Fine Asymptotics for Models with Gamma Type Moments and Rates of Convergence on Wiener Space

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Dissertation

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1 Introduction

This thesis deals with Gamma approximation for elements of a Wiener chaos (chapter 2), fine asymptotics for certain models that have moments of Gamma type (chapter 3 and 4), as well as moment estimates of Rosenthal type with applications to a variety of models (chapter 4).

Chapter 2 of this thesis combines the techniques of Stein’s method and Malliavin calculus to obtain quantitative non-central limit theorems for certain functionals of Gaussian fields.

Stein’s Method was first introduced by Charles Stein in his paper [103], and then further developed in [104]. His crucial observation was that a real-valued random variable \(N\) follows a standard normal distribution \(N(0, 1)\) if and only if
\[
\mathbb{E}[Nf(N)] = \mathbb{E}[f'(N)]
\]
for a certain class of test functions \(f\). This is also known as Stein’s lemma. He then heuristically concluded, that if the quantity \(\mathbb{E}[Xf(X) − f'(X)]\) is close to zero for some class of test functions \(f\), then the random variable \(X\) must be close to a standard normal random variable. The proximity of two random variables \(X\) and \(Y\) is usually described in terms of the quantity \(\sup_{h \in \mathcal{F}} |\mathbb{E}[h(X)] − \mathbb{E}[h(Y)]|\), where \(\mathcal{F}\) is a class of test functions. Stein’s method for normal approximation now considers the following ordinary differential equation, the so-called Stein equation
\[
f'(x) − xf(x) = h(x) − \mathbb{E}[h(N)]. \tag{1.1}
\]
It is clear that when plugging in a random variable \(X\) for \(x\) and taking expectations on both sides, the right hand side is exactly the term we are interested in. If for any given \(h\) from a class of test functions, we are able to find bounds on the solution \(f_h\) to (1.1), then we are able to bound the distance between our random variable \(X\) and the standard normal target \(N\). Note that the left hand side does not involve the target random variable \(N\) anymore. Interestingly enough, it is typically easier to bound than the right hand side.

Stein’s method has since become a popular tool for showing quantitative limit theorems and has been applied in many fields of probability theory. For a comprehensive overview on Stein’s method for normal approximation, the reader is referred to the textbook [18].

Stein’s method for gamma approximation has first been considered by Luk [70] and...
then been further refined by Pickett [95] in their Ph.D. theses. The Stein equation we will be using has been introduced by Döbler and Pecatti in [24]. We use this equation, because it avoids the positive part that is present in the Stein equation that was used e.g. in [82], and because the solution has better smoothness properties.

Malliavin calculus of variations, on the other hand, is an infinite dimensional differential calculus that was first introduced by Malliavin in [72]. Its operators, some of which we will introduce in the preliminaries of the second chapter, act on functionals of Gaussian Processes (e.g. a Brownian motion). For a detailed overview on Malliavin calculus, the reader is referred to the textbook [87].

In 2005, Nualart and Peccati proved the famous fourth moment theorem using Malliavin calculus (see [88]). It states that for a sequence of certain functionals of a Gaussian field, a central limit theorem is equivalent to convergence of the third moments. More precisely, if \((X_n)_{n \in \mathbb{N}}\) is a sequence of centered random variables with unit variance that lies inside a fixed finite sum of Wiener chaoses, and \(N \sim \mathcal{N}(0,1)\), then

\[ X_n \xrightarrow{D} N \quad \text{if and only if} \quad \mathbb{E}[X_n^4] \xrightarrow{n \to \infty} 3 = \mathbb{E}[N^4]. \]

This can be seen as a substantial simplification of the method of moments/cumulants, where instead of just the fourth moment, one would have to check convergence of all corresponding moments or, equivalently, cumulants.

A combination of Stein’s method and Malliavin calculus was first exploited by Nourdin and Peccati in [82]. Merging these two techniques, the authors were able to provide explicit quantitative bounds in the fourth moment theorem in terms of the fourth moment / fourth cumulant of the sequence. In particular, they showed that if \(F\) is Malliavin derivable, then

\[ d_W(F, N) \leq \sqrt{\mathbb{E}[1 - \Gamma_1(F)]}, \]

where \(d_W\) denotes the 1-Wasserstein distance (see section 2.1.2) and \(\Gamma_1\) is the iterated Gamma operator introduced in section 2.2.1. The authors then expressed the right hand side in terms of contraction norms, which can in turn be bounded by the fourth cumulant.

In the recent paper [85], Nourdin and Peccati showed that the exact convergence rate (in total variation distance) in the fourth moment theorem is determined by the fourth and third cumulant. More precisely, if \((F_n)_{n \in \mathbb{N}}\) is an element of a fixed Wiener chaos converging in distribution to \(N \sim \mathcal{N}(0,1)\), then there exist two positive constants \(c < C\), not depending on \(n\), such that

\[ c \times \max\{|\kappa_3(F_N)|, |\kappa_4(F_N)|\} \leq d_{TV}(F_n, N) \leq C \times \max\{|\kappa_3(F_N)|, |\kappa_4(F_N)|\}. \]

Note that whilst the third cumulant comes into play, the square root from the previous result has been removed in the upper bound. The proof is based on iterating both Stein’s method and the Malliavin integration-by-parts formula. To this end, the authors
employed the notion of iterated Gamma operators, albeit different ones than those that were used in previous publications.

Similar results are also available when the target distribution is replaced by a centered Gamma distribution (see section 2.1.1). A four moments theorem (not “fourth”) for Gamma approximation has already been considered in [81], [82] and [86]. For a quantitative version, we shall refer to Theorem 1.7 of [24], where the authors proved a bound in 1-Wasserstein distance, namely

$$d_W(F, G) \leq \max \left(1, \frac{2}{\nu}\right) \mathbb{E}\left[ (2F + 2\nu - \Gamma_1(F))^2 \right]^{1/2}.$$ 

Here $G(\nu)$ denotes a centered Gamma distribution with parameter $\nu > 0$. From [86, Theorem 3.6], for any random variable $F$ in the $q$-th Wiener chaos with $\mathbb{E}[F^2] = 2\nu$, we have the estimate

$$\mathbb{E}\left[ (2F + 2\nu - \Gamma_1(F))^2 \right] \leq \frac{q-1}{3q} |\kappa_4(F) - \kappa_4(G(\nu))| - 12\kappa_3(F) + 12\kappa_3(G(\nu))$$

$$\leq \text{const.} \times \max \left\{|\kappa_3(F) - \kappa_3(G(\nu))|, |\kappa_4(F) - \kappa_4(G(\nu))|\right\}.$$ 

Combining these two results, we obtain an upper bound similar to the one in the fourth moment theorem, but worse by a whole square root, namely

$$d_W(F, G) \leq \text{const.} \times \max \left\{|\kappa_3(F) - \kappa_3(G(\nu))|, |\kappa_4(F) - \kappa_4(G(\nu))|\right\}^{1/2}. \quad (1.2)$$

A natural question that arises, and that will be the main focus of the second chapter is if, and under which conditions, we can remove the square root employing techniques similar to the ones used in [85].

The second chapter is organized as follows. Section 2.1 introduces our target of interest, the centered Gamma distribution, as well as the various probability metrics we will consider. We then provide a basic account of the two main techniques we are using: Stein’s method for Gamma approximation and Malliavin calculus.

In section 2.2, we first introduce the concept of iterated Gamma operators. These operators, which have been used in the proof of the main theorem of [85], have not been discussed in full detail before, so we take a closer look at them. In particular, we point out the difference between these new $\Gamma$-operators and the classical ones. With all these tools at hand, we are then able to proof our main upper bound in terms of Gamma operators, which is Theorem 2.2.2. We then briefly discuss why in general it is almost impossible to translate these bounds into cumulants as it has been done in [83] for a Gaussian target.

Section 2.3 then focuses solely on the case, where the examined sequence lies in the second Wiener chaos. Here, we have additional techniques at our disposal, namely that we can consider the eigenvalues of the so-called Hilbert-Schmidt operator associated to
the random element of the second Wiener chaos. After some toy examples, it becomes clear, that the square root in the upper bound in 1.2 cannot always be removed using our Stein-Malliavin bound from the previous section. In order to do so, we would need two variance estimates, both of which are considered separately. We will see that the first one always holds, but in order to show the second one, we need an additional condition on the underlying sequence of random variables, or rather the eigenvalues of the corresponding Hilbert-Schmidt operators. Under this additional condition, we are then able to state and prove our optimal Theorem 2.3.18. In the next part of this section, we then consider a technical condition on the Hilbert-Schmidt operator itself, under which our optimality theorem continues to hold. With this we are even able to state a non-asymptotic version of it. In the final part of this section, we come back to our toy example, where the removal of the square root could not be achieved. Using explicit computations with the corresponding densities, we are able to show that in total variation, the desired upper bound indeed does hold.

The next section 2.4 goes back to the general setting and removes the restriction of being in the second Wiener chaos. Using techniques that date back to Tikhomirov ([105]), we apply the Stein equation on the level of characteristic functions to derive an upper bound for convergence rate in Kolmogorov distance. This result is completely independent from the findings of the previous sections.

Finally, section 2.5 presents an outlook on what could be done in the future to remove the technical condition in our optimality theorem. Chapter two is based on the preprint


Chapter 3 presents fine asymptotics for random variables with moments of Gamma type. In the survey [57] (see also [58]), a positive random variable $X$ is defined to have moments of Gamma type if, for $s$ in some interval,

$$\mathbb{E}[X^s] = CD^s \frac{\prod_{j=1}^{J} \Gamma(a_j s + b_j)}{\prod_{k=1}^{K} \Gamma(a'_k s + b'_k)}$$

for some integers $J, K \geq 0$ and some real constants $C, D > 0$, $a_j, b_j, a'_k, b'_k$. In [57] and [58] a rich class of examples was presented. This includes many standard distributions, among them the Gamma distribution, the Beta distribution, stable distributions, the Mittag-Leffler distribution, extreme value distributions, the Fejér distribution, and many more probability laws. For our purposes we will choose $J = K = p$, which might depend on a parameter, say $p(n)$. Moreover, we choose $b_j = b'_j = \alpha(j + l)$ for some $l$ which might depend on $n$ and $p(n)$ and a constant $\alpha$, and we will choose $a'_j = 0$. The random variables with moments of Gamma type that we focus on in this chapter are mainly determinants of random matrices, as well as volumes of random parallelotopes.
and simplices.

In order to obtain our results for these models, we use the framework of mod-$\phi$ convergence. Mod-Gaussian convergence (a special case of mod-$\phi$ convergence) was first studied in [53], inspired by theorems and conjectures in random matrix theory and number theory concerning moments of values of characteristic polynomials or zeta functions. The basic idea is to look for a natural renormalization of the characteristic functions of a sequence of random variables that does not converge in distribution. This sequence of characteristic functions then converges to some non-trivial limit.

Mod-Gaussian convergence immediately implies a central limit theorem for a properly rescaled version of the sequence under consideration. But in fact, there is much more encoded in mod-$\phi$ convergence. Under some extra conditions, one can show an extended central limit theorem, precise deviations, local limit theorems, large and moderate deviation principles and Berry-Esseen bounds, see Theorems 3.1.4, 3.1.5, 3.1.9 and 3.1.10 in section 3.1.

Recently, in [16], second-order refinements of central limit theorems for log-determinants of certain random matrix ensembles were considered. The authors provide an asymptotic expansion of the Laplace transforms of the log-determinants and apply the framework of mod-Gaussian convergence. Their results include mod-Gaussian convergence, extended central limit theorems, precise moderate deviations, Berry-Esseen bounds, as well as local limit theorems. Moreover, they were able to apply the techniques to random characteristic polynomials evaluated at 1 for circular and circular Jacobi beta ensembles.

In this chapter, we study precise asymptotics for log-determinants of $\beta$-Laguerre ensembles for $p(n) \times p(n)$ random matrices $A^t A$, where $A$ is a certain $p(n) \times n$ matrix and $A^t$ denotes the transpose, the Hermitian conjugate or the dual of $A$ when $A$ is real, complex and quaternion respectively. In mathematical statistics, $p(n)$ typically is the number of variables, each of which is observed $n$ times. Therefore, it is natural to consider the case $n \neq p(n)$. The case $n = p(n)$ has already been covered in [16]. Depending on the behavior of the sequence $n - p(n)$, we observe mod-Gaussian convergence or mod-stable convergence on $i\mathbb{R}$ (see Theorem 3.5.1).

An important observation of our findings is that the asymptotic behavior of the determinants of $\beta$-Laguerre ensembles for varying dimensions is sufficient to be able to study the asymptotics of determinants of $\beta$-Jacobi ensembles, of Ginibre ensembles, of 7 further matrix ensembles within the tenfold way of mesoscopic physics, and of the determinant of certain Gram matrices with respect to certain distributions in $\mathbb{R}^n$, representing the volume of parallelotopes. Hence we will provide similar fine asymptotics for all of these models.

The outline of chapter 3 is as follows. Section 3.1 establishes the concept of mod-$\phi$ convergence and presents the various results it entails such as extended central limit
theorems, precise deviations, local limit theorems and Berry-Esseen bounds. Section 3.2 then introduces all the different models we are considering. Based on Selberg integration, we provide explicit formulas for the Mellin transform of the corresponding random determinant. Inspired by the $\beta$-Laguerre case, we define our function $L(p, l, \alpha; z)$, and notice that it also appears in all of the other models. This is due to the fact that all of these determinants have moments of Gamma type.

In section 3.3, we prove our main theorem, which is nothing but an expansion of our function $L(p, l, \alpha; z)$. In section 3.4, we use this expansion to analyze the asymptotic behavior of $L(p(n), r(n), \beta/2; z)$ as $n$ tends to infinity. This is enough to cover all the models we consider.

Sections 3.5 and 3.6 then gathers all the mod-$\phi$ results for our models with all their consequences. Note that in some particular cases, no mod-$\phi$ converges is observed, for example in the $\beta$-Jacobi ensemble when $p(n)$ is fixed. In other cases we observe non-Gaussian mod-$\phi$ convergence on $i \mathbb{R}$, for instance in the $\beta$-Laguerre ensemble when the number of variables $p(n)$ is fixed.

Chapter four is based on the preprint


In chapter 4, we deduce inequalities of Rosenthal type under a certain condition on the cumulants of a given sequence of random variables $(Z_n)_{n \in \mathbb{N}}$. This kind of bounds on the cumulants we impose on our random variables are classic and have been studied before, for instance in [102], [37] and [25]. For independent random variables, the Rosenthal inequalities relate moments of order higher than 2 of partial sums of random variables to the variance of partial sums. In [96] it was proved that for $(X_k)_{k \in \mathbb{N}}$ being an independent and centered sequence of real valued random variables with finite moments of order $p$, $p \geq 2$, one obtains for every positive integer $n$

$$E \left[ \left| \sum_{j=1}^{n} X_j \right|^p \right] \leq \sum_{j=1}^{n} E \left[ |X_j|^p \right] + \left( \sum_{j=1}^{n} E[X_j^2] \right)^{\frac{p}{2}}.$$ 

Here $a_n \ll b_n$ means that there exists a numerical constant $C_p$, depending only on $p$ (neither on the underlying random variables nor on $n$), such that $a_n \leq C_p b_n$ for all positive integers $n$. A first Rosenthal-type inequality for weakly dependent random variables was derived in [28]. In [29] cumulant estimates are employed for deriving inequalities of Rosenthal type for weakly dependent random variables. Our abstract result, Theorem 4.2.1 is motivated by this work. We will prove moment estimates for a couple of statistics applying Theorem 4.2.1.

The chapter is structured as follows. Section 4.1 recalls the notion of cumulants and lists
some of their most important properties. We also present some of the classical results that follow from the kind of cumulant bounds we are considering.

Then next section 4.2 then presents our main Theorem 4.2.1 which is an upper bound on the difference between the moments of our considered random variable and standard Gaussian variable. As an initial example, we apply our theorem to a partial sum of independent, non-identically distributed random variables.

Section 4.3 then presents applications to several models: Dependency graphs, weighted dependency graphs, non-degenerate U-statistics, characteristic polynomials in the circular ensembles and determinants of random matrix ensembles and random simplices. Note that the last class of examples has already been studied in the previous chapter. Nonetheless, we would like to point out that these kinds of moment estimates do not follow from the asymptotics derived in chapter 2.

Chapter four is based on the preprint

2 Gamma Approximation on Wiener Chaos

2.1 Preliminaries

2.1.1 The Centered Gamma Distribution

In this section we introduce the main object of interest in this chapter – the centered Gamma distribution. In order to do so, we briefly recall the definition the classical Gamma distribution.

A continuous, real-valued random variable \( X \) is said to be Gamma-distributed with shape parameter \( \alpha > 0 \) and rate parameter \( \beta > 0 \) if it has probability density function

\[
f_{\alpha, \beta}(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}
\]

Here, \( \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt \) is the gamma function. We write \( X \sim \text{Gamma}(\alpha, \beta) \). As a convention, we sometimes write the density as

\[
f_{\alpha, \beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} 1_{\{x > 0\}}(x),
\]

(2.1)

where undefined terms, such as \((-1)^{1/2}\), are ignored as soon as the indicator function is zero.

Note that there is also an alternative parametrization of the Gamma distribution, using a shape parameter \( k = \alpha \) and a scale parameter \( \theta = \beta^{-1} \). Since we will always choose \( \beta = \theta = 1 \) in this thesis, this should not cause any confusion.

We say that a random variable \( G(\nu) \) follows a centered Gamma distribution with parameter \( \nu > 0 \), if it has the form \( G(\nu) = 2F(\nu/2) - \nu \), where \( F(\nu/2) \sim \text{Gamma}(\nu/2, 1) \). In this case, we write \( G(\nu) \sim \text{CenteredGamma}(\nu) \) (in \([23]\) this law is denoted by \( \bar{\Gamma}(\nu) \)).

Elementary computations yield that the density of \( G(\nu) \) is given by

\[
g_\nu(x) = \frac{2^{-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} (x + \nu)^{-\frac{\nu}{2}} e^{-\frac{x}{2} - \frac{\nu}{2}} 1_{\{x > -\nu\}}(x).
\]

(2.2)

When \( \nu \) is an integer, then \( G(\nu) \) is a centered \( \chi^2 \)-distribution with \( \nu \) degrees of freedom.

Note that when \( \nu < 2 \), then the density is not bounded. \( g_2 \) is bounded, but not continuous at \(-2\). For \( \nu > 2 \), the density function is continuous (see Figure [2.1]). These observations will become important later on in the proof of Lemma [2.4.4].
Figure 2.1: Pdf of $G(0.5), G(1), G(2)$ and $G(2.5)$
2.1.2 Probabilistic Metrics

The distance between two probability measures $\mu_1$ and $\mu_2$ is often measured by the quantity

$$d_D(\mu_1, \mu_2) = \sup_{f \in D} \left| \int f \, d\mu_1 - \int f \, d\mu_2 \right|,$$

for some class of measurable functions $D$.

Similarly, if $X$ and $Y$ are two real valued random variables, and $\mathcal{F}$ is a class of measurable functions $f : \mathbb{R} \to \mathbb{R}$, we define

$$d_\mathcal{F}(X,Y) = \sup_{f \in \mathcal{F}} \left| \mathbb{E}[f(X)] - \mathbb{E}[f(Y)] \right|,$$

whenever the expectations are well-defined. Some well-known examples include

- $\mathcal{F} = \{1_B : B \text{ is a Borel set}\}$ : The total variation distance $d_{TV}$,
- $\mathcal{F} = \{1_{(-\infty,a)} : a \in \mathbb{R}\}$ : The Kolmogorov distance $d_{Kol}$,
- $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is Lipschitz continuous with Lipschitz constant } L \leq 1\}$ : The $L$-Wasserstein distance $d_W$.

When $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, we write $f \in \text{Lip}(\mathbb{R})$. Such an $f$ is differentiable almost everywhere (Rademacher’s theorem, see e.g. [40, Theorem 3.1.6]). We write

$$\|f'\|_\infty = \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|f(x) - f(y)|}{|x - y|}$$

(2.3)

to denote the smallest Lipschitz constant of $f$. Note that this is not a norm, but a semi-norm on the vector space of Lipschitz continuous functions.

As a generalization of the $L$-Wasserstein distance $d_W$, we also define the following probability metrics. For $k \geq 1$, let

$$\mathcal{H}_k := \{h \in C^{k-1}(\mathbb{R}) : h^{(k-1)} \in \text{Lip}(\mathbb{R}) \text{ and } \|h^{(1)}\|_\infty \leq 1, \ldots, \|h^{(k)}\|_\infty \leq 1\}.$$

The corresponding distance between two random variables $X$ and $Y$ is then defined as

$$d_k(X,Y) := \sup_{h \in \mathcal{H}_k} \left| \mathbb{E}[h(X)] - \mathbb{E}[h(Y)] \right|. \quad (2.4)$$

Note that $d_1 = d_W$. Also note that all of these distances are convergence determining, meaning that convergence under any of these metrics implies weak convergence (see e.g. [84, Proposition C.3.1]).
2.1.3 Stein’s Method for Gamma Approximation

Stein’s method for the Gamma distribution was first studied in [70]. The author observed that a random variable $X$ follows a $\text{Gamma}(\alpha, \beta)$ distribution, if and only if

$$\mathbb{E}[X f'(X) + (\alpha - \beta X)f(X)] = 0$$

for a certain class of test functions $f$. If $X \sim \text{Gamma}(\alpha, \beta)$ and $h : \mathbb{R} \to \mathbb{R}$ is a given test function, then the Stein equation corresponding to $h$ is the first order ordinary differential equation

$$xf'(x) + (\alpha - \beta x)f(x) = h(x) - \mathbb{E}[h(X_{\alpha,\beta})], \quad x \in \mathbb{R}. \quad (2.5)$$

If $G(\nu) \sim \text{CenteredGamma}(\nu)$, then the Stein equation for the centered Gamma distribution is given by

$$2(x + \nu)f'(x) - xf(x) = h(x) - \mathbb{E}[h(G(\nu))], \quad x \in \mathbb{R}. \quad (2.6)$$

Until the recent work of Döbler and Peccati [24], the Stein equations (2.5) and (2.6) were usually considered only on the support of the corresponding distributions, i.e. $(0, \infty)$ and $(-\nu, \infty)$ respectively. In [82] and [91], the following slightly different Stein equation for the centered Gamma distribution was used:

$$2(x + \nu)f'(x) - xf(x) = h(x) - \mathbb{E}[h(G(\nu))], \quad x \in \mathbb{R}. \quad (2.6)$$

Here $(x + \nu)_+ = \max\{x + \nu, 0\}$ denotes the positive part. As we will see later on, this Stein equation would not have worked in the proof of our Theorem 2.2.2, since we cannot separate the $x$ from the $\nu$.

We will also provide an explicit integral representation of the solution to (2.6). Since this is of independent interest, and not explicitly used in our proofs, the formula can be found in the Appendix 2.6.2. Here, we only list some important properties of the solution, which are due to [24, Theorem 2.3].

**Theorem 2.1.1.** (a) Let $h : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Then there exists a unique Lipschitz-continuous solution $f$ of (2.6) with

$$\|f_h\|_{\infty} \leq \|h'\|_{\infty} \quad \text{and} \quad \|f'_h\|_{\infty} \leq \max\left\{1, \frac{2}{\nu}\right\}\|h'\|_{\infty}.$$  

(b) If in addition $h$ is continuously differentiable on $\mathbb{R}$ and both $h$ and $h'$ are Lipschitz, then $f_h$ is also continuously differentiable, and $f'_h$ is Lipschitz-continuous with smallest Lipschitz constant

$$\|f''_h\|_{\infty} \leq \max\left\{1, \frac{2}{\nu}\right\}\|h'\|_{\infty} + \|h''\|_{\infty}.$$  

Note that the uniqueness of the solution is not stated in [24, Theorem 2.3], but follows immediately from the fact that it is unique on both $(-\infty, -\nu)$ and $(-\nu, \infty)$ (see [24, Lemma 2.6]).
2.1.4 Gaussian Analysis and Malliavin Calculus

In this section, we are going to introduce all the main concepts and operators from Gaussian analysis and Malliavin calculus that are relevant to this thesis. This is intended to be a concise introduction. For a more self-contained approach and proofs, the reader is referred to the textbooks [87] and [84] upon which this section is based.

As a starting point, we consider a separable Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle_H$. An isonormal Gaussian process $X = \{X(h) : h \in \mathcal{H}\}$ is a family of centered, jointly Gaussian random variables, defined on some probability space $(\Omega, \mathcal{F}, P)$, with covariance structure $E[X(h_1)X(h_2)] = \langle h_1, h_2 \rangle_H$, $h_1, h_2 \in \mathcal{H}$. We assume that $\mathcal{F}$ is the $\sigma$-algebra generated by $X$.

For $q \in \mathbb{N}_0$, the $q$-th Hermite polynomial $H_q$ is given by

$$H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q} e^{-x^2/2}.$$

We are now able to define the important notion of Wiener chaos.

**Definition 2.1.2.** For $q \geq 0$, we denote by $\mathcal{H}_q$ the closed linear subspace of $L^2(\Omega)$ generated by $\{H_q(X(h)) : h \in \mathcal{H} \text{ with } \|h\|_H = 1\}$. This is called the $q$-th Wiener chaos of $X$.

Note that $\mathcal{H}_0 = \mathbb{R}$ and $\mathcal{H}_1 = X$. We have the following crucial observation that $L^2(\Omega)$ can be written as the direct sum of the Wiener chaoses:

**Theorem 2.1.3 (Wiener-Itô chaotic decomposition).**

$$L^2(\Omega) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q.$$

This means that for any $F \in L^2(\Omega)$, there exists a uniquely determined sequence $(F_n)_{n \in \mathbb{N}_0}$ with $F_0 = E[F] \in \mathcal{H}_0$ and $F_q \in \mathcal{H}_q$ for $q \geq 1$, such that $F = \sum_{q=0}^{\infty} F_q$, and the sum converges in $L^2(\Omega)$.

We write the uniquely determined projection of $F$ onto the $q$-th Wiener chaos as $F_q = \text{Proj}(F \mid \mathcal{H}_q)$.

We denote by $\mathcal{S}$ the set of smooth random variables, i.e. all random variables of the form $F = g(X(\phi_1), \ldots, X(\phi_n))$, where $n \geq 1$, $\phi_1, \ldots, \phi_n \in \mathcal{H}$ and $g : \mathbb{R}^n \to \mathbb{R}$ is a $C^\infty$-function, whose partial derivatives have at most polynomial growth.

**Definition 2.1.4.** Let $F \in \mathcal{S}$ be of the form $F = g(X(\phi_1), \ldots, X(\phi_n))$. We define the Malliavin derivative of $F$ with respect to $X$ as the $\mathcal{H}$-valued random element $DF \in L^2(\Omega, \mathcal{H})$

$$DF = \sum_{i=1}^{\infty} \frac{\partial g}{\partial x_i}(X(\phi_1), \ldots, X(\phi_n)) \phi_i.$$
When \( f \in \mathfrak{A} \) and \( g \in \mathfrak{B} \) are elements from two Hilbert spaces \( \mathfrak{A} \) and \( \mathfrak{B} \), we write \( f \otimes g \in \mathfrak{A} \otimes \mathfrak{B} \) to denote their tensor product. We write \( \mathfrak{H}^\otimes q \) for the \( q \)-fold tensor product of \( \mathfrak{H} \) with itself. Likewise, we write \( \mathfrak{H}^\oslash p \) for the symmetric tensor product. For more details on tensor products on Hilbert spaces, see Appendix B of [56] and Appendix E of [34].

**Definition 2.1.5.** Let \( F \in \mathcal{F} \) be of the form \( F = g(X(\phi_1), \ldots, X(\phi_n)) \) and \( p \in \mathbb{N} \). We define the \( p \)-th Malliavin derivative of \( F \) with respect to \( X \) as the \( \mathfrak{H}^\oslash p \)-valued random element \( D^p F \in L^2(\Omega, \mathfrak{H}^\otimes p) \)

\[
D^p F = \sum_{i_1, \ldots, i_p=1}^{m} \frac{\partial^p g}{\partial x_{i_1} \ldots \partial x_{i_p}}(X(h_1), \ldots, X(h_n)) \ h_{i_1} \otimes \ldots \otimes h_{i_p}.
\]

Let \( q \geq 1 \) and \( p \in \mathbb{N} \). Denote by \( \mathcal{D}^{p,q} \) the closure of \( \mathcal{F} \) with respect to the Norm

\[
\|F\|_{\mathcal{D}^{p,q}} = \left( \mathbb{E}[|F|^q] + \sum_{j=1}^{p} \mathbb{E}[\|D^j F\|_{\mathfrak{H} \otimes j}^q] \right)^{1/q}.
\]

This is the *domain of \( D^p \) in \( L^q(\Omega) \).* Using a closure argument, we can extend the definition of \( D^p \) to elements in \( \mathcal{D}^{p,q} \). The Malliavin derivative satisfies the following product rule.

**Proposition 2.1.6** (Product rule). For \( F, G \in \mathcal{D}^{1,4} \) we have that \( FG \in \mathcal{D}^{1,2} \) and

\[
D(FG) = FD(G) + D(F)G.
\]  

(2.7)

Furthermore, we have the following chain rule.

**Proposition 2.1.7** (Chain rule). If \( \varphi : \mathbb{R}^m \to \mathbb{R} \) is a continuously differentiable function with bounded partial derivatives and \( F = (F_1, \ldots, F_m) \) is a vector of elements of \( \mathcal{D}^{1,q} \) for some \( q \), then \( \varphi(F) \in \mathcal{D}^{1,q} \) and

\[
D\varphi(F) = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i} (F) \ DF_i.
\]

(2.8)

Note that the conditions on \( \varphi \) are not optimal and can be weakened.

We now define the divergence operator \( \delta^p \) as the adjoint of \( D^p : \mathcal{D}^{p,2} \to L^2(\Omega, \mathfrak{H}^\otimes p) \). First, we specify its domain:

**Definition 2.1.8.** Let \( p \in \mathbb{N} \). \( \text{Dom} \delta^p \) is the subset of \( L^2(\Omega, \mathfrak{H}^\otimes p) \) consisting of those elements \( u \) such that there exists a constant \( c > 0 \) with

\[
\left| \mathbb{E} \left[ (D^p F, u)_{\mathfrak{H} \otimes p} \right] \right| \leq c \sqrt{\mathbb{E}[F^2]} \quad \text{for all } F \in \mathcal{F}.
\]
For \( u \in \text{Dom} \delta^p \) this implies that \( F \mapsto \mathbb{E}[\langle D^p F, u \rangle_{\mathcal{H}^\otimes p}] \) is a continuous linear functional from \( \mathcal{S} \) equipped with the \( L^2(\Omega) \) norm to \( \mathbb{R} \). Since \( \mathcal{S} \) is dense in \( L^2(\Omega) \), we can extend this mapping to a continuous linear operator from \( L^2(\Omega) \) to \( \mathbb{R} \). Now by the Riesz representation theorem, there exists a unique element \( g \in L^2(\Omega) \) such that 
\[ E[\langle D^p F, u \rangle_{\mathcal{H}^\otimes p}] = E[Fg] \] for all \( F \in \mathcal{S} \). Thus we have the following definition:

**Definition 2.1.9.** For \( u \in \text{Dom} \delta^p \), \( \delta^p(u) \) is the unique element of \( L^2(\Omega) \) such that the following integration by parts formula holds:

\[ E[\langle D^p F, u \rangle_{\mathcal{H}^\otimes p}] = E[I_p F] \] for all \( F \in S \).

Thus we have the following definition:

**Definition 2.1.9.** For \( u \in \text{Dom} \delta^p \), \( \delta^p(u) \) is the unique element of \( L^2(\Omega) \) such that the following integration by parts formula holds:

\[ E[\langle D^p F, u \rangle_{\mathcal{H}^\otimes p}] = E[I_p F] \] for all \( F \in S \).

The operator \( \delta^p : \text{Dom} \delta^p \rightarrow L^2(\Omega) \) is called the **multiple divergence operator** of order \( p \).

If \( u = f \) for some \( f \in \mathcal{S}^\otimes p \) is deterministic, we call \( \delta^p(u) \) the \( p \)-th multiple integral of \( f \).

**Definition 2.1.10.** Let \( p \in \mathbb{N} \) and \( f \in \mathcal{S}^\otimes p \). Then the **\( p \)-th multiple integral integral** of \( f \) with respect to \( X \) is defined by

\[ I_p(f) = \delta^p(f) \].

**Remark 2.1.11.** If \( \mathcal{S} = L^2(\mathbb{R}) \) then \( \delta \) (and thus the operator \( I \)) coincides with the Itô integral for adapted processes with respect to a two-sided Brownian motion. Moreover, we can show that \( I_p \) coincides with the so-called \( p \)-th multiple Wiener-Itô integral or Skorohod integral. Hence the name multiple integral. For more details see [87].

We state the following useful propositions. The first states that the Malliavin derivative of a multiple integral is again a (Hilbert space valued) multiple integral. The second is an isometry property for multiple integrals.

**Proposition 2.1.12.** If \( f \in \mathcal{S}^\otimes p \) for some \( p \in \mathbb{N} \), then for all \( q \in [1, \infty) \) we have that \( I_p(f) \in D_{\infty,q} \). Furthermore, for all \( r \in \mathbb{N} \)

\[ D^r(I_p(f)) = \begin{cases} \frac{p!}{(p-r)!} I_{p-r}(f) & \text{if } r \leq p \\ 0 & \text{if } r > p. \end{cases} \]

For the exact meaning of \( I_q(f) \), when \( f \in \mathcal{S}^\otimes p \) with \( q < p \), see the chapter 2.6 of [84] about Hilbert space valued divergences.

**Proposition 2.1.13** (Isometry property of multiple integrals). Let \( p, q \in \mathbb{N} \) with \( p \geq q \) and \( f \in \mathcal{S}^\otimes p \) and \( g \in \mathcal{S}^\otimes q \). Then we have

\[ E[I_p(f)I_q(g)] = \begin{cases} p! \langle f, g \rangle_{\mathcal{H}^\otimes r} & \text{if } p = q \\ 0 & \text{otherwise}. \end{cases} \]  \( (2.9) \)

We have the following reformulation of the Wiener-Itô chaos decomposition.
Theorem 2.1.14 (Wiener-Itô chaotic decomposition revisited). For any \( f \in \mathcal{H} \) with \( \|f\|_{\mathcal{H}} = 1 \) and any \( p \in \mathbb{N} \), we have that

\[
H_p(X(f)) = I_p(f^{\otimes p})
\]

Hence the linear operator \( I_p \) provides an isometry from \((\mathcal{H}^{\otimes p}, \| \cdot \|_{L^2(\Omega)})\) onto \((\mathcal{H}_p, \| \cdot \|_{L^2(\Omega)})\) i.e. the \( p \)-th Wiener chaos equipped with the \( L^2 \)-norm. As a consequence, we have that every \( F \in L^2(\Omega) \) can be expanded as

\[
F = \mathbb{E}[F] + \sum_{p=1}^{\infty} I_p(f_p),
\]

where the kernels \( f_p \in \mathcal{H}^{\otimes p}, p \geq 1 \) are uniquely determined.

The last important result involving multiple integrals is the so-called product formula.

Theorem 2.1.15 (Product formula). Let \( p, q \in \mathbb{N} \). If \( f \in \mathcal{H}^{\otimes p} \) and \( g \in \mathcal{H}^{\otimes q} \) then

\[
I_p(f) I_q(g) = \sum_{r=0}^{\min(p,q)} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g).
\]  

(2.10)

Here, \( f \otimes_r g \) denotes the \( r \)-th symmetric contraction of \( f \) and \( g \), which can be defined as follows. Simply, when \( \mathcal{H} = L^2(A, \mathcal{A}, \mu) \), where \((A, \mathcal{A}, \mu)\) is a non-atomic measure space, then for \( 1 \leq r \leq p \leq q \) and \( f \in L^2(A^p, \mathcal{A}^{\otimes p}, \mu^{\otimes p}) \) and \( g \in L^2(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q}) \), we have

\[
f \otimes_r g(a_1, \ldots, a_{p+q-2r}) = \int_{A^r} f(x_1, \ldots, x_r, a_1, \ldots, a_{p-r})
\]

\[
g(x_1, \ldots, x_r, a_{p-r+1}, \ldots, a_{p+q-2r}) d\mu(x_1) \ldots d\mu(x_r),
\]

and \( f \otimes_r g = \widetilde{f \otimes_r g} \), where

\[
\widetilde{h}(y_1, \ldots, y_n) = \frac{1}{n!} \sum_{\sigma \in S_n} h(y_{\sigma(1)}, \ldots, y_{\sigma(n)})
\]

is the canonical symmetrization of \( h \). For a definition of contractions on general Hilbert spaces, see Appendix B.3 of [85].

Finally, we introduce some operator \( L \) and its so-called pseudo inverse \( L^{-1} \) and state the crucial integration by parts formula, which will be one of the main tools in the proof of our Theorem 2.2.2.

Definition 2.1.16. Let

\[
\text{Dom}(L) := \left\{ F \in L^2(\Omega) : \sum_{p=1}^{\infty} p^2 \mathbb{E}[\text{Proj}(F | \mathcal{H}^p)^2] < \infty \right\}.
\]

For \( F \in \text{Dom}(L) \), we define

\[
LF = -\sum_{p=1}^{\infty} p \text{Proj}(F | \mathcal{H}_p).
\]
$L$ is referred to as the \textit{infinitesimal generator of the Ornstein-Uhlenbeck semigroup}. We also define its so-called \textit{pseudo-inverse} $L^{-1}$.

**Definition 2.1.17.** Let $F \in L^2(\Omega)$. We define

$$L^{-1}F = -\sum_{p=1}^{\infty} \frac{1}{p} \Proj (F \mid \mathcal{H}_p).$$

It is easy to see that for any $F \in L^2(\Omega)$, we have $LL^{-1}(F) = F - \mathbb{E}[F]$. Hence, if $F$ is centered, $L^{-1}$ is indeed the inverse of $L$.

**Theorem 2.1.18** (Integration by parts formula). Let $F, G \in \mathbb{D}^{1,2}$. Then

$$\mathbb{E}[FG] = \mathbb{E}[F] \mathbb{E}[G] + \mathbb{E}[\langle DG, -DL^{-1}F \rangle_{\mathcal{B}}].$$

(2.11)

### 2.2 Gamma Approximation on General Wiener Chaos

Before we present our upper bound in the general setting, we will first discuss the operators $\Gamma$ in more detail. In particular, we examine how they are related to the classical $\Gamma_{\text{alt}}$-operators. This is important, because to our best knowledge, these new Gamma operators have not been discussed in full detail in the literature before. They were however used in the proof of Theorem 1.2 of [85].

#### 2.2.1 Gamma Operators and Cumulants

Let $F$ be a random variable with characteristic function $\varphi_F(t) = \mathbb{E}[e^{itF}]$. We define its $n$-th cumulant, denoted by $\kappa_n(F)$, as

$$\kappa_n(F) = \frac{1}{i^n} \frac{\partial^n}{\partial t^n} \log \varphi_F(t) \bigg|_{t=0},$$

if it exists. If the moment generating function $M_X(t) = \mathbb{E}[e^{tX}]$ exists, we define its cumulant generating function as $K_X(t) = \log M_X(t)$. Then

$$\kappa_n(F) = \frac{\partial^n}{\partial t^n} K_X(t) \bigg|_{t=0}. $$

Let $F$ be a random variable with a finite chaos expansion. We define the operators $\Gamma_i$, $i \in \mathbb{N}_0$ via

$$\Gamma_0(F) := F$$

and

$$\Gamma_{i+1}(F) := \langle D \Gamma_i(F), -DL^{-1}F \rangle_{\mathcal{B}}, \quad \text{for } i \geq 0. \quad (2.12)$$

These are the Gamma operators used in the proof of the main theorem in [85], although they are defined differently there. Before that, another family of Gamma operators was used, which we shall refer to as \textit{classical} or \textit{alternative Gamma operators}. These can be
found frequently in this framework, see for example Definition 8.4.1 in [84] or Definition 3.6 in [15]. We shall denote them by the additional subscript \( \text{alt} \) (meaning alternative). They are defined via

\[
\Gamma_{\text{alt},0}(F) := F \quad \text{and} \quad \Gamma_{\text{alt},i+1}(F) := \langle DF, DL^{-1} \Gamma_{\text{alt},i}(F) \rangle_{\mathcal{S}}, \quad \text{for } i \geq 0.
\]

(2.13)

The classical Gamma operators are related to the cumulants of \( F \) by the following identity from [83]: For all \( j \geq 0 \), we have

\[
E[\Gamma_{\text{alt},j}(F)] = \frac{1}{j!} \kappa_{j+1}(F).
\]

If \( j \geq 3 \), this does not hold anymore for our new Gamma operators. Instead, we will list some useful relations between the classical and the new Gamma operators.

**Proposition 2.2.1.** Let \( F \) be a centered random variable admitting a finite chaos expansion. Then

(a) \( \Gamma_1(F) = \Gamma_{\text{alt},1}(F) \),

(b) if \( j = 1 \) or \( j = 2 \), then

\[
E[\Gamma_j(F)] = E[\Gamma_{\text{alt},j}(F)] = \frac{1}{j!} \kappa_{j+1}(F),
\]

(c) \( E[\Gamma_3(F)] = 2E[\Gamma_{\text{alt},3}(F)] - \text{Var} (\Gamma_1(F)) = \frac{1}{3} \kappa_4(F) - \text{Var} (\Gamma_1(F)) \),

(d) When \( F = I_2(f) \), for some \( f \in \mathcal{S} \otimes^2 \), is an element of the second Wiener chaos, then

\[
\Gamma_j(F) = \Gamma_{\text{alt},j}(F) \quad \text{for all } j \geq 1.
\]

The proof of these statements can be found in the appendix along with an explicit representation of the Gamma operators in terms of contractions (Lemma 2.6.1).

### 2.2.2 The Malliavin-Stein Upper Bound

In the following, we will use centered versions of the Gamma-operators, i.e.

\[
\Gamma_j(F) := \Gamma_j(F) - E[\Gamma_j(F)].
\]

**Theorem 2.2.2.** Let \( F \) be a centered random variable admitting a finite chaos expansion with \( \text{Var}(F) = 2\nu \). Let \( G(\nu) \sim \text{CenteredGamma}(\nu) \). Then there exists a constant \( C > 0 \) (only depending on \( \nu \)), such that

\[
d_2(F, G(\nu)) \leq C \left\{ \mathbb{E} \left[ (2F - \Gamma_1(F))^2 \right] + \mathbb{E} \left[ (\Gamma_2(F) - 2\Gamma_1(F))^2 \right]^{1/2} \mathbb{E} \left[ (2F - \Gamma_1(F))^2 \right]^{1/2} \right. \\
+ \mathbb{E} \left[ (\Gamma_3(F) - 2\Gamma_2(F))^2 \right]^{1/2} \right. \\
+ |\kappa_3(F) - \kappa_3(G(\nu))| + |\kappa_4(F) - \kappa_4(G(\nu))| \right\}. \tag{2.14}
\]
To simplify computations, we begin with the following Lemma.

**Lemma 2.2.3.** Let $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, where $g$ and $g'$ are bounded by a constant only depending on $\nu > 0$. Consider the solution $\varphi$ to the Stein equation $g(x) - \mathbb{E}[g(G(\nu))] = 2(x + \nu)\varphi'(x) - x\varphi(x)$. Then

(a) $\varphi$ is again a Lipschitz function, where $\varphi$ and $\varphi'$ are bounded by a constant only depending on $\nu$.

(b) If $F \in \mathcal{D}^\infty$ is a centered random variable with variance $\mathbb{E}[F^2] = 2\nu$, then for any $r \in \mathbb{N}$:

$$
\mathbb{E}[g(F)\left(\Gamma_r(F) - 2\Gamma_{r-1}(F)\right)] = -\mathbb{E}[\varphi'(F)\left(\Gamma_r(F) - 2\Gamma_{r-1}(F)\right)\left(\Gamma_1(F) - 2F\right)]
$$

Proof. Part (a) is a consequence of Theorem [2.1] part (a). For part (b), note that $2\nu = \mathbb{E}[\Gamma_1(F)]$. Thus

$$
\mathbb{E}[g(F)\left(\Gamma_r(F) - 2\Gamma_{r-1}(F)\right)] = \mathbb{E}\left[\left(g(F) - \mathbb{E}[g(G(\nu))]\right)\left(\Gamma_r(F) - 2\Gamma_{r-1}(F)\right)\right]
$$

$$
= \mathbb{E}\left[\left(2(F + \nu)\varphi'(F) - F\varphi(F)\right)\left(\Gamma_r(F) - 2\Gamma_{r-1}(F)\right)\right]
$$

$$
= 2\mathbb{E}[F\varphi'(F)\Gamma_r(F)] + \mathbb{E}[\Gamma_1(F)]\mathbb{E}[\varphi'(F)\Gamma_r(F)] - \mathbb{E}[F\varphi(F)\Gamma_r(F)]
$$

$$
- 4\mathbb{E}[F\varphi'(F)\Gamma_{r-1}(F)] - 2\mathbb{E}[\Gamma_1(F)]\mathbb{E}[\varphi'(F)\Gamma_{r-1}(F)] + 2\mathbb{E}[F\varphi(F)\Gamma_{r-1}(F)]
$$

$$
=: \sum_{i=1}^{6} T_i.
$$

Now, we use the integration by parts formula (2.11) in combination with the chain rule (2.8) and the product rule (2.7) to obtain

$$
T_3 + T_2 = -\mathbb{E}[F\varphi(F)\Gamma_r(F)] + \mathbb{E}[\Gamma_1(F)]\mathbb{E}[\varphi'(F)\Gamma_r(F)]
$$

$$
= -\mathbb{E}[\Gamma_1(F)\Gamma_r(F)\varphi'(F)] - \mathbb{E}[\varphi(F)\Gamma_{r+1}(F)] + \mathbb{E}[\Gamma_1(F)]\mathbb{E}[\varphi'(F)\Gamma_r(F)]
$$

$$
= -\mathbb{E}[\Gamma_1(F)\Gamma_r(F)\varphi'(F)] - \mathbb{E}[\varphi(F)\Gamma_{r+1}(F)],
$$

and

$$
T_6 + T_5 = 2\mathbb{E}[F\varphi(F)\Gamma_{r-1}(F)] - 2\mathbb{E}[\Gamma_1(F)]\mathbb{E}[\varphi'(F)\Gamma_{r-1}(F)]
$$

$$
= 2\mathbb{E}[\Gamma_1(F)\Gamma_{r-1}(F)\varphi'(F)] + 2\mathbb{E}[\varphi(F)\Gamma_r(F)] - 2\mathbb{E}[\Gamma_1(F)]\mathbb{E}[\varphi'(F)\Gamma_{r-1}(F)]
$$

$$
= 2\mathbb{E}[\Gamma_1(F)\Gamma_{r-1}(F)\varphi'(F)] + 2\mathbb{E}[\varphi(F)\Gamma_r(F)].
$$

Hence, putting everything together, the result follows.
Proof of Theorem 2.2.2. As a starting point, we use the Stein equation (2.7) from [24]. Let \( h \in \mathcal{H}_2 \) be a test function and denote by \( f_h \) the solution to the Stein equation \( 2(x + \nu)f_h'(x) - xf_h(x) = h(x) - \mathbb{E}[h(G(\nu))] \). Then by using the integration by parts formula (2.11), as well as the fact that \( \mathbb{E}[\Gamma_1(F)] = \kappa_2(F) = 2\nu \), we get

\[
|\mathbb{E}[h(F)] - \mathbb{E}[h(G(\nu))]| = |\mathbb{E}[2(F + \nu)f_h'(F) - Ff_h(F)]|
= |\mathbb{E}[2(F + \nu)f_h'(F) - f_h'(F)(DF, -DL^{-1}F)_B]|
= |\mathbb{E}[f_h'(F)(2F - \Gamma_1(F))]|.
\]

Now set \( g := f_h' \). Then \( g \) is a bounded Lipschitz function whose derivative \( g' \) is bounded by a constant only depending on \( \nu \), see part (b) of Theorem 2.1.1. Denote by \( \varphi \) the solution to the Gamma Stein equation \( g(x) - \mathbb{E}[g(G(\nu))] = 2(x + \nu)\varphi'(x) - x\varphi(x) \), and by \( \psi \) the solution to \( \varphi(x) - \mathbb{E}[\varphi(G(\nu))] = 2(x + \nu)\psi'(x) - x\psi(x) \). By Lemma 2.2.3 (a), both \( \varphi \) and \( \psi \) are Lipschitz, where the functions themselves, as well as their almost everywhere defined derivatives are bounded by a constant only depending on \( \nu \). Now apply Lemma 2.2.3 (b) twice, to get

\[
\mathbb{E}[g(F)(2F - \Gamma_1(F))] = \mathbb{E}[\varphi'(F)(\Gamma_1(F) - 2F)^2] + \mathbb{E}[\varphi(F)(\Gamma_2(F) - 2\Gamma_1(F))]
= \mathbb{E}[\varphi'(F)(\Gamma_1(F) - 2F)^2] - \mathbb{E}[\varphi(F)]\left(\frac{1}{2}\kappa_3(F) - 2\kappa_2(F)\right)
+ \mathbb{E}[\varphi(F)(\Gamma_2(F) - 2\Gamma_1(F))]
= \mathbb{E}[\varphi'(F)(\Gamma_1(F) - 2F)^2] - \mathbb{E}[\varphi(F)]\left(\frac{1}{2}\kappa_3(F) - 2\kappa_2(F)\right)
- \mathbb{E}[\psi'(F)(\Gamma_2(F) - 2\Gamma_1(F))(\Gamma_1(F) - 2F)]
- \mathbb{E}[\psi'(F)(\Gamma_3(F) - 2\Gamma_2(F))(\Gamma_1(F) - 2F)]
+ \mathbb{E}[\psi(F)]\left(\mathbb{E}[\Gamma_3(F)] - \kappa_3(F)\right).
\]

Note that we cannot translate \( \mathbb{E}[\Gamma_3(F)] \) directly into the fourth cumulant, but instead by Proposition 2.2.1 part (c) we have \( \mathbb{E}[\Gamma_3(F)] = \frac{1}{3}\kappa_4(F) - \text{Var}(\Gamma_1(F)) \). The variance term can be written as

\[
\text{Var}(\Gamma_1(F)) = \text{Var}(\Gamma_1(F) - 2F) - 4\kappa_2(F) + 4\mathbb{E}[F\Gamma_1(F)]
= \text{Var}(\Gamma_1(F) - 2F) - 4\kappa_2(F) + 4\mathbb{E}[\Gamma_2(F)]
= \text{Var}(\Gamma_1(F) - 2F) - 4\kappa_2(F) + 2\kappa_3(F).
\]

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Putting everything together, we obtain
\[
\mathbb{E}\left[g(F)\left(2F - \Gamma_1(F)\right)^2\right] = \mathbb{E}\left[\varphi'(F)\left(\Gamma_1(F) - 2F\right)^2\right] - \mathbb{E}\left[\psi'(F)\left(\Gamma_2(F) - 2\Gamma_1(F)\right)\left(\Gamma_1(F) - 2F\right)\right] \\
- \mathbb{E}\left[\psi(F)\left(\Gamma_3(F) - 2\Gamma_2(F)\right)\right] - \mathbb{E}[\varphi(F)]\left(\frac{1}{2}\kappa_3(F) - 2\kappa_2(F)\right) \\
- \mathbb{E}[\psi(F)]\text{Var}(\Gamma_1(F) - 2F) + \mathbb{E}[\psi(F)]\left(\frac{1}{3}\kappa_4(F) - 3\kappa_3(F) + 4\kappa_2(F)\right).
\]

The result follows by applying Cauchy-Schwarz inequality, as well as using the fact that \(\kappa_2(G(\nu)) = \kappa_2(F) = 2\nu\), \(\kappa_3(G(\nu)) = 8\nu\) and \(\kappa_4(G(\nu)) = 48\nu\), see (2.18).

\textbf{Remark 2.2.4.} 
(i) The argument based on iterating Stein’s method, instead of applying Cauchy-Schwarz inequality after using the Malliavin integration by parts formula once, implemented in the proof of Theorem 2.2.2, is similar to the main result from [83, p. 3129].

(ii) A natural framework in which to apply our main Theorem 2.2.2 is when the candidate random variable \(F\) is chaotic, meaning that \(F = I_q(f)\) for some \(q \geq 2\), and kernel \(f \in \mathcal{S}^q\). In this framework, it is well-known (e.g. [86]) that the first summand in the RHS of estimate (2.14) can be further controlled by using the third and fourth cumulants, namely that
\[
\mathbb{E}\left[(2F - \Gamma_1(F))^2\right] = \text{Var}(\Gamma_1(F) - 2F) \leq \frac{q-1}{3q} \left\{\kappa_4(F) - 12\kappa_3(F) + 48\nu\right\}.
\]
We emphasize that, when \(q \geq 4\) and \(F\) is chaotic, the linear combination of the cumulants \(\kappa_4(F) - 12\kappa_3(F) + 48\nu\) is positive, see [81, Corollary 4.4].

(iii) In order to interpret our upper bound in Theorem 2.2.2 in the terms of cumulants, analogous to the result achieved in [85], for a chaotic random variable \(F = I_q(f)\) with \(q \geq 2\), we need cumulant-type inequalities comparable to Proposition 4.3 in [15] for the remaining terms in the RHS of (2.14):
\[
\text{Var}(\Gamma_2(F) - 2\Gamma_1(F)) \quad \text{and} \quad \text{Var}(\Gamma_3(F) - 2\Gamma_2(F)).
\]

This is a deep problem and for the time being, such inequalities are difficult to tackle in full generality using the available techniques such as \textit{contraction operators} or the recent machinery of \textit{Markov triplets} [66, 7, 9]. For example, a suitable cumulant counterpart for studying the variance of the iterated Gamma quantity \(\Gamma_3(F)\) is the 8th cumulant \(\kappa_8(F)\). There exists an explicit representation (see Lemma 2.6.1) of the quantity \(\Gamma_3(F)\) in terms of appropriate contractions, involving the kernel function \(f\). However, due to several zero-contractions appearing in the cumulant side \(\kappa_8(F)\) which do not show up in the Wiener chaotic representation.
of $\Gamma_3(F)$, such a comparison does not seem feasible. The second major obstacle is that one needs to control the variance of an explicit linear structure of Gamma operators in terms of an “efficient” linear combination of cumulants. Here with “efficient” we mean that when plugging in the target random variable $G(\nu)$, the introduced cumulant combination vanishes. For instance, one has to note that $\kappa_4(G(\nu)) - 12\kappa_3(G(\nu)) + 48\nu = 0$. Thus, in Section 2.3 in order to analyze these variance quantities, we will focus on the case of $F$ belonging to the second Wiener chaos, where we have explicit representations of $F$ in terms of the eigenvalues of the corresponding Hilbert-Schmidt operator.

As we will see later on, when focusing on second Wiener chaos, the most critical term to analyze is $\text{Var} (\Gamma_3(F) - 2\Gamma_2(F))$. If we choose our test function $h$ to be smoother, we can deduce an upper bound in the smoother probability metric $d_3$ (see (2.4) for definition) without the need of iterating Stein’s method. We simply apply the integration by parts formula three times and use the classical Gamma operators.

**Proposition 2.2.5.** Let $F$ be a centered random variable admitting a finite chaos expansion with $\text{Var}(F) = 2\nu$. Let $G(\nu) \sim \text{CenteredGamma}(\nu)$. Then there exists a constant $C > 0$ (only depending on $\nu$), such that

$$d_3(F, G(\nu)) \leq C \left\{ \sqrt{\text{Var}(\Gamma_{\text{alt},3}(F) - 2\Gamma_{\text{alt},2}(F)) + |\kappa_3(F) - \kappa_3(G(\nu))| + |\kappa_4(F) - \kappa_4(G(\nu))|} \right\}.$$ (2.15)

**Proof.** Let $h \in \mathcal{H}_3$ be a test function and denote by $f$ the solution to the Stein equation $h(x) - \mathbb{E}[h(G(\nu))] = 2(x + \nu)f'(x) - xf(x)$. Now we use the Malliavin integration by parts formula (2.11) a total number of three times to get

$$\mathbb{E}[h(F)] - \mathbb{E}[h(G(\nu))] = \mathbb{E}[2(F + \nu)f'(F) - Ff(F)]$$

$$= \mathbb{E}[f'(F)(2F - \Gamma_{\text{alt},1}(F))]$$

$$= \mathbb{E}[f''(F)(2\Gamma_{\text{alt},1}(F) - \Gamma_{\text{alt},2}(F))] + \mathbb{E}[f''(F)] \left[ 2\mathbb{E}[\Gamma_{\text{alt},1}(F)] - \mathbb{E}[\Gamma_{\text{alt},2}(F)] \right]$$

$$= \mathbb{E}[f''(F)(2\Gamma_{\text{alt},2}(F) - \Gamma_{\text{alt},3}(F))] + \mathbb{E}[f''(F)] \left( \frac{1}{2}\kappa_3(F) - 4\nu \right)$$

$$= \mathbb{E}[f''(F)(2\Gamma_{\text{alt},3}(F) - \Gamma_{\text{alt},4}(F))] + \mathbb{E}[f''(F)] \left( \frac{1}{2}\kappa_3(F) - 4\nu \right)$$

$$= \mathbb{E}[f''(F)(2\Gamma_{\text{alt},4}(F) - \Gamma_{\text{alt},5}(F))] + \mathbb{E}[f''(F)](\kappa_3(F) - 8\nu)$$

$$+ \mathbb{E}[f''(F)] \left( 8\nu - \frac{1}{6}\kappa_4(F) \right) + \mathbb{E}[f''(F)] \left( \frac{1}{2}\kappa_3(F) - 4\nu \right).$$

We know that $\mathbb{E}[\kappa_3(G(\nu))] = 8\nu$ and $\mathbb{E}[\kappa_4(G(\nu))] = 48\nu$. Combining this with the
boundedness of $f''$, $f'''$ and Cauchy-Schwarz inequality, we obtain

$$\begin{align*}
|\mathbb{E}[h(F)] - \mathbb{E}[h(G(\nu))]| \\
\leq C \left\{ \mathbb{E} \left[ \left| (\Gamma_{alt,3}(F) - 2\Gamma_{alt,2}(F)) \right| \right] + |\kappa_3(F) - \kappa_3(G(\nu))| + |\kappa_4(F) - \kappa_4(G(\nu))| \right\} \\
\leq C \left\{ \sqrt{\text{Var}(\Gamma_{alt,3}(F) - 2\Gamma_{alt,2}(F))} + |\kappa_3(F) - \kappa_3(G(\nu))| + |\kappa_4(F) - \kappa_4(G(\nu))| \right\}.
\end{align*}$$

\[ \square \]

**Remark 2.2.6.** Here, we have used the traditional Gamma operators as defined in (2.13). This way, we get a simple proof for the upper bound in a smoother integral probability metric. In the next section, we will focus only on random elements $F$ belonging to the second Wiener chaos, where the two different notions of Gamma operators coincide (see part (d) of Proposition 2.2.1).

### 2.3 The Case of Second Wiener Chaos

Throughout this section we assume that $F = I_2(f)$, for some $f \in \mathcal{H}^{\otimes 2}$, belongs to the second Wiener chaos. It is a classical result (see [84, Section 2.7.4]) that these kind of random variables can be analyzed through the associated Hilbert-Schmidt operator

$$A_f : \mathcal{H} \to \mathcal{H}, \quad g \mapsto f \otimes_1 g.$$  

Denote by $\{c_{f,i} : i \in \mathbb{N}\}$ the set of eigenvalues of $A_f$. We also introduce the following sequence of auxiliary kernels

$$\left\{ f \otimes_1^{(p)} f : p \geq 1 \right\} \subset \mathcal{H}^{\otimes 2},$$

defined recursively as $f \otimes_1^{(1)} f = f$, and for $p \geq 2$,

$$f \otimes_1^{(p)} f = \left( f \otimes_1^{(p-1)} f \right) \otimes_1 f.$$

**Proposition 2.3.1.** (see e.g. [84, p. 43])

1. The random variable $F$ admits the representation

$$F = \sum_{i=1}^{\infty} c_{f,i} \left( N_i^2 - 1 \right),$$

where the $N_i$ are i.i.d. $\mathcal{N}(0,1)$ and the series converges in $L^2(\Omega)$ and almost surely.
2. For every \( p \geq 2 \)

\[
\kappa_p(F) = 2^{p-1}(p - 1)! \sum_{i=1}^{\infty} c_{f,i}^p = 2^{p-1}(p - 1)! \langle f, f \otimes (p-1) f \rangle_{\mathcal{B}}
\]

\[
= 2^{p-1}(p - 1)! \text{Tr} \left( A_f^p \right)
\]

where \( \text{Tr}(A_f^p) \) stands for the trace of the \( p \)-th power of the operator \( A_f \).

When \( \nu \) is an integer, i.e. \( G(\nu) \) is a centered chi-squared random variable with \( \nu \) degrees of freedom, then (2.16) shows us that \( G(\nu) \) is itself an element of the second Wiener chaos, where \( \nu \)-many of the eigenvalues are 1 and the remaining ones are 0. Hence, in this case, we deduce from (2.17) that 

\[
\kappa_p(G(\nu)) = 2^{p-1}(p - 1)! \nu.
\]

Perhaps not surprisingly, this is also the case, when \( \nu \) is any positive real number.

**Lemma 2.3.2.** Let \( \nu > 0 \) and \( G(\nu) \sim \text{CenteredGamma}(\nu) \). Then

\[
\kappa_p(G(\nu)) = \begin{cases} 
0, & p = 1; \\
2^{p-1}(p - 1)! \nu, & p \geq 2.
\end{cases}
\]

**Proof.** Since the cumulant generating function of a Gamma random variable is well-known, we can easily compute that of \( G(\nu) \) to be

\[
K(t) = \frac{\nu}{2} \log \left( \frac{1}{1 - 2t} \right) - \nu t.
\]

By simple induction over \( p \), we obtain

\[
\frac{d^p K}{dt^p}(t) = \begin{cases} 
-\nu + \frac{\nu}{1 - 2t}, & p = 1; \\
\nu \frac{2^p(p - 1)!}{(1 - 2t)^{p+1}}, & p \geq 2.
\end{cases}
\]

The result now follows by letting \( t = 0 \).

**Lemma 2.3.3.** Let \( F = I_2(f) \), for some \( f \in \mathcal{S}^2 \), and denote by \( A_f \) the corresponding Hilbert-Schmidt operator with eigenvalues \( \{c_{f,i} : i \geq 1\} \). Then for all \( \alpha, \beta \in \mathbb{R} \) and \( r \geq 1 \)

\[
\mathbb{E} \left[ \left( \alpha \overline{\Gamma}_r(F) + \beta \overline{\Gamma}_{r-1}(F) \right)^2 \right] = \alpha^2 \frac{\kappa_{2r+2}(F)}{(2r+1)!} + 2\alpha\beta \frac{\kappa_{2r+1}(F)}{(2r)!} + \beta^2 \frac{\kappa_{2r}(F)}{(2r - 1)!},
\]

and

\[
\mathbb{E} \left[ \left( \alpha \overline{\Gamma}_{r+1}(F) + \beta \overline{\Gamma}_{r-1}(F) \right)^2 \right] = \alpha^2 \frac{\kappa_{2r+4}(F)}{(2r+3)!} + 2\alpha\beta \frac{\kappa_{2r+2}(F)}{(2r + 1)!} + \beta^2 \frac{\kappa_{2r}(F)}{(2r - 1)!}.
\]

**Proof.** From [10] equation (24), which follows by induction on \( r \), we have the representation

\[
\overline{\Gamma}_r(F) = 2^r I_2 \left( f \otimes (r+1) f \right).
\]
Using the isometry property (2.9), we obtain
\[
\text{Var} \left( \alpha \Gamma_r(F) + \beta \Gamma_{r-1}(F) \right) = 2^{2r+1} \| \alpha (f \otimes_1^{(r+1)} f) + \frac{\beta}{2} (f \otimes_1^{(r)} f) \|^2_{B\otimes^2} \\
= 2^{2r+1} \left( \alpha^2 (f, f \otimes_1^{(2r+1)} f)_{B\otimes^2} + \alpha \beta \langle f, f \otimes_1^{(2r)} f \rangle_{B\otimes^2} + \frac{\beta^2}{4} \langle f, f \otimes_1^{(2r-1)} f \rangle_{B\otimes^2} \right) \\
= 2^{2r+1} \left( \alpha^2 \text{Tr} (A_f^{2r+2}) + \alpha \beta \text{Tr} (A_f^{2r+1}) + \frac{\beta^2}{4} \text{Tr} (A_f^{2r}) \right).
\]
The result now follows with (2.17). (2.20) can be shown similarly.

**Corollary 2.3.4.** Let \( F = I_2(f) \) as in Lemma 2.3.3. Then for all \( r \geq 1 \)
\[
\mathbb{E} \left[ \left( \Gamma_r(F) - 2 \Gamma_{r-1}(F) \right)^2 \right] = \frac{\kappa_{2r+2}(F)}{(2r+1)!} - 4 \frac{\kappa_{2r+1}(F)}{(2r)!} + 4 \frac{\kappa_{2r}(F)}{(2r-1)!} \\
= 2^{2r+1} \sum_{i=1}^{\infty} c_{f,i}^2 (c_f, i - 1)^2, \tag{2.22}
\]
and
\[
\mathbb{E} \left[ \left( \Gamma_{r+1}(F) - 2 \Gamma_r(F) \right) \left( \Gamma_r(F) - 2 \Gamma_{r-1}(F) \right) \right] = \frac{\kappa_{2r+3}(F)}{(2r+2)!} - 4 \frac{\kappa_{2r+2}(F)}{(2r+1)!} + 4 \frac{\kappa_{2r+1}(F)}{(2r)!}. \tag{2.23}
\]

**Proof.** (2.22) follows from (2.19) by letting \( \alpha = 1 \) and \( \beta = -2 \), as well as using (2.17) to express the cumulants in terms of the eigenvalues. (2.23) follows by considering
\[
\mathbb{E} \left[ \left( \Gamma_{r+1}(F) - 2 \Gamma_r(F) \right) \left( \Gamma_r(F) - 2 \Gamma_{r-1}(F) \right) \right] = \frac{1}{4} \mathbb{E} \left[ (\Gamma_{r+1}(F) + \Gamma_r(F))^2 \right] - \frac{1}{4} \mathbb{E} \left[ (\Gamma_{r+1}(F) - \Gamma_r(F))^2 \right] + \mathbb{E} \left[ (\Gamma_r(F) + \Gamma_{r-1}(F))^2 \right] \\
- \mathbb{E} \left[ (\Gamma_r(F) - \Gamma_{r-1}(F))^2 \right] - \frac{1}{2} \mathbb{E} \left[ (\Gamma_{r+1}(F) + \Gamma_{r-1}(F))^2 \right] \\
+ \frac{1}{2} \mathbb{E} \left[ (\Gamma_{r+1}(F) - \Gamma_{r-1}(F))^2 \right] - 2 \mathbb{E} \left[ (\Gamma_r(F) + 0 \Gamma_{r-1}(F))^2 \right],
\]
and plugging the right coefficients \( \alpha, \beta \) into (2.19) and (2.20) to evaluate each term.

**Assumption 2.3.5.** Since the order of the eigenvalues does not influence the distribution of the corresponding random variable, it will be handy to order them by descending absolute value. If any eigenvalue occurs with both positive and negative sign, we take the positive value first to make the representation unique. Hence for any element of the second Wiener chaos \( F = I_2(f) \), we have a canonical representation
\[
F \overset{D}{=} \sum_{i=1}^{\infty} c_{f,i} \left( N_i^2 - 1 \right),
\]
where \( |c_{f,1}| \geq |c_{f,2}| \ldots \) and if \( |c_{f,i}| = |c_{f,i+1}| \) for some \( i \in \mathbb{N} \), then \( c_{f,i} \geq c_{f,i+1} \).

\[25\]
2.3.1 Motivating Examples

Let $\nu > 0$. Assume that $(F_n)_{n \geq 1}$ is a sequence of random elements in the second Wiener chaos such that $\mathbb{E}(F_n^2) = 2\nu$ for all $n \geq 1$. The main Theorem 2.2.2 reads that there exists a general constant $C$ (only depending on $\nu$), such that

$$d_2(F_n, G(\nu)) \leq C \left\{ \operatorname{Var}(\Gamma_1(F_n) - 2F_n) \\
+ \sqrt{\operatorname{Var}(\Gamma_2(F_n) - 2\Gamma_1(F_n)) \times \operatorname{Var}(\Gamma_1(F_n) - 2F_n)} \\
+ \sqrt{\operatorname{Var}(\Gamma_3(F_n) - 2\Gamma_2(F_n)) + |\kappa_3(F_n) - \kappa_3(G(\nu))| + |\kappa_4(F_n) - \kappa_4(G(\nu))|} \right\}.$$  

As a consequence, in order for the square root in the upper bound in (1.2) to be removed, it is sufficient to verify the following statement.

There exists a constant $C$ (independent of $n$, but possibly depending on the sequence $(F_n)_{n \geq 1}$), such that the following variance-estimates hold:

$$\operatorname{Var}(\Gamma_2(F_n) - 2\Gamma_1(F_n)) \leq C \operatorname{Var}(\Gamma_1(F_n) - 2F_n), \quad (2.24)$$
$$\operatorname{Var}(\Gamma_3(F_n) - 2\Gamma_2(F_n)) \leq C \operatorname{Var}^2(\Gamma_1(F_n) - 2F_n) \quad (2.25)$$

Our main aim in the present section is to show that

(i) The variance-estimate (2.24) is universal in the sense that it holds for any random variable $F$ in the second Wiener chaos having second moment $\mathbb{E}(F^2) = 2\nu$. In particular it is not a matter of the fact whether the sequence $F_n$ converges in distribution towards a centered Gamma target $G(\nu)$.

(ii) The second variance-estimate (2.25) has a completely different flavor, and that occasionally holds too, meaning that it can be seen as a Gamma characterization estimate. By this we mean that the central assumption that the sequence $F_n$ converges in distribution towards the Gamma target distribution $G(\nu)$ is heavily used to establish the estimate.

In order to classify the convergence rate of a sequence, we introduce the following notation: When $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ are two real number sequences converging to zero. We write $a_n \approx b_n$ if

$$0 < \liminf_{n \to \infty} \frac{|a_n|}{|b_n|} \leq \limsup_{n \to \infty} \frac{|a_n|}{|b_n|} < \infty. \quad (2.26)$$

In particular, this is the case whenever $\lim_{n \to \infty} |a_n/b_n| = C$ for some constant $C > 0$. In the language of big $O$ notation this is usually written as $a_n = \Theta(b_n)$. Moreover, we write $a_n = o(b_n)$, when

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 0.$$  

Here are a few toy examples.
Example 2.3.6. Let \( \alpha_n, \beta_n \) be two sequences of positive real numbers converging to zero as \( n \to \infty \) and assume that \( (1 - \alpha_n)^2 + \beta_n^2 = 1 \) for all \( n \geq 1 \). Consider the following sequence in the second Wiener chaos

\[
F_n = c_{1,n}(N_1^2 - 1) + c_{2,n}(N_2^2 - 1) := (1 - \alpha_n)(N_1^2 - 1) - \beta_n(N_2^2 - 1)
\]

\( \overset{D}{\to} G(1) \), as \( n \to \infty \).

Note that the second moment assumption \( \mathbb{E}(F_n^2) = 2(1 - \alpha_n)^2 + 2\beta_n^2 = 2 \) implies that \( \beta_n \approx \sqrt{\alpha_n} \). Hence, using (2.22), we get

\[
\text{Var}\left(\Gamma_1(F_n) - 2\Gamma_2(F_n)\right) = 2^7[(1 - \alpha_n)^6\alpha_n^2 + \beta_n^6(1 - \beta_n)2] \approx \alpha_n^2,
\]

\[
\text{Var}\left(\Gamma_1(F_n) - 2F_n\right) = 2^4[(1 - \alpha_n)^2\alpha_n^2 + \beta_n^2(1 - \beta_n)^2] \approx \alpha_n.
\]

Also

\[
\text{Var}\left(\Gamma_2(F) - 2\Gamma_1(F)\right) = 2^5\sum_{i=1}^{2} c_{i,n}^4(1 - c_{i,n})^2 \leq 4\left(2^3\sum_{i=1}^{2} c_{i,n}^2(1 - c_{i,n})^2\right) = 4\text{Var}\left(\Gamma_1(F) - 2F\right).
\]

Therefore, for some constant \( C \) (independent of \( n \)), both estimates \( (2.24), (2.25) \) take place. Therefore, our main theorem 2.2.2 yields that

\[
d_2(F_n, G(1)) \leq C \max\{ |\kappa_3(F_n) - \kappa_3(G(1))|, |\kappa_4(F_n) - \kappa_4(G(1))| \}.
\]

Later we will see that this is always the case when the target is \( G(1) \).

Example 2.3.7. In this example, instead, we consider the following sequence

\[
F_n = c_{1,n}(N_1^2 - 1) + c_{2,n}(N_2^2 - 1) := \sqrt{1 + \frac{1}{n}(N_1^2 - 1)} + \sqrt{1 - \frac{1}{n}(N_2^2 - 1)}
\]

\( \overset{D}{\to} G(2) \), as \( n \to \infty \).

Here, we have that

\[
\text{Var}\left(\Gamma_3(F_n) - 2\Gamma_2(F_n)\right) = 2^7[\epsilon_{1,n}^6(1 - c_{1,n})^2 + \epsilon_{2,n}^6(1 - c_{2,n})^2] \approx \frac{1}{n^2},
\]

\[
\text{Var}\left(\Gamma_1(F_n) - 2F_n\right) = 2^3[\epsilon_{1,n}^2(1 - c_{1,n})^2 + \epsilon_{2,n}^2(1 - c_{2,n})^2] \approx \frac{1}{n^2}.
\]

Hence our estimate (2.25) does not hold and thus we cannot remove the square root in the convergence rate. Indeed using (2.17), we have

\[
\kappa_4(F_n) - \kappa_4(G(2)) = 2^3 3! [(c_{1,n}^4 - 1) + (c_{2,n}^4 - 1)] \approx \frac{1}{n^2},
\]

\[
\kappa_3(F_n) - \kappa_3(G(2)) = 2^2 2! [(c_{1,n}^3 - 1) + (c_{2,n}^3 - 1)] \approx \frac{1}{n^2},
\]

and thus \( \max\{ |\kappa_3(F_n) - \kappa_3(G(2))|, |\kappa_4(F_n) - \kappa_4(G(2))| \} \) converges faster than \( \text{Var}(\Gamma_3(F_n) - 2\Gamma_2(F_n))^{1/2} \).
2.3.2 Variance Estimates

**Variance Estimate:** \( \text{Var} \left( \Gamma_2(F_n) - 2\Gamma_1(F_n) \right) \leq C \text{Var} \left( \Gamma_1(F_n) - 2F_n \right) \)

We start with variance estimate (2.24). We make use of a recent discovery in [10] that the second Wiener chaos is stable under the Gamma operators, meaning that for any element \( F \) in the second Wiener chaos, the resulting centered random variable \( \overline{\Gamma}_r(F) \) remains inside the second Wiener chaos for all \( r \in \mathbb{N} \).

**Lemma 2.3.8.** Let \( \nu > 0 \), and let \( F = I_2(f) \), for some \( f \in \mathcal{F}_{\nu}^2 \) be a random variable in the second Wiener chaos such that \( \mathbb{E}[F^2] = 2\nu \). Then, for every \( r \geq 1 \), we have

\[
\text{Var} \left( \Gamma_{r+1}(F) - 2\Gamma_r(F) \right) \leq 4\nu \times \text{Var} \left( \Gamma_r(F) - 2\Gamma_{r-1}(F) \right).
\] (2.27)

In particular

\[
\text{Var} \left( \Gamma_2(F) - 2\Gamma_1(F) \right) \leq 4\nu \times \text{Var} \left( \Gamma_1(F) - 2F \right).
\]

Furthermore, for every \( r \geq 1 \), and with constant \( C_r = (4\nu)^r \), we have the following variance estimate

\[
\text{Var} \left( \Gamma_{r+1}(F) - 2\Gamma_r(F) \right) \leq C_r \times \text{Var} \left( \Gamma_1(F) - 2F \right).
\] (2.28)

**Proof.** We first prove estimate (2.27). Then estimate (2.28) follows by induction. Let \( r \geq 1 \). Denote by \( A_f \) the associated Hilbert-Schmidt operator. As in the proof of Lemma 2.3.3, the variance of the random quantity \( \Gamma_{r+1}(F) - 2\Gamma_r(F) \) can be rewritten as

\[
\text{Var} \left( \Gamma_{r+1}(F) - 2\Gamma_r(F) \right) = 2^{2r+3} \text{Tr} \left( (A_f^{r+2} - A_f^{r+1})^2 \right)
\]

\[
= 2^{2r+3} \text{Tr} (A_f^2 (A_f^{r+1} - A_f^r)^2)
\]

\[
\leq 2^{2r+3} \text{Tr}(A_f^2) \times \text{Tr} \left( (A_f^{r+1} - A_f^r)^2 \right)
\]

\[
= 2 \left( 2 \text{Tr}(A_f^2) \right) \times \text{Tr} \left( (A_f^{r+1} - A_f^r)^2 \right)
\]

\[
= 2\kappa_2(F) \times \text{Tr} \left( (A_f^{r+1} - A_f^r)^2 \right)
\]

\[
= 4\nu \times \text{Var} \left( \Gamma_r(F) - 2\Gamma_{r-1}(F) \right),
\]

where in the third step, we have used the following trace inequality for non-negative operators (see [69]),

\[
\text{Tr}(AB) \leq \text{Tr}(A) \text{Tr}(B) \quad \text{for} \quad A, B \geq 0.
\]

\( \square \)

**Remark 2.3.9.** A direct consequence of Lemma 2.3.8 is that for a random element \( F \) in the second Wiener chaos with \( \text{Var} \left( \Gamma_1(F) - 2F \right) = 0 \) (and therefore \( F = G(\nu) \) in distribution), we necessarily obtain for \( r \geq 2 \),

\[
0 = \text{Var} \left( \Gamma_{r+1}(F) - 2\Gamma_r(F) \right)
\]

\[
= \frac{1}{(2r+3)!} \kappa_{2r+4}(F) - \frac{4}{(2r+2)!} \kappa_{2r+3}(F) + \frac{4}{(2r+1)!} \kappa_{2r+2}(F).
\] (2.29)
**Variance Estimate:** \( \text{Var} \left( \Gamma_3(F_n) - 2\Gamma_2(F_n) \right) \leq C \text{Var}^2 \left( \Gamma_1(F_n) - 2F_n \right) \)

We begin with the following important observation, namely that a sequence in the second Wiener chaos can only converge to a centered chi-squared \( (\chi^2) \) distribution, not to any other centered Gamma distribution.

**Proposition 2.3.10.** Let \( \nu > 0 \), and let \( (F_n = I_2(f_n))_{n \geq 1} \) be a sequence of random variables in the second Wiener chaos such that \( \lim_{n \to \infty} \mathbb{E}[F_n^p] = 2\nu \). Denote by \( c_{j,n} \) the \( j \)-th eigenvalue of the Hilbert-Schmidt operator \( A_{f_n} \) associated with \( F_n \). Without loss of generality, assume that \( |c_{1,n}| \geq |c_{2,n}| \geq \ldots \) (see Assumption 2.3.5). Then \( F_n \) converges in distribution (even in \( L^2(\Omega) \)) to \( G(\nu) \sim \text{CenteredGamma}(\nu) \) if and only if

(a) \( \nu \) is an integer, and

(b) \( c_{j,n} \xrightarrow{n \to \infty} 0 \) for \( j > \nu \).

**Proof.** Assume that \( F_n \xrightarrow{D} G(\nu) \) for some \( \nu > 0 \). Since this implies convergence of all cumulants, (2.17) and (2.18) imply that

\[
2(p-1)! \sum_{i=1}^{\infty} c_{p,i,n} \to 2(p-1)! \nu \quad \text{as } n \to \infty
\]

\[
\iff \sum_{i=1}^{\infty} c_{p,i,n} \to \nu \quad \text{as } n \to \infty,
\]

(2.30)

for all \( p \geq 2 \). Furthermore, \( F_n \xrightarrow{D} G(\nu) \) implies \( \text{Var}(\Gamma_1(F_n) - 2F_n) \to 0 \), see e.g. [81], Theorem 1.2 condition (v). Hence by Lemma 2.3.3, we have for all \( j \in \mathbb{N} \) that

\[
c_{j,n}(c_{j,n} - 1)^2 \leq \sum_{i=1}^{\infty} c_{i,n}^2(c_{i,n} - 1)^2 = \frac{1}{23} \text{Var} \left( \Gamma_1(F_n) - 2F_n \right) \to 0 \quad \text{as } n \to \infty.
\]

From this we deduce that for all \( j \), the sequence \( (c_{j,n})_{n \geq 1} \) is bounded and can only have accumulation points 0 and 1.

First, consider \( (c_{1,n})_{n \geq 1} \). Assume there exists a subsequence \( (c_{1,n_k})_{k \geq 1} \) that converges to 0. Then using the ordering of the eigenvalues, we get

\[
\sum_{i=1}^{\infty} c_{i,n_k}^4 \leq c_{2,n_k}^2 \sum_{i=1}^{\infty} c_{i,n_k}^2 \xrightarrow{k \to \infty} 0 \times \nu = 0,
\]

which contradicts (2.30). Hence \( \lim_{n \to \infty} c_{1,n} = 1 \). What remains is

\[
\lim_{n \to \infty} \sum_{i=2}^{\infty} c_{i,n}^p = \nu - 1 \quad \text{for all } p \geq 2.
\]

(2.31)

From here, we continue inductively, each time subtracting 1 from the right hand side of (2.31). Since the right hand side cannot be negative, we conclude that \( \nu \) must be an integer and that \( \lim_{n \to \infty} c_{1,n} = \ldots = \lim_{n \to \infty} c_{\nu,n} = 1 \).
Now we are left with
\[ \lim_{n \to \infty} \sum_{i=\nu+1}^{\infty} c_{i,n}^p = 0 \quad \text{for all } p \geq 2, \] (2.32)
from which we deduce for all \( j \geq \nu + 1 \) that
\[ c_{j,n}^2 \leq \sum_{i=\nu+1}^{\infty} c_{i,n}^2 \to 0, \]
and thus \( \lim_{n \to \infty} c_{j,n} = 0 \).

Conversely, suppose that \( \nu \) is an integer and \((b)\) holds. The target has the representation
\[ G(\nu) = \sum_{i=1}^{n} (N_i^2 - 1). \]
Therefore, the \( L^2 \)-distance between \( F_n \) and \( G(\nu) \) is given by
\[ \mathbb{E}[(F_n - G(\nu))^2] = 2 \sum_{i=1}^{\nu} (c_{i,n} - 1)^2 + 2 \sum_{i=\nu+1}^{\infty} c_{i,n}^2. \]
The first term goes to zero since there are only finitely many summands. For the second term, the assumption \( \lim_{n \to \infty} \mathbb{E}[F_n^2] = 2\nu \) yields
\[ \lim_{n \to \infty} \sum_{i=1}^{\infty} c_{i,n}^2 = \nu \quad \implies \quad \lim_{n \to \infty} \sum_{i=\nu+1}^{\infty} c_{i,n}^2 = 0. \]
Hence \( F_n \to G(\nu) \) in \( L^2 \).

**Remark 2.3.11.**

(i) For the implication \( F_n \overset{D}{\to} G(\nu) \implies (a) \) and \( (b) \), we can drop the assumption \( \lim_{n \to \infty} \text{Var}(F_n) = 2\nu \), but not the ordering of the eigenvalues. Take for example the sequence
\[ F_n = \sum_{i=1}^{\infty} c_{i,n}(N_i^2 - 1) = N_n^2 - 1, \]
i.e. \( c_{i,n} = 1 \{ i = n \} \). Then obviously \( F_n \overset{D}{=} G(1) \) for all \( n \), but \( \lim_{n \to \infty} c_{i,n} = 0 \) for all \( i \in \mathbb{N} \).

(ii) For the converse, \( (a) \) and \( (b) \) \( \implies \) \( F_n \overset{D}{\to} G(\nu) \), we do not need to order the eigenvalues (by descending absolute value), but we cannot drop the variance assumption \( \lim_{n \to \infty} \text{Var}(F_n) = 2\nu \). Take for example the sequence
\[ F_n = (N_1^2 - 1) + \sum_{i=2}^{n+1} \frac{1}{\sqrt{2n}} (N_i^2 - 1) := (N_1^2 - 1) + S_n. \]
The sum \( S_n \) is independent of \( N_1^2 - 1 \), and thus by the central limit theorem \( F_n \overset{D}{\to} G(1) + N \), where \( G(1) \sim \text{CenteredGamma}(1) \) and \( N \) is an independent \( \mathcal{N}(0,1) \) variable. Here \( c_{1,n} \to 1 \) and \( c_{i,n} \to 0 \) for \( i \geq 2 \), so \( \nu = 1 \). However, for all \( n \), we have \( \text{Var}(F_n) = 3 \neq 2\nu \).
Because of Proposition 2.3.10, from now on, we will only focus on cases where \( \nu \) is an integer. Also recall that on second Wiener chaos \( \Gamma_j = \Gamma_{alt,j} \) for all \( j \), so we will always use the notation without the additional subscript.

To show variance estimate (2.25), we discuss two separate cases. First we consider the case where there are only finitely many eigenvalues, meaning that there exists \( M \in \mathbb{N} \), such that \( c_{i,n} = 0 \) for all \( i > M \) and all \( n \in \mathbb{N} \). Then we discuss the case where each sequence \( (c_{i,n})_{n \in \mathbb{N}} \) has non-zero elements.

**Proposition 2.3.12.** \((The case of finitely many eigenvalues)\) Let \( \nu > 0 \) and \( M \geq 2 \). For each \( n \geq 1 \), let \( c_{1,n}, \ldots, c_{M,n} \) be ordered by descending absolute value (see Assumption 2.3.5) and assume that 
\[
F_n := c_{1,n}(N_1^2 - 1) + c_{2,n}(N_2^2 - 1) + \ldots + c_{M,n}(N_M^2 - 1)
\]
\( \overset{D}{\rightarrow} G(\nu) \),
where \( G(\nu) \) is a centered Gamma random variable, and \( \{N_i\}_{1 \leq i \leq M} \) is a family of independent \( \mathcal{N}(0,1) \) random variables. Then \( \nu \in \{1, 2, \ldots, M\} \) is an integer, and therefore the target \( G(\nu) \) is a centered \( \chi^2 \) random variable with \( \nu \) degrees of freedom. Set
\[
\omega(n) := \max \{|1 - c_{i,n}| : i \in \{1, \ldots, \nu\}\}, \quad \text{and} \quad \vartheta(n) := \sum_{i=\nu+1}^{M} c_{i,n}^2.
\]
(2.33)
Then \( F_n \overset{L^2}{\rightarrow} G(\nu) \), as \( n \to \infty \), and the rate of the convergence in the square mean is \( \max\{\omega(n)^2, \vartheta(n)\} \). Furthermore,

(a) the asymptotic assertion
\[
\text{Var} \left( \Gamma_3(F_n) - 2\Gamma_2(F_n) \right) \approx \text{Var}^2 \left( \Gamma_1(F_n) - 2F_n \right)
\]
holds if and only if \( \vartheta(n) \approx \omega(n) \). The latter assumption holds whenever the degree of freedom \( \nu \) is equal to 1.

(b) the asymptotic assertion
\[
\text{Var} \left( \Gamma_3(F_n) - 2\Gamma_2(F_n) \right) \approx \text{Var} \left( \Gamma_1(F_n) - 2F_n \right)
\]
holds if and only if \( M \) is equal to the degree of freedom, i.e \( M = \nu \), or \( \vartheta(n) \approx \omega(n)^2 \), or \( \vartheta(n) = o(\omega(n)^2) \).

**Proof.** The first part follows immediately from Proposition 2.3.10. Now, the second moment assumption \( \mathbb{E}(F_n^2) = 2 \sum_{1 \leq i \leq M} c_{i,n}^2 = 2\nu \) implies that
\[
\sum_{i=1}^{\nu} (1 - c_{i,n})^2 = \sum_{i=1}^{\nu} (1 + c_{i,n}^2 - 2c_{i,n}) = \sum_{i=1}^{\nu} (1 + c_{i,n}^2) + \sum_{i=\nu+1}^{M} c_{i,n}^2 - 2 \sum_{i=1}^{\nu} c_{i,n} - \sum_{i=\nu+1}^{M} c_{i,n}^2
\]
\[= 2 \sum_{i=1}^{\nu} (1 - c_{i,n}) - \sum_{i=\nu+1}^{M} c_{i,n}^2. \quad (2.34)\]
Therefore,
\[ \mathbb{E}[(F_n - G(\nu))^2] = 2 \sum_{i=1}^{\nu} (1 - c_{i,n})^2 + 2 \sum_{i=\nu+1}^{M} c_{i,n}^2 = 4 \sum_{i=1}^{\nu} (1 - c_{i,n}) \to 0, \quad (2.35) \]
and the rate of convergence in \( L^2 \) can be retrieved from the from the first equality in \( (2.35) \).

**Proof of (a)**: when \( \nu = M \), then for all \( 1 \leq i \leq M \), the coefficients \( c_{i,n} \to 1 \), as \( n \to \infty \). Hence,
\[
\text{Var}(\Gamma_3(F_n) - 2\Gamma_2(F_n)) = 2^7 \sum_{i=1}^{M} c_{i,n}^6 (1 - c_{i,n})^2 \approx \omega(n)^2, \\
\text{Var}(\Gamma_1(F_n) - 2F_n) = 2^3 \sum_{i=1}^{M} c_{i,n}^2 (1 - c_{i,n})^2 \approx \omega(n)^2.
\]
Note that in this case \( \vartheta(n) = 0 \) and thus \( \vartheta(n) \approx \omega(n) \) is not possible. Here we necessarily have
\[
\text{Var}(\Gamma_3(F_n) - 2\Gamma_2(F_n)) \approx \text{Var}(\Gamma_1(F_n) - 2F_n).
\]
Hence, we assume that \( \nu < M \). Then \( c_{1,n}, \ldots, c_{\nu,n} \to 1 \), and \( c_{\nu+1,n}, \ldots, c_{M,n} \to 0 \) as \( n \to \infty \). Note that by \( (2.34) \), we have
\[
0 \leq \sum_{i=1}^{\nu} (1 - c_{i,n})^2 = 2 \sum_{i=1}^{\nu} (1 - c_{i,n}) - \sum_{i=\nu+1}^{M} c_{i,n}^2 \iff \theta(n) \leq 2 \sum_{i=1}^{\nu} (1 - c_{i,n}), \quad (2.36)
\]
which implies \( \theta(n) \leq 2\nu \omega(n) \). From this we can deduce that \( \sum_{i=\nu+1}^{M} c_{i,n}^6 = o(\omega(n)^2) \).

Hence
\[
\text{Var}(\Gamma_3(F_n) - 2\Gamma_2(F_n)) = 2^7 \sum_{i=1}^{M} c_{i,n}^6 (1 - c_{i,n})^2 \\
= 2^7 \left( \sum_{i=1}^{\nu} c_{i,n}^6 (1 - c_{i,n})^2 + \sum_{i=\nu+1}^{M} c_{i,n}^6 (1 - c_{i,n})^2 \right) \approx \max \{ \omega(n)^2, o(\omega(n)^2) \} \approx \omega(n)^2.
\]
Also,
\[
\text{Var}(\Gamma_1(F_n) - 2F_n) = 2^3 \left( \sum_{i=1}^{\nu} c_{i,n}^2 (1 - c_{i,n})^2 + \sum_{i=\nu+1}^{M} c_{i,n}^2 (1 - c_{i,n})^2 \right) \\
\approx \max \{ \omega(n)^2, \vartheta(n) \}. \quad (2.37)
\]
Hence,
\[
\text{Var}(\Gamma_3(F_n) - 2\Gamma_2(F_n)) \approx \text{Var}^2(\Gamma_1(F_n) - 2F_n) \quad \text{if and only if} \quad \vartheta(n) \approx \omega(n).
\]
When \( \nu = 1 \), then \( \omega(n) = |1 - c_{1,n}| \) and by (2.34) we have

\[
\theta(n) = \sum_{i=2}^{M} c_{i,n}^2 = 2(1 - c_{1,n}) - (1 - c_{1,n})^2 = (1 - c_{1,n})(1 + c_{1,n})
\]

and hence \( \lim_{n \to \infty} \frac{\theta(n)}{\omega(n)} = 2 \), so \( \theta(n) \approx \omega(n) \) is in order whenever the degree of freedom \( \nu \) is equal to 1.

**Proof of (b):** This can be proven using similar arguments. \( \Box \)

**Remark 2.3.13.** By (2.36) we always have \( \vartheta(n) \leq 2\nu \omega(n) \). Taking this into account together with

\[
\var{\Gamma_2(F_n) - 2\Gamma_1(F_n)} = 2^5 \sum_{i=1}^{M} c_{i,n}^4 (1 - c_{i,n})^2 \approx \frac{1}{n^2}
\]

one can conclude that the asymptotic estimate

\[
\var{\Gamma_2(F_n) - 2\Gamma_1(F_n)} \approx \var{\Gamma_3(F_n) - 2\Gamma_2(F_n)}
\]

takes place as soon as the sequence \( F_n \) in the second Wiener chaos converges in distribution towards a centered Gamma distribution \( G(\nu) \) without any further assumptions.

The following simple example shows that, in general, many things can happen.

**Example 2.3.14.** Let \( \delta \in [0,1] \), and consider the sequence \( F_n = \sum_{i=1}^{5} c_{i,n} (N_i^2 - 1) \) in the second Wiener chaos, where the coefficients \( c_{i,n} \) are given as

\[
\begin{align*}
  c_{1,n} &= \sqrt{1 + \frac{1}{n}}, & c_{2,n} &= \sqrt{1 - \frac{1}{n}}, \\
  c_{3,n} &= \sqrt{1 - \frac{1}{n^{1+\delta}}}, & c_{4,n} &= \sqrt{\frac{1}{2n^{1+\delta}}}, & c_{5,n} &= \sqrt{\frac{1}{2n^{1+\delta}}}.
\end{align*}
\]

After some simple computations, we arrive at

\[
\begin{align*}
  \var{\Gamma_3(F_n) - 2\Gamma_2(F_n)} &= 2^7 \sum_{i=1}^{5} c_{i,n}^\delta (1 - c_{i,n})^2 \approx \frac{1}{n^2} \quad \text{and} \\
  \var{\Gamma_1(F_n) - 2F_n} &= 2^3 \sum_{i=1}^{5} c_{i,n}^2 (1 - c_{i,n})^2 \approx \frac{1}{n^{1+\delta}}.
\end{align*}
\]

Therefore, when \( \delta = 0 \), then our desirable estimate

\[
\var{\Gamma_3(F_n) - 2\Gamma_2(F_n)} \approx \var^2 \left( \Gamma_1(F_n) - 2F_n \right)
\]

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takes place, and when \( \delta = 1 \), then
\[
\text{Var} \left( \Gamma_2(F_n) - 2\Gamma_1(F_n) \right) \approx \text{Var} \left( \Gamma_3(F_n) - 2\Gamma_2(F_n) \right) \approx \text{Var} \left( \Gamma_1(F_n) - 2F_n \right).
\]

In general
\[
\text{Var} \left( \Gamma_3(F_n) - 2\Gamma_2(F_n) \right) \approx \left( \text{Var} \left( \Gamma_1(F_n) - 2F_n \right) \right)^{\frac{2}{1+\delta}}.
\]

One can also consider more involved intermediate rates such as \( \vartheta(n) \approx \omega(n)^{1+5} \log^\gamma(\omega(n)) \) for some \( \delta, \gamma \geq 0 \).

**Corollary 2.3.15.** Let \( M \geq 2 \) and \( \nu > 0 \). Consider a sequence \((F_n)_{n \geq 1}\) of random elements in the second Wiener chaos such that \( \mathbb{E}(F_n^2) = 2\nu \) for all \( n \geq 1 \), possessing the representation
\[
F_n = \sum_{i=1}^{M} c_{i,n}(N_i^2 - 1), \quad n \geq 1,
\]
where \(|c_{1,n}| \geq \ldots \geq |c_{M,n}| \) for each \( n \geq 1 \) (see Assumption 2.3.5). Also, we assume that \( F_n \) converges in distribution towards a centered Gamma distribution with parameter \( \nu > 0 \). Then there exist two constants \( 0 < C_1 < C_2 \) (may depend on sequence \( F_n \), but independent of \( n \)), such that for all \( n \geq 1 \),

(i) if \( \nu = 1 \), or \( \vartheta(n) \approx \omega(n) \), then
\[
C_1 \text{Var}(\Gamma_1(F_n) - 2F_n) \leq \text{Var} \left( \Gamma_3(F_n) - 2\Gamma_2(F_n) \right) \leq C_2 \text{Var}(\Gamma_1(F_n) - 2F_n);
\]

(ii) if \( \nu = M \), or \( \vartheta(n) \approx \omega(n)^2 \), or \( \vartheta(n) \approx o(\omega(n)^2) \), then
\[
C_1 \text{Var}(\Gamma_1(F_n) - 2F_n) \leq \text{Var} \left( \Gamma_3(F_n) - 2\Gamma_2(F_n) \right) \leq C_2 \text{Var}(\Gamma_1(F_n) - 2F_n).
\]

**Remark 2.3.16. (Case \( \nu = M \))** Let \( M \geq 2 \) and \( \nu > 0 \). Assume that
\[
F_n = \sum_{i=1}^{M} c_{i,n}(N_i^2 - 1)
\]
is a sequence in the second Wiener chaos converging in distribution to a centered \( \chi^2 \) distribution \( G(M) \) with \( M \) degrees of freedom. Furthermore assume that \( \mathbb{E}(F_n^2) = 2\nu \) for all \( n \geq 1 \). By (2.34), we have that
\[
\sum_{i=1}^{M} (1 - c_{i,n})^2 = 2 \sum_{i=1}^{M} (1 - c_{i,n}) \geq 0.
\]
On the other hand,

\[ |\kappa_3(F_n) - \kappa_3(G(\nu))| = 8 \left| \sum_{i=1}^{M} (c_{i,n}^3 - 1) \right| = 8 \left| \sum_{i=1}^{M} (c_{i,n} - 1)(c_{i,n}^2 + c_{i,n} + 1) \right| \]

\[ = 8 \left| \sum_{i=1}^{M} (c_{i,n} - 1) \left( (c_{i,n} - 1)^2 + 3c_{i,n} \right) \right| \]

\[ = 8 \left| \sum_{i=1}^{M} (c_{i,n} - 1) \left( (c_{i,n} - 1)^2 + 3(c_{i,n} - 1) + 3 \right) \right| \]

\[ = 8 \left| 3 \sum_{i=1}^{M} (c_{i,n} - 1) + 3 \sum_{i=1}^{M} (c_{i,n} - 1)^2 + \sum_{i=1}^{M} (c_{i,n} - 1)^3 \right| \]

\[ = 8 \left| 9 \sum_{i=1}^{M} (c_{i,n} - 1) + \sum_{i=1}^{M} (c_{i,n} - 1)^3 \right| \]

\[ \approx \left| \sum_{i=1}^{M} (c_{i,n} - 1) \right|, \quad (2.38) \]

which in general is less than the rate \( \max \{ |1 - c_{i,n}| : i = 1, \ldots, M \} \) due to cancellation that may occur. Similarly,

\[ |\kappa_4(F_n) - \kappa_4(G(\nu))| \approx \left| \sum_{i=1}^{M} (c_{i,n} - 1) \right|. \quad (2.39) \]

Hence, the following remarks of independent interest are in order.

(i) Observations (2.38) and (2.39) reveal that in this special case, the difference of the third cumulants is of the same order as the difference of the fourth cumulant.

(ii) It is worth mentioning that if \( M = \nu \geq 5 \), then [111, Theorem 1.2] yields that in fact, in total variation distance \( d_{TV} \), there exists a constant \( C \) (which may depend on sequence \( (F_n)_{n \geq 1} \), but not on \( n \)), such that for all \( n \geq 1 \),

\[ d_{TV}(F_n, G(\nu)) \leq C \omega(n) = C \max \{ |1 - c_{i,n}| : i = 1, \ldots, M \}. \]

Hence,

\[ d_{TV}(F_n, G(\nu)) \leq C \sqrt{\max \left\{ \left| \kappa_3(F_n) - \kappa_3(G(\nu)) \right|, \left| \kappa_4(F_n) - \kappa_4(G(\nu)) \right| \right\}}. \]

**Proposition 2.3.17.** (The case of ultimately infinitely many non-zero eigenvalues) Let \( \nu > 0 \), and \( (M_n)_{n \geq 1} \subset \mathbb{N} \cup \{ \infty \} \) be a sequence such that \( M_n \uparrow \infty \), meaning that \( (M_n)_{n \geq 1} \) is increasing and \( \lim_{n \to \infty} M_n = \infty \). Consider a sequence \( (F_n)_{n \geq 1} \) of random elements in
the second Wiener chaos such that $\mathbb{E}[F_n^2] = 2\nu$ for all $n \geq 1$, possessing the following representation

$$F_n = \sum_{i=1}^{M_n} c_{i,n} (N_i^2 - 1), \quad n \geq 1,$$

where for each $n \geq 1$, it holds that $|c_{1,n}| \geq \ldots \geq |c_{M_n,n}|$ (see Assumption 2.3.3). Also, we assume that $F_n$ converges in distribution towards a centered Gamma distribution $G(\nu)$ with parameter $\nu \geq 1$. Then, the asymptotic relation

$$\text{Var} (\Gamma(F_n) - 2\Gamma(F_n)) \approx \text{Var}^2 (\Gamma_1(F_n) - 2F_n)$$

holds if and only if $\vartheta(n) \approx \omega(n)$, where $\vartheta(n)$ and $\omega(n)$ are as defined in (2.33), except that here $M = \infty$.

Consequently, whenever the aforementioned asymptotic condition takes place, there exist two constants $0 < C_1 < C_2$ (possibly depending on the sequence $F_n$, but independent of $n$) such that for all $n \geq 1$,

$$C_1 \text{Var}^2 (\Gamma_1(F_n) - 2F_n) \leq \text{Var} (\Gamma_2(F_n) - 2\Gamma_2(F_n)) \leq C_2 \text{Var}^2 (\Gamma_1(F_n) - 2F_n).$$

Proof. First note that since $M_n \uparrow \infty$, we have $M_n > \nu$ for large enough values of $n$. So without loss of generality, we assume $M_n = \infty$ for all $n \geq 1$. Using Proposition 2.3.10 we deduce that as $n \to \infty$, $c_{1,n}, \ldots, c_{\nu,n} \to 1$ and $c_{i,n} \to 0$ for all $i \geq \nu + 1$. Then relation (2.37) yields that

$$\text{Var} (\Gamma_1(F_n) - 2F_n) \approx \omega(n)$$

if and only if $\vartheta(n) \approx \omega(n)$. Note that there are infinitely many coefficients tending to zero. We claim that

$$\sum_{i=\nu+1}^{\infty} c_{i,n}^6 = o(\omega(n)^2).$$

To this end, define

$$x_{m,r}(n) := \sum_{i=\nu+1}^{\nu+1+m} c_{i,n}^r.$$

Then for each $m \in \mathbb{N}$, the estimate $x_{m,2}(n) \leq \sum_{i=\nu+1}^{\nu+1} c_{i,n}^2 = \vartheta(n) \leq 2\nu \omega(n)$ holds. So the above analysis, together with the fact that the sum in $x_{m,r}(n)$ is finite for $m \geq 1$, tells us that

$$x_{m,6}(n) = o(\omega(n)^2), \quad \forall m \geq 1.$$

Now, taking into account that $x_{m,6} \to x_{\infty,6}(n) := \sum_{i=\nu+1}^{\infty} c_{i,n}^6$, as $m \to \infty$, and each $x_{m,6}(n) = o(\omega(n)^2)$, a direct application of the monotone convergence theorem implies

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that $x_{\infty,6} = o(\omega(n)^2)$. Therefore,

$$\text{Var} \left( \Gamma_3(F_n) - 2\Gamma_2(F_n) \right) = 2^7 \sum_{i=1}^{\infty} c_{i,n}^6 (1 - c_{i,n})^2$$

$$= 2^7 \left\{ \sum_{i=1}^\nu c_{i,n}^6 (1 - c_{i,n})^2 + \sum_{i=\nu+1}^{\infty} c_{i,n}^6 (1 - c_{i,n})^2 \right\}$$

$$\approx \max \{|1 - c_{i,n}|^2 : i = 1, \ldots, \nu \} \approx \omega(n)^2.$$

Hence the claim follows. \qed

### 2.3.3 An Optimal Theorem

Now we are ready to state our main theorem providing an optimal rate of convergence in terms of the third and the fourth cumulants. The following result provides an analogous counterpart to the same phenomenon in the case of normal approximation on arbitrary Wiener chaos, see [85, Theorem 1.2].

**Theorem 2.3.18.** Let $\nu > 0$. Assume that $$(F_n)_{n \geq 1} = \left( \sum_{i \geq 1} c_{i,n}(N_i^2 - 1) \right)_{n \geq 1}$$
is a sequence of elements in the second Wiener chaos such that $|c_{1,n}| \geq |c_{2,n}| \geq \ldots$ (see Assumption 2.3.5) and $E(F_n^2) = 2\sum_{i \geq 1} c_{i,n}^2 = 2\nu$ for all $n \geq 1$. In addition, assume that as $n \to \infty$,

$$\text{Var} \left( \Gamma_1(F_n) - 2F_n \right) \to 0. \tag{2.40}$$

Then $F_n$ converges in distribution towards a centered Gamma distribution $G(\nu)$ with parameter $\nu$. Furthermore, when $\vartheta(n) \approx \omega(n)$, where $\vartheta(n)$ and $\omega(n)$ are as in (2.33), then there exist two constants $0 < C_1 < C_2$ (possibly depending on the sequence $F_n$, but independent of $n$) such that for all $n \geq 1$,

$$C_1 \mathcal{M}(F_n) \leq d_2(F_n, G(\nu)) \leq C_2 \mathcal{M}(F_n), \tag{2.41}$$

where as before

$$\mathcal{M}(F_n) := \max \{|\kappa_3(F_n) - \kappa_3(G(\nu))|, |\kappa_4(F_n) - \kappa_4(G(\nu))|\}.$$

**Proof.** The asymptotic relation (2.40) implies that $F_n$ converges in distribution towards a centered Gamma distribution $G(\nu)$, which is a well known fact, see for example [81].

(Upper bound): This is a direct application of Theorem 2.2.2, Corollary 2.3.15, and Proposition 2.3.17.

(Lower bound): Fix a real number $\rho > 0$ whose range of values will be determined
follow the integral representations.

In this case \( \varphi_{F_n} \) is regular in the strip \( \{ z \in \mathbb{C} : |\text{Im} z| < \alpha \} \). By the hypercontractivity property, we have that \( F_n \) has moments of all order and that (see [84, (2.7.2)])

\[
E[|F_n|^k]^{1/k} \leq (k - 1) \times E[F_n^2]^{1/2} = \sqrt{2\nu}(k - 1).
\]

Now using the Stirling formula, we obtain

\[
\limsup_{k \to \infty} \left( \frac{|E[F_n^k]|}{k!} \right)^{1/k} \leq \limsup_{k \to \infty} \frac{\sqrt{2\nu}(k - 1)}{k^{1/k}} = \sqrt{2\nu} e.
\]

The same applies to \( \varphi_{G(\nu)} \). Hence the characteristic functions \( \varphi_{F_n} \) and \( \varphi_{G(\nu)} \) are analytic inside the strip \( \Delta_\nu := \{ z \in \mathbb{C} : |\text{Im} z| < \frac{1}{\sqrt{2\nu} e} \} \), and in the strip of regularity \( \Delta_\nu \) they follow the integral representations.

\[
\varphi_{F_n}(z) = \int_{\mathbb{R}} e^{ixz} \mu_n(dx) \quad \text{and} \quad \varphi_{G(\nu)}(z) = \int_{\mathbb{R}} e^{ixz} \mu_\nu(dx),
\]

where \( \mu_n \) and \( \mu_\nu \) stand for the probability measures of \( F_n \) and \( G(\nu) \) respectively. Recall that all elements in the second Wiener chaos have exponential moments, see [84, Proposition 2.7.13, item (iii)]. Denote by \( \Omega_{\rho,\nu} \) the domain

\[
\Omega_{\rho,\nu} := \left\{ z = t + iy \in \mathbb{C} : |\text{Re} z| < \rho, |\text{Im} z| < \min\{(\sqrt{2\nu}e)^{-1}, e^{-1}\} \right\}.
\]

Then for any \( z \in \Omega_{\rho,\nu} \), together with a Fubini’s argument, we have that

\[
|\varphi_{F_n}(z) - \varphi_{G(\nu)}(z)| = \left| \int_{\mathbb{R}} e^{itx-xy} (\mu_n - \mu_\nu)(dx) \right| \leq \sum_{k \geq 0} \frac{(-y)^k}{k!} \int_{\mathbb{R}} x^k e^{ixt}(\mu_n - \mu_\nu)(dx) \leq \sum_{k \geq 0} \frac{e^{-k}}{k!} \varphi_F^{(k)}(t) - \varphi_G^{(k)}(t) \leq \sum_{k \geq 0} \frac{e^{-k}}{k!} \rho^{k+1} d_2(F_n, G(\nu)) = \rho e^{\rho e^{-1}} d_2(F_n, G(\nu)).
\]

Hence \( |\varphi_{F_n}(z) - \varphi_{G(\nu)}(z)| \leq C_{\rho,\nu} d_2(F_n, G(\nu)) \) for every \( z \in \Omega_{\rho,\nu} \). Let \( R > 0 \) such that the disk \( D_R \subset \mathbb{C} \) with the origin as center and radius \( R \) is contained in the domain \( \Omega_{\rho,\nu} \) (note that \( R \) depends only on \( \nu \), since \( \rho \) is a free parameter. For example, one can choose \( \min\{(\sqrt{2\nu}e)^{-1}, e^{-1}\} < \rho < 2 \min\{(\sqrt{2\nu}e)^{-1}, e^{-1}\} \)). Now for any \( z \in D_R \), and using the fact that

\[
\frac{1}{\varphi_{G(\nu)}(z)} = (e^{2iz(1 - 2iz)})^\nu,
\]

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one can readily conclude that the function \( \varphi_{G(\nu)}(z) \) is bounded away from 0 on the disk \( D_R \). Also, for any \( r \geq 2 \),

\[
|\kappa_r(F_n)| \leq 2^{r-1}(r-1)!\sum_{i \geq 1} c_{i,n}^r \leq 2^{r-1}(r-1)!\max c_{i,n}^{r-2}\sum_{i \geq 1} c_{i,n}^2
\]  

\[
\leq 2^{r-2}(r-1)!\sqrt{\nu}r^2 \mathbb{E}(F_n^2) = 2^{r-2}(r-1)!\sqrt{\nu}r. 
\]

Therefore, for any \( z \in D_R \),

\[
\left| 1 - \frac{\varphi_{F_n}(z)}{\varphi_{G(\nu)}(z)} \right| \leq \exp \left\{ \sum_{r \geq 2} \frac{|\kappa_r(F_n)|}{r!}|z|^r \right\} \leq \exp \left\{ \sum_{r \geq 2} \frac{2^{r-2}(r-1)!\sqrt{\nu}r^2}{r!}|z|^r \right\} 
\]

\[
\leq \exp \left\{ \sum_{r \geq 2} \frac{2^{r-2}(r-1)!\sqrt{\nu}r^2}{r!}R^r \right\} := C_{R,\nu} < \infty. 
\]

Hence the function \( \varphi_{F_n}(z) \) is also bounded away from 0 on the disk \( D_R \). Also, relation (2.42) implies that the following power series (complex variable) converge to some analytic function as soon as \( |z| < R \);

\[
\sum_{r \geq 1} \frac{\kappa_r(F_n)}{r!}(iz)^r, \quad \sum_{r \geq 1} \frac{\kappa_r(G(\nu))}{r!}(iz)^r. 
\]

Thus we come to the conclusion that the functions \( \varphi_{G(\nu)}(z) \) and \( \varphi_{F_n}(z) \) are analytic on the disk \( D_R \). Moreover, there exists a constant \( c > 0 \) such that \( \left| \varphi_{G(\nu)}(z) \right|, \left| \varphi_{F_n}(z) \right| \geq c > 0 \) for every \( z \in D_R \). This implies that on the disk \( D_R \) there exist two analytic functions \( g_n \) and \( g_\nu \) such that

\[
\varphi_{F_n}(z) = e^{g_n(z)}, \quad \varphi_{G(\nu)}(z) = e^{g_\nu(z)},
\]

i.e. \( g_n(z) = \log(\varphi_{F_n}(z)) \) and \( g_\nu(z) = \log(\varphi_{G(\nu)}(z)) \), for \( z \in D_R \). In fact, the functions \( g_n \) and \( g_\nu \) are given by the power series (2.43). Since the derivative of the analytic branch of the complex logarithm is \( (\log z)' = \frac{1}{z} \) (see [21 Corollary 2.21]), one can infer that for some constant \( C \), whose value may differ from line to line, and for every \( z \in D_R \), we have

\[
\left| \sum_{r \geq 2} \frac{\kappa_r(F_n) - \kappa_r(G(\nu))}{r!}(iz)^r \right| = \left| \log(\varphi_{F_n}(z)) - \log(\varphi_{G(\nu)}(z)) \right| 
\]

\[
\leq C \left| \varphi_{F_n}(z) - \varphi_{G(\nu)}(z) \right| \leq C d_2(F_n, G(\nu)). 
\]

Now, using Cauchy’s estimate for the coefficients of analytic functions, for any \( r \geq 3 \), we obtain that

\[
|\kappa_r(F_n) - \kappa_r(G(\nu))| \leq r! R^r \sup_{|z| \leq R} \log \varphi_{F_n}(z) - \log \varphi_{G(\nu)}(z). 
\]

Therefore,

\[
\max \left\{ |\kappa_3(F_n) - \kappa_3(G(\nu))|, |\kappa_4(F_n) - \kappa_4(G(\nu))| \right\} \leq C d_2(F_n, G(\nu)).
\]

\( \square \)
To demonstrate the power of Theorem 2.3.18, we consider a second order U-statistic with degeneracy order 1. The following example is taken from [6, Section 3.1].

**Example 2.3.19.** Let \( \{h_i\}_{i \geq 1} \) be an orthonormal basis of \( \mathcal{H} \) and for \( i \geq 1 \) set \( Z_i := I_1(h_i) \). For \( a \neq 0 \) consider

\[
U_n = \frac{2a}{n(n-1)} \sum_{1 \leq i < j \leq n} Z_i Z_j = I_2 \left( \frac{2a}{n(n-1)} \sum_{1 \leq i < j \leq n} h_i \otimes h_j \right).
\]

Then \( nU_n \overset{D}{\rightarrow} a(Z_1^2 - 1) \) as \( n \to \infty \). Since the target is only distributed according to a centered Gamma distribution if \( a = 1 \), we will restrict ourselves to this case and write \( G(1) \) for the target. Furthermore, in our setting, we need to fix the variance of our sequence to 2. Hence we consider

\[
W_n := \sqrt{\frac{n-1}{n}} nU_n = I_2 \left( \frac{2}{\sqrt{n(n-1)}} \sum_{1 \leq i < j \leq n} h_i \otimes h_j \right) = I_2(f_n)
\]

We consider the associated Hilbert-Schmidt operator \( A_{f_n} g = f_n \otimes_1 g \). Using the fact that \((h_i \otimes h_j) \otimes_1 h_k = \langle h_i, h_k \rangle h_j\) we can explicitly compute the non-zero eigenvalues \( c_{1,n}, \ldots, c_{n,n} \) of \( A_{f_n} \). They are

\[
c_{1,n} = \sqrt{\frac{n-1}{n}}, \quad \text{and} \quad c_{2,n} = \ldots = c_{n,n} = -\frac{1}{\sqrt{n(n-1)}}.
\]

Since our target has 1 degree of freedom, the assumptions of Theorem 2.3.18 are in order (see Proposition 2.3.12(a)) and thus the optimality result (2.41) holds for \( W_n \). Also, with the eigenvalues given above and Lemma 2.3.3, one may verify manually that \( \text{Var}(\Gamma_3(W_n) - 2\Gamma_2(W_n)) \approx \text{Var}^2(\Gamma_1(W_n) - 2W_n) \approx \frac{1}{n^2} \). As a consequence

\[
d_2(W_n, G(1)) \approx \left| \kappa_3(W_n) - \kappa_3(G(1)) \right| \approx \left| \kappa_4(W_n) - \kappa_4(G(1)) \right| \approx \frac{1}{n}.
\]

### 2.3.4 Trace Class Operators

**Lemma 2.3.20.** Let \( F = I_2(f) \) be a random element in the second Wiener chaos such that \( A_f^4 - A_f^3 \geq 0 \) (or \( \leq 0 \)) is a non-negative (or non-positive) operator, where \( A_f \) is the associated Hilbert-Schmidt operator. Then

\[
\text{Var} \left( \Gamma_3(F) - 2\Gamma_2(F) \right) \leq 2 \times 3!^2 \left( \kappa_4(F) - 6\kappa_3(F) \right)^2.
\]
Proof. Using relation (2.29) and (2.17), as well as the main result of [69], one can write

\[
\text{Var} \left( \Gamma_3(F) - 2\Gamma_2(F) \right) = \frac{1}{7!} \kappa_6(F) - \frac{4}{6!} \kappa_7(F) + \frac{4}{5!} \kappa_6(F)
\]

\[
= 2^7 \text{Tr}(A_f^7) - 2^8 \text{Tr}(A_f^7) + 2^7 \text{Tr}(A_f^6) = 2^7 \text{Tr}(A_f^8 - 2A_f^7 + A_f^6)
\]

\[
= 2^7 \text{Tr} \left( (A_f^4 - A_f^3)^2 \right) \leq 2^7 \left( \text{Tr}(A_f^7 - A_f^6) \right)^2 = 2 \times 3^2 \left( \kappa_4(F) - 6\kappa_3(F) \right)^2.
\]

Now, we can state the following non asymptotic version of the optimal rate of convergence towards the centered Gamma distribution \(G(\nu).\)

**Proposition 2.3.21.** Let \(\nu > 0.\) Assume that \(F = I_2(f)\) is a random element in the second Wiener chaos such that \(E(F^2) = 2\nu.\) Moreover, assume that \(A_f^4 - A_f^3 \geq 0\) (or \(\leq 0\)), where \(A_f\) is the Hilbert-Schmidt operator associated with \(F.\) Then there exist two constants \(0 < C_1 < C_2,\) such that

\[
C_1 M(F) \leq d_2(F, G(\nu)) \leq C_2 M(F), \quad (2.44)
\]

where, as before, \(M(F) := \max \{|\kappa_3(F) - \kappa_3(G(\nu))|, |\kappa_4(F) - \kappa_4(G(\nu))|\}.\)

**Proof.** For the upper bound, combine Theorem 2.2.2 together with Lemma 2.3.8 and Lemma 2.3.20. The lower bound is derived from Theorem 2.3.13.

We close this section with two lemmas of independent interests. The first lemma gathers some non-asymptotic variance-estimates. The second lemma displays that differences of all higher cumulants can be controlled from above by the quantity \(M(F).\)

**Lemma 2.3.22.** Let \(F = I_2(f)\) be a general element in the second Wiener chaos. Then, for \(r \geq 1,\) the following estimates hold.

\[
\text{Var}^2 \left( \Gamma_{r+1}(F) - 2\Gamma_r(F) \right) \leq C \text{Var} \left( \Gamma_r(F) - 2\Gamma_{r-1}(F) \right) \times \text{Var} \left( \Gamma_{r+2}(F) - 2\Gamma_{r+1}(F) \right), \quad (2.45)
\]

\[
\text{Var}^{2r} \left( \Gamma_2(F) - 2\Gamma_1(F) \right) \leq C \text{Var}^{2r-1} \left( \Gamma_1(F) - 2F \right) \times \text{Var} \left( \Gamma_{2r+1}(F) - 2\Gamma_{2r}(F) \right), \quad (2.46)
\]

where the general constant \(C\) is independent of \(F.\) In particular,

\[
\text{Var}^2 \left( \Gamma_2(F) - 2\Gamma_1(F) \right) \leq C \text{Var} \left( \Gamma_1(F) - 2F \right) \times \text{Var} \left( \Gamma_3(F) - 2\Gamma_2(F) \right).
\]

Moreover,

\[
\text{Var} \left( \Gamma_{2r+1}(F) - 2\Gamma_{2r}(F) \right) \leq C \text{Var}^2 \left( \Gamma_r(F) - 2\Gamma_{r-1}(F) \right)
\]

\[
\leq C \text{Var} \left( \Gamma_{r-1}(F) - 2\Gamma_{r-2}(F) \right) \times \text{Var} \left( \Gamma_{r+1}(F) - 2\Gamma_r(F) \right). \quad (2.47)
\]

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Proof. This is a direct application of \[31\] Corollary 1 with $P = (A_f^{r+1} - A_f^r)^2$, $C = A_f^2$, and the fact that, for $r \geq 0$, we have

$$\text{Var} \left( \Gamma_{r+1}(F) - 2 \Gamma_r(F) \right) = 2^{2r+3} \text{Tr} \left( (A_f^{r+2} - A_f^{r+1})^2 \right).$$

The estimate (2.46) is also an application of \[32\] Corollary 1 with $P = (A_f^2 - A_f)^2$, and the convex function $f(x) = x^{2r}$. \[\square\]

**Lemma 2.3.23.** Let $\nu > 0$, and $F = I_2(f)$ in the second Wiener chaos so that $\mathbb{E}[F^2] = 2\nu$. Then, for every $r \geq 1$, there exists a constant $C$ (depending only on $\nu$, and $r$) such that

$$\left| \mathbb{E}[\Gamma_{r+1}(F)] - 2 \mathbb{E}[\Gamma_r(F)] \right| \leq C \times M(F), \quad (2.48)$$

and also,

$$\left| \kappa_r(F) - \kappa_r(G(\nu)) \right| \leq C \times M(F). \quad (2.49)$$

**Proof.** We proof estimate (2.48) by induction on $r$. Obviously (2.48) holds for $r = 1, 2$, so we assume that $r \geq 3$. Note that, for some constant $C > 0$ whose value may change from line to line,

$$\left| \mathbb{E}[\Gamma_{r+2}(F)] - 2 \mathbb{E}[\Gamma_{r+1}(F)] \right| = \left| \frac{\kappa_{r+3}(F)}{(r+2)!} - 2 \frac{\kappa_{r+2}(F)}{(r+1)!} \right| \leq C \left( \frac{\kappa_{r+3}(F)}{(r+2)!} - 4 \frac{\kappa_{r+2}(F)}{(r+1)!} + 4 \frac{\kappa_{r+1}(F)}{r!} \right) + \left| \frac{\kappa_{r+2}(F)}{(r+1)!} - 2 \frac{\kappa_{r+1}(F)}{r!} \right|$$

$$= C \left( \frac{\kappa_{r+3}(F)}{(r+2)!} - 4 \frac{\kappa_{r+2}(F)}{(r+1)!} + 4 \frac{\kappa_{r+1}(F)}{r!} \right) + \left| \mathbb{E}[\Gamma_{r+1}(F)] - 2 \mathbb{E}[\Gamma_r(F)] \right|.$$

The second summand on the right hand side is where we apply the induction hypothesis. For the first summand, we have two possibilities. If $r = 2s + 1$, for some $s \geq 1$, then by (2.22) and (2.28),

$$\frac{\kappa_{r+3}(F)}{(r+2)!} - 4 \frac{\kappa_{r+2}(F)}{(r+1)!} + 4 \frac{\kappa_{r+1}(F)}{r!} = \text{Var} \left( \Gamma_{s+1}(F) - 2 \Gamma_s(F) \right) \leq C \text{Var} \left( \Gamma_1(F) - 2F \right),$$

and so we are done. Otherwise, $r = 2s$ for some $s \geq 2$. Hence, using Cauchy-Schwarz inequality and (2.23), we obtain that

$$\left| \frac{\kappa_{r+3}(F)}{(r+2)!} - 4 \frac{\kappa_{r+2}(F)}{(r+1)!} + 4 \frac{\kappa_{r+1}(F)}{r!} \right| = \left| \frac{\kappa_{2s+3}(F)}{(2s+2)!} - 4 \frac{\kappa_{2s+2}(F)}{(2s+1)!} + 4 \frac{\kappa_{2s+1}(F)}{(2s)!} \right|$$

$$= \left| \mathbb{E} \left( (\Gamma_{s+1}(F) - 2 \Gamma_s(F)) \times (\Gamma_s(F) - 2 \Gamma_{s-1}(F)) \right) \right| \leq \sqrt{\text{Var} \left( \Gamma_{s+1}(F) - 2 \Gamma_s(F) \right)} \times \sqrt{\text{Var} \left( \Gamma_s(F) - 2 \Gamma_{s-1}(F) \right)}$$

$$\leq C \text{ Var} \left( \Gamma_1(F) - 2F \right).$$
For the estimate (2.49), note that \( \kappa_{r+1}(G(\nu)) = 2r\kappa_r(G(\nu)). \) Therefore,
\[
|\kappa_{r+1}(F) - \kappa_{r+1}(G(\nu))| \leq |\kappa_{r+1}(F) - 2r\kappa_r(F)| + |2r\kappa_r(F) - \kappa_{r+1}(G(\nu))| \\
\leq C \left[ \mathbb{E}[\Gamma_r(F)] - 2\mathbb{E}[\Gamma_{r-1}(F)] \right] + |\kappa_r(F) - \kappa_r(G(\nu))|.
\]
Now, the estimate (2.49) can be derived similarly by induction. \( \square \)

2.3.5 A Further Example: Optimal Rate in Total Variation Distance

In this section we introduce a concrete example of a sequence within the second Wiener chaos. The corresponding Hilbert-Schmidt will operator only have two non-zero eigenvalues, both of which are converging to 1. A crucial observation is that although the presented example lies out of the favorable regimes discussed in Section 2.3.2 the optimal rate \( M(F_n) \) insists to hold in total variation distance.

**Proposition 2.3.24.** Consider the sequence \( \{F_n = c_{1,n}(N_1^2 - 1) + c_{2,n}(N_2^2 - 1)\}_{n \geq 1} \) in the second Wiener chaos where \( c_{1,n} = \sqrt{1 + \frac{1}{n}} \) and \( c_{2,n} = \sqrt{1 - \frac{1}{n}}. \) Then
\[
d_{TV}(F_n, G(2)) \approx \max \left\{ |\kappa_3(F_n) - \kappa_3(G(2))|, |\kappa_4(F_n) - \kappa_4(G(2))| \right\} \approx \frac{1}{n^2}.
\]

**Proof.** First note that
\[
\kappa_4(F_n) - \kappa_4(G(2)) = 48 \sum_{j=1}^2 (c_{j,n}^4 - 1) = 48 \frac{2}{n^2} \approx \frac{1}{n^2}.
\]
Similarly \( \kappa_3(F_n) - \kappa_3(G(2)) = 8 \sum_{j=1}^2 (c_{j,n}^3 - 1) \approx \frac{1}{n^2}. \) To shorten notation, we write \( c_1 \) and \( c_2 \) instead of \( c_{1,n} \) and \( c_{2,n}. \) We start by computing \( \varphi_n, \) the probability density function of \( F_n. \) The density of \( F_n \) is given by
\[
\varphi_n(x) = \frac{1}{2\pi \sqrt{c_1 c_2}} \int_{-c_1}^{x+c_2} e^{-\frac{1}{2} \left( \frac{x^2 + c_1 + x^2 + c_2}{c_1} \right)} \sqrt{t + c_1} \sqrt{t^2 - t + c_2} \, dt \, \mathbb{1}_{\{x > -c_1 - c_2\}}(x) \\
= \frac{1}{2\pi \sqrt{c_1 c_2}} e^{-\frac{x^2}{2c_2}} \int_{-c_1}^{x+c_2} e^{\frac{x^2}{2c_2} - \frac{1}{2c_1}} \sqrt{t + c_1} \sqrt{t^2 - t + c_2} \, dt \, \mathbb{1}_{\{x > -c_1 - c_2\}}(x).
\]
Substituting \( t = (c_1 + c_2 + c)u - c_1, \) we get
\[
\varphi_n(x) = \frac{1}{2\pi \sqrt{c_1 c_2}} e^{-\frac{x^2}{2c_2}} \int_0^1 \frac{e^{\frac{(c_1 + c_2)(c_1 + c_2 + x)}{2c_1 c_2} - \frac{c_1 - c_2}{2c_2} (x + c_1 + c_2)}}{u(x + c_1 + c_2) \sqrt{(1 - u)(x + c_1 + c_2)}} \, du \, \mathbb{1}_{\{x > -c_1 - c_2\}}(x) \\
= \frac{1}{2\sqrt{c_1 c_2}} e^{-\frac{x^2}{2c_2}} \frac{1}{c_1 - c_2} \int_0^1 \frac{e^{\frac{(c_1 - c_2)(c_1 + c_2 + x)}{2c_1 c_2}}}{\sqrt{u(1 - u)}} \, u \, \frac{1}{\sqrt{1 - u}} \, du \, \mathbb{1}_{\{x > -c_1 - c_2\}}(x) \\
= \frac{1}{2\sqrt{c_1 c_2}} e^{-\frac{x^2}{2c_2}} \frac{1}{c_1 - c_2} \times 1_{F_1} \left( \frac{1}{2}, 1, \frac{c_1 - c_2}{2c_1 c_2} (c_1 + c_2 + x) \right) \times \mathbb{1}_{\{x > -c_1 - c_2\}}(x).
\]
Here, \(_1F_1\) is the confluent hypergeometric function, which can be represented as
\[
_1F_1(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zu}u^{a-1}(1-u)^{b-a-1} du
\]
for \(\text{Re}(b) > \text{Re}(a) > 0\). Note that if \(a = 1/2\) and \(b = 1\), we get
\[
_1F_1\left(\frac{1}{2}, 1, z\right) = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})^2} \int_0^1 e^{zu} \frac{1}{\sqrt{u}} \frac{1}{\sqrt{1-u}} du = \frac{1}{\pi} \int_0^1 e^{zu} \frac{1}{\sqrt{u}} \frac{1}{\sqrt{1-u}} du.
\]
Also note that the roles of \(c_1\) and \(c_2\) are completely interchangeable. It is just a matter of how we write down the convolution. Thus we can also write
\[
\varphi_n(x) = \frac{1}{2\sqrt{c_1c_2}} e^{-\frac{x + \sqrt{c_1} + \sqrt{c_2}}{2c_1}} \times _1F_1\left(\frac{1}{2}, 1, -\frac{c_1 - c_2}{2c_1c_2}(c_1 + c_2 + x)\right) \times 1_{\{x > -c_1 - c_2\}}(x).
\]
Also recall that the density of the target \(G(2)\) is given by
\[
\psi(x) = \frac{1}{2} e^{-\frac{x}{2}} 1_{\{x > -2\}}(x).
\]
The next step is to explicitly write down the total variation distance in terms of the density functions:
\[
d_{TV}(F_n, G(2)) = \frac{1}{2} \int_{-\infty}^\infty |\varphi_n(x) - \psi(x)| \, dx
\]
\[
= \frac{1}{2} \int_{-\infty}^{-c_1-c_2} \psi(x) \, dx + \frac{1}{2} \int_{-c_1-c_2}^\infty \varphi_n(x) - \psi(x) \, dx
\]
\[
= \frac{1}{2} \left(1 - e^{c_1 + c_2 - 1}\right) + \frac{1}{2} \int_{-c_1-c_2}^\infty \varphi_n(x) - \psi(x) \, dx
\]
\[
= \frac{1}{2} \left(\alpha_1(n) + \alpha_2(n)\right).
\]
One can readily check that \(\alpha_1(n) \approx \frac{1}{n^2}\). To examine the asymptotic behavior of \(\alpha_2(n)\), we write
\[
\varphi_n(x) - \psi(x) = \frac{1}{2} e^{-\frac{x}{2}} \left[\frac{1}{\sqrt{c_1c_2}} e^{-\frac{\sqrt{c_1} + \sqrt{c_2}}{2c_1}} e^x\left(\frac{1}{2} - \frac{1}{c_1}\right) _1F_1\left(\frac{1}{2}, 1, -\frac{c_1 - c_2}{2c_1c_2}(c_1 + c_2 + x)\right) - 1\right],
\]
and find a series expansion for the term inside the square brackets. Expanding \(_1F_1\) as a series (see e.g. [2] p. 504), we get
\[
_1F_1\left(\frac{1}{2}, 1, -\frac{c_1 - c_2}{2c_1c_2}(c_1 + c_2 + x)\right) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma\left\{\frac{1}{2} + k\right\}}{\Gamma(1 + k)} \frac{\left[-\frac{c_1 - c_2}{2c_1c_2}(c_1 + c_2 + x)\right]^k}{k!}.
\]
On the other hand, we can expand the exponential around $-c_1 - c_2$ as

$$e^{x(\frac{1}{2} - \frac{1}{2x_1})} = e^{-(c_1 - c_2)(\frac{1}{2} - \frac{1}{2x_1})} \sum_{k=0}^{\infty} \left(\frac{1}{2} - \frac{1}{2x_1}\right)^k \frac{(x + c_1 + c_2)^k}{k!}.$$ 

Thus, we obtain the following series expansion

$$e^{x(\frac{1}{2} - \frac{1}{2x_1})} \times _1F_1\left(\frac{1}{2}, 1, -\frac{c_1 - c_2}{2c_1 + c_2}(c_1 + c_2 + x)\right)$$

$$= \frac{e^{-(c_1 - c_2)(\frac{1}{2} - \frac{1}{2x_1})}}{\sqrt{\pi}} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{\Gamma(1 + \ell)}{\Gamma(1 + \ell + \ell)\ell!} \left[-\frac{c_1 - c_2}{2c_1 + c_2}(c_1 + c_2 + x)\right]^\ell \left(\frac{1}{2} - \frac{1}{2x_1}\right)^{k-\ell} (x + c_1 + c_2)^k$$

$$= \sum_{k=0}^{\infty} A_k(c_1, c_2)(x + c_1 + c_2)^k.$$ 

Now

$$2 \alpha_2(n) = \int_{-c_1 - c_2}^{\infty} e^{-\frac{k}{2}} \left[ \frac{1}{\sqrt{c_1 c_2}} e^{-\frac{c_1 + c_2}{2x_1}} \left( \sum_{k=0}^{\infty} A_k(c_1, c_2)(x + c_1 + c_2)^k \right) - e^{-1} \right] dx$$

$$= \left( e^{-\frac{1}{2}(c_1 + c_2)} - e^{-1} \right) \int_{-c_1 - c_2}^{\infty} e^{-\frac{k}{2}} dx$$

$$+ \sum_{k=1}^{\infty} \frac{1}{\sqrt{c_1 c_2}} A_k(c_1, c_2) \int_{-c_1 - c_2}^{\infty} e^{-\frac{k}{2}} (x + c_1 + c_2)^k dx.$$ 

Using the fact that $\int_{-c_1 - c_2}^{\infty} \exp(-\frac{k}{2})(x + c_1 + c_2)^k = k! 2^{k+1} \exp\left(\frac{c_1 + c_2}{2}\right)$ for all $k \in \mathbb{N}_0$, and setting

$$B_1(c_1, c_2) := 2 e^{\frac{c_1 + c_2}{2}} \quad \text{and}$$

$$B_2(c_1, c_2, k) := \frac{1}{\sqrt{c_1 c_2}} e^{-\frac{c_1 + c_2}{2x_1}} \times k! 2^{k+1} e^{\frac{c_1 + c_2}{2}} \times \frac{e^{-(c_1 - c_2)(\frac{1}{2} - \frac{1}{2x_1})}}{\sqrt{\pi} 2^k} = \frac{2k!}{\sqrt{c_1 c_2} \sqrt{\pi}},$$

we get

$$2 \alpha_2(n) = B_1(c_1, c_2) \left( e^{-\frac{1}{2}(c_1 + c_2)} - e^{-1} \right)$$

$$+ \sum_{k=1}^{\infty} B_2(c_1, c_2, k) \left( \sum_{\ell=0}^{k} \frac{\Gamma(\ell+1)}{\Gamma(1+\ell)\ell!} \left(-\frac{c_1 - c_2}{c_1 + c_2}\right)^\ell \left(1 - \frac{1}{c_1}\right)^{k-\ell} \right).$$
where $B_1(c_1, c_2)$ and $B_2(c_1, c_2, k)$ converge (for fixed $k$) to a positive constant as $n \to \infty$, and thus do not contribute to the rate of convergence. One can easily check that 

$$e^{-\frac{1}{2} \left(\frac{c_1 + c_2}{\sqrt{c_1 c_2}}\right)} - e^{-1} \approx \frac{1}{n^2}$$

as $n \to \infty$. All the other terms are of the form “something that converges to a constant” $\times$ “a polynomial in $c_1$ and $c_2$”. However, the terms for $k = 1$ and $k = 2$ also have the same rate of convergence, whereas the terms for $k \geq 3$ converge to zero at a faster rate. More precisely, we have

$$2 \alpha_2(n) = 2 \left(\frac{1}{\sqrt{c_1 c_2}} - e^{\frac{c_1 + c_2}{\sqrt{c_1 c_2}}} - 1\right) + \frac{1}{\sqrt{c_1 c_2 c_1 c_2}} (2c_1 c_2 - c_1 - c_2)$$

$$+ \frac{1}{4\sqrt{c_1 c_2 c_1 c_2}} \left(8c_1^2 c_2^2 - 8c_1^2 c_2 + 3c_1^2 - 8c_1 c_2^2 + 2c_1 c_2 + 3c_2^2\right)$$

$$\sum_{k=3}^{\infty} B_2(c_1, c_2, k) \left(\sum_{\ell=0}^{k} \Gamma\left(\frac{1}{2} + \ell\right) \frac{\left(-\frac{c_1 - c_2}{c_1 c_2}\right)^{\ell} \left((1 - \frac{1}{c_1})^{k-\ell}}{\ell! (k-\ell)!}\right)\right)$$

$$=: C(c_1, c_2) + \sum_{k=3}^{\infty} B_2(c_1, c_2, k) \left(\sum_{\ell=0}^{k} \Gamma\left(\frac{1}{2} + \ell\right) \frac{\left(-\frac{c_1 - c_2}{c_1 c_2}\right)^{\ell} \left((1 - \frac{1}{c_1})^{k-\ell}}{\ell! (k-\ell)!}\right)\right).$$

After some computations, we see that, as $n \to \infty$, $\frac{C(c_1, c_2)}{1/n^2} \to 1$, whereas the remaining terms converge faster.

\[\square\]

### 2.4 A Bound in Kolmogorov Distance

We would like to point out, that from now on, we drop the assumption that our sequence of random variables lies in the second Wiener chaos, as well as the requirement that $\nu$ be an integer.

In this section, we use techniques that date back to Tikhomirov from 1981 [105], who used Stein’s equation on the level of characteristic functions in order to present a result for Gamma approximation in terms of the Kolmogorov distance.

The starting point is the following classical Berry-Esseen lemma as stated in [94, p. 104]. For a more general version of the lemma, the reader is referred to Zolotarev [112].

**Lemma 2.4.1.** Let $F$ and $G$ be two cumulative distribution functions with corresponding characteristic functions $\varphi_F$, and $\varphi_G$. Then for every positive number $T > 0$, and every
\( b > 1/2\pi \), the estimate
\[
d_{\text{Kol}}(F,G) := \sup_{x \in \mathbb{R}} |F(x) - G(x)| \leq b \int_{-T}^{T} \left| \frac{\varphi_F(t) - \varphi_G(t)}{t} \right| dt + bT \sup_x \int_{|y| \leq c(b)} |G(x+y) - G(x)| dy,
\]
(2.50)
takes place, where \( c(b) \) is a constant depending only on \( b \), and it is given by the root of the following equation
\[
\int_0^{\frac{\pi}{4}} \sin^2 u \, du = \frac{\pi}{4} + \frac{1}{8b}.
\]
In particular, if \( \sup_x |G'(x)| \leq K \), then
\[
d_{\text{Kol}}(F,G) := \sup_{x \in \mathbb{R}} |F(x) - G(x)| \leq b \int_{-T}^{T} \left| \frac{\varphi_F(t) - \varphi_G(t)}{t} \right| dt + c(b) K \frac{T}{T}.
\]
(2.51)

In order to prove a Kolmogorov bound, we need an estimate on the difference of the characteristic functions and the distribution functions. The first is done in the following Lemma:

**Lemma 2.4.2.** Let \( \nu > 0 \) and let \( F \) be a random variable admitting a finite chaos expansion with variance \( \mathbb{E}[F^2] = 2\nu \). Let \( G(\nu) \sim \text{CenteredGamma}(\nu) \). Define
\[
D(t) := \varphi_F(t) - \varphi_G(\nu)(t) = \mathbb{E}[e^{itF}] - \mathbb{E}[e^{itG(\nu)}], \quad t \in \mathbb{R}.
\]
Then the following estimates take place:
\[
|D(t)| \leq \frac{1}{2} |t| \mathbb{E}|2(F + \nu) - \Gamma_1(F)| \leq \frac{1}{2} |t| \sqrt{\text{Var}(\Gamma_1(F) - 2F)}.
\]
(2.52)

**Proof.** We consider the Stein operator associated to a centered Gamma random variable \( G(\nu) \) (see (2.6)):
\[
\mathcal{L}f(x) = 2(x + \nu)f'(x) - xf(x).
\]
Using the integration by parts formula (2.11), we get for all \( f \in C^1 \) with bounded derivative
\[
\mathbb{E}[\mathcal{L}f(F)] = \mathbb{E}\left[f'(F)\{2(F + \nu) - \Gamma_1(F)\}\right].
\]
(2.53)
Also, for all \( C^1 \) functions \( f : \mathbb{R} \to \mathbb{R} \), such that the expectation exists (e.g. if \( f \) is polynomially bounded), we have
\[
\mathbb{E}[\mathcal{L}f(G(\nu))] = 0.
\]
(2.54)

By considering real and imaginary part separately and using linearity, we can extend (2.53) and (2.54) to complex valued functions \( f : \mathbb{R} \to \mathbb{C} \). Thus letting \( f(x) = e^{itx} \) for \( t \in \mathbb{R} \), we obtain
\[
\mathbb{E}[\mathcal{L}f(F)] = i t \mathbb{E}\left[e^{itF}\{2(F + \nu) - \Gamma_1(F)\}\right].
\]
Therefore
\[ it \mathbb{E}[e^{itF} \{2(F + \nu) - \Gamma_1(F)\}] = \mathbb{E}[\mathbb{E}f(F)] = \mathbb{E}[\mathbb{E}f(F)] - 0 = \mathbb{E}[\mathbb{E}f(F)] - \mathbb{E}[\mathbb{E}f(G(\nu))] \]

\[ = it \times 2\nu \left( \mathbb{E}[e^{itF}] - \mathbb{E}[e^{itG(\nu)}] \right) - (1 - 2it) \left( \mathbb{E}[Fe^{itF}] - \mathbb{E}[G(\nu)e^{itG(\nu)}] \right) \]

\[ = it \times 2\nu D(t) + (2t + i)D'(t). \]

So \( D \) satisfies the differential equation
\[ (1 - 2ti)D'(t) + 2\nu t D(t) = e(t), \quad \text{where } e(t) := t \mathbb{E}[e^{itF} \{2(F + \nu) - \Gamma_1(F)\}]. \] (2.55)

Using the fact that \( D(-t) = \overline{D(t)} \) and \(|D(t)| = |\overline{D(t)}|\), we focus only on \( t \geq 0 \). The solution of the ordinary differential equation (2.55) with initial condition \( D(0) = 0 \) is given by
\[ D(t) = e^{-a(t)} \int_0^t \frac{e(s)}{1 - 2si} e^{a(s)} \, ds, \]
where
\[ a(t) = \int \frac{2\nu t}{1 - 2ti} dt = \frac{\nu}{4} \log(4t^2 + 1) + it \left( t\nu - \frac{\nu}{2} \arctan(2t) \right). \]

Note that
\[ |e^{a(t)}| = (4t^2 + 1)^{\frac{\nu}{4}} \quad \text{and} \quad |e^{-a(t)}| = (4t^2 + 1)^{-\frac{\nu}{4}}. \]

Thus we can estimate
\[ |D(t)| \leq |e^{-a(t)}| \int_0^t \frac{1}{1 - 2si} |e(s)| |e^{a(s)}| \, ds \]
\[ \leq (4t^2 + 1)^{-\frac{\nu}{4}} \int_0^t \frac{1}{\sqrt{4s^2 + 1}} |e(s)| \, ds \]
\[ \leq \mathbb{E}\left[ 2(F + \nu) - \Gamma_1(F) \right] (4t^2 + 1)^{-\frac{\nu}{4}} \int_0^t s (4s^2 + 1)^{\frac{\nu}{4} - \frac{1}{2}} \, ds \]
\[ = \mathbb{E}\left[ 2(F + \nu) - \Gamma_1(F) \right] (4t^2 + 1)^{-\frac{\nu}{4}} \left( \frac{1}{2(\nu + 2)} \left( (4t^2 + 1)^{\frac{\nu}{4} + \frac{1}{2}} - 1 \right) \right) \]
\[ = \mathbb{E}\left[ 2(F + \nu) - \Gamma_1(F) \right] \frac{1}{2(\nu + 2)} \left( \sqrt{4t^2 + 1} - (4t^2 + 1)^{-\nu/4} \right) \]
\[ \leq \frac{1}{2} t \mathbb{E}\left[ 2(F + \nu) - \Gamma_1(F) \right]. \]

The last estimate is due to Lemma 2.4.3 below. The second inequality in (2.52) is just Cauchy Schwarz.

**Lemma 2.4.3.** For any \( \nu > 0 \) and \( t \geq 0 \), we have that
\[ \sqrt{4t^2 + 1} - (4t^2 + 1)^{-\nu/4} \leq (2 + \nu) \times t. \]
Proof. We make use of the following well-known inequalities:

\[ \sqrt{x+y} \leq \sqrt{x} + \sqrt{y}, \text{ for all } x, y \geq 0; \]  
\[ 1 - e^{-x} \leq x, \text{ for all } x \geq -1; \]  
\[ \log(x) \leq 2(\sqrt{x} - 1), \text{ for all } x > 0. \]  

With this we get

\[ \sqrt{4t^2+1} - (4t^2+1)^{-\nu/4} \leq 2t + 1 - e^{-\frac{\nu}{4} \log(4t^2+1)} \]  
\[ \leq 2t + \frac{\nu}{4} \log(4t^2 + 1) \]  
\[ \leq 2t + \frac{\nu}{2} (\sqrt{4t^2+1} - 1) \]  
\[ \leq (2 + \nu) \times t. \]  

In order to estimate the second term in the Esseen-Lemma (2.50), we need to study the cumulative distribution function (CDF) of a centered Gamma random variable \( G(\nu) \). We show that it satisfies a Hölder condition, where the exponent is depending on \( \nu \).

Lemma 2.4.4. Let \( \nu > 0 \) and \( G(\nu) \sim \text{CenteredGamma}(\nu) \). Denote by \( G_\nu \) its distribution function. Then there exists a constant \( K > 0 \), such that for all \( a, b \in \mathbb{R} \) we have

\[ |G_\nu(a) - G_\nu(b)| \leq K |a - b|, \text{ if } \nu \geq 2, \]  
and

\[ |G_\nu(a) - G_\nu(b)| \leq K |a - b|^{\nu/2}, \text{ if } \nu \in (0, 2). \]  

Proof. Denote by \( g_\nu \) the probability density function (PDF) of \( G(\nu) \). Recall that it is given by

\[ g_\nu(x) = 2^{-\frac{\nu}{2}} \Gamma \left( \frac{\nu}{2} \right)^{-1} (x + \nu)^{\frac{\nu}{2} - 1} e^{-\frac{x}{2} - \frac{\nu}{2}} \mathbb{1}_{\{x > -\nu\}}(x), \]

where we used the convention introduced in equation (2.1) to write the density function with indicators. If \( \nu > 2 \), then \( g_\nu \) is continuous and hence \( G_\nu \) is differentiable on the whole real line. One can readily verify that \( g_\nu \) is bounded with \( K := \sup_{x \in \mathbb{R}} |g_\nu(x)| = g_\nu(-2) \). So (2.59) is just an application of the mean value theorem.

When \( \nu = 2 \), then

\[ g_\nu(x) = \frac{1}{2} e^{-\frac{x}{2} - 1} \mathbb{1}_{\{x > -2\}}(x). \]

In this case \( G_\nu \) is not differentiable in \( x = -2 \). However, because of the monotonicity, \( g_\nu \) is bounded by \( K := \sup_{x \in \mathbb{R}} |g_\nu(x)| = \lim_{x \to -2} g_\nu(x) = 1/2 \). Therefore (2.59) holds for all \( a, b \in (-\infty, -2) \) and all \( a, b \in (-2, \infty) \). Using the continuity of \( G_\nu \), we can easily
show that (2.59) extends to the whole real line.

When $0 < \nu < 2$, then $g_\nu$ is not bounded (see Figure 2.1), in fact $\lim_{x \to -\nu} g_\nu(x) = \infty$. Without loss of generality, assume that $b > a$. We split the proof into three cases:

**Case 1 ($a < b \leq -\nu$):** Here (2.60) holds, since $G_\nu(a) = G_\nu(b) = 0$.

**Case 2 ($-\nu < a < b$):** Define $C := 2^{-\nu/2} \Gamma(\nu/2)$. Then we have

$$g_\nu(x) \leq C \times (x + \nu)^{\frac{\nu}{2} - 1} \quad \forall x > -\nu.$$ 

We compute (note that $G_\nu$ is increasing):

$$G_\nu(b) - G_\nu(a) = \int_a^b g_\nu(t) \, dt \leq C \int_a^b (t + \nu)^{\frac{\nu}{2} - 1} \, dt = \frac{2C}{\nu} [(b + \nu)^{\frac{\nu}{2}} - (a + \nu)^{\frac{\nu}{2}}] \leq \frac{2C}{\nu} (b - a)^{\frac{\nu}{2}}.$$ 

Here, the last inequality follows from the well-known inequality $x^p + y^p \geq (x + y)^p$ for all $x, y \geq 0$ and $p \in (0, 1)$ by letting $x = a + \nu, y = b - a$ and $p = \frac{\nu}{2}$.

**Case 3 ($a \leq -\nu < b$):** Using the continuity of $G_\nu$, we get:

$$G_\nu(b) - G_\nu(a) = G_\nu(b) - G_\nu(-\nu) + \lim_{\varepsilon \searrow 0} G_\nu(b) - G_\nu(-\nu + \varepsilon)$$

$$\leq \lim_{\varepsilon \searrow 0} \frac{2C}{\nu} (b + \nu - \varepsilon)^{\frac{\nu}{2}} = \frac{2C}{\nu} (b + \nu)^{\frac{\nu}{2}} \leq \frac{2C}{\nu} (b - a)^{\frac{\nu}{2}}.$$ 

**Remark 2.4.5.** Now let $a = -1$ and $b \in (-1, 0)$. With similar arguments as above, this time using the upper bound $g_1(t) \geq \frac{e^{-1/2}}{\sqrt{2\pi}} \times \frac{1}{\sqrt{t+1}}$, we can show that

$$G_1(b) - G_1(-1) \geq \frac{\sqrt{2} e^{-1/2}}{\sqrt{\pi}} \times \sqrt{b - (-1)}.$$ 

Thus in a vicinity of $-1$, estimate (2.60) is actually the best we can do when $\nu = 1$.

Now we have all the ingredients to show the following theorem.

**Theorem 2.4.6.** Let $\nu > 0$, and let $F$ be a centered random variable admitting a finite chaos expansion, such that $\mathbb{E}[F^2] = 2\nu$. Let $G(\nu) \sim \text{CenteredGamma}(\nu)$. Then

$$d_{\text{Kol}}(F, G(\nu)) \leq \begin{cases} C \times \left( \Gamma_1(F) - 2F \right)^{\frac{1}{4}}, & \text{if } \nu \geq 2 \\ C \times \left( \Gamma_1(F) - 2F \right)^{\frac{1}{2\nu+2}}, & \text{if } \nu \in (0, 2), \end{cases}$$

where $C > 0$ is a constant only depending on $\nu$. 

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Proof. If $\nu \geq 2$ then putting the bounds from Lemma 2.4.2 and Lemma 2.4.4 into the Berry-Esseen lemma (2.50), we get for every $T > 0$

\[
d_{\text{Kol}}(F, G(\nu)) \leq b \int_{-T}^{T} \frac{\varphi_F(t) - \varphi_{G(\nu)}(t)}{t} dt + bT \sup_{x \in \mathbb{R}} \int_{|y| \leq c(b)} |G_{\nu}(x + y) - G_{\nu}(x)| dy
\]

\[
\leq bT \sqrt{\text{Var} \left( \Gamma_1(F) - 2F \right)} + bK \frac{c(b)^2}{T}
\]

\[
=: c_1 T \sqrt{\text{Var} \left( \Gamma_1(F) - 2F \right)} + c_2.
\]

The minimum is achieved at

\[
T_{\text{min}} = \left( \frac{c_2}{c_1} \right)^{1/2} \text{Var} \left( \Gamma_1(F) - 2F \right)^{-1/4},
\]

and is given by

\[
2 \sqrt{c_1 c_2} \text{Var} \left( \Gamma_1(F) - 2F \right)^{1/4}.
\]

If $0 < \nu < 2$, then instead we get

\[
d_{\text{Kol}}(F, G(\nu)) \leq b \int_{-T}^{T} \frac{\varphi_F(t) - \varphi_{G(\nu)}(t)}{t} dt + bT \sup_{x \in \mathbb{R}} \int_{|y| \leq c(b)} |G_{\nu}(x + y) - G_{\nu}(x)| dy
\]

\[
\leq bT \sqrt{\text{Var} \left( \Gamma_1(F) - 2F \right)} + \frac{4b}{\nu + 2} c(b)^{2+\nu} K T^{-\frac{\nu}{2}}
\]

\[
=: \hat{c}_1 T \sqrt{\text{Var} \left( \Gamma_1(F) - 2F \right)} + \hat{c}_2 T^{-\frac{\nu}{2}}.
\]

Again, minimizing over $T > 0$ yields

\[
T_{\text{min}} = \left( \frac{4 \hat{c}_1^2}{\hat{c}_2^2 \nu^2} \text{Var}(\Gamma_1(F) - 2F) \right)^{\frac{1}{\nu + 2}}
\]

and thus the minimum is

\[
(\nu + 2) \nu \frac{\nu}{\nu + 2} \hat{c}_1^{\frac{\nu}{\nu + 2}} \hat{c}_2^{\frac{\nu}{\nu + 2}} \text{Var}(\Gamma_1(F) - 2F) \frac{\sqrt{\nu + 1}}{\nu + 2}.
\]

\[\square\]

Remark 2.4.7. The theorem for $\nu \geq 2$ is also a consequence of previous results. In fact, using Theorem 1.7 from [24], as well as the fact that

\[
d_{\text{Kol}}(F, G) \leq C \sqrt{d_1(F, G)},
\]

whenever the density of $G$ is bounded, we immediately retrieve the case $\nu \geq 2$. Here, we presented an original proof. To our best knowledge, when $\nu < 2$, our result is new, as in this case the corresponding density $g_{\nu}$ is not bounded.
2.5 Outlook

In this section, we will briefly discuss some means to possibly remove the technical condition \( \theta(n) \approx \omega(n) \) in Theorem 2.3.18. From Example 2.3.7 it should be clear, that using the upper bound (2.14), we cannot do this in general. Hence, one might be needing a different upper bound. Looking at (2.47), it is clear that it would be beneficial to create a term

\[
\text{Var}^{1/2} \left( \Gamma_3(F) - 2\Gamma_2(F) \right) - 2(\Gamma_2(F) - 2\Gamma_1(F))
\]

instead of

\[
\text{Var}^{1/2} \left( \Gamma_3(F_n) - 2\Gamma_2(F_n) \right).
\]

This is the basis for the following idea.

For constants \( a, b \in \mathbb{R}^+ \cup \{\infty\} \), denote by

\[
\mathcal{B}_{a,b} := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ Lipschitz} : \|f\|_\infty < a, \|f'\|_\infty < b \right\},
\]

where \( \|f'\|_\infty \) denotes the smallest Lipschitz constant as defined in (2.3). We set \( \mathcal{B} := \mathcal{B}_{\infty,\infty} \) and equip it with the norm \( \|f\|_\mathcal{B} := \|f\|_\infty + \|f'\|_\infty \). Then \( \mathcal{B} \) is a Banach space, the so-called Lipschitz-space (see e.g. [107]).

In our proof of Theorem 2.2.2, we start with a function \( g \in \mathcal{B}_{c_1,c_2} \), where

\[
c_1 = \max \left\{ 1, \frac{2}{\nu} \right\} \quad \text{and} \quad c_2 = \max \left\{ 1, \frac{2}{\nu} \right\} + 1.
\]

We now make a crucial assumption that will allow us to remove the technical condition:

**Assumption 2.5.1.** Suppose that for every \( g \in \mathcal{B} \) there exists a unique \( h \in \mathcal{B} \) such that

\[
g = h + 2S(h),
\]

where \( S(h) \) denotes the unique bounded solution to the Stein equation

\[
h(x) - \mathbb{E}[h(G(\nu))] = 2(x + \nu)S(h)'(x) - xS(h)(x).
\]

It is not hard to see, that the map \( \mathbf{L} : \mathcal{B} \to \mathcal{B}, \ h \mapsto g + 2S(h) \) is a bounded linear operator. The linearity of \( \mathbf{L} \) follows immediately from the linearity of \( S : \mathcal{B} \to \mathcal{B} \), which in turn is a direct consequence of the Stein equation (2.62). For the boundedness, apply Theorem 2.1.1 part (a), to get

\[
\|h + 2S(h)\|_\mathcal{B} \leq \|h\|_\infty + \|h'\|_\infty + 2\|S(h)\|_\infty + 2\|S(h)'\|_\infty \leq \|h\|_\infty + \|h'\|_\infty + 2\|h'\|_\infty + 2c_1 \|h'\|_\infty \leq (3 + 2c_1) \|h\|_\mathcal{B}.
\]

The bounded inverse theorem now states that there exist universal constants \( \tilde{c}_1, \tilde{c}_2 \), only depending on \( \nu \), such that for each \( g \in \mathcal{B}_{c_1,c_2} \), the solution \( h = \mathbf{L}^{-1}(g) \in \mathcal{B}_{\tilde{c}_1,\tilde{c}_2} \).
Now applying our Lemma 2.2.3 twice to \( E[h(F)(2F - \Gamma_1(F))] \), we obtain the term

\[-E \left[ S(S(h))(F) \left( \Gamma_3(F) - 2\Gamma_2(F) \right) \right]\]

and applying it once to \( E[2S(h)(F)(2F - \Gamma_1(F))] \), we get

\[2E \left[ S(S(h))(F) \left( \Gamma_2(F) - 2\Gamma_1(F) \right) \right].\]

In summary, for some constant \( C_\nu \) only depending on \( \nu \), we get

\[
\left| E \left[ S(S(h))(F) \left( \Gamma_3(F) - 2\Gamma_2(F) - 2\left( \Gamma_2(F) - 2\Gamma_1(F) \right) \right) \right] \right| \\
\leq C_\nu E \left[ \Gamma_3(F) - 2\Gamma_2(F) - 2\left( \Gamma_2(F) - 2\Gamma_1(F) \right) \right] \\
\leq C_\nu \sqrt{\text{Var} \left( \Gamma_3(F) - 2\Gamma_2(F) - 2\left( \Gamma_2(F) - 2\Gamma_1(F) \right) \right)}.
\]

When \( F \) is in the second Wiener chaos, this term can be bounded by \( \text{Var}(\Gamma_1(F) - 2F) \), see (2.47).

### 2.6 Appendix

#### 2.6.1 The Solution to the Non-Centered Gamma Stein Equation

Since it might be useful for further computations, we present here an explicit integral representation of the solution to the centered Gamma Stein equation. The following is based on the representation of the solution of the non-centered Gamma Stein equation due to [24], which we shall first recall.

If \( X_r \sim \text{Gamma}(r, 1) \) for some \( r > 0 \), then the corresponding Stein equation is

\[ xg'(x) + (r - x)g(x) = h(x) - \mathbb{E}[h(X_r)], \quad x \in \mathbb{R}. \tag{2.63} \]

If \( h \in \text{Lip}(\mathbb{R}) \), then there exists a unique Lipschitz-continuous function \( g_h \) on \( \mathbb{R} \) solving (2.63), given by

\[ g_h(x) = \begin{cases} 
  g_h^-(x), & x \leq 0, \\
  g_h^+(x), & x \geq 0.
\end{cases} \]

Here,

\[ g_h^-(x) = \frac{1}{xq_l(x)} \int_0^x \left( h(t) - \mathbb{E}[h(X_r)] \right) q_l(t) \, dt, \quad x < 0, \]

where

\[ q_l(x) = -(-x)^{r-1}e^{-x}, \]
and
\[ g_h^+(x) = \frac{1}{xp_r(x)} \int_0^x \left( h(t) - \mathbb{E}[h(X_r)] \right) p_r(t) \, dt, \quad x > 0, \]
where \( p_r \) is the density of \( X_r \) given by
\[ p_r(x) = \begin{cases} \frac{1}{r^\nu} x^{r-1} e^{-x}, & \text{if } x > 0, \\ 0, & \text{otherwise}. \end{cases} \]

Note that we can extend \( g_h^- \) and \( g_h^+ \) continuously by setting
\[ g_h^-(0) := g_h^+(0) := \frac{h(0) - \mathbb{E}[h(X_r)]}{r}. \]

In summary, we have
\[ g_h(x) = \int_0^x \left( \frac{q_l(t)}{xq_l(x)} \mathbbm{1}_{\{x<0\}}(x) + \frac{p_r(t)}{xp_r(x)} \mathbbm{1}_{\{x>0\}}(x) \right) \left( h(t) - \mathbb{E}[h(X_r)] \right) dt. \]

### 2.6.2 The Solution to the Centered Gamma Stein Equation

For \( \nu > 0 \), set \( G(\nu) = 2X_{\nu/2} - \nu \), where \( X_r \sim \text{Gamma}(r, 1) \). The Stein equation for the centered Gamma distributed random variable \( G(\nu) \) is
\[ 2(x + \nu)f'(x) - xf(x) = h(x) - \mathbb{E}[h(G(\nu))]. \tag{2.64} \]

If we define \( h_1(x) := h(2x - \nu) \). If \( g_h \) is the solution of \( (2.63) \) \( (r = \nu/2) \), where \( h \) is replaced by \( h_1 \), then
\[ f_h(x) := \frac{1}{2} g_h \left( \frac{x + \nu}{2} \right) \]
solves \( (2.64) \). In particular, we have
\[ g_h(x) = \int_0^x \left( \frac{q_l(t)}{xq_l(x)} \mathbbm{1}_{\{x<0\}}(x) + \frac{p_r(t)}{xp_r(x)} \mathbbm{1}_{\{x>0\}}(x) \right) \left( h(2t - \nu) - \mathbb{E}[h(G(\nu))] \right) dt, \]
and
\[ f_h(x) = \frac{1}{2} \int_0^{x + \nu/2} \left( \frac{q_l(t)}{x + \nu} q_l \left( \frac{x + \nu}{2} \right) \mathbbm{1}_{\{x<0\}}(x) + \frac{p_{\nu/2}(t)}{x + \nu} p_{\nu/2} \left( \frac{x + \nu}{2} \right) \mathbbm{1}_{\{x>0\}}(x) \right) \left( h(2t - \nu) - \mathbb{E}[h(G(\nu))] \right) dt \]
\[ = \int_0^{x + \nu/2} \left( \frac{q_l(t)}{(x + \nu)q_l \left( \frac{x + \nu}{2} \right)} \mathbbm{1}_{\{x<0\}}(x) + \frac{p_{\nu/2}(t)}{(x + \nu)p_{\nu/2} \left( \frac{x + \nu}{2} \right)} \mathbbm{1}_{\{x>0\}}(x) \right) \left( h(2t - \nu) - \mathbb{E}[h(G(\nu))] \right) dt \]
\[ = \frac{1}{2} \int_{-\nu}^x \left( \frac{q_l(t)}{(x + \nu)} q_l \left( \frac{x + \nu}{2} \right) \mathbbm{1}_{\{x<0\}}(x) + \frac{p_{\nu/2}(t)}{(x + \nu)p_{\nu/2} \left( \frac{x + \nu}{2} \right)} \mathbbm{1}_{\{x>0\}}(x) \right) \left( h(t) - \mathbb{E}[h(G(\nu))] \right) dt. \]
Recall that the density $\hat{p}_\nu$ of $G(\nu)$ is

$$\hat{p}_\nu(x) = \frac{1}{2} p_{\nu/2} \left( \frac{x + \nu}{2} \right) = \begin{cases} \frac{2^{-\frac{\nu}{2}} \Gamma \left( \frac{\nu}{2} \right)^{-1} (x + \nu)^{\frac{\nu}{2} - 1} e^{-\frac{x+\nu}{2}}, & x > -\nu \\ 0, & x \leq -\nu. \end{cases}$$

Furthermore, set

$$\hat{q}(x) := \frac{1}{2} q_{\nu} \left( \frac{x + \nu}{2} \right) = -2^{-\frac{\nu}{2}} (-(x + \nu)^{\frac{\nu}{2} - 1} e^{-\frac{x+\nu}{2}}).$$

Then

$$f_h(x) = \int_{-\nu}^{x} \left( \frac{\hat{q}(t)}{2(x+\nu)\hat{q}(x)} 1_{\{x < -\nu\}}(x) + \frac{\hat{p}_\nu(t)}{2(x+\nu)\hat{p}_\nu(x)} 1_{\{x > -\nu\}}(x) \right) \left( h(t) - \mathbb{E}\left[h(G(\nu))\right] \right) dt.$$

Note that when $x = -\nu$, we have

$$f_h(-\nu) = \frac{h(-\nu) - \mathbb{E}\left[h(G(\nu))\right]}{\nu}.$$

### 2.6.3 Gamma Operators

The following lemma provides an explicit representation of the new Gamma operators used in this thesis in terms of contractions. Recall that these are not the same as e.g. in [83], but rather the new ones introduced in (2.12).

**Lemma 2.6.1.** For $q \geq 1$, let $F = I_q(f)$, for some $f \in \mathcal{H}^\otimes q$ be an element of the $q$-th Wiener chaos. Then

$$\Gamma_s(F) = \sum_{r_1=1}^{q} \ldots \sum_{r_s=1}^{\lfloor sq - 2r_1 - \ldots - 2r_{s-1} \rfloor \land q} c_q(r_1, \ldots, r_s) I_{\{r_1 < q\}} \ldots I_{\{r_1 + \ldots + r_{s-1} < \frac{sq}{2}\}} \times I_{(s+1)q - 2r_1 - \ldots - 2r_s} \left( \left( \ldots (f \otimes_{r_1} f) \otimes_{r_2} f \ldots f \right) \otimes_{r_s} f \right),$$

where the constants $c_q$ are defined as in (2.66) and (2.67).

**Proof.** Without loss of generality, we assume that $\mathcal{H} = L^2(\mathcal{A}, \mathcal{B}, \mu)$, where $(\mathcal{A}, \mathcal{B})$ is a measurable space and $\mu$ a $\sigma$-finite measure without atoms.

Note that for $s = 1$, the product $I_{\{r_1 < q\}} \ldots I_{\{r_1 + \ldots + r_{s-1} < \frac{sq}{2}\}}$ is empty, i.e. 1. In this case (2.65) reads:

$$\Gamma_1(F) = \sum_{r=1}^{q} c_q(r) I_{2q - 2r} (f \otimes_{r} f).$$
We show this using the product formula:

\[
\Gamma_1(F) = \langle DF, -DL^{-1}F \rangle_{\mathcal{H}} = \frac{1}{q} \langle DF \rangle^2_{\mathcal{H}} = q \int_{\mathcal{H}} I_{q-1}(f(\cdot, a))^2 \mu(da)
\]

\[
= q \sum_{r=0}^{q-1} (q-1)^2 \left( \int_{\mathcal{H}} f(\cdot, a) \otimes_r f(\cdot, a) \mu(da) \right)
\]

\[
= q \sum_{r=0}^{q-1} r! \left( \begin{array}{c} q-1 \\ r \end{array} \right)^2 I_{2q-2r-2}(f \otimes_{r+1} f)
\]

\[
= q \sum_{r=1}^{q} (r-1)! \left( \begin{array}{c} q-1 \\ r-1 \end{array} \right) I_{2q-2r}(f \otimes_r f).
\]

We now show the induction step \( s-1 \rightarrow s \). Note that we have

\[-DL^{-1}(F)(a) = -DL^{-1}(I_q(f))(a) = -D \left( -\frac{1}{q} I_q(f) \right)(a) = I_{q-1}(f(\cdot, a))\]

and

\[DI_p((f))(a) = \mathbf{1}_{\{p>0\}} p I_{p-1}(f(\cdot, a)).\]

Therefore

\[D\Gamma_{s-1}(F)(a) = \sum_{r_1=1}^{q} \cdots \sum_{r_{s-1}=1}^{[(s-1)q-2r_1-\cdots-2r_{s-1}]q} c_q(r_1, \ldots, r_{s-1}) \mathbf{1}_{\{r_1<q\}} \cdots \mathbf{1}_{\{r_1+\cdots+r_{s-2}<\frac{(s-1)q}{2}\}} \times \mathbf{1}_{\{r_1+\cdots+r_{s-1}<\frac{s(q-1)}{2}\}} (sq - 2r_1 - \cdots - 2r_{s-1})
\]

\[\times I_{sq-2r_1-\cdots-2r_{s-1}-1} \left( \left[ \cdots \left[ f \otimes_{r_1} f \otimes_{r_2} f \cdots f \right] \otimes_{r_{s-1}} f \right] \right)(\cdot, a),\]
and thus
\[
\Gamma_s(F) = \langle D\Gamma_{s-1}(F), -DL^{-1}F \rangle_B
\]
\[
= \sum_{r_1=1}^q \sum_{r_{s-1}=1}^{[(s-1)q - 2r_1 - \cdots - 2r_{s-2}] / q} c_q(r_1, \ldots, r_{s-1}) \mathbb{1}_{\{r_1 < q\}} \cdots \mathbb{1}_{\{r_1 + \cdots + r_{s-2} < \frac{(s-1)q}{2}\}}
\]
\[
\times \mathbb{1}_{\{r_1 + \cdots + r_{s-1} < \frac{sq}{2}\}} (sq - 2r_1 - \cdots - 2r_{s-1})
\]
\[
\times \int_{\mathbb{H}} I_{q-1}(f(\cdot, a)) I_{sq-2r_1-\cdots-2r_{s-1}-1} \left( \left( \left[ \left[ \cdots [f \otimes_{r_1} f] \otimes_{r_2} f \right] \cdots f \right] \otimes_{r_{s-1}} f \right)(\cdot, a) \right) \mu(da)
\]
\[
= \sum_{r_1=1}^q \sum_{r_{s-1}=1}^{[(s-1)q - 2r_1 - \cdots - 2r_{s-2}] / q} c_q(r_1, \ldots, r_{s-1}) \mathbb{1}_{\{r_1 < q\}} \cdots \mathbb{1}_{\{r_1 + \cdots + r_{s-2} < \frac{(s-1)q}{2}\}}
\]
\[
\times \mathbb{1}_{\{r_1 + \cdots + r_{s-1} < \frac{sq}{2}\}} (sq - 2r_1 - \cdots - 2r_{s-1})
\]
\[
\times \sum_{r_{s-1}=1}^{r_1 + \cdots + r_{s-1} < \frac{sq}{2}} (r_{s-1} - 1)! \left( sq - 2r_1 - \cdots - 2r_{s-1} - 1 \right) \left( \frac{q - 1}{r_{s-1}} \right)
\]
\[
I_{(s+1)q-2r_1-\cdots-2r_{s-1}-2r_s} \left( \int_{\mathbb{H}} \left( \left[ \left[ \cdots [f \otimes_{r_1} f] \otimes_{r_2} f \right] \cdots f \right] \otimes_{r_{s-1}} f \right)(\cdot, a) \otimes_{r_s} f(\cdot, a) \mu(da) \right)
\]
\[
= \sum_{r_1=1}^q \sum_{r_{s-1}=1}^{[(s-1)q - 2r_1 - \cdots - 2r_{s-2}] / q} c_q(r_1, \ldots, r_{s-1}) \mathbb{1}_{\{r_1 < q\}} \cdots \mathbb{1}_{\{r_1 + \cdots + r_{s-2} < \frac{(s-1)q}{2}\}}
\]
\[
\times \mathbb{1}_{\{r_1 + \cdots + r_{s-1} < \frac{sq}{2}\}} (sq - 2r_1 - \cdots - 2r_{s-1})
\]
\[
\times \sum_{r_{s-1}=1}^{r_1 + \cdots + r_{s-1} < \frac{sq}{2}} (r_{s-1} - 1)! \left( sq - 2r_1 - \cdots - 2r_{s-1} - 1 \right) \left( \frac{q - 1}{r_{s-1}} \right)
\]
\[
I_{(s+1)q-2r_1-\cdots-2r_{s-1}-2r_s} \left( \left[ \left[ \cdots [f \otimes_{r_1} f] \otimes_{r_2} f \right] \cdots f \right] \otimes_{r_{s-1}} f \right) \otimes_{r_s} f .
\]

The constants are recursively defined via
\[
c_q(r) = q \left( r - 1 \right)! \left( \frac{q - 1}{r - 1} \right)^2,
\]
(2.66)

and
\[
c_q(r_1, \ldots, r_s) = (sq - 2r_1 - \cdots - 2r_{s-1})(r_s - 1)! \left( sq - 2r_1 - \cdots - 2r_{s-1} - 1 \right) \left( \frac{q - 1}{r_s - 1} \right) c_q(r_1, \ldots, r_{s-1}).
\]
(2.67)

With this, we are able to proof Proposition 2.2.1.
Proof of Proposition 2.2.1. Part (a) is clear from the definition. Part (b) for \( j = 1 \) is also trivial. For \( j = 2 \), we use the fact that \( \Gamma_1 = \Gamma_{\text{alt},1} \), as well as the integration by parts formula (2.11), to get

\[
\mathbb{E}[\Gamma_2(F)] = \mathbb{E}\left[\langle D\Gamma_1(F), -DL^{-1}F\rangle_B\right] = \mathbb{E}[\Gamma_1(F)F] = \mathbb{E}[F\Gamma_{\text{alt},1}(F)] = \mathbb{E}\left[\langle DF, -DL^{-1}\Gamma_{\text{alt},1}(F)\rangle_B\right] = \mathbb{E}[\Gamma_{\text{alt},2}(F)].
\]

For part (c), consider

\[
\mathbb{E}[\Gamma_3(F)] = \mathbb{E}\left[\langle D\Gamma_2(F), -DL^{-1}F\rangle_B\right] = \mathbb{E}[F\Gamma_2(F)] = \mathbb{E}\left[DF\Gamma_1(F), -DL^{-1}F\rangle_B\right] = \mathbb{E}\left[DF\Gamma_1(F)\right] - \mathbb{E}\left[\Gamma_1(F)\langle DF, -DL^{-1}F\rangle_B\right] = \mathbb{E}\left[F^2\Gamma_{\text{alt},1}\right] - \mathbb{E}[\Gamma_{\text{alt},1}(F)^2] = \mathbb{E}\left[F^2\left[\Gamma_{\text{alt},1}(F)\right]\right] + \mathbb{E}\left[2F\langle DF, -DL^{-1}\Gamma_{\text{alt},1}(F)\rangle_B\right] - \mathbb{E}[\Gamma_{\text{alt},1}(F)^2] = \mathbb{E}\left[\Gamma_{\text{alt},1}(F)^2\right] - 2\mathbb{E}[\Gamma_{\text{alt},1}(F)]^2 - \mathbb{E}[\Gamma_{\text{alt},1}(F)^2] = -\text{Var}\left(\Gamma_{\text{alt},1}(F)\right).
\]

For part (d), we consider the representation of \( \Gamma_{\text{alt},s} \) given in equation (5.25) of [83]. The representation is exactly the same as for \( \Gamma_s \) (Lemma 2.6.1), except for the recursive formula of the constants \( c_q \). For \( \Gamma_{\text{alt},j} \), they are given by

\[
c_{\text{alt},q}(r) = c_q(r) = q(r - 1)! \left(\frac{q - 1}{r - 1}\right)^2
\]

and

\[
c_{\text{alt},q}(r_1, \ldots, r_s) = q(r_s - 1)! \left(\frac{sq - 2r_1 - \cdots - 2r_{s-1} - 1}{r_s - 1}\right) \left(\frac{q - 1}{r_s - 1}\right) c_q(r_1, \ldots, r_{s-1}).
\]

Comparing this with our formula (2.67), we see that only the first factor is different, namely \( q \) instead of \( (sq - 2r_1 - \ldots - 2r_{s-1}) \). But now for \( q = 2 \), the indicator \( 1_{\{r_1 + \cdots + r_{s-1} < \frac{sq}{2}\}} \) dictates that \( r_1 = \ldots = r_{s-1} = 1 \) and hence

\[
q = 2 = 2s - 2r_1 - \ldots - 2r_{s-1}.
\]

Therefore, the two notions of Gamma operators coincide when \( q = 2 \). \( \square \)
3 Fine Asymptotics for Models with Gamma Type Moments

3.1 Preliminaries: Mod-φ Convergence and Precise Deviations

The notion of mod-φ convergence has been studied and developed in the articles [53] and [22] – see the textbook [42] for more references. The main idea was to look for a natural renormalization of the characteristic functions of random variables which do not converge in law, instead of renormalization of the random variables themselves. We will use the following definition of mod-φ convergence, see [42, Definition 1.1.1] and [45, Definition 1.1]:

Definition 3.1.1. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of real-valued random variables, and let us denote by

\[ \varphi_n(z) = \mathbb{E}[e^{z X_n}] \]

their moment generating functions (Laplace transforms), all of which we assume to exist in a subset \(D \subset \mathbb{C}\) containing the origin. Let \(\phi\) be a non-constant infinitely divisible probability law and assume that its moment generating function is well defined on \(D\) and can be written as

\[ \int e^{zx} \phi(dx) = e^{\eta(z)}, \quad z \in D, \]

for some \(\eta\) which is called the Lévy exponent. Let \(\psi\) be an analytic function that does not vanish on \(D \cap \{z \in \mathbb{C} : \text{Im}(z) = 0\}\) such that locally uniformly on \(D\)

\[ e^{-t_n \eta(z)} \varphi_n(z) \xrightarrow{n \to \infty} \psi(z), \]

(3.1)

where \((t_n)_{n \in \mathbb{N}}\) is some sequence going to \(\infty\). We then say that \((X_n)_{n \in \mathbb{N}}\) converges mod-φ over \(D\) with parameters \((t_n)_{n \in \mathbb{N}}\) and limiting function \(\psi : D \to \mathbb{C}\). In the following we denote by

\[ \psi_n(z) := e^{-t_n \eta(z)} \varphi_n(z). \]

(3.2)

Remark 3.1.2. For the most part, we will consider two subsets \(D\): a strip

\[ S_{(c,d)} = \{ z \in \mathbb{C} : c < \text{Re} z < d \} \]

with \(c, d \in \mathbb{R} \cup \{-\infty, \infty\}\) and \(c < 0 < d\), or \(D = i \mathbb{R}\). Mod-φ convergence on \(D = i \mathbb{R}\) corresponds to \(\lim_{n \to \infty} \psi_n(i \xi) = \psi(i \xi)\) uniformly in \(\xi\) on compact subsets of \(\mathbb{R}\).
Remark 3.1.3. Recall that mod-$\phi$ convergence on an open subset $D$ of $\mathbb{C}$ containing 0 can only occur when the characteristic function of $\phi$ is analytic around 0. Among the class of stable distributions, only Gaussian laws satisfy this property. Mod-$\phi$ convergence on $D = i \mathbb{R}$ can however be considered for any stable distribution $\phi$.

It is easy to see that mod-Gaussian convergence on $S_{c,d}$ implies a central limit theorem for a proper renormalization of $(X_n)_{n \in \mathbb{N}}$, if $(t_n)_{n \in \mathbb{N}}$ goes to infinity. We consider

$$Y_n := \frac{X_n - t_n \eta'(0)}{\sqrt{t_n \eta''(0)}} = \frac{X_n}{\sqrt{t_n}}.$$ 

Indeed, for all $\xi \in \mathbb{R}$, we have that the characteristic function of $Y_n$ is given by

$$E[e^{i \xi Y_n}] = E[e^{\frac{i \xi X_n}{\sqrt{t_n}}}] = \varphi_n \left( \frac{i \xi}{\sqrt{t_n}} \right) e^{-\frac{\eta'(0)^2}{2} \xi^2} \eta''(0),$$

thanks to the uniform convergence of $\psi_n$ to $\psi$. But in fact there is much more information encoded in mod-Gaussian convergence. For instance, one can show that the normality zone is of order $o(\sqrt{t_n})$, this means that the limit

$$\lim_{n \to \infty} \frac{P(Y_n \geq x)}{P(N(0, 1) \geq x)} = 1$$

holds true for any $x = o(\sqrt{t_n})$. At the edges of such zone this approximation breaks down and the residue $\psi$ describes how to correct the Gaussian approximation of the tails. This is made more precise in the next two theorems.

Theorem 3.1.4. (Extended central limit theorem, Theorem 4.3.1 and Proposition 4.4.1 in [42])

Consider a sequence $(X_n)_{n \in \mathbb{N}}$ that converges mod-$\phi$ on a band $S_{(c,d)}$ with limiting distribution $\psi$ and parameters $t_n$, where $\phi$ is an infinitely divisible law that is absolutely continuous with respect to the Lebesgue measure. Let $x = o(\sqrt{t_n})^{1/6}$, then

$$P(X_n \geq t_n \eta'(0) + \sqrt{t_n \eta''(0)} x) = P(N(0, 1) \geq x) (1 + o(1)).$$

In the case of mod-Gaussian convergence the normality zone is $o(\sqrt{t_n})$.

For the next result we need the definition and some properties of the Legendre-Fenchel transform, a classical object in large deviation theory. The Legendre-Fenchel transform of a function $\eta$ is defined by

$$F(x) = \sup_{h \in \mathbb{R}} \left( hx - \eta(h) \right).$$

Now assume that $\eta$ is the logarithm of the moment generating function of a random variable (called cumulant generating function). Then $\eta$ is convex. Furthermore, $F$ is a non-negative function and the unique $h$ maximizing $hx - \eta(h)$, if it exists, is defined by the implicit equation $\eta'(h) = x$. Here $h$ depends on $x$. One obtains $F(x) = xh - \eta(h)$ and $F'(x) = h$ and $F''(x) = h'(x) = \frac{1}{\eta''(h)}$. 

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Theorem 3.1.5. (Precise deviations, Theorem 4.2.1 in [42]) Consider a sequence \((X_n)_{n \in \mathbb{N}}\) that converges mod-\(\phi\) on a band \(S_{(c,d)}\) with limiting distribution \(\psi\) and parameters \(t_n\), where \(\phi\) is a non-lattice infinitely divisible law that is absolutely continuous w.r.t. the Lebesgue measure. Denote by \(\eta\) its cumulant generating function and by \(F\) the Legendre-Fenchel transform of \(\eta\). Let \(x \in (\eta'(0), \eta'(d))\), then
\[
P(X_n \geq t_n x) = \frac{\exp(-t_n F(x))}{h \sqrt{2 \pi t_n \eta''(h)}} \psi(h)(1 + o(1))
\]
where \(h = h(x)\) is given implicitly by \(\eta'(h) = x\).

By applying the result to \((-X_n)_{n \in \mathbb{N}}\) one similarly gets
\[
P(X_n \leq t_n x) = \frac{\exp(-t_n F(x))}{|h| \sqrt{2 \pi t_n \eta''(h)}} \psi(h)(1 + o(1))
\]
for \(x \in (\eta'(c), \eta'(0))\).

In case of mod-Gauss convergence we obtain \(\eta(x) = x^2/2\) and therefore \(h = x\) and \(F(x) = x^2/2\) and \((\eta'(0), \eta'(d)) = (0, d)\).

Mod-Stable Convergence

In [45], estimates for the speed of convergence towards a limiting stable law in the setting of mod-\(\phi\) convergence are given. The notion of a zone of control was introduced in [45] in the context of mod-stable convergence. For the following characterization of stable distributions, see e.g. [101] or [80].

Definition 3.1.6. We say that a law \(\phi_{c,\alpha,\beta}\) is stable with scale parameter \(c > 0\), stability parameter \(\alpha \in (0, 2]\) and skewness parameter \(\beta \in [-1, 1]\) if \(Z \sim \phi_{c,\alpha,\beta}\) has characteristic function
\[
\mathbb{E}[e^{i\xi Z}] = \begin{cases} 
\exp\left(-|c\xi|^\alpha(1 - i\beta \tan(\pi \frac{\alpha}{2}) \text{sign}(\xi))\right) & , \alpha \neq 1 \\
\exp\left(-c|\xi|(1 + i\beta \frac{\alpha}{2} \text{sign}(\xi) \log|c\xi|)\right) & , \alpha = 1.
\end{cases}
\]

Any stable law \(\phi_{c,\alpha,\beta}\) has a density \(p_{c,\alpha,\beta}\) with respect to the Lebesgue measure. Some famous examples include
- the standard normal law: \(c = \frac{1}{\sqrt{2}}, \alpha = 2\) and \(\beta = 0\),
- the standard Cauchy distribution: \(c = 1, \alpha = 1\) and \(\beta = 0\),
- the standard Levy distribution: \(c = 1, \alpha = \frac{1}{2}\) and \(\beta = 1\).

If \((X_n)_{n \in \mathbb{N}}\) converges mod-\(\phi_{c,\alpha,\beta}\) on \(i \mathbb{R}\) with parameters \(t_n\), then \(\frac{X_n}{(t_n)^{1/\alpha}}\) converges to \(\phi_{c,\alpha,\beta}\), if \(\alpha \neq 1\). In the case \(\alpha = 1\)
\[
\frac{X_n}{t_n} = \frac{2c\beta}{\pi} \log t_n
\]
converges to \(\phi_{c,\alpha,\beta}\), see [45] Proposition 1.3].
**Definition 3.1.7.** Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of real random variables, \(\phi_{c,\alpha,\beta}\) a stable law, and \((t_n)_{n \in \mathbb{N}}\) a growing sequence. Consider the following assertions:

(a) Fix \(v \geq 0, w > 0\) and \(\gamma \in \mathbb{R}\). There exists a zone of convergence \([-D t_n^\gamma, D t_n^\gamma]\), \(D > 0\), such that for all \(\xi \in \mathbb{R}\) in this zone,

\[
|\psi_n(i\xi) - 1| \leq K_1 |\xi|^v \exp(K_2|\xi|^w)
\]

for some positive constants \(K_1\) and \(K_2\) that are independent of \(n\). Here, \(\psi_n\) is given by \(\psi_n(z) = e^{-t_n \eta_{c,\alpha,\beta} E[e^{zX_n}]}, \) where \(\eta_{c,\alpha,\beta}\) denotes the Lévy exponent of \(\phi_{c,\alpha,\beta}\) (cf. (3.2)).

(b) One has

\[
\alpha \leq w, \quad -\frac{1}{\alpha} \leq \gamma \leq \frac{1}{w - \alpha}, \quad 0 < D \leq \left(\frac{e^\alpha}{2K_2}\right)^{\frac{1}{\alpha}}.
\]

If (a) holds for some parameters \(\gamma > -\frac{1}{\alpha}\) and \(v, w, D, K_1, K_2\), then (b) can always be forced by increasing \(w\), and then decreasing \(D\) and \(\gamma\) in the bound of condition (a). If conditions (a) and (b) are satisfied, we say that \((X_n)_{n \in \mathbb{N}}\) has a zone of control \([-D t_n^\gamma, D t_n^\gamma]\) and index of control \((v, w)\).

The terminology of a zone of control does not mention the reference law \(\phi_{c,\alpha,\delta}\) although it depends on the law. The law is considered to be fixed.

**Remark 3.1.8.** If \((X_n)_{n \in \mathbb{N}}\) has a zone of control \([-D t_n^\gamma, D t_n^\gamma]\) and index of control \((v, w)\), and if \(-\frac{1}{\alpha} < \gamma\), then for \(\alpha \neq 1\) the sequence \(\frac{X_n}{t_n^{\gamma/\alpha}}\), and for \(\alpha = 1\) the sequence in (3.3) converges to the law \(\phi_{c,\alpha,\delta}\), see [45, Proposition 2.3]. In the definition of zone of control, one does not assume the mod-\(\phi_{c,\alpha,\delta}\) convergence of \((X_n)_{n \in \mathbb{N}}\). However in our examples we shall indeed have mod-\(\phi\) convergence with the same parameters \(t_n\).

We then speak of mod-\(\phi\) convergence with a zone of control \([-D t_n^\gamma, D t_n^\gamma]\) and index of control \((v, w)\).

The following result can be found in [45, Theorem 2.16]:

**Theorem 3.1.9.** (Rate of convergence) Fix a reference stable distribution \(\phi_{c,\alpha,\delta}\) and consider a sequence \((X_n)_{n \in \mathbb{N}}\) of random variables with a zone of control \([-D t_n^\gamma, D t_n^\gamma]\) and index of control \((v, w)\). Assume in addition that \(\gamma \leq \frac{v-1}{\alpha}\). If \(Y\) denotes a random variable with law \(\phi_{c,\alpha,\delta}\), then there exists a constant \(C(D, v, K_1, \alpha, c)\) such that

\[
d_{Kol}(Y_n, Y) \leq C(D, v, K_1, \alpha, c) \frac{1}{t_n^{\gamma + \frac{1}{\alpha}}},
\]

where \(d_{Kol}(\cdot, \cdot)\) denotes the Kolmogorov distance and \(Y_n\) is \(\frac{X_n}{t_n^\gamma}\) if \(\alpha \neq 1\) and the random variable in (3.3) if \(\alpha = 1\).
For an explicit form of the constant $C(D,v,K_1,\alpha,c)$, see [15] Theorem 2.16. In the Gaussian case we have

$$C\left(D,v,K_1,2,\frac{1}{\sqrt{2}}\right) = \frac{3}{2\pi} \left(2^{\nu-1}\Gamma(\nu/2)K_1 + \frac{7}{D}\sqrt{\pi/2}\right).$$

(3.4)

In [17] the authors proved the following local limit theorem:

**Theorem 3.1.10.** (Local limit theorem) Fix a reference stable distribution $\phi_{c,\alpha,\delta}$ and consider a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables with a zone of control $[-Dt_n^\gamma, Dt_n^\gamma]$ and index of control $(v,w)$. Let $Y_n = \frac{X_n}{t_n^\gamma}$ if $\alpha \neq 1$, and the random variable in (3.3) if $\alpha = 1$. Let $x \in \mathbb{R}$ and $B$ be a fixed Jordan measurable subset with strictly positive Lebesgue-measure $m(B)$. Then for every exponent $\mu \in (0, \gamma + 1)$,

$$\lim_{n \to \infty} (t_n)^\mu P\left(Y_n - x \in \frac{1}{t_n^\gamma}B\right) = m(B)p_{c,\alpha,\delta}(x).$$

### 3.2 Preliminaries: Classes of Models

In this section we will collect all classes of examples we are interested in.

#### 3.2.1 Wishart Matrices / Laguerre Ensembles

As our first motivating example let us consider the following prototype of a random matrix ensemble from mathematical statistics. The study of sample covariance matrices is fundamental in multivariate statistics. Typically, one thinks of $p(n)$ variables $y_k$ with each variable measured or observed $n$ times. One is interested in analyzing the covariance matrix $A^t A$, with $A$ being the $n \times p(n)$ matrix with $p(n) \leq n$, and entries $y_k^{(j)}$ for $j = 1, \ldots, n$ and $k = 1, \ldots, p(n)$. If $A$ is chosen to be a Gaussian matrix over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, the distribution of the $p(n) \times p(n)$ random matrix $A^t A$ is called *Laguerre* real, complex or symplectic ensemble. Here $A^t$ denotes the transpose, the Hermitian conjugate or the dual of $A$ accordingly, when $A$ is real, complex or quaternion. The eigenvalues $(\lambda_1, \ldots, \lambda_{p(n)})$ are real and non-negative and it is a well known fact that the joint density function on the set $(0, \infty)^{p(n)}$ is

$$\frac{1}{Z_{n,p(n),\beta}} \prod_{1 \leq j < k \leq p(n)} |\lambda_j - \lambda_k|^\beta \prod_{k=1}^{p(n)} \left(\lambda_k^{\frac{\beta}{2}(n-p(n)+1)-1} e^{-\lambda_k}\right)$$

for $\beta = 1, 2, 4$ respectively, see for example [43, Proposition 3.2.2]. Tridiagonal models for the $\beta$-Laguerre Ensembles have been constructed in [33]. Using Selberg integration from [75, (17.6.5)], we obtain

$$Z_{n,p(n),\beta} = 2^{\beta np(n)-p(n)} \prod_{k=1}^{p(n)} \frac{\Gamma(1 + \frac{\beta}{2}k)\Gamma(\frac{\beta}{2}(n-p(n)) + \frac{\beta}{2}k)}{\Gamma(1 + \frac{\beta}{2})}.$$
Using this Selberg formula, one obtains directly that

\[ E \left[ \left( \det W_{n,p(n)}^{L,\beta} \right)^z \right] = 2^{p(n)z} \prod_{k=1}^{p(n)} \frac{\Gamma \left( \frac{\beta}{2} (n - p(n) + k) + z \right)}{\Gamma \left( \frac{\beta}{2} (n - p(n) + k) \right)} \]

\[ = 2^{p(n)z} \prod_{k=1+1-n-p(n)}^{n} \frac{\Gamma \left( \frac{\beta}{2} k + z \right)}{\Gamma \left( \frac{\beta}{2} k \right)}, \]

where \( W_{n,p(n)}^{L,\beta} \) denotes the \( \beta \)-Laguerre distributed random matrix of dimension \( p(n) \times p(n) \). This object is called the Mellin transform of the determinant, which is defined for any \( z \in \mathbb{C} \) with \( \text{Re}(z) > -\frac{\beta}{2} \).

We introduce the notion

\[ L(p, l, \alpha; z) = \log \left( \prod_{k=1}^{p} \frac{\Gamma(\alpha(k + l) + z)}{\Gamma(\alpha(k + l))} \right), \quad (3.5) \]

with \( p, l \geq 1 \) and \( z \in \mathbb{C} \) with \( \text{Re}(z) > -\alpha \) and \( \alpha \in \mathbb{R} \) and obtain

\[ \log E \left[ \left( \det W_{n,p(n)}^{L,\beta} \right)^z \right] = zp(n) \log 2 + L(p(n), n - p(n), \beta/2; z). \quad (3.6) \]

In the case \( p(n) = n \) of \( n \times n \) matrices, asymptotic expansions of (3.6) have been considered in [16, Theorem 5.1]. From a point of view of mathematical statistics, the number of variables \( p(n) \) and the number of measurements or observations \( n \) are typically different. One aim of this chapter is to study arbitrary Wishart matrices. We will consider \( n - p(n) \) converging to zero, or converging to a constant \( c > 0 \), or \( n - p(n) \) is growing at a certain rate with \( n \). Moreover we will analyze the case of a fixed number of variables \( p \). This case will behave differently, see Theorem 3.4.3 and Theorem 3.5.1. A good overview of results for \( \beta \)-Laguerre ensembles is [11] and [43]. In [60] one can find a very early result: the author proved a central limit theorem for \( \log \det W_{n,n}^{L,1} \), which is

\[ \frac{\log \det W_{n,n}^{L,1} + n + \frac{3}{2} \log n}{\sqrt{2 \log n}} \rightarrow \mathcal{N}(0, 1), \]

where \( \mathcal{N}(0, 1) \) denotes the standard Gaussian distribution. We will add the second order analysis.

### 3.2.2 Jacobi Ensembles / Correlation Coefficients

Let \( A_1 \) and \( A_2 \) be \( n_1 \times p(n) \) and \( n_2 \times p(n) \) Gaussian matrices over \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \) with \( p(n) \leq \min(n_1, n_2) \). The distribution of the matrix

\[ A_1^\dagger A_1 (A_1^\dagger A_1 + A_2^\dagger A_2)^{-1} \]
is called Jacobi ensemble. The model can be generalized to all $\beta > 0$ as in the previous matrix models, see [33] and [62] for the corresponding tridiagonal models. The joint density of the eigenvalues on the set $(0, 1)^{p(n)}$ is given by

$$
\frac{1}{Z_{p(n),n_1,n_2,\beta}} \prod_{1 \leq j < k \leq p(n)} |\lambda_j - \lambda_k|^{\beta} \prod_{k=1}^{p(n)} \lambda_k^{\beta(n_1-p(n)+1)-1} (1 - \lambda_k)^{\beta(n_2-p(n)+1)-1}.
$$

One use of this joint density relates to correlation coefficients in multivariate statistics, see [43, Section 3.6.1] for details.

Using Selberg integration from [75, (17.1.3)], we obtain

$$
Z_{p(n),n_1,n_2,\beta} = \prod_{k=1}^{p(n)} \frac{\Gamma(\frac{\beta}{2}(n_1 - p(n) + k) + z) \Gamma(\frac{\beta}{2}(n_1 + n_2 - p(n) + k))}{\Gamma(\frac{\beta}{2}(n_1 - p(n) + k)) \Gamma(\frac{\beta}{2}(n_1 + n_2 - p(n) + k) + z)}.
$$

Using this Selberg formula, one obtains for the corresponding Mellin transform

$$
\mathbb{E}\left[ \left( \det W_{p(n),n_1,n_2}^{J,\beta} \right)^z \right] = \prod_{k=1}^{p(n)} \frac{\Gamma(\frac{\beta}{2}(n_1 - p(n) + k) + z) \Gamma(\frac{\beta}{2}(n_1 + n_2 - p(n) + k))}{\Gamma(\frac{\beta}{2}(n_1 - p(n) + k)) \Gamma(\frac{\beta}{2}(n_1 + n_2 - p(n) + k) + z)},
$$

where $W_{p(n),n_1,n_2}^{J,\beta}$ denotes the $\beta$-Jacobi distributed random matrix of dimension $p(n) \times p(n)$. Hence with (3.5) we have

$$
\log \mathbb{E}\left[ \left( \det W_{p(n),n_1,n_2}^{J,\beta} \right)^z \right] = L(p(n), n_1 - p(n), \beta/2; z) - L(p(n), n_1 + n_2 - p(n), \beta/2; z).
$$

(3.7)

Asymptotic expansions in the case $p(n) = n_1 = \lfloor n\tau_1 \rfloor$ and $n_2 = \lfloor n\tau_2 \rfloor$, for some $\tau_1, \tau_2 > 0$, were considered in [16, Theorem 4.5, 4.7, 4.9, 4.11 and 4.13].

The aim of this chapter is to study the asymptotic behavior under less restrictive assumptions. The main results are given in Theorem 3.5.12 and 3.5.13.

### 3.2.3 Ginibre Ensembles

We now consider an arbitrary $n \times n$ matrix $A$ whose entries are independent real or complex Gaussian random variables with mean zero and variance one. Using the Selberg identity of the Laguerre ensemble, one obtains

$$
\log \mathbb{E}\left[ \left( \det W_{n,\beta}^{G,\beta} \right)^z \right] = \frac{nz}{2} \log \left( \frac{2}{\beta} \right) + \log \prod_{k=1}^{n} \frac{\Gamma(\frac{\beta}{2}k + z)}{\Gamma(\frac{\beta}{2}k)} = \frac{nz}{2} \log \left( \frac{2}{\beta} \right) + L(n, 0, \beta/2; z),
$$

(3.8)

see for example [26, Section 3, (33)]. For a central limit theorem as $n \to \infty$ with a Berry-Esseen bound see [79]. For certain refinements in the Gaussian case see [26, Section 3]. Second order asymptotics are included in [16], see Section 6.3.
3.2.4 The Threefold Way due to Dyson and Fixed-Trace Ensembles

The most classical ensembles are the Hermite ensembles. Let $A$ be an $n \times n$ Gaussian matrix over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. The distribution of $\frac{A + A^\dagger}{2}$ is called the Gaussian orthogonal (GOE), unitary (GUE), and symplectic ensemble (GSE) respectively. The joint density function of the eigenvalues is given by

$$
\frac{1}{Z_n^H(\beta)} \prod_{1 \leq j < k \leq n} |\lambda_j - \lambda_k|^\beta \prod_{k=1}^n \exp \left( -\frac{\lambda_k^2}{2} \right),
$$

for $\beta = 1, 2, 4$. Using Selberg integration from [75, (17.6.7)], we obtain

$$
Z_n^H(\beta) = (2\pi)^{n/2} n \prod_{k=1}^n \Gamma \left( 1 + \frac{k}{\beta} \right) / \Gamma \left( 1 + \frac{\beta}{2} \right).
$$

Using a tridiagonal reduction algorithm, in [33], a matrix model for other choices of $\beta > 0$ was proved. But the lack of Selberg-integrals makes it much harder to investigate these models. In [16], the study was restricted to the GUE model. We shall denote by $W_n^H$ a GUE random matrix. Here one can use the following formula of the Mellin transform of the absolute value of the determinant, computed in [76]:

$$
\mathbb{E} \left[ |\det W_n^H|^z \right] = 2^{nz/2} \prod_{k=1}^n \frac{\Gamma \left( \frac{z+1}{2} + \left\lfloor \frac{k}{2} \right\rfloor \right)}{\Gamma \left( \frac{z}{2} + \left\lfloor \frac{k}{2} \right\rfloor \right)}, \quad (3.9)
$$

which is well defined for any $z \in \mathbb{C}$ with $\mathrm{Re}(z) > -1$. The three ensembles GOE, GUE, GSE are also called *Wigner-Dyson ensembles A, AI and AII*. The background is the following. The classical period of random matrix theory began in the late 1950s and early 1960s, when Wigner and Dyson proposed to model the discrete part of the spectrum of the Hamiltonian of a complicated quantum system by the spectrum of a suitable random matrix ensemble. To be a good model, this ensemble had to share certain symmetries with the quantum system. In his famous paper [34], Dyson adopted a set of symmetry assumptions, which was motivated by the framework of classical quantum mechanics, and classified those spaces of matrices which are compatible with the given symmetries. He ended up with the threefold way of hermitian matrices with real, complex, and quaternion entries, i.e., precisely with those spaces on which the familiar Gaussian orthogonal, unitary, and symplectic ensembles (GOE, GUE, GSE) of classical random matrix theory are supported. Dyson’s threefold way is established in geometrical terms, without reference to probability measures on the matrix spaces in question. In structural terms, the space of hermitian, real symmetric and quaternion real matrices can be viewed as tangent spaces to, or infinitesimal versions of Riemannian symmetric spaces of certain types. A good overview of results for the Dyson ensembles is [5].

*Fixed-trace ensembles* of random matrices were first considered by Rosenzweig and Bronk, see [75, Chapter 19]. Universal limits for the eigenvalue correlation functions in the bulk of the spectrum in trace-fixed matrix ensembles are considered in [36] [47].
Let us consider the GUE model. The maximum of $\prod_{k=1}^{n} |\lambda_k|$, subject to $\sum_{k=1}^{n} \lambda_k^2 = 1$, is $n^{-n/2}$. Hence we consider the rescaled determinant

$$n^{n/2} |\text{det} \, W_n^{H,ft}|,$$

where $\text{det} \, W_n^{H,ft}$ is the product of the eigenvalues of the fixed-trace GUE ensemble. Here one can use the following formula of the Mellin transform of $n^{n/2} |\text{det} \, W_n^{H,ft}|$, computed in [65]: consider [65, (20)] with $\beta = 2$ at the value $z + 1$, see also [26, page 256/257]:

$$E \left[ |n^{n/2} \det \, W_n^{H,ft}|^z \right] = \prod_{k=1}^{n} \frac{\Gamma \left( \frac{z+1}{2} + \lfloor \frac{k}{2} \rfloor \right)}{\Gamma \left( \frac{1}{2} + \lfloor \frac{k}{2} \rfloor \right)} \left( \prod_{k=1}^{n} \frac{\Gamma \left( \frac{z+n}{2} + \frac{k-1}{n} \right)}{\Gamma \left( \frac{n}{2} + \frac{k-1}{n} \right)} \right)^{-1}. \quad (3.10)$$

In [16], asymptotic expansions of (3.9) have been considered. We will add a precise asymptotic description for (3.10). This includes a central limit theorem, precise moderate deviations and Berry-Esseen bounds. All these results are new and presented in Section 3.5.5.

3.2.5 The Tenfold Way

In the 1990s, in the field of condensed matter physics, it was observed that random matrix models for so-called mesoscopic normal-superconducting hybrid structures must be taken from the infinitesimal versions of further symmetric spaces, different from the path of the threfold way of Dyson. In [51], a classification similar to Dyson’s way was developed, based on less restrictive assumptions, thus taking care of the needs of modern mesoscopic physics. Their list is in one-to-one correspondence with the infinite families of Riemannian symmetric spaces as classified by Cartan. In [38], the corresponding random matrix theory was introduced, with a special emphasis on large deviation principles. The seven new ensembles are listed in [43] in Section 3.1 and Section 3.3, as well as in [38]. All these new ensembles can be described by a certain matrix configuration (see the table on page 104 in [43]). For any new ensemble, the joint density of the positive eigenvalues is of the form

$$\frac{1}{Z_{n,p(n),\beta,\mu}} \prod_{1 \leq j < k \leq p(n)} |\lambda_j^2 - \lambda_k^2|^{\beta} \prod_{k=1}^{p(n)} \lambda_k^{\beta \mu} \exp \left( -\beta \lambda_k^2 / 2 \right), \quad (3.11)$$

for some $\mu \in \mathbb{R}$.

Chiral ensembles

The three Chiral ensembles are defined in [43, Definition 3.1.2]. Here we have to choose $\beta \in \{1, 2, 4\}$ and $\mu_{\text{chiral}} = n - p(n) + 1 - \frac{1}{\beta}$.

Bogoliubov- de Gennes ensembles

The other four ensembles are collected in Subsection 3.3.2 of [43]. Here, for the first ensemble, we have to choose $n = p(n)$, $\beta = 1$ and $\mu = 1$. For the second (only if $n$
is even) \( p(n) = n/2 \), \( \beta = 2 \) and \( \mu = 0 \). The third is given by \( p(n) = n/2 \), \( \beta = 4 \) and \( \mu = 1/4 \) if \( n \) is even, and by \( p(n) = (n-1)/2 \), \( \beta = 2 \) and \( \mu = 5/4 \) if \( n \) is odd. Finally, the fourth ensemble is given by \( p(n) = n \), \( \beta = 2 \) and \( \mu = 1 \).

We will discuss the precise asymptotic behavior of the seven ensembles. Therefore we have to calculate \( Z_{n,p(n),\beta,\mu} \). We start at \([75\text{ (17.6.6)}]) with \( \gamma = \beta/2 \):

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1 \leq j < k \leq p(n)} |\lambda_j^2 - \lambda_k^2|^\beta \prod_{k=1}^{p(n)} |\lambda_k|^{2v-1} e^{-\beta \lambda_k^2} d\lambda_1 \cdots d\lambda_k
\]

\[
= \left( \frac{2}{\beta} \right)^{v p(n)+\frac{2}{p(n)}(p(n)-1)} 2^{-p(n)} \prod_{k=1}^{p(n)} \frac{\Gamma\left(1 + \frac{\beta}{2} k\right) \Gamma\left(v + \frac{\beta}{2} (k-1)\right)}{\Gamma\left(1 + \frac{\beta}{2}\right)}.
\]

Note that the integrand is an even function in \( \lambda_k \), so with \( 2v - 1 = \beta \mu \), we obtain that

\[
Z_{n,p(n),\beta,\mu} = \left( \frac{2}{\beta} \right)^{\frac{\beta}{2} + \frac{1}{2}} 2^{p(n)(p(n)-1)} \prod_{k=1}^{p(n)} \frac{\Gamma\left(\frac{\beta}{2} + \frac{1}{2} + \frac{z+1}{2} + \frac{\beta}{2} (k-1)\right)}{\Gamma\left(\frac{\beta}{2} + \frac{1}{2} + \frac{\beta}{2} (k-1)\right)}.
\]

Let us denote by \( W_{n,p(n)}^{\beta,\mu} \) a random matrix with joint eigenvalue distribution \((3.11)\). Using this Selberg formula, we obtain the corresponding Mellin transform

\[
\mathbb{E}\left[\left( \det W_{n,p(n)}^{\beta,\mu} \right)^z \right] = \prod_{k=1}^{p(n)} \frac{\Gamma\left(\frac{\beta}{2} (n-p(n)+k) + \frac{z+1}{2} \right)}{\Gamma\left(\frac{\beta}{2} (n-p(n)+k) \right)} = \exp\left( L\left(p(n), n-p(n), \beta/2; \frac{z+1}{2}\right) \right).
\]

Hence for the three chiral ensembles \( (\beta \in \{1, 2, 4\}) \) we have

\[
\mathbb{E}\left[\left( \det W_{n,p(n)}^{\beta,\mu,\text{chiral}} \right)^z \right] = \prod_{k=1}^{p(n)} \frac{\Gamma\left(\frac{1}{2} (n-p(n)+k) + \frac{z+1}{2} \right)}{\Gamma\left(\frac{1}{2} (n-p(n)+k) \right)} = \exp\left( L\left(n, 1, 1/2; \frac{z+1}{2}\right) \right).
\]

For the Bogoliubov- de Gennes ensembles, since \( \mu \) is constant, the behavior is different. Let us consider the case \( n = p(n) \), \( \beta = 1 \) and \( \mu = 1 \). Here we have

\[
\mathbb{E}\left[\left( \det W_{n,n}^{1,1} \right)^z \right] = \prod_{k=1}^{n} \frac{\Gamma\left(\frac{1}{2} (k+1) + \frac{z+1}{2} \right)}{\Gamma\left(\frac{1}{2} (k+1) \right)} = \exp\left( L\left(n, 1, 1/2; \frac{z+1}{2}\right) \right).
\]

Summarizing, the asymptotic behavior of the log-determinant of a Laguerre ensemble will lead to the asymptotic behavior of the product of non-negative eigenvalues in the seven new ensembles. All our results for the sum of the logarithms of the positive eigenvalues are new and will be presented in Section \[3.5.4\].

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3.2.6 Gram Ensembles and Random Simplices

If for \( p(n) \leq n \), \( X_1, \ldots, X_{p(n)+1} \) are independent random points in \( \mathbb{R}^n \) which are distributed according to a multivariate Gaussian distribution with density

\[
    f(x) = (2\pi)^{-n/2} \exp\left(-\frac{1}{2}|x|^2\right), \quad x \in \mathbb{R}^n,
\]

we denote by \( VP_{n,p(n)} \) the \( p(n) \)-dimensional volume of the parallelotope spanned by the points \( X_1, \ldots, X_{p(n)} \). This is the determinant of the corresponding Gram matrix. It is known, see [73], that for all \( m \geq 0 \) the moments of order \( 2m \) of the volume satisfy

\[
    \log \mathbb{E}\left[(VP_{n,p(n)})^{2m}\right] = mp(n) \log 2 + \log \prod_{k=1}^{p(n)} \frac{\Gamma\left(\frac{1}{2}(n - p(n) + k) + m\right)}{\Gamma\left(\frac{1}{2}(n - p(n) + k)\right)}.
\]

The formula is a consequence of the so-called Blaschke-Petkantschin formula from integral geometry. With (3.5), hence we will study the asymptotics of

\[
    \log \mathbb{E}\left[(VP_{n,p(n)})^z\right] = \frac{z}{2} p(n) \log 2 + L\left(p(n), n - p(n), 1/2; z/2\right), \quad (3.15)
\]

which is exactly the same as studying the asymptotic behavior of the log-determinant of a Laguerre ensemble in the case \( \beta = 1 \) for \( z/2 \) instead of \( z \), see (3.6). Interestingly enough, the application of the Blaschke-Petkantschin formula is an alternative proof of the moment identity (3.6), which in random matrix theory is proved with the help of Selberg integrals.

**Remark 3.2.1.** In the theory of random matrices it is quite natural to consider a matrix size of \( p(n) \) growing with \( n \). In models from random geometry, the number of points \( p(n) \) in \( \mathbb{R}^n \) might not depend on \( n \) and it might be a challenge to let \( p(n) \) and \( n \) tend to infinity simultaneously. Our results for \( L(p(n), n - p(n), 1/2; z/2) \) will imply this type of phenomena in high dimensions for free.

In [74], the author studied the moments of order \( 2m \) of \( VP_{n,p(n)} \) if the random points are distributed according to three other distributions, which are called the Beta model, the Beta prime model and the spherical model. In the Beta model with parameter \( \nu > 0 \), the i.i.d. points in the ball of radius 1 are distributed with respect to the density

\[
    f(|x|) = \frac{1}{\pi^{n/2}} \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} (1 - |x|^2)^{(\nu-2)/2}, \quad x \in \mathbb{R}^n, |x| < 1.
\]

Here for \( z = 2m \) we have

\[
    \log \mathbb{E}\left[(VP_{n,p(n)})^z\right] = p(n) \log \Gamma\left(\frac{n + \nu}{2}\right) - p(n) \log \Gamma\left(\frac{n + \nu}{2} + \frac{z}{2}\right)
    + L\left(p(n), n - p(n), 1/2; z/2\right), \quad (3.16)
\]
In the Beta prime model with parameter $\nu > 0$ the density is given by

$$f(|x|) = \frac{1}{\pi^{n/2}} \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \left(1 + |x|^2\right)^{-\left(\nu+n\right)/2}, \quad x \in \mathbb{R}^n,$$

and for $z = 2m$ we have

$$\log \mathbb{E}[(VP_{n,p(n)}|^z)] = p(n) \log \Gamma\left(\frac{\nu}{2} - \frac{z}{2}\right) - p(n) \log \Gamma\left(\frac{\nu}{2}\right) + L(p(n), n - p(n), 1/2; z/2). \quad (3.17)$$

Finally, in the spherical model, the points are uniformly distributed on the sphere of radius 1 centered around the origin of $\mathbb{R}^n$. We have

$$\log \mathbb{E}[(VP_{n,p(n)}|^z)] = p(n) \log \Gamma\left(\frac{n}{2}\right) - p(n) \log \Gamma\left(\frac{n + z}{2}\right) + L(p(n), n - p(n), 1/2; z/2) \quad (3.18)$$

for any $z = 2m$ with $m \in \mathbb{N}$, which is the Beta model with $\nu = 0$. These identities are exactly given in [97, Section 2.2.2, page 189]. The second order analysis of the log-volume was already studied in [16], here in the case $p(n) = n$. The authors established mod-Gaussian convergence in Theorem 4.5, and extended results in Theorem 4.9, 4.11 and 4.13. Our results for $p(n) \neq n$ are collected in Section 3.6.

If we denote by $VS_{n,p(n)}$ the $p(n)$-dimensional volume of the simplex with vertices $X_1, \ldots, X_{p(n)+1}$, the moment formulas are very similar. The following formulas were proved using the affine Blaschke-Petkantschin formula, see [78] and [20]. In the Gaussian model one obtains

$$\log \mathbb{E}[(p(n)!VS_{n,p(n)}|^z)] = \frac{z}{2} \log \left(p(n) + 1\right) + \log \mathbb{E}[(VP_{n,p(n)}|^z)], \quad (3.19)$$

where $\log \mathbb{E}[(VP_{n,p(n)}|^z)]$ is defined in (3.15).

In the Beta model we have

$$\log \mathbb{E}[(p(n)!VS_{n,p(n)}|^z)] = \log f(n, p(n), \nu, z) + \log \mathbb{E}[(VP_{n,p(n)}|^z)], \quad (3.20)$$

where $\log \mathbb{E}[(VP_{n,p(n)}|^z)]$ is defined in (3.16) and where

$$f(n, p(n), \nu, z) = \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{\Gamma\left(\frac{n+\nu}{2} + \frac{z}{2}\right)} \frac{\Gamma\left(\frac{p(n)(n+\nu-2)+(n+\nu)}{2}\right)}{\Gamma\left(p(n)(n+\nu-2)+(n+\nu)\right)} \frac{(p(n) + 1)^z}{(p(n) + \frac{z}{2})}.$$ 

In the Beta prime model we have

$$\log \mathbb{E}[(p(n)!VS_{n,p(n)}|^z)] = \log g(n, p(n), \nu, z) + \log \mathbb{E}[(VP_{n,p(n)}|^z)], \quad (3.21)$$

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where \( \log \mathbb{E}[(VP_{n,p(n)})^z] \) is defined in (3.17) and where

\[
g(n, p(n), \nu, z) = \frac{\Gamma\left(\frac{\nu}{2} - \frac{z}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{\Gamma\left(\frac{(p(n)+1)\nu}{2} - p(n)\frac{z}{2}\right)}{\Gamma\left(\frac{(p(n)+1)\nu}{2} - (p(n) + 1)\frac{z}{2}\right)}.
\]

Finally, in the spherical model we obtain

\[
\log \mathbb{E}\left[(p(n)! V_{S_{n,p(n)}})^z\right] = \log h(n, p(n), \nu, z) + \log \mathbb{E}[(VP_{n,p(n)})^z], \quad (3.22)
\]

where \( \log \mathbb{E}[(VP_{n,p(n)})^z] \) is defined in (3.18) and where

\[
h(n, p(n), \nu, z) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2} + \frac{z}{2}\right)} \cdot \frac{\Gamma\left(\frac{p(n)(n-2)+n}{2} + (p(n) + 1)\frac{z}{2}\right)}{\Gamma\left(\frac{p(n)(n-2)+n}{2} + p(n)\frac{z}{2}\right)},
\]

which is the same as the Beta model with \( \nu = 0 \).

Summarizing, the asymptotic behavior of the log volume of random simplices is given by an expansion of \( L(p(n), n - p(n), \frac{1}{2}; \frac{z}{2}) \), as well as the asymptotic analysis of additional summands of the type

\[
\log \Gamma\left(m(n, \nu) + z\right) - \log \Gamma\left(m(n, \nu)\right), \quad (3.23)
\]

with certain functions \( m(n, \nu) \). Like in [16], proof of Lemma 4.2, the additional summands asymptotically behave like a polynomial of degree 2 in \( z \), see Proposition 3.6.1. Hence these terms only modify the mean of the log-volume, as well as the limiting function in the mod-Gaussian convergence, adding a term of the form \( e^{c \text{const. } z^2} \).

### 3.3 The Main Theorem

In the following sections, we will frequently make use of Big-O or Landau notation. For the sake of completeness, we add a brief definition here.

**Definition 3.3.1.** Let \( a_n \) and \( b_n \) be two sequences of real numbers. Then we write

\[
a_n = o(b_n) \quad \text{if} \quad \lim_{n \to \infty} \left|\frac{a_n}{b_n}\right| = 0,
\]

and

\[
a_n = O(b_n) \quad \text{if} \quad \limsup_{n \to \infty} \left|\frac{a_n}{b_n}\right| < \infty.
\]
In particular, \( a_n = o(1) \) means that \( \lim_{n \to \infty} a_n = 0 \), and \( a_n = O(b_n) \) means that there exists a constant \( C > 0 \) such that for all sufficiently large \( n \in \mathbb{N} \): \( a_n \leq C b_n \).

All the representations of the moments of Gamma type motivate to consider the following key asymptotic expansion, which is a generalization of Theorem 5.1 in [16]. Denote by

\[ S_\alpha := \{ z \in \mathbb{C} : -\alpha < \text{Re}(z) \} . \]

**Theorem 3.3.2.** For all \( p, l \geq 1 \) and any \( z \in S_\alpha \) with \( \alpha > 0 \) and \( |z| < \text{const.} \alpha(p+l)^{1/6} \), we have

\[
L(p, l, \alpha; z) = \log \left( \prod_{k=1}^{p} \frac{\Gamma(\alpha(k+l) + z)}{\Gamma(\alpha(k+l))} \right) = \sum_{i=1}^{3} T_i(p, l, \alpha; z) + T_4(l, \alpha; z) + T_5(l, \alpha; z) + R(p, l, \alpha; z),
\]

where \( T_i(p, l, \alpha; z) \), for \( i = 1, 2, 3 \), are defined in (3.26), (3.27), (3.31), \( T_4(l, \alpha; z) \) is defined in (3.28), \( T_5(l, \alpha; z) \) is defined in (3.32), and \( R(p, l, \alpha; z) \) is defined in (3.29).

**Proof.** The proof is a generalization of the proof of Theorem 5.1 in [16] applying the Abel-Plana formula (see Theorem 3.7.1 in the Appendix) which allows to evaluate even non-convergent sums, which cannot be handled applying Taylor expansion.

For a complex number \( z = |z| e^{i \text{arg}(z)} \), \( z \neq 0 \), \( \text{arg } z \in (-\pi, \pi] \) we define the principal branch of the logarithm as \( \log(z) := \log(|z|) + i \text{arg}(z) \). As is customary in this framework, every equation involving complex logarithms is to be read mod \( 2\pi i \). This way, the classical identities \( \log(xy) = \log(x) + \log(y) \) and \( \log(x^y) = y \log(x) \) remain true for complex arguments \( x \) and \( y \).

We have to consider the asymptotical behavior of

\[
\sum_{k=1}^{p} \left[ \log \Gamma(\alpha(k+l) + z) - \log \Gamma(\alpha(k+l)) \right]
\]

as \( p = p(n) \) or \( l = l(n) \) goes to infinity with \( n \to \infty \). The complex Gamma function for \( z \in \mathbb{C} \) with \( \text{Re}(z) > 0 \) is given by

\[
\Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} \, dt.
\]

The first Binet’s formula for the logarithm of the Gamma function (see (3.67)) is given by

\[
\log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + 1 + \int_{0}^{\infty} \varphi(s)(e^{-sz} - e^{-s}) \, ds, \quad \text{Re}(z) > 0.
\]

Here the function \( \varphi \) is given by \( \varphi(s) = \left( \frac{1}{2} - \frac{1}{s} + \frac{1}{e^{s-1} - 1} \right) \frac{1}{z} \) and, for every \( s > 0 \), satisfies
0 < \varphi(s) \leq \lim_{s \to 0} \varphi(s) = \frac{1}{12}$. Applying Binet’s formula leads to

\[
\sum_{k=1}^p \left( \log \Gamma(\alpha(k + l) + z) - \log \Gamma(\alpha(k + l)) \right)
\]

\[
= \sum_{k=1}^p \left\{ \left[ \alpha(k + l) + z - \frac{1}{2} \right] \log \left( \alpha(k + l) + z \right) - \alpha(k + l) + z + 1 \\
+ \int_0^\infty \varphi(s) \left( e^{-s(\alpha(k+1)+z)} - e^{-s} \right) \, ds \\
- \left[ \alpha(k+1) - \frac{1}{2} \right] \log \left( \alpha(k+1) \right) + \alpha(k+1) - 1 - \int_0^\infty \varphi(s) \left( e^{-s(\alpha(k+1))} - e^{-s} \right) \, ds \right\}
\]

\[
= \sum_{k=1}^p \left\{ z + \int_0^\infty \varphi(s) \left( e^{-s(\alpha(k+1)+z)} - e^{-s(\alpha(k+1))} \right) \, ds \right\}
\]

\[
+ \sum_{k=1}^p \left\{ \left[ \alpha(k + l) + z - \frac{1}{2} \right] \log \left( \alpha(k + l) + z \right) - \left( \alpha(k + l) - \frac{1}{2} \right) \log \left( \alpha(k + l) \right) \right\}
\]

\[
= T_1(p, l, \alpha; z) + S_1(p, l, \alpha, z),
\]

where

\[
S_1(p, l, \alpha, z) := \sum_{k=1}^p \left\{ \left[ \alpha(k + l) + z - \frac{1}{2} \right] \log \left( \alpha(k + l) + z \right) - \left( \alpha(k + l) - \frac{1}{2} \right) \log \left( \alpha(k + l) \right) \right\},
\]

and

\[
T_1(p, l, \alpha; z) := -pz + \sum_{k=1}^p \int_0^\infty \varphi(s) \left( e^{-s(\alpha(k+1)+z)} - e^{-s \alpha(k+1)} \right) \, ds
\]

\[
= -pz - \int_0^\infty \varphi(s) \left( e^{-sz} - 1 \right) e^{-s \alpha(p+1)} ds + \int_0^\infty \varphi(s) \left( e^{-sz} - 1 \right) e^{-s \alpha l} ds.
\]

(3.26)

Here, the last equality is due to evaluating the geometric sum \( \sum_{k=1}^p e^{-\alpha k} = \frac{1-e^{-pz}}{e^{\alpha}-1} \).

Considering the term \( S_1(p, l, \alpha, z) \) we obtain

\[
S_1(p, l, \alpha, z) = \sum_{k=1}^p \left( \alpha(k + l) + z - \frac{1}{2} \right) \log \left( 1 + \frac{z}{\alpha(k + l)} \right) + z \sum_{k=1}^p \log \left( \alpha(k + l) \right)
\]

\[
= \sum_{k=0}^{p-1} f_1(k) + T_2(p, l, \alpha; z),
\]

where

\[
T_2(p, l, \alpha; z) := z \log \left( \alpha^p(1 + l)(2 + l) \cdots (p + l) \right),
\]

(3.27)

and where

\[
f_1(s) := \left( \alpha(s + 1 + l) + z - \frac{1}{2} \right) \log \left( 1 + \frac{z}{\alpha(s + 1 + l)} \right).
\]
As in [16], we apply the Abel-Plana formula (Theorem 3.7.1) and obtain
\[
\sum_{k=0}^{p-1} f_l(k) = \int_0^p f_l(s) \, ds + \frac{1}{2} f_l(0) - \frac{1}{2} f_l(p) + \int_0^\infty \frac{f_l(is) - f_l(-is)}{e^{2\pi s} - 1} \, ds
\]
\[
- i \int_0^\infty \frac{f_l(p + is) - f_l(+ - is)}{e^{2\pi s} - 1} \, ds
\]
\[
= T_3(p, l, \alpha; z) + S_2(p, l, \alpha, z)
\]
with
\[
T_3(p, l, \alpha; z) = \int_1^{p + 1} f_l(s - 1) \, ds - \frac{1}{2} f_l(p)
\]
\[
= \int_1^{1+p} (\alpha(s + l) + z - \frac{1}{2}) \log \left(1 + \frac{z}{\alpha(s + l)}\right) \, ds - \frac{1}{2} (\alpha(p + 1 + l) + z - \frac{1}{2}) \log \left(1 + \frac{z}{\alpha(p + 1 + l)}\right),
\]
and where \(S_2(p, l, \alpha, z)\) is the remainder term in the Abel-Plana formula:
\[
S_2(p, l, \alpha, z) := T_4(l, \alpha; z) + T_5(l, \alpha; z) - R(p, l, \alpha, z),
\]
with
\[
T_4(l, \alpha; z) := \frac{1}{2} f_l(0) = \frac{1}{2} \left(\alpha(1 + l) + z - \frac{1}{2}\right) \log \left(1 + \frac{z}{\alpha(1 + l)}\right),
\]
(3.28)
and
\[
T_5(l, \alpha; z) := i \int_0^\infty \frac{f_l(is) - f_l(-is)}{e^{2\pi s} - 1} \, ds,
\]
and
\[
R(p, l, \alpha; z) := i \int_0^\infty \frac{f_l(p + is) - f_l(p - is)}{e^{2\pi s} - 1} \, ds.
\]
(3.29)
Adapting the proof in [16] we are able to show that
\[
R(p, l, \alpha; z) = \mathcal{O} \left(\frac{|z| + |z|^2}{p + l}\right),
\]
(3.30)
for some implied constant, depending only on \(\alpha\). The term \(T_3(p, l, \alpha; z)\) can be computed.
explicitly via integration by parts:

\[ T_3(p, l, \alpha; z) = \int_{1+l}^{1+p+l} (\alpha s + z - \frac{1}{2}) \log \left( 1 + \frac{z}{\alpha s} \right) ds \]

\[ - \frac{1}{2} (\alpha(p + 1 + l) + z - \frac{1}{2}) \log \left( 1 + \frac{z}{\alpha(p + 1 + l)} \right) \]

\[ = \left( \alpha \frac{s^2}{2} + (z - \frac{1}{2}) s \right) \log \left( 1 + \frac{z}{\alpha s} \right) \bigg|_{1+l}^{1+p+l} + \frac{z}{2} \int_{1+l}^{1+p+l} \frac{\alpha s + 2z - 1}{\alpha s + z} ds \]

\[ - \frac{1}{2} (\alpha(p + 1 + l) + z - \frac{1}{2}) \log \left( 1 + \frac{z}{\alpha(p + 1 + l)} \right). \]

Next we obtain that

\[ \int_{1+l}^{1+p+l} \frac{\alpha s + 2z - 1}{\alpha s + z} ds = \int_{1+l}^{1+p+l} \left( 1 + \frac{z - 1}{\alpha s + z} \right) ds \]

\[ = p + \frac{z - 1}{\alpha} \log \left( s + \frac{z}{\alpha} \right) \bigg|_{1+l}^{1+p+l} \]

\[ = p + \frac{z - 1}{\alpha} \log(1 + p + l) - \frac{z - 1}{\alpha} \log(1 + l) \]

\[ + \frac{z - 1}{\alpha} \log \left( 1 + \frac{z}{\alpha(1 + p + l)} \right) - \frac{z - 1}{\alpha} \log \left( 1 + \frac{z}{\alpha(1 + l)} \right). \]

Summarizing, we obtain

\[ T_3(p, l, \alpha; z) = \]

\[ \left( \alpha \frac{(1 + p + l)(p + l)}{2} + (z - \frac{1}{2})(p + l + \frac{1}{2}) + \frac{z(z - 1)}{2\alpha} \right) \log \left( 1 + \frac{z}{\alpha(1 + p + l)} \right) \]

\[ - \left( \alpha \frac{(1 + l)^2}{2} + (z - \frac{1}{2})(1 + l) + \frac{z(z - 1)}{2\alpha} \right) \log \left( 1 + \frac{z}{\alpha(1 + l)} \right) \]

\[ + \frac{pz}{2} + \frac{z(z - 1)}{2\alpha} \log \left( 1 + \frac{p}{1 + l} \right). \]  

(3.31)

Depending on whether \( p \) or \( l \) or both depend on \( n \) we will apply an expansion on the logarithm at this point. The last point is – in adaption of the proof of Theorem 5.1 in [16] – to represent the term \( T_5(l, \alpha; z) \) in terms of nice functions. By definition of \( f_l \) we obtain

\[ f_l(is) - f_l(-is) = i\alpha(s + l) \log \left( 1 + \left( 1 + \frac{z}{\alpha} \right)^2(s + l)^{-2} \right) \]

\[ - i\alpha(s + l) \log \left( 1 + (s + l)^{-2} \right) \]

\[ + 2i(\alpha + z - \frac{1}{2}) \left( \arctan \left( \frac{s + l}{1 + \frac{z}{\alpha}} \right) - \arctan(s + l) \right), \]
where we used the relation $\log\left(\frac{1+ix}{1-ix}\right) = 2i \arctan(x)$. Hence we obtain
\[
T_5(l, \alpha; z) = i \int_0^\infty \frac{f_l(is) - f_l(-is)}{e^{2\pi s} - 1} \, ds
\]
\[
= -\alpha \int_1^\infty \log \left(1 + (1 + \frac{z}{\alpha})^2 s^{-2}\right) \frac{s}{e^{2\pi s} - 1} \, ds
\]
\[
+ \alpha \int_1^\infty \log \left(1 + s^{-2}\right) \frac{s}{e^{2\pi s} - 1} \, ds
\]
\[- \left( \alpha + z - \frac{1}{2} \right) \int_1^\infty \arctan \frac{s}{1 + \frac{z}{\alpha}} \frac{1}{e^{2\pi s} - 1} \, ds
\]
\[+ \left( \alpha + z - \frac{1}{2} \right) \int_1^\infty \arctan s \frac{1}{e^{2\pi s} - 1} \, ds. \tag{3.32}
\]

This is the desired representation for our applications.

\[\square\]

### 3.4 The Key Asymptotics

In all our classes of examples, $L(p(n), r(n), \beta/2; z)$ has to be considered, see (3.5). Here, $(p(n))_{n \in \mathbb{N}}$ will be an increasing sequence of natural numbers, whereas $(r(n))_{n \in \mathbb{N}}$ is a sequence of real numbers. We will assume that
\[
|z| < \text{const} \frac{\beta}{2} \max\{p(n), r(n)\}^{1/6} \tag{3.33}
\]
and $z \in S_{\beta/2}$. As a consequence of Theorem 3.3.2 with (3.26) we obtain
\[
T_1(p(n), r(n), \beta/2; z) = -p(n)z + \int_0^\infty \frac{\varphi(s)(e^{-sz} - 1)e^{-s(\beta/2)(p(n) + r(n))}}{e^{s\beta/2} - 1} \, ds
\]
\[+ O\left(\frac{|z|}{p(n) + r(n)}\right). \tag{3.34}
\]

This follows by applying the inequalities $e^x - 1 \geq x$ and $|e^z - 1| \leq |z| |e^{|z|}$, for any $x \geq 0$ and $z \in \mathbb{C}$ respectively, to be able to bound
\[
\left|\int_0^\infty \frac{\varphi(s)(e^{-sz} - 1)}{e^{s\beta/2} - 1} e^{-s(\beta/2)(p(n) + r(n))} \, ds\right| \leq \frac{1}{12} \int_0^\infty \frac{|e^{-sz} - 1|}{|e^{s\beta/2} - 1|} e^{-s(\beta/2)(p(n) + r(n))} \, ds
\]
\[
\leq \frac{1}{12} \int_0^\infty s|z| e^{s|z|} e^{-s(\beta/2)(p(n) + r(n))} \, ds
\]
\[
\leq \frac{|z|}{6\beta} \int_0^\infty e^{-s(\beta/4)(p(n) + r(n))} \, ds
\]
\[= \frac{1}{6(\beta/2)^2} |z| \left(\frac{1}{p(n) + r(n)}\right),
\]
as soon as $|z| \leq (\beta/4)(p(n) + r(n))$, which is compatible with the assumption (3.33), see Theorem 3.3.2. Precisely this estimate is presented in [16].
For (3.27) we apply $\log n = n \log n - n + \frac{1}{2} \log(2\pi n) + O(1/n)$ and get

$$T_2(p(n), r(n), \beta/2; z) = z \left\{ p(n) \log \frac{\beta}{2} + (p(n) + r(n)) \log (p(n) + r(n)) - p(n) \right\} - r(n) \log r(n) + \frac{1}{2} \log \left( 2\pi \left( 1 + \frac{p(n)}{r(n)} \right) \right) + O\left( \frac{|z|}{p(n) + r(n)} \right). \tag{3.35}$$

Expanding the logarithm in the first summand of $T_3(p(n), r(n), \beta/2; z)$ in (3.31), we obtain

$$T_3(p(n), r(n), \beta/2; z) = \frac{r(n)}{2} + zp(n) + \frac{2z^2}{\beta} - \frac{z}{\beta} + O\left( \frac{|z| + |z|^2 + |z|^3}{p(n) + r(n)} \right) - \left( \frac{\beta(1 + r(n))^2}{4} + (z - \frac{1}{2})(1 + r(n)) + \frac{z(z - 1)}{\beta} \right) \log \left( 1 + \frac{z}{\beta/2(1 + r(n))} \right) + \frac{z(z - 1)}{\beta} \log \left( \frac{1 + p(n) + r(n)}{1 + r(n)} \right). \tag{3.36}$$

Next, (3.28) reads

$$T_4(r(n), \beta/2; z) = \frac{1}{2} \left( \beta/2(1 + r(n)) + z - \frac{1}{2} \right) \log \left( 1 + \frac{2z}{\beta(1 + r(n))} \right). \tag{3.37}$$

Finally, $T_5(r(n), \beta/2; z)$ is given by (3.32), and

$$R(p(n), r(n), \beta/2; z) = O\left( \frac{|z| + |z|^2}{p(n) + r(n)} \right).$$

For the most part, we will apply the expansions for $r(n) \to \infty$ as $n \to \infty$. In this case, we expand the logarithm and observe after some small computations

$$T_3(p(n), r(n), \beta/2; z) + T_4(p(n), r(n), \beta/2; z) = zp(n) + \frac{z(z - 1)}{\beta} \log \left( \frac{1 + p(n) + r(n)}{1 + r(n)} \right) + O\left( \frac{|z| + |z|^2 + |z|^3}{p(n) + r(n)} \right). \tag{3.38}$$

In the statement of the Theorems below, $G$ denotes the Barnes $G$-function, see the Appendix. Moreover, we define

$$\Phi_\alpha(z) := \alpha \log G\left( \frac{z}{\alpha} + 1 \right) - \left( z - \frac{1}{2} \right) \log \Gamma\left( \frac{z}{\alpha} + 1 \right) + \int_0^\infty \left( \frac{1}{2s} - \frac{1}{s^2} + \frac{1}{s(e^s - 1)} \right) \frac{e^{-sz} - 1}{e^{s\alpha} - 1} ds + \frac{3z^2}{4\alpha} + \frac{z}{2}. \tag{3.39}$$

**Remark 3.4.1.** In [16], $\Phi_{\beta/2}$ was introduced at the beginning of section 4. Checking the proof of [16, Theorem 5.1], we are sure that the penultimate summand has to be $\frac{3z^2}{2\pi}$. 

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Remark 3.4.2. In Lemma 7.1 of [16], the authors proved that $\Phi_\alpha$ can be written as a finite sum of log-Gamma and log-Barnes-$G$-functions. These expressions are simpler, since they do not depend on the integral
\[
\int_0^\infty \left( \frac{1}{2s} - \frac{1}{s^2} + \frac{1}{s(e^s - 1)} \right) \frac{e^{-sz} - 1}{e^{s\alpha} - 1} \, ds,
\]
which does not have a closed formula for all $\alpha > 0$. We only mention two examples. If $\beta = 2$, one has for all $n \geq 1$ and any $z \in S_\alpha$ with $|z| < \frac{1}{2}n^{1/6}$ that
\[
\Phi_1(z) = \Phi_{\beta/2}(z) = \frac{z}{2} \log(2\pi) - \log G(1 + z).
\]
If $\beta = 1$, for the same $z$ it holds that
\[
\Phi_{1/2}(z) = z \left( -\frac{1}{2} \log \frac{1}{2} + \frac{1}{2} \log(2\pi) - \frac{1}{2} \log G(1 + 2z) - \frac{1}{2} \left( \log \Gamma \left( \frac{1}{2} \right) - \log \Gamma \left( \frac{1}{2} + z \right) \right) \right).
\]

We consider formula (3.6) for the moments of the determinant of a $\beta$-Laguerre ensemble and will obtain the following results, depending on the growth of the sequence $(n - p(n))_{n \in \mathbb{N}}$. Interestingly enough, in most of the cases, we will observe mod-Gaussian convergence. In some cases no mod-$\phi$ or a non-Gaussian mod-$\phi$ convergence occurs.

Theorem 3.4.3. $L(p(n), n - p(n), \beta/2; z)$, defined in (3.5), satisfies the following asymptotic expansion locally uniformly on $S_{\beta/2}$:

(a) $p(n) = n$:
\[
L(n, 0, \beta/2; z) = z\mu_1(n, n) + \frac{z^2}{\beta} \log n + \Phi_{\beta/2}(z) + o(1)
\]
with $\Phi_{\beta/2}(z)$ given by (3.39) and $\mu_1(n, n)$ defined by
\[
\mu_1(n, n) := \left( \frac{1}{2} - \frac{1}{\beta} \right) \log n - n \log n + n \log \frac{\beta}{2}.
\]

(b) Case $n - p(n) \to 0$ as $n \to \infty$:
\[
L(p(n), n - p(n), \beta/2; z) = z\mu_1(p(n), n) + \frac{z^2}{\beta} \log n + \Phi_{\beta/2}^{n,p(n)}(z) + o(1),
\]
where
\[
\mu_1(p(n), n) := \left( \frac{1}{2} - \frac{1}{\beta} \right) \log \left( \frac{n}{n - p(n)} \right) + \frac{n}{2} - \frac{3p(n)}{2} + n \log n - (n - p(n)) \log(n - p(n)) + p(n) \log \left( \frac{\beta}{2} \right),
\]

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and $\Phi^{{n, p(n)}}_{\beta/2}(z)$ is a function depending on $n$ and $p(n)$ such that

$$\lim_{n \to \infty} \Phi^{{n, p(n)}}_{\beta/2}(z) = \Phi_{\beta/2}(z)$$

for all $z \in \mathbb{C}$ we are considering, and $\Phi_{\beta/2}(z)$ given by (3.39).

(c) Case $n - p(n) = c$ with $c \in \mathbb{N}$ fixed:

$$L(p(n), c, \beta/2; z) = z \mu_2(p(n), c) + \frac{z^2}{\beta} \log \left( \frac{p(n) + 1 + c}{1 + c} \right) + \Phi^c_{\beta/2}(z) + o(1)$$

with

$$\mu_2(p(n), c) = \frac{1}{2} \log(p(n) + c) - \frac{1}{\beta} \log(p(n) + 1 + c)$$

$$+ (p(n) + c) \log(p(n) + c) + p(n) \log \left( \frac{\beta}{2} \right) - p(n).$$

Here $\Phi^c_{\beta/2}(z)$ is defined in (3.50).

(d) Case $n - p(n) \to \infty$ as $n \to \infty$:

$$L(p(n), n - p(n), \beta/2; z) = z \mu_3(p(n), n) + \frac{z^2}{\beta} \log \left( \frac{n}{n - p(n)} \right) + o(1)$$

with

$$\mu_3(p(n), n) := \left( \frac{1}{2} - \frac{1}{\beta} \right) \log \left( \frac{n}{n - p(n)} \right) + \frac{1}{2} \log(2\pi)$$

$$+ n \log n - (n - p(n)) \log(n - p(n)) + p(n) \log \left( \frac{\beta}{2} \right) - p(n).$$

Here $\Phi_{\beta/2}(z)$ is defined in (3.50).

(e) Case $p(n) = p$ for some $p \in \mathbb{N}$ fixed:

$$L(p, n - p, \beta/2; z) = z \mu_4(p, n) + z \left( p \log \left( \frac{\beta}{2} \right) - p + \frac{1}{2} \log(2\pi) \right) + o(1)$$

with

$$\mu_4(p, n) := \left( \frac{1}{2} - \frac{1}{\beta} \right) \log n - \left( \frac{1}{2} - \frac{1}{\beta} \right) \log(n - p) + n \log n - (n - p) \log(n - p).$$

Remark 3.4.4. We notice that the corresponding $\mu'$s are the expectations of the log-determinants up to a constant.
Proof.

(a) Case \( p(n) = n \): We will apply Theorem 3.3.2 with \( p = p(n) = n \), \( l = l(n) = n - p(n) = 0 \) and \( \alpha = \beta/2 \). Now we are exactly in the situation of [16, Theorem 5.1]. It is not obvious to obtain this result directly from the representation in Theorem 3.3.2. Therefore we give the proof. From (3.34) we obtain

\[
T_1(n, 0, \beta/2; z) = -nz + \int_0^\infty \frac{\varphi(s)(e^{-sz} - 1)}{e^{\beta s/2} - 1} ds + O\left(\frac{|z|}{n}\right),
\]

as soon as \(|z| \leq \beta/4n\), which is compatible with our assumption. Moreover, from (3.35) we obtain

\[
T_2(n, 0, \beta/2; z) = z\left(n \log \beta/2 + n \log n - n + \frac{1}{2} \log(2\pi n)\right) + O\left(\frac{|z|}{n}\right).
\]

From (3.36) it follows that

\[
T_3(n, 0, \beta/2; z) = nz + \frac{z(z-1)}{\beta} \log n - \frac{z}{\beta} + \frac{2z^2}{\beta} - \left(\frac{\beta}{4} + z - \frac{1}{2} + \frac{z(z-1)}{\beta}\right) \log \left(1 + \frac{2z}{\beta}\right) + O\left(\frac{|z| + |z|^2 + |z|^3}{n}\right).
\]

Moreover, (3.37) leads to

\[
T_4(0, \beta/2; z) = \frac{1}{2} \left(\frac{\beta}{2} + z - \frac{1}{2}\right) \log \left(1 + \frac{2z}{\beta}\right).
\]

A nice fact is that \( T_5(0, \beta/2; z) \) can be represented in terms of the Barnes G function and the Gamma function, which was presented in [16, page 20]. Applying (3.68) and (3.73), we obtain

\[
T_5(0, \beta/2; z) = \frac{\beta}{2} \log G\left(1 + \frac{2z}{\beta}\right) - \left(z - \frac{1}{2}\right) \log \Gamma\left(1 + \frac{2z}{\beta}\right)
+ \log \left(1 + \frac{2z}{\beta}\right) \left(\frac{z^2}{\beta} - \frac{z}{\beta} + \frac{z}{2} - \frac{1}{4}\right) - \frac{z^2}{2\beta} + \frac{z}{\beta} + \frac{z}{2} - \frac{z}{2} \log(2\pi).
\]

Putting all terms together, we conclude as on page 20 in [16]. Note that \( \frac{z}{2} \log(2\pi) \) is \( T_2 \) canceled by the last summand in \( T_5 \).

In all other cases \( p(n) \neq n \), and we choose \( l(n) = r(n) = n - p(n) \) and \( \alpha = \beta/2 \) and observe from (3.34)

\[
T_1(p(n), n - p(n), \beta/2; z) = -p(n)z + \int_0^\infty \frac{\varphi(s)(e^{-sz} - 1)}{e^{\beta s/2} - 1} ds + O\left(\frac{|z|}{n}\right),
\]

and from (3.35)

\[
T_2(p(n), n - p(n), \beta/2; z) = z \left\{ p(n) \log(\beta/2) + n \log n - p(n) - (n - p(n)) \log(n - p(n)) + \frac{1}{2} \log(2\pi) + \frac{1}{2} \log\left(\frac{n}{n - p(n)}\right)\right\} + O\left(\frac{|z|}{n}\right).
\]

(3.45)
and with (3.36):
\[
T_3(p(n), n - p(n), \beta/2; z) = z \frac{n}{2} + \frac{2z^2}{\beta} - \frac{z}{\beta} + \mathcal{O}\left(\frac{|z| + |z|^2 + |z|^3}{n}\right)
- \left(\frac{\beta(1 + n - p(n))^2}{4} + (z - \frac{1}{2})(1 + n - p(n)) + \frac{z(z - 1)}{\beta}\right) \log\left(1 + \frac{z}{\beta(1 + n - p(n))}\right)
+ \frac{p(n)z}{2} + \frac{z(z - 1)}{\beta} \log\left(\frac{1 + n}{1 + n - p(n)}\right).
\] (3.48)

Moreover, we have
\[
T_4(n - p(n), \beta/2; z) = \frac{1}{2}\left(\frac{\beta}{2}(1 + n - p(n)) + z - \frac{1}{2}\right) \log\left(1 + \frac{2z}{\beta(1 + n - p(n))}\right).
\] (3.49)

\( T_5(n - p(n), \beta/2; z) \) is defined in (3.32), and
\[
R(p(n), n - p(n), \beta/2; z) = \mathcal{O}\left(\frac{|z| + |z|^2}{n}\right).
\]

Now we are prepared to prove the other cases.

(b) Case \( n - p(n) \to 0 \): Intuitively, we will expect the same asymptotic behavior as in the case \( n = p(n) \). First, we collect in \( T_1, \ldots, T_5 \) the \( n \)-dependent prefactors of \( z \) to obtain the size of the expected value of the log-determinant. It is \(-p(n)\) in \( T_1 \) and
\[
p(n) \log(\beta/2) + n \log n - p(n) + \frac{1}{2} \log(2\pi) - (n - p(n)) \log(n - p(n)) + \frac{1}{2} \log\left(\frac{n}{n - p(n)}\right)
\]
in \( T_2 \). The \( n \)-dependent prefactor of \( z \) in \( T_3 \) is
\[
\frac{n}{2} + \frac{p(n)}{2} - \frac{1}{\beta} \log\left(\frac{n}{n - p(n)}\right),
\]
see (3.48). We obtain \( \mu_1(p(n), n) \) in (3.41). The \( n \) dependent prefactor of \( z^2 \) is \( \frac{\log n}{\beta} \), see (3.48). The sum of the remaining terms (without the \( \mathcal{O} \)-terms) are defined to be \( \Phi_{n,p(n)}(z) \) which converges to \( \Phi_{\beta/2}(z) \) as \( n \to \infty \) (see case (a)). This can be shown easily and the details are left to the reader. Notice that \( \frac{z}{2} \log(2\pi) \) is canceled by the last summand of the limit of \( T_5 \).

(c) Case \( n - p(n) = c \) for some fixed \( c \in \mathbb{N} \): Obviously the sums of all \( n \)-dependent prefactors of \( z \) and \( z^2 \) in \( T_1, \ldots, T_5 \) are \( \mu_2(p(n), c) \), and \( \frac{1}{\beta} \log\left(\frac{p(n)+1+c}{1+c}\right) \) respectively. The terms which do not depend on \( n \) are
\[
U_1(c, \beta/2; z) = \int_0^\infty \frac{\varphi(s)(e^{-sz} - 1)e^{-s\beta c/2}}{e^{(\beta/2)s} - 1} \, ds
\]
from $T_1$, $U_2(c, \beta/2; z) = z \left( \frac{1}{2}(\log \left( \frac{2\pi}{c} \right) - c \log(c) \right)$ from $T_2$,

$$U_3(c, \beta/2; z) = \frac{cz}{2} + \frac{2z^2}{\beta} - \frac{z}{\beta} + \frac{z}{\beta} \log(1 + c)$$

$$- \left( \frac{\beta(1 + c)^2}{4} + (z - \frac{1}{2})(1 + c) + \frac{z(z - 1)}{\beta} \right) \log \left(1 + \frac{z}{\beta}(1 + c)\right)$$

from $T_3$,

$$U_4(c, \beta/2; z) = \frac{1}{2} \left( \frac{\beta}{2}(1 + c) + z - \frac{1}{2} \right) \log \left(1 + \frac{z}{\beta}(1 + c)\right)$$

from $T_4$ and $U_5(c, \beta/2; z) = T_5(c, \beta/2; z)$. The result follows with

$$\Phi_{c, \beta/2}^5(z) := \sum_{j=1}^{5} U_j(c, \beta/2; z).$$

(3.50)

(d) Case $n - p(n) \to \infty$ as $n \to \infty$: In this case we obtain

$$T_1(p(n), n - p(n), \beta/2; z) = -p(n)z + \mathcal{O}\left( \frac{|z|}{n} + \frac{|z|}{n - p(n)} \right)$$

and

$$T_2(p(n), n - p(n), \beta/2; z) = zp(n) \log(\beta/2) + zn \log n - zp(n) + z \frac{1}{2} \log(2\pi n)$$

$$- z(n - p(n)) \log(n - p(n)) - z \frac{1}{2} \log(2\pi(n - p(n))) + \mathcal{O}\left( \frac{|z|}{n} + \frac{|z|}{n - p(n)} \right).$$

With the notions of the proof of Theorem 3.3.2, we obtain that

$$\sum_{k=0}^{p(n)-1} f_{n-p(n)}(k) = \sum_{k=0}^{n-1} f_0(k) - \sum_{k=0}^{n-p(n)-1} f_0(k).$$

Since

$$\sum_{k=0}^{n-1} f_0(k) = T_3(n, 0, \beta/2; z) + T_4(0, \beta/2; z) + T_5(0, \beta/2; z) - R(n, 0, \beta/2; z)$$

and

$$\sum_{k=0}^{n-p(n)-1} f_0(k) = T_3(n - p(n), 0, \beta/2; z) + T_4(0, \beta/2; z) + T_5(0, \beta/2; z) - R(n - p(n), 0, \beta/2; z),$$
we obtain
\[
\sum_{k=0}^{p(n)-1} f_{n-p(n)}(k) = T_3(n, 0, \beta/2; z) - T_3(n - p(n), 0, \beta/2; z)
\]
\[
+ \mathcal{O}\left(\frac{|z| + |z|^2}{n}\right) + \mathcal{O}\left(\frac{|z| + |z|^2}{n - p(n)}\right).
\]

(3.51)

With (3.45), we get
\[
T_3(n, 0, \beta/2; z) - T_3(n - p(n), 0, \beta/2; z) = zp(n) + \frac{z(z - 1)}{\beta} \log \left(\frac{n}{n - p(n)}\right),
\]
and the result follows.

(e) Case \(p(n) = p\) for a fixed \(p \in \mathbb{N}\): From the formulas in the proof of the previous case \((n - p(n) \to \infty)\), we observe that
\[
T_1(p, n - p, \beta/2; z) = -pz + \mathcal{O}\left(\frac{|z|}{n}\right)
\]
and
\[
T_2(p, n - p, \beta/2; z) = zp \log(\beta/2) + zn \log n - zp + \frac{1}{2} \log(2\pi n)
\]
\[
- z(n - p) \log(n - p) - z \frac{1}{2} \log(2 \pi (n - p)) + \mathcal{O}\left(\frac{|z|}{n}\right).
\]
Moreover,
\[
T_3(n, 0, \beta/2; z) - T_3(n - p, 0, \beta/2; z) = zp + \frac{z(z - 1)}{\beta} \log \left(\frac{n}{n - p}\right).
\]
Combining these terms as in the case before \((n - p(n) \to \infty)\), we obtain that the expectation of the log-determinant is of size \(\mu_3(p, n)\), and the result follows.

### 3.5 Random Matrix Ensembles

#### 3.5.1 \(\beta\)-Laguerre Ensembles

A direct consequence of Theorem 3.4.3 is mod-\(\phi\) convergence for the shifted log-determinants of the considered \(\beta\)-Laguerre ensemble. We observe that mod-\(\phi\) convergence sometimes fails, e.g. when \(n - p(n) \to \infty\), but \(p(n)/n \to c \in [0, 1)\) as \(n \to \infty\). Recall that by (3.6), we have
\[
\log \mathbb{E} \left[ \exp \left( z \log \left( \det W_{n,n}^{L,\beta} \right) \right) \right] = zp(n) \log 2 + L(p(n), n - p(n), \beta/2; z).
\]
Theorem 3.5.1 (Mod-$\phi$ convergence for the log-determinant of $\beta$-Laguerre ensembles).

(a) **Case $p(n) = n$:** The sequence

\[
(X^L_1(n) := \log \left( \det W_{n,n}^{L,\beta} \right) - \mu_1(n, n) - n \log 2)_{n\in\mathbb{N}}
\]

converges mod-Gaussian on $S_{\beta/2}$ with parameter $t_n = \frac{2}{\beta} \log n$ and limiting function $\Psi(z) = \exp(\Phi_{\beta/2}(z))$. Here $\mu_1(n, n)$ is defined in (3.40).

(b) **Case $n - p(n) \to 0$ as $n \to \infty$:** The sequence

\[
(X^L_2(n) := \log \left( \det W_{n,p(n)}^{L,\beta} \right) - \mu_1(p(n), n) - p(n) \log 2)_{n\in\mathbb{N}}
\]

converges mod-Gaussian on $S_{\beta/2}$ with parameter $t_n = \frac{2}{\beta} \log n$ and limiting function $\Psi(z) = \exp(\Phi_{\beta/2}(z))$. Here $\mu_1(p(n), n)$ is defined in (3.41).

(c) **Case $n - p(n) = c$ with $c \in \mathbb{N}$ fixed:** The sequence

\[
(X^L_3(n) := \log \left( \det W_{n,p(n)}^{L,\beta} \right) - \mu_2(p(n), c) - p(n) \log 2)_{n\in\mathbb{N}}
\]

converges mod-Gaussian on $S_{\beta/2}$ with parameter $t_n = \frac{2}{\beta} \log \left( \frac{n}{n-p(n)} \right)$ and limiting function $\Psi(z) = \exp(\Phi_{\beta/2}(z))$. Here $\mu_2(p(n), c)$ is defined in (3.42).

(d) **Case $n - p(n) \to \infty$ as $n \to \infty$:** The sequence

\[
(X^L_4(n) := \log \left( \det W_{n,p(n)}^{L,\beta} \right) - \mu_3(p(n), n) - p(n) \log 2)_{n\in\mathbb{N}}
\]

converges mod-Gaussian on $S_{\beta/2}$ with parameter $t_n = \frac{2}{\beta} \log \left( \frac{n}{n-p(n)} \right)$ and limiting function $\Psi(z) = \exp(\Phi_{\beta/2}(z))$. Here $\mu_3(p(n), n)$ is defined in (3.43). Hence for any sequence $p(n)$ with $\frac{p(n)}{n} \to c \in [0, 1)$, no mod-$\phi$ convergence takes place. A non-central limit theorem appears, see Chapter 2 in [109].

(e) **Case $p(n) = p$ for a fixed $p \in \mathbb{N}$:** The sequence

\[
(\log \left( \det W_{n,p}^{L,\beta} \right) - \mu_4(p, n) - p \log 2)_{n\in\mathbb{N}}
\]

does not converge in the sense of mod-$\phi$ convergence. But the sequence

\[
(n \log \left( \det W_{n,p}^{L,\beta} \right) - n(p \log n - p))_{n\in\mathbb{N}}
\]

converges mod-$\phi$ on $i\mathbb{R}$ with parameter $t_n = p n$ and limiting function

\[
\psi(z) = \left( 1 + \frac{2z}{\beta} \right)^{-\frac{\beta(p-1)^2}{4} - \frac{z}{\beta}}.
\]

Here $\phi$ is such that the Lévy exponent is

\[
\eta(z) = \log \int_\mathbb{R} e^{zx} \phi(dx) = -\frac{\beta}{2} \log \beta + \left( z + \frac{\beta}{2} \right) \log 2 + \left( z + \frac{\beta}{2} \right) \log \left( z + \frac{\beta}{2} \right). \quad (3.52)
\]
Summarizing, we obtain mod-ϕ convergence for the centered version of \( \log(\det W_{n,p}^{L,\beta}) \):

<table>
<thead>
<tr>
<th>Condition</th>
<th>Centered Version of</th>
<th>mod-ϕ</th>
<th>( t_n )</th>
<th>Limiting Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(n) = n )</td>
<td>( \log(\det W_{n,p}^{L,\beta}) )</td>
<td>( \text{mod-}\mathcal{N}(0,1) )</td>
<td>( \frac{2}{\beta} \log n )</td>
<td>( \exp(\Phi_{\beta/2}(z)) )</td>
</tr>
<tr>
<td>( n - p(n) \to 0 )</td>
<td>( \log(\det W_{n,p}^{L,\beta}) )</td>
<td>( \text{mod-}\mathcal{N}(0,1) )</td>
<td>( \frac{2}{\beta} \log n )</td>
<td>( \exp(\Phi_{\beta/2}(z)) )</td>
</tr>
<tr>
<td>( n - p(n) = c )</td>
<td>( \log(\det W_{n,p}^{L,\beta}) )</td>
<td>( \text{mod-}\mathcal{N}(0,1) )</td>
<td>( \frac{2}{\beta} \log \left( \frac{p(n)+1+c}{\beta} \right) )</td>
<td>( \exp(\Phi_{\beta/2}(z)) )</td>
</tr>
<tr>
<td>( n - p(n) = o(n) )</td>
<td>( \log(\det W_{n,p}^{L,\beta}) )</td>
<td>( \text{mod-}\mathcal{N}(0,1) )</td>
<td>( \frac{2}{\beta} \log \left( \frac{n}{n-p(n)} \right) )</td>
<td>1</td>
</tr>
<tr>
<td>( p(n) = p )</td>
<td>( n \log(\det W_{n,p}^{L,\beta}) )</td>
<td>mod-ϕ on ( i\mathbb{R} )</td>
<td>( pn )</td>
<td>( (1 + \frac{z}{\beta}) \frac{2p-1+p}{2} - \frac{z}{\beta} )</td>
</tr>
</tbody>
</table>

**Proof.** The results, in most of the cases, follow directly from Theorem 3.4.3. The only fact which has to be proven is case (e). We need to prove that \( n \mu_3(p) \) converges mod-ϕ. Note that for any ansatz considering \( n \alpha \log(\det W_{n,p}^{L,\beta}) - n \mu_3(p) \) for some \( \alpha \leq 1 \), only the choice \( \alpha = 1 \) leads to an appropriate asymptotic expansion. Interestingly enough, we will not apply Theorem 3.3.2. The reason is that \( nR(p,n-p,z) \equiv O(|z| + |z|^2) \) will not be sufficient to achieve convergence. For the finite number of \( p \) factors, we alternatively apply Stirling’s formula, which reads

\[
\Gamma(az + b) = \sqrt{2\pi} \exp(-az)(az)^{az+b-\frac{1}{2}} \left(1 + O(1/az)\right)
\]

as \( |z| \to \infty, a > 0, b \in \mathbb{R} \) and \( |\arg z| < \pi \), see [11, page 257]. Applying Stirling’s formula for \( \Gamma\left(\frac{\beta}{2}(n-p+k) + nz\right) \) and \( \Gamma\left(\frac{\beta}{2}(n-p+k)\right) \) leads to

\[
2^{p\beta n} \prod_{k=1}^{n} \frac{\Gamma\left(\frac{\beta}{2}(n-p+k) + nz\right)}{\Gamma\left(\frac{\beta}{2}(n-p+k)\right)}
= 2^{p\beta n} e^{-pz} n^{\beta(n-p+k) + nz} \prod_{k=1}^{n} \frac{\left(\beta/2 + z\right)^{\beta/2 + \beta(k-p) - 1/2}}{\left(\beta/2\right)^{\beta/2 + \beta(k-p) - 1/2}} + o(1)
= 2^{p\beta n} e^{-pz} n^{\beta/2 + z} \frac{(\beta/2 + z)^{\beta(p-1)/4 - p} - (\beta/2)^{\beta(p-1)/4 - p}}{\beta/2 - (\beta/2)^{-\beta(p-1)/4 - p}} + o(1).
\]

Hence

\[
\log \left( \mathbb{E} \left[ \det W_{n,p}^{L,\beta} \right] \right) = nz(p \log n - p)
+ pn \left( -\frac{\beta}{2} \log \beta + (z + \frac{\beta}{2}) \log 2 + (z + \frac{\beta}{2}) \log \left( z + \frac{\beta}{2} \right) \right)
+ \left( -\frac{\beta p(p-1)}{4} - \frac{p}{2} \right) \log \left( 1 + \frac{2z}{\beta} \right) + o(1).
\]

This is true for any \( z \in \mathbb{C} \) with \( |\arg z| < \pi \), especially for all \( z = i\xi \) with \( \xi \in \mathbb{R} \). Next we discuss \( \phi \). We observe that \( \phi \) is a non-constant infinitely divisible distribution. Moreover,
one can find a tilted, totally skewed 1-stable distribution such that the corresponding Lévy exponent $\eta$ is \([3.52]\). This means that $\phi = \phi_{c,1,-1}$ for a certain $c$, depending on $\beta$ in the sense of the definition given in section \([3.1]\). These distributions are known to be infinitely divisible. For details, see [100, Section 1.2], particularly Proposition 1.2.12 as well as [101, Chapter 2]. Hence we have proved mod-stable convergence on $i \mathbb{R}$.

A consequence of Theorem \([3.5.1]\) case (e) is:

**Corollary 3.5.2** (Weak convergence of the log-determinant of $\beta$-Laguerre ensembles if $p(n) = p$ for a fixed $p \in \mathbb{N}$).

Consider

$$Y^L_n(\beta, p, c) := \frac{1}{p} \log \left( \det W^L_{n,p} \right) + \frac{2c}{\pi} \log(pn).$$

We conclude that $Y^L_n(\beta, p, c)$ converges weakly to $\phi_{c,1,-1}$.

A direct consequence of Theorems \([3.1.4]\) and \([3.1.5]\) are the following two results:

**Theorem 3.5.3** (Extended central limit theorems for log-determinants of $\beta$-Laguerre ensembles).

In all cases (a)-(d) (in case (d) only if $n - p(n) = o(n)$) in Theorem \([3.5.1]\) for $y = o(\sqrt{\log n})$, we observe

$$P\left( X^L_i(n) \geq y \sqrt{\frac{2\log n}{\beta}} \right) = P\left( N(0,1) \geq y \right) \left( 1 + o(1) \right)$$

for $i = 1, \ldots, 4$.

**Theorem 3.5.4** (Precise deviations for log-determinants of $\beta$-Laguerre ensembles).

In case (a) and (b) in Theorem \([3.5.1]\) for $x > 0$, we obtain

$$P\left( X^L_i(n) \geq x \frac{2 \log n}{\beta} \right) = \frac{e^{-x^2 \frac{2 \log n}{\beta}}}{x \sqrt{\frac{4 \pi \log n}{\beta}}} \exp \left( \Phi_{\beta/2}(x) \right) \left( 1 + o(1) \right)$$

for $i = 1, 2$. In case (c), we obtain for all $x > 0$ that

$$P\left( X^L_3(n) \geq \frac{2x \log n}{\beta} \right) = \frac{e^{-x^2 \frac{2 \log n}{\beta}}}{x \sqrt{\frac{4 \pi \log n}{\beta}}} \exp \left( \Phi_{\beta/2}^c(x) \right) \left( 1 + o(1) \right).$$

In case (d), if $n - p(n) = o(n)$, we obtain for all $x > 0$ that

$$P\left( X^L_4(n) \geq \frac{2x \log n}{\beta} \right) = \frac{e^{-x^2 \frac{2 \log n}{\beta}}}{x \sqrt{\frac{4 \pi \log n}{\beta}}} \left( 1 + o(1) \right).$$
For the next result, recall the definition of a large deviation principle, see [23, Section 1.2].

**Corollary 3.5.5** (Large and moderate deviations principles for log-determinants of $\beta$-Laguerre ensembles).

1. In all cases (a)-(d) (in case (d) only if $n - p(n) = o(n)$) in Theorem 3.5.1, the sequence
$$\left( \frac{X_i^L(n)}{2 \log n / \beta} \right)_{n \in \mathbb{N}}$$
satisfies a large deviation principle with speed $\log n$ and rate function $\frac{x^2}{2}$.

2. In all cases (a)-(d) (in case (d) only if $n - p(n) = o(n)$) in Theorem 3.5.1, for any sequence $(a_n)_{n \in \mathbb{N}}$ with $a_n = o(\sqrt{\log n})$, the sequence
$$Y_{n,\beta}^L := \frac{X_i^L(n)}{a_n \sqrt{2 \log n / \beta}}$$
satisfies a large deviation principle with speed $a_n^2$ and rate function $\frac{x^2}{2}$.

**Proof.** (a) In all cases we apply the mod-$\phi$ convergence of Theorem 3.5.1, here Theorem 3.5.4, combined with the Theorem of Gärtner-Ellis, see [23, Theorem 2.3.6].

(b) Now we apply Theorem 3.5.3 for $y = t a_n$ with $t \in \mathbb{R}$. Hence we have
$$P(Y_{n,\beta}^L \geq t) = P(N(0, 1) \geq t a_n) \left(1 + o(1)\right).$$

A famous result, called Mill’s ratio, tells us:
$$\frac{1}{\sqrt{2\pi}} \frac{t a_n}{1 + (t a_n)^2} \exp \left( - \frac{(t a_n)^2}{2} \right) \leq P\left(N(0, 1) \geq t a_n\right) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{t a_n} \exp \left( - \frac{(t a_n)^2}{2} \right).$$

Now we take the logarithm and apply the condition $a_n = o(\sqrt{\log n})$. To obtain the full principle of large deviations, proceed as in [23, Proof of Theorem 1.4] applying Theorem 4.1.11 in [23].

**Corollary 3.5.6.** In case (d), now we assume that $p(n)/n \to c \in [0, 1)$ as $n \to \infty$. Then we obtain that
$$\frac{1}{p(n)} \left( \log \det W_{n,p(n)}^{L,\beta} - p(n)(\log n - 1) \right)$$

satisfies an LDP with speed $p(n) n$ and rate function $I$ which is the Legendre-Fenchel transform of
$$\eta(z) = z \log 2 + \left(z + \frac{\beta}{2} \left(1 - \frac{c}{2}\right) \right) \log \left( \frac{\beta}{2} (1 - c) + z \right) - \frac{\beta}{2} \left(1 - \frac{c}{2}\right) \log \left( \frac{\beta}{2} (1 - c) \right).$$

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If \( c = 0 \), the rate function \( I \) is given by

\[
I(x) = \sup_{a \in \mathbb{R}} (ax - \eta(a)) = \exp(x - \log 2 - 1) - \frac{\beta}{2} x + \frac{\beta}{2} \log \beta, \tag{3.55}
\]

where \( \eta \) is given by \((3.52)\).

**Proof.** Assume that \( p(n)/n \to c \in [0, 1) \). We apply \((3.53)\) for \( \Gamma\left(\frac{\beta}{2}(n - p(n) + k) + nz\right) \) and \( \Gamma\left(\frac{\beta}{2}(n - p(n) + k)\right) \), which leads to

\[
2^{p(n)} n \prod_{k=1}^{p(n)} \frac{\Gamma\left(\frac{\beta}{2}(n - p(n) + k) + nz\right)}{\Gamma\left(\frac{\beta}{2}(n - p(n) + k)\right)}
\]

\[
\sim 2^{z p(n)} e^{-z p(n) n} n^{z p(n)} \prod_{k=1}^{p(n)} \frac{\left(\frac{\beta}{2}\left(1 - \frac{p(n)}{n}\right) + z\right)^{(n-p(n))\frac{\beta}{2} + n z + \frac{\beta}{2} k - \frac{1}{2}}}{\left(\frac{\beta}{2}\left(1 - \frac{p(n)}{n}\right)\right)^{(n-p(n))\frac{\beta}{2} + \frac{\beta}{2} k - \frac{1}{2}}}
\]

\[
= 2^{z p(n)} n e^{-z p(n) n} n^{z p(n)} \prod_{k=1}^{p(n)} \left(\frac{\beta}{2}\left(1 - \frac{p(n)}{n}\right) + z\right)^{p(n)(n-p(n))\frac{\beta}{2} + p(n) n z + \frac{\beta p(n)(p(n)+1)}{4} - \frac{p(n)}{2}} \frac{\left(\frac{\beta}{2}\left(1 - \frac{p(n)}{n}\right)\right)^{p(n)(n-p(n))\frac{\beta}{2} + \frac{\beta p(n)(p(n)+1)}{4} - \frac{p(n)}{2}}}{\left(\frac{\beta}{2}\left(1 - \frac{p(n)}{n}\right)\right)^{p(n)(n-p(n))\frac{\beta}{2} + \frac{\beta p(n)(p(n)+1)}{4} - \frac{p(n)}{2}}}.\]

Hence

\[
\log \left( \mathbb{E}\left[ \left( \det W_{n,p(n)}^{L,\beta} \right)^{nz} \right] \right) \sim nz(p(n) \log n - p(n)) + np(n) z \log 2
\]

\[
+ np(n) z \log \left(\frac{\beta}{2}\left(1 - \frac{p(n)}{n}\right) + z\right) + np(n) \frac{\beta}{2} \log \left(1 + \frac{z}{\frac{\beta}{2}\left(1 - \frac{p(n)}{n}\right)}\right)
\]

\[
+ \left(\frac{\beta p(n)(p(n)+1)}{4} - \frac{p(n)}{2} - \frac{p(n)^2 \beta}{2}\right) \log \left(1 + \frac{z}{\frac{\beta}{2}\left(1 - \frac{p(n)}{n}\right)}\right). \tag{3.56}
\]

It follows that

\[
\frac{1}{np(n)} \log \mathbb{E}\left[ \exp \left(z n \left( \log \det W_{n,p(n)}^{L,\beta} - p(n)(\log n - 1) \right) \right) \right]
\]

\[
\sim z \log 2 + \left(z + \frac{\beta}{2}\right) \log \left(\frac{\beta}{2}\left(1 - \frac{p(n)}{n}\right) + z\right) - \frac{\beta}{2} \log \left(\frac{\beta}{2}\left(1 - \frac{p(n)}{n}\right)\right)
\]

\[
+ \left(\frac{\beta p(n)(p(n)+1)}{4n} - \frac{1}{2n} - \frac{p(n) \beta}{2n}\right) \log \left(1 + \frac{z}{\frac{\beta}{2}\left(1 - \frac{p(n)}{n}\right)}\right),
\]

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and thus
\[
\lim_{n \to \infty} \frac{1}{np(n)} \log \mathbb{E} \left[ \exp \left( zn \left( \log \det W_{n,p(n)}^{L,\beta} - p(n)(\log n - 1) \right) \right) \right] \\
= z \log 2 + \left( z + \frac{\beta}{2} \left( 1 - \frac{c}{2} \right) \right) \log \left( \frac{\beta}{2} (1 - c) + z \right) \\
- \frac{\beta}{2} \left( 1 - \frac{c}{2} \right) \log \left( \frac{\beta}{2} (1 - c) \right).
\]

Now the statement follows with [23, Theorem 2.3.6].

**Remark 3.5.7.** Our computations in (3.56) show that there is no hope to observe mod-\(\phi\) convergence for the sequence \(n \log \det W_{n,p(n)}^{L,\beta} - n(\log n - 1)\).

**Theorem 3.5.8** (Rate of convergence for the log-determinant of \(\beta\)-Laguerre ensembles). In cases (a), (c) and (d) in Theorem 3.5.1, we obtain
\[
d_{Kol} \left( X_{\beta}^L(n) \sqrt{\frac{\beta^2}{2 \log n}}, N(0,1) \right) \leq C \left( D, 1, K_1, \sqrt{\frac{2}{\pi}} \right) \left( \frac{1}{t_n} \right)^{1/2},
\]
where the constant is given by (3.4) with \(D\) and \(K_1\) depending only on \(\beta\).

**Proof.** Case (a): If \(p(n) = n\), the statement is Theorem 4.11 in [16].

Case (c): Assume that \(n - p(n) = c\) for a fixed \(c \in \mathbb{N}\). We adapt the techniques and methods of the proof of Theorem 4.11 in [16]. We have to check that condition (a) in Definition 3.1.7 is satisfied, which is finding a bound on \(|\psi_n(i \xi) - 1|\). Inspired by the proof of Lemma 4.1 in [16], we first consider another representation of the limiting function in the mod-Gaussian convergence.

With the definition of the Barnes \(G\)-function as the solution of \(G \left( \frac{s}{2} \right) = G \left( \frac{s}{2} + 1 \right) \Gamma \left( \frac{s}{2} \right)\), we obtain
\[
\prod_{k=1}^{p(n)} \frac{\Gamma \left( \frac{\beta}{2} (k + c) + z \right)}{\Gamma \left( \frac{\beta}{2} (k + c) \right)} = \frac{G \left( \frac{\beta}{2} (p(n) + c) + z + 1 \right) G \left( \frac{\beta}{2} (1 + c) \right)}{G \left( \frac{\beta}{2} (1 + c) + z \right) G \left( \frac{\beta}{2} (p(n) + c) + 1 \right)}.
\]

To handle the factor which depends on \(p(n)\), we use the estimate of Proposition 17 in [63], which holds true for \(|z| \leq \frac{1}{2} p^{1/6}\) and gives
\[
\frac{G(1 + z + p)}{G(1 + p)} = (2\pi)^{z/2} e^{-(p+1)z} (1 + p)^{z^2/2 + p z} S_p(z), \tag{3.57}
\]
with \(\log S_p(z) = O \left( \frac{|z| + |z|^2}{p} \right)\). Hence we obtain
\[
L(p(n), c, \beta/2; z) = \log G \left( \frac{\beta}{2} (1 + c) \right) - \log G \left( \frac{\beta}{2} (1 + c) + z \right) \\
+ z \log(\sqrt{2\pi}) - z \left( \frac{\beta c}{2} + 1 \right) - z \frac{\beta}{2} p(n) + \frac{z^2}{2} \log \left( \frac{\beta}{2} (p(n) + c) + 1 \right) \\
+ z \frac{\beta}{2} (p(n) + c) \log \left( \frac{\beta}{2} (p(n) + c) + 1 \right) + O \left( \frac{|z| + |z|^2}{p(n)} \right).
\]

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We arrive at
\[ \Phi_{\frac{\beta}{2}}(z) := \log G\left(\frac{\beta}{2}(1 + c)\right) - \log G\left(\frac{\beta}{2}(1 + c) + z\right) + z\left(\log(\sqrt{2\pi}) - \left(\frac{\beta c}{2} + 1\right)\right) \]
Consequently, we get
\[ \psi_n(z) = \exp\left(\Phi_{\frac{\beta}{2}}(z) + r_n(z)\right), \]
with
\[ r_n(z) = O\left(\frac{|z| + |z|^2}{p(n)}\right), \]
as soon as \( z \in \mathcal{S}_{\frac{\beta}{2}} \) and \( |z| \leq \frac{\beta}{p(n)^{1/6}} \). Therefore, there exists a constant \( C \) such that for every \( n \geq 1 \) and \( |\xi| \leq \frac{\beta}{p(n)^{1/6}} \)
\[ |r_n(i\xi)| \leq C|\xi| + |\xi|^2 \leq C|\xi| e^{\xi}, \]
and further using that \( |\xi| \leq \frac{\beta}{p(n)^{1/6}} \),
\[ |r_n(i\xi)| \leq C. \]
The constant might depend on \( \beta \) and \( c \). With the inequality \( |e^z - 1| \leq |z| e^{|z|} \) for \( z \in \mathbb{C} \), we have
\[ |\psi_n(i\xi) - 1| \leq |\Phi_{\frac{\beta}{2}}(i\xi) + r_n(i\xi)| e^{\Phi_{\frac{\beta}{2}}(i\xi) + r_n(i\xi)} \]
\[ \leq e^{C\left(|\Phi_{\frac{\beta}{2}}(i\xi)| + C|\xi| e^{\xi}\right)} e^{\Phi_{\frac{\beta}{2}}(i\xi)}. \]
To achieve our goal, it is sufficient to bound \( |\Phi_{\frac{\beta}{2}}(i\xi)| \). For our purposes, in (3.58), we can neglect the constant \( \log G\left(\frac{\beta}{2}(1 + c)\right) \). Hence we try to find a bound for
\[ f(\xi) := -\log G\left(\frac{\beta}{2}(1 + c) + i\xi\right) + i\xi\left(\log(\sqrt{2\pi}) - \left(\frac{\beta c}{2} + 1\right)\right). \]
By Theorem 5.19 in [99], we have \( |f(\xi) - f(0)| \leq |\xi| \sup_{t \in (0,2)} |f'(t)| \). Now
\[ f'(t) = \frac{G'(\frac{\beta}{2}(1 + c) + it)}{G\left(\frac{\beta}{2}(1 + c) + it\right)} + i\left(\log(\sqrt{2\pi}) - \left(\frac{\beta c}{2} + 1\right)\right). \]
With (3.69) and (3.70), we have \( |\Psi\left(\frac{\beta}{2}(1 + c) + it\right)| \leq a_1|t| + a_2 \) for some positive constants \( a_1, a_2 \). Indeed we apply the inequality \( |\log(c + it)| \leq |t|, c > 0 \) (using \( \log(c + i\xi) = \int_0^1 \frac{\xi}{i(t + \xi) + c} dt \), which is valid for \( z \in \mathbb{C} \setminus (-\infty, -c] \) and let \( z = i\xi \)). With (3.71) we get
\[ |f'(t)| \leq b_1 t^2 + b_2 |t| + b_3 \]
for some positive constants \( b_1, b_2, b_3 \). Summarizing, we have
\[ |f(i\xi)| \leq c_1|\xi|^3 + c_2|\xi|^2 + c_3|\xi| \]
\[ \leq c_4|\xi| e^{c_5|\xi|}. \]
In addition to that, we use the fact that there exists a \( c_6 > 0 \) such that for every \( x \geq 0 \)
\[
c_1 x^3 + c_2 x^2 + c_3 x \leq c_6 (x^3 + 1).
\] (3.62)

We now consider (3.59) and successively plug in (3.61), (3.60) and (3.62) to obtain
\[
|\psi_n(i\xi) - 1| \leq K_1 |\xi| \exp(K_2 |\xi|^3).
\]

Therefore, if \( n - p(n) = c \), we have checked that the sequence of log-determinants of the \( \beta \)-Laguerre ensembles converges mod-Gaussian with zone of control \([-D_{tn}, D_{tn}]\) with \( D \leq (4K_2)^{-1} \) and index of control \((1, 3)\). The result follows considering \( \gamma = \min\{1, \frac{v-1}{2}\} \).

Case (d): Let us assume that \( n - p(n) = o(n) \). From Theorem 3.5.1 we obtain that
\[
\psi_n(z) = \exp(r_n(z))
\]
with
\[
r_n(z) = O\left( \frac{|z| + |z|^2}{p(n)} \right) + O\left( \frac{|z| + |z|^2}{n - p(n)} \right),
\]
as soon as \( z \in S_{\beta/2} \) and \( |z| \leq \frac{\beta}{8} \max\{p(n), n - p(n)\}^{1/4} \). Therefore, there exists a constant \( C \) such that for every \( n \geq 1 \) and \( |\xi| \leq \frac{\beta}{8} \max\{p(n), n - p(n)\}^{1/4} \)
\[
|r_n(i\xi)| \leq C\left( \frac{|\xi| + |\xi|^2}{p(n)} + \frac{|\xi| + |\xi|^2}{n - p(n)} \right) \leq C|\xi|e^{\xi^2},
\]
and further using that \( |\xi| \leq \frac{\beta}{8} \max\{p(n), n - p(n)\}^{1/4} \),
\[
|r_n(i\xi)| \leq C.
\]

The constant \( C \) might depend on \( \beta \). Therefore it is enough to apply the same trick as in the case \( n - p(n) = c \) to obtain \( |\psi_n(i\xi)| \leq K_1 |\xi|e^{K_2 |\xi|^3} \). Thus, if \( n - p(n) = c(n) \), we have checked that the sequence of log-determinants of the \( \beta \)-Laguerre ensembles converges mod-Gaussian with zone of control \([-D_{tn}, D_{tn}]\) for some \( D > 0 \) with index of control \((1, 3)\). The result follows considering \( \gamma = \min\{1, \frac{v-1}{2}\} \).

**Theorem 3.5.9** (Local limit theorem for log-determinants of \( \beta \)-Laguerre ensembles).

In case (a), (c) and (d) in Theorem 3.5.1, we obtain for any \( \mu \in (0, \frac{3}{2}) \)
\[
P\left( \frac{X_{\beta}^L(n)}{\sqrt{2/n}} - x \in (\log n)^{-\mu} B \right) \simeq (\log n)^{-\mu} e^{-x^2/2} m(B).
\]

**Proof.** The result follows immediately from the proof of Theorem 3.1.9.

**Remark 3.5.10.** The speed of convergence and the statement of a local theorem in the case of mod-stable convergence \( p(n) = p \) is not available. The reason for this is that the estimate (3.57) is not sharp enough with respect to the order of the approximation-error \( \log S_p(nz) \).
3.5.2 $\beta$-Jacobi Ensembles

A direct consequence of Theorem 3.4.3 is mod-$\phi$ convergence for the shifted log-determinants of the $\beta$-Jacobi ensembles. Remember that by (3.47) we have

$$
\log E \left[ \exp \left( z \log \left( \det W_{p(n),n_1,n_2}^{J,\beta} \right) \right) \right] = L(p(n), n_1 - p(n), \beta/2; z) - L(p(n), n_1 + n_2 - p(n), \beta/2; z).
$$

Although in subsection 3.5 the asymptotic behavior of $L(p(n), n_1 - p(n), \beta/2; z)$ has been analyzed completely, we have to add a case by case analysis of $L(p(n), n_1 + n_2 - p(n), \beta/2; z)$:

**Proposition 3.5.11.** $L(p(n), n_1 + n_2 - p(n), \beta/2; z)$ defined in (3.5) satisfies the following asymptotic expansion locally uniformly on $S_{\beta/2}$:

$$
L(p(n), n_1 + n_2 - p(n), \beta/2; z) = z \mu(p(n), n_1, n_2) + \frac{z^2}{\beta} \log \left( \frac{n_1 + n_2}{n_1 + n_2 - p(n)} \right) + o(1),
$$

where

$$
\mu(p(n), n_1, n_2) := p(n) \log(\beta/2) + (n_1 + n_2) \log(n_1 + n_2)
$$

$$
- (n_1 + n_2 - p(n)) \log(n_1 + n_2 - p(n))
$$

$$
+ \frac{1}{2} \log(2\pi) + \left( \frac{1}{2} - \frac{1}{\beta} \right) \log \left( \frac{n_1 + n_2}{n_1 + n_2 - p(n)} \right).
$$

(3.63)

**Proof.** The proof follows after some easy computations. Therefore, we will use (3.34), (3.35) as well as the expansion of $\sum_{k=0}^{p(n)-1} f_{n_1+n_2-p(n)}(k)$ as in (3.51), together with Theorem 3.3.2.

Now the combination of Theorem 3.4.3 and Proposition 3.5.11 leads to the following mod-$\phi$ structure for the log-determinant of the Jacobi-ensembles.

**Theorem 3.5.12** (Mod-$\phi$ convergence for the log-determinant of $\beta$-Jacobi ensembles). We observe the following results on $S_{1/2}$:

<table>
<thead>
<tr>
<th>Condition</th>
<th>Centered Version of $\log(\det W_{p(n),n_1,n_2}^{J,\beta})$</th>
<th>mod-$\phi$</th>
<th>$t_n$</th>
<th>Limiting Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(n) = n_1$</td>
<td>$\log(\det W_{n_1,n_1,n_2}^{J,\beta})$</td>
<td>mod-$\mathcal{N}(0,1)$</td>
<td>$\frac{z^2}{2} \log \left( \frac{n_1+n_2}{n_1+n_2-p(n)} \right)$</td>
<td>$\exp(\Phi_{\beta/2}(z))$</td>
</tr>
<tr>
<td>$n_1 - p(n) \to 0$</td>
<td>$\log(\det W_{p(n),n_1,n_2}^{J,\beta})$</td>
<td>mod-$\mathcal{N}(0,1)$</td>
<td>$\frac{z^2}{2} \log \left( \frac{n_1+n_2}{n_1+n_2-p(n)} \right)$</td>
<td>$\exp(\Phi_{\beta/2}(z))$</td>
</tr>
<tr>
<td>$n_1 - p(n) = c$</td>
<td>$\log(\det W_{p(n),n_1,n_2}^{J,\beta})$</td>
<td>mod-$\mathcal{N}(0,1)$</td>
<td>$\frac{z^2}{2} \log \left( \frac{n_1+n_2}{n_1+n_2-p(n)} \right)$</td>
<td>$\exp(\Phi_{\beta/2}(z))$</td>
</tr>
<tr>
<td>$n_1 - p(n) = o\left( \frac{n_1+n_2}{n_1+n_2-p(n)} \right)$</td>
<td>$\log(\det W_{p(n),n_1,n_2}^{J,\beta})$</td>
<td>mod-$\mathcal{N}(0,1)$</td>
<td>$\frac{z^2}{2} \log \left( \frac{n_1+n_2}{n_1+n_2-p(n)} \right)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$p(n) = p$</td>
<td>$\log(\det W_{p(n),n_1,n_2}^{J,\beta})$</td>
<td>no mod-$\phi$</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
Here the limit is the limit as \( n_1 \) and \( n_2 \) tend to \( \infty \) simultaneously.

Moreover, we obtain that
\[
\mu_1^d = \mu_1(n_1, n_1) - \mu(n_1, n_1, n_2)
\]
(defined in (3.41) and (3.63) respectively) is the expectation if \( p(n) = n_1 \).
If \( n_1 - p(n) \to 0 \) as \( n \to \infty \), it is
\[
\mu_2^d = \mu_1(p(n), n_1) - \mu(p(n), n_1, n_2).
\]
If \( n_1 - p(n) = c \) for a fixed \( c \in \mathbb{N} \), the expectation is
\[
\mu_3^d = \mu_2(p(n), n_1) - \mu(p(n), n_1, n_2)
\]
(defined in (3.42) and (3.63) respectively).
Finally, for \( n_1 - p(n) = o\left(\frac{n_1 n_2}{n_1 + n_2}\right) \), we obtain that
\[
\mu_4^d = \mu_3(p(n), n_1) - \mu(p(n), n_1, n_2)
\]
(defined in (3.43) and (3.63) respectively) is the correct expectation.

**Proof.** If \( n_1 - p(n) \to \infty \), we have to assume in addition that \( n_1 - p(n) = o\left(\frac{n_1 n_2}{n_1 + n_2}\right) \) to ensure that the parameter sequence \((t_n)_{n \in \mathbb{N}}\) is increasing. The results follow immediately from Theorem 3.4.3 and Proposition 3.5.11.

**Theorem 3.5.13.** We fix \( \tau_1, \tau_2 > 0 \) and assume that \( n_1 = \lfloor n \tau_1 \rfloor \) and \( n_2 = \lfloor n \tau_2 \rfloor \). In this regime we obtain for the centered version of \( \log \left( \det W_{p(n), \lfloor n \tau_1 \rfloor, \lfloor n \tau_2 \rfloor} \right) \) on \( S_{1/2} \):

<table>
<thead>
<tr>
<th>condition</th>
<th>mod-( \phi )</th>
<th>( t_n )</th>
<th>limiting function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(n) = \lfloor n \tau_1 \rfloor )</td>
<td>mod-( \mathcal{N}(0, 1) )</td>
<td>( \frac{2}{\beta} \log n )</td>
<td>( \exp \left( \frac{\epsilon^2}{\beta^2} \log \left( \frac{\lfloor n \tau_2 \rfloor}{\lfloor n \tau_1 + \tau_2 \rfloor} \right) + \Phi_{\beta/2}(z) \right) )</td>
</tr>
<tr>
<td>( \lfloor n \tau_1 \rfloor - p(n) \to 0 )</td>
<td>mod-( \mathcal{N}(0, 1) )</td>
<td>( \frac{2}{\beta} \log \left( n + \frac{n \tau_1 - p(n)}{\tau_2} \right) )</td>
<td>( \exp \left( \frac{\epsilon^2}{\beta^2} \log \left( \frac{\lfloor n \tau_2 \rfloor}{\lfloor n \tau_1 + \tau_2 \rfloor} \right) + \Phi_{\beta/2}(z) \right) )</td>
</tr>
<tr>
<td>( \lfloor n \tau_1 \rfloor - p(n) = c )</td>
<td>mod-( \mathcal{N}(0, 1) )</td>
<td>( \frac{2}{\beta} \log \left( n + \frac{c}{\tau_2} \right) )</td>
<td>( \exp \left( \frac{\epsilon^2}{\beta^2} \log \left( \frac{\lfloor n \tau_2 \rfloor}{\lfloor n \tau_1 + \tau_2 \rfloor} \right) + \Phi_{\beta/2}(z) \right) )</td>
</tr>
<tr>
<td>( \lfloor n \tau_1 \rfloor - p(n) = o(n) )</td>
<td>mod-( \mathcal{N}(0, 1) )</td>
<td>( \frac{2}{\beta} \log \left( \frac{n}{\lfloor n \tau_1 - p(n) \rfloor} + \frac{1}{\tau_2} \right) )</td>
<td>( \frac{\lfloor n \tau_2 \rfloor}{\lfloor n \tau_1 + \tau_2 \rfloor} \exp \left( \frac{\epsilon^2}{\beta^2} \right) )</td>
</tr>
</tbody>
</table>

**Case** \( p(n) = p \) **for a fixed** \( p \in \mathbb{N} \): The centered version of the sequence
\[
\log \left( \det W_{p, \lfloor n \tau_1 \rfloor, \lfloor n \tau_2 \rfloor} \right)_{n \in \mathbb{N}}
\]
does not converge in the sense of mod-\( \phi \) convergence. However, the sequence
\[
\left( n \log \left( \det W_{p, \lfloor n \tau_1 \rfloor, \lfloor n \tau_2 \rfloor} \right) \right)_{n \in \mathbb{N}}
\]

converges \textit{mod-}φ on \(i\mathbb{R}\) with parameter \(t_n = pn\) and limiting function

\[
\psi(z) = \left( \frac{\tau_1 + \tau_2}{\tau_1} \right)^{\frac{\tau_1 \beta + 2z}{(\tau_1 + \tau_2) \beta + 2z}}. 
\]

Here \(\phi\) is such that the Lévy exponent is given by

\[
\eta(z) = \log \int_{\mathbb{R}} e^{xz} \phi(dx) = \frac{\beta}{2} (\tau_1 + \tau_2) \log \left( \frac{\beta}{2} (\tau_1 + \tau_2) \right) - \frac{\beta}{2} \tau_1 \log \left( \frac{\beta}{2} \tau_1 \right) \\
+ \left( z + \tau_1 \frac{\beta}{2} \right) \log \left( z + \tau_1 \frac{\beta}{2} \right) - \left( z + (\tau_1 + \tau_2) \frac{\beta}{2} \right) \log \left( z + (\tau_1 + \tau_2) \frac{\beta}{2} \right).
\]

\textbf{Proof.} First notice that the choices \(n_1 = \lfloor n \tau_1 \rfloor\) and \(n_2 = \lfloor n \tau_2 \rfloor\) lead to the extra summand

\[
\frac{z^2}{\beta} \log \left( \frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \right)
\]

in the limiting function. The only case we have to consider is the case \(p(n) = p\) for a fixed \(p \in \mathbb{N}\). Here, we apply Stirling’s formula like in the proof of Theorem 3.5.1 case (e), see (3.54). We apply the formula to \(L(p, |n \tau_1| - p, \beta/2; nz)\), as well as \(L(p, |n \tau_1| + |n \tau_2| - p, \beta/2; nz)\). The computations are similar to (3.54) and are left to the reader. \(\square\)

We do not formulate the corresponding extended central limit theorems, precise deviations, large and moderate deviation principles and Berry-Esseen bounds for the log-determinants of the Jacobi ensemble, because these results can be stated as in Theorems 3.5.1 3.5.3 3.5.4 Corollary 3.5.5 and Theorem 3.5.8.

### 3.5.3 Ginibre Ensembles

As a corollary of Theorem 3.4.3 case (a), we obtain for the log-determinants of the Ginibre ensemble with (3.8)

\[
\log \mathbb{E} \left[ \left( \det W_n^{G,\beta} \right)^z \right] = z \left( \frac{n}{2} \log \left( \frac{2}{\beta} \right) \right) + \left( \frac{n}{2} \log \left( \frac{2}{\beta} \right) \right) + L(n, 0, \beta/2; z) \\
= z \mu^G(n) + \frac{z^2}{\beta} \log n + \Phi_{\beta/2}(z) + o(1),
\]

where \(\mu^G(n) = \frac{n}{2} \log \left( \frac{2}{\beta} \right) + \mu_1(n, n)\), and \(\mu_1(n, n)\) is defined in (3.41). Hence we obtain mod-Gaussian convergence, an extended central limit theorem, precise deviations, large and moderate deviation principles and Berry-Esseen bounds for the log-determinants of the Ginibre ensemble as in case (a) in Theorems 3.5.1 3.5.3 3.5.4 Corollary 3.5.5 and Theorem 3.1.9. Notice that these results follow directly from the results in [10].

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3.5.4 Ensembles in Mesoscopic Physics

Let us first consider the chiral ensembles. Here we obtained in (3.13) that
\[ E \left( \det W_{n, p(n)}^{\beta, \mu_{\text{chiral}}} \right) = L \left( p(n), n - p(n), \beta/2; \frac{z + 1}{2} \right) \]
for \( \beta \in \{1, 2, 4\} \). Hence we obtain mod-Gaussian convergence, an extended central limit theorem, precise deviations, large and moderate deviation principles and Berry-Esseen bounds for the logarithm of the product of the positive eigenvalues of the chiral ensemble as in Theorems 3.5.1, 3.5.3, 3.5.4, Corollary 3.5.5 and Theorem 3.1.9.

Next we consider one of the Bogoliubov-de Gennes ensembles. We proved in (3.14) that
\[ E \left( \det W_{n, n}^{1, 1} \right) = L \left( n, 1/2; \frac{z + 1}{2} \right) = \prod_{k=1}^{n} \frac{\Gamma \left( \frac{1}{2} (k + 1) + \frac{z + 1}{2} \right)}{\Gamma \left( \frac{1}{2} (k + 1) \right)}. \]
With the same steps as in the proof of Theorem 3.1.9 we find
\[ E \left( \det W_{n, n}^{1, 1} \right) = G \left( \frac{1}{2} (n + 1) + \frac{z + 1}{2} \right) G \left( 1 + \frac{z + 1}{2} \right). \]
Now we adapt the calculations in the proof of Theorem 3.1.9 and obtain mod-Gaussian convergence with \( t_n = \log \left( \frac{1}{2} (n + 1) + 1 \right) \sim \log n \) and limiting function \( \exp \left( \Phi_{1/2} \left( \frac{z + 1}{2} \right) \right) \), where \( \Phi_{1/2} \) is defined in (3.58).

An extended central limit theorem, precise deviations, large and moderate deviation principles and Berry-Esseen bounds for the log-determinants of the Bogoliubov-de Gennes ensembles are consequences of case (c) in Theorems 3.5.3, 3.5.4, Corollary 3.5.5 and Theorem 3.5.8.

3.5.5 Trace-fixed GUE

Recall that in (3.10), we obtained the identity
\[ E \left[ n^n / \det W_n^{H, H^*} \right] = \prod_{k=1}^{n} \frac{\Gamma \left( \frac{z + 1}{2} + \frac{k}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{k}{2} \right)} \left( \prod_{k=1}^{n} \frac{\Gamma \left( \frac{z}{2} + \frac{n}{2} + \frac{k-1}{n} \right)}{\Gamma \left( \frac{1}{2} + \frac{k}{2} \right)} \right)^{-1}. \]
Let us consider the case where \( n \) is an odd number. If \( n \) is even, one can show similarly that this case leads to the same asymptotics. From Lemma 4.1 in [16] we know that locally uniformly on the band \( S_1 \)
\[ \log \prod_{k=1}^{n} \frac{\Gamma \left( \frac{z + 1}{2} + \frac{k}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{k}{2} \right)} = z \mu_n^H + \frac{z^2}{4} \log \left( \frac{n}{2} \right) + \Phi^H(z) + o(1) \]
with \( \mu_n^H = \frac{1}{2} \log (2\pi) - n + \frac{n}{2} \log n \) (we have to adapt the result in [16] by a summand \( -\frac{n}{2} \)). The function \( \Phi^H(z) \) is defined as
\[ \Phi^H(z) = \log \left( \frac{\Gamma \left( \frac{1}{2} \right) G \left( \frac{1}{2} + \frac{z+1}{2} \right)}{\Gamma \left( \frac{z+1}{2} \right) G \left( \frac{z+1}{2} \right)^2} \right). \]
Moreover, we have
\[
\prod_{k=1}^{n} \frac{\Gamma\left(\frac{z}{2} + \frac{n}{2} + \frac{k-1}{n}\right)}{\Gamma\left(\frac{z}{2} + \frac{k-1}{n}\right)} = G\left(\frac{z}{2} + 1 + \left(\frac{n}{2} + 1 - \frac{1}{n}\right)\right) G\left(\frac{n}{2} - 1 + 1\right) G\left(\frac{z}{2} + 1 + \frac{n}{2} - 1\right) G\left(\left(\frac{z}{2} + 1 - \frac{1}{n}\right) + 1\right).
\]

Now we apply (3.57) twice for \(|z| \leq \frac{1}{2}\left(\frac{n}{2}\right)^{1/6}\) and \(z \in S_1\) to obtain
\[
\log \prod_{k=1}^{n} \frac{\Gamma\left(\frac{z}{2} + \frac{n}{2} + \frac{k-1}{n}\right)}{\Gamma\left(\frac{z}{2} + \frac{n}{2} + \frac{k-1}{n}\right)} = z f(n) + \frac{z^2}{4} \log \left(1 + \frac{4}{n} - \frac{2}{n^2}\right) + \mathcal{O}\left(\frac{|z| + |z|^2}{n}\right),
\]
where
\[
f(n) = \frac{1}{2n} - \frac{3}{4} + \frac{1}{2} \left(\frac{n}{2} + 1 - \frac{1}{n}\right) \log \left(\frac{n}{2} + 2 - \frac{1}{n}\right) - \frac{1}{2} \left(\frac{n}{2} - 1\right) \log \left(\frac{n}{2}\right).
\]

Summarizing, we observe mod-Gaussian convergence with rate
\[
t_n = \frac{1}{2} \log \left(\frac{n}{2}\right) - \frac{1}{2} \log \left(1 + \frac{4}{n} - \frac{2}{n^2}\right).
\]
limiting function \(\exp(\Psi^H(z))\) and expectation of order \(\mu_n^H + f(n)\). We skip the formulation of an extended central limit theorem, precise deviations, large and moderate deviation principles and Berry-Esseen bounds for the sum of the log-eigenvalues in the GUE fixed-trace ensemble. It can be stated similarly to the statements in Theorems 3.5.3, 3.5.4, Corollary 3.5.5 and Theorem 3.5.8.

### 3.6 Gram Ensembles, Random Parallelotopes and Simplices

To be able to prove the results for the log-volume of random parallelotopes and random simplices, we will prove the following result:

**Proposition 3.6.1.** Let \(m(n, \nu)\) be a sequence in \(n\), with values in \(\mathbb{R}\), where \(\nu > 0\) is a real number. Assume that \(m(n, \nu)\) is increasing in \(n\) with \(m(n, \nu) \leq c(\nu)n\) with a constant \(c(\nu)\) depending only on \(\nu\). Then
\[
\log \Gamma(m(n, \nu) + z) - \log \Gamma(m(n, \nu)) = z \left(\log m(n, \nu) - \frac{1}{2m(n, \nu)}\right) + \frac{z^2}{m(n, \nu)} + \mathcal{O}\left(\frac{|z| + |z|^2 + |z|^3}{m(n, \nu)}\right) \tag{3.64}
\]
for any \(z \in S_c(\nu)\) with \(|z| < (c(\nu)/4) m(n, \nu)^{1/6}\).

**Proof.** This is an easy application of the first Binet’s formula (3.67) as well as expanding the logarithm and using the estimate in the proof of (3.34). \(\square\)
For the log-volume of the parallelotope spanned by random points \(X_1, \ldots, X_{p(n)}\), as well as for the log-volume of the simplex with vertices \(X_1, \ldots, X_{p(n)+1}\) (see Section 2.6), we obtain the following results:

**Theorem 3.6.2** (Gaussian model, parallelotope and simplex).

The logarithm of the \(p(n)\)-dimensional volume of the parallelotope (see (3.15)) satisfies the results of Theorem 3.5.1, where one has to replace \(z\) by \(z/2\).

The logarithm \(\log(p(n)!V_{S_{n,p(n)}})\) of the \(p(n)\)-dimensional volume of the simplex satisfies the same mod-\(\phi\) convergence properties. We only have to change the expectations: add \(\log(p(n)+1)\) to the expectations in the parallelotope case — see (3.19).

**Proof.** The case of a parallelotope follows from (3.15) and Theorem 3.5.1. For the case of a simplex, see (3.20) and Theorem 3.5.1.

**Theorem 3.6.3** (Beta and spherical model, parallelotope). The logarithm of the \(p(n)\)-dimensional volume of the parallelotope in the beta model (see (3.16)) and in the spherical model (see (3.18)) satisfy the following results on \(S_{1/2}\):

<table>
<thead>
<tr>
<th>Condition</th>
<th>Centered Version of</th>
<th>mod-(\phi)</th>
<th>(t_n)</th>
<th>Limiting Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p(n) = n)</td>
<td>(\log(n!V_{P_{n,n}}))</td>
<td>(\text{mod-}\mathcal{N}(0,1))</td>
<td>(\frac{2}{3} \log n)</td>
<td>(\exp(\Phi_{1/2}(z) - \frac{z^2}{2}))</td>
</tr>
<tr>
<td>(n - p(n) \to 0)</td>
<td>(\log(p(n)!V_{P_{n,n}}))</td>
<td>(\text{mod-}\mathcal{N}(0,1))</td>
<td>(\frac{2}{3} \log n)</td>
<td>(\exp(\Phi_{1/2}(z) - \frac{z^2}{2}))</td>
</tr>
<tr>
<td>(n - p(n) = c)</td>
<td>(\log(p(n)!V_{P_{n,c}}))</td>
<td>(\text{mod-}\mathcal{N}(0,1))</td>
<td>(\frac{2}{3} \log n)</td>
<td>(\exp(-z^2/2))</td>
</tr>
<tr>
<td>(n - p(n) = o(n))</td>
<td>(\log(p(n)!V_{P_{n,c}}))</td>
<td>(\text{mod-}\mathcal{N}(0,1))</td>
<td>(\frac{2}{3} \log \left(\frac{n}{n-p(n)}\right))</td>
<td>(\exp(-z^2/2))</td>
</tr>
<tr>
<td>(p(n) = p)</td>
<td>(n \log(p!V_{P_{n,p}}))</td>
<td>(\text{mod-}\phi) (\text{in } i\mathbb{R})</td>
<td>(p n)</td>
<td>((1 + z)^{-\frac{p+n}{2}}\frac{y}{p(\frac{z}{p})})</td>
</tr>
</tbody>
</table>

The corresponding expectations of the log-volumes are

\[
\mu_1(n,n) + \frac{n}{n+\nu} - n \log \left(\frac{n+\nu}{2}\right),
\]

(with \(\mu_1(n,n)\) defined in (3.41)) in the case \(p(n) = n\).

If \(n - p(n) \to 0\) as \(n \to \infty\), it is

\[
\mu_1(p(n),n) + \frac{p(n)}{n+\nu} - p(n) \log \left(\frac{n+\nu}{2}\right).
\]

If \(n - p(n) = c\) for a fixed \(c \in \mathbb{N}\), the expectation is

\[
\mu_2(p(n),n) + \frac{p(n)}{n+\nu} - p(n) \log \left(\frac{n+\nu}{2}\right).
\]
Finally, for \( n - p(n) = o(n) \), we have

\[
\mu_3(p(n), n) + \frac{p(n)}{n + \nu} - p(n) \log \left( \frac{n + \nu}{2} \right).
\]

Here \( \mu_2(p(n), n) \) and \( \mu_3(p(n), n) \) are defined in (3.42) and (3.43).

If \( p(n) = p \), we obtain the mod-\( \phi \) on \( i\mathbb{R} \) result with Lévy exponent

\[
\eta(z) = \frac{z}{2} \log \left( \frac{1}{2} \right)
\]

and expectation \( \mu_4 = p \log 2 \). The case \( \nu = 0 \) leads to the spherical model.

**Proof.** The proof is the same as the proof of Lemma 4.2 in [16]. We apply Proposition 3.6.1 with \( m(n, \nu) = \frac{n + \nu}{2} \). We get

\[
p(n)( - \log \Gamma(m(n, \nu) + z/2) + \log \Gamma(m(n, \nu))) = \frac{z}{2} p(n) \left( \frac{1}{n + \nu} - \log \left( \frac{n + \nu}{2} \right) \right) - \frac{p(n)}{2(n + \nu)} z^2 + O\left( \frac{|z| + |z|^2 + |z|^3}{n} \right).
\]

The results follow with (3.16) and (3.18), together with Theorem 3.4.3 and 3.5.1. Finally, we have to consider the case \( p(n) = p \). From (3.54) we observe that

\[
L\left( p, n - p, \frac{1}{2}, \frac{n z}{2} \right) \sim n \frac{z}{2} \left( p \log n - p \right) + p n \left( \frac{1}{2} \log 2 + \left( \frac{z}{2} + \frac{1}{2} \right) \log \left( \frac{z}{2} + \frac{1}{2} \right) \right)
\]

\[+ \left( - \frac{p(p - 1)}{4} - \frac{p}{2} \right) \log (1 + z). \quad (3.65)\]

With (3.53) we have

\[
p \log \Gamma\left( \frac{n + \nu}{2} \right) - p \log \Gamma\left( \frac{n + \nu}{2} + \frac{n z}{2} \right) \sim n \frac{z}{2} \left( p \log \frac{n}{2} \right)
\]

\[+ p n \left( - \left( \frac{z}{2} + \frac{1}{2} \right) \log (1 + z) \right) - p \left( \frac{\nu - 1}{2} \right) \log (1 + z).
\]

Hence the statement is proven. \( \square \)

**Remark 3.6.4.** Recall that \( \Phi_{1/2}(\cdot) \) has the following expression, see [16] Lemma 7.1 (2)]:

\[
\Phi_{1/2}(z) = z \left( - \frac{1}{2} \log \frac{1}{2} + \frac{1}{2} \log(2\pi) \right) - \frac{1}{2} \log G(1 + 2z) - \frac{1}{2} \left( \log \Gamma\left( \frac{1}{2} \right) - \log \Gamma\left( \frac{1}{2} + z \right) \right).
\]
Theorem 3.6.5 (Beta and spherical model, simplex). The logarithm of the \( p(n) \)-dimensional volume of the simplex in the beta model (see (3.20)) and in the spherical model (see (3.22)) satisfies the following results on \( S_{1/2} \):

<table>
<thead>
<tr>
<th>Condition</th>
<th>Centered Version of</th>
<th>( \text{mod-} \phi )</th>
<th>( t_n )</th>
<th>Limiting Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(n) = n )</td>
<td>( \log(n!VS_{n,n}) )</td>
<td>( \text{mod-}N'(0,1) )</td>
<td>( \frac{3}{p} \log n )</td>
<td>( \exp(\Phi_{1/2}(z) - \frac{z^2}{2}) )</td>
</tr>
<tr>
<td>( n - p(n) \to 0 )</td>
<td>( \log(p(n)!VS_{n,p(n)}) )</td>
<td>( \text{mod-}N(0,1) )</td>
<td>( \frac{3}{p} \log n )</td>
<td>( \exp(\Phi_{1/2}(z) - \frac{z^2}{2}) )</td>
</tr>
<tr>
<td>( n - p(n) = c )</td>
<td>( \log(p(n)!VS_{n,p(n)}) )</td>
<td>( \text{mod-}N(0,1) )</td>
<td>( \frac{3}{p} \log n )</td>
<td>( \exp(\Phi_{1/2}(z) - \frac{z^2}{2}) )</td>
</tr>
<tr>
<td>( n - p(n) = o(n) )</td>
<td>( \log(p(n)!VS_{n,p(n)}) )</td>
<td>( \text{mod-}N(0,1) )</td>
<td>( \frac{3}{p} \log \left( \frac{n}{-p(n)} \right) )</td>
<td>( \exp(-z^2/2) )</td>
</tr>
<tr>
<td>( p(n) = p )</td>
<td>( n \log(p!VS_{n,p}) )</td>
<td>( \text{mod-} \phi ) in ( i \mathbb{R} )</td>
<td>( p n )</td>
<td>( \psi^p(z) )</td>
</tr>
</tbody>
</table>

The corresponding expectations of the log-volumes are

\[
\mu_1(n,n) + \frac{n+1}{n+\nu} - (n+1) \log \left( \frac{n+\nu}{2} \right) + 2 \log n - \log 2
\]

(with \( \mu_1(n,n) \) defined in (3.41)) in the case \( p(n) = n \).
If \( n - p(n) \to 0 \) as \( n \to \infty \), it is

\[
\mu_1(p(n),n) + \frac{p(n)+1}{n+\nu} - (p(n)+1) \log \left( \frac{n+\nu}{2} \right) + \log \left( n \, p(n) \right) - \log 2.
\]

If \( n - p(n) = c \) for a fixed \( c \in \mathbb{N} \), the expectation is

\[
\mu_2(p(n),n) + \frac{p(n)+1}{n+\nu} - (p(n)+1) \log \left( \frac{n+\nu}{2} \right) + \log \left( n \, p(n) \right) - \log 2,
\]

with \( \mu_2(p(n),n) \) defined in (3.42).
Finally for \( n - p(n) = o(n) \), we obtain that

\[
\mu_3(p(n),n) + \frac{p(n)+1}{n+\nu} - (p(n)+1) \log \left( \frac{n+\nu}{2} \right) + \log \left( n \, p(n) \right) - \log 2,
\]

with \( \mu_3(p(n),n) \) defined in (3.43), is the correct expectation.

If \( p(n) = p \) we obtain the mod-\( \phi \) on \( i \mathbb{R} \) result with the Lévy exponent

\[
\eta(z) = (p+1)\left( \frac{z+1}{2} \right) \log \left( \frac{(z+1)(p+1)}{(z+1)p+1} \right) - \frac{pz}{2} \log 2 - \left( \frac{z+1}{2} \right) \log(z+1),
\]

expectation \( \nu_4 = p \log 2 \) and limiting function

\[
\psi^p(z) = (1+z)^{-\frac{p(p+1)}{2}-\frac{z}{2}(p+1)} \left( \frac{z+1}{2} \right)^{\frac{p(p+1)}{2}} \left( \frac{(z+1)(p+1)}{(z+1)p+1} \right)^{p(p-2)+p-1}.
\]

The case \( \nu = 0 \) leads to the spherical model.
Proof. With (3.20), we first have to consider

\[(p(n) + 1) \left( - \log \Gamma \left( m(n, \nu) + \frac{z}{2} \right) + \log \Gamma (m(n, \nu)) \right) \]

This term can be handled using Proposition 3.6.1 with \(m(n, \nu) = \frac{n + \nu}{2}\) and is equal to

\[z \left( \frac{1}{n + \nu} - \log \left( \frac{n + \nu}{2} \right) \right) - \frac{p(n) + 1}{2(n + \nu)} z^2 + O \left( \frac{|z| + |z|^2 + |z|^3}{n} \right) \]

Moreover, with Proposition 3.6.1 we obtain

\[\log \Gamma \left( \frac{p(n)(n+\nu-2)+(n+\nu)}{2} + (p(n) + 1) \frac{z}{2} \right) = \frac{z}{2} \left( \log(p(n) n) - \log 2 \right) + o(1) \]

The results follow with (3.20) and (3.22), together with Theorem 3.6.3. Finally, we have to consider the case \(p(n) = p\). With (3.53) we have

\[(p + 1) \log \Gamma \left( \frac{n + \nu}{2} \right) - (p + 1) \log \Gamma \left( \frac{n + \nu}{2} + \frac{n z}{2} \right) \sim n \frac{z}{2} \left( (p + 1) \left( 1 - \log \frac{n}{2} \right) \right) \\
+ (p + 1) n \left( - \left( \frac{z}{2} + \frac{1}{2} \right) \log (1 + z) \right) - (p + 1) \left( \frac{\nu}{2} - \frac{1}{2} \right) \log (1 + z) \]

Similarly we get

\[\log \Gamma \left( \frac{n}{2} (z + 1) (p + 1) + \frac{p(\nu - 2) + \nu}{2} \right) - \log \Gamma \left( \frac{n}{2} (z + 1) p + 1 + \frac{p(\nu - 2) + \nu}{2} \right) \]
\[\sim n \frac{z}{2} \left( \log \frac{n}{2} - 1 \right) + \frac{n}{2} (z + 1) (p + 1) \log \left( \frac{z + 1 (p + 1)}{(z + 1) p + 1} \right) \]
\[+ \frac{p(\nu - 2) + \nu}{2} - \frac{1}{2} \right) \log \left( \frac{z + 1 (p + 1)}{(z + 1) p + 1} \right) \]

Hence the statement is proven. \(\square\)

Interestingly enough, the Beta prime model (see (3.17) and (3.21)) behaves differently. The reason for this is that in (3.17), the first summand is

\[p(n) \left( \log \Gamma \left( \frac{\nu}{2} - \frac{z}{2} \right) - \log \Gamma \left( \frac{\nu}{2} \right) \right) =: p(n) g(z) \]

where \(g(z)\) can be represented, for example, with the help of Binet’s first formula (3.25).

However, if \(p(n)\) depends on \(n\), this term is never part of the expectation of the log-volume of the random parallelootope. Moreover, \(L(p(n), n - p(n), 1/2; z/2)\) leads to the parameter sequence \(t_n = 2 \log n\), which does not correspond to the sequence \(p(n)\). Hence \(g(z)\) is not part of the \(\eta\)-function in the definition of mod-\(\phi\) convergence. With (3.21) the same is true for the log-volume of a random simplex in the Beta prime model. The only case where we expect mod-\(\phi\) convergence is for a fixed \(p\).
Theorem 3.6.6 (Beta prime model, parallelotope and simplex).

Consider the sequence
\[ n \left( \log(p!VP_{n,p}) - (p \log n - p) \right). \]
It converges mod-\( \phi \) on \( i \mathbb{R} \) with parameters \( t_n = pn \) and limiting function
\[ \psi(z) = e^p(1 + z) - \frac{\Gamma \left( \frac{p}{2} - \frac{z}{2} \right)}{\Gamma \left( \frac{p}{2} \right)} \cdot \frac{\Gamma \left( \frac{\nu(p+1)}{2} - \frac{p}{2} \right)}{\Gamma \left( \frac{\nu(p+1)}{2} - (p+1) \frac{z}{2} \right)}. \]
Here the Lévy exponent is
\[ \eta(z) = \frac{1}{2} \log 2 + \left( \frac{z + 1}{2} \right) \log \left( \frac{z + 1}{2} \right). \tag{3.66} \]

Next, consider the sequence
\[ n \left( \log(p!VS_{n,p}) - (p \log n - p) \right). \]
It converges mod-\( \phi \) on \( i \mathbb{R} \), with parameters \( t_n = pn \), limiting function
\[ \psi(z) = e^{p+1}(1 + z) - \frac{\Gamma \left( \frac{p}{2} - \frac{z}{2} \right)}{\Gamma \left( \frac{p}{2} \right)} \cdot \frac{\Gamma \left( \frac{\nu(p+1)}{2} - \frac{p}{2} \right)}{\Gamma \left( \frac{\nu(p+1)}{2} - (p+1) \frac{z}{2} \right)}, \]
and the same Lévy exponent (3.66).

Proof. This is an immediate consequence of (3.17) and (3.21) combined with (3.65). \( \square \)

We do not formulate the corresponding extended central limit theorems, precise deviations, large and moderate deviation principles and Berry-Esseen bounds for the log-volumes of random parallelotopes and simplices, because these results can be stated as in Theorems 3.5.1, 3.5.3, 3.5.4, Corollary 3.5.5 and Theorem 3.5.8.

3.7 Appendix

In the appendix we are collecting some well known facts about the Gamma function and the Barnes \( G \) function. For \( z \in \mathbb{C} \) with \( \text{Re}(z) > 0 \), the complex Gamma function is given by
\[ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \]
The first Binet’s formula states that
\[ \log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + 1 + \int_0^\infty \varphi(s)(e^{-sz} - e^{-s}) ds, \quad \text{Re}(z) > 0, \tag{3.67} \]
see [108] p.242. Here, the function $\varphi$ is given by

$$\varphi(s) = \left(\frac{1}{2} - \frac{1}{s} + \frac{1}{e^s - 1}\right) \frac{1}{s},$$

and for every $s \geq 0$ satisfies $0 < \varphi(s) \leq \lim_{s \to 0} \varphi(s) = \frac{1}{12}$. The second Binet’s formula is

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log(2\pi) + 2 \int_0^\infty \frac{\arctan(s/z)}{e^{2\pi s} - 1} \, ds, \quad \text{Re}(z) > 0, \quad (3.68)$$

where $\arctan y := \int_0^y \frac{dt}{1+t^2}$ for any complex $y$, with integration along a straight line, see [108] p.243.

Differentiating (3.67), the derivative of the logarithm of the Gamma function $\psi(z)$, called the Digamma function, can be represented as

$$\Psi(z) = \log z - \int_0^\infty e^{-sz}\left(s \varphi(s) + \frac{1}{2}\right) \, ds, \quad (3.69)$$

and

$$0 < s \varphi(s) + \frac{1}{2} < 1. \quad (3.70)$$

The Barnes $G$ function is defined as the solution of

$$G(z + 1) = G(z) \Gamma(z).$$

Its derivative satisfies (see [108] p.258])

$$\frac{G'(z)}{G(z)} = (z - 1)\Psi(z) - z + \frac{1}{2} \log(2\pi) + \frac{1}{2}. \quad (3.71)$$

It is known that

$$\log G(z + 1) = \frac{z(z - 1)}{2} + \frac{z}{2} \log(2\pi) + z \log \Gamma(z) - \int_0^z \log \Gamma(x) \, dx, \quad \text{Re}(z) > 0, \quad (3.72)$$

see [13]. If we put the second Binet’s formula (3.68) into (3.72), we obtain for all $z \in \mathbb{C}$ with $\text{Re}(z) > 0$

$$\log G(z + 1) = \frac{z^2}{2} - \frac{3z^2}{2} + z \log(2\pi) + z\left(z - \frac{1}{2}\right) \log z - \frac{z}{2} \left(1 - \frac{z}{2} + (z - 1) \log z\right)$$

$$+ \frac{z^2}{2} - \frac{z}{2} \log(2\pi) - 2 \int_0^\infty \frac{\frac{1}{2} \log(z^2 + s^2) - s \log s}{e^{2\pi s} - 1} \, ds,$$

which gives

$$\log G(z + 1) = \frac{z^2}{2} \log z - \frac{3z^2}{4} + \frac{z}{2} \log(2\pi) - \int_0^\infty \log(1 + z^2 s^{-2}) \frac{s \, ds}{e^{2\pi s} - 1}. \quad (3.73)$$

We apply the famous Abel-Plana summation formula, see [89] p. 290: 102
Theorem 3.7.1. Let $f$ be a holomorphic function on the strip $\{z \in \mathbb{C} : 0 \leq \operatorname{Re}(z) \leq n\}$. Suppose that $f(z) = o(e^{2\pi|\operatorname{Im}(z)|})$ as $\operatorname{Im}(z) \to \pm \infty$, uniformly with respect to $\operatorname{Re}(z) \in [0,n]$. Then

$$\sum_{k=0}^{n-1} f(k) = \int_0^n f(s) \, ds + \frac{1}{2} f(0) - \frac{1}{2} f(n) + i \int_0^\infty \frac{f(is) - f(-is)}{e^{2\pi s} - 1} \, ds$$

$$- i \int_0^\infty \frac{f(n + is) - f(n - is)}{e^{2\pi s} - 1} \, ds.$$
4 Moment Estimates of Rosenthal Type via Cumulants

4.1 Preliminaries: Cumulants

In this section, we come back to cumulants. We will state some of their most important properties and will mention their connection to moments. As a reference, see e.g. chapter 3 of \[90\].

Let $X$ be a real-valued random variable having finite absolute moments of all orders. Recall from section 2.2.1 that the $j$-th cumulant of $X$ (also known as the $j$-th semi-invariant) is defined as

$$
\kappa_j(X) = (-i)^j \left. \frac{\partial^j}{\partial t^j} \log \mathbb{E}[e^{itX}] \right|_{t=0}.
$$

It is easy to verify that

$$
\kappa_1(X) = \mathbb{E}[X] \quad \text{and} \quad \kappa_2(X) = \text{Var}(X).
$$

Furthermore, we have the following properties

**Proposition 4.1.1.** Let $X$ and $Y$ be independent real-valued random variables and $c \in \mathbb{R}$. Then

(i) $\kappa_1(X + c) = \kappa_1(X) + c$ and $\kappa_j(X + c) = \kappa_j(X)$ for $j \geq 2$,

(ii) $\kappa_j(cX) = c^j \kappa_j(X)$ for all $j \geq 1$,

(iii) $\kappa_j(X + Y) = \kappa_j(X) + \kappa_j(Y)$ for all $j \geq 1$.

Now if $N_{\mu, \sigma} \sim \mathcal{N}(\mu, \sigma^2)$ is a Gaussian random variable, then its characteristic function is given by

$$
\mathbb{E}[e^{itN_{\mu, \sigma}}] = \exp \left( i\mu t - \frac{1}{2} \sigma^2 t^2 \right),
$$

and hence $\kappa_1(N_{\mu, \sigma}) = \mu$, $\kappa_2(N_{\mu, \sigma}) = \sigma^2$ and $\kappa_j(N_{\mu, \sigma}) = 0$ for all $j \geq 3$. In fact, the normal distribution is the only law having the property that all cumulants of order greater than 2 vanish, making this a characterization of the normal distribution. Additionally, we have the following characterization of a central limit theorem.
Theorem 4.1.2. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of real-valued random variables having finite moments of all orders and \(N \sim \mathcal{N}(0, 1)\). Then

\[ X_n \overset{D}{\to} N \quad \text{if and only if} \quad \kappa_2(X_n) \overset{n \to \infty}{\to} 1 \quad \text{and} \quad \kappa_j(X_n) \overset{n \to \infty}{\to} 0 \quad \text{for all} \quad j \neq 2. \]

This is an immediate consequence of the usual method of moments and the fact that moments can be expressed in terms of cumulants and vice versa. More precisely, we have the following identity due to Leonov and Shiryaev [68]. Let \(X\) be real-valued random variable with finite moments up to order \(k\). Then

\[ \mathbb{E}[X^k] = \sum_{j=1}^{k} \frac{1}{j!} \sum_{k_1 + \cdots + k_j = k} \frac{k!}{k_1! \cdots k_j!} \kappa_{k_1}(X) \cdots \kappa_{k_j}(X). \quad (4.1) \]

With a similar identity we can express the \(k\)-th cumulant as a polynomial of degree \(k\) in the first \(k\) moments, see e.g. equation (3.2.19) in [90].

Assume that \((Z_n)_{n \in \mathbb{N}}\) is a sequence of centered random variables with unit variance having finite moments of all orders. Furthermore, assume that \((Z_n)_{n \in \mathbb{N}}\) satisfies a central limit theorem, which is equivalent to saying that all cumulants of order greater than two converge to zero. If in addition one knows exact bounds on the cumulants, one is able to describe the asymptotic behavior more precisely. If

\[ |\kappa_j(Z_n)| \leq \frac{(j!)^{1+\gamma}}{\Delta_n^{1+2\gamma}} \quad (4.2) \]

for all \(j = 3, 4, \ldots\) and all \(n \geq 1\) for fixed \(\gamma \geq 0\) and \(\Delta_n > 0\), one obtains the following bound in Kolmogorov distance (see [102, Corollary 2.2])

\[ \sup_{x \in \mathbb{R}} \left| P(Z_n \leq x) - \Phi(x) \right| \leq c_\gamma \Delta_n^{-\frac{1}{1+2\gamma}}, \]

where \(c_\gamma\) is a constant only depending on \(\gamma\) and \(\Phi\) is the standard normal distribution function

\[ \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt. \]

By considering (4.2) for example for \(j = 3\), it is clear that \(\Delta_n \to \infty\) as \(n \to \infty\) and hence the distribution function \(F_n\) of \(Z_n\) converges uniformly to \(\Phi\) as \(n \to \infty\). Therefore, when \(x = O(1)\), we have

\[ \lim_{n \to \infty} \frac{1 - F_n(x)}{1 - \Phi(x)} = 1. \quad (4.3) \]

One is interested to have – under additional conditions – such a relation in the case when \(x_n\) tends to \(\infty\) as \(n \to \infty\). In particular, one is interested in conditions for which the relation (4.3) holds in the interval \(0 \leq x \leq f(n)\), where \(f(n)\) is a non-decreasing function such that \(f(n) \to \infty\). If the relation holds in such an interval, we call the interval a zone of normal convergence.

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For i.i.d. partial sums, the classical result due to Cramér is that if $\mathbb{E} e^{t |X_1|^{1/2}} < \infty$ for some $t > 0$, then (4.3) holds with $f(n) = o(n^{1/6})$. In [102, Chapter 2], relations of large deviations of the type (4.3) were proven under the condition (4.2) on cumulants with a zone of normal convergence of size proportional to $\Delta_n^{1+2\gamma}$, see Lemma 2.3 in [102].

4.2 The Main Result

Theorem 4.2.1. For any $n \in \mathbb{N}$, let $Z_n$ be a centered random variable with unit variance and existing absolute moments, which satisfies

$$|\kappa_j(Z_n)| \leq \frac{C_{j,\gamma}}{\Delta_n^{j-2}} \text{ for all } j = 3, 4, \ldots$$

(4.4)

for a constant $C_{j,\gamma}$ depending on $j$ and a fixed $\gamma > 0$ and $\Delta_n > 0$. Let $N \sim \mathcal{N}(0, 1)$ be a standard normally distributed random variable. Then for any $k = 3, 4, \ldots$ we obtain

$$|\mathbb{E}[Z_n^k] - \mathbb{E}[N^k]| \leq \sum_{1 \leq j \leq \left\lfloor \frac{k}{2} - 1 \right\rfloor} A_{j,k} \frac{1}{\Delta_n^{k-2j}}$$

with

$$A_{j,k} := \frac{1}{j!} \sum_{k_1 + \cdots + k_j = k, k_i \geq 2} C_{k_1,\gamma} \cdots C_{k_j,\gamma} \frac{k!}{k_1! \cdots k_j!}.$$

Here $\left\lceil \cdot \right\rceil$ denotes the ceiling function, meaning that $\left\lceil \frac{k}{2} - 1 \right\rceil = \frac{k}{2} - 1$ if $k$ is even and $\left\lceil \frac{k}{2} - 1 \right\rceil = \frac{k}{2} - \frac{1}{2}$ when $k$ is odd. For an even $k = 2l$, assuming that $\Delta_n \to \infty$ we obtain

$$|\mathbb{E}[Z_n^k] - \mathbb{E}[N^k]| \leq C_1(k, \gamma) \frac{1}{\Delta_n^{2}}$$

with a constant $C_1(k, \gamma)$ only depending on $k$ and $\gamma$. For an odd $k = 2l + 1$, assuming that $\Delta_n \to \infty$ we obtain

$$|\mathbb{E}[Z_n^k]| \leq C_2(k, \gamma) \frac{1}{\Delta_n^{2}}$$

with a constant $C_2(k, \gamma)$ only depending on $k$ and $\gamma$.

Corollary 4.2.2. For any $n \in \mathbb{N}$, let $Z_n$ be a centered random variable with variance one and existing absolute moments, which satisfies

$$|\kappa_j(Z_n)| \leq \frac{(j!)^{1+\gamma} \tilde{C}_j}{\Delta_n^{j-2}} \text{ for all } j = 3, 4, \ldots$$

(4.5)

Then for any $k = 3, 4, \ldots$ we obtain

$$|\mathbb{E}[Z_n^k] - \mathbb{E}[N^k]| \leq (k!)^{1+\gamma} \sum_{1 \leq j \leq \left\lfloor \frac{k}{2} \right\rfloor} \tilde{A}_{j,k} \frac{1}{\Delta_n^{k-2j}}$$
with
\[
\tilde{A}_{j,k} := \frac{1}{j!} \sum_{k_1+\cdots+k_j = k, k_i \geq 2} \tilde{C}_{k_1} \cdots \tilde{C}_{k_j} \frac{k!}{k_1! \cdots k_j!}.
\]

**Remark 4.2.3.** In our result, the rate of convergence of moments only depends on \(\Delta_n\) but *not* on the value \(\gamma\). The value \(\gamma\) only influences the size of the constants \(C_i(k, \gamma)\). This is remarkable, since under condition (4.5) the zone of normal convergence is of size \(\Delta_n^{1+2\gamma}\), heavily depending on \(\gamma\).

**Proof of Theorem 4.2.1.** By our assumptions, we have \(\mathbb{E}[Z_n] = \kappa_1(Z_n) = 0\) and \(\text{Var}(Z_n) = \kappa_2(Z_n) = 1\). Using formula (4.1) to express moments of order \(k\) in terms of the cumulants \(\kappa_1(Z_n), \ldots, \kappa_k(Z_n)\), we get
\[
\mathbb{E}[Z_n^k] = \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{1}{j!} \sum_{k_1+\cdots+k_j = k, k_i \geq 2, i=1, \ldots, j} \frac{k!}{k_1! \cdots k_j!} \kappa_{k_1}(Z_n) \cdots \kappa_{k_j}(Z_n),
\]  
(4.6)

Note that the sum only runs up to \(\lfloor k/2 \rfloor\), because \(X\) is centered. If \(j > k/2\), then one of the \(k_i\) is necessarily equal to 1 so that the corresponding term vanishes. Furthermore, we can restrict the inner sum to indices such that \(k_i \geq 2\) for all \(i\).

First, let us assume that \(k\) is an even number. Now the summand with \(j = k/2\) on the right hand side of (4.6) is equal to
\[
\frac{k!}{2^{k/2} (k/2)!} \left(\kappa_2(Z_n)\right)^{k/2} = \mathbb{E}[N^k].
\]

Now we apply (4.4) and obtain
\[
|\kappa_{k_1}(Z_n) \cdots \kappa_{k_j}(Z_n)| \leq C_{k_1, \gamma} \cdots C_{k_j, \gamma} \frac{1}{\Delta_n^{k-2j}}.
\]

With the definition of \(A_{j,k}\) we obtain the result
\[
|\mathbb{E}[Z_n^k] - \mathbb{E}[N^k]| \leq \sum_{1 \leq j \leq k/2 - 1} A_{j,k} \frac{1}{\Delta_n^{k-2j}}.
\]

If \(k\) is an odd number, then \(\mathbb{E}[N^k] = 0\), and hence with the same arguments we observe
\[
|\mathbb{E}[Z_n^k] - \mathbb{E}[N^k]| \leq \sum_{1 \leq j \leq k/2 - 1} A_{j,k} \frac{1}{\Delta_n^{k-2j}}.
\]

If \(k\) is even, the leading term in the bound is the summand with \(j = k/2 - 1\) yielding \(\frac{1}{\Delta_n^{k/2-1}}\). If \(k\) is odd, the leading term in the bound is the summand with \(j = k/2 - 1\) yielding \(\frac{1}{\Delta_n^{k/2}}\).
Proof of Corollary 4.2.2. With (4.5), we apply Hölder’s inequality to the Gamma function to see that \((k_j!)^{1/j} \leq k!\). Hence \(k_1! \cdots k_j! \leq k^{j! - 1} \frac{k^{j!}}{k^{j-1}} = k!\). Summarizing we obtain
\[
|\kappa_{k_1}(Z_1) \cdots \kappa_{k_j}(Z_j)| \leq k!^{1+\gamma} \tilde{C}_{k_1} \cdots \tilde{C}_{k_j} \frac{1}{\Delta n^{2j}}.
\]
With (4.1) the proof is the same as for Theorem 4.2.1. □

In [106, Theorem 4], a first result on the convergence of moments for a partial sum of independent random variables was obtained. The results were improved in [49] and [50]. Results from [14, p. 208] can be used to derive a rate of convergence in the classical central limit theorem for moments: let \((X_i)\) be an i.i.d. sequence of random variables with zero mean and unit variance, and let \(Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i\). If \(0 < p < 4\) and \(E[X_i^4] < \infty\), and \(X_1\) satisfies Cramér’s continuity condition \(\limsup_{t \to \infty} |E[e^{itX_1}]| < \infty\), then Theorem 20.1 in [14] implies
\[
E[|Z_n|^p] = E[|N|^p] + c_p \frac{1}{n} + o(n^{-1})
\]
as \(n \to \infty\), where the constant \(c_p\) depends only on \(p\) and the first four moments of \(X_1\).

Our Theorem 4.2.1 opens up the possibility to prove moment estimates for a wide range of dependent random variables. Before we proceed, we start with a warm up: we consider a partial sum of independent, non-identically distributed random variables.

**Theorem 4.2.4.** Let \((X_i)_{i \geq 1}\) be a sequence of independent real-valued random variables with expectation zero and variances \(\sigma_i^2 > 0\), \(i \geq 1\), and let us assume that there exist \(\gamma \geq 0\) and \(K > 0\) such that for all \(i \geq 1\)
\[
|E[X_i^j]| \leq (j!)^{1+\gamma} K^{j-2} \sigma_i^2 \quad \text{for all } j = 3, 4, \ldots. \tag{4.7}
\]
Let \(Z_n := \frac{1}{\sqrt{\sum_{i=1}^{n} \sigma_i}} \sum_{i=1}^{n} X_i\). Then we obtain for all \(k \geq 2\)
\[
|E[Z_n^{2k}] - E[N^{2k}]| \leq C_1(k) \frac{4 \max \left\{ K^2, \max_{1 \leq i \leq n} \{ \sigma_i^2 \} \right\} \sum_{i=1}^{n} \sigma_i^2}{\sum_{i=1}^{n} \sigma_i^2},
\]
and
\[
|E[Z_n^{2k+1}]| \leq C_2(k) \frac{2 \max \left\{ K, \max_{1 \leq i \leq n} \{ \sigma_i \} \right\}}{(\sum_{i=1}^{n} \sigma_i^2)^{1/2}}.
\]

Remark that condition (4.7) is a generalization of the classical Bernstein condition \((\gamma = 0)\).
Proof. Using a relation between moments and cumulants, condition \((4.7)\) implies that the \(j\)-th cumulant of \(X_i\) can be bounded by \((j!)^{1+\gamma}(2 \max \{K, \sigma_i\})^{j-2}\sigma_i^2\). Hence it follows from the independence of the random variables \(X_i, i \geq 1\), that the \(j\)-th cumulant of \(Z_n\) possesses the bound

\[
|\kappa_j(Z_n)| \leq (j!)^{1+\gamma} \left( \frac{2 \max \{K, \max_{1 \leq i \leq n} \{\sigma_i\}\}}{\sqrt{\sum_{i=1}^n \sigma_i^2}} \right)^{j-2},
\]

for details see for example [102, Theorem 3.1]. Thus for \(Z_n\), the condition of Theorem 4.2.1 holds with

\[
\Delta_n = \frac{\sqrt{\sum_{i=1}^n \sigma_i^2}}{2 \max \{K, \max_{1 \leq i \leq n} \{\sigma_i\}\}}.
\]

The result follows from Theorem 4.2.1.

Remark 4.2.5. If Cramér’s condition holds, that is there exists \(\lambda > 0\) such that \(\mathbb{E} e^{\lambda |X_i|} < \infty\) holds for all \(i \in \mathbb{N}\), then \(X_i\) satisfies Bernstein’s condition, which is the bound \((4.7)\) with \(\gamma = 0\), see for example [110, Remark 3.6.1]. This implies \((4.8)\) and we can apply Theorem 4.2.1 as above. Therefore Theorem 4.2.1 requires less restrictions on the random sequence than Cramér’s condition.

4.3 Applications

4.3.1 Uniform Control on Cumulants and Dependency Graphs

Let us start with the definition of a dependency graph due to [54]:

Definition 4.3.1. Let \(\{X_\alpha\}_{\alpha \in \mathcal{I}}\) be a family of random variables defined on a common probability space. A dependency graph for \(\{X_\alpha\}_{\alpha \in \mathcal{I}}\) is any graph \(L\) with vertex set \(\mathcal{I}\) that satisfies the following condition: For any two disjoint subsets of vertices \(V_1\) and \(V_2\) such that there is no edge from any vertex in \(V_1\) to any vertex in \(V_2\), the corresponding collections of random variables \(\{X_\alpha\}_{\alpha \in V_1}\) and \(\{X_\alpha\}_{\alpha \in V_2}\) are independent, see [54].

Let the maximal degree of a dependency graph \(L\) be the maximum number of edges coinciding at one vertex of \(L\). The idea behind the usefulness of dependency graphs is that if the maximal degree is not too large, one expects a central limit theorem for the partial sums of the family \(\{X_\alpha\}_{\alpha \in \mathcal{I}}\).

Example 4.3.2. A standard situation is that there is an underlying family of independent random variables \(\{Y_i\}_{i \in \mathcal{A}}\), and each \(X_\alpha\) is a function of the variables \(\{Y_i\}_{i \in \mathcal{A}_\alpha}\), for some \(\mathcal{A}_\alpha \subset \mathcal{A}\). With \(\mathcal{S} = \{\mathcal{A}_\alpha : \alpha \in \mathcal{I}\}\), the graph \(L = L(\mathcal{S})\) with vertex set \(\mathcal{I}\) and edge set \(\{\alpha \beta : \mathcal{A}_\alpha \cap \mathcal{A}_\beta \neq \emptyset\}\) is a dependency graph for the family \(\{X_\alpha\}_{\alpha \in \mathcal{I}}\). As a special case of this example, we will consider subgraphs in the Erdős-Rényi random graph model.
that is $G$ has vertex set $\{1, \ldots, n\}$, and it has an edge between $i$ and $j$ with probability $p_n$, all these events being independent from each other. Let $\mathcal{I}$ be the set of 3-element subsets of $\{1, \ldots, n\}$, and if $\alpha = \{i, j, k\} \in \mathcal{I}$, let $X_\alpha$ be the indicator function of the event the graph $G$ contains the triangle with vertices $i, j$ and $k$. Let $L$ be the graph with vertex set $\mathcal{I}$ and the following edge set: $\alpha$ and $\beta$ are linked if $|\alpha \cap \beta| = 2$ (that is, if the corresponding triangles share an edge in $G$). Then $L$ is a dependency graph for the family $\{X_\alpha\}_{\alpha \in \mathcal{I}}$.

Dependency graphs are used in geometric random graphs, see [92], and in geometric probability for statistics like the nearest-neighbor graph, the Delaunay triangulations and the Voronoi diagramm of random point configurations, see [93]. More recently is has been used to prove asymptotic normality of pattern counts in random permutations in [52]. Another context, outside the scope of the present work, in which dependency graphs are used is the Lovász local lemma, see [4].

We will consider the following setting:

**Assumption 4.3.3** (Dependency-graph model). From now on we consider the following model: Suppose that for each $n$, $\{X_{n,i}, 1 \leq i \leq N_n\}$ is a family of bounded random variables, $|X_{n,i}| \leq A_n$ a.s. Suppose, in addition, that $L_n$ is a dependency graph for this family and let $D_n - 1$ be the maximal degree of $L_n$. Let $Y_n := \sum_{i=1}^{N_n} X_{n,i}$ and $\sigma_n^2 := \text{Var}(Y_n)$.

Precise normality criteria for $(Y_n)_{n \in \mathbb{N}}$ using dependency graphs have been given in [54], [12] and [77]. In [54] the following normality criterion was proved. Assume that there exists an integer $s$ such that $(N_n A_n) \frac{\sigma_n}{D_n} \to 0$ as $n \to \infty$. Then for the dependency graph model in 4.3.3, $X_{n,i} \Rightarrow \text{Normal}(0, A_n)$ converges in distribution to a standard normally distributed random variable.

In the following, for two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$, we write $a_n \asymp b_n$ instead of $a_n = \Theta(b_n)$. In section 2.3 this was denoted by $a_n \approx b_n$, and the exact definition can be found in equation (2.26).

**Example 4.3.4.** We consider the $G(n, p_n)$-model in Example 4.3.2 and take $Y_n$ to be the number of triangles. Let $p_n$ be bounded away from 1. One has $N_n \asymp n^3$, $D_n \asymp n$ and $M_n = 1$. Since $\sigma_n^2 \asymp \max(n^3 p_n^2, n^5 p_n^2)$ (see [59], Lemma 3.5), the criterion is fulfilled if $p_n \gg n^{-1/3+\epsilon}$ for some $\epsilon > 0$. The asymptotic normality is in fact true under the less restrictive hypothesis $p_n \gg n^{-1}$, see [98].

A uniform control on cumulants of $(Y_n)_n$ from Assumption 4.3.3 was first considered in [54]: Under Assumption 4.3.3 one has that

\[ |\kappa_j(Y_n)| \leq C_j N_n D_n^{j-1} A_j \]  

(4.9)
for some universal constant $C_j$ and any $j \geq 3$. Here it is assumed that $|X_{n,i}| \leq A$ for all $i$ and $n$, a.s. In [27] was proved that one can take $C_j = (2e)^j (j!)^3$. The results was improved in [42, Theorem 9.1.7]: one can take $C_j = 2j^{-1}j^{j-2}$ giving uniform bounds on cumulants.

**Definition 4.3.5.** A sequence $(Y_n)_{n \in \mathbb{N}}$ of real valued random variables admits a uniform control on cumulants with DNA $(D_n, N_n, A)$, if $D_n = o(N_n)$, $N_n \to \infty$ as $n \to \infty$ and for all $j \geq 2$

$$|\kappa_j(Y_n)| \leq C_j N_n D_n^{j-1} A^j.$$  

(4.10)

Here $A$ is a constant and $C_j$ is a constant only depending on $j$.

**Remark 4.3.6.** The setting of Assumption [4.3.3] is an example for a uniform control on cumulants with DNA, see (4.9).

**Theorem 4.3.7.** Assume that a sequence $(Y_n)_{n}$ of real valued random variables admits a uniform control on cumulants with DNA $(D_n, N_n, A)$. Assume moreover that $\lim_{n \to \infty} \kappa_2(Y_n) / N_n D_n = \sigma^2$.  

(4.11)

Consider $Z_n := \frac{Y_n}{\sigma_n}$ with $\sigma_n^2 := \text{Var}(Y_n)$. Then we obtain, if $k$ is even,

$$|E[Z_n^k] - E[N^k]| \leq C_1(k) \frac{D_n}{N_n}.$$  

For odd $k$ we have

$$|E[Z_n^k]| \leq C_2(k) \left( \frac{D_n}{N_n} \right)^{1/2}.$$  

**Proof.** By assumption the cumulant bounds are of the form in Theorem [4.2.1] with $\gamma = 0$, $C_{j,0} = C_j A^j$ and with

$$\Delta_n^{j-2} = \frac{\sigma_n^j}{N_n D_n^{j-1}}.$$  

Hence we have $\Delta_n^2 = \left( \frac{\sigma_n^2}{N_n D_n^{j-1}} \right)^{j/2}$, which is depending on $j$. But with $\sigma_n^2 \asymp N_n D_n$, by assumption (4.11), we have $\Delta_n^2 \asymp \frac{N_n}{D_n}$. Now we can apply Theorem 4.2.1. \hfill $\square$

**Example 4.3.8** (Number of triangles in Erdős-Rényi random graphs). In the model of Example [4.3.2] we take $p \in (0, 1)$ being fixed. With $\sigma_n^2 \asymp \max(n^3 p_n^3, n^4 p_n^5)$, we obtain $\sigma_n^2 \asymp n^4$. With $N_n \asymp n^3$ and $D_n \asymp n$ we obtain that condition (4.11) holds. Hence we can apply Theorem 4.3.7 for even $k$ we have

$$|E[Z_n^k] - E[N^k]| \leq C_1(k) \frac{1}{n^2},$$  

and

$$|E[Z_n^k]| \leq C_2(k) \frac{1}{n}$$  

for $k$ being odd.
Example 4.3.9 (Number of subgraphs in Erdős-Rényi random graphs). Now we like to count the number of subgraphs isomorphic to a fixed graph $H$ with $k$ edges and $l$ vertices. As a special case of Example 4.3.2, let $\{H_\alpha\}_{\alpha \in I}$ be given subgraphs of the complete graph $K_n$ and let $I_\alpha$ be the indicator that $H_\alpha$ appears as a subgraph in $G(n, p_n)$, that is $I_\alpha = 1_{\{H_\alpha \subset G(n, p_n)\}}$, $\alpha \in I$. Then $L(S)$ with $S = \{e_{H_\alpha} : \alpha \in I\}$ is a dependency graph with edge set $\{\alpha \beta : e_{H_\alpha} \cap e_{H_\beta} \neq \emptyset\}$. Here we take the family of subgraphs of $K_n$ that are isomorphic to a fixed graph $H$, denoted by $\{G_\alpha\}_{\alpha \in I}$. Consider $X_\alpha = I_\alpha - IE(I_\alpha)$ and define the graph $L$ by connecting every pair of indices $\alpha$ and $\beta$ such that the corresponding graphs $G_\alpha$ and $G_\beta$ have a common edge. This is evidently a dependency graph for $(X_\alpha)_{\alpha \in A}$, see [59, Example 6.19]. The subgraph count statistic $Y$ is the sum of all $X_\alpha$. We prevent the dependence on $|I|$ in our notion. Again we only consider a fixed $p \in (0, 1)$ to guarantee condition (4.11): notice that for $p$ being fixed we have

$$\text{const. } n^{2l-2} p^{2k-1} (1 - p) \leq \text{Var } Y \leq \text{const. } n^{2l-2} p^{2k-1} (1 - p) \quad (4.12)$$

by [98, 2nd section, page 5]. Moreover we have

$$D_n \leq k(n-2)l-2 - 1 \leq kn^{l-2} - 1$$

(see [27, page 369, last estimate]). The number $N_n$ of the subgraphs in $K_n$ which are isomorphic to $H$ satisfies the inequality

$$\left(\begin{array}{c} n \\ l \end{array}\right) \leq N_n \leq n_l = n(n-1)\cdots(n-l-1).$$

Hence $N_n \asymp n^l$ and condition (4.11) is fulfilled. Summarizing, the cumulants of $Y$ can be bounded as follows: for any $j \geq 3$

$$|\kappa_j(Y)| \leq j!C_j n^l (kn^{l-2})^{j-1}.$$  

With the lower bound (4.12) we can bound the cumulants of $Z := \frac{X}{\sqrt{\text{Var}(X)}}$ for $j \geq 3$ as follows:

$$|\kappa_j(Z)| \leq j!C_j \frac{1}{n^{j-2}}.$$  

Here the constant $C_j$ is also depending on $k$ and $l$. See also the proof of Theorem 2.3 in [27]. Summarizing, applying Corollary 4.2.2 we obtain for fixed $p$ and for any subgraph $H$ with $k$ edges and $l$ vertices the bounds

$$|\mathbb{E}[Z^m] - \mathbb{E}[N^m]| \leq C_1(m, l, k) \frac{1}{n^2}$$

for $m$ being even and

$$|\mathbb{E}[Z_n^m]| \leq C_2(m, l, k) \frac{1}{n}$$

for $m$ being odd.
4.3.2 Weighted Dependency Graphs

Very recently, in [41] the concept of weighted dependency graphs was introduced. The concept includes the possibility of having small weights \( w_e \in [0, 1] \) on the edges of the graph, which encode the dependency structure. Here a weight 0 is the same as no edge. The examples are sums of pairwise dependent random variables. For such families, the only usual dependency graph is the complete graph and the standard theory of dependency graphs is useless. Informally speaking, that a family of random variables \( \{X_{n,i}, 1 \leq i \leq N_n\} \) admits a weighted graph \( G \) as weighted dependency graph means, that \( G \) has vertex-set of size \( N_n \), and the smaller the weight of an edge \( \{a, b\} \) is, the closer to independent \( X_{n,a} \) and \( X_{n,b} \) should be. In particular, an edge of weight 0 means that \( X_{n,a} \) and \( X_{n,b} \) are independent. Formally, this closeness to independence is not only measured by a bound on the covariance, but also involves bounds on higher order cumulants, see [41, Definition 4.5].

To cut the story short, for each \( n \), we consider a family \( \{X_{n,i}, 1 \leq i \leq N_n\} \) of random variables with finite moments defined on the same probability space. We assume that for each \( n \) one has a (\( \Psi_n, C \)) weighted dependency graph \( L_n \) for \( \{X_{n,i}, 1 \leq i \leq N_n\} \) in the sense of Definition 4.5 in [41], and we let \( Y_n = \sum_{i=1}^{N_n} X_{n,i} \) and \( \sigma_n^2 = \text{Var}(Y_n) \), and we assume that this sequence admits a uniform control on cumulants with DNA \( (Q_n, R_n, 1) \). We assume that \( Q_n = o(R_n) \), \( R_n \to \infty \) as \( n \to \infty \) and for all \( j \geq 1 \),

\[
|\kappa_j(Y_n)| \leq C_j R_n Q_n^{j-1},
\]

(4.13)

with a constant \( C_j \) only depending on \( j \). Although models with a corresponding weighted dependency graph are much more intricate concerning the dependency structure, the article [41] has successfully found examples, where the uniform control of the cumulants can be checked. As noticed in [41, Section 4.3], in the special case \( \Psi_n \equiv 1 \), the quantities \( R_n \) and \( Q_n \) in [4.13] can be replaced by \( N_n \) (the number of vertices) and \( D_n \) (the maximal weighted degree plus 1). In the following three examples, we restrict ourselves to this case.

**Example 4.3.10** (Crossings in random pair partitions). A pair partition of \([2n] := \{1, 2, \ldots, 2n\}\) is a set \( H \) of disjoint 2-element subsets of \([2n]\) whose union is \([2n]\). For each \( i \) in \([2n]\) there is a unique \( j \neq i \) such that \( \{i, j\} \) is in \( H \), the partner of \( i \). A uniform random pair partition of \([2n]\) can be constructed as follows: Take \( i_1 \) arbitrarily and choose its partner \( j_1 \) uniformly at random among numbers different from \( i_1 \), i.e. each number different from \( i_1 \) is taken with probability \( 1/(2n-1) \). Then take \( i_2 \) arbitrarily, different from \( i_1 \) and \( j_1 \), and choose its partner \( j_2 \) uniformly at random among numbers different from \( i_1, j_1 \) and \( i_2 \) (with probability \( 1/(2n-3) \)) and so on. A crossing in a pair partition \( H \) is a quadruple \( (i, j, k, l) \) with \( i < j < k < l \) such that \( \{i, k\} \) and \( \{j, l\} \) belong to \( H \). Now let \( A_n \) be the set of two element subsets of \([2n]\). For \( \{i, j\} \in A_n \) we define a random variable \( X_{i,j} \) such that \( X_{i,j} = 1 \), if \( \{i, j\} \) belongs to the random pair partition \( H_n \), and 0 otherwise. Let \( A'_n \) be the set of quadruples \( (i, j, k, l) \) of elements of \([2n]\) with \( i < j < k < l \). For \( (i, j, k, l) \in A'_n \) we set \( X_{i,j,k,l} := X_{i,k} X_{j,l} \). Hence this random variable
has value 1 if \((i,j,k,l)\) is a crossing in the random pair partition \(H_n\), and 0 otherwise. We consider the number of crossings in the random pair partition \(H_n\)

\[
Y_n := \sum_{i<j<k<l} X_{i,j,k,l}.
\]

In [41, Theorem 6.5], a CLT for \(Z_n := (Y_n - \mathbb{E}Y_n)/\sqrt{\text{Var}Y_n}\) was proved using the weighted dependency structure of this random variable. See [19] and references therein for numerous results on crossings. It was proven by showing that (4.13) holds true with a certain constant \(C_j\), with \(R_n \sim n^2\) (see [41, (6.3)]) and \(Q_n = n\). Moreover, the variance of \(Y_n\) was computed in [41, Appendix B.1], and we see that \(\text{Var}Y_n \sim n^3\). Hence assumption (4.11) is fulfilled and we obtain the bounds

\[
\left| \mathbb{E}(Z_n^k) - \mathbb{E}(N^k) \right| \leq C_1(k) \frac{1}{n}
\]

for \(k\) being even and

\[
\left| \mathbb{E}(Z_n^k) \right| \leq C_2(m, l, k) \frac{1}{n^{1/2}}
\]

for \(k\) being odd.

**Example 4.3.11** (Subgraph counts in Erdős-Rényi model \(G(n,m_n)\)). For each \(n\), let \(m_n\) be an integer between 0 and \(\binom{n}{2}\). We now consider the Erdős-Rényi graph model \(G(n,m_n)\), i.e. \(G\) is a graph with vertex set \(V = [n]\) and an edge set \(E\) of size \(m_n\), chosen uniformly at random among all possible edge sets of size \(m_n\). We set \(p_n := m_n/\binom{n}{2}\). For any 2-element subset \(\{i,j\}\) of \(V\), we define \(X_{i,j}\) such that \(X_{i,j} = 1\) if the edge \(\{i,j\}\) belongs to the random graph \(G\), and 0 otherwise. The value is 1 with probability \(p_n\). However, unlike in \(G(n,p_n)\), these random variables are not independent. In [41], a weighted dependency graph in \((G(n,m_n))\) for the family \((X_{i,j})\) is presented.

Now fix a graph \(H\) with at least one edge, and let \(A^H_n\) be the set of subgraphs \(H'\) of the complete graph \(K_n\) on vertex set \([n]\) that are isomorphic to \(H\). Let \(G\) be a random graph with the distribution of the model \(G(n,m_n)\). For \(H'\) we write

\[
X_{H'} = \prod_{\{i,j\} \in E_{H'}} X_{i,j},
\]

and denote by

\[
Y^H_n = \sum_{H' \in A^H_n} X_{H'}
\]

the number of subgraphs of \(G\) that are isomorphic to \(H\) (subgraph count statistic). In [41], Proposition 7.2, a weighted dependency graph for the family \((X_{H'})_{H' \in A^H_n}\) was constructed. If \(v_H\) denotes the number of vertices and \(e_H\) the number of edges of \(H\), we write

\[
\Phi_H := \min_{K \subset H, e_K > 0} \frac{v^K_H}{n^{v_K} p_n^{e_K}}
\]
and

$$\tilde{\Phi}_H := \min_{K \subset H, eK > 1} n^{v_K} p_n e^K.$$ 

In Theorem 7.5, it was observed that (4.13) holds true with a certain constant $C_j$, with $R_n \approx n^{v_H} p_n e^H$ (see (7.3)) and $Q_n = \frac{n^{v_H} p_n e^H}{\Phi_H}$. Moreover we use the following estimate for the variance given in Lemma 7.3:

$$\text{Var}(Y_n^H) \geq \frac{C(n^{v_H} p_n e^H)^2}{\Phi_H} (1 - p_n)^2,$$ (4.14)

for some constant $C > 0$ and whenever $n(1 - p_n)^2 \gg 1$ and $n$ is sufficiently large. Note that the variance of $Y_n^H$ has a different order of magnitude than in the independent model $G(n, p_n)$, which was already observed in [55].

**Assumption 4.3.12.** To be able to verify assumption (4.11), we assume that $p \in (0, 1)$ is fixed and $m_n \approx p(n^2)$. Moreover we assume that $H$ has a component with three vertices and two edges (a path $P_2$).

The assumption implies that $\Phi_H \approx \tilde{\Phi}_H \approx n^3$. Moreover we know that $\text{Var}(Y_n^H) \approx n^{2v_H - 3}$ (whereas $\text{Var}(Y_n^H) \approx n^{2v_H - 2}$ in the $G(n, p_n)$ random graph), see [59, Example 6.55]. We conclude that under Assumption 4.3.12 we have

$$\frac{\text{Var}(Y_n^H)}{R_n Q_n} \approx \text{const.},$$

and hence Assumption (4.11) is verified. Moreover we observe that

$$|\kappa_j(Y_n^H)| \leq C_j (n^{v_H} p_n e^H)^2 \frac{1}{\Phi_H^{j-1}}.$$

With the estimate (4.14), we have with $Z_n^H = \frac{Y_n^H - \mathbb{E}(Y_n^H)}{\sqrt{\text{Var}(Y_n^H)}}$ that

$$|\kappa_j(Z_n^H)| \leq C_j(p) \frac{\tilde{\Phi}_H^{j/2}}{\Phi_H^{j-1}} \leq C_j(p) \left(\frac{n^3}{2}\right)^{j/2}.$$

With Theorem 4.2.1 or Theorem 4.3.7 we have proven:

**Theorem 4.3.13.** Let $p \in (0, 1)$ be fixed and $m_n \approx p(n^2)$ and consider a random graph $G$ taken with Erdős-Rényi distribution $G(n, m_n)$. Fix some graph $H$ that contains $P_2$. We denote by $Y_n^H$ the number of copies of $H$ in the random graph $G$. Then with $Z_n^H = \frac{Y_n^H - \mathbb{E}(Y_n^H)}{\sqrt{\text{Var}(Y_n^H)}}$ we have for any $k \geq 3$

$$|\mathbb{E}[(Z_n^H)^k] - \mathbb{E}[N^k]| \leq C_1(k) \frac{1}{n^3}$$

for $k$ being even and

$$|\mathbb{E}[(Z_n^H)^k]| \leq C_2(k) \frac{1}{n^2}$$

for $k$ being odd.
Example 4.3.14 (Spins in the $d$-dimensional Ising model). The Ising model on a finite subset $\Lambda$ of $\mathbb{Z}^d$ is given by the Gibbs distribution

$$
\mu_{\Lambda, \beta, h}(\omega) = \frac{1}{Z_{\Lambda, \beta, h}} e^{-H_{\Lambda, \beta, h}}
$$

with

$$
H_{\Lambda, \beta, h} := -\beta \sum_{\{i,j\} \in E_\Lambda} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i \in \Lambda} \sigma_i(\omega)
$$

for each $\omega = (\sigma_i(\omega))_{i \in \Lambda}$ with $\sigma_i(\omega) \in \{-1, +1\}$. Here $h \in \mathbb{R}$ is called the magnetic field and $\beta > 0$ the inverse temperature, and $E_\Lambda := \{\{i,j\} \subset \Lambda : \|i - j\|_1 = 1\}$ is the set of nearest neighbor pairs in $\Lambda$, measured in the graph distance $\|\cdot\|_1$ in $\mathbb{Z}^d$. $Z_{\Lambda, \beta, h}$ is called the partition function. All the quantities are with free boundary conditions so far, which means that the value of the spins outside of $\Lambda$ is not taken into consideration. Fixing the partition function. All the quantities are with free boundary conditions so far, which means that the value of the spins outside of $\Lambda$ is not taken into consideration. Fixing a spin configuration $\eta \in \{-1, +1\}^{\mathbb{Z}^d}$, we define a spin configuration in $\Lambda$ with boundary condition $\eta$ as an element of the set $\Omega_\Lambda^\eta := \{\omega \in \{-1, +1\}^{\mathbb{Z}^d} : \omega_i = \eta_i \forall i \notin \Lambda\}$. Then the Hamiltonian is given by

$$
H_{\Lambda, \beta, h}^\eta := -\beta \sum_{\{i,j\} \in E_\Lambda^b} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i \in \Lambda} \sigma_i(\omega)
$$

with $E_\Lambda^b := \{\{i,j\} \subset \Lambda : \|i - j\|_1 = 1, \{i,j\} \subset \Lambda \neq \emptyset\}$. The most classical boundary conditions are the $+$ boundary condition, where $\eta_i = +1$ for all $i \in \mathbb{Z}^d$, and the $-$ boundary condition, where $\eta_i = -1$ for all $i \in \mathbb{Z}^d$. Quantities with $+$ (resp. $-$) boundary condition are denoted with a superscript $+$ (or $-$ respectively), e.g. $\mu_{\Lambda, \beta, h}^\eta$.

We now take an increasing sequence $\Lambda_n$ of finite sets with $\bigcup_{n \geq 1} \Lambda_n = \mathbb{Z}^d$. It is well known that the sequence $(\mu_{\Lambda_n, \beta, h}^\eta)_n$ converges in the weak sense to a measure denoted by $\mu_{\beta, h}^\eta$, as $n \to \infty$, see [14 Chapter 3]. In a high temperature regime with $\beta < \beta_1(d)$ and $h = 0$ (meaning that there exists a $\beta_1(d)$) or in the presence of a magnetic field $h \neq 0$, the limiting measure is independent of the choice of the boundary conditions. At low temperature $\beta > \beta_2(d)$ and $h = 0$, the limiting measure depends on the boundary conditions. Here, we restrict ourselves to $+$ boundary conditions to have a well defined limiting measure in all cases. We drop the superscript $+$ and denote the limiting measure by $\mu_{\beta, h}$.

The decay of joint cumulants of the spins under $\mu_{\beta, h}$ has been studied in a few research articles. A good summary is [30 Theorem 1.1] and reads as follows. For random variables $X_1, \ldots, X_j$ with finite moments, consider the joint cumulant as

$$
\Gamma(X_1, \ldots, X_j) = \left[t_1, \ldots, t_j\right] \log \mathbb{E} \left[ \exp(t_1 X_1 + \cdots + t_j X_j) \right].
$$

Here $[t_1, \ldots, t_j]F$ stands for the coefficient of $t_1 \cdots t_j$ in the series expansion of $F$ in positive powers of $t_1, \ldots, t_j$. Note that $\kappa_j(X) = \Gamma(X,\ldots, X)$.

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Theorem 4.3.15. For the Ising model on $\mathbb{Z}^d$ with parameters $(\beta, h)$, there exist positive constants $\epsilon(d) < 1, \beta_1(d), \beta_2(d)$ and $h(d)$ depending on the dimension $d$ with the following property. Assume that we are in one of the regimes $h > h(d)$, or $h = 0$ and $\beta < \beta_1(d)$, or $h = 0$ and $\beta > \beta_2(d)$. Then for any $j \geq 1$, there exists a constant $C_j$ such that for all $A = \{i_1, \ldots, i_j\} \subset \mathbb{Z}^d$, one has

$$\Gamma_j^{\beta,h}(\sigma_{i_1}, \ldots, \sigma_{i_j}) \leq C_j \epsilon(d)^{l_T(A)}.$$ 

Here we consider the joint cumulants with respect to the measure $\mu_{\beta,h}$ and $l_T(A)$ denotes the minimum length of a tree connecting vertices of $A$.

The bounds on joint cumulants had been translated in terms of weighted dependency graphs for the spin variables in [30, Theorem 1.2]:

Theorem 4.3.16. Let $\omega = (\sigma_i(\omega))_{i \in \mathbb{Z}^d}$ be a spin configuration according to $\mu_{\beta,h}$, where either $h > h(d)$, or $h = 0$ and $\beta < \beta_1(d)$, or $h = 0$ and $\beta > \beta_2(d)$. Let $G$ be the complete weighted graph with vertex set $\mathbb{Z}^d$, such that every edge $e = (i, j)$ has weight $w_e = \epsilon(d)^{\frac{\|i-j\|_1}{2}}$, where $\epsilon$ comes from Theorem 4.3.15. Then $G$ is a $C$-weighted dependency graph (see [30, Definition 2.3]) for the family $\{\sigma_i, i \in \mathbb{Z}^d\}$ and some $C = (C_r)_r$.

We now consider $\Lambda_n := [-n, n]^d$ the $d$-dimensional cube centred at 0 of side length $2n$, and we consider the magnetization $S_n = \sum_{i \in \Lambda_n} \sigma_i$, and

$$Z_n := \frac{S_n - \mathbb{E}[S_n]}{\sqrt{\text{Var}(S_n)}}.$$ 

With [30, Lemma V.7.1] we know that $\sigma^2 := \lim_{n \to \infty} \frac{\text{Var}(S_n)}{|\Lambda_n|}$ exists as an extended real number. Moreover, it is known that $\sigma^2 > 0$, and that it is finite in the three regimes of Theorem 4.3.15 see [30, Corollary 4.4 and the proof of Theorem 4.2]. With Theorem 4.3.16 the number of vertices of the weighted dependency graph on $\Lambda_n$ is $|\Lambda_n| = (2n+1)^d$. The maximal weighted degree is

$$D_n - 1 = \max_{i \in \Lambda_n} \sum_{j \in \Lambda_n} \epsilon^{\frac{\|i-j\|_1}{2}}.$$ 

As presented in [30], this object is bounded by a constant. Hence we can apply Theorem 4.3.7 – condition (4.11) is satisfied. We have proved the result:

Theorem 4.3.17. Consider the Ising model on $\mathbb{Z}^d$, with inverse temperature $\beta$ and magnetic field $h$, such that either $h > h(d)$, or $h = 0$ and $\beta < \beta_1(d)$, or $h = 0$ and $\beta > \beta_2(d)$. Then for even $k$ with $k \geq 4$, we have

$$|\mathbb{E}_{\beta,h}[Z_n^k] - \mathbb{E}[N^k]| \leq C_1(k) \frac{1}{(2n+1)^d}.$$
Remark 4.3.18. As was pointed out in [30], local and global patterns of spins in the Ising model satisfy a central limit theorem as well. For details see Theorem 1.3 and 1.4 in [30]. For local patterns the result of Theorem 4.3.17 can be proved. For global patterns of size \( m \), at least in the case where the patterns consist of positive spins only, the same result follows from [30, proof of Theorem 4] with a constant \( C_1(k, m) \), which is depending on the size \( m \) as well. The details are omitted.

4.3.3 Non-degenerate \( U \)-statistics

Let \( X_1, \ldots, X_n \) be independent and identically distributed random variables with values in a measurable space \( \mathcal{X} \). For a measurable and symmetric function \( h : \mathcal{X}^2 \to \mathbb{R} \) we define

\[
U_n(h) := \frac{1}{\binom{n}{2}} \sum_{1 \leq i_1 < i_2 \leq n} h(X_{i_1}, X_{i_2}),
\]

where symmetric means invariant under any permutation of its arguments. \( U_n(h) \) is called a \( U \)-statistic with kernel \( h \) and degree 2. Define the conditional expectation by

\[
h_1(x_1) := \mathbb{E}[h(x_1, X_2)] = \mathbb{E}[h(X_1, X_m)|X_1 = x_1]
\]

and the variance by \( \sigma_1^2 := \text{Var}[h_1(X_1)] \). A \( U \)-statistic is called non-degenerate if \( \sigma_1^2 > 0 \). We consider \( U \)-statistics which are assumed to be non-degenerate. Assume that \( 0 < \sigma_1^2 < \infty \), and suppose that there exist constants \( \gamma \geq 1 \) and \( C > 0 \) such that

\[
\mathbb{E}[|h(X_1, X_2)|^j] \leq C^j(j!)^\gamma
\]

for all \( j \geq 3 \). According to [3], see [102, Lemma 5.3], the cumulants of \( U_n \) can be bounded by

\[
|\kappa_j(U_n)| < 2e^{2(j-2)} \frac{2^j - 1}{j} C^j(j!)^{1+\gamma} \frac{1}{n^{j-1}}
\]

for all \( j = 1, 2, \ldots, n-1 \) and \( n \geq 7 \). The quite involved proof is presented in [102]. The variance for the non-degenerate \( U \)-statistic is given by \( \text{Var}(U_n) = \frac{4\sigma_1^2 n - 2}{n(n-1)} + \frac{2\sigma_1^2}{n(n-1)} \), see Theorem 3 in [67, chapter 1.3]. Hence there exists an \( n_0 \geq 7 \) large enough such that \( \sqrt{\text{Var}(U_n)} \geq \frac{\sigma_1}{\sqrt{2n}} \). The following bound holds for the cumulants of \( Z_n := \frac{U_n}{\sqrt{\text{Var}(U_n)}} \):

\[
|\kappa_j(Z_n)| \leq (j!)^{1+\gamma} \left( \frac{2\sqrt{2eC(\sigma_1)}}{\sqrt{n}} \right)^{j-2}
\]

for all \( j = 3, \ldots, n-1 \) and \( n \geq n_0 \). Applying Theorem 4.2.1 we have for any \( k \geq 3 \)

\[
|\mathbb{E}[Z_n^k] - \mathbb{E}[X^k]| \leq C_1(k) \frac{1}{n},
\]

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when \( k \) is even and
\[
|\mathbb{E}[Z_n^k]| \leq C_2(k) \frac{1}{n^2},
\]
if \( k \) is odd.

### 4.3.4 Characteristic Polynomials in the Circular Ensembles

Consider the characteristic polynomial \( Z(\theta) := Z(U, \theta) = \det(I - U e^{-i\theta}) \) of a unitary \( n \times n \) matrix \( U \). The matrix \( U \) is considered as a random variable in the circular unitary ensemble (CUE), that is the unitary group \( U(n) \) equipped with the unique translation-invariant (Haar) probability measure. In [61], exact expressions for any matrix size \( n \) are derived for the moments of \(|Z|\), and from these the asymptotics of the value distribution and cumulants of the real and imaginary parts of \( \log Z \) as \( n \to \infty \) are obtained. In the limit, these distributions are independent and Gaussian. In [61] the results were generalized to the circular orthogonal (COE) and the circular symplectic (CSE) ensembles. Let us consider the representation of \( Z(U, \theta) \) in terms of the eigenvalues \( e^{i\theta_k} \) of \( U \):

\[
Z(U, \theta) = \det(I - U e^{-i\theta}) = \prod_{k=1}^{n} (1 - e^{i(\theta_k - \theta)}).
\]

Now let \( Z \) represent the characteristic polynomial of an \( n \times n \) matrix \( U \) in either the CUE (\( \beta = 2 \)), the COE (\( \beta = 1 \)), or the CSE (\( \beta = 4 \)). The \( C\beta E \) average can then be performed using the joint probability density for the eigenphases \( \theta_k \):

\[
\langle |Z|^s \rangle_{\beta} = \frac{\Gamma((\beta/2)n)\prod_{1 \leq j < m \leq n} |e^{i\theta_j} - e^{i\theta_m}|^\beta}{(n\beta/2)!(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} d\theta_1 \cdots d\theta_n \prod_{1 \leq j < m \leq n} |e^{i\theta_j} - e^{i\theta_m}|^\beta \times \left| \prod_{k=1}^{n} (1 - e^{i(\theta_k - \theta)}) \right|^s.
\]

This integral can be evaluated using Selberg’s formula, see [75], which leads to

\[
\langle |Z|^s \rangle_{\beta} = \prod_{j=0}^{n} \frac{\Gamma(1 + j\beta/2)\Gamma(1 + s + j\beta/2)}{\Gamma(1 + s/2 + j\beta/2)},
\]

where \( \Gamma \) is the Gamma function. Hence \( \log \langle |Z|^s \rangle_{\beta} \) has a simple form and, at the same time, by definition equals \( \sum_{j=1}^{n} \frac{\Gamma_j(\beta)}{\beta^j} s^j \), where \( \kappa_j(\beta) = \kappa_j(\Re \log Z) \) denotes the \( j \)-th cumulant of the distribution of the real part of \( Z \) under \( C\beta E \). Differentiating \( \log \langle |Z|^s \rangle_{\beta} \) one obtains

\[
\kappa_j(\beta) = \frac{2^j - 1}{2^{j-1}} \sum_{k=0}^{n-1} \psi^{(j-1)}(1 + k\beta/2),
\]
where
\[ \psi^{(j)}(z) := \frac{d^{j+1} \log \Gamma(z)}{dz^{j+1}} = (-1)^{j+1} \int_0^{\infty} t^j e^{-zt} \frac{dt}{1 - e^{-t}}, \]
for \( z \in \mathbb{C} \) with \( \text{Re} \, z > 0 \), are the polygamma functions. In [27, Section 4] the authors proved that
\[ \left| \kappa_j \left( \frac{\text{Re} \log(Z)}{\sigma_{n,\beta}} \right) \right| \leq (j!)^{1/2} \frac{1}{\sigma_{n,\beta}} \begin{cases} 2 & \text{for } \beta = 1 \\ \frac{4 \pi^2}{\beta} & \text{for } \beta = 2 \\ \frac{8 \pi^2}{6} & \text{for } \beta = 4 \end{cases} \]
for all \( j \geq 3 \), hence equation (4.14) is satisfied for \( \gamma = 0 \) and \( \Delta_n = \sigma_{n,\beta} \). The \( j \)-th cumulant of the distribution of the imaginary part of \( \log Z \) can be bounded by the \( j \)-th cumulant of the distribution of the real part of \( \log Z \) for all \( j \geq 3 \), see [61, eq. (62)].

For \( \beta = 2 \) we know that \( \sigma_{n,2}^2 \approx \frac{1}{2} \log n \), see [61, eq. (45)]. Hence we have proved that for any \( k \geq 3 \) and \( Z_n = \frac{\text{Re} \log(Z)}{\sigma_{n,2}} \) we have
\[ |E[Z_n^k] - E[N^k]| \leq C_1(k) \frac{1}{\log n} \]
for \( k \) being even, and
\[ |E[Z_n^k]| \leq C_2(k) \left( \frac{1}{\log n} \right)^{1/2} \]
for \( k \) being odd.

### 4.3.5 Determinants of Random Matrix Ensembles and Random Simplices

In this section we consider random determinants of certain random matrix ensembles.

**Laguerre ensemble**

Let us come back to the Laguerre Ensemble already discussed in section [3.2.1]. Recall that if \( W_{n,p(n)}^{L,\beta} \) is a \( \beta \)-Laguerre distributed random matrix of dimension \( p(n) \times p(n) \), then we have
\[ \log \mathbb{E} \left[ \exp \left( z \log (\det W_{n,p(n)}^{L,\beta}) \right) \right] = zp(n) \log 2 + L(p(n), n - p(n), \beta/2; z), \]
where
\[ L(p, l, \alpha; z) = \log \left( \prod_{k=1}^{p} \frac{\Gamma(\alpha(k + l) + z)}{\Gamma(\alpha(k + l))} \right). \]

It follows that
\[ \kappa_j \left( \log \det W_{n,p(n)}^{L,\beta} \right) = \frac{d^j}{dz^j} L(p(n), n - p(n), \beta/2; z) \bigg|_{z=0} + 1_{(j=1)} p(n) \log 2. \]
Our aim is to analyze the asymptotic behavior of the first and second cumulant, and to bound higher order cumulants. With respect to random determinants of random matrix ensembles, this goes back to [64]. For further details see [26]. In [48] the results of [26] were applied to study volumes of random simplices.

From now on we only consider the case $\beta = 1$. For $\beta \neq 1$ the asymptotic behavior (in $n$ and $p(n)$) of all cumulants of $\det W_{n,p(n)}^{L,\beta}$ only differs by some constants depending on $\beta$.

The digamma function is defined as $\psi(z) = \psi^{(0)}(z) := \frac{d}{dz} \log \Gamma(z)$, and the polygamma functions

$$\psi^{(j)}(z) := \frac{d^j}{dz^j} \psi(z) = \frac{d^{j+1}}{dz^{j+1}} \log \Gamma(z), \quad j \in \mathbb{N}.$$  

First we analyze the expectation of $\det W_{n,p(n)}^{L,\beta}$. For $j = 1$, we have

$$\left. \frac{d}{dz} L(p(n), n - p(n), \frac{1}{2}; z) \right|_{z=0} = \sum_{k=1}^{p(n)} \psi \left( \frac{1}{2} (k + n - p(n)) \right) = \sum_{k=1}^{n} \psi \left( \frac{k}{2} \right) - \sum_{k=1}^{n-p(n)} \psi \left( \frac{k}{2} \right).$$

As $n \to \infty$, one has $\sum_{k=1}^{n} \psi \left( \frac{k}{2} \right) \sim n \log n$, see for example [26, relation (2.10) and (2.19)]. Hence

$$\mathbb{E} \left[ \log \det W_{n,p(n)}^{L,1} \right] \sim \begin{cases} n \log n + p(n) \log 2 & \text{for } n - p(n) = o(n), \\ p(n) \log(2n) & \text{for } p(n) = o(n), \\ cn \log(2n) & \text{for } p(n) \sim cn \text{ for some } c \in (0,1). \end{cases}$$

Next we analyze the variance of $\log \det W_{n,p(n)}^{L,\beta}$. We obtain

$$\left. \frac{d^2}{dz^2} L(p(n), n - p(n), \frac{1}{2}; z) \right|_{z=0} = \sum_{k=1}^{p(n)} \psi^{(1)} \left( \frac{1}{2} (k + n - p(n)) \right).$$

We collect some asymptotic relations and bounds for polygamma functions.

**Lemma 4.3.19.** Let $j \in \mathbb{N}$. Then as $|z| \to \infty$ in $|\arg z| < \pi - \epsilon$, one has

$$\psi^{(j)}(z) = (-1)^{j-1} \frac{(j-1)!}{z^j} + O \left( \frac{1}{z^{j+1}} \right),$$

and for all $z > 0$,

$$|\psi^{(j)}(z)| \leq \frac{(j-1)!}{z^j} + \frac{j!}{z^{j+1}}.$$  

Moreover we have

$$\sum_{k=1}^{n} \psi^{(1)} \left( \frac{k}{2} \right) = 2 \log n + c + o(1)$$

with an explicit constant $c = 2(\gamma + 1 + \frac{\pi^2}{8})$ with the Euler-Mascheroni constant $\gamma$.  

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Proof. The first asymptotic relation can be found in [1], pp. 259-260. The representation of $\Gamma(z)^{-1}$ due to Weierstrass is
\[
\Gamma(z)^{-1} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)e^{-\frac{z}{k}}.
\]
Differentiating $-\log \Gamma(z)$ leads to
\[
\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{z + k}\right) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{z + n}\right).
\]
Therefore one obtains
\[
\psi^{(j)}(z) = (-1)^{j+1} j! \sum_{k=0}^{\infty} \frac{1}{(z + k)^{j+1}}.
\]
(4.21)

It follows that
\[
|\psi^{(j)}(z)| \leq \frac{j!}{z^{j+1}} + j! \int_{z}^{\infty} \frac{dx}{x^{j+1}} = \frac{j!}{z^{j+1}} + \frac{(j-1)!}{z^j},
\]
which is (4.19). The last asymptotic relation (4.20) can be found in [26, relations (2.14) and (2.21)]. □

With (4.20) we obtain
\[
\sum_{k=1}^{p(n)} \psi^{(1)}\left(\frac{1}{2}(k + n - p(n))\right) = 2 \log n - 2 \log(n - p(n) + 1) + O(1) \sim 2 \log \frac{n}{n - p(n) + 1}
\]
in the case $n - p(n) = o(n)$. If $p(n) = o(n)$, we apply (4.18) to see that
\[
\sum_{k=1}^{p(n)} \psi^{(1)}\left(\frac{1}{2}(k + n - p(n))\right) \sim 2 \frac{p(n)}{n}.
\]
Finally, with $p(n) \sim cn$, we apply (4.20) to see
\[
\sum_{k=1}^{p(n)} \psi^{(1)}\left(\frac{1}{2}(k + n - p(n))\right) = 2 \log n + c - 2 \log \left(n - p(n)\right) - c + o(1)
\]
\[
= \log \frac{1}{1 - c} + o(1).
\]

Hence
\[
\text{Var} \left( \log \det W_{n,p(n)}^{L,1} \right) \sim \begin{cases}
2 \log \frac{n}{n-p(n)+1} & \text{for } n - p(n) = o(n), \\
2 \frac{p(n)}{n} & \text{for } p(n) = o(n), \\
2 \log \frac{1}{1-c} & \text{for } p(n) \sim cn \text{ for some } c \in (0, 1).
\end{cases}
\]
(4.22)
Finally we will bound the higher order cumulants. To this end we will combine results of [25] and [48]. By (4.21), \(|\psi^{(j-1)}(\cdot)|\) is decreasing, and therefore for \(j \geq 3\):

\[
|\kappa_j(\log \det W_{n,p(n)}^{L,1})| = \left| \sum_{k=1}^{p(n)} \psi^{(j-1)}\left(\frac{1}{2}(k + n - p(n))\right) \right| \\
\leq p(n) \left| \psi^{(j-1)}\left(\frac{1}{2}(1 + n - p(n))\right) \right|.
\]

With (4.19) we have \(|\psi^{(j-1)}(z)| \leq 2(j-1)!z^{1-m}\) for \(z \geq 1\). Hence

\[
|\kappa_j(\log \det W_{n,p(n)}^{L,1})| \leq 2^j d^{j-1} p(n) (j-1)! n^{1-j},
\]

where \(d\) is a constant such that \(\frac{n-p(n)+1}{2} > \frac{n}{d}\), which is possible to choose in the cases \(p(n) = o(n)\) and \(p(n) \sim c n\). The constant might depend on \(c\), but is does not depend on \(n\) or \(p(n)\). There is a very general bound for the higher order cumulants, which is valid for every choice of \(p(n)\). For \(j \geq 3\) we have

\[
|\kappa_j(\log \det W_{n,p(n)}^{L,1})| = \left| \sum_{k=1}^{p(n)} \psi^{(j-1)}\left(\frac{1}{2}(k + n - p(n))\right) \right| \\
\leq \sum_{k=1}^{n} \psi^{(j-1)}\left(\frac{k}{2}\right).
\]

With (4.19) it follows that for any \(j \geq 3\)

\[
|\kappa_j(\log \det W_{n,p(n)}^{L,1})| \leq 2^j \sum_{k \geq 1} \left(\frac{(j-1)!}{k^j} + \frac{(j-1)!}{4k^{j-1}}\right) \\
\leq 2^j \left(\zeta(3) + \frac{1}{4}\zeta(2)\right)(j - 1)! < 2^{j+1}(j - 1)!,
\]

where we used \((j - 2)! \leq \frac{1}{2}(j - 1)!\), and where \(\zeta\) denotes the Riemann zeta function. Summarizing we obtain

\[
|\kappa_j(\log \det W_{n,p(n)}^{L,1})| \leq \begin{cases} 
2^j d^{j-1} p(n) (j-1)! n^{1-j} & \text{for } p(n) = o(n) \text{ or } p(n) \sim c n, \\
2^{j+1}(j - 1)! & \text{for arbitrary } p(n).
\end{cases} \tag{4.23}
\]

Now we consider

\[
Z_{n,p(n)}^L := \log \det W_{n,p(n)}^{L,1} - \mathbb{E}\left[ \log \det W_{n,p(n)}^{L,1} \right] / \sqrt{\text{Var} \left( \log \det W_{n,p(n)}^{L,1} \right)},
\]

and with (4.22) and (4.23), we get, for some constants \(C_1(j)\) and \(C_2(j)\), that

\[
|\kappa_j(Z_{n,p(n)}^L)| \leq \begin{cases} 
C_1(j)(j - 1)! \left(\frac{1}{\sqrt{p(n)n}}\right)^{j-2} & \text{for } p(n) = o(n) \text{ or } p(n) \sim c n, \\
C_2(j)(j - 1)! \left(\frac{1}{\sqrt{\log n - p(n)+1}}\right)^{j-2} & \text{for } n - p(n) = o(n).
\end{cases} \tag{4.24}
\]

Now we can apply Corollary 4.2.2 to obtain:
Theorem 4.3.20. For the log-determinant of the Laguerre ensemble with $\beta = 1$, we obtain the bounds

$$\left| \mathbb{E}\left[ (Z_{n,p(n)}^L)^k \right] - \mathbb{E}[N^k] \right| \leq C_1(k) \frac{1}{p(n) n}$$

if $k$ is even and $p(n) = o(n)$ or $p(n) \sim cn$ for a fixed $c \in (0,1)$, and

$$\left| \mathbb{E}\left[ (Z_{n,p(n)}^L)^k \right] - \mathbb{E}[N^k] \right| \leq C_1(k) \frac{1}{\log n} n^{-p(n)+1}$$

if $k$ is even and $n - p(n) = o(n)$, including the case $n = p(n)$.

Further random matrix ensembles

In chapter 3 we observed that many other random matrix models can be analyzed knowing the behavior of $L$ in (4.16).

Recall from section 3.2.2, equation (3.7) that

$$\log \mathbb{E}\left[ \left( \det W_{p(n),n_1,n_2}^{J,\beta} \right)^z \right] = L(p(n), n_1 - p(n), \beta/2; z) - L(p(n), n_1 + n_2 - p(n), \beta/2; z),$$

where $W_{p(n),n_1,n_2}^{J,\beta}$ denotes the $\beta$-Jacobi distributed random matrix of dimension $p(n) \times p(n)$. Hence bounds on cumulants can be obtained starting with

$$\kappa_j \left( \log \det W_{p(n),n_1,n_2}^{J,\beta} \right) = \left. \frac{d^j}{dz^j} L(p(n), n_1 - p(n), \beta/2; z) - L(p(n), n_1 + n_2 - p(n), \beta/2; z) \right|_{z=0}.$$

For the Ginibre ensemble (starting with an arbitrary $n \times n$ matrix $A$ whose entries are independent real or complex Gaussian random variables with mean zero and variance one, see section 3.2.3), we observed in (3.8) that

$$\log \mathbb{E}\left[ \left( \det W_n^{G,\beta} \right)^z \right] = \frac{n z}{2} \log \left( \frac{2}{\beta} \right) + L(n, 0, \beta/2; z).$$

Here bounds on cumulants can be obtained starting with

$$\kappa_j \left( \log \det W_n^{G,\beta} \right) = \left. \frac{d^j}{dz^j} L(n, 0, \beta/2; z) \right|_{z=0} + 1_{\{j=1\}} \frac{n}{2} \log \frac{2}{\beta}.$$

More models were considered in sections 3.2.4 and 3.2.5. As we can see from (3.9) and (3.13), all models can be analyzed considering the $L$ in (4.16).
Random simplices

Recall from section 3.2.6 that if for \( p(n) \leq n \), \( X_1, \ldots, X_{p(n)+1} \) are independent random points in \( \mathbb{R}^n \) which are distributed according to a multivariate Gaussian distribution, then the \( p(n) \)-dimensional volume of the parallelotope spanned by the points \( X_1, \ldots, X_{p(n)} \), denoted by \( VP_{n,p(n)} \), is given by the determinant of the corresponding Gram matrix.

By (3.15), and with the definition of (4.16), we have

\[
\log \mathbb{E}[(VP_{n,p(n)})^z] = \frac{z}{2} p(n) \log 2 + L(p(n), n - p(n), 1/2; z/2).
\]

Recall that this is exactly the same as studying the asymptotic behavior of the log-determinant of a Laguerre ensemble in the case \( \beta = 1 \) for \( z/2 \) instead of \( z \), see (4.17). We obtain

\[
\kappa_j \left( \log \mathbb{E}[(VP_{n,p(n)})^z] \right) = \left. \frac{d^j}{dz^j} L(p(n), n - p(n), 1/2; z/2) \right|_{z=0} + 1_{\{j=1\}} \frac{p(n)}{2} \log 2.
\]

The only difference to our results in the \( \beta = 1 \) Laguerre case is that we have to use the identity

\[
\left. \frac{d^j}{dz^j} L(p(n), n - p(n), 1/2; z/2) \right|_{z=0} = \frac{1}{2j} \sum_{k=1}^{p(n)} \psi(j-1)\left( \frac{1}{2} (k + n - p(n)) \right).
\]

Therefore we only have to deal with the pre-factor \( \frac{1}{2j} \), which only changes the constants \( C_1(j) \) and \( C_2(j) \) in Theorem 4.3.20.

In the case of \( VS_{n,p(n)} \), the \( p(n) \)-dimensional volume of the simplex with vertices \( X_1, \ldots, X_{p(n)+1} \), we have by (3.19), that

\[
\log \mathbb{E}((p(n)! VS_{n,p(n)})^z) = \frac{z}{2} \log(p(n) + 1) + \log \mathbb{E}((VP_{n,p(n)})^z),
\]

where \( VP_{n,p(n)} \) is the volume of the corresponding parallelotope as defined previously. Again we can prove the same bounds as in Theorem 4.3.20.

Finally, in [74], the author studied the moments of order \( 2m \) of \( VP_{n,p(n)} \) and of \( VS_{n,p(n)} \), respectively, if the random points are distributed according to three other distributions, which are called the Beta model, the Beta prime model and the spherical model. All these models can be considered in the same way. Cumulant bounds can be found in [48], given case by case. The order of the bounds are the same and hence one can observe the same results as in Theorem 4.3.20.
Bibliography


