University of Heidelberg

Department of Economics

Discussion Paper Series | No. 493

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December 2009

POPULATION AGING AND THE DIRECTION OF TECHNICAL CHANGE

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Abstract: An analytical framework is developed to study the repercussions between endogenous capital- and labor-saving technical change and population aging. Following an intuition often attributed to Hicks (1932), I ask whether and how population aging affects the relative scarcity of factors of production, relative factor prices, and the direction of induced technical change. Aging is equivalent to an increase in the old-age dependency ratio of an OLG-economy with two-period lived individuals. In this framework aging increases the relative scarcity of labor with respect to capital. Therefore, there will be more labor- and less capital-saving technical change. Unless there are contemporaneous knowledge spillovers across innovating firms technical change induced by a small increase in the old-age dependency ratio has no first-order effect on current GDP. The presence of capital-saving technical change is shown to imply that the economy's steady-state growth rate is independent of its age structure.

Keywords: Demographic Transition, Capital Accumulation, Direction of Technical Change.

JEL-Classification: D91, D92, O33, O41,

This Version: December 10, 2009.

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I am grateful to Hendrik Hakenes, Burkhard Heer, Johanna Kuehnel, Edgar Vogel and audiences at Mannheim, Frankfurt, Heidelberg, and at the CESifo Area Conference - Public Sector Economics, May 2009, for helpful suggestions. Moreover, I would like to thank Christoph Lipponer for competent research assistance.

1 Introduction

Population aging, i. e., the process by which older individuals become a proportionally larger fraction of the total population, is an enduring phenomenon in many of today's developed and developing countries. Table 1 shows actual data and predictions of the United Nations concerning the old-age dependency ratio for several countries and regions.[1] Roughly speaking, between 2005 and 2050 this ratio is estimated to double in Europe and Northern America. In China, India, and Japan its predicted increase is even more pronounced. Such drastic demographic developments have become a severe burden for these economies. To meet this challenge it is essential to understand the economic consequences of population aging.

This paper studies the effect of population aging on innovation incentives and economic growth in an environment where the direction of technical change is endogenous, i. e., firms may undertake innovation investments that generate capital- or labor-saving technical change. My starting point is Hicks' contention according to which innovation incentives depend on relative factor prices reflecting the relative relative scarcity of these factors. In his book, *The Theory of Wages*, he asserts that

> "A change in the relative prices of the factors of production is itself a spur to invention, and to invention of a particular kind - directed to economizing the use of a factor which has become relatively expensive. ..." (Hicks (1932), p. 124)

Since the process of population aging tends to reduce the labor force, the factor labor becomes, ceteris paribus, scarcer relative to the factor capital. Accordingly, its relative price may increase and labor-saving innovations become more profitable. For the same reason, the incentive to direct innovation investments towards capital-saving innovations declines. Hence, if Hicks' contention applies, population aging may affect the direction of technical change.

However, from the perspective of a dynamic general equilibrium, the validity of Hicks' contention does not only hinge on the evolution of the labor force. It also depends on the ability and the willingness of an aging population to save and to accumulate capital. Fewer workers and/or a declining real rate of return may reduce aggregate savings. Hence, to understand the effect of population aging on the

[1]These numbers appear in United Nations (2008) as the 'medium variant' prediction. The old-age dependency ratio is the ratio of the population aged 65 or over to the population aged 15-64. This ratio is stated as the number of dependants per 100 persons of working age (15-64).

Table 1: Old-Age Dependency Ratios in Selected Countries and Regions.

Year	Europe	Northern America	China	India	Japan
2005	23	19	11	7	30
2050	47	36	38	20	74

direction of technical change one has to disentangle the intricate relationship between partial and general equilibrium effects. It is a primary purpose of the present paper to accomplish this. Building on the insights of my analysis, I inquire into the implications of population aging for steady-state and transitional growth.

To address these questions, I devise a new neoclassical growth model with endogenous capital- and labor-saving technical change. The model is set up in discrete time with two-period lived overlapping generations as in Allais (1947), Samuelson (1958), or Diamond (1965). This framework allows for a straightforward representation of population aging as an increase in the old-age dependency ratio. Both, a decline in the growth rate of the labor force and/or an increase in life expectancy parameterized by a probability to live through the period of old age, augment this ratio.

The production side of the economy under scrutiny is *neoclassical* since it maintains the assumptions of perfect competition, of an aggregate production function with constant returns to scale and positive and diminishing marginal products, and of capital accumulation. It features *endogenous growth* since economic growth results from innovation investments undertaken by profit-maximizing firms. To allow for innovation investments in *capital- and labor-saving technical change*, I introduce two intermediate-good sectors, one producing a capital-intensive intermediate, the other a labor-intensive intermediate. Innovation investments increase the productivity of capital and labor at the level of these intermediate-good firms. Moreover, they feed into aggregate productivity indicators that evolve cumulatively, i. e., in a way often referred to as 'standing on the shoulders of giants'.

In this framework, I derive the following major results. First, I establish the existence of a unique steady state with finite state variables. The properties of this steady state are consistent with Kaldor's famous facts (Kaldor (1961)). There is no capital-saving technical progress in the steady-state. Per-capita variables such as consumption, savings, or the real wage grow at the (net) growth rate of labor-saving technical knowledge.[2] More importantly, this steady-state growth rate is pinned down by the production side of the economy alone. Parameter changes reflecting an improvement in the quality of the institutional framework or in the efficiency of the innovation technology increase the steady-state growth rate. However, population aging has no effect on the steady-state growth rate.

Second, I prove that the steady state is locally stable and study the economic consequences of demographic change on transitional dynamics. Although population aging has no steady-state growth effect, it affects the economy's growth rate along the transition. Arguably, with a period representing approximately 30 years, these effects are quite relevant for those generations living through the transition. Taking a steady state as the starting point, I capture population aging by a once and for all decline in the growth rate of the labor force and/or by a once and for all increase in life expectancy. In both cases, the economy leaves its steady state and embarks on a trajectory with an increased speed of capital deepening, i. e., labor becomes scarcer relative to capital. In line with Hicks' contention, there is more labor- and less capital-saving technical change.

Third, I study the effect of population aging on the evolution of gross domestic product (GDP) both in absolute and per-capita terms. I find that a decline in the growth rate of the labor force may reduce GDP for two reasons. First, if the generation with fewer offsprings rationally anticipates a lower rate of return, it may save less. Second, a declining work force reduces aggregate output by more than aggregate innovation investments. For reasonable parameter values, these effects also imply a decline of per-capita GDP in response to population aging. However, there is no first-order effect of induced technical change on GDP unless I allow for contemporaneous knowledge spillovers as in Romer (1986). This confirms in a full-fledged endogenous growth model an intuition derived in Acemoglu (2007) and Acemoglu (2009) according to which the technology choice in a competitive economy maximizes GDP given factor endowments unless there is an externality.

My analysis builds on and contributes to several strands of the literature. First, it makes a contribution to the theoretical literature on the causal effect of demographic trends on physical capital accumulation and economic performance in models with overlapping generations. Hitherto, this literature has not looked at the implications

[2]The findings concerning steady-state growth rates are also in line with the so-called Steady-State Growth Theorem of Uzawa (1961). See Irmen (2009) for a discussion of the relationship between the production side of the economy studied here and Uzawa's theorem.

of demographic change on the direction of technical change. Recent contributions include d'Albis (2007) and Futagami and Nakajima (2001).[3]

These papers base their analysis on the continuous time OLG model with finite lifetime of Cass and Yaari (1967) and Tobin (1967). While d'Albis (2007) studies the monotonic relationship between the population growth rate and the steady-state capital intensity in an economy without technical change, the analysis in Futagami and Nakajima (2001) adds endogenous growth following Romer (1986). In fact, Futagami and Nakajima find a positive effect of longevity on the steady-state growth rate of the economy. Here, increasing longevity is equivalent to an increase in the finite and deterministic individual lifespan. This result relies i) on the complex relationship between the economy's aggregate savings rate and its steady-state age distribution, and ii) on the AK-type aggregate production function. Since a longer lifespan increases the economy's savings rate, it also accelerates steady-state growth. Contrary to this result, my analysis suggests that the link between aging and the aggregate savings rate leaves the steady-state growth rate unaffected once one allows for capital-saving technical change.

Second, the present paper extends and complements the literature on endogenous economic growth in competitive economies that started with Bester and Petrakis (2003), Hellwig and Irmen (2001), and Irmen (2005).[4] These contributions allow for endogenous economic growth through innovation investments that increase the productivity of labor. However, the competitive framework is also well suited for the analysis of capital- versus labor-saving technical change in an aging economy. Its neoclassical properties facilitate the identification of the role of both types of endogenous technical change relative to the neoclassical growth model of Solow (1956) and Swan (1956) with or without exogenous labor-saving technical change.

Third, my model makes a contribution to the theory of endogenous economic growth that also explains the direction of technical change.[5] The focus of this literature is on the interaction between changing factor endowments and the incentives to engage in innovation investments that may increase the productivity of these factors

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[3]There is also a theoretical literature concerned with the role of demographic change for human capital accumulation and economic performance in OLG models. Recent contributions include de la Croix and Licandro (1999), Boucekkine, de la Croix, and Licandro (2002), Heijdra and Romp (2009), and Ludwig and Vogel (2009).

[4]An alternative way to model endogenous economic growth in a competitive framework has been proposed by Boldrin and Levine (1999), and Boldrin and Levine (2008).

[5]This theory has its roots in the so-called 'induced innovations' literature of the 1960s. It is comprehensively surveyed in Acemoglu (2003a). Funk (2002) provides a critique of this literature and a microfoundation in a perfectly competitive setup.

differently. This literature has been initiated by the works of Daron Acemoglu (see, e. g., Acemoglu (1998), Acemoglu (2002), Acemoglu (2003b), Acemoglu (2007), and Acemoglu (2009)).[6] Similar to these studies, the direction of technical change in my model is determined by innovation decisions of intermediate-good firms. However, to allow for a balanced growth path with population growth I dispense with the scale effect both at the level of intermediate-good firms and at the level of the innovation technology.

This paper is organized as follows. Section 2 presents the details of the model. Section 3 studies the intertemporal general equilibrium and establishes the existence and the stability properties of the steady state. In Section 4, I consider increases in the old-age dependency ratio due to a decline in the labor force growth rate. The focus of my analysis is on the implications for the evolution of GDP and for the direction of technical change. Section 5 extends the analysis in two directions. First, following Romer (1986), I allow for contemporaneous knowledge spillovers across innovating firms in Section 5.1. Second, I study the case where the old-age dependency ratio increases because of a higher life expectancy in Section 5.2. Section 6 concludes. Proofs are relegated to Appendix A. Details on the phase diagrams are given in Appendix B. Appendix C presents a calibration exercise. It shows that the model is consistent with the recent growth experience of today's industrialized countries.

2 The Basic Model

The economy has a household sector, a final-good sector, and two intermediate-good sectors in an infinite sequence of periods $t = 1, 2, ..., \infty$. The household sector comprises two-period lived individuals. There are five objects of exchange. The *manufactured final good* can be consumed or invested. If invested it may either become future capital or serve as an input in current innovation activity undertaken by intermediate-good firms. Intermediate-good firms produce one of two types of intermediates and sell it to the final-good sector. The production of the *labor-intensive intermediate good* uses labor as the sole input, the only input in the production of the *capital-intensive intermediate good* is capital. Intermediate-good firms either belong to the sector that produces the labor- or the capital-intensive intermediate. Accordingly, I shall refer to the labor- and the capital-intensive intermediate-good sector. Labor- and capital-saving technical change is the result of innovation investments undertaken by intermediate-good firms of the respective sector. Households

[6] Other important studies on this topic include Lloyd-Ellis (1999), Kiley (1999), Galor and Moav (2000), or Krusell, Ohanian, Rios-Rull, and Violante (2000).

supply *labor* and *capital* to the intermediate-good sectors. Labor is "owned" by the young, the old own the capital stock. Capital is the only asset in the economy.[7] Each period has markets for all five objects of exchange. The final good serves as numéraire.

2.1 Households

The two-period lived individuals work and save when young, retire when old, and consume during both periods of their life. At t, there are L_t young and L_{t-1} old individuals. Individual labor supply when young is exogenous and normalized to one. I denote $\lambda > (-1)$ the exogenous growth rate of the labor force. Population aging in the sense of an increasing old-age dependency ratio defined as L_{t-1}/L_t occurs whenever λ declines between two adjacent periods.

Preferences of a member of cohort t are homothetic and defined over the level of consumption when young and old, c_t^y and c_{t+1}^o, respectively. Lifetime utility is

$$U_t = u\left(c_t^y\right) + \beta u\left(c_{t+1}^o\right), \tag{2.1}$$

where $u : \mathbb{R}_{++} \to \mathbb{R}$ is a per-period utility function. It is $\mathcal{C}^2$ and satisfies $u'(c) > 0 > u''(c)$ for all $c > 0$ as well as $\lim_{c \to 0} u'(c) = \infty$. To ensure the uniqueness of the equilibrium, I assume that preferences are such that the intertemporal elasticity of substitution is greater or equal to unity. Moreover, $\beta \in (0,1)$ is the discount factor.

The maximization of (2.1) is subject to the per-period budget constraints $c_t^y + s_t = w_t$ and $c_{t+1}^o = s_t\, R_{t+1}$, where s_t denotes savings, $w_t > 0$ the real wage at t, and $R_{t+1} > 0$ the real rental rate of capital between t and $t+1$, expected under perfect foresight. The optimal plan of a member of cohort t, $\left(c_t^y, s_t, c_{t+1}^o\right)$, results from the Euler condition

$$-u'\left(c_t^y\right) + \beta\, R_{t+1} u'\left(c_{t+1}^o\right) = 0 \tag{2.2}$$

in conjunction with the two budget constraints. Given our assumptions on preferences, this plan involves a continuous and partially differentiable function that

[7] My setup is mute on the question as to who owns the infinitely-lived firms in the economy. Moreover, I consider competitive equilibria where per-period profits are zero such that the expected present discounted value of dividends associated with any ownership share is zero, too. It is well known that these considerations are not sufficient to exclude equilibria with bubbles since the number of potential traders is infinite in the OLG-framework (Tirole (1985)). In what follows, I disregard this possibility and focus on equilibria without bubbles.

relates savings to current income, the expected rental rate of capital, and the discount factor

$$s_t = s\left(R_{t+1}, \beta\right) w_t, \tag{2.3}$$

where $s\left(R_{t+1}, \beta\right) \in (0, 1)$ is the marginal propensity to save out of wage income with partial derivatives $s_R\left(R_{t+1}, \beta\right) \geq 0$ and $s_\beta\left(R_{t+1}, \beta\right) > 0$.

2.2 The Final-Good Sector

The final-good sector produces with a production function $F : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$

$$Y_t = F\left(Y_{K,t}, Y_{L,t}\right), \tag{2.4}$$

where Y_t is aggregate output in t, $Y_{K,t}$ is the aggregate amount of the capital-intensive intermediate input used in t, and $Y_{L,t}$ denotes the aggregate amount of the labor-intensive intermediate input. The function F is $\mathcal{C}^2$ with $F_1 > 0 > F_{11}$ and $F_2 > 0 > F_{22}$. Moreover, it exhibits constant-returns-to-scale with respect to both inputs. To include, e. g., the CES production function, I make no assumptions on the limits of the function F and its derivatives for $Y_K \to 0$, $Y_L \to 0$, $Y_K \to \infty$, and $Y_L \to \infty$.

Under these assumptions, I may study the competitive behavior of the final-good sector in terms of a single representative firm. In units of the final good of period t as numéraire the profit in t of the final-good sector is

$$Y_t - p_{K,t}Y_{K,t} - p_{L,t}Y_{L,t}, \tag{2.5}$$

where $p_{j,t}$, $j = K, L$, is the price of the respective intermediate.

The final-good sector takes the sequence $\{p_{K,t}, p_{L,t}\}_{t=1}^{\infty}$ of factor prices as given and maximizes the sum of the present discounted values of profits in all periods. Since it simply buys both intermediates in each period, its maximization problem is equivalent to a series of one-period maximization problems. Let κ_t denote the period-t factor intensity in the final-good sector, i. e.,

$$\kappa_t = \frac{Y_{K,t}}{Y_{L,t}}. \tag{2.6}$$

Then, the production function in intensive form is $F\left(\kappa_t, 1\right) \equiv f\left(\kappa_t\right)$. The respective profit-maximizing first-order conditions for $t = 1, 2, ...$ are

$$Y_{K,t} \; : \; p_{K,t} = f'\left(\kappa_t\right), \tag{2.7}$$

$$Y_{L,t} \; : \; p_{L,t} = f\left(\kappa_t\right) - \kappa_t f'\left(\kappa_t\right). \tag{2.8}$$

2.3 The Intermediate-Good Sector

There are two different sets of intermediate-good firms, each represented by the set $\mathbb{R}_+$ of nonnegative real numbers with Lebesgue measure. Intermediate-good firms either belong to the labor- or to the capital-intensive intermediate-good sector.

2.3.1 Technology

At any date t, all firms of a sector have access to the same sector-specific technology with production function

$$y_{l,t} = \min\{1, a_t l_t\} \quad \text{and} \quad y_{k,t} = \min\{1, b_t k_t\}, \tag{2.9}$$

where $y_{l,t}$ and $y_{k,t}$ is output, 1 a capacity limit,[8] a_t and b_t denote a firm's labor and capital productivity in period t, l_t and k_t is the labor and the capital input. While labor, l_t, is hired at t, capital, k_t, must be installed in $t-1$. The firms' respective labor and capital productivity is equal to

$$a_t = A_{t-1}(1-\delta+q_t^A) \quad \text{and} \quad b_t = B_{t-1}(1-\delta+q_t^B); \tag{2.10}$$

here A_{t-1} and B_{t-1} denote aggregate indicators of the level of technological knowledge to which innovating firms in period t have access for free. Naturally, $\delta \in (0,1)$ is the rate of depreciation of technological knowledge in both sectors, and q_t^A and q_t^B are indicators of (gross) productivity growth at the firm level. Without loss of generality, capital fully depreciates after one period.

To achieve a productivity growth rate $q_t^j > 0$, $j = A, B$, a firm must invest $i(q_t^j)$ units of the final good in period t. The function $i : \mathbb{R}_{++} \to \mathbb{R}_{++}$ is the same for both sectors, time invariant, $\mathcal{C}^2$, increasing and strictly convex. Hence, higher rates of productivity growth require ever larger innovation investments. Moreover, with the notation $i'(q^j) \equiv di(q^j)/dq^j$ for $j = A, B$, it satisfies

$$\lim_{q^j \to 0} i(q^j) = \lim_{q^j \to 0} i'(q^j) = 0, \quad \text{and} \quad \lim_{q^j \to \infty} i\left(q^j\right) = \lim_{q^j \to \infty} i'\left(q^j\right) = \infty. \tag{2.11}$$

If a firm innovates, the assumption is that an innovation in period t is proprietary knowledge of the firm only in t, i. e., in the period when it is made. Subsequently, the

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[8] The analysis is easily generalized to allow for an endogenous capacity choice requiring additional capacity investments, with investment outlays being a strictly convex function of capacity. In such a setting profit-maximizing behavior implies that a larger innovation investment is accompanied by a larger capacity investment (see, Hellwig and Irmen (2001) for details). Thus, the simpler specification treated here abstracts from effects on firm size in an environment with changing levels of innovation investments. Otherwise, it does not affect the generality of my results.

innovation becomes embodied in the sector specific productivity indicators (A_t, B_t), (A_{t+1}, B_{t+1}), ..., with no further scope for proprietary exploitation. The evolution of these indicators will be specified below. If firms decide not to undertake an innovation investment in period t, then they have access to the production technique represented by A_{t-1} and B_{t-1} such that $a_t = A_{t-1}(1-\delta)$ and $b_t = B_{t-1}(1-\delta)$, respectively.

2.3.2 Profit Maximization and Zero-Profits

Per-period profits in units of the current final good are

$$\pi_{L,t} \;=\; p_{L,t}y_{l,t} - w_t l_t - i(q_t^A), \quad \pi_{K,t} = p_{K,t}y_{k,t} - R_t k_t - i(q_t^B), \tag{2.12}$$

where $p_{L,t}y_{l,t}$, $p_{K,t}y_{k,t}$ is the respective firm's revenue from output sales, $w_t l_t$, $R_t k_t$ its wage bill at the real wage rate w_t and its capital cost at the real rental rate of capital R_t, and $i(q_t^j)$, $j = A, B$, its outlays for innovation investment.

Firms choose a production plan $(y_{l,t}, l_t, q_t^A)$ and $(y_{k,t}, k_t, q_t^B)$ taking the sequence $\{p_{L,t}, p_{K,t}, w_t, R_t\}_{t=1}^{\infty}$ of real prices and the sequence $\{A_{t-1}, B_{t-1}\}_{t=1}^{\infty}$ of aggregate productivity indicators as given. They choose a production plan that maximizes the sum of the present discounted values of profits in all periods. Because production choices for different periods are independent of each other, for each period t, they choose the plan $(y_{l,t}, l_t, q_t^A)$ and $(y_{k,t}, k_t, q_t^B)$ that maximizes the profit $\pi_{L,t}$ and $\pi_{K,t}$, respectively.

If a firm innovates, it incurs an investment cost $i(q_t^j) > 0$ that is associated with a given innovation rate $q_t^j > 0$ and is independent of the output $y_{l,t}$ or $y_{k,t}$. An innovation investment is only profit-maximizing if the firm's margin is strictly positive, i. e., if $p_{L,t} > w_t/a_t$ or $p_{K,t} > R_t/b_t$. Then, there is a positive scale effect, namely if the firm innovates, it wants to apply the innovation to as large an output as possible and produces at the capacity limit, i. e., $y_{l,t} = 1$ or $y_{k,t} = 1$. The choice of (l_t, q_t^A) and (k_t, q_t^B) must then minimize the costs of producing the capacity output, i. e., assuming $w_t > 0$ and $R_t > 0$ these input combinations must satisfy

$$l_t = \frac{1}{A_{t-1}(1-\delta+q_t^A)}, \quad k_t = \frac{1}{B_{t-1}(1-\delta+q_t^B)}, \tag{2.13}$$

and

$$\begin{aligned} \hat{q}_t^A \;&\in\; \arg\min_{q^A \geq 0} \left[\frac{w_t}{A_{t-1}(1-\delta+q^A)} + i(q^A) \right], \\ \hat{q}_t^B \;&\in\; \arg\min_{q^B \geq 0} \left[\frac{R_t}{B_{t-1}(1-\delta+q^B)} + i(q^B) \right]. \end{aligned} \tag{2.14}$$

Given the convexity of the innovation cost function and the fact that $\lim_{q^j \to 0} i'(q^j) = 0$, $j = A, B$, the following conditions determine a unique level $\hat{q}_t^A > 0$ and $\hat{q}_t^B > 0$ as the solution to the first-order conditions

$$\frac{w_t}{A_{t-1}(1-\delta+\hat{q}_t^A)^2} = i'\left(\hat{q}_t^A\right) \quad \text{and} \quad \frac{R_t}{B_{t-1}(1-\delta+\hat{q}_t^B)^2} = i'\left(\hat{q}_t^B\right). \tag{2.15}$$

These conditions relate the marginal reduction of a firm's wage bill/capital cost to the marginal increase in its investment costs. Upon dividing one by the other and rearranging reveals that Hicks' conjecture mentioned in the Introduction is essentially about the incentives to minimize costs. Indeed, one obtains

$$\frac{w_t}{R_t} = \frac{A_{t-1}}{B_{t-1}} \left(\frac{(1-\delta+\hat{q}_t^A)^2\, i'\left(\hat{q}_t^A\right)}{(1-\delta+\hat{q}_t^B)^2\, i'\left(\hat{q}_t^B\right)} \right). \tag{2.16}$$

Since the numerator of the right-hand side increases in $\hat{q}_t^A$ and the denominator increases in $\hat{q}_t^B$, an increase in the relative price of labor induces, ceteris paribus, relatively more labor-saving technical change, i. e., a hike in w_t/R_t means a greater ratio $\hat{q}_t^A/\hat{q}_t^B$. However, to pin down the latter ratio one has to know the relative factor prices per efficiency unit, i. e., the ratio $(w_t/A_{t-1}) / (R_t/B_{t-1})$, where "efficiency" refers to the level of technology before innovation investments are undertaken. Changes in this ratio determine the direction of technical change.

If a firm's margin is not strictly positive, i. e., $p_{L,t} \leq w_t/a_t$ or $p_{K,t} \leq R_t/b_t$, then it will not invest. In case of a zero-margin, any plan $(y_{l,t}, l_t, 0)$ with $y_{l,t} \in [0, 1]$ and $l_t = y_{l,t}/A_{t-1}(1-\delta)$ or $(y_{k,t}, l_t, 0)$ with $y_{k,t} \in [0, 1]$ and $k_t = y_{k,t}/B_{t-1}(1-\delta)$ maximizes π_t^L or π_t^K, respectively. Without loss of generality and to simplify the notation, I assume for this case that intermediate-good firms still produce the capacity output. If a firm faces a strictly negative margin, it won't produce and $(0, 0, 0)$ is the optimal plan.

2.4 Consolidating the Production Sector

Turning to the implications for the general equilibrium, recall that the set of each sector of intermediate-good firms is $\mathbb{R}_+$ with Lebesgue measure. Consider equilibria where both intermediates are produced. Then, the maximum profit of any intermediate-good firm producing the labor- or the capital-intensive intermediate must be zero at any t. Indeed, since the supply of labor and capital is bounded in each period, the set of intermediate-good firms employing more than some $\varepsilon > 0$ units of labor or capital must have bounded measure and hence must be smaller than the set of all intermediate-good firms. Given that inactive intermediate-good firms must be maximizing profits just like the active ones, I need that maximum profits of all active intermediate-good firms at equilibrium prices are equal to zero.

Using (2.12), (2.13), and (2.15), I find for profit-maximizing intermediate-good firms earning zero profits in equilibrium that

$$p_{L,t} = (1 - \delta + \hat{q}_t^A) i'(\hat{q}_t^A) + i(\hat{q}_t^A), \quad p_{K,t} = (1 - \delta + \hat{q}_t^B) i'(\hat{q}_t^B) + i(\hat{q}_t^B), \tag{2.17}$$

i. e., the price is equal to variable cost plus fixed costs when w_t/a_t and R_t/b_t are consistent with profit-maximization as required by (2.15). Upon combining the equilibrium conditions of the final-good sector and both intermediate-good sectors the following proposition holds.

Proposition 1 *If (2.7), (2.8), and (2.17) hold for all producing firms at t, then there are maps, $g^A : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $g^B : \mathbb{R}_{++} \to \mathbb{R}_{++}$, such that $\hat{q}_t^A = g^A(\kappa_t)$ and $\hat{q}_t^B = g^B(\kappa_t)$ satisfy*

$$g_\kappa^A(\kappa_t) > 0 \quad \textit{and} \quad g_\kappa^B(\kappa_t) < 0 \quad \textit{for all } \kappa_t > 0. \tag{2.18}$$

Proposition 1 states a key property of the production sector. The equilibrium incentives to engage in labor- and capital-saving technical change depend on the factor intensity of the final-good sector. This is due to the properties of the production function F. Under constant returns to scale and positive, yet decreasing marginal products, both intermediate-good inputs are complements in the production of the final good. Hence, having relatively more of $Y_{K,t}$, i. e., a higher κ_t, increases $p_{L,t}$ and decreases $p_{K,t}$. These price movements increase $\hat{q}_t^A$ and decrease $\hat{q}_t^B$ in accordance with (2.17).

2.5 Evolution of Technological Knowledge

The evolution of the economy's level of technological knowledge is given by the evolution of the aggregate indicators A_t and B_t. An important question is then how these indicators are linked to the innovation investments of individual intermediate-good firms.

Labor- and capital-saving productivity growth occurs at those intermediate-good firms that produce at t. Denoting the measure of the firms producing the labor- and the capital-intensive intermediate in t by n_t and m_t, respectively, their contribution to A_t and B_t is equal to the highest level of labor and capital productivity attained by one of them, i. e.,

$$\begin{aligned} A_t &= \max\{a_t(n) = A_{t-1}\left(1-\delta+q_t^A(n)\right) \mid n \in [0, n_t]\} \\ B_t &= \max\{b_t(m) = B_{t-1}\left(1-\delta+q_t^B(m)\right) \mid m \in [0, m_t]\}. \end{aligned} \tag{2.19}$$

Since in equilibrium $q_t^A(n) = q_t^A$ and $q_t^B(m) = q_t^B$, I have $a_t = A_{t-1}\left(1-\delta+q_t^A\right)$ and $b_t = B_{t-1}\left(1-\delta+q_t^B\right)$. Hence, for all $t = 1, 2, ...$

$$A_t = A_{t-1}\left(1-\delta+q_t^A\right) \quad \text{and} \quad B_t = B_{t-1}\left(1-\delta+q_t^B\right) \tag{2.20}$$

with $A_0 > 0$ and $B_0 > 0$ as initial conditions.

3 Intertemporal General Equilibrium

3.1 Definition

A *price system* corresponds to a sequence $\{p_{L,t}, p_{K,t}, w_t, R_t\}_{t=1}^{\infty}$. An *allocation* is a sequence $\{c_t^y, s_t, c_t^o, Y_t, Y_{K,t}, Y_{L,t}, n_t, m_t, y_{L,t}, y_{K,t}, q_t^A, q_t^B, a_t, b_t, l_t, k_t, L_{t-1}, K_t\}_{t=1}^{\infty}$ that comprises a strategy $\{c_t^y, s_t, c_{t+1}^o\}_{t=1}^{\infty}$ for all cohorts, consumption of the old at $t = 1$, c_1^o, a strategy $\{Y_t, Y_{K,t}, Y_{L,t}\}_{t=1}^{\infty}$ for the final-good sector, measures n_t and m_t of intermediate-good firms active at t producing the capacity output $y_{l,t} = y_{k,t} = 1$ with input choices (l_t, q_t^A), and (k_t, q_t^B) resulting in the respective productivity levels (a_t, b_t), and demanding the aggregate supply of labor and capital, L_t and K_t.

For an exogenous evolution of the labor force, $L_t = L_0 (1+\lambda)^t$ with $L_0 > 0$ and $\lambda > (-1)$, a given initial level of capital, $K_1 > 0$, and initial values of technological knowledge, $A_0 > 0$ and $B_0 > 0$, an *intertemporal general equilibrium* corresponds to a price system, an allocation, and a sequence $\{A_{t-1}, B_{t-1}\}_{t=1}^{\infty}$ of indicators for the level of aggregate technological knowledge that satisfy the following conditions for all $t = 1, 2, ..., \infty$:

(E1) The young of each period save according to (2.3) and supply L_t units of labor.

(E2) The production sector satisfies Proposition 1.

(E3) The market for the final good clears, i. e.,

$$L_{t-1}c_t^o + L_t c_t^y + I_t^K + I_t^A + I_t^B = Y_t, \tag{3.1}$$

where I_t^K is capital investment, I_t^A and I_t^B denote aggregate innovation investments in labor- and capital-saving technical change.

(E4) The market for both intermediates clears, i. e.,

$$Y_{L,t} = n_t \quad \text{and} \quad Y_{K,t} = m_t. \tag{3.2}$$

(E5) There is full employment of labor and capital, i. e.,

$$n_t l_t = L_t \quad \text{and} \quad m_t k_t = K_t. \tag{3.3}$$

(E6) The productivity indicators A_t and B_t evolve according to (2.20).

(E1) guarantees optimal behavior of all households under perfect foresight. Since the old at $t = 1$ own the capital stock, their consumption is $L_0 c_1^o = R_1 K_1$. (E2) assures optimal behavior of all firms. Due to constant returns to scale in final-good production, there are no profits in equilibrium. The resource constraint (3.1) of (E3) reflects the fact that capital fully depreciates after one period.

Market clearing for each intermediate good (E4), full employment of labor and capital (E5), (2.13), and the updating condition (E6) imply that $n_t = A_t L_t$ and $m_t = B_t K_t$. Hence, in equilibrium, we have

$$Y_{L,t} = A_t L_t \qquad \text{and} \qquad Y_{K,t} = B_t K_t, \tag{3.4}$$

$$I_t^A = A_t L_t i\left(q_t^A\right) \qquad \text{and} \qquad I_t^B = B_t K_t i\left(q_t^B\right), \tag{3.5}$$

i. e., aggregate output of each intermediate good is equal to the respective input in efficiency units, and aggregate investment in labor- and capital-saving technical change is proportionate to the respective input in efficiency units. Observe that (2.6) and (3.4) imply an equilibrium factor intensity

$$\kappa_t \quad = \quad \frac{B_{t-1}\left(1-\delta+g^B\left(\kappa_t\right)\right) K_t}{A_{t-1}\left(1-\delta+g^A\left(\kappa_t\right)\right) L_t} \tag{3.6}$$

Thus, in equilibrium κ_t is the intensity of efficient capital per unit of efficient labor. For short, I shall henceforth refer to κ_t as the 'efficient capital intensity' as opposed to the 'capital intensity', i. e., the amount of capital per worker K_t/L_t. With this in mind Proposition 1 may be reinterpreted from a perspective of the general equilibrium.

Corollary 1 *At all t, the equilibrium incentives to engage in capital- and labor-saving technical change depend on the ratio $B_{t-1}K_t/A_{t-1}L_t$. If this ratio increases (decreases), then $\hat{q}_t^A$ increases (decreases) and $\hat{q}_t^B$ decreases (increases).*

Corollary 1 links Hicks' conjecture about relative factor prices as stated in condition (2.16) to relative factor endowments. Moreover, it emphasizes that the correct measure of relative scarcity is the amount of efficient capital per unit of efficient labor, where 'efficient' refers to the technological level at the beginning of the respective period, i. e., before innovation investments are being undertaken. If $B_{t-1}K_t/A_{t-1}L_t$ increases, then κ_t also increases. Since $g_\kappa^A(\kappa_t) > 0 > g_\kappa^B(\kappa_t)$, $\hat{q}_t^A$ increases and $\hat{q}_t^B$ decreases. It is in this sense that the relative scarcity of factors of production induces technical change.

3.2 The Dynamical System

The equilibrium conditions (E1), (E2), (E3), (E5) and (E6), require savings to equal capital investment, i. e.,

$$I_t^K = s_t L_t = K_{t+1} \quad \text{for } t = 1, 2, ..., \infty. \tag{3.7}$$

The evolution of the economy can then be characterized by means of two state variables, namely the efficient capital intensity of the final-good sector, κ_t, and the level of capital-saving technological knowledge, B_t.

For further reference it proves useful to introduce some additional variables. I denote the real wage per efficiency unit and the real rental rate of capital per efficiency unit by $w_t/A_t \equiv \tilde{w}(\kappa_t)$ and $R_t/B_t \equiv \tilde{R}(\kappa_t)$. Then, the equilibrium real rental rate of capital and the equilibrium real wage become respectively $R(\kappa_t) = B_t \tilde{R}(\kappa_t)$ and $w(\kappa_t) = A_t \tilde{w}(\kappa_t)$.

Proposition 2 *(Dynamical System)*

Given $(K_1, L_0, A_0, B_0) > 0$ as initial conditions, there is a unique equilibrium sequence $\{\kappa_t, B_t\}_{t=1}^{\infty}$ determined by

$$\frac{s(R(\kappa_{t+1}), \beta)}{1+\lambda} \tilde{w}(\kappa_t) = \frac{\kappa_{t+1}}{B_t} \frac{1-\delta+g^A(\kappa_{t+1})}{1-\delta+g^B(\kappa_{t+1})}, \tag{3.8}$$

and

$$B_{t+1} = B_t \left(1-\delta+g^B(\kappa_{t+1})\right), \tag{3.9}$$

where κ_1 and B_1 satisfy

$$\kappa_1 = \frac{B_0\left(1-\delta+g^B(\kappa_1)\right)K_1}{A_0\left(1-\delta+g^A(\kappa_1)\right)L_0(1+\lambda)} > 0, \quad B_1 = B_0\left(1-\delta+g^B(\kappa_1)\right) > 0. \tag{3.10}$$

According to Proposition 2, the dynamical system is a two-dimensional system of first-order, autonomous, non-linear difference equations. The equation of motion for the efficient capital intensity is (3.8). It restates the condition for savings to equal capital investment (3.7) where K_{t+1} is replaced by an update of (3.6). Since preferences are homothetic and w_t is proportionate to A_t, the latter drops out on both sides of the equation. Observe further that (3.8) collapses to Diamond's difference equation for the capital intensity if we eliminate capital- and labor-saving technical. Equation (3.9) determines the evolution of the level of capital-saving technological knowledge. For any given pair $(\kappa_t, B_t) \in \mathbb{R}_{++}$, (3.8) assigns a unique value κ_{t+1} which is then used to find B_{t+1} with (3.9). Since K_1, L_0, A_0, and B_0 are initial conditions and $L_1 = (1+\lambda)L_0$, κ_1 is pinned down by (3.6) for $t = 1$.

Define a steady state as a trajectory along which all variables grow at a constant rate. From (3.9) I deduce that a trajectory with $B_{t+1}/B_t - 1 = const.$ requires $\kappa_t = \kappa_{t+1} = \kappa^*$. Moreover, according to (3.8), the latter needs $B_{t+1} = B_t = B^*$. Hence, a steady state is a solution to

$$\frac{s\left(R\left(\kappa^*\right), \beta\right)}{1+\lambda} \tilde{w}\left(\kappa^*\right) = \frac{\kappa^*}{B^*}\left(1 - \delta + g^A\left(\kappa^*\right)\right). \tag{3.11}$$

$$g^B\left(\kappa^*\right) = \delta \tag{3.12}$$

Proposition 3 *(Steady State)*

1. *There is a unique steady state involving $\kappa^* \in (0, \infty)$ and $B^* \in (0, \infty)$ if and only if*

$$\lim_{\kappa \to 0} f'(\kappa) > i'(\delta) + i(\delta) > \lim_{\kappa \to \infty} f'(\kappa). \tag{3.13}$$

2. *The steady-state growth rate of the economy is $g^* \equiv A_{t+1}/A_t - 1 = g^A\left(\kappa^*\right) - \delta$. Moreover, along a steady-state path, I have*

   a) $$\frac{a_{t+1}}{a_t} = \frac{w_{t+1}}{w_t} = \frac{c^y_{t+1}}{c^y_t} = \frac{c^o_{t+1}}{c^o_t} = \frac{s_{t+1}}{s_t} = 1 + g^*,$$

   b) $$\frac{Y_{t+1}}{Y_t} = \frac{Y_{K,t+1}}{Y_{K,t}} = \frac{Y_{L,t+1}}{Y_{L,t}} = \frac{K_{t+1}}{K_t} = \frac{n_{t+1}}{n_t} = \frac{m_{t+1}}{m_t} = \left(1+g^*\right)(1+\lambda),$$

   c) $$B_t = b_t = B^*, \quad R^* = B^*\left[f'\left(\kappa^*\right) - i(\delta)\right],$$

   d) $$p^*_L = f\left(\kappa^*\right) - \kappa^* f'\left(\kappa^*\right), \quad p^*_K = f'\left(\kappa^*\right), \quad k^* = \frac{1}{B^*}, \quad \frac{l_{t+1}}{l_t} = \frac{1}{1+g^*}.$$

According to Proposition 3, a steady state exists if the marginal product of the capital-intensive intermediate good in final-good production is sufficiently high (low) when $Y_K \to 0$ ($Y_K \to \infty$). Then, condition (3.13) assures a value $\kappa \in (0, \infty)$ such that the zero-profit condition of intermediate-good firms of the capital-intensive sector is satisfied at $\hat{q}^B = \delta$.[9] At this value $p_K = f'(\kappa^*)$ is the revenue whereas $i'(\delta) + i(\delta)$ is the total cost of such a firm. Clearly, condition (3.13) would always hold if I had imposed the usual Inada conditions on the final-good production function F. On the other hand, a violation of (3.13) implies that B_t either becomes unbounded or vanishes in the limit as $t \to \infty$.

Contrary to the one-sector neoclassical growth model where multiple steady states with $\kappa^* \in (0, \infty)$ may exist if capital and labor are poor substitutes (see, e. g., Galor (1996)), the steady state is unique in my setting. The reason is that here the steady-state capital intensity κ^* is not determined by the difference equation for capital accumulation. Instead, κ adjusts such that additions to the stock of capital-saving technological knowledge just offset depreciation. Then, given κ^*, there is a unique steady-state level of capital-saving technological knowledge B^* that satisfies (3.11).

Statement 2 of Proposition 3 gives a complete list of the steady-state evolution. The steady-state growth rate of the economy is equal to the growth rate of the stock of labor-saving technological knowledge. Labor productivity, the real wage, and individual consumption and savings grow at this rate. Aggregate variables such as Y_t or K_t grow at rate $g^* + \lambda$. There is no growth of capital-saving technological knowledge. The rental rate of capital and intermediate-good prices are constant. With these results at hand, it is straightforward to see that the steady state is consistent with Kaldor's facts (Kaldor (1961)) as long as $g^* > 0$.

For the upcoming analysis it is important to keep in mind that the determination of the steady-state efficient capital intensity as well as the steady-state growth rate is independent of the household sector. In fact, they only hinge on parameters that characterize the production sector. For instance, if $F = \Gamma \left(Y_{K,t}\right)^{1-\gamma} \left(Y_{L,t}\right)^{\gamma}$, $\Gamma > 0$, $0 < \gamma < 1$, and $i = v_0 \left(q_t^j\right)^2$, $v_0 > 0$, $j = A, B$, I find

$$\kappa^* = \left(\frac{\Gamma(1-\gamma)}{\delta(2+\delta)v_0}\right)^{\frac{1}{\gamma}} \quad \text{and} \quad g^A(\kappa^*) = -\frac{1-\delta}{3} + \sqrt{\frac{(1-\delta)^2}{9} + \frac{\Gamma\gamma}{3v_0}(\kappa^*)^{1-\gamma}}. \tag{3.14}$$

Quite intuitively, κ^* increases in the productivity parameter Γ and decreases in the innovation cost parameter v_0. The steady-state growth rate $g^* = g^A(\kappa^*) - \delta$

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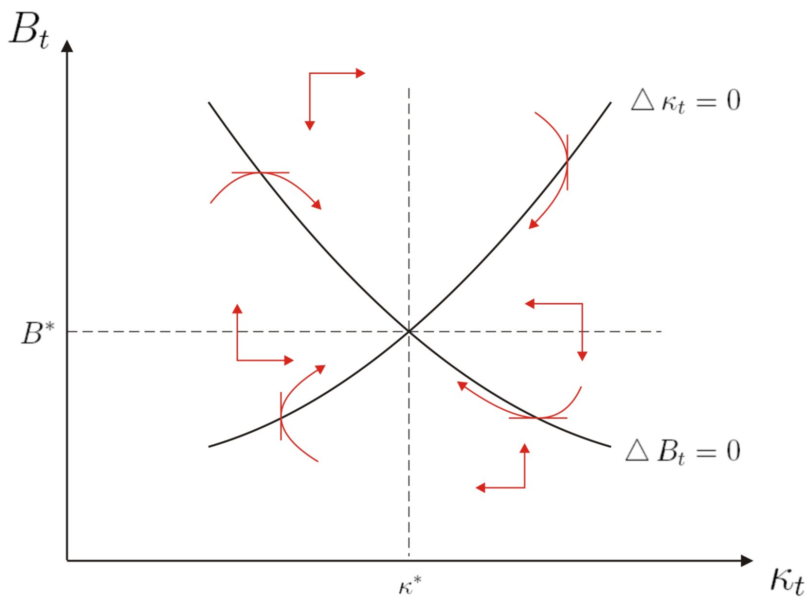
[9] In light of Proposition 1, condition (3.13) may also be stated as $\lim_{\kappa \to 0} g^B(\kappa) > \delta > \lim_{\kappa \to \infty} g^B(\kappa)$. Hence, diminishing returns to (efficient) capital still play a key role here since they imply $g^B_\kappa(\kappa_t) < 0$.

Figure 3.1: The Phase-Diagram of the Locally Stable Steady State (κ^*, B^*).



inherits these comparative statics. In Appendix C, I show that the steady-state growth rate g^* is broadly consistent with the long-run growth performance of today's industrialized countries for reasonable parameter values.

To get an idea of the transitional dynamics involved, consider the phase diagram in the (B_t, κ_t) – plane shown in Figure 3.1 (see Appendix B for details). Based on (3.8), I denote the locus of all pairs (B_t, κ_t) for which the evolution of κ_t is at a point of rest by $\Delta\kappa_t = 0 \equiv \{(B_t, \kappa_t)\,|\kappa_{t+1} - \kappa_t = 0\}$. I assume this locus to be stable in a sufficiently small neighborhood of the steady state. This assumption may be justified with reference to the analysis of Diamond (1965) where the focus is also confined to locally stable steady states. As a consequence, the horizontal arrows point towards the $\Delta\kappa_t = 0$ – locus. Moreover, this locus is strictly increasing. Intuitively, $\Delta\kappa_t = 0$ requires $d\kappa_{t+1} = d\kappa_t$ whereas local stability means here that $d\kappa_{t+1} < d\kappa_t$. Then, from (3.8), $dB_t/d\kappa_t > 0$ is necessary to reestablish $d\kappa_{t+1} = d\kappa_t$ following a change $d\kappa_t \neq 0$.

The locus $\Delta B_t = 0 \equiv \{(B_t, \kappa_t)\,|B_{t+1} - B_t = 0\}$ gives all pairs (B_t, κ_t), for which the evolution of B_t is at a point of rest. In the vicinity of the steady state, this locus is strictly decreasing and stable. Intuitively, $\Delta B_t = 0$ requires $\delta = g^B(\kappa_{t+1})$, i. e., κ_{t+1} must not change as I vary κ_t. Since any change $d\kappa_t > 0$ increases the left-hand side of (3.8), I need $dB_t < 0$ to offset this effect. Hence, the negative slope. The stability of this locus may also be deduced from (3.8). Since $dB_t > 0$ ($dB_t < 0$) implies an increase (decrease) in κ_{t+1}, I have $\delta > g^B(\kappa_{t+1})$ above and $\delta < g^B(\kappa_{t+1})$ below the

$\Delta B_t = 0$ – locus. Accordingly, the vertical arrows point towards $\Delta B_t = 0$.

The continuous time representation of the trajectories in Figure 3.1 does not substitute for a discrete time analysis which is provided by the following lemma.

Lemma 1 *If the locus $\Delta \kappa_t = 0$ is stable in a vicinity of (κ^*, B^*), then the steady state is either a stable node or a clockwise spiral sink.*

4 Population Aging, Labor Force Growth, and the Direction of Technical Change

This section studies the effect of a permanent decline in the growth rate of the labor force λ on the direction of technical change. This is equivalent to an increase in the old-age dependency ratio at t defined as $L_{t-1}/L_t = (1+\lambda)^{-1}$. Increases in the latter are meant to capture the tendencies shown in Table 1. For ease of notation, I consider an economy in the steady state when a change in λ occurs. Proposition 4 has the main result of this section.

Proposition 4 *(Labor Force Growth, Comparative Statics and Dynamics)*

Consider an economy in the steady state in period $t = 1$. Then, it experiences a small and permanent decline in its labor force growth rate such that $L_t = L_1(1+\lambda')^{t-1}$ with $\lambda > \lambda' > (-1)$ for all $t = 2, 3, ..., \infty$. Denote variables associated with an evolution under λ' by an apostrophe such that $(\kappa^{*\prime}, B^{*\prime})$ is the steady state under λ'.

1. *It holds that $\kappa_2' > \kappa^*$ and $B_2' < B^*$.*
2. *It holds that $\kappa^{*\prime} = \kappa^*$ and $B^{*\prime} < B^*$.*

According to the first claim of Proposition 4 an increase in the old-age dependency ratio leads to a higher efficient capital intensity and a lower level of capital-saving technical knowledge. This is the result of two opposing forces on the capital intensity of $t = 2$. On the one hand, the labor force will be smaller under λ' than under λ. From Corollary 1, I know that this leads, ceteris paribus, to a higher efficient capital intensity. On the other hand, under rational expectations, generation 1 may reduce its savings rate to a lower rental rate of capital.[10] The point of the first claim is that the former effect dominates the latter. Therefore, between $t = 1$ and $t = 2$ the capital intensity grows faster than along the steady-state trajectory with λ. Following Cutler, Poterba, Sheiner, and Summers (1990), I refer to this acceleration in capital deepening as the *Solow effect*. In view of Corollary 1, the Solow effect induces *Hicks effects*: since $\kappa_2' > \kappa^*$ there is more labor-saving and less capital-saving technical change. In other words, $g^A(\kappa_2') > g^A(\kappa^*)$ and $g^B(\kappa_2') < \delta = g^B(\kappa^*)$ where the latter implies $B_2' < B^*$. Hence, population aging affects the relative scarcity of factors of production and becomes a determinant of the direction of technical change.

How does aging affect the evolution of gross domestic product (GDP) between $t = 1$ and $t = 2$? To address this question I use (2.4), (3.4) and (3.5) and express GDP and per-capita GDP at t as

$$
\begin{aligned}
GDP_t(K_t, L_t, B_t, A_t) &\equiv F(B_t K_t, A_t L_t) - A_t L_t i\left(g^A(\kappa_t)\right) - B_t K_t i\left(g^B(\kappa_t)\right) \\
gdp_t(K_t, L_t, B_t, A_t) &\equiv \frac{GDP_t(K_t, L_t, B_t, A_t)}{L_{t-1} + L_t}.
\end{aligned}
\tag{4.1}
$$

To assess the impact of aging on these variables, I consider the differential evolution of GDP_2 and gdp_2 under λ' and λ and evaluate this difference at the evolution under λ, i.e.,

$$
\begin{aligned}
dGDP_2(K_2, L_2, A_2, B_2) &= GDP_2(K_2', L_2', A_2', B_2') - GDP_2(K_2, L_2, A_2, B_2), \\
dgdp_2(K_2, L_2, A_2, B_2) &= gdp_2(K_2', L_2', A_2', B_2') - gdp_2(K_2, L_2, A_2, B_2),
\end{aligned}
\tag{4.2}
$$

where $B_2 = B^*$, $A_2 = A_1\left(1 - \delta + g^A(\kappa^*)\right)$, $A_2' = A_1\left(1 - \delta + g^A(\kappa_2')\right)$, and $B_2' = B^*\left(1 - \delta + g^B(\kappa_2')\right)$.

Corollary 2 *(Aging, GDP, and per-capita GDP)*

1. *It holds that*

$$
dGDP_2(K_2, L_2, A_2, B_2) \approx R_2(K_2' - K_2) + w_2(L_2' - L_2) < 0. \tag{4.3}
$$

---

[10] There are at least two cases where this second channel is mute since savings do not adjust to a lower rental rate of capital. First, if the intertemporal elasticity of substitution is equal to one, then savings are independent of it. Second, if expectations of generation 1 are not rational but "myopic" (Michel and de la Croix (2000)) such that the marginal propensity to save out of wage income at t becomes $s(R_t, \beta) = s(R(\kappa^*), \beta)$.

2. *It holds that*

$$dgdp_2\left(K_2, L_2, A_2, B_2\right) \approx \left(\frac{w_2 L_2}{GDP_2(\cdot)}\frac{2+\lambda}{1+\lambda} - 1\right)\frac{GDP_2(\cdot)}{\left(L_1+L_2\right)^2}\left(L_2' - L_2\right) + \frac{R_2}{L_1+L_2}\left(K_2' - K_2\right). \tag{4.4}$$

According to Corollary 2 aging reduces GDP_2 because there is less capital and less labor in $t = 2$. The former effect is the result of a lower expected rental rate of capital, the latter is the direct effect of aging on the evolution of the labor force. Both effects are proportionate to the respective factor price. More importantly, induced technical change does not play a role here. In fact, as emphasized by Acemoglu (2007), the choice of technology in a competitive economy is such that it maximizes GDP_t at given factor endowments (K_t, L_t). Therefore, the induced changes of q_2^A and q_2^B have no first-order effect on GDP_2.

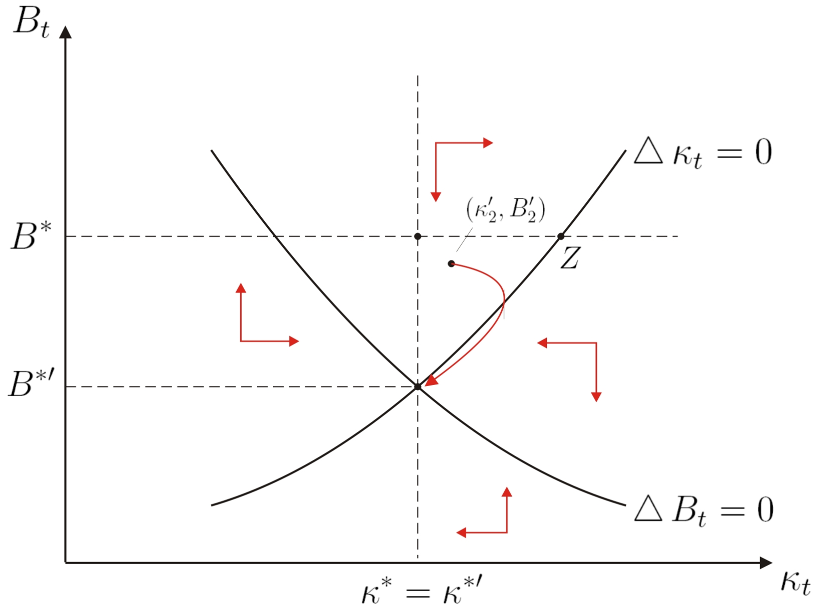
The effect of aging on per-capita GDP is indeterminate in general. This reflects the fact that less labor may reduce GDP by more than population. The first term in parenthesis of (4.4) captures this tension. However, if the labor share w_2L_2/GDP_2 is close to 2/3 and λ close to zero, the effect on aggregate output is likely to dominate. Then, as $K_2' \leq K_2$, per-capita GDP falls in response to population aging.

Finally, it is worth noting that Claim 1 of Proposition 4 as well as Corollary 2 hold for any pair of adjacent periods that experience a small decline in the labor force growth rate. Hence, their applicability is not restricted to the steady state as the starting point.

Let me get back to Claim 2 of Proposition 4 which has the steady-state effects of a permanent increase in the old-age dependency ratio. In accordance with the interpretation of Proposition 3 the steady-state efficient capital intensity remains unaffected and so does the steady-state growth rate. However, there is a level effect on capital-saving technological knowledge. From (3.11), I deduce that B^* declines at the same rate as the growth factor of the labor force.

To understand the role of capital-saving technical change for the steady-state effects consider the phase diagram of Figure 4.1 (again, see Appendix B for details). At B^*, the one time and permanent decline in λ shifts the $\Delta\kappa_t = 0$ – locus to the right since savings per unit of next period's workers increase and (κ^*, B^*) is locally stable. In a world without capital-saving technical change, the new steady state would be at point Z since the evolution of κ_t is stable around κ^* given B^*. Intuitively, this shift reflects the *Solow effect* and the *Hicks effect* on labor-saving technical change. As a consequence, such a framework predicts that a permanent decline in λ induces

Figure 4.1: Comparative Statics and Dynamics for a Permanent Decline in the Growth Rate of the Labor Force. The Case of a Stable Node.



faster steady-state growth due to the Hicks effect on labor-saving technical change. Once the direction of technical change is endogenous, point Z cannot be a steady state: to the right of κ^*, the growth rate of B_t is strictly negative.

5 Extensions

5.1 Contemporaneous Knowledge Spillovers

What is the effect of population aging on the direction of technical change in a competitive economy with contemporaneous knowledge spillovers? To allow for such an externality I replace (2.10) by

$$\begin{aligned} a_t(n) &= A_{t-1}\left(1-\delta+q_t^A(n)+\eta\left[\frac{1}{n_t}\int_0^{n_t} q_t^A(n)dn\right]\right), \\ b_t(m) &= B_{t-1}\left(1-\delta+q_t^B(m)+\eta\left[\frac{1}{m_t}\int_0^{m_t} q_t^B(m)dm\right]\right), \end{aligned} \tag{5.1}$$

where $\eta \in \mathbb{R}_+$, $n \in [0, n_t]$, and $m \in [0, m_t]$. The terms in squared brackets represent the knowledge externality that is external to the respective intermediate-good firm.

It is proportionate to the average of the net productivity growth rates achieved by all firms of the respective sector.

The incorporation of contemporaneous knowledge spillovers requires some modifications that, however, do not invalidate the qualitative results established so far except for those of Corollary 2. To get an idea of the necessary changes, observe that the reasoning that led to Proposition 1 delivers now a symmetric equilibrium configuration with $q_t^A(n) = q_t^A$, $q_t^B(m) = q_t^B$, $a_t = A_{t-1}\left(1-\delta+(1+\eta)q_t^A\right)$, and $b_t = B_{t-1}\left(1-\delta+(1+\eta)q_t^B\right)$. Moreover, there are maps $g^A : \mathbb{R}_{++}^2 \to \mathbb{R}_{++}$ and $g^B : \mathbb{R}_{++}^2 \to \mathbb{R}_{++}$, such that $\hat{q}_t^A = g^A(\kappa_t, \eta)$ and $\hat{q}_t^B = g^B(\kappa_t, \eta)$ with $g_\kappa^A(\kappa_t, \eta) > 0 > g_\kappa^B(\kappa_t, \eta)$, and $g_\eta^i(\kappa_t, \eta) < 0$, $i = A, B$. Substituting these functions in (3.8) - (3.10) delivers the dynamical system with contemporaneous spillovers. Its steady-state efficient capital intensity satisfies $g^B(\kappa^*, \eta) = \delta/(1+\eta)$. Hence, $\kappa^* = \kappa^*(\eta)$ and $B^* = B^*(\eta)$. A unique steady state with $\kappa^*(\eta) \in (0, \infty)$ and $B^*(\eta) \in (0, \infty)$ exists if and only if $\lim_{\kappa\to 0} f'(\kappa) > i'(\delta/(1+\eta)) + i(\delta/(1+\eta)) > \lim_{\kappa\to\infty} f'(\kappa)$, which replaces (3.13). Moreover, the steady-state growth rate of the economy becomes $g^*(\eta) = (1+\eta)g^A(\kappa^*(\eta), \eta) - \delta$.

To understand the role of contemporaneous knowledge spillovers for the effect of aging on the economy, consider its GDP_t and gdp_t of (4.1) replacing $g^i(\kappa_t)$ by $g^i(\kappa_t, \eta)$, $i = A, B$. In analogy to (4.2), I denote by $dGDP_2(K_2, L_2, A_2, B_2; \eta)$ and $dgdp_2(K_2, L_2, A_2, B_2; \eta)$ the differential effect of a small decline in the work force in $t = 2$ on the evolution of GDP and per-capita GDP if $\eta > 0$.

Corollary 3 *(Spillovers, Aging, GDP, and per-capita GDP)*

Consider an environment as in Proposition 4 with $\eta > 0$ and let

$$E(\cdot) \equiv \left[\sum_{i=A,B} \frac{\partial GDP_2(\cdot)}{\partial q_2^i} g_\kappa^i(\kappa_2, \eta)\right] d\kappa_2(\cdot), \tag{5.2}$$

where the evaluation is at $(K_2, L_2, A_2, B_2; \eta)$.

1. *It holds that*

$$dGDP_2(K_2, L_2, A_2, B_2; \eta) \approx R_2(K_2' - K_2) + w_2(L_2' - L_2) + E(\cdot). \tag{5.3}$$

2. *It holds that*

$$\begin{aligned} dgdp_2(K_2, L_2, A_2, B_2; \eta) \approx & \left(\frac{w_2 L_2}{GDP_2(\cdot)}\frac{2+\lambda}{1+\lambda} - 1\right)\frac{GDP_2(\cdot)}{(L_1+L_2)^2}(L_2' - L_2) \\ & + \frac{R_2}{L_1+L_2}(K_2' - K_2) + \frac{E(\cdot)}{L_1+L_2}. \end{aligned} \tag{5.4}$$

Corollary 3 extends the findings of Corollary 2 to the case where $\eta > 0$. The new feature is the presence of $E(\cdot)$ in (5.3) and (5.4). It captures the effect of population aging on the evolution of GDP and gdp through induced technical change.

According to (5.2), the expression $E(\cdot)$ consists of two factors. First, there is the effect of a changing capital intensity, on the efficient capital intensity κ_2. In light of Proposition 4, I have $K_2'/L_2' > K_2/L_2$, hence, $d\kappa_2(\cdot) = \kappa_2' - \kappa^* > 0$. The second factor, i. e., the term in brackets of (5.2), has the effect of induced technical change on GDP_2. Unlike in the scenario without contemporaneous knowledge spillovers, this effect does not vanish. In fact, due to the externality there is too little innovation in the competitive economy such that $sign\left[\partial GDP_2\left(\cdot\right)/\partial q_2^i\right] > 0$, $i = A, B$. However, since $g_\kappa^A\left(\kappa_2, \eta\right) > 0 > g_\kappa^B\left(\kappa_2, \eta\right)$, the sign of the term in brackets is indeterminate in general. Hence, for the results of Corollary 2 to go through, I have to assume that contemporaneous knowledge spillovers are weak such that η is sufficiently close zero.

5.2 Increasing Life-Expectancy

Thus far, I studied aging as a decline in the growth rate of the labor force. However, only a slight reinterpretation of the analytical framework is necessary to allow for the incorporation of the effect of an increasing life expectancy on the direction of technical change. To accomplish this, suppose that each individual faces a probability to die at the onset of old age equal to $(1 - \nu) \in (0, 1)$. Let $\beta_\nu \in (0, 1)$ denote the discount factor that the individual applies in the presence of a probability to die. Moreover, normalize $u(0) = 0$ to be the utility after death. Under these assumptions, we may interpret U_t of (2.1) as the expected utility of generation t with $\beta \equiv \beta_\nu \nu$ as the effective discount factor. The old-age dependency ratio at t becomes $\nu L_{t-1}/L_t$ and increases in ν.

There is a perfect annuity market for insurance against survival risk. Then the economy inherits the properties established in Proposition 1, 2, 3, and Lemma 1. However, since the savings rate becomes a function of life expectancy, the following proposition holds.[11]

Proposition 5 *(Life Expectancy, Comparative Statics and Dynamics)*

Consider an economy in a steady state in period $t = 1$. Then, it experiences a small and permanent increase in its life expectancy such that the effective discount

[11] Without a perfect annuity market there are unintended bequests to deal with and the effect of an increase in life-expectancy on aggregate savings is indeterminate in general (see, e. g., Sheshinski (2007) for a discussion).

factor increases, i. e., $\beta' > \beta$ for all generations $t = 1, 2, 3, ..., \infty$. Denote variables associated with an evolution under β' by an apostrophe such that $(\kappa^{*\prime}, B^{*\prime})$ is the steady state under β'.

1. *If generation 1 anticipates the increase in its life expectancy, then $\kappa_2' > \kappa^*$ and $B_2' < B^*$. If this change is anticipated by all generations $t = 2, 3, ..., \infty$ and not by generation 1, then $\kappa_2' = \kappa^*$, $B_2' = B^*$ and $\kappa_3' > \kappa^*$, $B_3' < B^*$.*
2. *It holds that $\kappa^{*\prime} = \kappa^*$ and $B^{*\prime} < B^*$.*

Intuitively, an anticipated higher life expectancy increases savings per next period's worker since the weight on expected old-age utility increases. This dominates the effect of a declining real rental rate of capital. Then, initially the effective capital intensity increases. However, contrary to the case of a permanent decline in the growth rate of the labor force, this effect may be delayed by one generation if generation 1 makes its plan anticipating an effective discount factor of β instead of β'. Arguably, this is the relevant case since expectations of one's own life expectancy are often myopic, i. e., coincide with the actual life expectancy of the previous generation. Moreover, unlike a reduction in a generation's number of offsprings, the choice of (c_1^y, s_1, c_2^o) is usually made before the change in the survival probability is experienced. As a consequence, the effect of a permanent increase in life expectancy on the direction of technical change may be delayed by one generation.

Since the steady-state efficient capital intensity is independent of the household sector, the qualitative finding on the long-run implications mimic those of Proposition 4: an increase in life expectancy has no effect on the steady-state growth rate, however, the steady-state rate of return on capital falls since $B^{*\prime} < B^*$.

The initial effect of an increasing life expectancy on GDP in absolute and per-capita terms may be directly deduced from Corollary 3. Since $K_2' \geq K_2$ and $L_2' = L_2$, (5.3) and (5.4) deliver $dGDP_2\left(K_2, L_2, A_2, B_2; \eta\right) \approx R_2(K_2' - K_2) + E\left(\cdot\right)$, which is positive if η is sufficiently small and $K_2' > K_2$. As to per-capita GDP, I find $dgdp_2\left(K_2, L_2, A_2, B_2; \eta\right) \approx \left[R_2\left(K_2' - K_2\right) + E(\cdot)\right] / \left[\nu L_1 + L_2\right] - gdp_2 L_1(\nu' - \nu) / \left[\nu L_1 + L_2\right]$. The sign is indeterminate even if η is small. It will be negative if a higher ν increases population by more than GDP_2.

6 Concluding Remarks

What is the role of population aging for the direction of technical change and economic performance ? I address this question in a competitive economy that exhibits endogenous capital- and labor-saving technical change. Population aging corresponds either to a decline in the growth rate of the labor force or to an increase in life-expectancy. Both phenomena increase the economy's old-age dependency ratio.

Contrary to the existing literature on aging and endogenous growth, I find that the steady-state growth rate is independent of the economy's age structure. However, population aging affects the transitional dynamics. Even if the current young reduce their savings in anticipation of a declining rental rate of capital, the relative scarcity of labor increases. This leads to more labor- and less capital- saving technical change. Unless there are contemporaneous knowledge spillovers, this change in the direction of technical change fails to have a first-order effect on the evolution GDP. If aging takes the form of a declining labor force growth rate, it reduces GDP because it lowers the growth of the work force and of the capital stock. Per-capita GDP is then likely to decline, too, since the effect on GDP more than outweighs the one on population growth.

An increase in life expectancy has the potential to raise GDP along the transition if it leads to more savings and capital accumulation. However, these effects may be delayed if a rise in life expectancy is not anticipated. Overall, the presence of contemporaneous knowledge spillovers makes it more difficult to come up with clear-cut predictions on the relationship between population aging, the direction of technical change, and economic performance.

The present analysis suggests several routes for future research. They include the following. First, one may want to understand the role of fiscal policy in an economy with endogenous capital- and labor-saving technical change. The fact that the steady-state growth rate is independent of the household sector suggests that steady-state growth effects that arise, e. g., in Saint-Paul (1992), may not obtain once endogenous capital-saving technical change is allowed for.

Second, the question arises to what extend the implications for the transitional dynamics hinge on the assumption of a household sector with two-period lived individuals. While this framework has the advantage of an intuitive representation of aging and of analytical tractability, it abstracts from facets that may impinge on the relationship between aging and the direction of technical change such as elements of intragenerational heterogeneity or intergenerational transfers. I leave these questions for future research.

A Proofs

A.1 Proof of Proposition 1

Upon substitution of (2.7) and (2.8) in the respective zero-profit condition of (2.17), we obtain

$$\begin{aligned} f(\kappa_t) - \kappa_t f'(\kappa_t) &= (1-\delta+\hat{q}_t^A)i'(\hat{q}_t^A) + i(\hat{q}_t^A), \\ f'(\kappa_t) &= (1-\delta+\hat{q}_t^B)i'(\hat{q}_t^B) + i(\hat{q}_t^B). \end{aligned} \tag{A.1}$$

Denote $RHS(q)$ the right-hand side of both conditions. In view of the properties of the function i given in (2.11), $RHS(q)$ is a mapping $RHS(q): \mathbb{R}_+ \to \mathbb{R}_+$ with $RHS'(q) > 0$ for $q \geq 0$ and $\lim_{q\to\infty} RHS(q) = \infty$. Moreover, the properties of the function $f(\kappa_t)$ imply that the left-hand side of both conditions is strictly positive for $\kappa_t > 0$. Hence, for each $\kappa_t > 0$ there is a unique $\hat{q}_t^j > 0$, $j = A, B$, that satisfies the respective condition stated in (A.1). I denote these maps by $\hat{q}_t^j = g^j(\kappa_t)$, $j = A, B$.

An application of the implicit function theorem to (A.1) gives

$$\begin{aligned} \frac{d\hat{q}^A}{d\kappa_t} &= \frac{-\kappa_t f''(\kappa_t)}{\left(1-\delta+\hat{q}_t^A\right) i''\left(\hat{q}_t^A\right) + 2i'\left(\hat{q}_t^A\right)} \equiv g_\kappa^A(\kappa_t) > 0, \\ \frac{d\hat{q}^B}{d\kappa_t} &= \frac{f''(\kappa_t)}{\left(1-\delta+\hat{q}_t^B\right) i''\left(\hat{q}_t^B\right) + 2i'\left(\hat{q}_t^B\right)} \equiv g_\kappa^B(\kappa_t) < 0. \end{aligned} \tag{A.2}$$

The respective signs follow from the properties of the functions f and i. ■

A.2 Proof of Corollary 1

Equation (3.6) is a fixed-point problem with a unique solution $\hat{\kappa}_t > 0$. To see this, write the right-hand side of (3.6) as $RHS(\kappa_t, B_{t-1}K_t/A_{t-1}L_t)$. Given the properties of $g^A(\kappa_t)$ and $g^B(\kappa_t)$ as stated in (2.18) and $B_{t-1}K_t/A_{t-1}L_t > 0$, the function $RHS(\kappa_t, B_{t-1}K_t/A_{t-1}L_t)$ is continuous, strictly decreasing and strictly positive for all $\kappa_t > 0$. Hence, $\lim_{\kappa_t\to 0} RHS(\kappa_t, \cdot) > 0$. Accordingly, there is a unique $\hat{\kappa}_t > 0$ such that $\hat{\kappa}_t = RHS(\hat{\kappa}_t, B_{t-1}K_t/A_{t-1}L_t)$.

Ceteris paribus, a greater $B_{t-1}K_t/A_{t-1}L_t$ shifts the function $RHS(\kappa_t, B_{t-1}K_t/A_{t-1}L_t)$ upwards. Therefore, $\hat{\kappa}_t$ is greater the greater $B_{t-1}K_t/A_{t-1}L_t$. Then, from (2.18), $\hat{q}_t^A = g^A(\kappa_t)$ increases whereas $\hat{q}_t^B = g^B(\kappa_t)$ decreases. Obviously, the opposite response obtains when $B_{t-1}K_t/A_{t-1}L_t$ falls. ■

A.3 Proof of Proposition 2

The proof consists of two steps. First, I show that the variables κ_t and B_t are indeed state variables of the economy at t. Second, I prove the existence of a unique equilibrium sequence $\{\kappa_t, B_t\}_{t=1}^{\infty}$.

1. Given L_{t-1}, L_t, and K_t, the equilibrium determines 22 variables belonging to period t. One readily verifies that there are also 22 conditions for each period. Since $\hat{q}^A = g^A(\kappa_t)$ and $\hat{q}^B = g^B(\kappa_t)$ according to Proposition 1, it is also straightforward to verify that all prices $\{p_{L,t}, p_{K,t}, w_t, R_t\}_{t=1}^{\infty}$ depend on κ_t. Moreover, $\{Y_t, Y_{K,t}, Y_{L,t}, n_t, m_t, a_t, b_t, l_t, k_t\}_{t=1}^{\infty}$ as well as $\{A_t, B_t\}_{t=1}^{\infty}$ depend on κ_t. In addition, $y_{L,t} = y_{K,t} = 1$. With this in mind, individual savings of (2.3) becomes $s_t = s(R(\kappa_{t+1}), \beta) w(\kappa_t)$, where

$$w(\kappa_t) \equiv A_{t-1}\left(1-\delta+g^A(\kappa_t)\right)\left[f(\kappa_t)-\kappa_t f'(\kappa_t)-i\left(g^A(\kappa_t)\right)\right] \tag{A.3}$$

and

$$R(\kappa_{t+1}) \equiv B_t\left(1-\delta+g^B(\kappa_{t+1})\right)\left[f'(\kappa_{t+1})-i\left(g^B(\kappa_{t+1})\right)\right]. \tag{A.4}$$

Both factor prices, $w(\kappa_t)$ and $R(\kappa_{t+1})$, are found from the respective zero-profit condition given in (2.12) and Proposition 1. The individual budget constraints deliver $c_t^y = (1-s(R(\kappa_{t+1}), \beta)) w(\kappa_t)$ and $c_t^o = R(\kappa_t)\, s(R(\kappa_t), \beta) w(\kappa_{t-1})$. For $t=1$, we have by assumption that $c_1^o = R(\kappa_1) K_1/L_0$.

2. To obtain equation (3.8) solve (3.6) for K_{t+1} and substitute the resulting expression into (3.7). This gives

$$s_t L_t = \kappa_{t+1}\frac{A_t\left(1-\delta+g^A(\kappa_{t+1})\right)}{B_t\left(1-\delta+g^B(\kappa_{t+1})\right)}L_{t+1}, \qquad \text{for } t=1,2,...,\infty. \tag{A.5}$$

Using $s_t = s(R(\kappa_{t+1}), \beta) w(\kappa_t)$ and $w_t/A_t \equiv \tilde{w}(\kappa_t)$ gives after some straightforward manipulations (3.8).

The second difference equation of the dynamical system states the evolution of B_{t+1} described by (2.20) where q_{t+1}^B is replaced by $g^B(\kappa_{t+1})$ in accordance with Proposition 1.

In the first period, κ_1 is pinned down by (3.6) for given initial values A_0, B_0, L_0, K_1 and $L_1 = (1+\lambda)L_0$. From the proof of Corollary 1, there is a unique solution $\kappa_1 > 0$.

Before we turn to the uniqueness of the sequence $\{\kappa_t, B_t\}_{t=1}^{\infty}$ it is useful to state and prove the following lemma.

Lemma 2 *Define*

$$\Psi(\kappa_t) \equiv \frac{1-\delta+g^A(\kappa_t)}{1-\delta+g^B(\kappa_t)}\frac{\kappa_t(1+\lambda)}{s(R(\kappa_t), \beta)}. \tag{A.6}$$

It holds for all $\kappa_t > 0$ that

$$\begin{aligned}
\tilde{w}(\kappa_t) &> 0 \quad \textit{and} \quad \tilde{w}'(\kappa_t) > 0, \\
\tilde{R}(\kappa_t) &> 0 \quad \textit{and} \quad \tilde{R}'(\kappa_t) < 0, \\
w(\kappa_t) &> 0 \quad \textit{and} \quad w'(\kappa_t) > 0, \\
R(\kappa_t) &> 0 \quad \textit{and} \quad R'(\kappa_t) < 0, \\
\Psi(\kappa_t) &> 0 \quad \textit{and} \quad \Psi'(\kappa_t) > 0 \quad \textit{with} \quad \lim_{\kappa_t\to 0}\Psi(\kappa_t) = 0 \quad \textit{and} \quad \lim_{\kappa_t\to\infty}\Psi(\kappa_t) = \infty.
\end{aligned} \tag{A.7}$$

Proof of Lemma 2

First, I note that (2.12), Proposition 1, and the updating conditions (2.20) deliver

$$\begin{aligned} w_t/A_t &= f(\kappa_t) - \kappa_t f'(\kappa_t) - i\left(g^A(\kappa_t)\right) \equiv \tilde{w}(\kappa_t), \\ R_t/B_t &= f'(\kappa_t) - i\left(g^B(\kappa_t)\right) \equiv \tilde{R}(\kappa_t). \end{aligned} \tag{A.8}$$

With (A.1), this gives

$$\begin{aligned} \tilde{w}(\kappa_t) &= \left(1-\delta+g^A(\kappa_t)\right) i'\left(g^A(\kappa_t)\right) > 0, \\ \tilde{R}(\kappa_t) &= \left(1-\delta+g^B(\kappa_t)\right) i'\left(g^B(\kappa_t)\right) > 0, \end{aligned} \tag{A.9}$$

where, for $\kappa_t > 0$, the signs follow from Proposition 1 and the properties of the function i given in (2.11). It follows that

$$\begin{aligned} \tilde{w}'(\kappa_t) &= g^A_\kappa(\kappa_t)\left[i'\left(g^A(\kappa_t) + \left(1-\delta+g^A(\kappa_t)\right) i''\left(g^A(\kappa_t)\right)\right)\right] > 0, \\ \tilde{R}'(\kappa_t) &= g^B_\kappa(\kappa_t)\left[i'\left(g^B(\kappa_t) + \left(1-\delta+g^B(\kappa_t)\right) i''\left(g^B(\kappa_t)\right)\right)\right] < 0. \end{aligned} \tag{A.10}$$

Again, for $\kappa_t > 0$, the signs follow from Proposition 1 and the properties of the function i given in (2.11).

The third and the forth claim of (A.7) follow immediately from (A.9) and (A.10) since the updating conditions (2.20) imply

$$\begin{aligned} w(\kappa_t) &= A_{t-1}\left(1-\delta+g^A(\kappa_t)\right)^2 i'\left(g^A(\kappa_t)\right), \\ R(\kappa_t) &= B_{t-1}\left(1-\delta+g^B(\kappa_t)\right)^2 i'\left(g^B(\kappa_t)\right). \end{aligned} \tag{A.11}$$

As to the fifth claim of (A.7), I note that $\Psi(\kappa_t) > 0$ and $\Psi'(\kappa_t) > 0$ follow immediately from the definition of Ψ given in (A.6), Proposition 1, and the fact that $s_R(\kappa_t, \beta) \geq 0$.

Since $g^A(\kappa_t)$ is increasing on $\mathbb{R}_{++}$ and bounded from below by zero, $\lim_{\kappa_t \to 0} g^A(\kappa_t)$ is finite and $\lim_{\kappa_t \to \infty} g^A(\kappa_t)$ is finite or infinite. Since $g^B(\kappa_t)$ is decreasing on $\mathbb{R}_{++}$ and bounded from below by zero, $\lim_{\kappa_t \to 0} g^B(\kappa_t)$ is either finite or infinite and $\lim_{\kappa_t \to \infty} g^B(\kappa_t)$ is finite. It follows that $\lim_{\kappa_t \to 0}\left(1-\delta+g^A(\kappa_t)\right)/\left(1-\delta+g^B(\kappa_t)\right)$ is finite. Moreover, $\lim_{\kappa_t \to \infty}\left(1-\delta+g^A(\kappa_t)\right)/\left(1-\delta+g^B(\kappa_t)\right)$ is either finite or infinite.

Since $s(R(\kappa_t), \beta) \in (0,1)$ for all $\kappa_t \in \mathbb{R}_{++}$ and $s_R(R(\kappa_t), \beta) \geq 0$, I find that $1 \geq \lim_{\kappa_t \to 0} s(R(\kappa_t), \beta) > 0$ and $1 > \lim_{\kappa_t \to \infty} s(R(\kappa_t), \beta) \geq 0$ if $s_R(R(\kappa_t), \beta) > 0$. Otherwise, $s(R(\kappa_t), \beta)$ is independent of κ_t. Hence, I have $\lim_{\kappa_t \to 0} \Psi(\kappa_t) = 0$ and $\lim_{\kappa_t \to \infty} \Psi(\kappa_t) = \infty$. ■

Rearranging (3.8) gives

$$B_t\, \tilde{w}(\kappa_t) = \Psi(\kappa_{t+1}). \tag{A.12}$$

According to Lemma 2, the left-hand side of (A.12) is strictly positive for any $(\kappa_t, B_t) \in \mathbb{R}^2_{++}$. Moreover, the right-hand side of (A.12) is increasing on $\mathbb{R}_{++}$ approaching zero as $\kappa_{t+1} \to 0$ and infinity as $\kappa_{t+1} \to \infty$. Hence, there is a unique κ_{t+1} that satisfies (3.8) given $(\kappa_t, B_t) \in \mathbb{R}^2_{++}$. With this value at hand, (3.9) delivers a unique $B_{t+1} > 0$. ■

A.4 Proof of Proposition 3

1. First, I prove the "if"-part before I turn to the "only if"-part.

   - "$\Rightarrow$": If condition (3.13) holds then (A.1) delivers for the capital-intensive intermediate-good sector $\lim_{\kappa\to 0} g^B(\kappa) > \delta > \lim_{\kappa\to\infty} g^B(\kappa)$. The latter is necessary and sufficient for (3.12) to have at least one solution $\kappa^* \in (0,\infty)$ given that $g^B(\kappa)$ is continuous according to Proposition 1.

     The existence of a steady state $(\kappa^*, B^*) \in \mathbb{R}^2_{++}$ follows from (3.11) and Lemma 2. They assure that a value $B^* = \Psi(\kappa^*)/w(\kappa^*) \in (0,\infty)$ exists since $\Psi(\kappa) > 0$ and $\tilde{w}(\kappa) > 0$ for any $\kappa \in (0,\infty)$.

     Uniqueness of κ^* follows from $g^B_\kappa(\kappa) < 0$ on $\mathbb{R}_{++}$. The uniqueness of B^* follows since the total differential of (A.12) reveals for any given κ_t that $d\kappa_{t+1}/dB_t > 0$. Hence, there is a unique B^* that satisfies (A.12) at $\kappa_t = \kappa_{t+1} = \kappa^*$.

   - "$\Leftarrow$": If $i'(\delta) + i(\delta) \geq \lim_{\kappa\to 0} f'(\kappa)$, then $\delta \geq \lim_{\kappa\to 0} g^B(\kappa)$. As a consequence, for all $\kappa \in (0,\infty)$ the sequence $\{B_t\}_{t=1}^{\infty}$ is decreasing for all t. Accordingly, $B_t \in [0, B_0]$ and $B_t \leq B_0 \left(1 - \delta + \lim_{\kappa\to 0} g^B(\kappa)\right)^t$. Therefore, $\lim_{t\to\infty} B_t = 0$. If $\lim_{\kappa\to\infty} f'(\kappa) \geq i'(\delta) + i(\delta)$ then $\lim_{\kappa\to\infty} g^B(\kappa) \geq \delta$. Accordingly, the sequence $\{B_t\}_{t=1}^{\infty}$ is increasing for all t and $B_t \geq B_0 \left(1 - \delta + \lim_{\kappa\to\infty} g^B(\kappa)\right)^t$. Hence, $\lim_{t\to\infty} B_t = \infty$.

2. Since (3.12) determines κ^*, I obtain from Proposition 1 and (2.20) that $A_{t+1}/A_t - 1 = g^A(\kappa^*) - \delta \equiv g^*$. The stated findings about the steady-state growth rate of a_t, w_t, c^y_t, c^o_t, and s_t are immediate from $a_t = A_t$, (A.8), (2.3) with $R^* = B^*[p^*_K - i(\delta)]$. Steady-state growth of K_t follows from (3.7). The evolution of k_t and l_t follow immediately from (2.13). All other growth factors under b) result from (2.4) in conjunction with (E4), (E5), and (3.4). From (2.7), I have $p^*_K = f'(\kappa^*)$. Then, (A.8) delivers $R^* = B^*[p^*_K - i(\delta)]$. ■

A.5 Proof of Lemma 1

The following Result introduces some necessary notation for the analysis of the local stability of the steady state.

Result 1 *With (κ_1, B_1) given by (3.10), the dynamical system can be stated as*

$$(\kappa_{t+1}, B_{t+1}) = \phi(\kappa_t, B_t) \equiv \left(\phi^\kappa(\kappa_t, B_t), \phi^B(\kappa_t, B_t)\right), \ t = 1, 2, ..., \infty, \tag{A.13}$$

where $\phi^j : \mathbb{R}_{++} \to \mathbb{R}_{++}$, $j = \kappa, B$, are continuously differentiable functions.

Proof of Result 1

Write (3.8) and (3.9) as $\kappa_{t+1} = h(\kappa_t, B_{t+1})$ and $B_{t+1} = z(\kappa_{t+1}, B_t)$, where h and z are some continuously differentiable functions. With B_0 as an initial condition and κ_1 determined by (3.10), B_1 is the equal to $z(B_0, \kappa_1)$. Hence, we may state the the dynamical system as

$$\kappa_{t+1} = h(\kappa_t, B_{t+1}), \quad B_{t+1} = z(\kappa_{t+1}, B_t) \quad t = 1, 2, ..., \infty, (\kappa_1, B_1) \text{ given.} \tag{A.14}$$

Then, for any (κ_t, B_t) these equations determine (κ_{t+1}, B_{t+1}). More precisely, upon substitution we obtain $\kappa_{t+1} = h(\kappa_t, z(\kappa_{t+1}, B_t))$ which implicitly defines $\kappa_{t+1} = \phi^\kappa(\kappa_t, B_t)$. In turn, using the latter, we find $B_{t+1} = z(\kappa_{t+1}, B_t) = z(\phi^\kappa(\kappa_t, B_t), B_t) \equiv \phi^B(\kappa_t, B_t)$. Hence, we may state the dynamical system as in (A.13). ■

The steady state is a fixed point of the system (A.13). To study the local behavior of the system around the steady state, we have to know the eigenvalues of the Jacobian matrix

$$D\phi(\kappa^*, B^*) \equiv \begin{pmatrix} \phi^\kappa_\kappa(\kappa^*, B^*) & \phi^\kappa_B(\kappa^*, B^*) \\ \phi^B_\kappa(\kappa^*, B^*) & \phi^B_B(\kappa^*, B^*) \end{pmatrix}. \tag{A.15}$$

We study each of the four elements of the Jacobian in turn.

1. First, consider $\phi^\kappa(\kappa_t, B_t)$. Consider (3.8), which we repeat here for convenience in the form of (A.12)

$$B_t\, \tilde{w}(\kappa_t) = \Psi(\kappa_{t+1}).$$

The implicit function theorem assures that a function $\kappa_{t+1} = \phi^\kappa(\kappa_t, B_t)$ exists if $\phi^\kappa_\kappa(\kappa_t, B_t) \equiv d\kappa_{t+1}/d\kappa_t$ and $\phi^\kappa_B(\kappa_t, B_t) \equiv d\kappa_{t+1}/dB_t$ exist. We show next that this is the case for all $(\kappa_t, B_t) > 0$.

(a) We start with $\phi^\kappa_\kappa(\kappa_t, B_t)$. Implicit differentiation of (3.8) gives

$$\phi^\kappa_\kappa(\kappa_t, B_t) = \frac{B_t\, \tilde{w}'(\kappa_t)}{\Psi'(\kappa_{t+1})} > 0. \tag{A.16}$$

Form Lemma 2 the numerator and the denominator are strictly positive. Hence, the derivative $\phi^\kappa(\kappa_t, B_t)$ exists and is strictly positive for all (κ_t, B_t).

Evaluated at the steady state, (A.16) simplifies to

$$\phi^\kappa_\kappa(\kappa^*, B^*) = \frac{\Psi(\kappa^*)\, w'(\kappa^*)}{\Psi'(\kappa^*)\, w(\kappa^*)}. \tag{A.17}$$

The assumption that the locus $\Delta\kappa_t = 0$ is stable in the vicinity of the steady state is equivalent to $\phi^\kappa_\kappa(\kappa^*, B^*) \in (0, 1)$.

(b) Next, I turn to the derivative $\phi^\kappa_B(\kappa_t, B_t)$. Total differentiation of (3.8) gives now

$$\phi^\kappa_B(\kappa_t, B_t) = \frac{\Psi(\kappa_{t+1})}{B_t\, \Psi'(\kappa_{t+1})} > 0. \tag{A.18}$$

By Lemma 2, the derivative exists for all $\kappa_{t+1} > 0$ and is strictly positive. Evaluated at the steady state, we have

$$\phi^\kappa_B(\kappa^*, B^*) = \frac{\Psi(\kappa^*)}{B^*\, \Psi'(\kappa^*)}. \tag{A.19}$$

2. Next, I turn to $\phi^B(\kappa_t, B_t)$. Consider (3.9) for $t+1$ and substitute $\kappa_{t+1} = \phi^\kappa(\kappa_t, B_t)$. This gives

$$B_{t+1} = B_t\left(1 - \delta + g^B(\kappa_{t+1})\right) = B_t\left(1 - \delta + g^B(\phi^\kappa(\kappa_t, B_t))\right) \equiv \phi^B(\kappa_t, B_t). \tag{A.20}$$

Since $\phi^\kappa(\kappa_t, B_t)$ exists for all $(\kappa_t, B_t) > 0$ so does $\phi^B(\kappa_t, B_t)$. We now characterize the partial derivatives of this function.

(a) First, consider $\phi^B_\kappa(\kappa_t, B_t) \equiv \partial B_{t+1}/\partial \kappa_t$. From (A.20) we have

$$\phi^B_\kappa(\kappa_t, B_t) = B_t\, g^B_\kappa(\phi^\kappa(\kappa_t, B_t))\, \phi^\kappa_\kappa(\kappa_t, B_t) < 0. \tag{A.21}$$

Evaluated at the steady state, this gives

$$\phi^B_\kappa(\kappa^*, B^*) = B^*\, g^B_\kappa(\kappa^*)\, \phi^\kappa_\kappa(\kappa^*, B^*). \tag{A.22}$$

(b) Next, we consider the derivative $\phi^B_B(\kappa_t, B_t) \equiv \partial B_{t+1}/\partial B_t$. From (A.20), we find

$$\phi^B_B(\kappa_t, B_t) = \big(1 - \delta + g^B(\phi^\kappa(\kappa_t, B_t))\big) + B_t\, g^B_\kappa(\phi^\kappa(\kappa_t, B_t))\, \phi^\kappa_B(\kappa_t, B_t).$$

Evaluated at the steady state, this gives

$$\phi^B_B(\kappa^*, B^*) = 1 + B^*\, g^B_\kappa(\kappa^*)\, \phi^\kappa_B(\kappa^*, B^*). \tag{A.23}$$

Observe that $\phi^B_B(\kappa^*, B^*) \in (0,1)$. Indeed, since $g^B_\kappa < 0$ and $\phi^\kappa_B(\kappa^*, B^*) > 0$, we have $\phi^B_B(\kappa^*, B^*) < 1$. Using (A.19) and the definition of Ψ of (A.6) I find

$$\phi^B_B(\kappa^*, B^*) > 0 \quad \Leftrightarrow \quad \big(1 - \delta + g^A(\kappa^*)\big)\left(1 - \frac{s_R\, R'(\kappa^*)}{s}\right) + \kappa^*\, g^A_\kappa(\kappa^*) > 0, \tag{A.24}$$

where s is evaluated at $(R(\kappa^*), \beta)$. The sign follows from Proposition 1.

Using the results of (A.17), (A.19), (A.22), and (A.23), the required Jacobian (A.15) can be written

$$D\,\phi(\kappa^*, B^*) = \begin{pmatrix} \phi^\kappa_\kappa(\kappa^*, B^*) & \phi^\kappa_B(\kappa^*, B^*) \\ B^* g^B_\kappa(\kappa^*)\, \phi^\kappa_\kappa(\kappa^*, B^*) & 1 + B^*\, g^B_\kappa(\kappa^*)\, \phi^\kappa_B(\kappa^*, B^*) \end{pmatrix}. \tag{A.25}$$

Denote μ_1 and μ_2 the eigenvalues of the Jacobian (A.25). With (A.17), (A.19), (A.22), and (A.23), they are given by

$$\mu_1 = \frac{\phi^\kappa_\kappa + \phi^B_B}{2} + \sqrt{\left(\frac{\phi^\kappa_\kappa + \phi^B_B}{2}\right)^2 - \phi^\kappa_\kappa},$$

$$\mu_2 = \frac{\phi^\kappa_\kappa + \phi^B_B}{2} - \sqrt{\left(\frac{\phi^\kappa_\kappa + \phi^B_B}{2}\right)^2 - \phi^\kappa_\kappa}.$$

Both eigenvalues are real if $\phi^B_B \geq 2\sqrt{\phi^\kappa_\kappa} - \phi^\kappa_\kappa \geq 0$, or

$$1 + B^*\, g^B_\kappa(\kappa^*)\, \phi^\kappa_B(\kappa^*, B^*) \geq 2\sqrt{\phi^\kappa_\kappa} - \phi^\kappa_\kappa.$$

One readily verifies that μ_1 is strictly increasing in ϕ^B_B with $\mu_1|_{\phi^B_B=1} = 1$. Moreover, $\mu_1|_{\phi^B_B=2\sqrt{\phi^\kappa_\kappa}-\phi^\kappa_\kappa} = \sqrt{\phi^\kappa_\kappa}$. Hence, $\mu_1 \in (0,1)$. On the other hand, μ_2 decreases in ϕ^B_B with $\mu_2|_{\phi^B_B=2\sqrt{\phi^\kappa_\kappa}-\phi^\kappa_\kappa} = \sqrt{\phi^\kappa_\kappa}$ and $\mu_2|_{\phi^B_B} = \phi^\kappa_\kappa$. Hence, $\mu_2 \in (0, \sqrt{\phi^\kappa_\kappa})$. Moreover, $\mu_1 \geq \mu_2$ with equality when $\phi^B_B = 2\sqrt{\phi^\kappa_\kappa} - \phi^\kappa_\kappa$.

If (A.26) is violated, then $D\phi$ has two distinct complex eigenvalues. Then, the steady state is a spiral sink since $det\,(D\phi) = \phi^\kappa_\kappa < 1$ (see, e. g., Galor (2007), Proposition 3.8). The stability of the loci $\Delta\kappa_t = 0$ and $\Delta B_t = 0$ imply the clockwise orientation of the spiral sink. ■

A.6 Proof of Proposition 4

1. Under rational expectations, generation 1 makes its plan (c_1^y, s_1, c_2^o) anticipating a real rental rate of capital equal to $R_2 = R(\kappa_2')$. Then, the capital stock at $t = 2$ is equal to $K_2 = s(R(\kappa_2'), \beta) w_1 L_1$. Moreover, κ_2' is given by (3.8), i. e.,

$$B^* \ \tilde{w}(\kappa^*) = \Psi(\kappa_2') \frac{1+\lambda'}{1+\lambda}. \tag{A.26}$$

   From Lemma 2, I know that $\Psi(\kappa)$ is increasing for all $\kappa > 0$. Hence, $\kappa_2' > \kappa^*$ as $\lambda > \lambda'$.

2. This follows immediately from (3.11) and (3.12) for λ and λ' and $\lambda > \lambda'$. ■

A.7 Proof of Corollary 2

Equations (4.3) and (4.4) are first-order Taylor approximations of $dGDP_2$ and $dgdp_2$ at the steady-state path under λ. I know from the proof of Proposition 4 that $\kappa_2' > \kappa^*$, hence $K_2' \leq K_2$. Moreover, $\lambda' < \lambda$ implies $L_2' < L_2$.

The total effect of a small change of L_2 and K_2 on GDP_2 at (K_2, L_2, A_2, B_2) may be written as

$$\frac{dGDP_2(\cdot)}{dL_2} = \frac{\partial GDP_2(\cdot)}{\partial L_2} + \left[\sum_{i=A,B} \frac{\partial GDP_2(\cdot)}{\partial q_2^i} g_\kappa^i(\kappa_2)\right] \frac{\partial \kappa_2(\cdot)}{\partial L_2} = w_2 > 0, \tag{A.27}$$

$$\frac{dGDP_2(\cdot)}{dK_2} = \frac{\partial GDP_2(\cdot)}{\partial K_2} + \left[\sum_{i=A,B} \frac{\partial GDP_2(\cdot)}{\partial q_2^i} g_\kappa^i(\kappa_2)\right] \frac{\partial \kappa_2(\cdot)}{\partial K_2} = R_2 > 0, \tag{A.28}$$

where $\partial\kappa_2(\cdot)/\partial L_2 < 0$ and $\partial\kappa_2(\cdot)/\partial K_2 > 0$ are implicitly given by (3.6) for $t = 2$. The partial effect of L_2 on GDP_2 is $\partial GDP_2(\cdot)/\partial L_2 = A_2\left[\partial F(\cdot)/\partial(A_2 L_2) - i\left(g^A(\kappa_2)\right)\right]$ following (4.1). It coincides with $w_2 = w(\kappa_2) = w(\kappa^*)$ using (A.3). Since $g_\kappa^i(\kappa_2) \neq 0$, $i = A, B$, (A.27) follows from the observation that

$$\begin{aligned}
\frac{\partial GDP_2(\cdot)}{\partial q_2^A} &= A_1 L_2 \left[\frac{\partial F(\cdot)}{\partial(A_2 L_2)} - \left(1 - \delta + g^A(\kappa_2)\right) i'\left(g^A(\kappa_2)\right) - i\left(g^A(\kappa_2)\right)\right] = 0, \\
\frac{\partial GDP_2(\cdot)}{\partial q_2^B} &= B_1 K_2 \left[\frac{\partial F(\cdot)}{\partial(B_2 K_2)} - \left(1 - \delta + g^B(\kappa_2)\right) i'\left(g^B(\kappa_2)\right) - i\left(g^B(\kappa_2)\right)\right] = 0.
\end{aligned} \tag{A.29}$$

The expressions in brackets coincide with equilibrium profits of intermediate-good firms at $t = 2$ and are therefore equal to zero in accordance with Proposition 1. The same reasoning proves the result for a small change in K_2 as given in (A.28). Hence, (4.3) holds.

As to (4.4), I use (A.27) and (A.28) to derive $dgdp_2/dL_2$ and $dgdp_2/dK_2$. Straightforward manipulations of these derivatives deliver (4.4). ■

A.8 Proof of Corollary 3

Equations (5.3) and (5.4) are first-order Taylor approximations of $dGDP_2$ and $dgdp_2$ at the steady-state path under λ. Since Proposition 4 also holds if $\eta > 0$, I have $\kappa_2' > \kappa^*(\eta)$, hence $d\kappa_2(\cdot) =$

$\partial\kappa_2(\cdot)/\partial L_2\left(L_2'-L_2\right)+\partial\kappa_2(\cdot)/\partial K_2\left(K_2'-K_2\right)>0$, where the evaluation is at $(K_2,L_2,A_2,B_2;\eta)$ and $\partial\kappa_2(\cdot)/\partial L_2<0$ and $\partial\kappa_2(\cdot)/\partial K_2>0$ are implicitly given by (3.6) for $t=2$ with $g_\kappa^i(\kappa_2,\eta)$ replacing $g_\kappa^i(\kappa_2)$, $i=A,B$.

The total effect of a small change of L_2 and K_2 on GDP_2 at $(K_2,L_2,A_2,B_2;\eta)$ may then be written as in (A.27) and (A.28). One readily verifies that $\partial GDP_2(\cdot)/\partial L_2=w_2=w(\kappa^*(\eta))$ and $\partial GDP_2(\cdot)/\partial K_2=R_2=R(\kappa^*(\eta))$.

However, with $\eta>0$, I have

$$\frac{\partial GDP_2(\cdot)}{\partial q_2^A}=A_1L_2\left[\pi_{L,2}+\eta\left(\frac{w_2}{A_2}\right)\right],\quad\text{and}\quad\frac{\partial GDP_2(\cdot)}{\partial q_2^B}=B_1K_2\left[\pi_{K,2}+\eta\left(\frac{R_2}{B_2}\right)\right],$$

since

$$\pi_{L,2}=\frac{\partial F(\cdot)}{\partial(A_2L_2)}-\left(1-\delta+(1+\eta)g^A(\kappa_2,\eta)\right)i'\left(g^A(\kappa_2,\eta)\right)-i\left(g^A(\kappa_2,\eta)\right),$$

$$\pi_{K,2}=\frac{\partial F(\cdot)}{\partial(B_2K_2)}-\left(1-\delta+(1+\eta)g^B(\kappa_2,\eta)\right)i'\left(g^B(\kappa_2,\eta)\right)-i\left(g^B(\kappa_2,\eta)\right),$$

and

$$\frac{w_2}{A_2}=\frac{\partial F(\cdot)}{\partial(A_2L_2)}-i\left(g^A(\kappa_2,\eta)\right),\quad\frac{R_2}{B_2}=\frac{\partial F(\cdot)}{\partial(B_2K_2)}-i\left(g^B(\kappa_2,\eta)\right).$$

As $\pi_{L,2}=\pi_{K,2}=0$ holds in equilibrium and factor prices are strictly positive, I arrive at $\partial GDP_2(\cdot)/\partial q_2^A>0$ and $\partial GDP_2(\cdot)/\partial q_2^B>0$.

In view of $g_\kappa^A(\kappa_2,\eta)>0$ and $g_\kappa^B(\kappa_2,\eta)<0$, the conclusion is that the effect of a declining labor force on $E(\cdot)$ is a sum of two terms of opposite sign with $E(K_2,L_2,A_2,B_2;\eta=0)=0$.

As to (5.4), I use the same reasoning as above to derive $dgdp_2/dL_2$ and $dgdp_2/dK_2$. Then, straightforward manipulations of the Taylor approximation of $dgdp_2(\cdot)$ deliver (5.4). ■

A.9 Proof of Proposition 5

In the presence of a perfect annuity market, an individual born at t chooses the plan $\left(c_t^y,s_t,c_{t+1}^o\right)$ to maximize lifetime utility (2.1) subject to $c_t^y+s_t=w_t$ and $c_{t+1}^o=s_tR_{t+1}/\nu$. Writing the problem like this uses the fact that an individual's assets at $t+1$ are equal to $s_t+(1-\nu)s_t/\nu$. Moreover, it incorporates the results of Yaari (1965) and Sheshinski and Weiss (1981) according to which individuals without a bequest motive want to annuitize all their wealth.

Since preferences are homothetic, the Euler condition delivers $c_{t+1}^o/c_t^y=h(\beta R_{t+1}/\nu)$, where $h:\mathbb{R}_{++}\to\mathbb{R}_{++}$ is a monotonically increasing function. Using $c_{t+1}^o=s_tR_{t+1}/\nu$ and the Euler equation in the budget constraint of period t, one has $s_t=w_t-c_{t+1}^o/h(\cdot)$ or

$$s_t=\frac{\nu h(\cdot)w_t}{\nu h(\cdot)+R_{t+1}}=s\left(R_{t+1},\beta_\nu,\nu\right)w_t,$$

where $s\left(R_{t+1},\beta_\nu,\nu\right)\equiv s\left(\nu,h\left(\beta_\nu R_{t+1}\right),R_{t+1}\right)$. Since the intertemporal elasticity of substitution is greater than or equal to unity, one has $ds_t/dR_{t+1}\geq 0$. Moreover,

$$\frac{ds_t}{d\nu}=\frac{\partial s}{\partial\nu}w_t>0.$$

1. (a) The increase in life expectancy is expected if generation 1 makes its plan (c_1^y, s_1, c_2^o) anticipating ν'. Then, $s_1 = s(R(\kappa_2), \beta_\nu, \nu') w(\kappa^*)$. Since $ds_t/d\nu > 0$, I deduce from (3.8), that $d\kappa_{t+1}/d\nu > 0$. Hence, $\kappa_2 > \kappa^*$. Moreover, (3.9) delivers $B_2 < B^*$.

   (b) The increase in life expectancy is unexpected if generation 1 makes its plan (c_1^y, s_1, c_2^o) anticipating no change in the effective discount factor. Then, $s_1 = s(R(\kappa^*), \beta_\nu, \nu) w(\kappa^*)$, hence $\kappa_2 = \kappa^*$ and $B_2 = B^*$. The only change that occurs at $t = 2$ concerns consumption of the old at $t = 2$. Since the insurance companies pay less, one has $c_{t+1}^o = s_t R_{t+1}/\nu'$. As generation 2 anticipates ν', I have $\kappa_3 > \kappa^*$ and $B_3 < B^*$.

2. From (3.12) it follows that $\kappa^* = \kappa^{*\prime}$. A higher life expectancy increases the left-hand side of (3.11). Hence, to satisfy this equation B^* must decrease. ■

B The Phase Diagram

B.1 The Phase-Diagram of Figure 3.1

I develop the phase diagram in the (B_t, κ_t) – plane.

First, consider the locus $\Delta\kappa_t \equiv \kappa_{t+1} - \kappa_t$. From (3.8) it follows that

$$\Delta\kappa_t = 0 \equiv \{(B_t, \kappa_t) \,|\kappa_{t+1} - \kappa_t = 0\} \quad \Leftrightarrow \quad B_t = \frac{\Psi(\kappa_t)}{\tilde{w}(\kappa_t)}. \tag{B.1}$$

By assumption, the locus $\Delta\kappa_t = 0$ is stable in the vicinity of the steady state. Then, from (A.17) I have $\Psi'(\kappa^*)/\Psi(\kappa^*) > \tilde{w}'(\kappa^*)/\tilde{w}(\kappa^*)$. Hence, for all pairs (B_t, κ_t) satisfying (B.1) near (B^*, κ^*) I have $dB_t/d\kappa_t > 0$.

Next, consider the locus $\Delta B_t = 0$. From (A.20) we have for $B_t > 0$

$$\Delta B_t = 0 \equiv \{(B_t, \kappa_t) \,|B_{t+1} - B_t = 0\} \quad \Leftrightarrow \quad \delta = g^B(\kappa_{t+1}) = g^B(\phi^\kappa(\kappa_t, B_t)). \tag{B.2}$$

By (A.18), this locus is stable with monotonic convergence. Moreover, all pairs $(\kappa_t, B_t) > 0$ that satisfy (B.2) are implicitly given by

$$\kappa^* = \phi^\kappa(\kappa_t, B_t). \tag{B.3}$$

Implicit differentiation of (B.3) reveals that $dB_t/d\kappa_t = -\phi^\kappa_\kappa/\phi^\kappa_B < 0$ for all pairs $(\kappa_t, B_t) > 0$. The sign follows since $\phi^\kappa_\kappa > 0$ and $\phi^\kappa_B > 0$ from Lemma 1.

B.2 The Phase-Diagram of Figure 4.1

This section proves the rightward shift of the $\Delta\kappa_t = 0$ – locus and the downward shift of the $\Delta B_t = 0$ – locus in the neighborhood of the locally stable steady state (κ^*, B^*) in response to a decline in the growth rate of the labor force.

For $\lambda' \neq \lambda$, equation (3.8) may be written as

$$B_t \tilde{w}(\kappa_t) = \Psi(\kappa_{t+1}) \frac{1+\lambda'}{1+\lambda}. \tag{B.4}$$

Local stability of (κ^*, B^*) implies for all $(\kappa_t, \kappa_{t+1}, B_t)$ sufficiently close that

$$\frac{d\kappa_{t+1}}{d\kappa_t} = \frac{B_t \tilde{w}'(\kappa_t)}{\Psi(\kappa_{t+1})} \frac{1+\lambda}{1+\lambda'} < 1. \tag{B.5}$$

The $\Delta\kappa_t = 0$ – locus associated with λ' satisfies (B.4) for $\kappa_t = \kappa_{t+1}$. Then, changes in λ' such that κ_t remains an element of the set $\Delta\kappa_t = 0$ satisfy

$$d\kappa_t = \frac{\Psi(\kappa_t)}{1+\lambda} \left[B_t \tilde{w}'(\kappa_t) - \Psi'(\kappa_t) \frac{1+\lambda'}{1+\lambda} \right]^{-1} d\lambda'. \tag{B.6}$$

From (B.5), the term in brackets is negative. Hence, given B_t and $d\lambda' < 0$, (B.6) gives $d\kappa_t > 0$.

As to the $\Delta B_t = 0$ – locus consider (B.3). In the neighborhood of (κ^*, B^*) I have $dB_t = -\phi^\kappa_\kappa(\kappa_t, B_t)/\phi^\kappa_B(\kappa_t, B_t)\, d\kappa_t$. Moreover, from (B.4), one finds $d\kappa_t = \Psi(\kappa_{t+1}) / [B_t \tilde{w}(\kappa_t)(1+\lambda)]\, d\lambda'$, given κ_{t+1}. Upon combining these two differentials, one readily verifies that $dB_t < 0$ for $d\lambda' < 0$.

C Three Numerical Examples

This appendix presents three calibration exercises. The focus is on the steady state and its stability properties.

I make the following three assumptions:

Assumption 1 *Per-period utility is logarithmic such that individual savings at* t *is* $s_t = \beta\, w_t/(1+\beta)$.

Assumption 2 *The production function of the final good* F *is CES, i. e.,*

$$F\left(Y_{K,t}, Y_{L,t}\right) = \Gamma\left[(1-\gamma)\, Y_{K,t}^{\frac{\varepsilon-1}{\varepsilon}} + \gamma Y_{L,t}^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{C.1}$$

where $\Gamma > 0$*,* $0 < \varepsilon < \infty$ *is the constant elasticity of substitution between both inputs, and* $0 < \gamma < 1$.

Assumption 3 *For* $j = A, B$*, the investment requirement function is given by*

$$i(q^j) = v_0\left(q^j\right)^v, \quad \text{with } v_0 > 0,\ v > 1. \tag{C.2}$$

Calibration The examples were computed with *mathematica*. All notebooks are available upon request.

I consider three environments parameterized by the elasticity of substitution: $\varepsilon = 1$, $\varepsilon = .9$, and $\varepsilon = 2$. Let a period represent 30 years. Then, I choose $\beta = .55$, a population growth rate $\lambda = .35$, and a depreciation rate of technological knowledge $\delta = .26$. These numbers correspond to per-annum values of .98, .01, and .01, respectively. The depreciation rate of capital is set equal to unity. The distribution parameter is equal to $\gamma = 2/3$ such that the share of output that accrues to the capital-intensive (labor-intensive) intermediate is 1/3 (2/3) if the elasticity of substitution is unity. The parameters of the input requirement function are $v_0 = 1/2$ and $v = 2$. Finally, I set $\Gamma = 2.1$ with the implication, that the steady-state annual growth rate of per-capita variables is 1.8% if $\varepsilon = 1$. This growth rate is approximately consistent with the trend growth rates of most of today's industrialized countries.

Key Results For Three Examples.

Elasticity of Substitution	Steady State (κ^*, B^*)	steady-state growth rate $g^* = g^A(\kappa^*) - \delta$	average annual growth rate	eigenvalues (μ_1, μ_2)
$\varepsilon = 1$	$(3.6776, 14.2964)$	0.71869	1.8%	$(0.778386, 0.409777)$
$\varepsilon = .9$	$(3.151871, 12.451)$	0.703	1.79%	$(0.711324, 0.46547)$
$\varepsilon = 2$	$(59.5637, 150.124)$	1.24951	2.7%	$(0.973239, 0.343434)$

Since both eigenvalues are strictly positive and smaller than unity, the steady state is a locally stable node. Moreover, observe that the growth rates increase in the elasticity of substitution, a finding consistent with the general results established in Irmen (2009).

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