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MINIMAL CODIMENSION ONE FOLIATION OF A SYMMETRIC SPACE BY DAMEK-RICCI SPACES

GERHARD KNIEPER, JOHN R. PARKER, AND NORBERT PEYERIMHOFF

ABSTRACT. In this article we consider solvable hypersurfaces of the form $N \exp(\mathbb{R}H)$ with induced metrics in the symmetric space $M = SL(3, \mathbb{C})/SU(3)$, where H a suitable unit length vector in the subgroup A of the Iwasawa decomposition $SL(3, \mathbb{C}) = NAK$. Since M is rank 2, A is 2-dimensional and we can parametrize these hypersurfaces via an angle $\alpha \in [0, \pi/2]$ determining the direction of H. We show that one of the hypersurfaces (corresponding to $\alpha = 0$) is minimally embedded and isometric to the non-symmetric 7-dimensional Damek-Ricci space. We also provide an explicit formula for the Ricci curvatures of these hypersurfaces and show that all hypersurfaces for $\alpha \in (0, \frac{\pi}{2}]$ admit planes of both negative and positive sectional curvature. Moreover, the symmetric space M admits a minimal foliation with all leaves isometric to the non-symmetric 7-dimensional Damek-Ricci space.

1. INTRODUCTION

The purpose of this article is to study homogeneous hypersurfaces in the 8-dimensional symmetric space $SL(3, \mathbb{C})/SU(3)$. This rank two symmetric space can be canonically identified with the solvable group $S = NA$ with left invariant metric, using the Iwasawa decomposition $SL(3, \mathbb{C}) = NAK$, $K = SU(3)$. A specific orthonormal basis of the associated two-dimensional Lie algebra $\mathfrak{a} \subset T_eS$ is given by

$$H_0 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad H_1 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{2\sqrt{3}} \end{pmatrix} \in \mathfrak{a}.$$

Details are explained in Section 2 below. We have the following result:

Theorem 1.1. *Let $S = NA$ be the symmetric space $SL(3, \mathbb{C})/SU(3)$ with isometrically embedded hypersurfaces $S_H = N \exp(\mathbb{R}H)$, $H = \cos(\alpha)H_0 + \sin(\alpha)H_1$, $\alpha \in [0, \pi/2]$.*

Then S_H is a simply connected constant mean curvature (CMC) hypersurface with mean curvature $-4\sin(\alpha)$. Moreover, S_{H_0} is minimally embedded in S and isometric to the 7-dimensional Damek-Ricci space. In particular, S_{H_0} is a harmonic manifold, and therefore Einstein, with non-positive sectional curvature admitting planes of zero curvature.

Moreover, the following are equivalent:

(a) *$S_H \subset S$ is minimally embedded;*
(b) *the Cheeger constant of S_H is maximal,*
(c) *$H = H_0$.*

Damek-Ricci spaces are particularly important since they provide counterexamples to the Lichnerowicz Conjecture. According to this conjecture, all simply connected harmonic manifolds should be either flat or rank one symmetric spaces. Harmonic manifolds are characterized by the property that all harmonic functions (i.e., $\Delta f = 0$) have the mean value property, that is, the average of f over any geodesic sphere agrees with the value of f at the center (see [17]). It is well known that harmonic manifolds are Einstein (see [3]). In the compact case, the Lichnerowicz Conjecture was settled affirmatively by Szabó [16]. It was shown by Knieper [12] that all non-flat non-positively curved harmonic manifolds are Gromov hyperbolic and have the Anosov property. Damek-Ricci spaces are non-compact homogeneous harmonic manifolds of non-positive curvature and cover all rank one symmetric spaces except for the real hyperbolic spaces. It was shown by Heber [8] that there are no other homogeneous harmonic manifolds than the ones mentioned above and it is not known whether there are non-homogeneous harmonic examples. Dotti [7] provided the first complete proof that Damek-Ricci spaces admit planes of vanishing curvature if and only if they are non-symmetric. The smallest non-symmetric Damek-Ricci space has dimension 7. In brief, the above theorem tells us that we can recover this 7-dimensional non-symmetric Damek-Ricci space as a minimal hypersurface of the specific rank two symmetric space $SL(3,\mathbb{C})/SU(3)$. For more information about Damek-Ricci spaces and recent results on harmonic manifolds see, e.g., [6] or the surveys [2, 15, 13].

Remark. There is an analogous result for homogeneous hypersurfaces in $SL(3,\mathbb{R})/SO(3)$. The corresponding subspaces S_H are then 4-dimensional, simply connected CMC hypersurfaces with mean curvature $-2\sin(\alpha)$ and S_{H_0} is minimally embedded and isometric to the complex hyperbolic plane $\mathbb{C}H^2$. Since irreducible symmetric spaces do not admit totally geodesic hypersurfaces unless they have constant

curvature (see [11] or, more generally [1]), note that there is no totally geodesic embedding of $\mathbb{C}H^2$ into $SL(3,\mathbb{R})/SO(3)$.

As a consequence of Theorem 1.1 we obtain that $SL(3,\mathbb{C})/SU(3)$ has a natural minimal codimension one foliation with leaves isometric to the 7-dimensional Damek-Ricci space:

Corollary 1.2. *Let $\alpha \in [0, \frac{\pi}{2}]$ and the flow $\{\phi_H^s : S \to S\}_{s \in \mathbb{R}}$ be defined by*

$$\phi_H^s(q) := q \cdot \exp(sT_H)$$

with $T_H = \sin(\alpha)H_0 - \cos(\alpha)H_1 \perp T_e S_H$. Then S admits a codimension one foliation with leaves $\{\phi_H^s(S_H)\}_{s \in \mathbb{R}}$. Moreover, the leaves of this foliation are pairwise equidistant and isometric to S_H.

In the particular case $\alpha = 0$, all leaves of this foliation are minimal and isometric to the Damek-Ricci space S_{H_0}, and ϕ_H^s is volume preserving both in S and as a map between the leaves.

Finally, we investigate curvature properties of the hypersurfaces S_H. To state the result, we need a suitable orthonormal basis of $T_e S_H$, given by $V, iV, W, iW, Z_0, iZ_0, H$ with

$$V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem 1.3. *Let $X = aV + bW + cZ_0 + tH \in T_e S_H$ with $a, b, c \in \mathbb{C}$ and $t \in \mathbb{R}$ be a unit vector, that is $|a|^2 + |b|^2 + |c|^2 + t^2 = 1$. Then the Ricci curvature of X is given by*

$$\mathrm{Ric}^{S_H}(X) =$$
$$- 3 + 4\sin(\alpha)\left(\sin\left(\alpha - \frac{\pi}{3}\right)|a|^2 + \sin\left(\alpha + \frac{\pi}{3}\right)|b|^2 + \sin(\alpha)|c|^2\right).$$

In particular, the space S_H has strictly negative Ricci curvature if and only if $\alpha \in [0, \frac{\pi}{3})$. S_H admits directions of vanishing Ricci curvature for $\alpha = \frac{\pi}{3}$ and directions of positive Ricci curvature for $\alpha \in (\frac{\pi}{3}, \frac{\pi}{2}]$. In particular, S_H is Einstein if and only if $\alpha = 0$.

With regards to sectional curvature, the hypersurfaces S_H have always planes of positive and negative curvature unless $\alpha = 0$. ($\alpha = 0$ implies that S_H is a non-positively curved Damek-Ricci space.)

The structure of this article is as follows: In Section 2 we introduce the hypersurfaces S_H, compute their second fundamental form and Cheeger constants. Section 3 is devoted to the proof of Theorem 1.1 and Corollary 1.2. The curvature results presented in Theorem 1.3 are

proved in Section 4 using Maple computations. The Maple code can be found in Appendix A.

2. Basic geometric properties of the hypersurfaces S_H

2.1. **The Riemannian manifolds S and S_H.** Henceforth, let $G = SL(3, \mathbb{C})$ and $K = SU(3)$ and $\pi : G \to M = G/K$, $\pi(g) = gK$ be the canonical projection with $x_0 = \pi(e)$.

We briefly recall the construction of a Riemannian metric which makes $M = G/K$ a symmetric space: A Cartan involution on $\mathfrak{g}$ is given by $\theta : \mathfrak{g} \to \mathfrak{g}$, $\theta(X) = -\bar{X}^\top$. The Killing form

$$B(X_1, X_2) = \text{Tr}(\text{ad}\, X_1 \circ \text{ad}\, X_2) = 12\, \text{Re}\, \text{Tr}(X_1 X_2),$$

gives rise to the following inner product on $\mathfrak{g}$:

$$(1) \quad \langle X_1, X_2 \rangle_\mathfrak{g} = -\frac{1}{6} B(X_1, \theta X_2) = 2\text{Re}\, \text{Tr}(X_1 \bar{X}_2^\top)$$

$$= 2\text{Re}\left(\sum_{i,j} (X_1)_{ij} \overline{(X_2)_{ij}} \right).$$

Since $\ker D\pi(e) = \mathfrak{k}$, the differential $D\pi(e)$ provides a canonical identification of $\mathfrak{p}$ and $T_{x_0}M$, where

$$(2) \quad \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}, \quad X \mapsto \frac{1}{2}(X - \theta(X)) + \frac{1}{2}(X + \theta(X))$$

is the Cartan decomposition with $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$ and $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$. The restriction of $\langle \cdot, \cdot \rangle_\mathfrak{g}$ to $\mathfrak{p}$ induces an inner product on $T_{x_0}M$. Left-translation induces a Riemannian metric on M such that M becomes a rank two symmetric space of non-compact type.

Alternatively, we can view $M = G/K$ as a solvable group S with left invariant metric: the Iwasawa decomposition $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C}) = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$

on the Lie algebra level is given by

$$(3) \qquad \mathfrak{n} = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \middle| a, b, c \in \mathbb{C} \right\},$$

$$\mathfrak{a} = \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \middle| t_1, t_2, t_3 \in \mathbb{R}, t_1 + t_2 + t_3 = 0 \right\},$$

$$\mathfrak{k} = \mathfrak{su}(3) = \{ X \in \mathfrak{sl}(3, \mathbb{C}) \mid X = -\bar{X}^\top \}.$$

Let $N, A \subset G$ be the Lie groups corresponding to $\mathfrak{n}$ and $\mathfrak{a}$. Then the restriction of $\pi : G \to M$ to the solvable group $S = NA$ defines a diffeomorphism $\pi|_S : S \to M$, $s \mapsto sK$. The pull-back of the Riemannian metric on M via this diffeomorphism equips S with a left-invariant metric. This left-invariant metric induces an inner product $\langle \cdot, \cdot \rangle_\mathfrak{s}$ on the Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a} = T_e S$ of S. Using (2) we have the following identifications:

$$T_e S = \mathfrak{s} \to \quad T_{x_0} M = \mathfrak{g}/\mathfrak{k} \quad \to \mathfrak{p},$$

$$X \mapsto \quad X + \mathfrak{k} \quad \mapsto \frac{1}{2}(X - \theta(X)) = \frac{1}{2}(X + \bar{X}^\top),$$

leading to the linear isometry $\phi : \mathfrak{s} \to \mathfrak{p}$, $\phi(X) = \frac{1}{2}(X + \bar{X}^\top)$. Our next aim is to calculate $\langle \cdot, \cdot \rangle_\mathfrak{s}$: Let $X_1 = Y_1 + H, X_2 = Y_2 + \tilde{H} \in \mathfrak{s}$ with $Y_1, Y_1 \in \mathfrak{n}$ and $H, \tilde{H} \in \mathfrak{a}$. We have

$$\langle X_1, X_2 \rangle_\mathfrak{s} = \langle \phi(X_1), \phi(X_2) \rangle_\mathfrak{g} = \frac{1}{4} \langle X_1 + \bar{X}_1^\top, X_2 + \bar{X}_2^\top \rangle_\mathfrak{g}.$$

Using (1), we obtain $\mathfrak{a} \perp \mathfrak{n}$ and $\mathfrak{n} \perp \mathfrak{n}^\top$ with respect to $\langle \cdot, \cdot \rangle_\mathfrak{g}$ and, therefore, we have

$$(4) \quad \langle Y_1 + H, Y_2 + \tilde{H} \rangle_\mathfrak{s} = \frac{1}{4} \langle Y_1 + \bar{Y}_1^\top + 2H, Y_2 + \bar{Y}_2^\top + 2\tilde{H} \rangle_\mathfrak{g}$$

$$= \frac{1}{2} \langle Y_1, Y_2 \rangle_\mathfrak{g} + \langle H, \tilde{H} \rangle_\mathfrak{g}$$

$$= \mathrm{Re}\left(\mathrm{Tr}(Y_1 \bar{Y}_2^\top) \right) + 2 \mathrm{Tr}(H \tilde{H}).$$

In particular, we have $\mathfrak{a} \perp \mathfrak{n}$ with respect to $\langle \cdot, \cdot \rangle_\mathfrak{s}$ and the matrices

$$(5) \qquad H_0 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad H_1 = \begin{pmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{2\sqrt{3}} \end{pmatrix}$$

form an orthonormal basis of the 2-dimensional vector space $\mathfrak{a}$. Any matrix in $\mathfrak{a}$ of unit length can then be expressed as

$$(6) \qquad H = \cos(\alpha)H_0 + \sin(\alpha)H_1,$$

and we define the corresponding hypersurface by

$$S_H = N\exp(\mathbb{R}H) \subset S = NA.$$

2.2. **The second fundamental form of S_H.** Next we want to compute the second fundamental form of $S_H \subset S$ explicitly. The vector $T_H = \sin(\alpha)H_0 - \cos(\alpha)H_1 \in \mathfrak{s}$ is a unit vector orthogonal to $T_e S_H = \mathfrak{s}_H = \mathfrak{n} \oplus \mathbb{R}H$ with respect to $\langle \cdot, \cdot \rangle_\mathfrak{s}$. Its left invariant extension along S_H provides a global unit normal vector field of $S_H \subset S$. Any $X \in \mathfrak{s}_H$ can be written as $X = aV + bW + cZ_0 + tH$ with $a, b, c \in \mathbb{C}$, $t \in \mathbb{R}$ and

$$(7) \qquad V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $V, iV, W, iW, Z_0, iZ_0, H, T_H$ form an orthonormal basis of $\mathfrak{s}$. Henceforth ∇^S denotes the Levi-Civita connection of S.

Proposition 2.1. *Let $H = \cos(\alpha)H_0 + \sin(\alpha)H_1$ with H_0, H_1 given in (5). Then the second fundamental form of S_H is given by*

$$\nabla^S_{aV+bW+cZ_0+tH} T_H =$$

$$a\left(\frac{\sqrt{3}}{2}\cos\alpha - \frac{\sin\alpha}{2}\right)V + b\left(-\frac{\sqrt{3}}{2}\cos\alpha - \frac{\sin\alpha}{2}\right)W - c\left(\sin\alpha\right)Z_0.$$

Moreover S_H is a CMC hypersurface in S with mean curvature

$$(8) \qquad M(\alpha) = -4\sin\alpha.$$

Remark. Note that the hypersurfaces S_H are horospheres iff $\alpha \in [\frac{\pi}{3}, \frac{\pi}{2}]$ (with the singular horosphere at $\alpha = \frac{\pi}{3}$ and the barycentric horosphere at $\alpha = \frac{\pi}{2}$) in which case the eigenvalues of the second fundamental form, given by $\pm\frac{\sqrt{3}}{2}\cos\alpha - \frac{\sin\alpha}{2}, -\sin(\alpha), 0$ are non-positive.

Proof. Using the canonical identification of $\mathfrak{s}$ with left invariant vector fields on S and applying Koszul's formula, we obtain

$$\langle \nabla^S_{X_1} X_2, X_3 \rangle_\mathfrak{s} = \frac{1}{2}\left(\langle X_1, [X_3, X_2]\rangle_\mathfrak{s} + \langle X_2, [X_1, X_3]\rangle_\mathfrak{s} + \langle X_3, [X_1, X_2]\rangle_\mathfrak{s}\right)$$

for $X_1, X_2, X_3 \in \mathfrak{s}$. This in particular implies,

$$(9) \qquad \langle \nabla^S_{X_1} T_H, X_2 \rangle_\mathfrak{s} = \frac{1}{2}\left(\langle X_1, [X_2, T_H]\rangle_\mathfrak{s} + \langle X_2, [X_1, T_H]\rangle_\mathfrak{s}\right),$$

since $\langle T_H, [X_1, X_2]\rangle_{\mathfrak{s}} = 0$ because of $[X_1, X_2] \in \mathfrak{s}_H$. A straightforward calculation shows

$$(10) \qquad [E_{ij}, T] = E_{ij}T - TE_{ij} = (t_j - t_i)E_{ij}$$

with E_{ij} a 3×3 matrix with all entries equals 0 except for one entry equals 1 at position (i, j) and T a diagonal matrix with diagonal entries (t_1, t_2, t_3). Since

$$T_H = \begin{pmatrix} \frac{\sin\alpha}{2} - \frac{\cos\alpha}{2\sqrt{3}} & 0 & 0 \\ 0 & \frac{\cos\alpha}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{\sin\alpha}{2} - \frac{\cos\alpha}{2\sqrt{3}} \end{pmatrix},$$

this implies that

$$[aV + bW + cZ_0 + tH, T_H] =$$
$$a\left(-\frac{1}{2}\sin\alpha + \frac{\sqrt{3}}{2}\cos\alpha\right)V + b\left(-\frac{1}{2}\sin\alpha - \frac{\sqrt{3}}{2}\cos\alpha\right)W - c(\sin\alpha)Z_0.$$

Consequently, $\nabla_\bullet T_H$ has diagonal structure with respect to $V, iV, W, iW,$ Z_0, iZ_0, H, and we have

$$\nabla_V^S T_H = \langle \nabla_V^S T_H, V\rangle_{\mathfrak{s}} V = \langle V, [V, T_H]\rangle_{\mathfrak{s}} V = \left(-\frac{1}{2}\sin\alpha + \frac{\sqrt{3}}{2}\cos\alpha\right)V,$$

and similarly for the other unit vectors. This finishes the proof of Proposition 2.1. $\qquad \square$

2.3. **The Cheeger constant of S_H.** The Cheeger isoperimetric constant $h(M)$ of a complete non-compact Riemannian manifold M is defined by

$$h(M) = \inf_{K \subset M} \frac{\text{area}(\partial K)}{\text{vol}(K)},$$

where K ranges over all connected, open submanifolds of M with compact closure and smooth boundary.

A formula for this constant was given in [14] for general solvable groups with left invariant metric. Since S_H is a solvable group, we obtain from this formula

$$(11) \qquad h(S_H) = \max_{X \in \mathfrak{s}_H, \|X\|_{\mathfrak{s}}=1} \text{Tr}(\text{ad}\, X),$$

where $\text{ad}\, X(\tilde{X}) = [X, \tilde{X}]$ is viewed as linear transformation on the 7-dimensional real vector space $\mathfrak{s}_H$ spanned by $V, iV, W, iW, Z_0, iZ_0, H$. This is the main ingredient of the proof of the following result:

Proposition 2.2. *Let* $H = \cos(\alpha)H_0 + \sin(\alpha)H_1$ *with* H_0, H_1 *given in* (5). *Then the Cheeger constant of* S_H *is given by*

$$h(S_H) = 4\cos\alpha.$$

In particular, S_{H_1} *has a vanishing Cheeger constant.*

Proof. In view of (11) we only have to calculate $\mathrm{Tr}(\mathrm{ad}\,X)$ for $X = aV + bW + cZ_0 + tH$ with $|a|^2 + |b|^2 + |c|^2 + t^2 = 1$ with $a, b, c \in \mathbb{C}$ and $t \in \mathbb{R}$. Using (10) we conclude for $e \in \{1, i\}$ that

$$[H, eV] = \left(\frac{\cos\alpha}{2} + \frac{\sqrt{3}}{2}\sin\alpha\right)eV,$$

$$[H, eW] = \left(\frac{\cos\alpha}{2} - \frac{\sqrt{3}}{2}\sin\alpha\right)eW,$$

$$[H, eZ_0] = (\cos\alpha)eZ_0.$$

Note that the traces of $\mathrm{ad}\,eV, \mathrm{ad}\,eW$ and $\mathrm{ad}\,eZ_0$ vanish since the matrix representations of these operators have zero for each diagonal entry. This implies that

$$h(S_H) = \max_{X \in \mathfrak{s}_H, \|X\|_{\mathfrak{s}} = 1} \mathrm{Tr}(\mathrm{ad}\,X) = \mathrm{Tr}(\mathrm{ad}\,H) =$$

$$2\left(\frac{\cos\alpha}{2} + \frac{\sqrt{3}}{2}\sin\alpha\right) + 2\left(\frac{\cos\alpha}{2} - \frac{\sqrt{3}}{2}\sin\alpha\right) + 2\cos\alpha = 4\cos\alpha.$$

$\square$

3. Proof of Theorem 1.1 and Corollary 1.2

For the reader's convenience, we recall Theorem 1.1 from the Introduction:

Theorem. *Let* $S = NA$ *be the symmetric space* $SL(3, \mathbb{C})/SU(3)$ *with isometrically embedded hypersurfaces* $S_H = N \exp(\mathbb{R}H)$, $H = \cos(\alpha)H_0 + \sin(\alpha)H_1$, $\alpha \in [0, \pi/2]$, *with* H_0, H_1 *given in* (5).

Then S_H *is a simply connected CMC hypersurface with mean curvature* $-4\sin(\alpha)$ *and* S_{H_0} *is minimally embedded in* S *and isometric to the 7-dimensional Damek-Ricci space. In particular,* S_{H_0} *is a harmonic manifold, and therefore Einstein, with non-positive sectional curvature admitting planes of zero curvature.*

Moreover, the following are equivalent:

(a) $S_H \subset S$ *is minimally embedded;*
(b) *the Cheeger constant of* S_H *is maximal,*
(c) $H = H_0$.

Proof. The solvable group S_{H_0} with left invariant metric is a Damek-Ricci space if the following properties of $(\mathfrak{s}_{H_0}, \langle \cdot, \cdot \rangle_{\mathfrak{s}})$ are satisfied:

(1) $\mathfrak{s}_{H_0} = \mathfrak{n} \oplus \mathbb{R} H_0$, $\mathfrak{n} \perp H_0$ and H_0 is a unit vector with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{s}}$;

(2) $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ with $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$ and $[\mathfrak{v}, \mathfrak{z}], [\mathfrak{z}, \mathfrak{z}] = \{0\}$ (that is $\mathfrak{n}$ is 2-step nilpotent);

(3) $\mathfrak{v} \perp \mathfrak{z}$ with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{s}}$;

(4) let $Z \in \mathfrak{z}$; then the map $J_Z \in \operatorname{End}(\mathfrak{v})$, defined by

$$\langle J_Z(U_1), U_2 \rangle_{\mathfrak{s}} = \langle Z, [U_1, U_2] \rangle_{\mathfrak{s}} \quad \text{for all } U_1, U_2 \in \mathfrak{v},$$

satisfies $J_Z^2 = -\|Z\|^2 \operatorname{id}_{\mathfrak{v}}$;

(5) $[H_0, U] = \frac{1}{2} U$ for all $U \in \mathfrak{v}$ and $[H_0, Z] = Z$ for all $Z \in \mathfrak{z}$.

We note that a Lie algeba $\mathfrak{n}$ satisfying properties (2), (3) and (4) is called a Lie algebra of Heisenberg type.

Properties (1), (2), (3) and (5) are obviously satisfied by choosing $\mathfrak{v} = \mathbb{C}V \oplus \mathbb{C}W$ and $\mathfrak{z} = \mathbb{C}Z_0$ since $V, iV, W, iW, Z_0, iZ_0, H_0$ are an orthonormal basis of $\mathfrak{s}_{H_0}$ with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{s}}$. For example, (2) follows from $[V, W] = VW - WV = Z_0$ and (5) follows from

$$\left[H_0, \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \right] =$$

$$\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{a}{2} & c \\ 0 & 0 & \frac{b}{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

To show (4), we define for $Z = zZ_0$, $z \in \mathbb{C}$,

$$J_Z \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = z \begin{pmatrix} 0 & -\bar{b} & 0 \\ 0 & 0 & \bar{a} \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we have

$$\left\langle J_{Z_0} \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \right\rangle_{\mathfrak{s}} = \operatorname{Re}(\overline{ad - bc}) =$$

$$\left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & ad - bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle_{\mathfrak{s}} = \left\langle Z_0, \left[\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \right] \right\rangle_{\mathfrak{s}}$$

and

$$J_{Z_0}^2 \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} = J_{Z_0} \begin{pmatrix} 0 & -\bar{b} & 0 \\ 0 & 0 & \bar{a} \\ 0 & 0 & 0 \end{pmatrix} = - \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that S_{H_0} is the 7-dimensional Damek-Ricci space which is, therefore, a harmonic manifold (see [6]). The space S_{H_0} cannot be a symmetric space since $\dim_{\mathbb{R}} \mathfrak{z} = 2$ and the centres of symmetric Damek-Ricci spaces must have dimension $1, 3$ or 7. It was shown independently by [4] and [5] that all Damek-Ricci spaces have non-positive sectional curvature and by [7] that these spaces admit planes of zero curvature if and only if they are non-symmetric.

Finally, the equivalences of (a), (b) and (c) follow immediately from Propositions 2.1 and 2.2. $\qquad\square$

Remark. In the case of the rank two symmetric space $\mathbb{H}^2 \times \mathbb{H}^2$ (where $\mathbb{H}^k$ denotes the k-dimensional real hyperbolic space) a similar analysis shows that S_{H_0} is of constant negative curvature, that is, agrees with $\mathbb{H}^3$ up to scaling. Here the direction H_0 in the flat $\mathfrak{a}$ is characterized by the fact that S_{H_0} is minimally embedded in $\mathbb{H}^2 \times \mathbb{H}^2$. It would be interesting to investigate which of the corresponding hypersurfaces in rank two symmetric spaces of non-compact type are harmonic manifolds.

Theorem 1.1 has the following consequence:

Corollary. *Let $\alpha \in [0, \frac{\pi}{2}]$ and the flow $\{\phi_H^s : S \to S\}_{s \in \mathbb{R}}$ be defined by*

$$\phi_H^s(q) := q \cdot \exp(sT_H).$$

Then S admits a codimension one foliation with leaves $\{\phi_H^s(S_H)\}_{s \in \mathbb{R}}$. Moreover, the leaves of this foliation are pairwise equidistant and isometric to S_H.

In the particular case $\alpha = 0$, all leaves of this foliation are minimal and isometric to the Damek-Ricci space S_{H_0}, and ϕ_H^s is volume preserving both in S and as a map between the leaves.

Proof. By abuse of notation, we extend $T_H \in \mathfrak{s} = T_e S$ to a global unit vector field on S, again denoted by T_H, orthogonal to S_H and given by

$$T_H(q) = \frac{d}{ds}\Big|_{s=0} q \exp(sT_H).$$

Then ϕ_H^s is the associated flow and its flow lines $s \mapsto \phi_H^s(q)$ are geodesics in S through q. This implies that the leaves are equidistant.

Next we show that all leaves are isometric to S_H: Let $F_H^s : S \to S$ be the isometry $F_H^s(q) = \exp(sT_H)q$. Then we have for all $q \in S_H$ that there exists $q' \in S_H$ with

(12) $$\phi_H^s(q) = F_H^s(q'),$$

and, therefore, $\phi_H^s(S_H)$ and $F_H^s(S_H)$ coincide as sets and are isometric to S_H. Indeed, if

$$q = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \exp(tH) \in S_H$$

and

$$\exp(sT_H) = \begin{pmatrix} e^{\tau_1} & 0 & 0 \\ 0 & e^{\tau_2} & 0 \\ 0 & 0 & e^{\tau_3} \end{pmatrix},$$

with suitable $\tau_1, \tau_2, \tau_3 \in \mathbb{R}$, then (12) is satisfied if

$$q' = \begin{pmatrix} 1 & e^{\tau_2-\tau_1}x & e^{\tau_3-\tau_1}z \\ 0 & 1 & e^{\tau_3-\tau_2}y \\ 0 & 0 & 1 \end{pmatrix} \exp(tH) \in S_H.$$

We know from Theorem 1.1 that S_{H_0} is a Damek-Ricci space and minimal in S. Since F_H^s is an isometry mapping leaves to leaves, the mean curvature is preserved for all leaves. Finally, the volume distortion of the flow ϕ_H^s on both S and as a map between the leaves is given by $e^{sM(\alpha)}$ with the mean curvature $M(\alpha) = -4\sin(\alpha)$ given in (8). Hence ϕ_H^s is volume preserving for $\alpha = 0$. $\qquad\square$

4. Curvature considerations for the hypersurfaces S_H

This section is devoted to the proof of Theorem 1.3 from the Introduction which states the following:

Theorem. *Let $X = aV + bW + cZ_0 + tH \in \mathfrak{s}_H$ with V, W, Z_0 given in (7) and H given in (6). We assume that X is a unit vector, that is $a, b, c \in \mathbb{C}$ and $t \in \mathbb{R}$ with $|a|^2 + |b|^2 + |c|^2 + t^2 = 1$. Then the Ricci curvature of X is given by*

$$\mathrm{Ric}^{S_H}(X) =$$
$$- 3 + 4\sin(\alpha)\left(\sin\left(\alpha - \frac{\pi}{3}\right)|a|^2 + \sin\left(\alpha + \frac{\pi}{3}\right)|b|^2 + \sin(\alpha)|c|^2\right).$$

In particular, the space S_H has strictly negative Ricci curvature if and only if $\alpha \in [0, \frac{\pi}{3})$. S_H admits directions of vanishing Ricci curvature for $\alpha = \frac{\pi}{3}$ and directions of positive Ricci curvature for $\alpha \in (\frac{\pi}{3}, \frac{\pi}{2}]$. In particular, S_H is Einstein if and only if $\alpha = 0$.

With regards to sectional curvature, the hypersurfaces S_H have always planes of positive and negative curvature unless $\alpha = 0$. ($\alpha = 0$ implies that S_H is a non-positively curved Damek-Ricci space.)

Before we enter the proof we like to make the following general remark.

Remark. The following result was shown in Heber [9, Theorem 4.18] (related to earlier work by Wolter [18]): Let $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ be a Lie algebra of Iwasawa type with inner product Q which is Einstein and $H_Q \in \mathfrak{s}$ be the vector defined by $Q(H_Q, X) = \operatorname{Tr} \operatorname{ad}_X$ for all $X \in \mathfrak{s}$. Then the metric subalgebra $(\mathfrak{a}' \oplus \mathfrak{n}, Q)$ with non-trivial subspace $\mathfrak{a}' \subset \mathfrak{a}$ is Einstein if and only if $H_Q \in \mathfrak{a}'$. In particular, $(\mathbb{R}H_Q \oplus \mathfrak{n}, Q)$ is Einstein.

Note that our Lie algebra $(\mathfrak{s}, \langle \cdot, \cdot \rangle_s)$ is Einstein since its corresponding Lie group with left invariant metric is a symmetric space and we can apply this result with $Q = \langle \cdot, \cdot \rangle_\mathfrak{s}$. A straightforward calculation yields then $H_Q = 4 \cdot H_0$ and Heber's result agrees with our result that amongst all hypersurfaces S_H with $H = \cos(\alpha)H_0 + \sin(\alpha)H_1$ only S_{H_0} is an Einstein manifold.

It would be interesting to investigate which of the homogeneous Einstein manifolds appearing in the more general setting of Heber are Damek-Ricci spaces.

Proof. Let R^S be the Riemannian curvature tensor of S given by

$$R^S(X_1, X_2)X_3 = \nabla^S_{X_1}\nabla^S_{X_2}X_3 - \nabla^S_{X_2}\nabla^S_{X_1}X_3 - \nabla^S_{[X_1, X_2]}X_3$$

and R^{S_H} be the corresponding curvature tensor of S_H.

The derivation of the expression (15) is based on the Gauss equation:

$$(13) \quad \langle R^{S_H}(X_1, X)X, X_1 \rangle_\mathfrak{s} = \langle R^S(X_1, X)X, X_1 \rangle_\mathfrak{s} +$$
$$\langle \nabla^S_{X_1}T_H, X_1 \rangle_\mathfrak{s} \langle \nabla^S_X T_H, X \rangle_\mathfrak{s} - \left(\langle \nabla^S_{X_1}T_H, X \rangle_\mathfrak{s} \right)^2,$$

where $X_1 \in \mathfrak{s}_H = \mathfrak{n} \oplus \mathbb{R}H = T_e S_H$ and $T_H = \sin(\alpha)H_0 - \cos(\alpha)H_1 \in \mathfrak{s}$. The ingredients in (13) are explicitly calculated using

$$\langle R^S(X_1, X_2)X_2, X_1 \rangle_\mathfrak{s} = -\langle [[\phi(X_1), \phi(X_2)], \phi(X_2)], \phi(X_1) \rangle_\mathfrak{g}$$

from the theory of symmetric spaces (see, e.g., [10, Theorem IV.4.2]) and the following consequence of Koszul's formula (see (9)):

$$\langle \nabla^S_{X_1}T_H, X_2 \rangle_\mathfrak{s} = \frac{1}{2} \left(\langle \Phi(X_1), \Phi([X_2, T_H]) \rangle_\mathfrak{g} + \langle \Phi(X_2), \Phi([X_1, T_H]) \rangle_\mathfrak{g} \right).$$

The Ricci curvature is then given by

$$(14) \quad \operatorname{Ric}^{S_H}(X) = \langle R^{S_H}(V, X)X, V \rangle_\mathfrak{s} + \langle R^{S_H}(iV, X)X, iV \rangle_\mathfrak{s} +$$
$$\langle R^{S_H}(W, X)X, W \rangle_\mathfrak{s} + \langle R^{S_H}(iW, X)X, iW \rangle_\mathfrak{s} +$$
$$\langle R^{S_H}(Z_0, X)X, Z_0 \rangle_\mathfrak{s} + \langle R^{S_H}(iZ_0, X)X, iZ_0 \rangle_\mathfrak{s} + \langle R^{S_H}(H, X)X, H \rangle_\mathfrak{s}.$$

The calculation of (14) in the case $X = aV + bW + CZ_0 + tH$ with $|a|^2 + |b|^2 + |c|^2 + t^2 = 1$ was done with Maple (see Appendix A) with the following result:

$$\mathrm{Ric}^{S_H}(X) = -2\sqrt{3}\sin(\alpha)\cos(\alpha)(|a|^2 - |b|^2)$$
$$- 2(|a|^2 + |b|^2 + 2|c|^2)\cos^2(\alpha) - 3t^2 - |a|^2 - |b|^2 + |c|^2,$$

which simplifies to

$$(15) \quad \mathrm{Ric}^{S_H}(X) =$$
$$- 3 + 4\sin(\alpha)\left(\sin\left(\alpha - \frac{\pi}{3}\right)|a|^2 + \sin\left(\alpha + \frac{\pi}{3}\right)|b|^2 + \sin(\alpha)|c|^2\right),$$

using $|a|^2 + |b|^2 + |c|^2 = 1 - t^2$.

In order to find the maximum of (15) for a given value of $\alpha \in [0, \frac{\pi}{2}]$, it is sufficient to assume that a, b, c are real with $a^2 + b^2 + c^2 \leq 1$. Let

$$f_\alpha(a, b, c) = 4\sin(\alpha)\left(\sin\left(\alpha - \frac{\pi}{3}\right)|a|^2 + \sin\left(\alpha + \frac{\pi}{3}\right)|b|^2 + \sin(\alpha)|c|^2\right).$$

Since $f_\alpha(a, b, c)$ is a homogeneous polynomial of degree 2, we have

$$\max_{a^2+b^2+c^2\leq 1} f_\alpha(a, b, c) = \max_{a^2+b^2+c^2=1} f_\alpha(a, b, c).$$

When $a^2 + b^2 + c^2 = 1$, it is obvious that the maximal value of f_α is equal to

$$4\sin(\alpha)\max\left\{\sin\left(\alpha - \frac{\pi}{3}\right), \sin\left(\alpha + \frac{\pi}{3}\right), \sin(\alpha)\right\},$$

and we obtain

$$-3 + \max_{a^2+b^2+c^2=1} f_\alpha(a, b, c) = \begin{cases} 4\sin(\alpha)\sin(\alpha + \frac{\pi}{3}) - 3 & \text{if } 0 \leq \alpha \leq \frac{\pi}{3}; \\ 4\sin^2(\alpha) - 3 & \text{if } \frac{\pi}{3} < \alpha \leq \frac{\pi}{2}. \end{cases}$$

This means that the maximum is strictly monotone in α and vanishes at $\alpha = \frac{\pi}{3}$, which implies the statements about the Ricci curvature signs.

Finally, we have $f_0(a, b, c) = -3$ and S_H is Einstein for $\alpha = 0$. For $\alpha \in (0, \frac{\pi}{2}]$, we have $f_\alpha(0, 0, c) = 4c^2\sin^2(\alpha) - 3$ which is non-constant since $c \in [-1, 1]$. This implies that S_H is not Einstein in this case.

Concerning sectional curvature, we consider the plane $\sigma \subset \mathfrak{s}_H$ spanned by the orthonormal vectors

$$X_1 = \sqrt{\frac{2}{3}}W + \frac{1}{\sqrt{3}}Z_0 \quad \text{and} \quad X_2 = -\sqrt{\frac{2}{3}}iW + \frac{1}{\sqrt{3}}iZ_0.$$

Using (13) we obtain again with the help of Maple (see Appendix A)

$$K^{S_H}(\sigma) = \langle R^{S_H}(X1, X2)X2, X1\rangle_\mathfrak{s} = \frac{4}{3\sqrt{3}}\sin(\alpha)\cos(\alpha) + \frac{1}{9}\sin(\alpha)^2.$$

This expression vanishes only if $\alpha = 0$ and is strictly positive for any $\alpha \in (0, \frac{\pi}{2}]$. Moreover, since $\mathrm{Ric}^{S_H}(H) = -3$ for all $\alpha \in [0, \frac{\pi}{2}]$, there are also planes of strictly negative curvature. $\qquad\square$

APPENDIX A. MAPLE CALCULATIONS

In this appendix, we discuss the Maple code for the calculation of Ricci curvature of hypersurface S_H within $SL(3, C)/SU(3)$ and the existence of planes with positive sectional curvatures.

The following lines guarantee that Maple treats α and t as real variables:

```
> with(LinearAlgebra):
> assume(alpha, 'real'):  assume(t, 'real'):
```

Next, we define the map $\phi : \mathfrak{s} \to \mathfrak{p}$ and the Lie bracket $[\cdot, \cdot]$ (in Maple denoted by $LB(\cdot, \cdot)$):

```
> Phi := X -> (1/2)*X+(1/2)*conjugate(Transpose(X)):
> LB := (X1, X2) -> X1.X2-X2.X1:
```

Now, we define the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ (in Maple denoted by $G(\cdot, \cdot)$) and the unit vectors H_0, H_1 and $H = \cos(\alpha)H_0 + \sin(\alpha)H_1, T_H = \sin(\alpha)H_0 - \cos(\alpha)H_1$, V, W, Z_0 in the tangent space $\mathfrak{s}_H = T_e S_H$ of the hypersurface S_H:

```
> G := (X1, X2) -> 2*Trace(X1.conjugate(Transpose(X2))):
> H0 := Matrix([[1/2, 0, 0], [0, 0, 0], [0, 0, -1/2]]):
> H1 := Matrix([[(1/6)*3^(1/2), 0, 0],
[0, -(1/3)*3^(1/2), 0], [0, 0, (1/6)*3^(1/2)]]):
> H := cos(alpha)*H0+sin(alpha)*H1:
> T_H := sin(alpha)*H0-cos(alpha)*H1:
> V := Matrix([[0, 1, 0], [0, 0, 0], [0, 0, 0]]):
> W := Matrix([[0, 0, 0], [0, 0, 1], [0, 0, 0]]):
> Z0 := Matrix([[0, 0, 1], [0, 0, 0], [0, 0, 0]]):
```

The Riemannian curvature tensor $\langle R^S(X_1, X_2)X_2, X_1 \rangle_{\mathfrak{g}}$ in the ambient space S (in Maple denoted by $R_S(X1, X2)$), the second fundamental form: $\langle \nabla^S_{X_1} T_H, X_2 \rangle_{\mathfrak{g}}$ (in Maple denoted by $SecFF(X1, X2)$), the curvature tensor $\langle R^{S_H}(X_1, X_2)X_2, X_1 \rangle_{\mathfrak{g}}$ in the hypersurface S_H (in

Maple denoted by $R_{SH}(X1, X2)$ and the Ricci curvature $\mathrm{Ric}^{S_H}(X)$ (in Maple denoted by $Ric_{SH}(X)$ are introduced via the following lines:

```
>   R_S := (X1, X2) -> -G(LB(LB(Phi(X1),Phi(X2)),
Phi(X2)), Phi(X1)):
>   SecFF := (X1, X2) -> (1/2)*G(Phi(X1),Phi(LB(X2,T_H)))
+ (1/2)*G(Phi(X2),Phi(LB(X1,T_H))):
>   R_SH := (X1, X2) -> R_S(X1, X2) + SecFF(X1, X1)*
SecFF(X2, X2) - (SecFF(X1, X2))^2:
>   Ric_SH := X -> R_SH(V, X) + R_SH(I*V, X) + R_SH(W, X)
+ R_SH(I*W, X) + R_SH(Z0, X) + R_SH(I*Z0, X) + R_SH(H, X):
```

The relevant results are now obtained via the following lines:

```
>   simplify(expand(Ric_SH(a*V+b*W+c*Z0+t*H)));
```

$$-2 * \cos(\alpha) * \sin(\alpha) * (|a| - |b|) * (|a| + |b|) * \sqrt{3}+$$

$$(-2 * |a|^2 - 2 * |b|^2 - 4 * |c|^2) * \cos(\alpha)^2 - 3 * t^2 - |a|^2 - |b|^2 + |c|^2$$

```
>   simplify(expand(R_SH((2/3)^(1/2)*W+(1/3)*3^(1/2)*Z0,
-(2/3)^(1/2)*I*W+(1/3)*3^(1/2)*I*Z0)));
```

$$(4/9) * \sin(\alpha) * \cos(\alpha) * \sqrt{3} + (1/9) * \sin(\alpha)^2$$

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