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COMPOSITION OF IRREDUCIBLE MORPHISMS IN COILS

CLAUDIA CHAIO AND PIOTR MALICKI

Abstract. We study the non-zero composition of n irreducible morphisms between modules lying in coils in relation with the powers of the radical of their module category.

1. Introduction and the main results

Throughout the paper, by an *algebra* we mean an artin algebra over a fixed commutative artin ring R. We denote by $\operatorname{mod} A$ the category of finitely generated right A-modules and by $\operatorname{ind} A$ a full subcategory of $\operatorname{mod} A$ consisting of one representative of each isomorphism class of indecomposable A-modules.

We denote the radical of the module category $\operatorname{mod} A$ by rad_A. We recall that, for $X, Y \in \operatorname{ind} A$ the ideal $\operatorname{rad}_A(X, Y)$ is the set of all non-isomorphisms between X and Y. Inductively, the powers of $\operatorname{rad}_A(X, Y)$ are defined. By $\operatorname{rad}_A^\infty(X, Y)$ we denote the intersection of all powers $\operatorname{rad}_A^i(X, Y)$ of $\operatorname{rad}_A(X, Y)$ with $i \geq 1$. Moreover, we denote by Γ_A the *Auslander-Reiten quiver* of A, and by τ_A and τ_A^{-1} the Auslander-Reiten translations $D\mathrm{Tr}$ and $\mathrm{Tr}D$, respectively. Recall that Γ_A is a valued translation quiver defined as follows: the vertices of Γ_A are the isomorphism classes $[X]$ of modules X in $\operatorname{ind} A$, we put an arrow from $[X] \to [Y]$ in Γ_A if there is an irreducible morphism from X to Y in $\operatorname{mod} A$. The valuation (d_{XY}, d'_{XY}) of an arrow $[X] \to [Y]$ in Γ_A is defined such that d_{XY} is the multiplicity of Y in the codomain of the minimal left almost split morphism for X and d'_{XY} is the multiplicity of X in the domain of the minimal right almost split morphism for Y. We shall not distinguish between an indecomposable A-module and the vertex of Γ_A corresponding to it. The valuation $(1, 1)$ of an arrow in Γ_A will be omitted and we will say that a component Γ of Γ_A has *trivial valuation* if all arrows in Γ have valuation $(1, 1)$. By a *component* of Γ_A we mean a connected component of the quiver Γ_A. In general, the Auslander-Reiten quiver Γ_A describes only the quotient category $\operatorname{mod} A/\operatorname{rad}_A^\infty$.

In the representation theory of algebras, a prominent role is played by the components called *stable tubes*, that is, the translation quivers $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ for $r \geq 1$ consisting of τ-periodic vertices of period r. It follows from Zhang's theorem [27] that an infinite

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Auslander-Reiten component $\mathscr{C}$ containing an oriented cycle is stable if and only if $\mathscr{C}$ is a stable tube. More generally, by Liu's theorem [19] an infinite Auslander-Reiten component $\mathscr{C}$ containing an oriented cycle is left stable (respectively, right stable) if and only if $\mathscr{C}$ is a *ray tube* (respectively, *coray tube*), that is, can be obtained from a stable tube by a finite number of ray (respectively, coray) insertions in the sense of D'Este and Ringel [16]. In [2, 3] Assem and Skowroński introduced and investigated a more general type of translation quivers called coils. We recall that a *coil* is a translation quiver obtained from one stable tube by an iterated application of admissible operations of types (ad 1)-(ad 3) and their dual (ad 1*)-(ad 3*) (see Section 3 for details). We mention that the coils have played a fundamental role in the representation theory of strongly simply connected algebras of polynomial growth established in [25]. For example, it was shown in [25] that a strongly simply connected algebra Λ over an algebraically closed field K is of polynomial growth if and only if every infinite component of Γ_Λ containing an oriented cycle is a standard coil. Let us mention that the class of coil algebras, which are the tame algebras with a separating family of coils, have played a fundamental role in the study of tame strongly simply connected algebras [8].

There is a close relationship between irreducible morphisms and the powers of the radical of its module category. In [6] Bautista proved that a morphism $f : X \to Y$ between two indecomposable modules X and Y in a module category $\operatorname{mod} A$ is irreducible if and only if $f \in \operatorname{rad}_A(X,Y) \setminus \operatorname{rad}_A^2(X,Y)$. This was generalized by Igusa and Todorov [17, Theorem 13.3] who proved that, for a sectional path

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

of irreducible morphisms between indecomposable modules in $\operatorname{mod} A$, we have

$$f_n \dots f_2 f_1 \in \operatorname{rad}_A^n(X_1, X_{n+1}) \setminus \operatorname{rad}_A^{n+1}(X_1, X_{n+1}).$$

In [18, 19] Liu introduced the notions of left and right degrees of irreducible morphisms of modules (see 2.2) and showed their importance for describing the shapes of the components of the Auslander-Reiten quivers of algebras of infinite representation type.

An important research direction towards understanding the structure of module categories is the study of compositions of irreducible morphisms between indecomposable modules. Recently, there has been many new results related to the subject of the composition of irreducible morphisms and their relation with the power of the radical of their module category. Most of them involving the concept of degree. For instance, see [9, 10, 11, 13].

In this paper we are interested in the non-zero composition of irreducible morphisms between indecomposable modules lying in infinite Auslander-Reiten components containing oriented cycles called coils in relation with the powers of the radical of their module category.

The main result of this article is the following theorem.

Theorem 1.1. *Let A be an artin algebra and $\mathscr{C}$ a coil in Γ_A. Let*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

be a path of irreducible morphisms with $X_i \in \mathscr{C}$ for $i = 1, \dots, n+1$. Then, $f_n \dots f_1 \in \mathrm{rad}_A^{n+1}(X_1, X_{n+1})$ if and only if $f_n \dots f_1 \in \mathrm{rad}_A^{\infty}(X_1, X_{n+1})$.

Recall that an algebra A is called *selfinjective* if A_A is an injective module, or equivalently, the projective modules in $\mathrm{mod}\, A$ are injective. As a consequence of Theorem 1.1 (see Corollary 4.12 and Remarks 4.13) we obtain the following fact which is a generalization of [13, Theorem A].

Corollary 1.2. *Let A be a selfinjective artin algebra and $\mathscr{C}$ an infinite component of Γ_A containing an oriented cycle. Let*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

be a path of irreducible morphisms with $X_i \in \mathscr{C}$ for $i = 1, \dots, n+1$. Then, $f_n \dots f_1 \in \mathrm{rad}_A^{n+1}(X_1, X_{n+1})$ if and only if $f_n \dots f_1 \in \mathrm{rad}_A^{\infty}(X_1, X_{n+1})$.

For basic background on representation theory of algebras we refer to [1], [5] and [26].

2. Preliminaries

2.1. Let A be an algebra, $X, Y \in \mathrm{ind}\, A$, and $f : X \to Y$ be an irreducible morphism in $\mathrm{mod}\, A$. If X is not injective, we shall denote by $\epsilon(X)$ the almost split sequence starting at X and by $\alpha(X)$ the number of indecomposable direct summands of the middle term of $\epsilon(X)$.

Dually, if X is not projective, we shall denote by $\epsilon'(X)$ the almost split sequence ending in X and by $\alpha'(X)$ the number of indecomposable direct summands of the middle term of $\epsilon'(X)$.

2.2. Let A be an algebra and let $f : X \to Y$ be an irreducible morphism in $\mathrm{mod}\, A$, with X or Y indecomposable. Following Liu [18], the *left degree* $d_l(f)$ of f is infinite, if for each integer $n \geq 1$, each module $Z \in \mathrm{mod}\, A$ and each morphism $g \in \mathrm{rad}_A^n(Z, X) \setminus \mathrm{rad}_A^{n+1}(Z, X)$ we have that $fg \notin \mathrm{rad}_A^{n+2}(Z, Y)$. Otherwise, the left degree of f is the smallest positive integer m such that there is an A-module Z and a morphism $g \in \operatorname{rad}_A^m(Z,X) \setminus \operatorname{rad}_A^{m+1}(Z,X)$ such that $fg \in \operatorname{rad}_A^{m+2}(Z,Y)$.

The *right degree* $d_r(f)$ of an irreducible morphism f is dually defined.

For the convenience of the reader we state below [18, Lemma 1.2], since we shall refer to it frequently throughout this paper.

Lemma 2.3. ([18, Lemma 1.2]). *Let $m \geq 1$ be an integer and let $p : X \to Y$ and $f : Y \to Z$ be morphisms in* $\operatorname{mod} A$*. Suppose that f is irreducible and Z indecomposable. If $p \notin \operatorname{rad}_A^{m+1}(X,Y)$ and $fp \in \operatorname{rad}_A^{m+2}(X,Z)$, then*

(i) *Z is not projective, and*
(ii) *if $0 \longrightarrow \tau_A Z \xrightarrow{(g,g')^t} Y \oplus Y' \xrightarrow{(f,f')} Z \longrightarrow 0$ is an almost split sequence, then there exists a morphism $q : X \to \tau_A Z$ in* $\operatorname{mod} A$ *such that $q \notin \operatorname{rad}_A^m(X, \tau_A Z)$, $p + gq \in \operatorname{rad}_A^{m+1}(X,Y)$ and $g'q \in \operatorname{rad}_A^{m+1}(X,Y')$.*

2.4. Let A be an algebra. By a *path* in Γ_A we mean a sequence of irreducible morphisms between indecomposable modules $Y_1 \to Y_2 \to \cdots \to Y_{n-1} \to Y_n$, and by a *non-zero path* (*zero-path*) we mean that the composition of the irreducible morphisms of the path does not vanish (vanishes).

In [6], Bautista defined the notion of sectional paths. A path $Y_1 \to Y_2 \to \cdots \to Y_{n-1} \to Y_n$ in Γ_A is said to be *sectional* if for each $i = 2, \ldots, n-1$ we have that $Y_{i+1} \not\simeq {\tau_A}^{-1} Y_{i-1}$.

Furthermore, in [17] Igusa and Todorov proved that if

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n$$

is a sectional path then the composition $f_n \ldots f_1 : X_0 \to X_n$ is such that $f_n \ldots f_1 \in \operatorname{rad}_A^n(X_0, X_n) \setminus \operatorname{rad}_A^{n+1}(X_0, X_n)$.

By a *cycle* in Γ_A we mean a sequence of irreducible morphisms between indecomposable modules of the form $Y_1 \to Y_2 \to \cdots \to Y_{n-1} \to Y_n \to Y_1$.

2.5. We recall the definition of depth of a morphism given in [12] for any artin algebra A. Let $f : M \to N$ be a morphism in $\operatorname{mod} A$. We say that the *depth* of f, denoted by $\operatorname{dp}(f)$, is infinite in case $f \in \operatorname{rad}_A^\infty(M,N)$; otherwise, it is the integer $n \geq 0$ for which $f \in \operatorname{rad}_A^n(M,N)$ but $f \notin \operatorname{rad}_A^{n+1}(M,N)$.

2.6. For the convenience of the reader we state [10, Lemma 2.1] and [10, Proposition 2.2] which we will use all through this paper. In fact, taking into account these results it is not hard to see that it is enough to study the irreducible morphisms satisfying the mesh relations of the components under consideration in order to have information on the irreducible morphisms of $\operatorname{mod} A$.

Lemma 2.7. ([10, Lemma 2.1]) *Let A be an artin algebra and Γ be a component of Γ_A with trivial valuation. Let $h_i : X_i \to X_{i+1}$ be an irreducible morphism with $X_i \in \Gamma$, for $i = 1, \dots, n$. Then, for any choice of irreducible morphisms $f_i : X_i \to X_{i+1}$ we have that $h_n \dots h_1 = \delta f_n \dots f_1 + \mu$ with $\delta \in \mathrm{Aut}(X_{n+1})$ and $\mu \in \mathrm{rad}_A^{n+1}(X_1, X_{n+1})$.*

Let $f : X \to Y$ be an irreducible morphism between indecomposable modules in $\mathrm{mod}\, A$. We set

$$\mathrm{Irr}_A(X,Y) = \mathrm{rad}_A(X,Y)/\mathrm{rad}_A^2(X,Y).$$

We recall that $\mathrm{Irr}_A(X,Y)$ is a $k_X - k_Y-$bimodule where

$$k_X = \mathrm{End}_A(X)/\mathrm{rad}_A(X,X) \quad \text{and} \quad k_Y = \mathrm{End}_A(Y)/\mathrm{rad}_A(Y,Y).$$

Moreover, k_Z is a division ring whenever Z is an indecomposable A-module.

Proposition 2.8. ([10, Proposition 2.2]) *Let A be an artin algebra and $X_i \in \mathrm{ind}\, A$ for $1 \leq i \leq n+1$. Assume that $\dim_{k_{X_i}} \mathrm{Irr}_A(X_i, X_{i+1}) = \dim_{k_{X_{i+1}}} \mathrm{Irr}_A(X_i, X_{i+1}) = 1$, for $i = 1, \dots, n$. Then, the following conditions are equivalent:*

(i) *There are irreducible morphisms $f_i : X_i \to X_{i+1}$ in $\mathrm{mod}\, A$, for $i = 1, \dots, n$ with $f_n \dots f_1 \notin \mathrm{rad}_A^{n+1}(X_1, X_{n+1})$.*

(ii) *Given any irreducible morphisms $h_i : X_i \to X_{i+1}$ in $\mathrm{mod}\, A$, for $i = 1, \dots, n$, then $h_n \dots h_1 \notin \mathrm{rad}_A^{n+1}(X_1, X_{n+1})$.*

3. Coils

We shall recall some basic facts on coils introduced by Assem and Skowroński in [2] and [3].

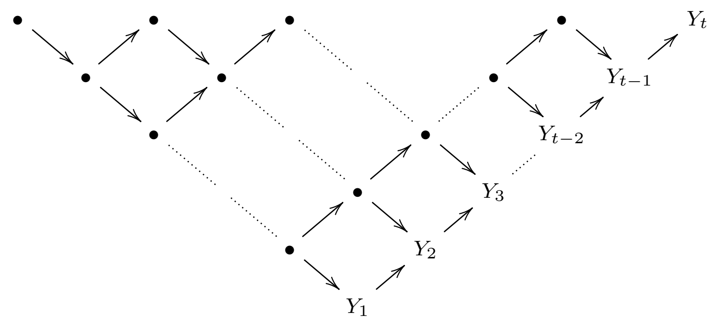
3.1. A translation quiver Γ is called a *tube* if it contains a cyclical path and if its underlying topological space is homeomorphic to $S^1 \times \mathbb{R}^+$ (where S^1 is the unit circle and $\mathbb{R}^+$ the set of non-negative real numbers). Tubes containing neither projective vertices nor injective vertices are called *stable*. The *rank* of a stable tube $\mathscr{T}$ is the least positive integer r such that $\tau^r X = X$ for all X in $\mathscr{T}$.

3.2. A coil is a translation quiver constructed inductively from a stable tube by a sequence of operations called admissible. Our first task is thus to define the latter. Let (Γ, τ) be a translation quiver with trivial valuations. For a vertex X in Γ, called the *pivot*, one defines three operations modifying (Γ, τ) to a new translation quiver (Γ', τ') depending on the shape of paths in Γ starting from X.

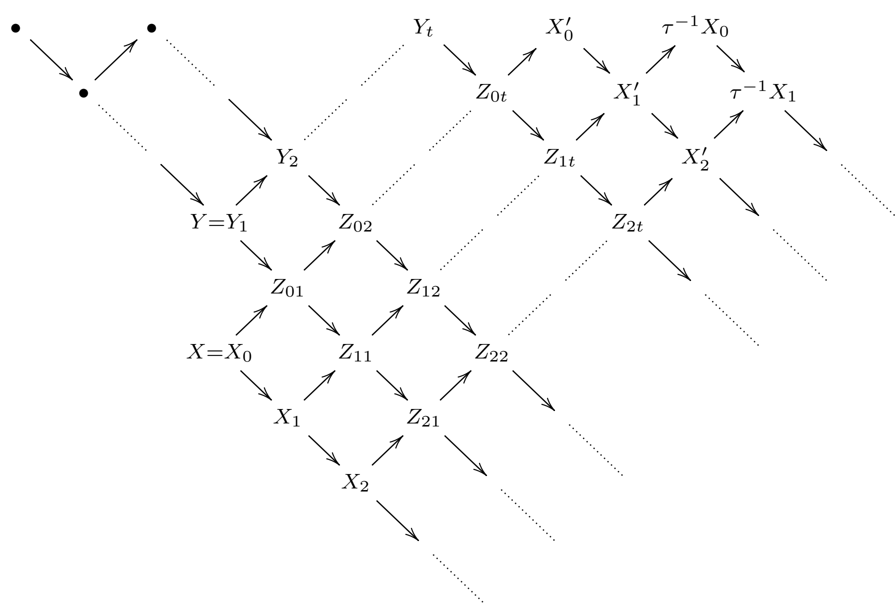
(ad 1) Suppose that Γ admits an infinite sectional path

$$X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

starting at X, and assume that every sectional path in Γ starting at X is a subpath of the above path. For $t \geq 1$, let Γ_t be the following translation quiver, isomorphic to the Auslander-Reiten quiver of the full $t \times t$ lower triangular matrix algebra,



We then let Γ' be the translation quiver having as vertices those of Γ, those of Γ_t, additional vertices Z_{ij} and X'_i (where $i \geq 0$, $1 \leq j \leq t$) and having arrows as in the figure below

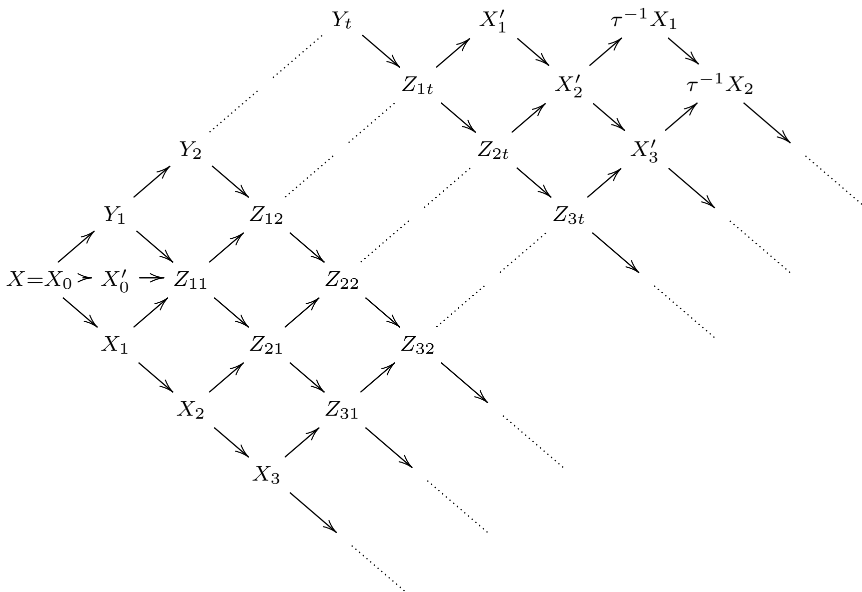


The translation τ' of Γ' is defined as follows: $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 1, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{0j} = Y_{j-1}$ if $j \geq 2, Z_{01}$ is projective, $\tau' X'_0 = Y_t, \tau' X'_i = Z_{i-1,t}$ if $i \geq 1, \tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_t, respectively. If $t = 0$, the new translation quiver Γ' is obtained from Γ by inserting only the sectional path consisting of the vertices X'_i, $i \geq 0$.

(ad 2) Suppose that Γ admits two sectional paths starting at X, one infinite and the other finite with at least one arrow

$$Y_t \longleftarrow \cdots \longleftarrow Y_2 \longleftarrow Y_1 \longleftarrow X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

such that any sectional path starting at X is a subpath of one of these paths and X_0 is injective. Then Γ' is the translation quiver having as vertices those of Γ, additional vertices denoted by X'_0, Z_{ij}, X'_i (where $i \geq 1$, $1 \leq j \leq t$), and having arrows as in the figure below



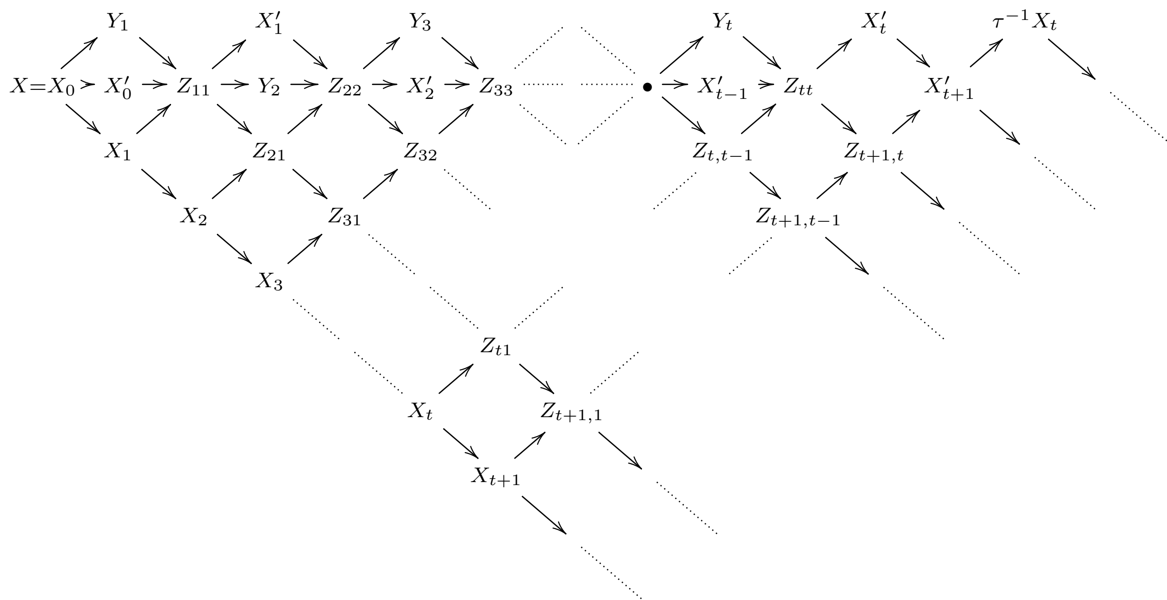
The translation τ' of Γ' is defined as follows: X'_0 is projective-injective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2, j \geq 2, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1, \tau' Z_{1j} = Y_{j-1}$ if $j \geq 2, \tau' X'_i = Z_{i-1,t}$ if $i \geq 2$, $\tau' X'_1 = Y_t$, $\tau'(\tau^{-1} X_i) = X'_i$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

(ad 3) Suppose that Γ admits a full translation subquiver

$$\begin{array}{ccccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow \cdots \longrightarrow & Y_t & & & & \\ \uparrow & & \uparrow & & \uparrow & & & & \\ X = X_0 & \longrightarrow & X_1 & \longrightarrow \cdots \longrightarrow & X_{t-1} & \longrightarrow & X_t & \longrightarrow & \cdots \end{array}$$

$t \geq 2$, X_{t-1} is injective, the paths $Y_1 \longrightarrow Y_2 \longrightarrow \cdots \longrightarrow Y_t$, $X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$ are sectional and every sectional path in Γ starting at X_0 (respectively, at Y_1) is a subpath of one of the paths $X_0 \longrightarrow Y_1$ or $X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$ (respectively, of $Y_1 \longrightarrow Y_2 \longrightarrow \cdots \longrightarrow Y_t$). Moreover, consider the subquiver of Γ obtained by deleting the arrows $Y_i \to \tau^- Y_{i-1}, 2 \leq i \leq t$, and assume that its connected component Γ^* containing the vertex X does not contain any of the vertices $\tau^- Y_{i-1}, 2 \leq i \leq t$. Then

Γ' is the translation quiver having as vertices those vertices of Γ^*, additional vertices denoted by X'_i, Z_{kj} (where $i \geq 0$, $1 \leq j \leq t$, $k \geq j$), and having arrows as in the figure below for t being an odd number



For t being an even number, we have to exchange Y_t with X'_{t-1} in the figure above (see Example 6.2 for $t = 4$). The translation τ' of Γ' is defined as follows: X'_0 is projective, $\tau' Z_{ij} = Z_{i-1,j-1}$ if $i \geq 2, 2 \leq j \leq t, \tau' Z_{i1} = X_{i-1}$ if $i \geq 1$, $\tau' X'_i = Y_i$ if $1 \leq i \leq t$, $\tau' X'_i = Z_{i-1,t}$ if $i \geq t+1, \tau' Y_j = X'_{j-2}$ if $2 \leq j \leq t$, $\tau'(\tau^{-1} X_i) = X'_i$, if $i \geq t$ provided X_i is not injective in Γ, otherwise X'_i is injective in Γ'. In both cases, X'_{t-1} is injective. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

Finally, together with each of the admissible operations (ad 1), (ad 2) and (ad 3), we consider its dual, denoted by (ad 1*), (ad 2*) and (ad 3*). These six operations are called the *admissible operations*.

Clearly, the admissible operations can be defined as operations on Auslander-Reiten components rather than on translation quivers. The definitions are done in an obvious manner (see [3, Section 2] or [22, Section 3] in a more general context).

Definition 3.3. A connected translation quiver Γ is said to be a *coil* if Γ can be obtained from a stable tube $\mathscr{T}$ by an iterated application of admissible operations (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3) or (ad 3*).

Observe that any stable tube is trivially a coil. A *tube* (in the sense of [16]) is a coil having the property that each admissible operation in the sequence defining it is of the form (ad 1) or (ad 1*). If we apply only operations of type (ad 1) (respectively, of type (ad 1*)) then such a coil is called a *ray tube* (respectively, a *coray tube*) (see

[19, Section 1] and [26, (4.6)]). Observe that a coil without injective (respectively, projective) vertices is a ray tube (respectively, a coray tube). A *quasi-tube* (in the sense of [24]) is a coil having the property that each of the admissible operations in the sequence defining it is of type (ad 1), (ad 1*), (ad 2) or (ad 2*).

3.4. Let A be an algebra. A component Γ of Γ_A is called *coherent* if the following two conditions are satisfied:

(C1) For each projective module P in Γ there is an infinite sectional path

$$P \;=\; X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow X_{i+2} \longrightarrow \cdots$$

(C2) For each injective module I in Γ there is an infinite sectional path

$$\cdots \longrightarrow Y_{j+2} \longrightarrow Y_{j+1} \longrightarrow Y_j \longrightarrow \cdots \longrightarrow Y_2 \longrightarrow Y_1 = I.$$

Further, a component Γ of Γ_A is called *almost cyclic* if all but finitely many modules of Γ lie on oriented cycles in Γ_A, so contained entirely in Γ. Note that in [14] and [15], the authors studied the finiteness of degrees of irreducible morphisms between indecomposable modules lying in (generalized standard) coherent almost cyclic components of Γ_A.

We note that any coil Γ is a coherent translation quiver with trivial valuations and its cyclic part ${}_c\Gamma$ (obtained from Γ by removing all acyclic vertices and the arrows attached to them) is infinite and cofinite in Γ, and so Γ is almost cyclic.

Let Γ be a connected component of Γ_A with trivial valuations. We denote by $|\Gamma|$ the geometric realization of Γ, as defined in [7, (4.1)], and by $\pi_1(|\Gamma|)$ the fundamental group $\pi_1(|\Gamma|, X)$ of $|\Gamma|$ at a fixed vertex X of Γ. Then we have the following characterization of components of Γ_A which are coils proved in [21, Corollary D].

Proposition 3.5. *Let A be an artin algebra and Γ a connected component of Γ_A. Then Γ is a coil if and only if Γ is coherent, almost cyclic, and $\pi_1(|\Gamma|)$ is an infinite cyclic group.*

We refer also to [23] for the structure of indecomposable modules lying in (generalized) standard coils.

3.6. It follows from the definition that coils share many properties with tubes. For instance, all but finitely many vertices in a coil belong to a cyclical path. A vertex X in a coil Γ is said to be belong to the *mouth* of Γ if X is the starting or ending, vertex of a mesh in Γ with a unique middle term. Also, Γ contains a (maximal) tube as a cofinite full translation subquiver. Arrows of this tube either point to the mouth or point to infinity. An infinite sectional path in Γ

$$X = X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{i-1}} X_i \xrightarrow{\alpha_i} X_{i+1} \xrightarrow{\alpha_{i+1}} \cdots$$

is called a *ray starting at* X if there exists $i_0 \geq 1$ such that, for all $i \geq i_0$, the arrow α_i points to infinity. Dually an infinite sectional path in Γ

$$\cdots \xrightarrow{\beta_{j+1}} X_{j+1} \xrightarrow{\beta_j} X_j \xrightarrow{\beta_{j-1}} \cdots \xrightarrow{\beta_2} X_2 \xrightarrow{\beta_1} X_1 = X$$

is called a *coray ending with* X if there exists $j_0 \geq 1$ such that, for all $j \geq j_0$, the arrow β_j points to the mouth.

3.7. In the next considerations we need the following notions. For an admissible operation (ad 1) (respectively, (ad 2) and (ad 3)), we consider the infinite set of vertices $\mathcal{R}_r^1 = \{X_i, X_i', Z_{ij} \mid i \geq 0, 1 \leq j \leq t\}$ (respectively, $\mathcal{R}_r^2 = \{X_i, X_i', Y_j, Z_{i+1,j} \mid i \geq 0, 1 \leq j \leq t\}$ and $\mathcal{R}_r^3 = \{X_i, X_i', Y_j, Z_{kj} \mid i \geq 0, 1 \leq j \leq t, k \geq j\}$) (see definitions of admissible operations). Then $\mathcal{R}_r^1$ (respectively, $\mathcal{R}_r^2$ and $\mathcal{R}_r^3$) is called the *right rectangle determined by (ad 1)* (respectively, (ad 2) and (ad 3)). One defines dually the *left rectangle* $\mathcal{R}_l^1$ (respectively, $\mathcal{R}_l^2$ and $\mathcal{R}_l^3$) *determined by (ad 1*)* (respectively, (ad 2*) and (ad 3*)).

4. The results

We start this section recalling the definition of compositions of morphisms behaving well from [13].

Definition 4.1. We say that a composition $\varphi_m \dots \varphi_1$ of morphisms (respectively, irreducible morphisms) φ_j, for $j = 1, \dots, m$, in $\operatorname{mod} A$ (respectively, in a component Γ of Γ_A) *behaves well* whenever $\operatorname{dp}(\varphi_j) = r_j$ with $r_j \geq 0$ then $\operatorname{dp}(\varphi_m \dots \varphi_1) = r_m + \cdots + r_1$.

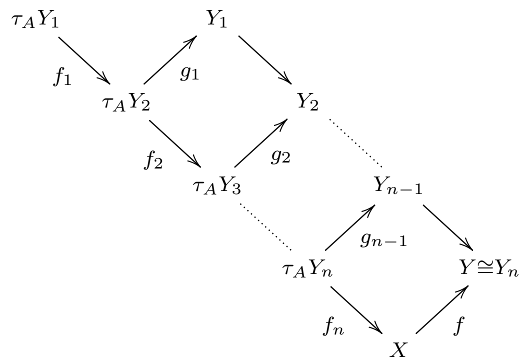
Next, we observe the following useful fact.

Remark 4.2. The composition of irreducible morphisms involving only irreducible morphisms with infinite left (right) degree behaves well. In fact, if the composition of n irreducible morphisms between indecomposable modules lies in $\operatorname{rad}_A^{n+1}$ then the path contains both an arrow of finite left degree and an arrow of finite right degree, see [18, p. 41].

As an immediate consequence of the proof of [15, Theorem A] and the fact that coils do not contain exceptional configurations of modules (see [15] for the definition) we obtain the following theorem.

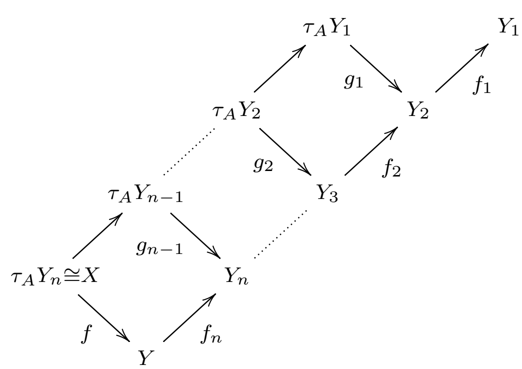
Theorem 4.3. *Let A be an artin algebra and $\mathscr{C}$ a coil in Γ_A. Let $f : X \to Y$ be an irreducible morphism in* $\operatorname{mod} A$ *with $X, Y \in \mathscr{C}$, and n a positive integer. Then, the following equivalences hold.*

(i) $d_l(f) = n$ *if and only if* $\mathscr{C}$ *admits a mesh-complete full subquiver of the form*



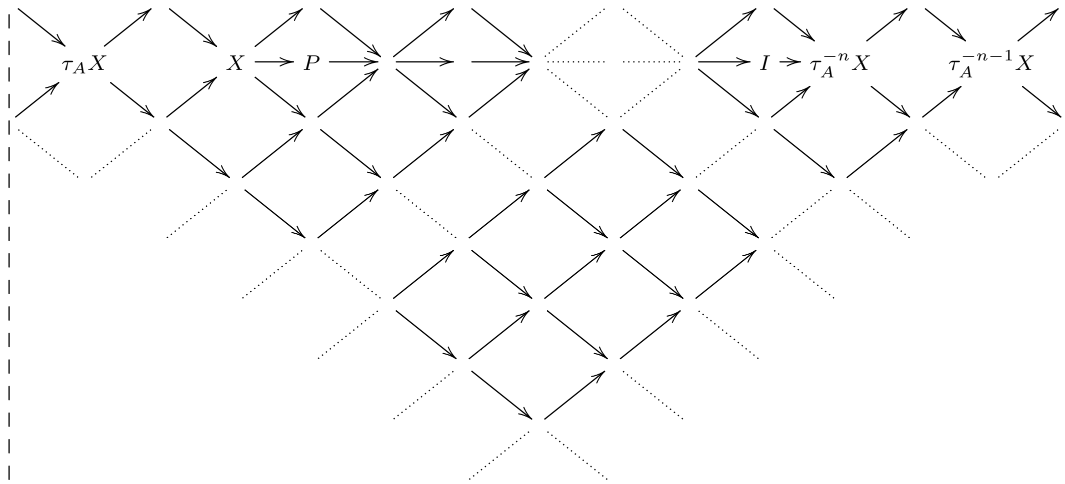
where $\tau_A Y_1 \xrightarrow{f_1} \tau_A Y_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} \tau_A Y_n \xrightarrow{f_n} X$ *is a sectional path of irreducible morphisms in* $\operatorname{mod} A$ *such that* $f f_n \dots f_1 = 0$ *and* $\alpha(Y_1) = 1$*. Moreover,* $d_l(g_i) = i$ *for* $i = 1, \dots, n-1$*.*

(ii) $d_r(f) = n$ *if and only if* $\mathscr{C}$ *admits a mesh-complete full subquiver of the form*



where $Y \xrightarrow{f_n} Y_n \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} Y_2 \xrightarrow{f_1} Y_1$ *is a sectional path of irreducible morphisms in* $\operatorname{mod} A$ *such that* $f_1 f_2 \dots f_n f = 0$ *and* $\alpha'(\tau_A Y_1) = 1$*. Moreover,* $d_r(g_i) = i$ *for* $i = 1, \dots, n-1$*.*

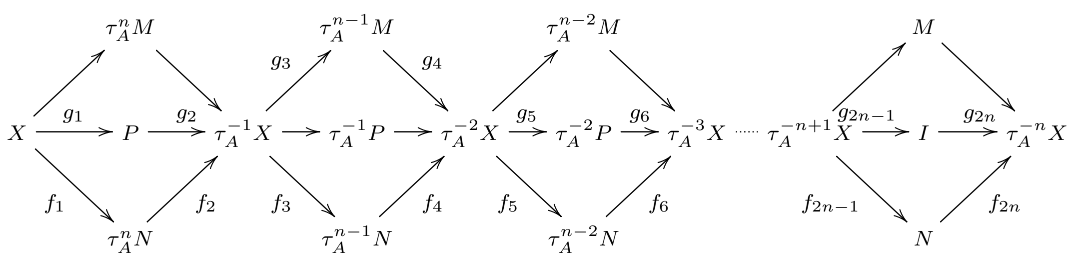
Lemma 4.4. *Let* A *be an artin algebra and* $\mathscr{C}$ *a coil in* Γ_A*. Consider the configuration of almost split sequences in* $\mathscr{C}$ *as follows:*



where P is projective, I is injective and n is a positive odd integer. Then, the following statements hold.

(i) *All non-zero compositions of irreducible morphisms from X to $\tau_A^{-n}X$ satisfying the mesh relations in $\mathscr{C}$ behave well.*
(ii) *All non-zero compositions of irreducible morphisms from $\tau_A X$ to $\tau_A^{-n-1}X$ satisfying the mesh relations in $\mathscr{C}$ behave well.*

Proof. (i) Using the additive function we can see that there are only two morphisms in $\mathscr{C}$ from X to $\tau_A^{-n}X$, the ones involving the almost split sequences with three indecomposable middle terms. We observe that, by construction, the other paths in $\mathscr{C}$ from X to $\tau_A^{-n}X$, vanish. Hence, it is enough to consider any two different paths modulo mesh. We illustrate the situation of the possible paths in $\mathscr{C}$ with the following diagram of almost split sequences with three indecomposable middle terms:



Without loss of generality, we can consider the following two paths

$$(1)\qquad X \xrightarrow{g_1} P \xrightarrow{g_2} \tau_A^{-1}X \xrightarrow{g_3} \tau_A^{n-1}M \xrightarrow{g_4} \tau_A^{-2}X \xrightarrow{g_5} \tau_A^{-2}P \xrightarrow{g_6}$$

$$\tau_A^{-3}X \longrightarrow \cdots \longrightarrow \tau_A^{-n+1}X \xrightarrow{g_{2n-1}} I \xrightarrow{g_{2n}} \tau_A^{-n}X$$

and

$$(2)\ X \xrightarrow{f_1} \tau_A^n N \xrightarrow{f_2} \tau_A^{-1} X \xrightarrow{f_3} \tau_A^{n-1} N \xrightarrow{f_4} \tau_A^{-2} X \xrightarrow{f_5} \tau_A^{n-2} N \xrightarrow{f_6}$$
$$\tau_A^{-3} X \longrightarrow \cdots \longrightarrow \tau_A^{-n+1} X \xrightarrow{f_{2n-1}} N \xrightarrow{f_{2n}} \tau_A^{-n} X$$

which are clearly different modulo mesh.

We start analyzing the first path. Assume that $g_{2n} \dots g_1 \in \mathrm{rad}_A^{2n+1}(X, \tau_A^{-n} X)$. By [15, Lemma 3.1], we know that $d_r(g_1) = \infty$ and $d_l(g_{2n}) = \infty$. Then, the fact that $g_{2n} \dots g_1 \in \mathrm{rad}_A^{2n+1}(X, \tau_A^{-n} X)$ implies that $g_{2n-1} \dots g_2 \in \mathrm{rad}_A^{2n-1}(P, I)$.

Now, assume that $\mathrm{dp}(g_{2n-2} \dots g_2) = 2n - 3$. Since

$$0 \longrightarrow \tau_A I \xrightarrow{t_{2n-2}} \tau_A^{-n+1} X \xrightarrow{g_{2n-1}} I \longrightarrow 0$$

is an almost split sequence, then by Lemma 2.3, there is a morphism $\varphi : P \to \tau_A I$ in $\mathrm{mod}\, A$ such that $\varphi \notin \mathrm{rad}_A^{2n-3}(P, \tau_A I)$. Using the additive function and the fact that the number of almost split sequences with three indecomposable middle terms are odd, is not hard to see that any morphism in $\mathscr{C}$ from P to $\tau_A I$ vanish, contradicting Lemma 2.3. Therefore, if $g_{2n-1} \dots g_2 \in \mathrm{rad}_A^{2n-1}(P, I)$ then $g_{2n-2} \dots g_2 \in \mathrm{rad}_A^{2n-2}(P, \tau_A^{-n+1} X)$.

On the other hand, since $d_l(g_{2n-2}) = \infty$ then we can consider that $g_{2n-3} \dots g_2 \in \mathrm{rad}_A^{2n-3}(P, \tau_A I)$. Iterating a finite number of times the same arguments as above, we get that $g_4 g_3 g_2 \in \mathrm{rad}_A^4(P, \tau_A^{-2} X)$. Since by [11, Lemma 2.1], we know that $g_3 g_2 \notin \mathrm{rad}_A^3(P, \tau_A^{n-1} M)$ and $g_4 g_3 \notin \mathrm{rad}_A^3(\tau_A^{-1} X, \tau_A^{-2} X)$ then we get a contradiction to [9, Theorem C]. Therefore, we prove that the composition $g_{2n} \dots g_1$ behaves well.

Now, if we consider the second path, then by [15, Lemma 3.1], we know that all the morphisms f_i are such that $d_l(f_i) = \infty$, for $i = 1, \dots, 2n$. Then, the result follows trivially by Remark 4.2.

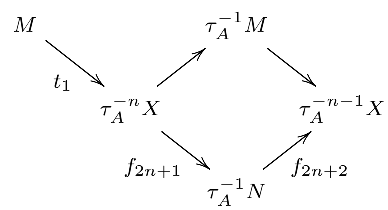
(ii) Note that we only have one morphism from $\tau_A X$ to $\tau_A^{-n-1} X$ in $\mathscr{C}$. Without loss of generality, we can consider the following path

$$\tau_A X \xrightarrow{f_{-1}} \tau_A^{n+1} N \xrightarrow{f_0} X \xrightarrow{f_1} \tau_A^n N \xrightarrow{f_2} \tau_A^{-1} X \xrightarrow{f_3} \tau_A^{n-1} N \xrightarrow{f_4}$$
$$\tau_A^{-2} X \xrightarrow{f_5} \tau_A^{n-2} N \xrightarrow{f_6} \tau_A^{-3} X \longrightarrow \dots \longrightarrow \tau_A^{-n+1} X \xrightarrow{f_{2n-1}} N \xrightarrow{f_{2n}}$$
$$\tau_A^{-n} X \xrightarrow{f_{2n+1}} \tau_A^{-1} N \xrightarrow{f_{2n+2}} \tau_A^{-n-1} X.$$

Assume that $f_{2n+2} f_{2n+1} f_{2n} \dots f_1 f_0 f_{-1} \in \mathrm{rad}_A^{2n+5}(\tau_A X, \tau_A^{-n-1} X)$. Since f_{-1} is an irreducible epimorphism of infinite right degree then the above assumption is equivalent to say that $f_{2n+2} f_{2n+1} f_{2n} \dots f_1 f_0 \in \mathrm{rad}_A^{2n+4}(\tau_A^{n+1} N, \tau_A^{-n-1} X)$.

By [9, Theorem C], we know that $\mathrm{dp}(f_2 f_1 f_0) = 3$. Moreover, by Theorem 4.3 we know that $d_r(f_i) = d_l(f_i) = \infty$, for $i = 3, \dots, 2n$. Then, we have that the composition $f_{2n} \dots f_1 f_0 f_{-1}$ behaves well. Furthermore, the irreducible morphism f_{2n+1} is such that $d_r(f_{2n+1}) = \infty$, then $f_{2n+1} f_{2n} \dots f_1 f_0 f_{-1}$ also behaves well.

Finally, again by Theorem 4.3, we know that $d_l(f_{2n+2}) < \infty$. Moreover, $d_l(f_{2n+2}) = 2$. Then, there is a configuration of almost split sequences as follows:

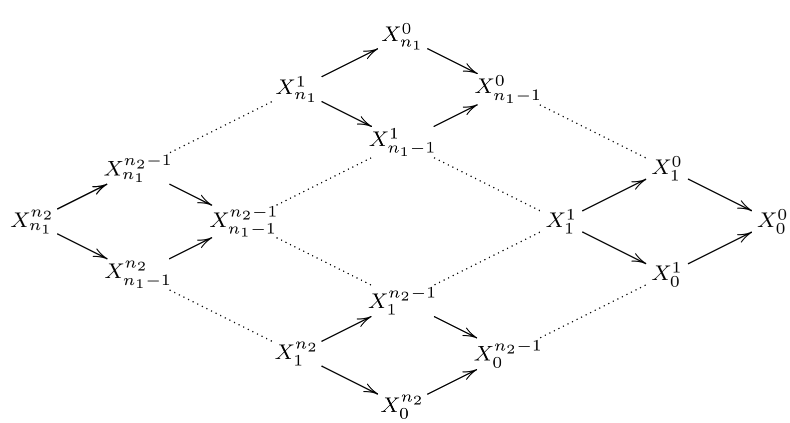


If $f_{2n+2}f_{2n+1}\dots f_1f_0 \in \mathrm{rad}_A^{2n+4}(\tau_A^{n+1}N, \tau_A^{-n-1}N)$ then by Lemma 2.3 there is a morphism $\varphi : \tau_A^{n+1}N \to M$ in $\mathrm{mod}\, A$, such that $\varphi \notin \mathrm{rad}_A^{2n+2}(\tau_A^{n+1}N, M)$. Using the additive function is not hard to see, because of the shape of the component, that there does not exist a non-zero morphism from $\tau_A^{n+1}N$ to M in $\mathscr{C}$. Moreover, by Proposition 2.8 we conclude that there are not non-zero morphisms from $\tau_A^{n+1}N$ to M in $\mathrm{mod}\, A$. Hence, $\mathrm{dp}(f_{2n+2}f_{2n+1}f_{2n}\dots f_1f_0) = 2n+3$, proving the result. □

Remark 4.5. Let us mention that in the formulations of our statements involving the admissible operation (ad 3) we use an odd number of almost split sequences with three indecomposable middle terms throughout all this paper (see the definition of (ad 3)). For an admissible operation (ad 3) with an even number of almost split sequences with three indecomposable middle terms the statements are the same and the proofs are analogues.

Because of Theorem 4.3, the result proved in [10, Proposition 2.3] can be generalized to coils components in Auslander-Reiten quivers, since the tools needed to prove it still hold true here. For the convenience of the reader we formulate the statement.

Lemma 4.6. [10, Proposition 2.3] *Let A be an artin algebra, $X_i^j \in \mathrm{ind}\, A$, for $i = 0, \dots, n_1$ and $j = 0, \dots, n_2$. Let $\mathscr{C}$ be a coil in Γ_A and assume that there is a configuration of almost split sequences in* $\mathrm{mod}\, A$ *as follows*



where $\alpha'(X_i^j) = 2$, *for* $i = 1, \dots, n_1$, $j = 1, \dots, n_2$. *Let* $g_i : X_i^{n_2} \to X_{i-1}^{n_2}$ *for* $i = 1, \dots, n_1$ *and* $f_j : X_0^j \to X_0^{j-1}$ *for* $j = 1, \dots, n_2$ *are irreducible morphisms satisfying the mesh relations, then the composition* $f_1 \dots f_{n_2} g_1 \dots g_{n_1} \notin \mathrm{rad}_A^{n_1+n_2+1}$. *Moreover, given any irreducible morphism* $h_0^j : X_0^j \to X_0^{j-1}$ *and* $h_i^{n_2} : X_i^{n_2} \to X_{i-1}^{n_2}$ *in* $\mathrm{mod}\, A$ *for* $i = 1, \dots, n_1$ *and* $j = 1, \dots, n_2$ *then* $\mathrm{dp}(h_0^1 \dots h_0^{n_2} h_1^{n_2} \dots h_{n_1}^{n_2}) = n_1 + n_2$.

The next result shows that it is enough to consider non-zero paths in a coil $\mathscr{C}$ in order to analyze if they behave well.

Lemma 4.7. *Let* A *be an artin algebra and* $\mathscr{C}$ *be a coil in* Γ_A. *Assume we have in* $\mathscr{C}$ *a zero-path of irreducible morphisms*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

with $X_i \in \mathscr{C}$ *for* $i = 1, \dots, n+1$ *of length* $n \geq 1$. *Then, any path from* X_1 *to* X_{n+1} *in* $\mathscr{C}$ *of length greater than* n *vanishes.*

Proof. Let A be an artin algebra and $\mathscr{C}$ be a coil in Γ_A. It follows from [21, Theorem E] that a coil $\mathscr{C}$ is a connected component of an Auslander-Reiten quiver Γ_A. By the definition of coil we know that $\mathscr{C}$ is a coherent component of Γ_A, that is, every projective module in $\mathscr{C}$ is the starting module of an infinite sectional path and every injective module in $\mathscr{C}$ is the ending module of an infinite sectional path. Since $\mathscr{C}$ is also almost cyclic component of Γ_A, that is, all but finitely many modules of $\mathscr{C}$ lie on oriented cycles in $\mathscr{C}$, applying [21, Theorem A], we infer that $\mathscr{C}$, considered as a translation quiver, can be obtained from a stable tube by an iterated application of admissible operations of type (ad 1), (ad 2), (ad 3) and their duals. Moreover, by [21, Theorem F], we obtain that $\mathscr{C}$ is a connected coherent infinite translation quiver with a positively valued additive length function ℓ such that there are convex subquiver $\mathscr{D}$ of the left stable part ${}_l\mathscr{C}$ of $\mathscr{C}$ and convex subquiver $\mathscr{E}$ of the right stable part ${}_r\mathscr{C}$ of $\mathscr{C}$ satisfying the conditions that $\mathscr{D}$ is cyclic coray tube, $\mathscr{E}$ is cyclic ray tube, and the vertices of $\mathscr{D}$, $\mathscr{E}$ exhaust all but finitely many vertices of $\mathscr{C}$. From [21, Corollary B], we also know that every arrow in $\mathscr{C}$ has the trivial valuation.

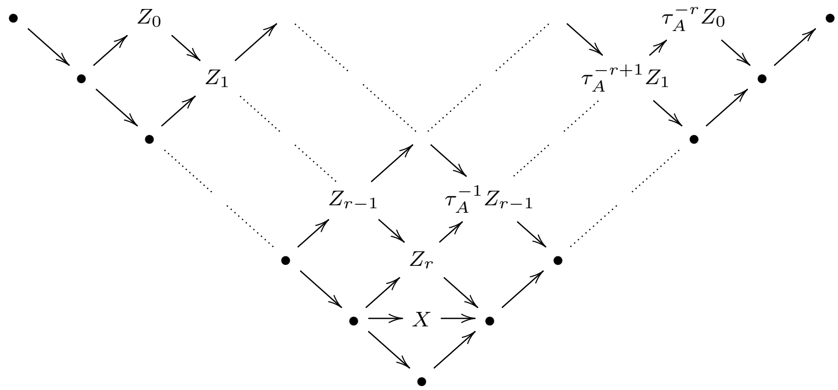
Now, let $\alpha : X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n \longrightarrow X_{n+1}$ be a zero path of irreducible morphisms in $\mathscr{C}$ such that the subpath $\beta : X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n$ of α is non-zero. Then, it follows from the definition of admissible operations that, we have three cases.

(a) The vertex X_{n+1} lies on a ray

$$Y_1 \longrightarrow Y_2 \longrightarrow \cdots \longrightarrow Y_i = X_{n+1} \longrightarrow Y_{i+1} \longrightarrow \cdots$$

starting at Y_1, where $i \geq 1$ and Y_1 belongs to the mouth of $\mathscr{C}$. Let s be the length of shortest path in $\mathscr{C}$ from X_1 to Y_1. Then, for arbitrary $i \geq 1$, any path from X_1 to Y_i in $\mathscr{C}$ of length $s+i-1$ vanishes. Therefore, any path of irreducible morphisms in $\mathscr{C}$ from X_1 to Y_i, for $i \geq 1$, is zero path.

(b) The vertex $X_{n+1} = Z_j$ for some $0 \leq j \leq r$, where Z_j belongs to the finite sectional path $Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_r$ in the full translation subquiver of $\mathscr{C}$ of the form

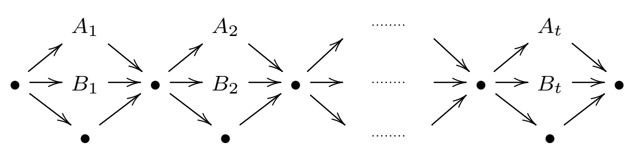


Note that for each $0 \leq j \leq r$, the vertex $\tau_A^{-j}Z_0$ belongs to the mouth of $\mathscr{C}$, and X is projective-injective. Let s be the length of shortest path in $\mathscr{C}$ from X_1 to Z_0. Then, for $0 \leq j \leq r$ and $0 \leq k \leq r-j$, any path from X_1 to $\tau_A^{-k}Z_j$ in $\mathscr{C}$ of length $s+2k+j$ vanishes. Moreover, in this case, there exists the ray in $\mathscr{C}$ of the form

$$\tau_A^{-r-2}Z_0 = Y_0 \longrightarrow \tau_A^{-r-2}Z_1 = Y_1 \longrightarrow \cdots \longrightarrow \tau_A^{-r-2}Z_r = Y_r \longrightarrow Y_{r+1} \longrightarrow \cdots$$

starting at the vertex $\tau_A^{-r-2}Z_0$ lying on the mouth of $\mathscr{C}$. Then, for arbitrary $i \geq 0$, any path from X_1 to Y_i in $\mathscr{C}$ of length $s+2r+4+i$ vanishes. Therefore, any path of irreducible morphisms in $\mathscr{C}$ from X_1 to Y_i, for $i \geq 0$, is zero path.

(c) The vertex X_{n+1} belongs to a mesh with exactly three middle terms and lies on the mouth of $\mathscr{C}$. More precisely, $X_{n+1} = A_j$ or $X_{n+1} = B_j$, for some $1 \leq j \leq t$, where A_j and B_j belong to the full translation subquiver of $\mathscr{C}$ of the form



$t \geq 2$, A_1 or B_1 is projective, and A_t or B_t is injective. Let s be the length of shortest path in $\mathscr{C}$ from X_1 to A_t, so also from X_1 to B_t. We know that at most one of A_t and B_t can be injective. Let Y be not injective, where Y is equal to A_t or B_t. Then, there exists the ray in $\mathscr{C}$ of the form

$$\tau_A^{-2}Y = Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \cdots$$

starting at the vertex $\tau_A^{-2}Y$ lying on the mouth of $\mathscr{C}$. Then, for arbitrary $i \geq 0$, any path from X_1 to Y_i in $\mathscr{C}$ of length $s+4+i$ vanishes. Therefore, any path of irreducible morphisms in $\mathscr{C}$ from X_1 to Y_i, for $i \geq 0$, is zero path.

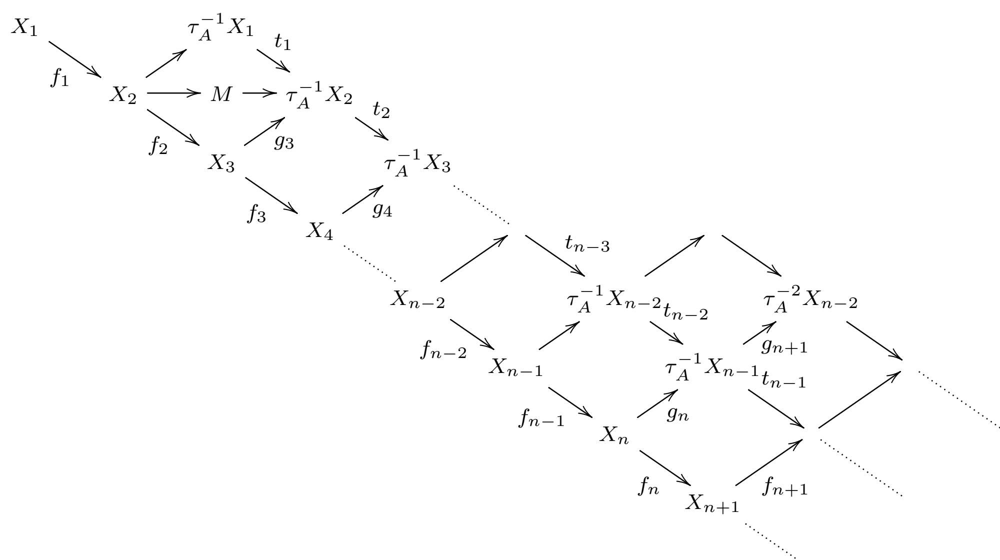
Therefore, any path in $\mathscr{C}$ from X_1 to X_{n+1} of length greater than n vanishes. □

Next, we observe some facts that we shall use through this paper.

Remarks 4.8. Let us note the following.

(a) We would like to stress that all presented facts can be formulated and proved for dual admissible operations (ad 1*), (ad 2*) and (ad 3*) in a similar way.
(b) Note that if we have a path φ of n irreducible morphisms between indecomposable modules which behaves well, then any subpath of φ also behave well.

Lemma 4.9. *Let A be an artin algebra and and $\mathscr{C}$ be a coil in Γ_A which contains the following mesh-complete full translation subquiver*



with M any module in $\mathscr{C}$, $\alpha(X_1) = 1$, $\alpha(X_2) = 3$, $\alpha(\tau_A^{-1}X_{n-2}) = 2$ and $\alpha(X_i) = 2$ for $i \geq 3$. Then, any non-zero composition of irreducible morphisms satisfying the mesh relations of the above subquiver behave well.

Proof. By Lemma 4.4 (i) and Remark 4.8 (b), we know that any non-zero path in $\mathscr{C}$ from X_1 to $\tau_A^{-1}X_2$ behave well. Using the additive function in $\mathscr{C}$, it is not hard to see that it is enough to analyze that the compositions

(1) $g_{n+1}g_nf_{n-1}\dots f_1$, and
(2) $t_{n-1}g_nf_{n-1}\dots f_1$

of irreducible morphisms satisfying the mesh relations behave well.

Consider the composition in (1). Since $f_{n-1}\dots f_1$ is a sectional path, we know that $\mathrm{dp}(f_{n-1}\dots f_1) = n-1$, by [17]. Moreover, by [15, Lemma 3.1], we have that $d_l(g_{n+1}) = d_l(g_n) = \infty$. Therefore, $\mathrm{dp}(g_{n+1}g_nf_{n-1}\dots f_1) = n+1$.

Now, if we consider the composition in (2), we observe that

$$t_{n-1}g_nf_{n-1}\dots f_1 = f_{n+1}f_nf_{n-1}\dots f_1$$

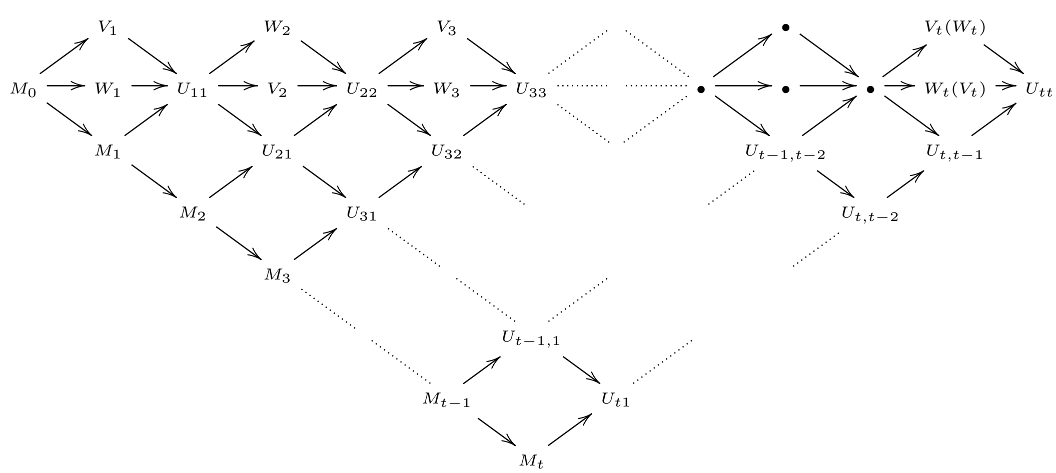
modulo mesh with $\mathrm{dp}(f_n\dots f_1) = n$, because the irreducible morphisms $f_1,\dots,f_n$ belong to a sectional path. Moreover, again by [15, Lemma 3.1], we have that $d_l(f_{n+1}) = \infty$. Hence, we get the result. □

Proposition 4.10. *Let A be an artin algebra and $\mathscr{C}$ be a coil in Γ_A obtained from a stable tube by the admissible operation* (ad 3)*. Then, any non-zero composition of n irreducible morphisms*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

between modules in the component $\mathscr{C}$ and satisfying the mesh relations are such that $\mathrm{dp}(f_n\dots f_1) = n$.

Proof. Let $\mathcal{M}$ be the set of all modules lying on the mouth in $\mathscr{C}$. We know that the non-zero paths from $X \in \mathcal{R}_r \cap \mathcal{M}$ are the ones in $\mathcal{R}_r$. Let $\Delta(M_0,t)$ be a mesh-complete full translation subquiver of $\mathscr{C}$ of the form



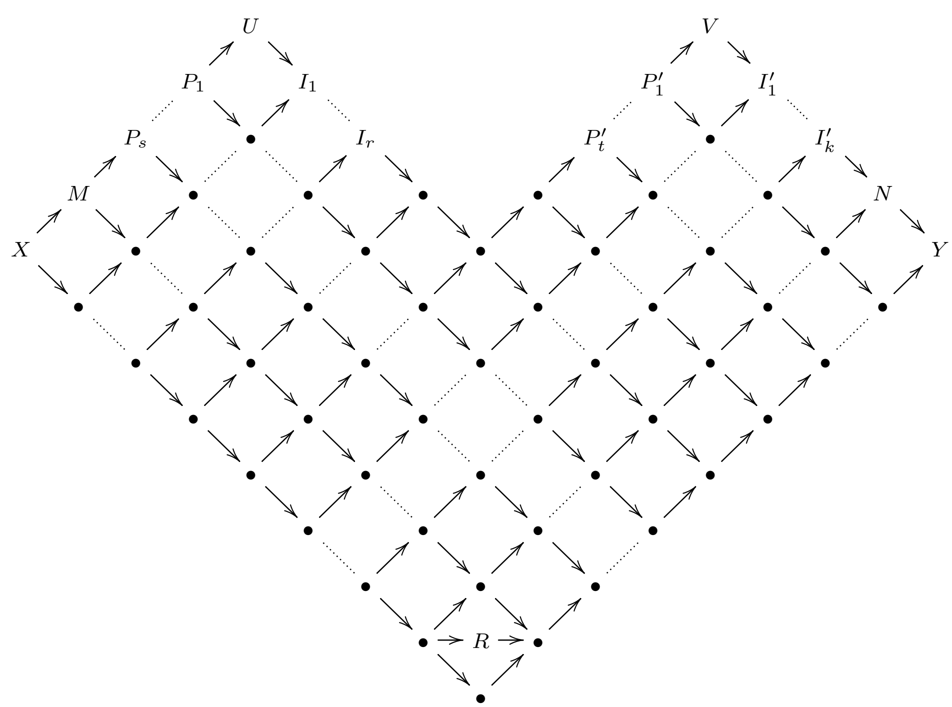
where $t \geq 2$, W_1 is projective and W_t is injective. By [15, Lemma 3.1], we know that if $f : X \to Y$ be an irreducible morphism in $\mathrm{mod}\,A$ with $X, Y \in \Delta(M_0,t)$ then

- $d_l(f) = \infty$ if and only if $d_l(f) \neq 1$, and
- $d_r(f) = \infty$ if and only if $d_r(f) \neq 1$.

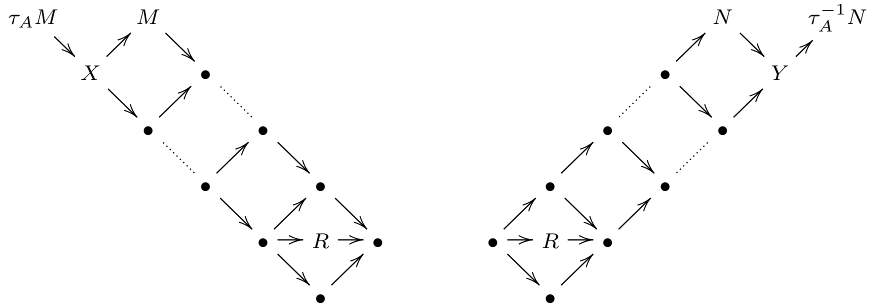
Hence, in $\Delta(M_0, t)$ the non-zero composition of irreducible morphisms between modules in $\mathscr{C}$ which belong to almost split sequences with exactly two indecomposable middle terms behave well. By [17], the compositions of irreducible morphisms which belong to sectional paths also behave well.

Finally, by Remarks 4.2, 4.5, Lemmas 4.4, 4.9, and also taking into account the shape of the component $\mathscr{C}$ we conclude that the non-zero compositions of irreducible morphisms in $\mathcal{R}_r$ behave well. □

Let A be an artin algebra. Following [13], a full translation subquiver of Γ_A of the form



with $P_1, \dots, P_s, P'_1, \dots, P'_t$, $s, t \geq 0$, projective, $I_1, \dots, I_r, I'_1, \dots, I'_k$, $r, k \geq 0$, injective and U, V, R projective-injective A-modules is said to be a *special configuration of modules*. We note that in [13] the concept of a special configuration of modules has been considered for $s = k = 0$. Now, let $\mathscr{C}$ be a coil containing the above special configuration of modules. Then, by the definition of admissible operations (ad 2) and (ad 2*) we know that there exist configurations of almost split sequences in $\mathscr{C}$ as follows:



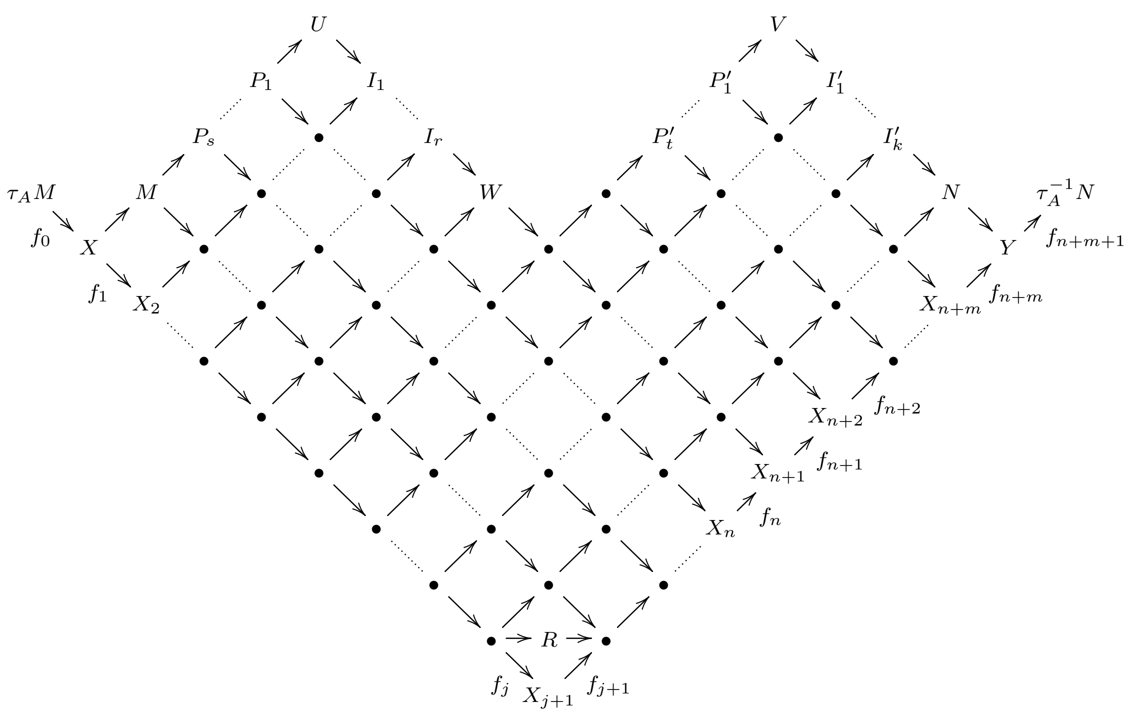
where $0 \to \tau_A M \to X \to M \to 0$ and $0 \to N \to Y \to \tau_A^{-} N \to 0$ are almost split sequences with one indecomposable middle term.

Lemma 4.11. *Let A be an artin algebra and $\mathscr{C}$ be a coil in Γ_A with a special configuration of modules. Then, the non-zero composition of n irreducible morphisms*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

between modules in the configuration and satisfying the mesh relations are such that $\mathrm{dp}(f_n \dots f_1) = n$.

Proof. Without loss of generality, we may assume that we have the following situation in $\mathscr{C}$.



where $s, r, t, k \geq 0$, $j = r+k+4$, $n = s+r+k+6$, $m = t+2$. Using the additive function we observe that there is only one non-zero morphism from X to Y in $\mathscr{C}$. Furthermore, there is also only one morphism from $\tau_A M$ to Y. Hence, we can consider any path from X to Y in $\mathscr{C}$. Let

$$X \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_j} X_{j+1} \xrightarrow{f_{j+1}} \cdots \xrightarrow{f_{n+m-1}} X_{n+m} \xrightarrow{f_{n+m}} Y \xrightarrow{f_{n+m+1}} \tau_A^{-1} N$$

be a non-zero path. In order to analyze if the composition $f_{n+m+1} \dots f_1$ behaves well, we consider the path

$$\tau_A M \xrightarrow{f_0} X \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_j} X_{j+1} \xrightarrow{f_{j+1}} \cdots$$

$$\cdots \xrightarrow{f_{n+m-1}} X_{n+m} \xrightarrow{f_{n+m}} Y \xrightarrow{f_{n+m+1}} \tau_A^{-1} N$$

and we prove that $\mathrm{dp}(f_{n+m+1} \dots f_1 f_0) = n + m + 2$. In fact, assume that $f_n \dots f_1 f_0 \in \mathrm{rad}_A^{n+1}(\tau_A M, X_{n+1})$. By Theorem 4.3, we have that the left degree $d_l(f_i) = \infty$ for $i = 0, 1, \dots, n$. Hence, $\mathrm{dp}(f_n \dots f_1 f_0) = n + 1$.

On the other hand, we know that $d_l(f_{n+1}) < \infty$. Then, there is a configuration of almost split sequences as in Theorem 4.3. By Lemma 2.3, there exists a morphism $q : \tau_A M \to W$ in $\mathrm{mod}\, A$ such that $q \notin \mathrm{rad}_A^{n+1}(\tau_A M, W)$. Again, using the additive function it is not hard to see that any morphism in $\mathscr{C}$ from $\tau_A M$ to W vanish. Therefore, by Proposition 2.8 any morphism in $\mathrm{mod}\, A$ from $\tau_A M$ to W do not behave well, getting a contradiction to Lemma 2.3.

Again, by Theorem 4.3 we have that $d_l(f_i) = \infty$ for $i = n+2, \dots, n+m$, proving that the depth $\mathrm{dp}(f_{n+m} \dots f_1 f_0) = n + m + 1$. Finally, assume that $f_{n+m+1} \dots f_1 f_0 \in \mathrm{rad}_A^{n+m+2}(\tau_A M, \tau_A^{-1} N)$. By [18, Proposition 1.12] we have that $d_l(f_{n+m+1}) = 1$, but any morphism in $\mathscr{C}$ from $\tau_A M$ to N vanish, getting a contradiction to Lemma 2.3. Therefore, we prove that $\mathrm{dp}(f_{n+m+1} \dots f_1 f_0) = n + m + 2$. Finally, by Remark 4.8 (b), we get the result. □

As a consequence we have the following corollary.

Corollary 4.12. *Let A be an artin algebra and $\mathscr{C}$ be a coil which is a quasi-tube in Γ_A. Let*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

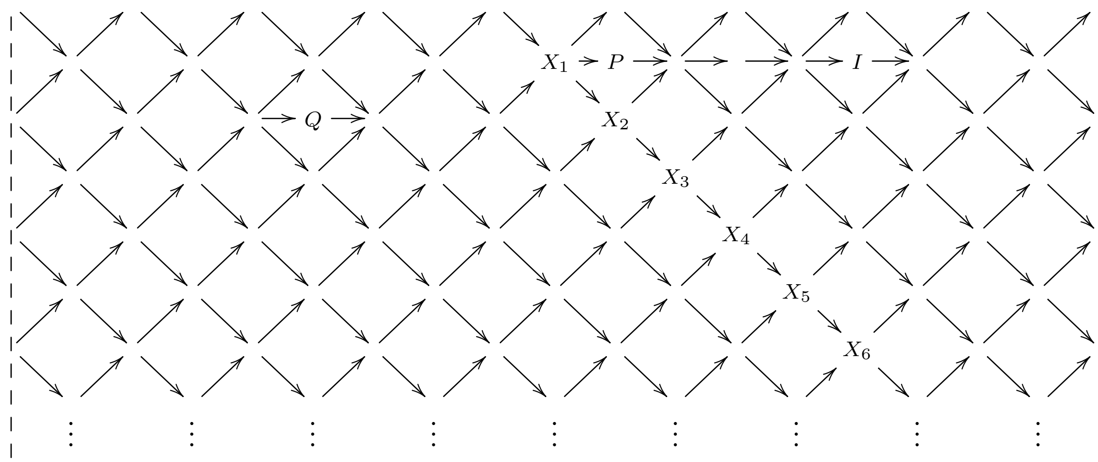
be a path of irreducible morphisms with $X_i \in \mathscr{C}$ for $i = 1, \dots, n+1$. Then, $f_n \dots f_1 \in \mathrm{rad}_A^{n+1}(X_1, X_{n+1})$ if and only if $f_n \dots f_1 \in \mathrm{rad}_A^{\infty}(X_1, X_{n+1})$.

Remarks 4.13. Let us note the following.

(a) Recall that a quasi-tube is a connected translation quiver obtained from a stable tube by an iterated applications of admissible operations (ad 1), (ad 1*), (ad 2) or (ad 2*).

(b) Let us note that Corollary 4.12 is a generalization of [13, Theorem A]. In fact, by [18] and [27], we know that if A is a self-injective artin algebra and Γ is an infinite component in Γ_A with an oriented cycle then Γ is a quasi-tube.

Remark 4.14. Observe that to prove the results it is enough to consider cases where the intersection $\mathcal{R}_r^i \cap \mathcal{R}_r^j$, $i, j \in \{1, 2, 3\}$, is an infinite sectional path pointing to infinity because the non-zero paths in the other cases ($\mathcal{R}_r^i \cap \mathcal{R}_r^j = \emptyset$) are considered when $\mathcal{R}_r^i \cap \mathcal{R}_r^j$ is above. We illustrate the situation for $i = 2$ and $j = 3$ as follows:



where Q is projective-injective, P is projective, I is injective vertex and

$$X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow X_4 \longrightarrow \cdots$$

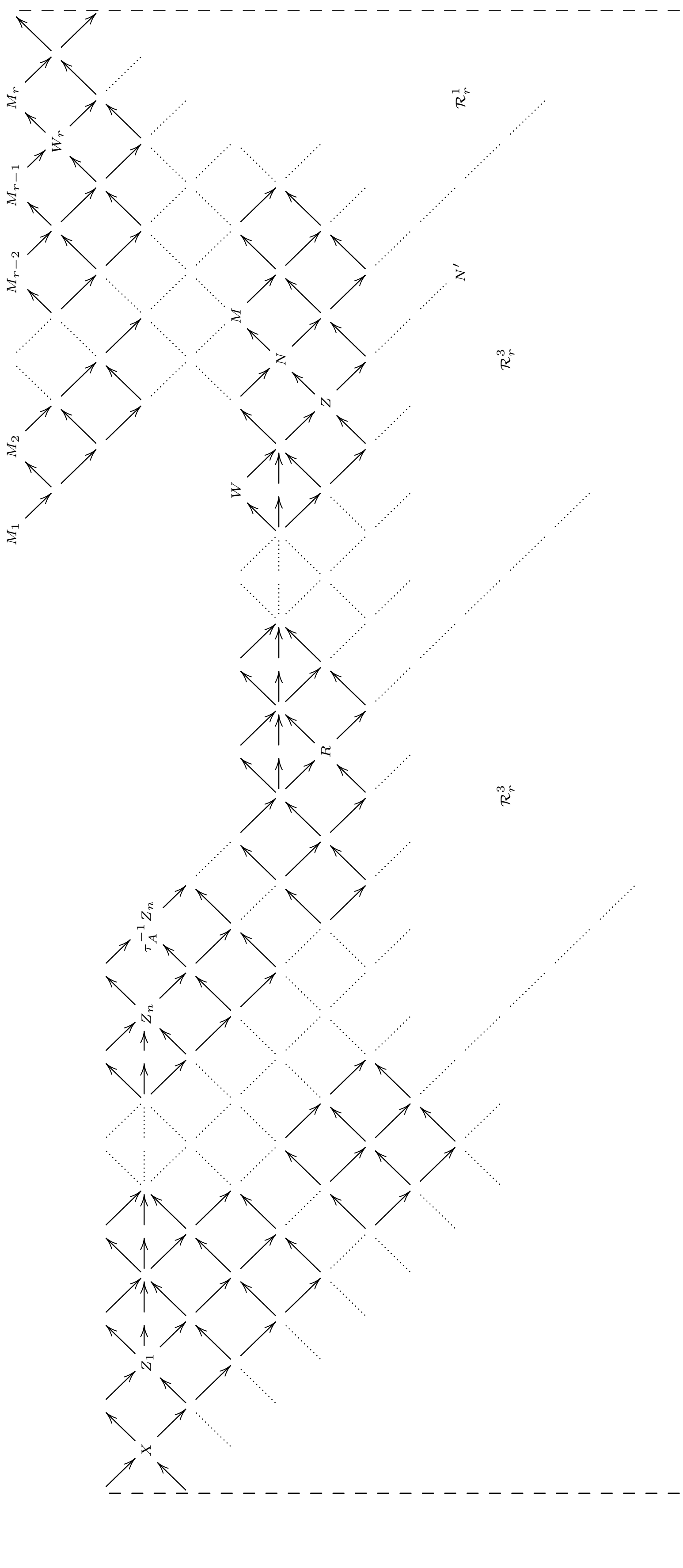
is the infinite sectional path belonging to $\mathcal{R}_r^2 \cap \mathcal{R}_r^3$.

Proposition 4.15. *Let A be an artin algebra and $\mathscr{C}$ be a coil in Γ_A obtained from a stable tube by the admissible operations* (ad 3), (ad 3) *and a finite number of admissible operations* (ad 1). *Then, any non-zero composition of n irreducible morphisms*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

between modules in the component $\mathscr{C}$ and satisfying the mesh relations are such that $\mathrm{dp}(f_n \dots f_1) = n$.

Proof. Without loss of generality we can assume that $\mathscr{C}$ has the following shape



M_1
M_2
M_{r-2}
M_{r-1}
W_r
M_r
$\mathcal{R}_r^1$
N'
M
N
Z
$\mathcal{R}_r^3$
W
R
$\mathcal{R}_r^3$
$\tau_A^{-1}Z_n$
Z_n
Z_1
X

observe that the non-zero paths starting at X and ending in Y in $\mathscr{C}$ are those for which $Y \in \mathcal{R}_r^3 \cup \mathcal{R}_r^3 \cup \mathcal{R}_r^1$. It is enough to analyze some non-zero paths in $\mathscr{C}$, which will give us a complete information if all non-zero paths in $\mathscr{C}$ behave well.

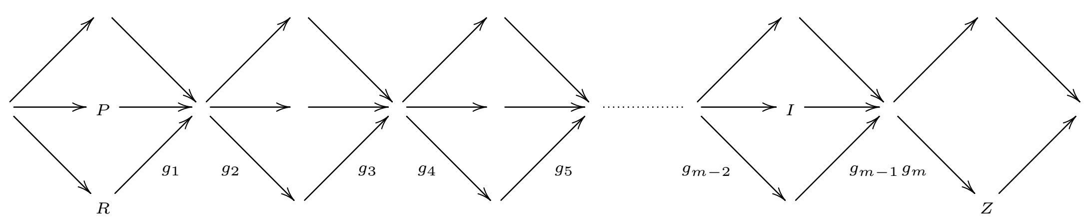
By Lemma 4.4, we know that any non-zero path from X to $\tau_A^{-1} Z_n$ and from Z_1 to Z_n behave well.

Using the additive function it is not hard to see that there are only one non-zero path in $\mathscr{C}$ from X to N and another from X to M. We claim that non-zero path in $\mathscr{C}$ behaves well. In fact, by Lemma 4.4 we know that there is only one path from X to $\tau_A^{-1} Z_n$ and this path behaves well. Observe that all arrows in the sectional path from $\tau_A^{-1} Z_n$ to R have infinite left degree. Hence, the path from X to R going through $\tau_A^{-1} Z_n$

$$X \longrightarrow \tau_A^{-1} Z_n \longrightarrow R$$

behaves well.

Now, without loss of generality we may consider the path $g_m \dots g_2 g_1 : R \to Z$,



since there is only one non-zero path from X to Z. By [15, Lemma 3.1] we have that $d_l(g_i) = \infty$ for $i = 1, \dots, m$. Therefore,

$$X \longrightarrow \tau_A^{-1} Z_n \longrightarrow R \xrightarrow{g_1} \cdots \xrightarrow{g_m} Z$$

behaves well. Any non-zero path

$$X \longrightarrow \tau_A^{-1} Z_n \longrightarrow R \xrightarrow{g_1} \cdots \xrightarrow{g_m} Z \to N'$$

behave well, where $Z \to N'$ is a sectional path pointing to infinity consisting of arrows with infinite left degree.

On the other hand, if we consider the path

$$X \longrightarrow \tau_A^{-1} Z_n \longrightarrow R \xrightarrow{g_1} \cdots \xrightarrow{g_m} Z \xrightarrow{t} N$$

we know that $d_l(t) < \infty$. By Lemma 2.3 we shall analyze if there is a non-zero morphism in $\mathscr{C}$ from X to W. But using the additive function it is not hard to see that any morphism in $\mathscr{C}$ from X to W vanish. Therefore, by Proposition 2.8 any morphism in mod A from X to W do not behave well, getting a contradiction to Lemma 2.3. Hence, any non-zero path from X to N behave well.

The arrow $N \longrightarrow M$ has infinite left degree. Clearly, all morphisms from X to M_i with $i = 1, \ldots, r-2$ vanish. Hence, we get a contradiction to Lemma 2.3 if we assume that the path does not behave well.

Finally, the only possibility to have a non-zero path is to consider a sectional path starting at W_r and pointing to infinity, but the arrows there have infinite left degree. Hence, the path behaves well. □

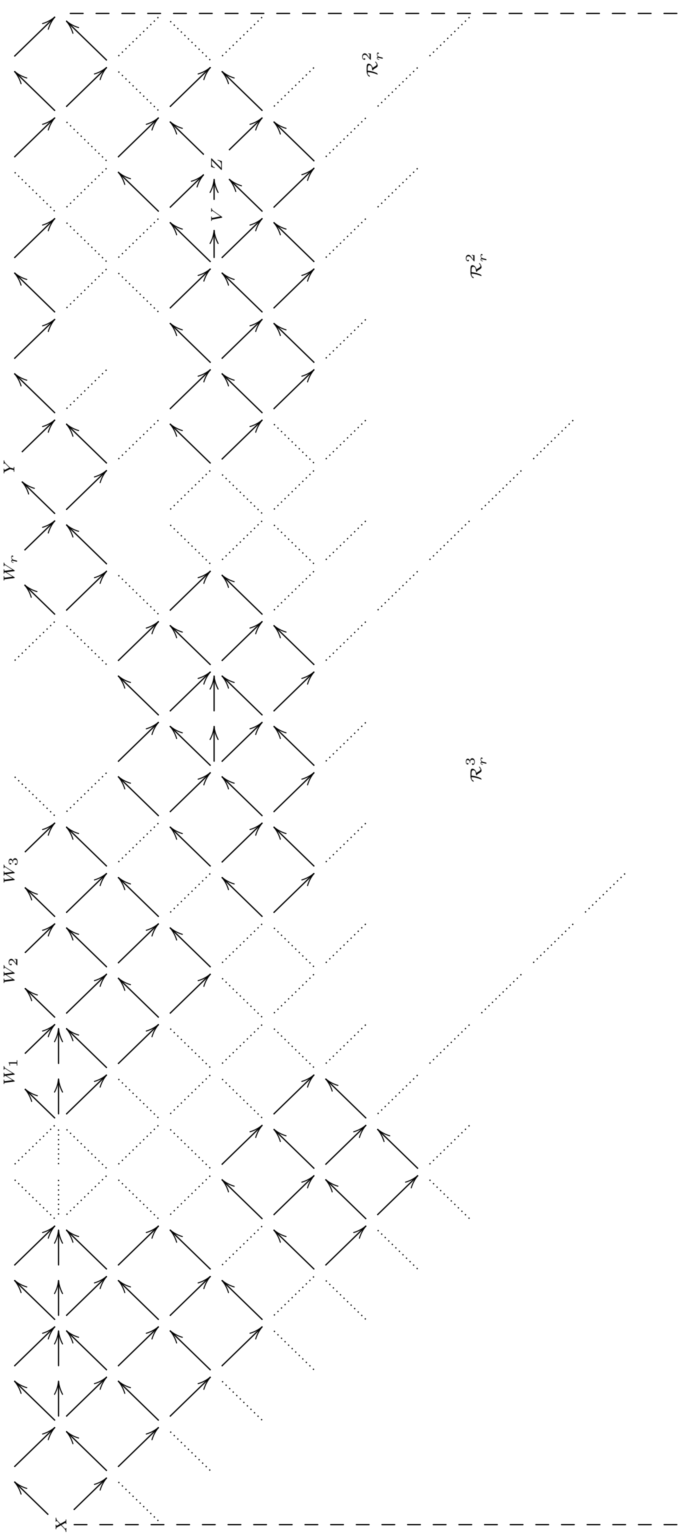
Remark 4.16. If we consider a coil $\mathscr{C}$ with finitely many (say $s \geq 2$) admissible operations (ad 3) with infinite rectangles $\mathcal{R}^3_{r_1}, \mathcal{R}^3_{r_2}, \ldots, \mathcal{R}^3_{r_s}$ such that $\mathcal{R}^3_{r_i} \cap \mathcal{R}^3_{r_{i+1}}$ for $i = 1, \ldots, s-1$ is the sectional infinite path pointing to infinity, then we get a similar result as in Proposition 4.15. The proof also involves the same arguments.

Proposition 4.17. *Let A be an artin algebra and $\mathscr{C}$ be a coil in Γ_A obtained from a stable tube by the admissible operations* (ad 3), (ad 2) *and* (ad 2). *Then, any non-zero composition of n irreducible morphisms*

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

between modules in the component $\mathscr{C}$ and satisfying the mesh relations are such that $\mathrm{dp}(f_n \ldots f_1) = n$.

Proof. Without loss of generality, we may assume that $\mathscr{C}$ is as follows:

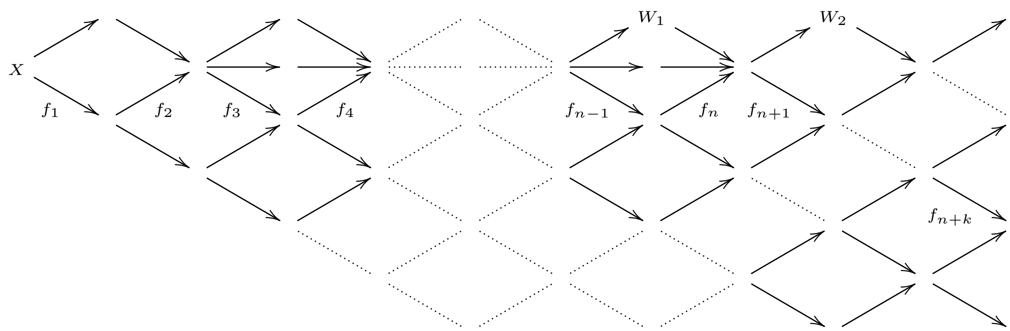

$\mathcal{R}_r^2$
$\mathcal{R}_r^2$
$\mathcal{R}_r^3$
Z
V
Y
W_r
W_3
W_2
W_1
X

By Lemmas 4.6 and 4.7, it is enough to analyze if any non-zero path

(i) from X to Y and
(ii) from X to Z,

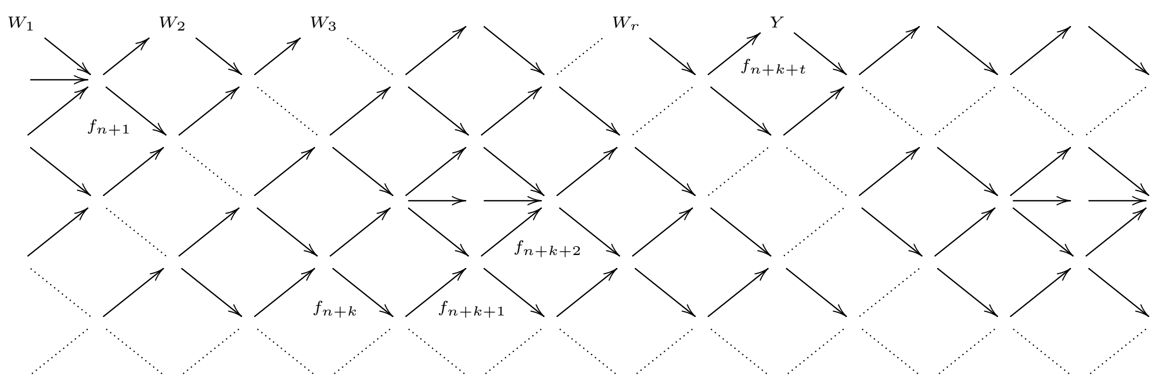
behave well, in order to conclude that all non-zero morphisms in $\mathscr{C}$ behave well.

(i) Note that there is only one non-zero path in $\mathscr{C}$ from X to Y by using the additive function. Assume that $f_{n+k+t}\dots f_1 \in \mathrm{rad}_A^{n+k+t+1}(X,Y)$, where $f_{n+k+t}\dots f_1 : X \to Y$ is any non-zero path in $\mathscr{C}$ form X to Y. Without loss of generality, we may assume that f_i for $i = 1,\dots,n+k$ are the following irreducible morphisms



By Lemma 4.4, we know that $\mathrm{dp}(f_{n+1}\dots f_1) = n+1$. Moreover, the left degree of $f_{n+2},\dots,f_{n+k}$ are infinite. Therefore, the composition $f_{n+k}\dots f_1$ behaves well.

Now, we choose the following morphisms,

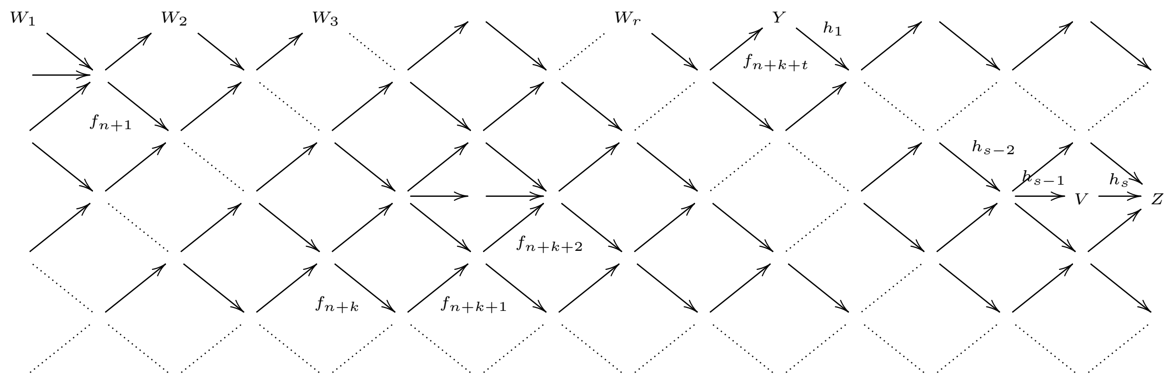


where $r = k+1$, since there is only one non-zero path from X to Y. By Theorem 4.3, we know that $d_l(f_{n+k+1}) < \infty$. Assume that $f_{n+k+1}f_{n+k}\dots f_1 \in \mathrm{rad}_A^{n+k+2}$. Then, by Lemma 2.3 there exists a morphism $\delta \in \mathrm{mod}\, A$ from X to W_1, $\delta \notin \mathrm{rad}_A^{n+k}$. Then, it follows by the additive function that from X to W_1 there is only the zero-morphism. Hence, $\delta = 0$, and therefore δ in $\mathrm{mod}\, A$ is such that $\delta \in \mathrm{rad}_A^{n+k}$. Then, $f_{n+k+1}f_{n+k}\dots f_1$ behaves well.

Now, by Theorem 4.3, we know that $d_l(f_{n+k+2}) = \infty$. Hence, the composition $f_{n+k+2}f_{n+k+1}\dots f_1$ behaves well. We also know by Theorem 4.3, that $d_l(f_{n+k+3}) < \infty$.

Suppose that $f_{n+k+3}f_{n+k+2}\dots f_1 \in \mathrm{rad}_A^{n+k+4}$. Iterating the same argument as above, we can prove that the only morphism in $\mathscr{C}$ from X to W_3 vanishes. Hence, we infer that $f_{n+k+3}f_{n+k+2}\dots f_1$ behaves well.

(ii) Now, we analyze that any non-zero morphism from X to Z behave well. Note that there is only one non-zero path from X to Z modulo mesh. Without loss of generality, we may consider the same path consider in (i), that is, $f_{n+k+t}\dots f_{n+k+2}\dots f_n \dots f_1 : X \to Y$ and compose it with the path $f_{n+k+s}\dots f_{n+k+t}$ from Y to Z going through V, where $s > t$. We denote $f_{n+k+t+1}, \dots, f_{n+k+s}$ by $h_1, \dots, h_s$, respectively. Next, we illustrate the chosen path $h_s \dots h_1$ in the next picture:



By Statement (i) we know that $\mathrm{dp}(f_{n+k+t}\dots f_1) = n + k + t$. Note that the path $f_{n+k+s-1}\dots f_{n+k+t}$ from Y to V are morphisms in a sectional path of infinite left degree. Therefore, $\mathrm{dp}(f_{n+k+s-1}\dots f_1) = n + k + s - 1$. Finally, by Theorem 4.3 we know that the left degree of the irreducible morphism f_{n+k+s} is infinite. Hence, we conclude that the considered composition behaves well, proving the result. □

Remark 4.18. If we consider a coil $\mathscr{C}$ with finitely many (say $s \geq 2$) admissible operations (ad 2) with infinite rectangles $\mathcal{R}^2_{r_1}, \mathcal{R}^2_{r_2}, \dots, \mathcal{R}^2_{r_s}$ such that $\mathcal{R}^2_{r_i} \cap \mathcal{R}^2_{r_{i+1}}$ for $i = 1, \dots, s-1$ is the sectional infinite path pointing to infinity, then we get a similar result as in Proposition 4.17. The proof also involves the same arguments.

Proposition 4.19. *Let A be an artin algebra and $\mathscr{C}$ be a coil in Γ_A obtained from a stable tube $\mathcal{T}$, by the admissible operations*

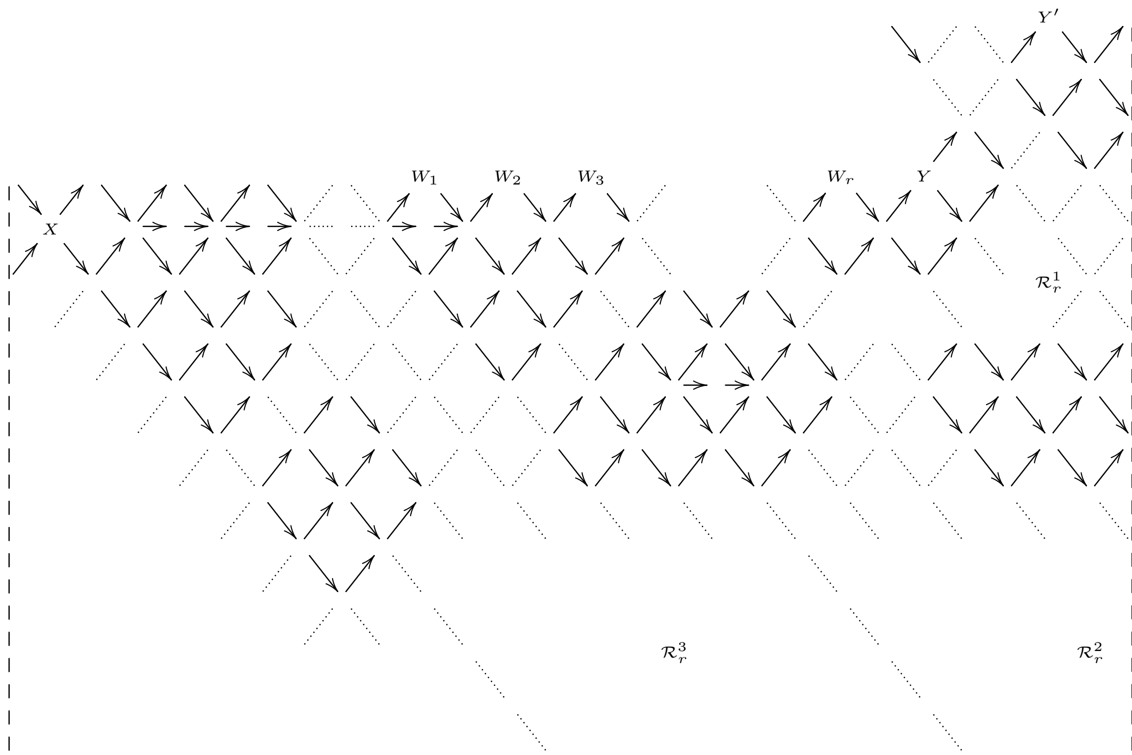
(i) (ad 3), (ad 2) *and a finite number of admissible operations* (ad 1), *or*
(ii) (ad 3), (ad 1) *and a finite number of admissible operations* (ad 2).

Then, any non-zero composition of n irreducible morphisms

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

between modules in the component $\mathscr{C}$ and satisfying the mesh relations are such that $\mathrm{dp}(f_n \dots f_1) = n$.

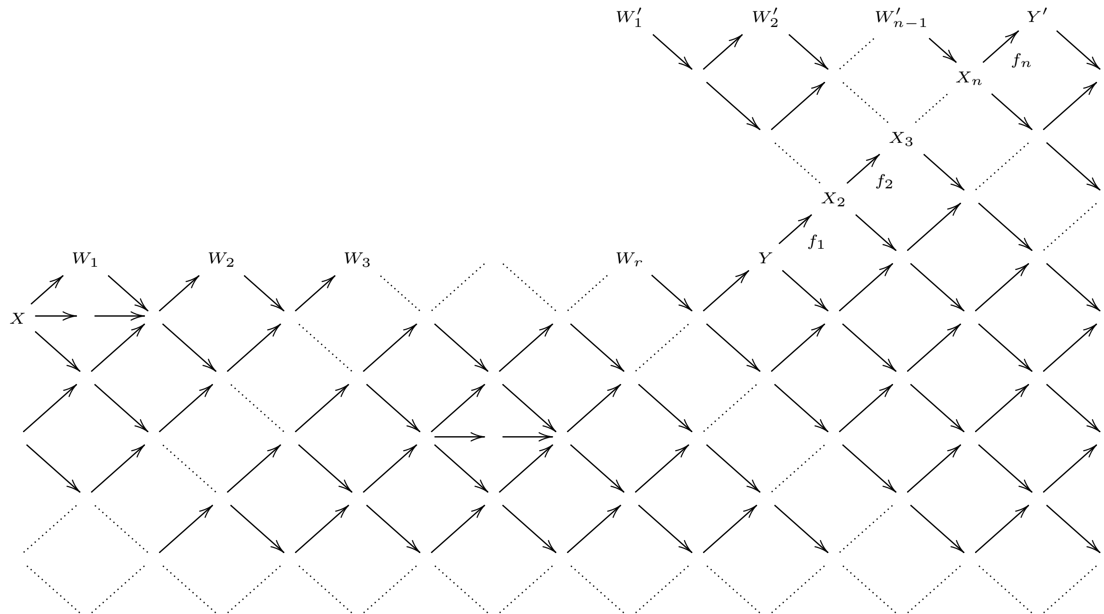
Proof. (i) Without loss of generality, we may assume that $\mathscr{C}$ has the following form:



Since there is only one non-zero path from X to Y' in $\mathscr{C}$, we can consider a non-zero path which is the composition of the path a path from X to Y and a path from Y to Y'. By Proposition 4.17, we know that any non-zero path from X to Y behave well. Then, it is enough to analyze the situation where the path is as follows:

$$X \xrightarrow{\varphi} Y \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} Y'$$

with $k \geq 1$ and φ any non-zero path from X to Y in $\mathscr{C}$ such that $\mathrm{dp}(\varphi) = k$. We illustrate the situation as follows:



Assume that $f_n \dots f_1\varphi \in \mathrm{rad}_A^{n+k+1}(X, Y')$, where $\mathrm{dp}(\varphi) = k$. By Theorem 4.3, $d_l(f_1) = \infty$. Then, $\mathrm{dp}(f_1\varphi) = k+1$. Now, again by Theorem 4.3 we know that $d_l(f_i) < \infty$ for $i = 2, \dots, n$. By Lemma 2.3, there exists a morphism $\delta \in \mathrm{mod}\, A$ from X to W_j', $\delta \notin \mathrm{rad}_A^{k+j}$ for $j = 1, \dots, n-1$. Observe that there are no morphisms in $\mathscr{C}$ from X to W_j', hence we get a contradiction to our assumption. Therefore, we prove that $\mathrm{dp}(f_i \dots f_1\varphi) = i + k$, that is, the composition behaves well, for all $i = 1, \dots, n$.

(ii) With similar arguments and techniques as we used in Statement (i), we can get the result for (ii). □

If we change the order of the admissible operations, the result still holds.

Proposition 4.20. *Let A be an artin algebra and $\mathscr{C}$ be a coil in Γ_A obtained from a stable tube $\mathcal{T}$, by the admissible operations*

(i) (ad 1), (ad 2) *and* (ad 3), *or*
(ii) (ad 1), (ad 3) *and* (ad 2), *or*
(iii) (ad 2), (ad 1) *and* (ad 3), *or*
(iv) (ad 2), (ad 3) *and* (ad 1).

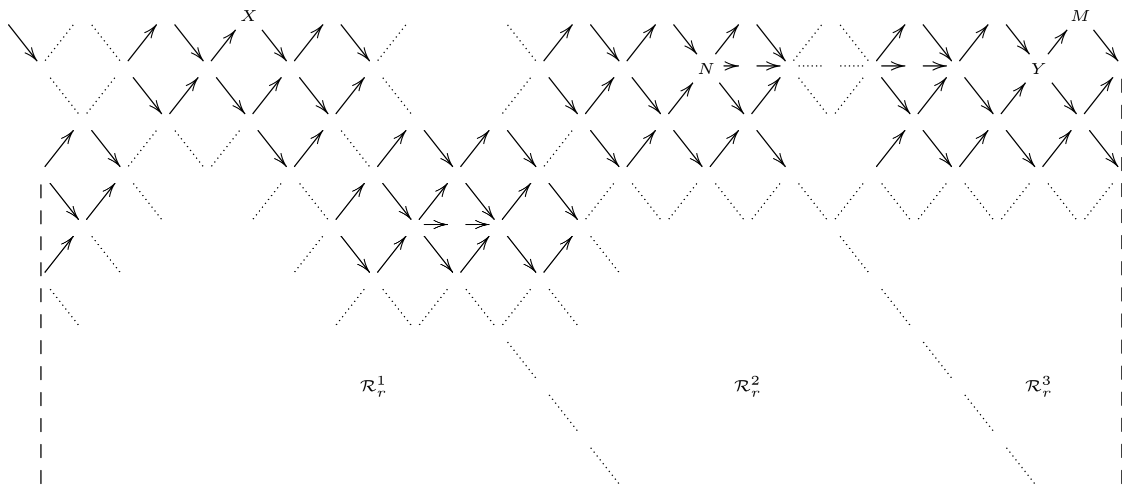
Then, any non-zero composition of n irreducible morphisms

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

between modules in the component $\mathscr{C}$ and satisfying the mesh relations are such that $\mathrm{dp}(f_n \dots f_1) = n$.

Proof. We only prove Statement (i), since the others statements can be proved with similar arguments.

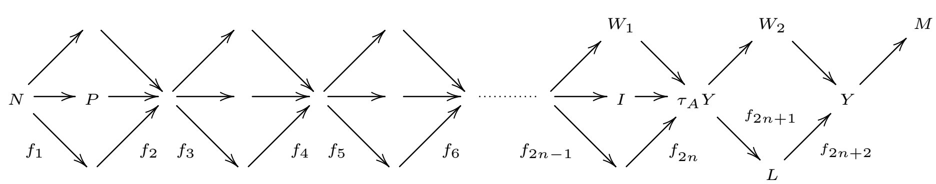
(i) Without loss of generality, we may assume that we have the following situation in $\mathscr{C}$:



Note that there is only one non-zero morphism from X to Y and that any morphism from X to M vanishes. It is enough to analyze a non-zero path from X to Y. We also observe that there are no non-zero morphisms from X to M.

The composition of the morphisms in the coray ending in X behaves well, see [17].

On the other hand, by Propositions 4.15 and 4.17 we know that any morphism from X to N behave well. By Theorem 4.3, the $d_l(f_i) = \infty$ for $i = 1, \dots, 2n+1$, where the irreducible morphisms f_i are the following:



Therefore, all non-zero morphisms from X to L behave well.

Finally, since $d_l(f_{2n+2}) < \infty$ then applying Lemma 2.3, it is not hard to see that there is not a non-zero morphism from X to W_1 in $\mathscr{C}$. Same analysis for a morphism from X to W_2. Therefore, the composition of non-zero paths behave well. □

5. Proof of Theorem 1.1

We only prove that, if $f_n \dots f_1 \in \mathrm{rad}_A^{n+1}(X_1, X_{n+1})$ then $f_n \dots f_1 \in \mathrm{rad}_A^{\infty}(X_1, X_{n+1})$ since the other implication is clear.

To analyze the composition of irreducible morphisms in $\mathscr{C}$ we start with the ones near the mouth of $\mathscr{C}$. It is enough to prove that all non-zero compositions behave well.

Note that for stable tubes the result is proved in [10, Theorem A] and for tubes in [13, Theorem 3.18]. By Lemma 4.11 and Corollary 4.12, the result also holds for quasi-tubes. In order to complete the proof we consider all the cases involving the admissible operations (ad 3) or (ad 3*). Moreover, by Remarks 4.5 and 4.8 (a) it is enough to consider (ad 3) with t odd. Therefore, by Propositions 4.10, 4.15, 4.17, 4.19 and 4.20 we conclude that the statement holds.

Now, it is enough to prove the result for zero paths in $\mathscr{C}$, because if we have a non-zero path

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1}$$

in $\mathscr{C}$ then, as we see above, $f_n \dots f_1$ behaves well, getting a contradiction with our assumption. Therefore, $f_n \dots f_1 = 0$.

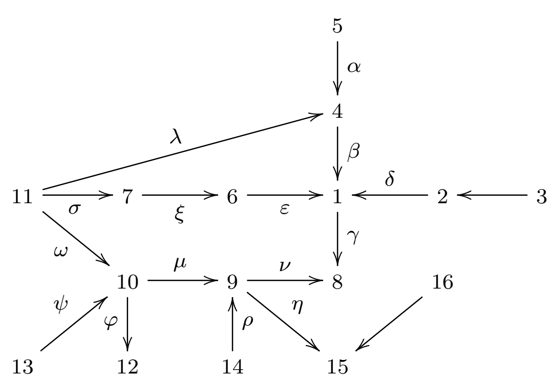
Further, any other composition of irreducible morphisms $h_i : X_i \to X_{i+1}$ for $i = 1, \dots, n$ is such that $h_n \dots h_1 = \delta f_n \dots f_1 + \mu$ with $\mu \in \mathrm{rad}_A^{n+1}(X_1, X_{n+1})$ and $\delta \in \mathrm{Aut}(X_{n+1})$. Hence, $h_n \dots h_1 \in \mathrm{rad}_A^{n+1}(X_1, X_{n+1})$.

Assume that $h_n \dots h_1 \notin \mathrm{rad}_A^{\infty}(X_1, X_{n+1})$, that is, the composition $h_n \dots h_1$ belongs to $\mathrm{rad}_A^m(X_1, X_{n+1}) \backslash \mathrm{rad}_A^{m+1}(X_1, X_{n+1})$ with $m > n$. Hence there is a non-zero path from X_1 to X_{n+1} of length longer than n, contradicting Lemma 4.7. The proof is completed.

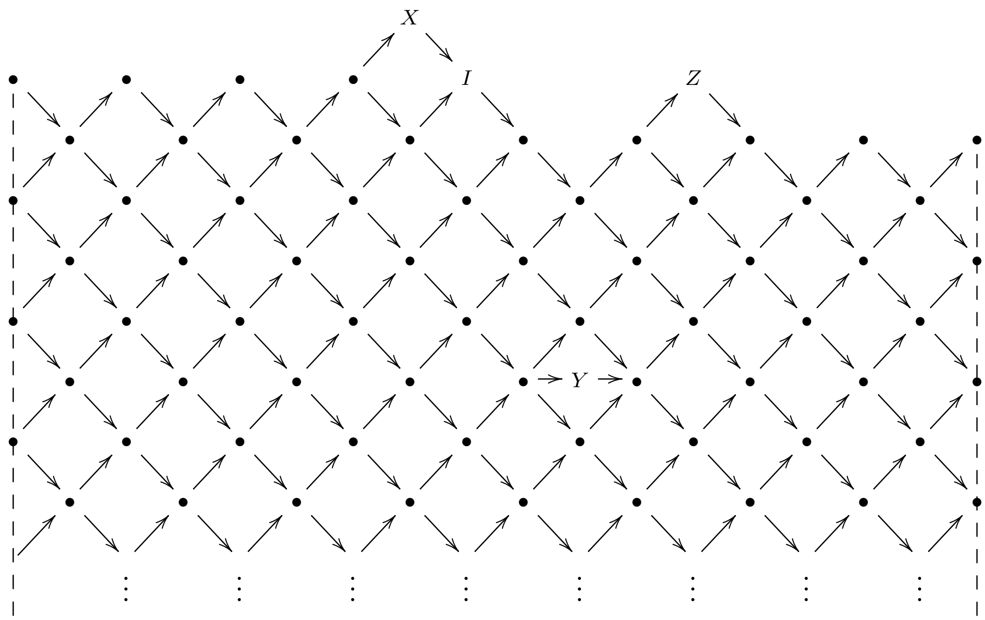
6. Examples

In this section we present some examples of coils.

Example 6.1. Let K be a field, Q the quiver of the form



I the ideal of KQ generated by the paths $\alpha\beta\gamma$, $\delta\gamma$, $\lambda\beta = \sigma\xi\varepsilon$, $\omega\mu\nu = \lambda\beta\gamma$, $\omega\varphi$, $\psi\mu$, $\rho\nu$, $\mu\eta$, and $A = KQ/I$ the associated bound quiver algebra. Then the Auslander-Reiten quiver Γ_A of A admits a coil $\mathscr{C}$ which is a quasi-tube of the following form



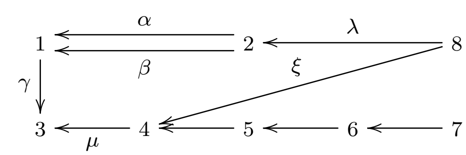
where X, Y, Z are projective-injective A-modules, I is injective A-module and the vertical dashed lines have to be identified in order to obtain the coil $\mathscr{C}$. Note that $\mathscr{C}$ contains a special configuration of modules (with $r = 1$ and $s = t = k = 0$). We will show that A is a coil enlargements (see [22, Section 3] or [4, Section 2] for finite-dimensional algebras over an algebraically closed field) of a concealed canonical algebra A_1. Indeed, let A_1 be the hereditary algebra of Euclidean type $\widetilde{\mathbb{E}}_6$ given by the vertices $1, 2, 3, 4, 5, 6, 7$. Let us denote vectors in $K_0(A)$:

$$\mathbf{a_1} = \begin{smallmatrix} 0 \\ 1 \\ 11100 \end{smallmatrix}, \;\; \mathbf{a_2} = \begin{smallmatrix} 0 \\ 1 \\ 11100 \\ 111 \end{smallmatrix}, \;\; \mathbf{a_3} = \begin{smallmatrix} 0 \\ 0 \\ 000000 \\ 100 \end{smallmatrix}, \;\; \mathbf{a_4} = \begin{smallmatrix} 0 \\ 0 \\ 000000 \\ 100 \\ 1 \end{smallmatrix},$$

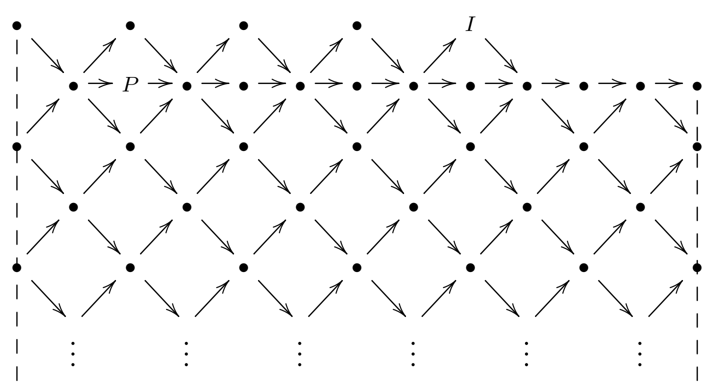
$$\mathbf{a_5} = \begin{smallmatrix} 0 \\ 0 \\ 000000 \\ 010 \\ 00 \end{smallmatrix}, \;\; \mathbf{a_6} = \begin{smallmatrix} 0 \\ 0 \\ 000000 \\ 010 \\ 001 \end{smallmatrix}, \;\; \mathbf{a_7} = \begin{smallmatrix} 0 \\ 0 \\ 000000 \\ 010 \\ 0011 \end{smallmatrix}.$$

We apply the admissible operation (ad 1^*) to A_1 with the pivot the simple regular A_1-module with vector $\mathbf{a_1}$, and with $t = 2$. The modified algebra A_2 is given by the quiver with the vertices $1, 2, \ldots, 10$ bound by $\alpha\beta\gamma$, $\delta\gamma$. Now, we apply (ad 2) to A_2 with the pivot the indecomposable A_2-module with vector $\mathbf{a_2}$, and with $t = 2$. The modified algebra A_3 is given by the quiver with the vertices $1, 2, \ldots, 11$ bound by $\alpha\beta\gamma$, $\delta\gamma$, $\lambda\beta = \sigma\xi\varepsilon$, $\omega\mu\nu = \lambda\beta\gamma$. Next, we apply (ad 1^*) to A_3 with the pivot the indecomposable A_3-module with vector $\mathbf{a_3}$, and with $t = 0$. The modified algebra A_4 is given by the quiver with the vertices $1, 2, \ldots, 12$ bound by $\alpha\beta\gamma$, $\delta\gamma$, $\lambda\beta = \sigma\xi\varepsilon$, $\omega\mu\nu = \lambda\beta\gamma$, $\omega\varphi$. In the next step, we apply the admissible operation (ad 1) to A_4 with the pivot the indecomposable A_4-module with vector $\mathbf{a_4}$, and with $t = 0$. The modified algebra A_5 is given by the quiver with the vertices $1, 2, \ldots, 13$ bound by $\alpha\beta\gamma$, $\delta\gamma$, $\lambda\beta = \sigma\xi\varepsilon$, $\omega\mu\nu = \lambda\beta\gamma$, $\omega\varphi$, $\psi\mu$. Now, we apply (ad 1) to A_5 with the pivot the indecomposable A_5-module with vector $\mathbf{a_5}$, and with $t = 0$. The modified algebra A_6 is given by the quiver with the vertices $1, 2, \ldots, 14$ bound by $\alpha\beta\gamma$, $\delta\gamma$, $\lambda\beta = \sigma\xi\varepsilon$, $\omega\mu\nu = \lambda\beta\gamma$, $\omega\varphi$, $\psi\mu$, $\rho\nu$. Next, we apply the admissible operation (ad 1^*) to A_6 with the pivot the indecomposable A_6-module with vector $\mathbf{a_6}$, and with $t = 0$. The modified algebra A_7 is given by the quiver with the vertices $1, 2, \ldots, 15$ bound by $\alpha\beta\gamma$, $\delta\gamma$, $\lambda\beta = \sigma\xi\varepsilon$, $\omega\mu\nu = \lambda\beta\gamma$, $\omega\varphi$, $\psi\mu$, $\rho\nu$, $\mu\eta$. Finally, we apply the admissible operation (ad 1^*) to A_7 with the pivot the indecomposable A_7-module with vector $\mathbf{a_7}$, and with $t = 0$. The modified algebra is equal to A.

Example 6.2. Let K be a field, Q the quiver of the form



I the ideal of KQ generated by the paths $\alpha\gamma$, $\lambda\alpha$, $\lambda\beta\gamma = \xi\mu$, and $A = KQ/I$ the associated bound quiver algebra. Then the Auslander-Reiten quiver Γ_A of A admits a component $\mathscr{C}$ which is a coil of the following form



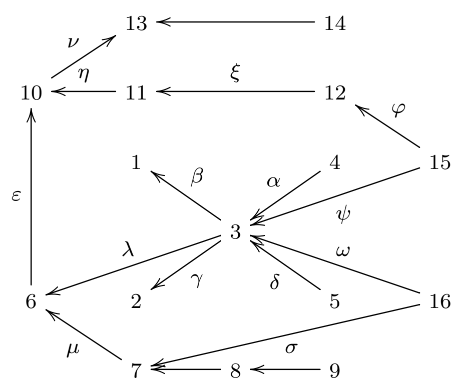
where P is projective, I is injective and the vertical dashed lines have to be identified in order to obtain the coil $\mathscr{C}$. We note that A is a coil enlargements (see [22, Section 3] or [4, Section 2] for finite-dimensional algebras over an algebraically closed field) of a concealed canonical algebra A_1. Indeed, let A_1 be the Kronecker algebra given by the vertices $1, 2$. Let us denote vectors in $K_0(A)$:

$$\mathbf{a_1} = {\scriptstyle 11}\,, \quad \mathbf{a_2} = {\scriptstyle \substack{1\ 1\\ 11000}}\,.$$

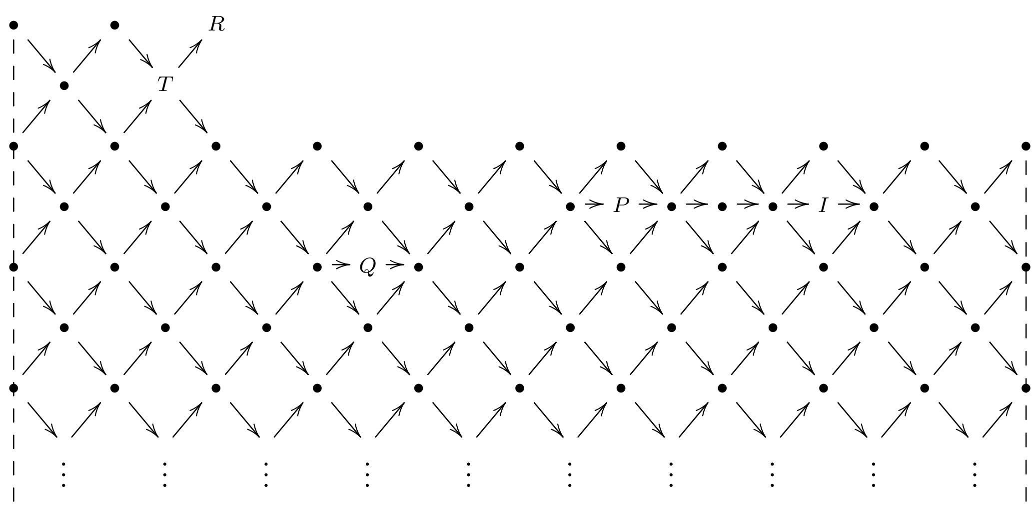
We apply the admissible operation (ad 1^*) to A_1 with the pivot the simple regular (homogeneous) A_1-module with vector $\mathbf{a_1}$, and with $t = 4$. The modified algebra A_2 is given by the quiver with the vertices $1, 2, \ldots, 7$ bound by $\alpha\gamma$. Finally, we apply the admissible operation (ad 3) to A_2 with the pivot the indecomposable A_2-module with vector $\mathbf{a_2}$, and with $t = 4$. The modified algebra is equal to A.

Observe that $\mathscr{C}$ is the coil in which each τ_A-orbit contains finitely many vertices. For an algebra Λ we refer the reader to [20] for a detailed description of the infinite components (coils) of an Auslander-Reiten quiver Γ_Λ having only finitely many vertices in each τ_Λ-orbit.

Example 6.3. Let K be a field, Q the quiver of the form



I the ideal of KQ generated by the paths $\alpha\lambda$, $\delta\lambda$, $\mu\varepsilon$, $\eta\nu$, $\psi\beta$, $\psi\gamma$, $\varphi\xi\eta\nu$, $\varphi\xi\eta = \psi\lambda\varepsilon$, $\omega\beta$, $\omega\gamma$, $\omega\lambda\varepsilon$, $\omega\lambda = \sigma\mu$, and $A = KQ/I$ the associated bound quiver algebra. Then the Auslander-Reiten quiver Γ_A of A admits a component $\mathscr{C}$ which is a coil of the following form



where R, T, I are injective, P is projective, Q is projective-injective and the vertical dashed lines have to be identified in order to obtain the coil $\mathscr{C}$. We note that A is a coil enlargements (see [22, Section 3] or [4, Section 2] for finite-dimensional algebras over an algebraically closed field) of a concealed canonical algebra A_1. Indeed, let A_1 be the hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_4$ given by the vertices $1, 2, 3, 4, 5$. Let us denote vectors in $K_0(A)$:

$$\mathbf{a_1} = \begin{smallmatrix} 0\ 0 \\ 1 \\ 0\ 0 \end{smallmatrix}, \ \mathbf{a_2} = \begin{smallmatrix} 0\ 0 \\ 1 \\ 10\ 0 \\ 000 \end{smallmatrix}, \ \mathbf{a_3} = \begin{smallmatrix} 10\ 0 \\ 0\ 0 \\ 1 \\ 10\ 0 \\ 000 \end{smallmatrix}, \ \mathbf{a_4} = \begin{smallmatrix} 0\ 0 \\ 11\ 1 \\ 0\ 0 \\ 1 \\ 10\ 0 \\ 000 \end{smallmatrix}, \ \mathbf{a_5} = \begin{smallmatrix} 0\ 0 \\ 00\ 0 \\ 0\ 00 \\ 1 \\ 10\ 0 \\ 000 \end{smallmatrix}.$$

We apply the admissible operation (ad 1^*) to A_1 with the pivot the simple regular A_1-module with vector $\mathbf{a_1}$, and with $t = 3$. The modified algebra A_2 is given by the quiver with the vertices $1, 2, \dots, 9$ bound by $\alpha\lambda$, $\delta\lambda$. Now, we apply (ad 1^*) to A_2 with the pivot the indecomposable A_2-module with vector $\mathbf{a_2}$, and with $t = 2$. The modified algebra A_3 is given by the quiver with the vertices $1, 2, \dots, 12$ bound by $\alpha\lambda$, $\delta\lambda$, $\mu\varepsilon$. Next, we apply (ad 1^*) to A_3 with the pivot the indecomposable A_3-module with vector $\mathbf{a_3}$, and with $t = 1$. The modified algebra A_4 is given by the quiver with the vertices $1, 2, \dots, 14$ bound by $\alpha\lambda$, $\delta\lambda$, $\mu\varepsilon$, $\eta\nu$. In the next step, we apply the admissible operation (ad 2) to A_4 with the pivot the indecomposable A_4-module with vector $\mathbf{a_4}$, and with $t = 2$. The modified algebra A_5 is given by the quiver with the vertices $1, 2, \dots, 15$ bound by $\alpha\lambda$, $\delta\lambda$, $\mu\varepsilon$, $\eta\nu$, $\psi\beta$, $\psi\gamma$, $\varphi\xi\eta\nu$, $\varphi\xi\eta = \psi\lambda\varepsilon$. Finally, we apply the admissible operation (ad 3) to A_5 with the pivot the indecomposable A_5-module with vector $\mathbf{a_5}$, and with $t = 3$. The modified algebra is equal to A.

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