus D(v)\} = W_v^m$ by c–convexity. In the remaining case $(P \cap (N \setminus D(v)) \neq \emptyset \neq Q \cap (N \setminus D(v))$ we proceed as follows:

(a) There is no $S \subseteq W_v^m$ with $S \cap P \neq \emptyset \neq S \cap Q$.
Conversely assume there is a minimal winning coalition S intersecting both P and Q. Then

$$1 = v(S) + v(\emptyset) > 0 = v(S_P) + v(S_Q),$$

a contradiction.

By (a) a minimal winning coalition is either contained in $S = P \setminus D(v)$ or in $T = Q \setminus D(v)$ and both S and T contain at least one minimal winning coalition by the assumption.

(b) $S, T \in W_v^m$.
Conversely assume w.l.o.g. $S \notin W_v^m$. Hence there is $S^0 \subsetneq S$ with $S^0 \in W_v^m$. Take $i \in S \setminus S^0$ and observe there is $S^1 \in W_v^m$ with $i \in S^1$ since $i \notin D(v)$. Therefore $S^1 \subseteq P$ and

$$2 = v(S^0)+v(S^1) > 1 = v(S^0 \cup S^1) = v((S^0 \cup S^1)_P+(S^0 \cap S^1)_Q)+v((S^0 \cap S^1)_P+(S^0 \cup S^1)_Q),$$

a contradiction.

Assume w.l.o.g. $N = \{1,...,n\}$, $m_1 \geq ... \geq m_n$, and $1 \in P$.

(c) There is $i_0 \in N$ such that $S = \{1,...,i_0\}$.
Conversely assume there is $i \in N \setminus S$ with $i + 1 \in S$. With $\tilde{S} = S \cap \{1,...,i-1\}$ we have

$$m(\tilde{S}) < \lambda \leq m(\tilde{S} + \{i,...,n\}).$$

Let r be minimal such that $\lambda \leq m(\tilde{S} + \{i,...r\})$ and observe that $\tilde{S} + \{i,...,r\}$ intersects both P and Q and is a minimal winning coalition, hence a contradiction is established in this case.

(d) $i_0 = 1$.

Conversely assume $i_0 > 1$. Again there is a minimal r with

$$m((S \setminus \{i_0\}) \cup \{i_0+1,...r\}) \geq \lambda,$$

hence $(S \setminus \{i_0\}) \cup \{i_0+1,...,r\}$ intersects both P and Q and is a minimal winning coalition.

Summarizing we have shown that $v(\{1\}) = 1$ and $W_v^m = \{\{1\},T\}$, hence v has the desired properties. q.e.d.

Like in the definition of classical convexity the c–convexity property can be expressed in terms of increasing marginal contributions of players.

Lemma 1.6:

Let (N,v) be a game and (P,Q) be a partition of N. Then the following properties are equivalent.

(i) v is c–convex w.r.t. (P,Q).

(ii)
$$v(S+\{i\}) - v(S) \leq v(T+\{i\}) - v(T) \text{ for } i \in P \setminus T, \tag{2}$$
$$v(T+\{j\}) - v(T) \leq v(S+\{j\}) - v(S) \text{ for } j \in Q \setminus S \tag{3}$$
hold true for $S,T \subseteq N$ with $S_P \subseteq T_P$, $T_Q \subseteq S_Q$.

(iii) For $S \subseteq N$, $i, i_0 \in P \setminus S$, $j, j_0 \in Q \setminus S$, $i \neq i_0$, $j \neq j_0$ the following properties hold:

(a) $$v(S+\{i,j\}) - v(S+\{j\}) \leq v(S+\{i\}) - v(S), \tag{4}$$

(b) $$v(S+\{i,j\}) - v(S+\{i\}) \leq v(S+\{j\}) - v(S), \tag{5}$$

(c) $$v(S+\{i,i_0\}) - v(S+\{i\}) \geq v(S+\{i_0\}) - v(S), \tag{6}$$

(d) $$v(S+\{j,j_0\}) - v(S+\{j\}) \leq v(S+\{j_0\}) - v(S). \tag{7}$$

Proof:

To verify that (i) implies (ii) and (ii) implies (iii) is straightforward and therefore skipped.

(iii) implies (ii):

Let $S_P \subseteq T_P$, $T_Q \subseteq S_Q$, $i \in P \setminus T$, $j \in Q \setminus S$. Therefore there are nonnegative integers k,r and $i_1,...,i_k \in P$, $j_1,...,j_r \in Q$ such that $S_P + \{i_1,...,i_k\} = T_P$, $T_Q + \{j_1,...j_r\} = S_Q$.

Inequalities (6) and (4) directly imply

$$
\begin{aligned}
v(S+\{i\}) - v(S) &\leq v(S+\{i,i_1\}) - v(S+\{i_1\}) \quad \text{(by (6))}\\
&\leq \ldots\\
&\leq v(S+\{i,i_1,\ldots i_k\}) - v(S+\{i_1,\ldots,i_k\}) \quad \text{(by (6))}\\
&\leq v(S\backslash\{j_1\}) + \{i,i,\ldots,i_k\}) - v((S\backslash\{j_1\}) + \{i_1,\ldots,i_k\}) \text{ (by (4))}\\
&\leq \ldots\\
&\leq v(T+\{i\}) - v(T), \quad \text{(by (4))}
\end{aligned}
$$

hence (2) is verified. Analogous considerations, replacing (4), (6) by (5), (7) show inequality (3).

(ii) implies (i):

Let $\tilde{S}$, $\tilde{T} \subseteq N$ satisfy $\tilde{S}_P \subseteq \tilde{T}_P$, $\tilde{T}_Q \subseteq \tilde{S}_Q$ and take $\tilde{P} \subseteq P \setminus \tilde{T}$, $\tilde{Q} \subseteq Q \setminus \tilde{S}$. Successive application of (2) and (3) respectively show that

$$v(\tilde{S} + \tilde{P}) - v(\tilde{S}) \leq v(\tilde{T} + \tilde{P}) - v(\tilde{T}), \tag{8}$$

$$v(\tilde{T} + \tilde{Q}) - v(\tilde{T}) \leq v(\tilde{S} + \tilde{Q}) - v(\tilde{S}) \tag{9}$$

hold. Adding (8) and (9) yields

$$v(\tilde{S} + \tilde{P}) + v(\tilde{T} + \tilde{Q}) \leq v(\tilde{S} + \tilde{Q}) + v(\tilde{T} + \tilde{P}). \tag{10}$$

Take S, T $\subseteq$ N and define

$$\tilde{S} = (S \cap T)_P + S_Q \,,\ \tilde{P} = (S \setminus T)_P$$

$$\tilde{T} = T_P + (S \cap T)_Q \,,\ \tilde{Q} = (T \setminus S)_Q.$$

Indeed, $\tilde{S}_P \subseteq \tilde{T}_P$, $\tilde{T}_Q \subseteq \tilde{S}_Q$, $\tilde{P} \subseteq P \setminus \tilde{T}$, $\tilde{Q} \subseteq Q \setminus \tilde{S}$, hence

$$
\begin{aligned}
v(S) + v(T) &= v(\tilde{S} + \tilde{P}) + v(\tilde{T} + \tilde{Q}) && \text{(by definition)}\\
&\leq v(\tilde{S} + \tilde{Q}) + v(\tilde{T} + \tilde{P}) && \text{(by (10))}\\
&= v((S \cap T)_P + (S \cup T)_Q) + v((S \cup T)_P + (S \cap T)_Q).
\end{aligned}
$$

Thus (1) is valid. q.e.d.

Lemma 1.7:

(i) If (N,v) is an assignment game w.r.t. (P,Q), then v is c–concave w.r.t. (P,Q).

(ii) Let (N,v) be an M2–game defined by the vectors m^1, $m^2 \in \mathbb{R}^N$. Then v is c–concave w.r.t. any (P,Q) satisfying

$$\{i \in N \mid m_i^1 > m_i^2\} \subseteq P \subseteq \{i \in N \mid m_i^1 \geq m_i^2\}$$

and Q = N \ P.

Proof:

ad (i): In [7] it is shown that an assignment game v satisfies all inequalities opposite to (4) – (7). Hence v^* satisfies these inequalities and v^* is c–convex w.r.t. (P,Q) by Lemma 1.6.

ad (ii): Let $S,T \subseteq N$. The inequalities

$$m^i(S)+m^i(T) = m^i((S\cap T)_P+(S\cup T)_Q)+m^i((S\cup T)_P+(S\cap T)_Q),$$

for i = 1,2 and

$$m^1(S)+m^2(T)$$

$$= m^1((S\cap T)_P+(S\cup T)_Q)+m^2((S\cup T)_P+(S\cap T)_Q)$$

$$+ (m^1((S\setminus T)_P)-m^2((S\setminus T)_P))+(m^2((T\setminus S)_Q)-m^1((T\setminus S)_Q))$$

$$\geq m^1((S\cap T)_P+(S\cup T)_Q)+m^2((S\cup T)_P+(S\cap T)_Q) \text{ (by definition of P,Q)}$$

directly imply c–concavity w.r.t. (P,Q) for M2–games. q.e.d.

Examples 1.8:

(i) Let $N = \{1,\dots,n\}$ with $n \in \mathbb{N}$, $n \geq 2$, and define $P = \{1\}$, $Q = N \setminus P$ and (N,v) via

$$v(S) = \begin{cases} 1 \text{ , if } P \subseteq S \text{ or } Q \subseteq S \\ 0 \text{ , otherwise} \end{cases}.$$

Then v is a weighted majority game with representation (n–1; n–1, 1,...,1), whereas v^* is a weighted majority game with representation (n; n–1, 1,...,1). Moreover, v^* is an assignment game (w.r.t. (P,Q)), defined by the $P \times Q$ matrix (1,...,1), and an M2–game defined by the vectors $(1,0,\dots,0), (0,1,\dots,1) \in \mathbb{R}^N$. Note that v^* is the (P,Q) glove game. A **glove game w.r.t.** disjoint finite nonvoid sets P,Q is the assignment game defined by the $P \times Q$ matrix A with $a_{ij} = 1$ for $i \in P$, $j \in Q$. It coincides with the M2–game defined by

$$m^1, m^2 \in \mathbb{R}^{P+Q}, \quad m^1_i = \begin{cases} 1, & i \in P \\ 0, & i \in Q \end{cases}, \; m^2_i = \begin{cases} 1, & i \in Q \\ 0, & i \in P \end{cases}.$$

(ii) A function $f: \mathbb{R}^2_{\geq 0} \to \mathbb{R}$, $f(0) = 0$, with continuous second derivatives is called **c–convex**, if it satisfies

(a) $\frac{\partial^2 f}{\partial x_1^2} \geq 0 \leq \frac{\partial^2 f}{\partial x_2^2}$ (convexity w.r.t. the canonical directions),

(b) $\frac{\partial^2 f}{\partial x_1 \partial x_2} \leq 0.$

A c–convex function f together with two vectors $m^P \in \mathbb{R}^P_{\geq 0}$, $m^Q \in \mathbb{R}^Q_{\geq 0}$ of two disjoint Euclidean spaces of finite dimension, not both of dimension 0 defines a game (P + Q, v) – v =: $v_f^{m^P, m^Q}$ – by

$$v(S) = f(m^P(S_P), m^Q(S_Q)) \text{ for } S \subseteq P + Q.$$

The following considerations show that v is c–convex w.r.t. (P,Q). Let $S \subseteq N$, i_0, $i \in P \setminus S$, $j \in Q \setminus S$. We proceed verifying inequalities (4), (6). The proof of (5), (7) is completely analogous and therefore dropped. With $\alpha = m^P(S_P)$, $\beta = m^Q(S_Q)$ it is to show that

$$f(\alpha+m_i,\beta+m_j) - f(\alpha,\beta+m_j) \leq f(\alpha+m_i,\beta) - f(\alpha,\beta) \tag{11}$$

and

$$f(a+m_i+m_{i_0},\beta) - f(\alpha-m_i,\beta) \geq f(\alpha+m_{i_0},\beta) - f(\alpha,\beta) \tag{12}$$

hold. Inequality (12) holds by convexity of f w.r.t. the first canonical direction, i.e. by $\frac{\partial^2 f}{\partial x_1^2} \geq 0$. In order to verify (11) define $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by

$$g(x) = f(\alpha+x, \beta+m_j) - f(\alpha+x,\beta) - f(\alpha,\beta+m_j) + f(\alpha,\beta).$$

Clearly $g(0) = 0$ and $g'(x) = \frac{\partial f}{\partial f_1}(\alpha+x, \beta+m_j) - \frac{\partial f}{\partial f_1}(\alpha+x,\beta)$, hence $g'(x) = \frac{\partial^2 f}{\partial x_1 \partial x_2}(\alpha+x,\beta+\delta) \cdot m_j \leq 0$ for every $x \geq 0$ and some $\delta = \delta(x)$ with $0 \leq \delta \leq x$. Consequently $g(x) \leq 0$ for all $x \geq 0$ and (11) is implied by the fact that

$$0 \geq g(m_i) = f(\alpha+m_i,\beta+m_j) - f(\alpha+m_i,\beta) - f(\alpha,\beta+m_j) + f(\alpha,\beta).$$

If (a) and (b) hold strictly, i.e. $\frac{\partial^2 f}{\partial x_1^2} > 0 < \frac{\partial^2 f}{\partial x_2^2}$, $\frac{\partial^2 f}{\partial x_1 dx_2} < 0$, and if m^P and m^Q are strictly positive, then the arising game $v_f^{m_P m_Q}$ is **strictly** c–convex w.r.t. (P,Q), i.e.

$$v(S) + v(T) \quad = v((S \cap T)_P + (S \cup T)_Q) + v((S \cup T)_P + (S \cap T)_Q)$$

iff

$$\{S,T\} \quad = \{(S \cap T_P) + (S \cup T)_Q, (S \cup T)_P + (S \cap T)_Q\}.$$

Finally a class of c–convex functions is defined by

$$\left\{ f : \mathbb{R}^2_{\geq 0} \to \mathbb{R} \;\middle|\; \begin{array}{l} f(x_1,x_2) = a_1 x_1^{b_1} + a_2 x_2^{b_2} - a_{12} x_1 x_2 \text{ for some} \\ a_1,\ a_2,\ a_{12} \geq 0,\ b_1,\ b_2 \geq 2 \end{array} \right\}.$$

2. The Least Core of C–Convex Games

In this section the extremal points of the least core in the c–convex case are described. Let (N,v) be a c–convex game w.r.t. (P,Q) for some nonvoid disjoint sets P and Q. The following notation is frequently used:

$$\gamma(v) = \frac{v(P)+v(Q)-v(N)}{2},\ \alpha(v) = \frac{v(P)-v(Q)+v(N)}{2},\ \beta(v) = \frac{v(Q)-v(P)+v(N)}{2}.$$

Note that $\gamma(v)$ is a lower bound for the maximal excess of v at an arbitrary preimputation. Indeed, if $x \in X(v)$, then

$$\begin{aligned} 2\mu(x,v) &\geq (v(P)-x(P))+(v(Q)-x(Q)) = v(P)+v(Q)-x(N) \\ &= v(P)+v(Q)-v(N) \qquad \text{(by } x \in X(v)) \\ &= 2\gamma(v). \end{aligned} \tag{1}$$

Moreover, (1) shows that there is a nontrivial coalition (P or Q) of nonnegative excess, since

$$v(N) = v(N)+v(\emptyset) \leq v(P)+v(Q) \qquad \text{(by c–convexity)},$$

hence $\gamma(v) \geq 0$.

Lemma 2.1: $\mathscr{LC}(v) = \{x \in X(v) \mid \mu(x,v) = \gamma(v)\}$.

Proof:

In view of (1) it is sufficient to show:

$$\{x \in X(v) \mid \mu(x,v) = \gamma(v)\} \neq \emptyset.$$

Define (P,u) by

$$\begin{aligned} u(S) &= \max\{v(S),\ v(S+Q)-\beta(v)\} - \gamma(v) \\ &= \max\{v(S)-\gamma(v),\ v(S+Q)-v(Q)\} \text{ for } S \subseteq P. \end{aligned}$$

Observe that

$$u(\emptyset) = \max\{-\gamma(v),0\} = 0,$$

hence (P,u) is a game. Moreover

$$\begin{aligned} u(P) &= \max\{v(P)-\gamma(v),\ v(N)-v(Q)\} \\ &= \max\{\alpha(v),\ v(N)-v(Q)\} \\ &= \alpha(v) \end{aligned}$$

holds true. The last equality is satisfied, since

$$\alpha(v)-(v(N)-v(Q)) = \gamma(v) \geq 0.$$

(a) u is convex.

Take $S,T \subseteq P$ and distinguish the following 4 cases:

(i) $u(S) = v(S)-\gamma(v)$, $u(T) = v(T)-\gamma(v)$. Then

$$\begin{aligned} u(S)+u(T) &= v(S)+v(T)-2\gamma(v) \leq v(S \cap T) + v(S \cup T)-2\gamma(v) \text{ (by c-convexity)} \\ &\leq u(S \cap T)+u(S \cup T). \end{aligned}$$

(ii) $u(S) = v(S+Q)-v(Q)$, $u(T) = v(T+Q)-v(Q)$. Then

$$\begin{aligned} u(S)+u(T) &= v(S+Q)+v(T+Q)-2v(Q) \\ &\leq (v((S \cap T)+Q)-v(Q))+(v((S \cup T)+Q)-v(Q)) \text{ (by c-convexity)} \\ &\leq u(S \cap T)+u(S \cup T). \end{aligned}$$

(iii) $u(S) = v(S)-\gamma(v)$, $u(T) = v(T+Q)-v(Q)$. Then

$$\begin{aligned} u(S)+u(T) &= v(S)+v(T+Q)-\gamma(v)-v(Q) \\ &\leq (v((S \cap T)+Q)-v(Q))+(v(S \cup T)-\gamma(v)) \text{ (by c-convexity)} \\ &\leq u(S \cap T) + u(S \cup T). \end{aligned}$$

(iv) The case $u(S) = v(S+Q)-v(Q)$, $u(T) = v(T)-\gamma(v)$ can be solved analogously to (iii) interchanging the roles of S and T.

Take any $x \in \text{Core}(u) = \{x \in X(v) \mid e(S,x,v) \leq 0 \text{ for } S \subseteq N\}$. Such x exists by the convexity of u. Now define a game (Q,w) on Q which depends on the choice of x:

$$w(S) = \max \{v(R+S)-x(R) \mid R \subseteq P\} - \gamma(v) \text{ for } S \subseteq Q.$$

Indeed, $w(\emptyset) = \max \{v(R)-x(R)\} - \gamma(v) \leq \max \{u(R)-x(R)\} = 0$, since $x \in \text{Core}(u)$. On the other hand

$$w(\emptyset) \geq v(P)-x(P)-\gamma(v) = 0,$$

hence $w(\emptyset) = 0$.

Moreover

$$\begin{aligned} w(Q) &= \max \{v(R+Q)-x(R) \mid R \subseteq P\}-\gamma(v) \\ &\leq \max \{u(R)-x(R)+\beta(v) \mid R \subseteq P\} \leq \beta(v), \\ w(Q) &\geq v(Q)-\gamma(v) = \beta(v), \end{aligned}$$

thus $w(Q) = \beta(v)$.

Again, convexity of w can be verified straightforward.

Take any $y \in \text{Core}(w)$ and define $z \in \mathbb{R}^N$ by

$$z_i = \begin{cases} x_i, & i \in P \\ y_i, & i \in Q \end{cases}$$

Then z is a preimputation as

$$v(N) = \alpha(v)+\beta(v) = x(P)+y(Q) = z(N).$$

Moreover observe that

$$\begin{aligned} v(S)-z(S) &\leq \max\{v(R+S_Q)-x(R)-y(S_Q) \mid R \subseteq P\} = w(S_Q)-y(S_Q)+\gamma(v) \\ &\leq \gamma(v) \text{ (by } y \in \text{Core}(w)) \end{aligned}$$

holds; thus

$$e(S,z,v) \leq \gamma(v) \text{ for } S \subseteq N.$$

q.e.d.

In the context of classical convexity the extremal points of the Core are strongly related to "tight" sequences of coalitions. In the context of c–convexity a similar construction is useful. A sequence $(S^1,...,S^n)$ – where n denotes the cardinality of N – is **(P,Q)–tight**, if

$$S^1=P,\ S^n=Q,\ S_P^i \supseteq S_P^{i+1},\ S_Q^i \subseteq S_Q^{i+1},\ S^i \neq S^{i+1}, \text{ and } |S_P^i \setminus S_P^{i+1}| \leq 1 \geq |S_Q^{i+1} \setminus S_Q^i|$$

for $i \in \{1,...,n-1\}$.

Lemma 2.2: There are exactly (n–1)! pq (P,Q)–tight sequences.

Proof:

Fix permutations π^1 of P and π^2 of Q. It is sufficient to determine the number of (P,Q)–tight sequences $(S^1,...,S^n)$ with the following property:

$$S_P^i = \{\pi_1^1,..., \pi^1_{|S_P^i|}\},\ S_Q^i = \{\pi_1^2,...,\pi^2_{|S_Q^i|}\}$$

for $1 \leq i \leq n$. The sequence is uniquely determined by the permutations and the vector $x \in \mathbb{R}^n$ defined by $x_i = |S_P^i| - |S_Q^i| + q$.

Clearly, x is strictly increasing, $x_1 = n$, $x_n = 0$ by definition. This means that there exists a unique i_0 with $x_{i_0} - x_{i_0+1} = 2$ (and $x_i - x_{i+1} = 1$ for all other $i \in \{1,...,n-1\} - \{i_0\}$).

For fixed i_0 there are exactly $\binom{n-2}{p-1}$ possible sequences $(S^1,...,S^n)$. Exactly n–1 locations for i_0 and p!q! pairs of permutations exist. As a consequence we get exactly

$$\binom{n-2}{p-1} (n-1)\, p!q! = (n-1)!\ p \cdot q$$

(P,Q)–tight sequences.

q.e.d.

Lemma 2.3:

If $(S^1,...,S^n)$ is a (P,Q)–tight sequence, then $\{1_{S^1},...,1_{S^n}\}$ is a (vector space) basis of $\mathbb{R}^N$.

Proof:

Let $(S^1,...,S^n)$ be (P,Q)–tight. In view of the proof of Lemma 2.2 there is a unique i_0

$(1 \le i_0 \le n)$ such that

$$S^{i_0+1}_P \subseteq S^{i_0}_P,\ S^{i_0}_Q \subseteq S^{i_0+1}_Q.$$

Automatically for all $i \in \{1,\dots,n-1\} \setminus \{i_0\}$

$$(S^i \setminus S^{i+1}) \cup (S^{i+1} \setminus S^i) = \{k_i\}$$

is true. Thus $\{1_{S^1},\dots,1_{S^n}\}$ generate n–2 canonical basis vectors

$$e^{k_i} = (0,\dots,0,\underset{k_i}{1},0,\dots0),\ 1 \le i < n,\ i \ne i_0.$$

Clearly k_i coincides with k_j, iff $i = j$. Moreover

$$p-1 = |\{k_i \in P \mid n \ne i \ne i_0\}|,\ q-1 = |\{k_i \in Q \mid n \ne i \ne i_0\}|.$$

With the help of $S^1 = P$ and $S^n = Q$ the remaining canonical basis vector can be constructed:

$$1_P - \sum_{k_i \in P} e^{k_i},\ 1_Q - \sum_{k_i \in Q} e^{k_i}.$$ q.e.d.

Theorem 2.4:

(i) If x is an extremal point of $\mathscr{LB}(v)$, then there is a (P,Q)–tight sequence $(S^1,\dots,S^n)$ satisfying $e(S^i,x,v) = \gamma(v)$ for $i \in \{1,\dots,n\}$.

(ii) A (P,Q)–tight sequence $(S^1,\dots,S^n)$ uniquely determines a preimputation $x \in X(v)$ satisfying $e(S^i,x,v) = \gamma(v)$ for $i \in \{1,\dots,n\}$.

Proof:

ad (ii): Clearly $(S^1,\dots S^n)$ uniquely determines $x \in \mathbb{R}^N$ with $e(S^i,x,v) = \gamma(v)$ (by Lemma 2.3).

It remains to show that x is **Pareto optimal**, i.e. $x(N) = v(N)$. Now $x(N) = x(P)+x(Q)$, but

	$x(P)$	$= v(P)-\gamma(v)$	(by $S^1 = P$)
		$= \alpha(v)$	(by definition)
and	$x(Q)$	$= v(Q)-\gamma(v)$	(by $S^n = Q$)
		$= \beta(v)$,	

thus x is Pareto optimal.

ad (i): Let x be an extremal point of $\mathscr{LB}(v)$. Let M be the set of coalitions of maximal excess, i.e. $M = \{S \mid S \subseteq N,\ e(S,x,v) = \mu(x,v)\}$. By Lemma 2.1 and (1) $\mu(x,v) = \gamma(v)$ and $P,Q \in M$.

Step 1: $\{1_S \mid S \in M\} = \tilde{M}$ generates $\mathbb{R}^N$.

Conversely, assume $\tilde{M}$ does not generate $\mathbb{R}^N$. Then there exists $z \in \mathbb{R}^N$, $z \neq 0$ such that $z(S) = 0$ for all $S \in M$, hence $z(N) = z(P)+z(Q) = 0$. Consequently there is $\epsilon > 0$ such that $x \pm \epsilon z \in \mathscr{LC}(v)$, a contradiction.

Take any sequence of coaltions $(S^1,\ldots,S^k)$ with $S^1 = P$, $S^k = Q$, $S^i \neq S^{i+1}$, $S^{i+1}_P \subsetneq S^i_P$, $S^i_Q \subsetneq S^{i+1}_Q$ for $1 \leq i < k$ such that k is maximal.

Claim: $(S^1,\ldots,S^k)$ contains a (P,Q)–tight sequence.

This claim will be a direct consequence of the following two steps.

Step 2: $|S^i_P \setminus S^{i+1}_P| \leq 1 \geq |S^{i+1}_Q \setminus S^i_Q|$ for $1 \leq i < k$. (2)

Assume, on the contrary, there is i such that (2) is not valid. Let us say $S^i_P \setminus S^{i+1}_P \supseteq \{p_1,p_2\}$ for some $p_1 \neq p_2$. Since $\tilde{M}$ generates $\mathbb{R}^N$ there is $S \in M$ such that $|S \cap \{p_1,p_2\}|$, let us say $p_2 \notin S \ni p_1$. Using c–consistency it directly turns out that both

$$T = (S^i \cap S)_P + (S^i \cup S)_Q \text{ and } R = (T \cup S^{i+1})_P + (T \cap S^{i+1})_Q$$

are members of M. The obvious facts $S^{i+1}_P \subsetneq R_P \subsetneq S^i_P$, $S^i_Q \subsetneq R_Q \subsetneq S^{i+1}_Q$ directly establish the desired contradiction.

Step 3: There is at most one i_0 such that

$$|S^{i_0}_P \setminus S^{i_0+1}_P| = 1 = |S^{i_0+1}_Q \setminus S^{i_0}_Q|.$$

Assume, on the contrary, there are different i_j $(j = 0,1)$ such that $S^{i_j}_P \setminus S^{i_j+1}_P = \{p_i\}$, $S^{i_j+1}_Q \setminus S^{i_j+1}_Q = \{q_i\}$, hence $p_0 \neq p_1$, $q_0 \neq q_1$. Define $z \in \mathbb{R}^N$ by

$$z_i = \begin{cases} 1, & i \in \{p_0,q_0\} \\ -1, & i \in \{P_1,q_1\}. \\ 0, & \text{otherwise} \end{cases}$$

Then $z(S^i) = 0$ for $1 \leq i \leq k$. Hence there exists $S \in M$ with

$(p_j,q_j \in S)$ or $(p_j,q_j \notin S)$ for some $j = 0,1$.

(a) $p_j,q_j \in S$. Then both

$$T = (S^{i_j} \cap S)_P + (S^{i_j} \cup S)_Q \text{ and } R = (T \cup S^{i_j+1})_P + (T \cap S^{i_j+1})_Q$$

are members of M. But the equalities $S^{i_j}_P = R_P$, $S^{i_j+1}_Q = R_Q$ directly generate a contradiction.

(b) $p_j, q_j \notin S$. Then

$$T = (S^{i_j+1} \cup S)_P + (S^{i_j+1} \cap S)_Q \text{ and } R = (S^{i_j} \cap T)_P + (S^{i_j} \cup T)_Q$$

are members of M. Observing $T_P = S^{i_j+1}_P$, $T_Q = S^{i_j}_Q$ directly yields a contradiction.

Hence either $k = n$ and $(S^1,...,S^n)$ is (P,Q)–tight or
$k = n+1$ and $|(S^i \setminus S^{i+1}) \cup (S^{i+1} \setminus S^i)| = 1$ for $1 \leq i \leq n$.

In the latter case each subsequence which arises from the initial sequence by deleting one coalition S^i, $2 \leq i \leq n$, is (P,Q)–tight. q.e.d.

Corollary 2.5: The least core of v has at most $(n-1)!\ p \cdot q$ extremal points.

The least core of v is the convex hull of its extremal points which can be computed along the following procedure: Compute all (P,Q)–tight sequences $(S^1,...,S^n)$ and to each one the unique vector x satisfying $x(S^i) = \gamma(v)$ for $1 \leq i \leq n$. Eliminate duplications and then vectors x for which there exists $S \subseteq N$ with $e(S,x,v) > \gamma(v)$.

A (P,Q)–tight sequence is **feasible for** v if it generates an extremal point of the least core of v via Theorem 2.4, Lemma 2.3. The following examples show that "feasibility" is not universal in the sense that a (P,Q)–tight sequence may or not be feasible for one or the other c–convex game w.r.t. a fixed partition (P,Q). Moreover it turns out that the number of extremal points of the least core may vary even in case (P,Q) is fixed and the games are strictly c–convex w.r.t. (P,Q).

Examples 2.6:

Let $P = \{1,2\}$, $Q = \{3,4\}$, $f: \mathbb{R}^2_{\leq 0} \to \mathbb{R}$ be defined by $f(x) = (x_1-x_2)^2$. Hence f is strictly c–convex. Let $m^P, \tilde{m}^P \in \mathbb{R}^P$, $m^Q, \tilde{m}^Q \in \mathbb{R}^Q$ be defined by

$$m^P_i = m^Q_{i+2} = 1,\ \tilde{m}^P_1 = \tilde{m}^Q_3 = 3,\ \tilde{m}^P_2 = \tilde{m}^Q_4 = 1.$$

Then both

$$v = v_f^{m^P, m^Q} \text{ and } w = v_f^{\tilde{m}^P, \tilde{m}^Q}$$

are strictly c–convex w.r.t. (P,Q) as shown in Section 1.

In view of the proof of Lemma 2.2 there are six (P,Q)–tight sequences, described as matrices, in which the rows are the indicator functions of the coalitions, arising from the unit permutations:

$$A_1 = \begin{bmatrix} 1\,1\,0\,0 \\ 1\,1\,1\,0 \\ 1\,0\,1\,0 \\ 0\,0\,1\,1 \end{bmatrix}\begin{matrix} \\ \\ * \\ \\ \end{matrix}, \quad A_2 = \begin{bmatrix} 1\,1\,0\,0 \\ 1\,0\,1\,0 \\ 0\,0\,1\,0 \\ 0\,0\,1\,1 \end{bmatrix}\begin{matrix} * \\ \\ \\ \\ \end{matrix}, \quad A_3 = \begin{bmatrix} 1\,1\,0\,0 \\ 1\,0\,0\,0 \\ 1\,0\,1\,0 \\ 0\,0\,1\,1 \end{bmatrix}\begin{matrix} \\ \\ * \\ \\ \end{matrix},$$

$$A_4 = \begin{bmatrix} 1\,1\,0\,0 \\ 1\,0\,1\,0 \\ 1\,0\,1\,1 \\ 0\,0\,1\,1 \end{bmatrix}\begin{matrix} * \\ \\ \\ \\ \end{matrix}, \quad A_5 = \begin{bmatrix} 1\,1\,0\,0 \\ 1\,1\,1\,0 \\ 1\,0\,1\,1 \\ 0\,0\,1\,1 \end{bmatrix}\begin{matrix} \\ * \\ \\ \\ \end{matrix}, \quad A_6 = \begin{bmatrix} 1\,1\,0\,0 \\ 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,1\,1 \end{bmatrix}\begin{matrix} \\ * \\ \\ \\ \end{matrix}.$$

The star at each matrix is a marker at row i_0 (for the definition of i_0 the proof of Lemma 2.2 is referred to). W.r.t. v and according to Theorem 2.4 A_1 and A_2 determine the vector (–1,1,–3,3), A_3 and A_4 generate (–3,3,–1,1), which both are extremal points of the least core of v. Finally both A_5 and A_6 generate (–3,3,–3,3), which does not belong to the least core of v. For obvious symmetry reasons the least core of v is the convex hull of its 8 extremal points.

All feasible (P,Q)–tight sequences for w can be computed to be

$$B_1 = \begin{bmatrix} 1\,1\,0\,0 \\ 0\,1\,1\,0 \\ 0\,0\,1\,0 \\ 0\,0\,1\,1 \end{bmatrix}\begin{matrix} * \\ \\ \\ \\ \end{matrix}, \quad B_2 = \begin{bmatrix} 1\,1\,0\,0 \\ 0\,1\,1\,0 \\ 0\,1\,1\,1 \\ 0\,0\,1\,1 \end{bmatrix}\begin{matrix} * \\ \\ \\ \\ \end{matrix}, \quad B_3 = \begin{bmatrix} 1\,1\,0\,0 \\ 1\,1\,0\,1 \\ 0\,1\,1\,1 \\ 0\,0\,1\,1 \end{bmatrix}\begin{matrix} \\ * \\ \\ \\ \end{matrix},$$

$$B_4 = \begin{bmatrix} 1\,1\,0\,0 \\ 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,1\,1 \end{bmatrix}\begin{matrix} \\ * \\ \\ \\ \end{matrix}, \quad B_5 = \begin{bmatrix} 1\,1\,0\,0 \\ 1\,0\,0\,0 \\ 1\,0\,0\,1 \\ 0\,0\,1\,1 \end{bmatrix}\begin{matrix} \\ \\ * \\ \\ \end{matrix}, \quad B_6 = \begin{bmatrix} 1\,1\,0\,0 \\ 1\,1\,0\,1 \\ 1\,0\,0\,1 \\ 0\,0\,1\,1 \end{bmatrix}\begin{matrix} \\ \\ * \\ \\ \end{matrix}.$$

which generate the extremal points

$$(5,-5,-7,7),\ (7,-7,-5,5),\ (7,-7,7,-7),\ (-7,7,-7,7),\ (-7,7,5,-5),\ (-5,5,7,-7)$$

of $\mathscr{LC}(w)$. B_3, B_4 are not feasible for v, whereas e.g., $A_1,\ldots,A_4$ are not feasible for w.

3. A Common Property of the Modified Least Core for Assignment and M2–Games

It is the aim of this section to show that the modified least core of an assignment game or an M2–game is a subset of the Least Core of the corresponding dual game if both P and Q are nonempty. If P or Q are empty, i.e. if the dual game is convex, then the modified least core is contained in the core of the dual game as shown in [10]. The following lemmata will be used in the proof of Theorems 3.5 and 3.6. Let

$$\mathscr{D}(x,v) = D(x,\mu(x,v),v)$$

(for the definitions of $D(\cdot,\cdot,\cdot)$ and $\mu(\cdot,\cdot)$ Section 1 is referred to) for a game (N,v) and $x \in \mathbb{R}^N$ denote the set of coalitions of maximal excess.

Lemma 3.1: Let (N,v) be a c–convex game w.r.t. (P,Q) and $x \in \mathbb{R}^N$. Then

(i) $e(S,x,v) + e(T,x,v) \leq e((S \cap T)_P + (S \cup T)_Q, x,v) + e((S \cup T)_P + (S \cap T)_Q,x,v)$ for $S,T \subseteq N$;

(ii) If $S,T \in \mathscr{D}(x,v)$, then $(S \cap T)_P + (S \cup T)_Q$, $(S \cup T)_P + (S \cap T)_Q$ are members of $\mathscr{D}(x,v)$;

(iii) $$S^L := \bigcap_{S \in \mathscr{D}(x,v)} S_P + \bigcup_{S \in \mathscr{D}(x,v)} S_Q \in \mathscr{D}(x,v),$$

$$S^R := \bigcup_{S \in \mathscr{D}(x,v)} S_P + \bigcap_{S \in \mathscr{D}(x,v)} S_Q \in \mathscr{D}(x,v).$$

For classical convex games, i.e. $P = \emptyset$ or $Q = \emptyset$, property (ii) of Lemma 3.1 is the near–ring property (see [2]). Therefore a set of coalitions satisfying (ii) of Lemma 3.1 is called **c–near–ring** here.

Proof:

(i) is a direct consequence of the definition of c–convexity.

(ii) is directly implied by (i), whereas (ii) implies (iii). q.e.d.

Lemma 3.2: Let $x \in \mathscr{M}\mathscr{L}\mathscr{C}(v)$ for some c–convex game (N,v) w.r.t. (P,Q). Then

$$(P \subseteq S^R \text{ and } Q \subseteq S^L) \text{ or} \tag{1}$$

$$(S^R \subseteq P \text{ and } S^L \subseteq Q) \tag{2}$$

where S^R, S^L are defined as in Lemma 3.1.

Proof:

Assume the contrary. W.l.o.g. $P \nsubseteq S^R$ (otherwise exchange the roles of P and Q). Two cases may occur.

Case 1: $S^R \subsetneqq P$.

Then $S^L \cap P \neq \emptyset$ (see(2)), but $S^L \cap P \subseteq S_P \subseteq S^R$ for $S \in \mathscr{D}(x,v)$ by definition of S^L, S^R.

Take $i \in S^L \cap P$, $j \in P \setminus S^R$, and a sequence $(\delta_{(S,T)})_{(S,T)\in\tilde{D}}$ of weakly balancing coefficients for $\tilde{D} = \tilde{D}(x,\mu(x,v)+\mu(x,v^*),v) = \mathscr{D}(x,v) \times \mathscr{D}(x,v^*)$, i.e.

$$\delta_{(S,T)} \geq 0 \text{ and } \sum_{(S,T)\in\tilde{D}} \delta_{(S,T)} (1_S+1_T) = 1_N. \tag{3}$$

For the existence of a weakly balancing sequence Lemma 1.3 is referred to. For $S \in \mathscr{D}(x,v)$ and $T \in \mathscr{D}(x,v^*)$ let $\delta_S = \sum_{T \in \mathscr{D}(x,v^*)} \delta_{(S,T)}$, $S_T^* = \sum_{S \in \mathscr{D}(x,v)} \delta_{(S,T)}$, hence $\delta_S \geq 0 \leq \delta_T^*$. Thus (3) can be rewritten to

$$\sum_{S \in \mathscr{D}(x,v)} \delta_S 1_S + \sum_{T \in \mathscr{D}(x,v^*)} \delta_T^* 1_T = 1_N, \tag{4}$$

$$\sum_{S \in \mathscr{D}(x,v)} \delta_S = \sum_{T \in \mathscr{D}(x,v^*)} \delta_T^*. \tag{5}$$

Therefore (4), applied to i, and the fact $i \in S$ for $S \in \mathscr{D}(x,v)$ implies

$$\sum_{S \in \mathscr{D}(x,v)} \delta_S \leq 1, \tag{6}$$

whereas (4), applied to $j \notin S$ for $S \in \mathscr{D}(x,v)$, implies

$$\sum_{T \in \mathscr{D}(x,v^*)} \delta_T^* \geq 1. \tag{7}$$

(5),(6),(7) are simultaneously true, thus

$$\sum_{S \in \mathscr{D}(x,v)} \delta_S = \sum_{T \in \mathscr{D}(x,v^*)} \delta_T^* = 1 \tag{8}$$

Define $\bar{D} := \{T \in \mathscr{D}(x,v^*) \mid \delta_T^* > 0\}$, hence $\bar{D} \neq \emptyset$ by (8). (4),(8) together with $S_Q \subsetneq S^L$ for $S \in \mathscr{D}(x,v)$ implies

$$Q \setminus S^L \subsetneq T \text{ for } T \in \bar{D}. \tag{9}$$

Claim: $\quad T \cap (S^L \cap P) \neq \emptyset$ for $T \in \bar{D}$. $\hfill (10)$

If, on the contrary, (10) is not valid, then $P \cap S^L \subsetneq U$, $U_Q \subsetneq S^L \cap Q$, where $U = N \setminus T$ is a coalition of minimal excess at x w.r.t. v (see Section 1). Lemma 3.1 directly implies

$$e(S^L,x,v) + e((U \setminus S_P^L),x,v) \leq e(S_Q^L,x,v) + e(U,x,v),$$

but $S_Q^L \notin \mathscr{D}(x,v)$ by definition of S^L, hence

$$e(U \setminus S_P^L,x,v) < e(U,x,v),$$

a contradiction against the fact that the excess of U is minimal.

Take $T \in \bar{D}$ and $i \in T \cap (S^L \cap P)$. Then by (8) and (4), applied to player i, we come up with

$$1 \geq \sum_{S \in \mathscr{D}(x,v)} \delta_S + \delta_T = 1 + \delta_T > 1,$$

which is impossible.

Case 2: $S^R \cap Q \neq \emptyset$.

Then $S^R \cap P \subsetneq P$ by the assumption. Moreover, $S^R \cap Q \subseteq S$ for $S \in \mathscr{D}(x,v)$.

The same procedure as in Case 1 establishes a contradiction. Indeed, using the notation of Case 1 each $T \in \bar{D}$ contains $P \setminus S^R$, hence intersects $S^R \cap Q$. q.e.d.

Up to the end of this section let P and Q be finite disjoint nonvoid sets. In what follows one interesting common property of many classical solution concepts for cooperative games is described.

Definition 3.3: Let $x \in \mathbb{R}^N$ and (N,v) be a game (not necessarily c–convex). Then x is said to be **reasonable (on both sides)** if each component of x is bounded from below by the minimal and from above by the maximal marginal contribution of the corresponding player, i.e., if

$$\min \{v(S+\{i\}) - v(S) \mid S \subseteq N \setminus \{i\}\} \leq x_i \leq \max \{v(S+\{i\})-v(S) \mid S \subseteq N \setminus \{i\}\}$$

for $i \in N$.

It is well–known that, e.g., the Shapley value and each element of the least core of a game are reasonable. In [10] it is verified that each element of the modified least core is reasonable, too. Nevertheless, a proof is given below.

Lemma 3.4: Let $x \in \mathscr{ML}\mathscr{C}(v)$ for some game (N,v). Then x is reasonable.

Proof:

Assume, on the contrary, there is $x \in \mathscr{ML}\mathscr{C}(v)$ being not reasonable. For $S \subseteq N \setminus \{i\}$

$$v(S+\{i\})-v(S) = v^*(N \setminus S)-v^*((N \setminus S) \setminus \{i\}),$$

hence

$$g_i := \min \left\{v(S+\{i\})-v(S) \mid S \subseteq N \setminus \{i\}\right\} = \min \left\{v^*(S+\{i\})-v^*(S) \mid S \subseteq N \setminus \{i\}\right\}$$

and

$$h_i := \max \left\{v(S+\{i\})-v(S) \mid S \subseteq N \setminus \{i\}\right\} = \max \left\{v^*(S+\{i\})-v^*(S) \mid S \subseteq N \setminus \{i\}\right\}$$

for $i \in N$.

Case 1: $x_i > h_i$ for some $i \in N$.

Then, for $S \subseteq N$ with $i \in S$,

$$e(S,x,v) < e(S \setminus \{i\},x,v),$$
$$e(S,x,v^*) < e(S \setminus \{i\},x,v^*),$$

hence $i \notin S$ for $S \in \mathscr{D}(x,v) \cup \mathscr{D}(x,v^*)$.

Therefore $\mathscr{D}(x,v) \times \mathscr{D}(x,v^*)$ cannot be weakly balanced, a contradiction.

Case 2: $x_i < g_i$ for some $i \in N$.

A similar argument as in Case 1 shows

$$i \in S \text{ for } S \in \mathscr{D}(x,v) \cup \mathscr{D}(x,v^*).$$

With weakly balancing coefficients $(\delta_{(S,T)})_{(S,T) \in \tilde{D}}$ of $\tilde{D} = \mathscr{D}(x,v) \times \mathscr{D}(x,v^*)$, i.e. $\delta_{(S,T)} \geq 0$ and $\sum_{(S,T) \in \tilde{D}} \delta_{(S,T)}(1_S + 1_T) = 1_N$ it turns out that $\sum_{(S,T) \in \tilde{D}} \delta_{(S,T)} = 1$ (by applying the last equality to player i), hence $N \in \mathscr{D}(x,v)$. On the other hand $e(\{i\},x,v) = v(\{i\}) - x_i > 0 = e(N,x,v)$, a contradiction. q.e.d.

Theorem 3.5: The modified least core of an assignment game w.r.t. (P,Q) is a subset of the least core of the dual game.

Proof:

Let (N,u) be an assignment game w.r.t. (P,Q) and A the defining matrix (see Definition 1.4 (ii)). Let $v = u^*$ be the corresponding dual game which is c–convex w.r.t. (P,Q) in view of Lemma 1.7. By Remark 1.2(i) it suffices to show $\mathscr{ML}\mathscr{C}(v) \subseteq \mathscr{L}\mathscr{C}(v)$, because $\mathscr{ML}\mathscr{C}(w) = \mathscr{ML}\mathscr{C}(w^*)$ for arbitrary games w.

Take $x \in \mathscr{ML}\mathscr{C}(u)$ and assume, on the contrary, $x \notin \mathscr{L}\mathscr{C}(v)$. Moreover, assume w.l.o.g. $x(Q) \geq \beta(v)$ (otherwise exchange the roles of P and Q) – for the definition of $\beta(\cdot)$ Section 2 is referred to.

By the assumptions $Q \notin \mathscr{D}(x,v)$, hence – in view of Lemma 3.2 – two cases may occur: $Q \subsetneqq S^L$ or $S^L \subsetneqq Q$.

1. Case: $Q \subsetneqq S^L$.

Then $P \subseteq S^R$ by (1). By definition of an assignment game $u(T) = 0$ for $T \subseteq P$, i.e.

$$v(S) = v(N) \text{ for } S \supseteq Q.$$

Hence there is $i \in S^L \cap P$ such that $x_i < 0$, since $e(Q,x,v) < \mu(x,v) = e(S^L,x,v)$. On the other hand u, and thus v, is a monotonic game, implying

$$v(S+\{i\}) - v(S) \geq 0 \text{ for } S \subseteq N \setminus \{i\},$$

hence $x_i \geq 0$ by reasonableness of x (see Lemma 3.4). These considerations imply a contradiction in this case.

2. Case: $S^L \subsetneqq Q$.

Hence $S^R \subseteq P$ (by (2)). For $S \subseteq N$, $i \in P \setminus S$, $j \in Q \setminus S$ it is well–known that

$$u(S+\{i,j\}) \geq u(S)+a_{ij} \tag{11}$$

holds true.

Moreover, let $\sigma(S)$ denote the set of **assignments of** S, i.e.

$$\sigma(S) := \left\{ (i_k,j_k)_{k=1}^{t} \;\middle|\; \begin{array}{l} i_k \in S_P,\, j_k \in S_Q, \\ \{i_k, j_k\} \cap \{i_r, j_r\} = \emptyset, \end{array} \quad \text{for } r \neq k \right\}$$

where $t = \min\{|S_P|, |S_Q|\}$. Then

$$u(S) = \max_{(i_k,j_k)_{k=1}^{t} \in \sigma(S)} \sum_{k=1}^{t} a_{i_k j_k}.$$

For these properties, e.g., [8] or [9] are referred to.

Let $T = N \setminus S^L$, hence T has minimal excess at x w.r.t. u. Let $j \in T_Q$ with $x_j>0$. Indeed, player j exists since otherwise $e(S^L+\{j\},x,v) \geq e(S^L,x,v)$ (by monotonicity of v) which is impossible.

Let $U \in \mathscr{D}(x,u)$ with $j \in U$. Such U exists because $\mathscr{D}(x,v) \times \mathscr{D}(x,u)$ is weakly balanced and $j \notin S$ for $S \in \mathscr{D}(x,v)$. Take an optimal assignment $(i_k, j_k)_{k=1}^{t} \in \sigma(U)$ for U, i.e.

$$u(U) = \sum_{k=1}^{t} a_{i_k j_k}.$$

If $j \notin \{j_k \mid 1 \leq k \leq t\}$, then $u(U \setminus \{j\}) = u(U)$, hence $e(U,x,u) < e(U \setminus \{j\},x,u)$, a contradiction.

If $j = j_k$ for some $1 \leq k \leq t$, then put $i = i_k$. Obviously $u(U \setminus \{i,j\}) = u(U)-a_{ij}$ is valid; hence

$$e(U,x,u)-e(U \setminus \{i,j\},x,u) = a_{ij}-x_i-x_j \geq 0. \tag{12}$$

Moreover, $u(T \setminus \{i,j\}) \leq u(T)-a_{ij}$ (by (11)), thus

$$\begin{array}{lll} & e(T \setminus \{i,j\},x,u) & \leq u(T)-a_{ij}-x(T)+x_i+x_j \\ = & e(T,x,u)-(a_{ij}-x_i-x_j) & \leq e(T,x,u) \text{ (by (12))}, \end{array}$$

hence all inequalities are equalities. Therefore $N \setminus (T \setminus \{i,j\}) = S^L+\{i,j\}$ has maximal excess which contradicts the definition of S^L. q.e.d.

Theorem 3.6: Let (N,u) be an M2–game defined by $u(S) = \min\{m^1(S), m^2(S)\}$ for some $m^1, m^2 \in \mathbb{R}^N$ such that $\{i \in N \mid m_i^k \geq m_i^{3-k}\}$ is nonvoid for $k = 1,2$. Then $\mathscr{M}\mathscr{L}\mathscr{C}(u) \subseteq \mathscr{L}\mathscr{C}(u^*)$ holds true.

Proof:

If $\{i \in N \mid m_i^k \geq m_i^{3-k}\} = N$ for some $k \in \{1,2\}$, then $u(S) = m^{3-k}(S)$ for $S \subseteq N$, thus u is additive and the assertion is valid by reasonableness of each $x \in \mathscr{M}\mathscr{L}\mathscr{C}(u)$ (see Lemma 3.4) and each $y \in \mathscr{L}\mathscr{C}(u)$ by $u = u^*$.

Therefore choose any $P \subseteq N$ with

$$\{i \in N \mid m^1_i > m^2_i\} \subseteq P \subseteq \{i \in N \mid m^1_i \geq m^2_i\},$$

define $Q = N \setminus P$ and assume that $P \neq \emptyset \neq Q$ holds true. Moreover, $m^1(N) \leq m^2(N)$ can be assumed w.l.o.g. (otherwise exchange the roles of m^1 and m^2). Let $v = u^*$ be the dual game. Then $\mathscr{M}\mathscr{L}\mathscr{C}(v) = \mathscr{M}\mathscr{L}\mathscr{C}(u)$. Moreover, by Lemma 1.7, v is c–convex w.r.t. (P,Q).

Assume, on the contrary, there is $x \in \mathscr{M}\mathscr{L}\mathscr{C}(v) \setminus \mathscr{L}\mathscr{C}(v)$. With $\epsilon = m^1(N)-m^2(N)$ (≤ 0) it is easy to verify that

$$v(S) = \max\{m^1(S), m^2(S)+\epsilon\} \text{ for } S \subseteq N. \tag{13}$$

Moreover, for every coalition S with $P \subseteq S$

$$\begin{aligned} & m^2(S)+\epsilon && = m^2(S)+m^1(N)-m^2(N) \\ = \quad & m^1(N)-m^2(N \setminus S) && \leq m^1(S) \text{ (by the choice of P,Q)}, \end{aligned}$$

hence

$$v(S) = m^1(S) \text{ for } S \supseteq P. \tag{14}$$

The fact that – for every $S \supseteq Q$ – $m^2(S)+m^1(N)-m^2(N) = m^1(N)-m^2(N \setminus S) \geq m^1(S)$ implies

$$v(S) = m^2(S)+\epsilon \text{ for } S \supseteq Q. \tag{15}$$

For $i \in P$ the inequalities

$$\begin{aligned} m^2_i &\leq \min\left\{v(S+\{i\})-v(S) \mid S \subseteq N \setminus \{i\}\right\}, \\ m^1_i &\geq \max\left\{v(S+\{i\})-v(S) \mid S \subseteq N \setminus \{i\}\right\}, \end{aligned}$$

are direct consequences of (13) and the definition of P,Q. Thus – by reasonableness of x –

$$m^2_i \leq x_i \leq m^1_i \text{ for } i \in P. \tag{16}$$

Analogously it turns out that

$$m^1_j \leq x_j \leq m^2_j \text{ for } j \in Q. \tag{17}$$

Now two cases can be distinguished:

Case 1: $x(Q) \geq \beta(v)$.

Then, by Lemma 3.2, $S^L \supsetneq Q$ or $S^L \subsetneq Q$.

(a) $S^L \subsetneq Q$.

Take $i \in S^L \cap P$. Then

$$\begin{aligned} & v(S^L) && = && m^2(S^L) + \epsilon \text{ (by (15))}, \\ & v(S^L \setminus \{i\}) && = && m^2(S^L \setminus \{i\}) + \epsilon \text{ (by (15))} \\ = \quad & v(S^L)-m^2_i && \geq && v(S^L)-x^2_i \text{ (by (16))} \end{aligned} \tag{18}$$

hold true, hence $e(S^L \setminus \{i\},x,v) \geq e(S^L,x,v)$, a contradiction.

(b) $S^L \subsetneq Q$.

If $v(S^L) = m^2(S^L)+\epsilon$, take $j \in Q \setminus S^L$ and observe that

$$v(S^L+\{j\}) = v(S^L)+m^2_{\cdot j} \geq v(S^L)+x_j \text{ (by (17))}.$$

Thus $e(S^L+\{j\},k,v) \geq e(S^L,x,v)$, which is impossible.

If $v(S^L) = m^1(S^L)$, then $v(S^L) \leq x(S^L)$ (by (17)), hence $\mu(x,v) \leq 0 \leq \gamma(v)$, a contradiction.

Case 2: $x(Q) < \beta(v)$.

Then, by Lemma 3.2, $S^R \supsetneq P$ or $S^R \subsetneq P$.

(a) $S^R \supsetneq P$.

Hence $v(S^R) = m^1(S^R)$ (by (14)) and for $j \in S^R \cap Q$

$$\begin{aligned} v(S^R \setminus \{j\} &= m^1(S^R)-m^1_j \text{ (by (14))} \\ &\geq v(S^R)-x_j \text{ (by (17))}, \end{aligned}$$

thus $e(S^R \setminus \{j\},x,v) \geq e(S^R,x,v)$, a contradiction.

(b) $S^R \subsetneq P$.

If $v(S^R) = m^1(S^R)$, take $i \in P \setminus S^R$ and observe that

$$v(S^R+\{i\}) \geq m^1(S^R)+m^1_i \geq v(S^R)+x_i \text{ (by (16))},$$

hence $e(S^R+\{i\},x,v) \geq e(S^R,x,v)$, a contradiction.

If $v(S^R) = m^2(S^R)+\epsilon$, then

$$\begin{aligned} v(S^R) &\leq x(S^R)+\epsilon \text{ (by (16))} \\ &\leq x(S^R) \text{ (since } \epsilon \leq 0), \end{aligned}$$

thus $\mu(x,v) \leq 0 \leq \gamma(v)$, a contradiction. q.e.d.

Elements of the least core or core are, vaguely formulated, determined by only looking at the worth of coalitions ($v(S)$, $S \subseteq N$), whereas the "blocking power" of a coalition S, i.e. the worth which S cannot be prevented from by the complement coalition – $v(N)-v(N \setminus S) = v^*(S)$ – is not taken into consideration. E.g., if P (or Q) form a "syndicate" in an assignment game, then P (or Q) can prevent the opposite group Q (or P) from any positive amount. In the modified solutions both the "power" of a coalition, i.e. $v(S)$, and the blocking power, i.e. $v^*(S)$, play a totally symmetric role in general. Theorem 3.5 says in the assignment game case that both groups P and Q are treated equally, get the same aggregated amounts, from each preimputation of the modified least core. For M2–games – P,Q defined as in Theorem 3.6 – both groups have the same excess w.r.t. the dual game for each element of the modified least core.

Both M2– and assignment games are **linear production** games in the sense of [4] (see also [5]) and, thus, possess nonempty cores. Nevertheless, the modified least core frequently does not intersect the core (as seen below).

Examples 3.7

(i) Let P,Q be two disjoint nonvoid finite sets and let (Q,v) be the glove game w.r.t. (P,Q). For the definition of glove games Example 1.8 is referred to. W.l.o.g. let the cardinality of P (p,q denote the cardinalities of P,Q respectively) be smaller than or equal to the cardinality of Q. Then, as long as $p = q$, the nucleoli are singletons, namely

$$\nu(v) = \psi(v) = (1,\ldots,1)/_2 = \varphi(v),$$

where φ denotes the Shapley value. If $p < q$, then

$$\nu_i(v) = \begin{cases} 1, & i \in P \\ 0, & i \in Q \end{cases}, \quad \psi_i(v) = \begin{cases} \frac{1}{2}, & i \in P \\ \frac{p}{2q}, & i \in Q \end{cases} \quad \text{for } i \in N = P+Q,$$

whereas $\varphi(v)$ is a pure convex combination of $\nu(v)$ and $\psi(v)$. Therefore the Shapley value can be seen, in some sense, as a compromise between the modified nucleolus and the (pre)nucleolus in this case. The modified nucleolus highly evaluates the blocking power of the groups P and Q, whereas the nucleolus does not.

(ii) Let $m^1 = (5,10,6)$, $m^2 = (2,4,10) \in \mathbb{R}^3$ and (N,v) – $N = \{1,2,3\}$ – be the corresponding M2–game (i.e. $P = \{1\}$, $Q = \{2,3\}$). The (modified) nucleoli can be computed as

$$\nu(v) = (2,4.5,9.5),\ \nu(v^*) = (3,5,8),$$
$$\psi(v) = (2.5,5.5,8) = \psi(v^*) =: \psi.$$

By Theorem 3.6 the maximal excesses of the nucleoli w.r.t. v^* have to coincide. Indeed

$$\mu(\nu(v^*),v^*) = \mu(\psi,v^*) = 2$$

holds true. Moreover $\nu(v)$ is and has to be a member of the core of v since v is a linear production game and $\nu(v)$ is a core selector for balanced games (games with nonvoid core). Finally

$$\mu(\nu(v),v^*) = 3.5 > 2,\ \mu(\nu(v^*),v) = 1 > 0,\ \mu(\psi,v) = 0.5.$$

Therefore

$$\mathcal{LC}(v^*) \cap \mathcal{MLC}(v) = \text{Core}(v) \cap \mathcal{MLC}(v) = \emptyset.$$

Moreover, $\nu(v^*) \notin \mathcal{MLC}(v)$, since

$$\mu(\nu(v^*),v) = 1 > 0.5 = \mu(\psi,v).$$

Final Remarks 3.8:

(i) Let (N,v) be a c–convex game w.r.t. a nontrivial partition (P,Q) and let $(S^1,\dots,S^n)$ be a (P,Q)–tight sequence. Then there exists an easy procedure to construct the vector x generated by this sequence, i.e. $x(S^i) = \gamma(v)$ for every i. (For the corresponding definitions Section 2 is referred to.) Namely, given i_0 as in the proof of Lemma 2.2 the successive comparison of $S^1,\dots,S^{i_0}$ first and then of $S^n,\dots,S^{i_0+1}$, immediately determines p–1 components of x in P and q–1 components of x in Q, hence – by $x(P) = \alpha(v)$, $x(Q) = \beta(v)$ – all components of x. Moreover, (without going into the details) it can be checked at each stage by a canonical comparison whether the components of x computed so far are or are not (in the latter case stop the procedure as $(S^0,\dots,S^n)$ cannot be feasible w.r.t. v) components of some $y \in \mathscr{LC}(v)$.
Finally many (P,Q)–tight sequences cannot simultaneously be feasible or have to generate the same vector, but a description (even w.r.t. cardinalities) of the arising subsets of (P,Q)–tight sequences which can be identified or dropped is not known yet.

(ii) It is not known whether the modified least core of every c–convex game (N,v) w.r.t. a nontrivial partition of the player set is contained in the classical least core of the game. This author conjectures that the answer should be affirmative, i.e. $\mathscr{MLC}(v) \subseteq \mathscr{LC}(v)$, which is equivalent to the assertion that both (1) and (2) simultaneously hold under the prerequisites of Lemma 3.2.

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