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Convex Distance Functions in 3-Space are Different
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Abstract

We investigate the bisector systems of convex distance functions in 3-space and show that there is a substantial difference to the Euclidean metric which cannot be observed in 2-space. Namely, more than one sphere can pass through four points in general position. We show that in the $L_4$-metric there exist quadrupels of points that lie on the surface of three $L_4$-spheres, and that this number does not decrease if the four points are disturbed independently within 3-dimensional neighborhoods. Moreover, for each $n \geq 2$ we construct a smooth and symmetric convex distance function $d$ and four points that are contained in the surface of exactly $n$ $d$-spheres. This result implies that there is no general upper bound to the complexity of the Voronoi diagram of four sites based on a convex distance function in 3-space.

Keywords: Convex distance functions, Voronoi diagram

1 Introduction

Let $S$ be a compact, convex body in 3-space containing the origin $O$ in its interior. For two points $p, a$, we translate $S$ by vector $p$ and consider the ray from $p$ through $a$. Let $v$ denote the unique point on the boundary of $S$ hit by this ray; see Figure 1. Then by

$$d(p,a) = \frac{|a-p|}{|v-p|}$$

a convex distance function $d$ is defined. Here $|a-p|$ denotes the Euclidean distance between $p$ and $a$. Clearly, $S$ is the "unit sphere" of all points $a$ satisfying $d(0,a) \leq 1$, equality holding only for the points on the boundary of $S$. The distance function $d$ is called smooth if the boundary of $S$ is smooth. Well-known examples of convex distance functions are the $L_p$-metrics, $1 \leq p \leq \infty$, defined by $|x|^p = \sqrt[p]{|x_1|^p + |x_2|^p + |x_3|^p}$, among them the Euclidean distance, $L_2$.

Given $m$ point sites, $p_1, \ldots, p_m$, the Voronoi diagram based on convex distance function $d$ can be defined in the usual way. With each site $p_i$, the Voronoi region containing all points $a$ satisfying $d(p_i,a) = \min\{d(p_j,a); 1 \leq j \leq m, j \neq i\}$ is associated. The boundary of the region of $p_i$ consists of pieces of bisectors $B(p_i,p_j) = \{a; d(p_i,a) = d(p_j,a)\}$ where $i \neq j$. For a survey on Voronoi diagrams we refer to Aurenhammer [1].

Voronoi diagrams based on convex distance functions in 3-space are interesting for several reasons. First, they can be used for planning translational motions of a convex robot;

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see Chew and Drysdale [4]. Second, since convex distance functions are a natural generalization of the Euclidean distance, investigating their Voronoi diagrams is a natural and necessary step towards a unifying theory on 3-dimensional Voronoi diagrams. In dimension 2, such a unifying approach is offered by the concept of abstract Voronoi diagrams; see Klein [8, 9], Mehlhorn et al. [16], and Klein et al. [10] for a structural analysis and efficient algorithms.

Two-dimensional Voronoi diagrams based on convex distance functions were first studied by Shamos and Hoey [18] for $L_2$, by Lee [13] for the other $L_p$-metrics, by Widmayer et al. [20] for distance functions defined by convex polygons, and, at the same time, by Chew and Drysdale [4] in the general case. Here, the diagram of $m$ point sites can still be computed within $O(m \log m)$ steps in the worst case.

The reason why in 2-space the classical divide-and-conquer algorithm for $L_2$ can be applied to convex distance functions almost without modifications is that the bisector systems with respect to a smooth convex distance function $d$ share important combinatorial properties with Euclidean bisectors. Namely, the bisector of two points is homeomorphic to a line, and for any three points $p, q, r$ there is at most one $d$-circle passing through them, so the intersection $B(p, q) \cap B(q, r)$ is either empty or consists of a single point; see Section 2. (However, if $S$ is not a Euclidean circle then there exists a set of four sites such that their diagram with respect to $d$ has a structure different from their Euclidean diagram, see [15].)

For a smooth convex distance function in 3-space we prove in Section 3 that

- each bisector $B(p, q)$ is homeomorphic to a plane,
- each intersection $B(p, q) \cap B(q, r)$ is either empty or homeomorphic to a line.

It is very tempting to assume that the similarity to the Euclidean metric goes further than this. Drysdale and Schaudt for example, in their recent paper [19], assume that for non-degenerate sets of point sites in $D$-space each intersection $B(p_1, p_2) \cap B(p_2, p_3) \cap \ldots \cap B(p_k, p_{k+1})$ is either empty or a connected $D - k$ surface. In this paper we show that this does not hold, not even in dimension 3. In contrast to the Euclidean metric, there can be four points $p, q, r, s$ in general position that lie on the surface of different $d$-spheres. Thus,
\(B(p,q) \cap B(q,r) \cap B(r,s)\) is not connected. Apparently, this fact has not been realized before. Only for Voronoi diagrams in Riemannian manifolds, Ehrlich and Im Hof [5] have hinted at possible differences between 2-space and 3-space.

In Section 4 we give an example of four points in 3-space that lie on the surface of exactly three different \(L_4\)-spheres. Also, we show that this is not a mere degeneracy: each of the four points can be moved independently within a small 3-dimensional neighborhood, and still (at least) three \(L_4\)-spheres are passing through the four disturbed points. In proving these facts we are using tools from computer algebra as implemented in MAPLE [3], like Buchberger’s Gröbner base algorithm, Sturm sequences, and the theorem on implicit functions.

In Section 5 we construct, for a given number \(n \geq 2\), a convex distance function \(d\), whose unit sphere \(S\) is smooth and symmetric about the origin, and four points \(p, q, r, s\) in 3-space such that the set \(B(p,q) \cap B(q,r) \cap B(r,s)\) consists of exactly \(n\) points. Since each of these points is a vertex in the Voronoi diagram of \(\{p, q, r, s\}\) with respect to \(d\), this result implies that there is no upper bound to the complexity of a Voronoi diagram of \(J\) sites based on an arbitrary convex distance function in 3-space. Our construction does not require algebraic tools but uses analytical methods.

We are well aware that this paper raises a lot of questions. Some of the most urgent ones are listed in Section 6.

## 2 Bisectors in 2-space

In this section we derive some properties of the bisectors of convex distance functions in 2-space that will be needed in studying the 3-dimensional case. These properties have so far tacitly been taken for granted, but never explicitly been proven in the literature. Independently of our work, Mazón and Recio have recently given proofs for some of the following facts, see [14, 15].

Let \(d\) be a 2-dimensional convex distance function defined by the convex set \(S\). If \(S'\) results from scaling \(S\) by factor \(c\) then its associated distance function is \(d' = \frac{1}{c}d\). Clearly, \(d\) and \(d'\) yield the same bisector systems, so we can always assume that \(S\) is "sufficiently small".

In order to exclude degeneracies resulting from 2-dimensional bisector pieces (like in \(L_1\)) we require that the unit circle \(S\) of convex distance function \(d\) be strictly convex, which means that the boundary of \(S\) does not contain a line segment. It is known that then the strict triangle inequality \(d(p,b) < d(p,a) + d(a,b)\) holds, unless \(p, a, b\) are collinear; see [8]. We say that \(S\) is smooth if it admits, at each point \(w\) of its boundary, a unique tangent sharing locally only \(w\) with \(S\). Strict convexity is implied by smoothness, but not vice versa.

Now let \(a_1 \neq a_2\) be two points in the plane, and let \(S_i\) be copies of \(S\) translated to \(a_i, i = 1, 2\), small enough so that \(S_1 \cap S_2 = \emptyset\). By \(T\) and \(L\) we denote the upper and the lower outer common tangents of \(S_1\) and \(S_2\), and by \(t_{12}, t_{21}, d_{12}, d_{21}\) their corresponding intersections with the boundaries of \(S_1\) and \(S_2\); see Figure 2.

### Lemma 1

The bisector \(B(a_1,a_2)\) is homeomorphic to a line. It is fully contained in the interior of the bent strip defined by the rays \(a_1t_{12}, a_2t_{21}, a_1d_{12}, \) and \(a_2d_{21}\).

### Proof.

Let \(p\) be a point in \(B(a_1,a_2)\) that lies strictly above the line through \(a_1\) and \(a_2\), and let \(v_i\) be the point where segment \(a_i \overline{p}\) intersects \(\partial S_i\), for \(i = 1, 2\). Since \(p\) lies in \(B(a_1,a_2)\), we have
Figure 2: Analyzing the bisector $B(a_1, a_2)$.

\[
\frac{|p - a_1|}{|v_1 - a_1|} = \frac{|p - a_2|}{|v_2 - a_2|} \tag{1}
\]

which implies that $v_1 a_2$ and $v_2 a_2$ are parallel. The line $l$ through $v_1$ and $v_2$ intersects the boundary of each $S_i$ in a second point, $u_i$. Clearly, $v_1$ and $v_2$ must be the innermost points of \{u_1, u_2, v_1, v_2\} on $l$, as depicted in Figure 2, or the rays $a_1 v_1$ and $a_2 v_2$ would not intersect. Therefore, $p$ lies in the open strip defined by $a_1 t_1, a_2 t_2$, and $a_1 a_2$. Conversely, let $l$ be an arbitrary line parallel to (but not incident with) $a_1 a_2$ that intersects the interior of both $S_1$ and $S_2$, and let $v_1, v_2$ denote the neighboring pair of intersection points in $\partial S_1 \cap l$ and $\partial S_2 \cap l$. Then the rays $a_1 v_1$ and $a_2 v_2$ must intersect in some point $p$, and $p$ belongs to $B(a_1, a_2)$, due to (1). The line segment $a_1 a_2$ contains exactly one point of $B(a_1, a_2)$. Therefore, the central projection $p \rightarrow v_1$ is a continuous mapping from $B(a_1, a_2)$ onto the open boundary piece of $S_1$ between $t_1$ and $d_1$ (which in turn is homeomorphic to a line).

We show that this mapping is also bijective. Namely, if there were two points, $p$ and $p'$, of $B(a_1, a_2)$ on the same ray $a_1 v_1$, then $a_2 p' \cap \partial S_2$ would consist of a point $v_2'$ different from $v_2$. Since $v_1 v_2$ and $v_1 v_2'$ are parallel, they must be incident. This is a contradiction to the strict convexity of $S_2$. Thus, the inverse mapping to $p \rightarrow v_1$ exists; it is given by the above construction of $p$, which is continuous in $v_1$. $\square$

Next, we turn to the intersections of bisectors.

**Lemma 2** The bisectors $B(a_1, a_2)$ and $B(a_2, a_3)$ have at most one point in common.

**Proof.** Assume that $p, p' \in B(a_1, a_2) \cap B(a_2, a_3)$. Then there are two $d$-circles, $S_1$ and $S_2$, of possibly different sizes, passing through $a_1, a_2, a_3$. Let $T$ and $L$ be the common outer tangents to $S_1$ and $S_2$, and let $t_1, t_1, d_1, d_2$ be their corresponding points of intersections with $\partial S_1$ and $\partial S_2$, as shown in Figure 3. Assume that $T$ and $L$ intersect in some point, $c$ (the case where $T$ and $L$ are parallel can be dealt with in the same way). Let $A_i, B_i, i = 1, 2$, denote the open segments of $\partial S_i$ between $t_i$ and $d_i$ such that $A_1$ is on the same side as $A_2$ and closer to $c$. Clearly, we have

\[
A_1 \cap A_2 = \emptyset, \quad A_1 \cap B_2 = \emptyset, \quad B_1 \cap B_2 = \emptyset.
\]
Thus, $\partial S_1 \cap \partial S_2 = A_2 \cap B_1$. The rays from $c$ through $S_1$ impose the same ordering on $A_2$ and $B_1$; let $p$ and $q$ denote the topmost resp. the bottommost point in $A_2 \cap B_1$. Since the open line segment $pq$ is contained in the interior of both $S_1$ and $S_2$ it separates the boundary segments of $A_2$ and $B_1$ that could form a third intersection. Hence, $|A_2 \cap B_1| = 2$, a contradiction to the assumed existence of $a_1, a_2, a_3$. \qed

Theorem 3

(i) If $a_1, a_2, a_3$ are collinear then $B(a_1, a_2) \cap B(a_2, a_3) = \emptyset$.

(ii) If $a_1, a_2, a_3$ are not collinear then $B(a_1, a_2) \cap B(a_2, a_3)$ consists of a single point, provided $d$ is a smooth convex distance function.

Proof. We translate a copy $S_i$ of $S$ to each of $a_i$, $i = 1, 2, 3$, and consider their intersection points with the common outer tangents of $S_1$ and $S_2$, and of $S_2$ and $S_3$; see Figure 4. Due to Lemma 1, $B(a_1, a_2)$ and $B(a_2, a_3)$ are confined to the interiors of the depicted strips. If $a_1, a_2, a_3$ are collinear then $S_1$, $S_2$, and $S_3$ have the same outer tangents, so the strips are disjoint. If $a_1, a_2, a_3$ are not collinear then the strips—hence the bisectors—cross, provided that $t_{21} \neq t_{23}$ and $d_{21} \neq d_{23}$ hold for the tangent points on $\partial S_2$. But this is guaranteed if $\partial S_2$ is smooth. Due to Lemma 2 we have

$$|B(a_1, a_2) \cap B(a_2, a_3)| = 1$$

in this case. \qed

Note that the smoothness assumption is necessary; for example, if $S$ results from a Euclidean circle by removing a parallel slice in the middle and gluing together the remaining two pieces, then the two resulting cusps could be the common tangent points for $S_1$ and $S_2$ as well as for $S_2$ and $S_3$, so that the strips are disjoint.

3 Bisectors in 3-space

Let $d$ be a convex distance function in 3-space defined by a strictly convex body, $S$. In this section we prove that the bisectors with respect to $d$ do have some important properties in common with Euclidean bisectors. Apparently, these results have not been proven before.

Lemma 4 The bisector $B(a_1, a_2)$ is homeomorphic to the plane.
Figure 4: The intersection of $B(a_1, a_2)$ and $B(a_2, a_3)$.

**Proof.** Let $S_i$ denote a copy of $S$ translated to $a_i, i = 1, 2$. Let $\Gamma_i$ be the set of all points on the surface of $S_i$ that admit a tangent parallel to $\overline{a_1a_2}$. Since $S_i$ is strictly convex, $\Gamma_i$ is a simple closed curve that cuts $\partial S_i$ into two open "half-spheres", each of which is homeomorphic to the plane. Let $D_1, D_2$ denote the half-spheres on $\partial S_1$ and $\partial S_2$, respectively, that are intersected by the segment $\overline{a_1a_2}$.

For $p \in B(a_1, a_2)$, let $v_1$ be the point where the ray $\overrightarrow{a_1p}$ intersects $\partial S_i, i = 1, 2$. We show that $p \mapsto v_1$ is a homeomorphism from $B(a_1, a_2)$ onto $D_1$. The proof is the same as in Lemma 1 because we can use the arguments from the 2-dimensional case. For example, if we assume that there are two points $p, p'$ of $B(a_1, a_2)$ on the ray $\overrightarrow{a_1v_1}$ then $a_1, a_2, v_1, p, p'$ are contained in the same plane $\pi$ where the restriction of $d$ to the plane $\pi$ has the unit circle $S^1$.

**Remark.** Actually, we have shown a bit more: $B(a_1, a_2)$ is the graph of the continuous function $v_1 \mapsto |p - a_1|$ in polar coordinates about $a_1$.

Finally, we prove a result analogous to Theorem 3.

**Theorem 5**

(i) If $a_1, a_2, a_3$ are collinear then $B(a_1, a_2) \cap B(a_2, a_3)$ is empty.

(ii) If $a_1, a_2, a_3$ are not collinear, then $B(a_1, a_2) \cap B(a_2, a_3)$ is a curve homeomorphic to a line, provided that $d$ is smooth.

**Proof.**

(i) If there was a point $p$ in $B(a_1, a_2) \cap B(a_2, a_3)$ then $p, a_1, a_2, a_3$ would be coplanar, contradicting Theorem 3, (i).

(ii) Let $S_i$ be a copy of $S$ translated to $a_i$, for $i = 1, 2, 3$, and let $t_i, d_i$ denote the intersection points with the upper and the lower common tangent planes (that are parallel to the plane $\pi(a_1, a_2, a_3)$); see Figure 5. By $l_i$ we denote the concatenation of the segments $\overline{t_ia_i}$ and $\overline{a_id_i}$, without the endpoints $t_i$ and $d_i$. We are proving that $B = B(a_1, a_2) \cap B(a_2, a_3)$ is homeomorphic to $l_i$. 

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To this end, let \( p \in B \). As before, we denote by \( v_i \) the intersection of \( \overrightarrow{a_i p} \) and \( \partial S_i \). The plane \( \pi(v_1, v_2, v_3) \) is parallel to \( \pi(a_1, a_2, a_3) \). We map \( p \) onto the point \( w_1 \) where \( l_1 \) intersects with \( \pi(v_1, v_2, v_3) \). Clearly, this mapping is continuous.

\[ \begin{align*}
\text{Figure 5: Analyzing the intersection } B(a_1, a_2) \cap B(a_2, a_3). \end{align*} \]

Conversely, let \( w_1 \in l_1 \), and consider the plane \( \pi \) through \( w_1 \) that is parallel to \( \pi(a_1, a_2, a_3) \), and let \( \{w_i\} = \pi \cap l_i \); for \( i = 2, 3 \). The triangle \( t(w_1, w_2, w_3) \) is homothetic to \( t(a_1, a_2, a_3) \). The set \( S'_1 = S_1 \cap \pi \) is smooth, and convex, and contains \( w_1 \) in its interior. Thus, it defines a smooth convex distance function, \( d' \). The sets \( S'_i = S_i \cap \pi, i = 2, 3 \), are translates of \( S'_1 \) with respect to the points \( w_i \). From Theorem 3, (ii), we infer that there is exactly one point \( r \) in \( \pi \) that lies in the \( d' \)-bisectors of \( w_1 \) and \( w_2 \), and of \( w_2 \) and \( w_3 \). Let \( \{v_i\} = \overrightarrow{w_i r} \cap \partial S_i \), for \( i = 1, 2, 3 \). From the discussion of the 2-dimensional case we know that \( \overrightarrow{v_i v_j} \) is parallel to \( \overrightarrow{w_i w_j}, i \neq j \). Therefore, \( t(v_1, v_2, v_3) \) is homothetic to \( t(w_1, w_2, w_3) \), hence to \( t(a_1, a_2, a_3) \). Consequently, the rays \( \overrightarrow{a_i v_i} \) meet in some point \( p \) which belongs to \( B \); see Figure 5. This construction of \( p \) is continuous in \( w_1 \), and the mappings \( w_1 \longrightarrow p \) and \( p \longrightarrow w_1 \) are inverse to each other.

4 The \( L_4 \)-Metric in 3-space

In the preceding section we have shown that bisector systems with respect to a smooth convex distance function in 3-space behave like Euclidean bisectors, as long as only three point sites are involved. Now we prove that this analogy ceases as soon as four sites are considered, even in smooth, symmetric convex distance functions as simple as the \( L_4 \)-norm.
defined by

$$|z|_4 = \sqrt[4]{x_1^4 + x_2^4 + x_3^4}.$$ 

Let \( p = (0,0,0) \), \( q = (1,\frac{1}{2},-2) \), \( r = (-1,-\frac{3}{2},\frac{1}{3}) \), \( s = (-3,-4,-\frac{1}{2}) \).

**Theorem 6**

(i) There are exactly three \( L_4 \)-spheres passing through the four points \( p, q, r, s \).

(ii) There are 3-dimensional neighborhoods \( U_p, U_q, U_r, U_s \) for \( p, q, r, \) and \( s \), correspondingly, such that for each choice of \( p' \in U_p, q' \in U_q, r' \in U_r, \) and \( s' \in U_s \) there are at least three \( L_4 \)-spheres passing through \( p', q', r', s' \).

**Proof.**

(i) We have to determine the cardinality of the intersection \( B(p, q) \cap B(q, r) \cap B(r, s) \) which is the set of zeroes of the three polynomials

$$f(p, q, X), \ f(p, r, X), \ f(p, s, X) \tag{2}$$

where

$$f(p, q, X) = (X_1 - p_1)^4 + (X_2 - p_2)^4 + (X_3 - p_3)^4$$

$$-(X_1 - q_1)^4 - (X_2 - q_2)^4 - (X_3 - q_3)^4,$$

and \( p_1, p_2, p_3, q_1, \ldots, s_3 \) denote the concrete coordinates of the four points \( p, q, r, s \) given above (note that these polynomials are of degree 3 since the forth powers cancel out).

Using the MAPLE implementation [3] of Buchberger's algorithm [2] we find that the ideal generated by the polynomials in (2) has a Gröbner basis

$$\{ aX_1 + g_1(X_3), \ bX_2 + g_2(X_3), \ g_3(X_3) \} \tag{3}$$

where \( ab \neq 0 \), \( g_1 \) and \( g_2 \) are polynomials in \( X_3 \) of degree 26, and \( g_3 \) is a polynomial in \( X_3 \) of degree 27. The system (3) has the same zeroes \( (X_1, X_2, X_3) \) as (2) does, but due to the diagonal form of (3), it is much easier to determine the zeroes.

An application of Sturm sequences [7], as implemented in MAPLE, yields that \( g_3 \) has exactly 3 real roots \( x_3^{(1)}, x_3^{(2)}, x_3^{(3)} \) (which lie in the intervals \((-16, -8), (-8, -4), (-4, 0)) \).

For each of the three values of \( x_3 \) one can, from system (3), uniquely determine the values of \( x_2 \) and of \( x_1 \). Let \( x^{(i)} \) be \( (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) \) for \( i = 1, 2, 3 \). Consequently, \( B(p, q) \cap B(q, r) \cap B(r, s) \) is of cardinality 3.

(ii) Consider the mapping \( H : \mathbb{R}^{12} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \) defined by

$$H(P, Q, R, S, X) = \begin{pmatrix} f(P, Q, X) \\ f(P, R, X) \\ f(P, S, X) \end{pmatrix}$$

For the four points \( p, q, r, s \) we have \( H(p, q, r, s, x^{(i)}) = 0 \) for \( i = 1, 2, 3 \). Clearly, \( H \) is continuously differentiable with respect to \( X \). Furthermore, the determinant of the Jacobean matrix of \( H \) fulfills

$$\det \frac{\partial H}{\partial X}(p, q, r, s, x^{(i)}) \neq 0$$

for \( i = 1, 2, 3 \). This can be shown without knowing exact representations of the roots \( x^{(i)} \) by computing the Gröbner base of the ideal generated by the three polynomials of
system (2) and the polynomial det $\frac{\partial H}{\partial X}(p, q, r, s, X)$. Namely, this Gröbner base includes the constant 1, proving that the four polynomials do not have a zero in common.

Now the theorem on implicit functions (cf. [12]) implies that there exist neighborhoods $U^{(i)}$ of $(p, q, r, s)$ in $\mathbb{R}^4$ and functions differentiable $\varphi^{(i)} : U^{(i)} \rightarrow \mathbb{R}^3$ such that for each $i, 1 \leq i \leq 3$,

$$\forall (p', q', r', s') \in U^{(i)} \quad H(p', q', r', s', \varphi^{(i)}(p', q', r', s')) = 0$$

holds. If we choose neighborhoods $U_p, U_q, U_r, U_s$ of $p, q, r, s$ small enough so that their Cartesian product is contained in each of the $U^{(i)}$ the assertion follows. □

Clearly, the correctness of this proof is relative to the correctness of the MAPLE implementations we have used. However, we have tested this result numerically by using rational approximations of the roots $x^{(i)}$ by 50-digit integers and computed the values of $f$ to be less than $10^{-47}$. To compute the Gröbner bases took about 4 hours on a SUN SparcStation 1+.

This result marks a substantial difference between $L_4$ and $L_2$ in 3-space: In $L_4$, the Voronoi diagram of only four point sites can have three Voronoi vertices, whereas the $L_2$-diagram contains at most one Voronoi vertex.

5 Any number of spheres can pass through four points

In this section we show that there is no general upper bound to the number of vertices of a Voronoi diagram based on a convex distance function, of four points in space.

**Theorem 7** For each $n > 1$ there exist a smooth, symmetric convex distance function $d$ in 3-space, and four points $p, q, r, s$ such that

$$|B(p, q) \cap B(q, r) \cap B(r, s)| = n$$

holds for the cardinality of the intersection of the $d$-bisectors.

**Proof.** We have to construct a smooth convex body, $S$, symmetric about the origin, and four points that lie on the surface of exactly $n$ differently scaled copies of $S$. Instead, we prove the following equivalent theorem. □

**Theorem 8** For each $n > 1$ there exist a smooth, symmetric convex body $S$, and four points $p, q, r, s$ such that there are exactly $n$ tetrahedra homothetic to the tetrahedron $T(p, q, r, s)$ whose vertices lie on the surface of $S$.

The rest of this section is devoted to the proof of Theorem 8, i.e. to the construction of $S$.

The idea is as follows. Suppose we have two symmetric convex functions, $f$ and $g$, in the $(x, y)$-plane and in the $(z, y)$-plane, respectively, that both pass through the origin; see Figure 6. Let $x_i = \frac{i}{n+1}, 1 \leq i \leq n$, and consider the tetrahedra $T_i = (p_i, q_i, r_i, s_i)$ defined by the vertices $p_i = (x_i, f(x_i), 0), r_i = (-x_i, f(x_i), 0), q_i = (0, g(x_i), x_i)$, and $s_i = (0, g(x_i), -x_i)$.

Then $T_i$ is homothetic to $T$ iff their edges are parallel. By construction, this holds for $p_i q_i$ and $p_i r_i$, and for $q_i s_i$ and $r_i s_i$. Due to

$$r_i - q_i = (-x_i, f(x_i) - g(x_i), -x_i)$$

$$= \left(-\frac{i}{n+1}, f\left(\frac{i}{n+1}\right) - g\left(\frac{i}{n+1}\right), -\frac{i}{n+1}\right)$$

$$r_i - q_i = (-\frac{1}{n+1}, f\left(\frac{1}{n+1}\right) - g\left(\frac{1}{n+1}\right), -\frac{1}{n+1})$$

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edge \( \overline{q_i r_i} \) is parallel to \( \overline{q_1 r_1} \) provided we choose

\[
g(x) = f(x) + \varepsilon|x|
\]  

(4)

for some constant \( \varepsilon \). The same holds for the other edges. For \( f \) we choose the lower half-circle

\[
f(x) = 1 - \sqrt{1 - x^2}.
\]

So far, each point \( p \) on \( f \) gives rise to a tetrahedron homothetic to \( T_1 \). Therefore, we shall replace \( f \) by a convex function \( F \) that shares with \( f \) the points \( p_i, r_i, 1 \leq i \leq n \), but otherwise runs between \( f \) and the polygonal chain \( p_n, p_{n-1}, \ldots, p_1, 0, r_1, \ldots, r_n \). Then \( T_2, \ldots, T_n \) are the only tetrahedra homothetic to \( T_1 \) whose vertices are on the curves \( F \) and \( g \). The lower part, \( S_d \), of body \( S \) is constructed by wrapping ellipses parallel to the \((x, z)\)-plane around the graphs of \( F \) and \( g \). The upper part, \( S_t \), of \( S \) is obtained by reflecting \( S_d \) at the point \((0, 1, 0)\), the center of \( S \).

Figure 6: Constructing homothetic tetrahedra.

There are some technical difficulties.

First, to render \( S \) smooth, the function \( g \) should have vertical tangents where it meets its mirror image in \( S_t \). Also, we want \( g \) to be smooth at the origin. In Section 5.1 we show how this can be achieved by patching \( g \) with pieces of ellipses.

In Section 5.2 we prove, that \( S \) is smooth and convex. It is interesting to observe that a solid like \( S \) that is constructed by wrapping ellipses around a skeleton of two closed convex curves is not automatically convex although all its lines of latitude (the ellipses) and of longitude are convex. In Klein and Ma [11] we have analyzed this general problem and provided a criterion sufficient for convexity, that applies to this case.
In Section 5.3 we show that $T_1, \ldots, T_n$, are the only tetrahedra homothetic to $T_1$ that have their vertices on the surface of $S$. This will conclude the proof of Theorem 8.

5.1 Defining a symmetric body $S$

Let 

$$f(x) = 1 - \sqrt{1 - x^2}$$

denote the lower half-circle with center at $(0,1)$, and let 

$$g_0(x) = f(x) + \varepsilon x$$

where $\varepsilon > 0$ can be made arbitrarily small. Finally, let 

$$x_1 = \frac{1}{n+1}, \quad x_n = \frac{n}{n+1}.$$  

We will first show how to patch $g_0$ with pieces of ellipses for values $x \leq x_1$ and $x_n \leq x$, as shown in Figure 7, in order to obtain a function $g$ suitable for our construction.

Lemma 9 Let 

$$a = \sqrt{\frac{x_n}{x_n + \varepsilon \sqrt{1 - x_n^2}}} \quad \quad b = \sqrt{1 - \frac{x_n \varepsilon}{\sqrt{1 - x_n^2}}}$$

$$d = \frac{(1 - \sqrt{1 - x_1^2})g_0(x_1)}{(1 - \sqrt{1 - x_1^2})^2 - \varepsilon x_1 \sqrt{1 - x_1^2}} \quad \quad c = \frac{x_1 d}{\sqrt{d^2 - (g_0(x_1) - d)^2}}$$
(i) We have \( a < 1, b < 1, c > 1, \text{ and } d > 1 \). As functions of \( \varepsilon \), \( a \) and \( b \) are strictly decreasing whereas \( c \) and \( d \) are strictly increasing, as \( \varepsilon \) increases. All, \( a, b, c, \text{ and } d \) tend to 1 as \( \varepsilon \) tends to 0.

(ii) The ellipses

\[
\frac{X^2}{\varepsilon^2} + \frac{(Y - d)^2}{d^2} = 1 \tag{5}
\]

\[
\frac{X^2}{a^2} + \frac{(Y - 1)^2}{b^2} = 1 \tag{6}
\]

pass through the points \((x_1, g_0(x_1))\) and \((x_n, g_0(x_n))\), correspondingly, where they have the same tangents as \(g_0\). The ellipse (5) passes through the origin and has a horizontal tangent there.

Proof.

(i) One sees directly that \( a < 1 \) and \( b < 1 \) hold. Next, straightforward computation shows that the derivatives of \( a \) and \( b \) with respect to \( \varepsilon \) are negative whereas \( d' \) is positive, for each positive value of \( \varepsilon \). Using the fact \( d > g_0(x_1) \) one can also verify that \( c' > 0 \) holds. If \( \varepsilon \) tends to 0 then \( a \) and \( b \) tend to 1, and so does \( d \), due to \( g_0(x_1) \to f(x_1) \). Consequently, also \( c \) tends to 1. Since \( c \) and \( d \) are strictly increasing in \( \varepsilon \), we infer that \( c > 1, d > 1 \) hold for each \( \varepsilon > 0 \).

(ii) Substituting \((x_1, g_0(x_1))\) for \((X, Y)\) in (5), and plugging in the expression for \( c \), shows that ellipse (5) passes through \((x_1, g_0(x_1))\). To prove that ellipse (5) has, at this point, the same tangent as \(g_0\), one has to show that

\[
\frac{d}{dX} \left( \frac{X^2}{\varepsilon^2} + \frac{(g_0(X) - d)^2}{d^2} \right) = 0
\]

holds. This is done by substituting the expression for \( c \), and by using the identity

\[
x_1 g'_0(x_1)(d - g_0(x_1)) = d^2 - (d - g_0(x_1))^2
\]

that can be verified by substituting the expression for \( d \) and the definition of \( g_0 \).
That ellipse (6) passes through \((x_n, g_0(x_n))\) and has the same tangent as \(g_0\) is shown by substituting the expressions for \( a \) and for \( b \).

Now we define

\[
g(x) = \begin{cases} 
  d(1 - \sqrt{1 - \frac{x^2}{a^2}}) & \text{, } |x| < x_1 \\
  f(x) + \varepsilon|x| & \text{, } x_1 \leq |x| \leq x_n \\
  1 - b\sqrt{1 - \frac{x^2}{a^2}} & \text{, } x_n < |x| \leq a 
\end{cases} \tag{7}
\]

This function is the rib in the \((y, z)\)-plane of our convex body, \(S\); see Figure 6. The following properties of \( g \) are obvious or consequences of Lemma 9.

Lemma 10 The function \( g \) has a horizontal tangent at the origin and vertical tangents at the points \((a, 1)\) and \((-a, 1)\). It is continuously differentiable. The second derivative of \( g \) exists and is strictly positive.

Now we define the rib of \(S\) in the \((x, y)\)-plane, \(F\), which is to replace \(f\). Let \( x_i = \frac{i}{n+1} \), \( 0 \leq i \leq n + 1 \), and \( p_i = f(x_i) \). Then

\[
l(x) = f(x_k) + \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}(x - x_k) \quad \text{for} \quad x \in [x_{k-1}, x_k], k = 1, \ldots, n + 1
\]
is the polygonal chain with vertices \(0, p_1, p_2, \ldots, p_n, 1\). For an integer \(m \geq 2\) yet to be specified we put

\[
M = \max_{x \in [0, 1]} (|l(x) - f(x)|), \quad \mu(x) = \frac{(l(x) - f(x))^2}{m^2 M^2}
\]

and define

\[
F(x) = \begin{cases}
  f(x), & x \in [0, x_1] \\
  \mu(x)(l(x) + (1 - \mu(x))f(x), & x \in (x_1, x_n) \\
  f(x), & x \in [x_n, 1] \\
  F(-x), & x \in [-1, 0)
\end{cases}
\]

Lemma 11. \(F\) is a convex function, and twice continuously differentiable. Its third derivative is continuous on \([x_1, x_n]\). For all \(x \in [0, x_1] \cup [x_n, 1] \cup \{x_1, \ldots, x_n\}\) we have \(F(x) = f(x)\). At all other points \(F\) runs strictly between \(l\) and \(f\). We also have \(F'(x_i) = f'(x_i)\) and \(F''(x_i) = f''(x_i)\).

**Proof.** For \(x \in [0, 1]\) we have \(l(x) = f(x)\) iff \(x = x_k\) for some \(k\). Only in this case is \(\mu(x) = 0\). Otherwise, \(F\) runs strictly between \(l(x)\) and \(f(x)\) for \(x \in (x_1, x_n)\). If \(x \in (x_{k-1}, x_k)\), where \(1 < k < n + 1\), then

\[
\mu'(x) = \frac{2}{m^2 M^2} (l'(x) - f'(x))(l(x) - f(x)),
\]

hence

\[
F'(x) = \mu'(x)(l(x) - f(x)) + \mu(x)(l'(x) - f'(x)) + f'(x) = 2\mu(x)(l'(x) - f'(x)) + \mu(x)(l'(x) - f'(x)) + f'(x) = 3\mu(x)(l'(x) + (1 - 3\mu(x))f'(x)
\]

which shows that \(F'\) is continuous. Since \(l''(x) = 0\),

\[
\mu''(x) = \frac{2}{m^2 M^2} ((l'(x) - f'(x))^2 - f''(x)(l(x) - f(x)))
\]

holds and consequently

\[
F''(x) = \mu''(x)(l(x) - f(x)) + 2\mu'(x)(l'(x) - f'(x)) + (1 - \mu(x))f''(x) = \frac{6}{m^2 M^2} (l'(x) - f'(x))^2 (l(x) - f(x)) + (1 - 3\mu(x))f''(x)
\]

which proves that \(F''\) is continuous. The convexity follows since \(F''(x) > 0\) holds.

From the last formula we obtain

\[
F'''(x) = f'''(x) - \frac{3}{m^2 M^2} (f'''(x)(l(x) - f(x))^2 + 6f''(x)(l'(x) - f'(x))(l(x) - f(x)) - 2(l'(x) - f'(x))^3).
\]

Now, we refer to Figure 8 and reflect \(F\) and \(g\) at the point \((0, 1, 0)\). For \(y \in [0, 2]\) let \(\varphi(y)\) and \(\gamma(y)\) denote the inverse images of \(y\) under \(F\) resp. \(g\) on the \(x\)-axis resp. on the \(z\)-axis. For each \(y \in (0, 2)\) we wrap around \(F\) and \(g\) the ellipse parallel to the \((x, z)\)-plane given by

\[
\frac{X^2}{\varphi(y)^2} + \frac{Z^2}{\gamma(y)^2} = 1.
\]

The body \(S\) is then defined as the union of all these ellipses, joined with the points \((0, 0, 0)\) and \((0, 2, 0)\). The following fact is obvious.
Lemma 12 The surface $A$ of $S$ is described by

$$A = \{(\varphi(y) \cos v, y, -\gamma(y) \sin v); y \in [0, 2), v \in [0, 2\pi]\}$$

By construction, $A$ is symmetric about the point $(0, 1, 0)$. In addition, $A$ is symmetric about the $y$-axis.

5.2 The smoothness and convexity of $S$

To show that $S$ is convex we use the parametrization of $\partial S$ given in Lemma 12.

Lemma 13 The surface $A = \partial S$ is smooth.

Proof. At the origin, the $(x,z)$-plane is the unique tangent plane of $A$. Now let $w = (\varphi(u) \cos v, u, -\gamma(u) \sin v)$ where $u \in (0,1)$ and $v \in [0,2\pi)$. The direction of the normal vector at $w$ is given by the vector product

$$\frac{\partial A_x \times \partial A_y}{\partial u \times \partial v} = -\gamma(u) \cos v, \gamma(u) \varphi'(u) \cos^2 v + \gamma'(u) \varphi(u) \sin^2 v, \varphi(u) \sin v$$

provided the resulting vector is not equal to zero, which is not the case. Since $\varphi : [0,1] \rightarrow [0,1], \gamma : [0,1] \rightarrow [0,a]$ are the inverse functions of $F$ and $g$, respectively, their derivatives $\varphi'(y) = (F'(\varphi(y)))^{-1}$ and $\gamma'(y) = (g'(\gamma(y)))^{-1}$ are continuous on $(0,1)$ and we have $\varphi'(1) = 0 = \gamma'(1)$. Thus, if $u = 1$ then $w$ has a normal vector of direction $(-a \cos v, 0, \sin v)$.

Consequently, the tangent plane of $w$ is parallel to the $y$-axis. □

A mentioned before, the convexity of $S$ is not granted by construction. However, we have shown in [11] that $S$ is convex if both $\varphi \varphi'$ and $\gamma \gamma'$ are strictly decreasing functions. This is ensured by the following lemma.
Lemma 14

(i) The function $\gamma(y)\gamma'(y)$ is strictly decreasing in $y$ if the number $\varepsilon$ in (7), the definition of $g$, is small enough.

(ii) The function $\varphi(y)\varphi'(y)$ is strictly decreasing in $y$ if the integer $m$ in (8) is large enough.

Proof.

(i) We compute the inverse functions of the three constituent pieces of $g$, see (7), and find that

$$\gamma' = \begin{cases} \frac{\sqrt{d^2 - (y - d)^2}}{d} \cdot \frac{d-y}{d^2 - (y-d)^2} = \frac{\varepsilon}{\Delta} (d - y) & \text{if } y \in [0, g(x_1)] \\ \frac{(\Delta - \varepsilon (1-y))(\varepsilon \Delta + 1-y)}{\Delta} & \text{if } y \in [g(x_1), g(x_n)] \\ \frac{\sqrt{2} - (1-y)^2}{\varepsilon} \cdot \frac{1-y}{\sqrt{2} - (1-y)^2} = \frac{\varepsilon}{\Delta} (1-y) & \text{if } y \in [g(x_n), 1] \end{cases}$$

where $\Delta = \sqrt{1 - (1-y)^2 + \varepsilon^2}$. Only the middle part needs further attention. Putting

$$T(y) = \gamma(y)\gamma'(y)(1 + \varepsilon^2)^2 = \left(1 - \frac{\varepsilon(1-y)}{\Delta}\right) \varepsilon \Delta + 1 - y$$

we find (using $\Delta^2 + (1-y)^2 = 1 + \varepsilon^2$) that

$$T'(y) = \varepsilon(1 + \varepsilon^2) \left(\frac{x}{\Delta x} + \frac{1-y}{\Delta^2}\right) - \left(1 - \frac{\varepsilon(1-y)}{\Delta}\right)^2$$

$$T''(y) = -\varepsilon(1 + \varepsilon^2) \left(\frac{2x(1-y)}{\Delta x} + \frac{1+\varepsilon^2+2x(1-y)}{\Delta^2}\right) - 2\varepsilon \left(1 - \frac{\varepsilon(1-y)}{\Delta}\right) \frac{1+\varepsilon^2}{\Delta^2}.$$ 

Since

$$1 - \frac{\varepsilon(1-y)}{\Delta} = \frac{(1 + \varepsilon^2)(1 - (1-y)^2)}{\Delta(\Delta + \varepsilon(1-y))} > 0$$

the function $T'$ is strictly decreasing; therefore, it takes on its maximum value in $[g(x_1), g(x_n)]$ at $g(x_1) = 1 - \sqrt{1 - x_1^2 + \varepsilon x_1}$. If $\varepsilon$ tends to 0 then $T'(g(x_1))$ tends to $-1$. Therefore, the middle part of $\gamma'\gamma'$ is also strictly decreasing if $\varepsilon$ is small enough.

(ii) According to the definition (9) of $F$, and since $f(x) = 1 - \sqrt{1 - x^2}$, we have $\varphi(y)\varphi'(y) = 1-y$ if $y \in [0, f(x_1)]$ or $y \in [f(x_n), 1]$. Again, the middle part, where $y \in (f(x_1), f(x_n))$, is more difficult. We have to show that

$$(\varphi(y)\varphi'(y))' = \varphi'(y)^2 + \varphi(y)\varphi''(y)$$

$$= \frac{1}{F''(\varphi(y))^2} - \varphi(y)F''(\varphi(y))^2$$

is negative, where $z = \varphi(y) \in [x_1, x_n]$. Let $G(x) = F'(x) - xF''(x)$; then $G'(x) = -xF'''(x)$. From the last equation in the proof of Lemma 11 we infer that $F'''(x) > 0$ holds in $[x_1, x_n]$ if $m$ is large enough. Therefore, function $G$ is strictly decreasing. The assertion follows because due to Lemma 11 we have

$$G(x_1) = F'(x_1) - x_1F''(x_1)$$

$$= f'(x_1) - x_1f''(x_1)$$

$$= \frac{x_1}{(1-x_1^2)\frac{1}{2}} - x_1 \frac{1}{(1-x_1^2)^{\frac{3}{2}}}$$

$$< 0.$$
5.3 Counting tetrahedra

Let $T_i = T_i(l, l, r, s)$ be the homothetic tetrahedra defined by $p_i = (x_i, f(x_i), 0)$, $r_i = (-x_i, f(x_i), 0)$, $l_i = (0, g(x_i), x_i)$, and $s_i = (0, g(x_i), -x_i)$ where $x_i = \frac{i}{n+1}$ for $i = 1, \ldots, n$. Since $f(x_i) = F(x_i)$ holds by Lemma 11, these $n$ tetrahedra have their vertices on $\partial S$.

**Lemma 15** $T_1, \ldots, T_n$ are the only tetrahedra homothetic to $T_1$ whose vertices lie on the surface of $S$.

**Proof.** Let $T = T(p, q, r, s)$ be such a tetrahedron. Since $T$ is homothetic to $T_1$, the line segment $\overline{pq}$ is parallel to the $x$-axis; see Figure 6. Moreover, $p$ and $r$ have identical $y$-coordinates, so they lie on the same ellipse. By symmetry, the midpoint of $\overline{pq}$ has $x$-coordinate 0. Similarly, the midpoint of $\overline{sq}$ has a zero $z$-coordinate. Since both midpoints differ only in their $y$-coordinates this implies that the vertices of $T$ lie on the curves defined by $F$ and $g$, that is $p = (x_0, F(x_0), 0)$, $r = (-x_0, F(x_0), 0)$, $q = (0, g(x_0), x_0)$, and $s = (0, g(x_0), -x_0)$.

Tetrahedron $T$ being homothetic to $T_1$, there must be a constant $c > 0$ such that $p - r = c(p_1 - r_1)$, $q - s = c(q_1 - s_1)$, $q - r = c(q_1 - r_1)$ etc. From the first and the second identity we obtain $x_0 = cx_1 = x_0$. Now the third equality implies

$$g(x_0) - F(x_0) = c(g(x_1) - f(x_1))$$

$$= cx_1 = x_0$$

If $x_0 \in [x_1, x_n]$ then $T$ must be one of the tetrahedra $T_1, \ldots, T_n$, by construction. It remains to deal with the pieces of ellipses in the definition of $g$.

If $x_0 \in [0, x_1)$ we consider the function

$$h(x) = g(x) - F(x) - \varepsilon x$$

due to the definitions (7) of $g$ and (9) of $F$. By definition, $h(0) = 0$ and $h(x_1) = 0$ hold; the mean value theorem implies that there exists a number $\xi \in (0, x_1)$ where the first derivative

$$h'(x) = \frac{d}{c \sqrt{c^2 - x^2}} - \frac{x}{\sqrt{1 - x^2}} - \varepsilon$$

becomes zero. Also, $h'(x_1) = 0$, by Lemma 10.

It is easy to see that

$$h''(x) = \frac{cd}{(c^2 - x^2)^{\frac{3}{2}}} - \frac{1}{(1 - x^2)^{\frac{3}{2}}}$$

has at most one positive zero. Thus, $\xi_1$ and $x_1$ are the only zeroes of $h'(x)$. Since $h'(0) = -\varepsilon$ we conclude that $h(x)$ stays negative in $(0, x_1)$, so $x_0$ cannot be contained in this interval.

Finally, we consider the possibility of $x_0 \in (x_n, a]$. Now we have

$$h(x) = g(x) - F(x) - \varepsilon x$$

Let

$$T(x) = g(x) - F(x)$$

$$= \sqrt{1 - x^2} - b\sqrt{1 - \frac{x^2}{a^2}}$$

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according to the definitions of $g$ and $F$. Then

$$T'(x) = \frac{x \left( x^2 - \frac{a^4 - b^2}{a^2 - b^2} \right) (a^2 - b^2)}{\sqrt{a^2 - x^2} \sqrt{1 - x^2} (a \sqrt{a^2 - x^2} + b \sqrt{1 - x^2}) a}$$

$$T''(x) = \frac{ab}{(a^2 - x^2)^{3/2}} \frac{1}{(1 - x^2)^{3/2}} = \frac{((ab)^{4/3}(1 - x^2)^2 + (a^2 - x^2)^2 + (ab)^{2/3}(1 - x^2)(a^2 - x^2))(1 - (ab)^{2/3})}{(1 - x^2)^{3/2}(a^2 - x^2)^{3/2}}.$$  

By plugging in the very definitions of $a$ and $b$ (see Lemma 9) one verifies that $b^2 < a^2$ holds for $n \geq 3$.

Case I: $b^2 < a^4 < a^2$, $n \geq 3$.

The roots of $T'(x)$ and $T''(x)$ are real. Moreover,

$$\frac{x_n^2 - a^4 - b^2}{a^2 - b^2} = \frac{(\sqrt{1 - x_n^2} - \varepsilon x_n)(2x_n + \varepsilon \sqrt{1 - x_n^2})}{(x_n + \varepsilon \sqrt{1 - x_n^2})^2(a^2 - b^2)} > 0$$

$$\frac{a^4 - b^2}{a^2 - b^2} - \frac{a^2 - (ab)^{2/3}}{1 - (ab)^{2/3}} = \frac{b^{2/3}(1 - a^2)(a^{8/3} - b^{4/3})}{(a^2 - b^2)(1 - (ab)^{2/3})} > 0.$$  

Thus, the roots of $T'$ and $T''$ lie to the left of $(x_n, a]$. $T'(x) > 0$, $T''(x) > 0$ holds for $x \in (x_n, a]$. Also $T(x)$ is strictly convex and strictly increasing on this interval. Due to

$$T(x_n) = \varepsilon x_n \text{ and } T'(x_n) = \varepsilon$$

the function $T(x)$ must have bigger values, in $(x_n, a]$, than the linear function $\varepsilon x$. Therefore, $h(x) = T(x) - \varepsilon x$ is positive in $(x_n, a]$.

Case II: $a^4 < b^2 < a^2$, $n \geq 3$.

It is easy to see that $T'(x) > 0$ and $T''(x) > 0$ hold for $x \in (0, a]$. Also $h(x)$ is positive in $(x_n, a]$.

Case III: $n = 2$.

$a^4 < a^2 < b^2$ hold for $n = 2$, if $\varepsilon$ is small enough. Moreover,

$$\frac{a^4 - b^2}{a^2 - b^2} - 1 = \frac{a^4 - b^2 - a^2 + b^2}{a^2 - b^2} = \frac{a^2(a^2 - 1)}{a^2 - b^2} > 0.$$  

Thus, $T'(x) > 0$ and $T''(x) > 0$ in $(0, a]$, we know that $h(x)$ is positive in $(x_n, a]$.  

This concludes the proof of Theorem 8.

### 6 Conclusion

We have shown that convex distance functions in 3-space are in fact different from both 2D convex distance functions and from the Euclidean metric in 3D. Our results raise a number of interesting questions.

In the comparably simple case of the $L_4$-metric considered in Section 4, what is the maximum number of spheres that can pass through four points in general position? How do the corresponding bisector surfaces intersect one another? Could the arrangement of the bisectors of four sites, $p, q, r,$ and $s$, look as in Figure 9 a cyclic sequence $(ABCD)^*$ of horizontal cuts, as seen from above?
What happens with respect to the other $L_p$-metrics? How does the maximum number of spheres passing through four points depend on $p$? Bezout's theorem seems to indicate a growth rate proportional to $(p-1)^3$, see [6], but this counts complex solutions in projective space whereas we are only interested in real valued solutions.

As to the convex distance function constructed in Section 5, it is hard to see what happens if the four points are disturbed in an arbitrary way. But even if for sites in general position the complexity of the Voronoi diagram turned out to be bounded (based on our $L_4$-example, we do not think so!) this would not help us in designing efficient algorithms as long as we do not know which configurations of sites are degenerate, for a given convex distance function, and what the maximum intersection complexity of the bisectors is.

Clearly we have to provide answers to (some of) the above questions before we can devise algorithms for Voronoi diagrams in higher dimensions that are based on distance functions other than the Euclidean metric.

References


