Some properties of Euler type integral operator involving generalized Bessel-Maitland function

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Abstract

The aim of the present research paper is to establish a new class of extended Beta type integral operators involving generalized Bessel-Maitland function, defined by Ghayasuddin and Khan [6]. Further, we derive some potentially useful special cases of our main results.

Keywords: Euler type integrals, Extended Beta function, Generalized Bessel-Maitland function.

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1 Introduction

In recent years, many integral formulas involving a variety of special functions have been developed by many authors (see [1], [2], [3], [4], [7], [8]). Several integral formulas involving product of Bessel functions have been developed and play an important role in several physical problems. In fact, Bessel functions are associated with a wide range of problems in diverse areas of mathematical physics. Here, we aim at presenting two generalized integral formulas involving the generalized Bessel-Maitland function, which are expressed in terms of generalized (Wright) hypergeometric functions. Some interesting special case of our main results are also considered.

The special function of the form defined by the series representation as:

\[ J^\mu_\nu(z) = \sum_{m=0}^{\infty} \frac{(-z)^m}{\Gamma(\nu + m + 1)} (\mu > 0; z \in \mathbb{C}) \]  \hspace{1cm} (1.1)

is known as Bessel-Maitland function, or the Wright generalized function (see [21]).

Singh et al. [20] introduced the following generalization of Bessel-Maitland function as follows:

\[ J^\mu_{\nu, \gamma}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(n\mu + \nu + 1) (\gamma)^n} \]  \hspace{1cm} (1.2)

where \( \mu, \nu, \gamma \in \mathbb{C}; \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0, \) and \( q \in (0, 1) \cup \mathbb{N} \) and \( (\gamma)^n = \frac{\Gamma(\gamma + qn)}{\Gamma(\gamma)} \) denotes the generalized Pochhammer symbol (see Rainville, [14]).

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Very recently, Ghayasuddin and Khan [6] introduced and investigate a new extension of Bessel-Maitland function as follows:

\[ J_{\nu,\gamma,\delta}^{\mu,q,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)^n}{\Gamma(n\mu + \nu + 1)(\delta)^n} (-z)^n, \]  

(1.3)

where \( \mu, \nu, \gamma, \delta \in \mathbb{C}; \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0, \Re(\delta) \geq 0; p, q > 0, \) and \( q < \Re(\alpha) + p. \)

We investigate some special case of the generalized Bessel-Maitland function (1.3) by particular values to the parameters \( \mu, \nu, \delta, \gamma, p, q. \)

Relation with Mittag-Leffler functions:

(i) On replacing \( \nu \) by \( \nu - 1 \) in (1.3), we get the following interesting relation:

\[ J_{\nu-1,\gamma,\delta}^{\mu,q,p}(-z) = E_{\nu,\gamma,\delta}^{\mu,q,p}(z), \]

(1.4)

where \( E_{\nu,\gamma,\delta}^{\mu,q,p}(z) \) is the Mittag-Leffler function defined by Salim and Faraj [18], [19].

(ii) On setting \( p = \delta = 1 \) and replacing \( \nu \) by \( \nu - 1\) in (1.3), we get

\[ J_{\nu-1,\gamma,1}^{\mu,q,1}(-z) = E_{\nu,\gamma}^{\mu,q,1}(z), \]

(1.5)

where \( E_{\nu,\gamma}^{\mu,q,1}(z) \) is the Mittag-Leffler function defined by Shukla and Prajapati [16].

(iii) On setting \( p = q = \delta = 1 \) and replacing \( \nu \) by \( \nu - 1\) in (1.3), we get

\[ J_{\nu-1,\gamma,1}^{\mu,1,1}(-z) = E_{\nu,\gamma}^{\mu,1,1}(z), \]

(1.6)

where \( E_{\nu,\gamma}^{\mu,1,1}(z) \) is the Mittag-Leffler function defined by Prabhakar [12].

(iv) On setting \( p = q = \delta = \gamma = 1 \) and replacing \( \nu \) by \( \nu - 1\) in (1.3), we get

\[ J_{\nu-1,1,1}^{\mu,1,1}(z) = E_{\nu,1}^{\mu,1,1}(z), \]

(1.7)

where \( E_{\nu,1}^{\mu,1,1}(z) \) is the Mittag-Leffler function defined by Wiman [22].

(v) On setting \( p = q = \delta = \gamma = 1 \) and \( \nu = 0\) in (1.3), we get

\[ J_{0,1,1}^{\mu,1,1}(z) = E_{0}^{\mu}(z), \]

(1.8)

where \( E_{0}^{\mu}(z) \) is the Mittag-Leffler function defined by Ghosta Mittag-Leffler [11].

The generalization of the generalized hypergeometric series \( \,_{p}F_{q} \) is due to Fox [5] and Wright ([23], [24], [25]) who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [17], p.21; see also [15]):

\[ \,_{p}F_{q} \left[ \begin{array}{c} (\alpha_{1}, A_{1}), \ldots, (\alpha_{p}, A_{p}); \\ (\beta_{1}, B_{1}), \ldots, (\beta_{q}, B_{q}); \end{array} \right] z = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_{j} + A_{j}k)}{\prod_{j=1}^{q} \Gamma(\beta_{j} + B_{j}k)} \frac{z^{k}}{k!}, \]

(1.9)

where the coefficients \( A_{1}, \ldots, A_{p} \) and \( B_{1}, \ldots, B_{q} \) are positive real numbers such that

\( i \quad 1 + \sum_{j=1}^{q} B_{j} - \sum_{j=1}^{p} A_{j} > 0 \) and \( 0 < |z| < \infty; z \neq 0. \)
\(1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j = 0 \) and \(0 < |z| < A_1^{-A_1} \ldots A_p^{-A_p} B_1^{B_1} \ldots B_q^{B_q}.\)  

(1.11)

A special case of (1.9) is

\[
pFq \left[ \begin{array}{c} \alpha_1, 1, \ldots, (\alpha_p, 1); \\ \beta_1, 1, \ldots, (\beta_q, 1); \\ \end{array} \right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} \; _pFq \left[ \begin{array}{c} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \\ \end{array} ; z \right],
\]

where \( _pFq \) is the generalized hypergeometric series defined by [14]

\[
\sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}
\]

where \((\lambda)_n\) is the Pochhammer’s symbol [14].

Now, we recall the classical beta function denoted by \(B(a, b)\) and is defined (see [10], see also [9]):

\[
B(a, b) = \int_{0}^{1} t^{a-1}(1-t)^{b-1} \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} , (\Re(a) > 0, \Re(b) > 0).
\]

(1.14)

In (1997), Chaudhary et al. [1] presented the following extension of Euler’s Beta function as follows:

\[
B(a, b, \sigma) = \int_{0}^{1} t^{a-1}(1-t)^{b-1} \exp \left[ -\frac{\sigma}{1-t} \right] \, dt.
\]

(1.15)

Prudnikov et al. [13] presented the following extension of integral formula defined as

\[
\int_{a}^{b} (t-a)^{\alpha-1}(b-t)^{\beta-1}(at+b)^{\gamma} dt = B(\alpha, \beta)(b-a)^{\alpha+\beta-1}(au+v)^{\gamma} _2F_1 \left[ \begin{array}{c} \alpha, -\gamma; \alpha + \beta; \\ -\frac{(b-a)u}{au+v}; \\ \end{array} \right]
\]

(1.16)

\((\Re(\alpha) > 0, \Re(\beta) > 0; |\text{arg}(\frac{bu+v}{au+v})| < \pi).\)

2 Euler type integral operator involving generalized Bessel-Maitland function

Theorem 2.1. If \(x, y, \alpha, \gamma, \delta, v, \mu \in \mathbb{C}, \Re(x) > 0, \Re(y) > 0, \Re(\alpha) > 0, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\mu) > 0, \Re(v) > 0, \Re(A) > 0, p, q > 0 \) and \( q \leq R(\alpha) + p \), then

\[
\int_{0}^{1} t^{\gamma-1}(1-t)^{y-1} \exp \left( -\frac{A}{t(1-t)} \right) dt = \sum_{n=0}^{\infty} \frac{(\gamma)_n (-z)^n}{\Gamma(\mu n + v + 1) (\delta)_n} B(x+n\alpha, y; A).
\]

(2.17)
Proof. In order to derive (2.17), we denote L.H.S. of (2.17) by $I_1$ and then expanding $J_{\nu,\gamma,\delta}^{\mu,q,p}(z\lambda^n)$ by using (1.3), to get

$$I_1 = \int_0^1 t^{\alpha-1}(1-t)^{\gamma-1} \exp \left( \frac{-A}{t(1-t)} \right) \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\mu n + \nu + 1)(\delta)^n} \, dt.$$ 

Now changing the order of summation and integration (which is guaranteed under the given conditions), to get

$$I_1 = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\nu + \mu + 1)(\delta)^n} \int_0^1 t^{\alpha n - 1}(1-t)^{\gamma-1} \exp \left( \frac{-A}{t(1-t)} \right) \, dt.$$ 

\[ \square \]

Corollary 2.1. For $A = 0$ in Theorem 2.1, we immediately deduce the following result:

$$\frac{\Gamma(\gamma)}{\Gamma(\delta)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-1} J_{\nu,\gamma,\delta}^{\mu,q,p}(z\lambda^n) \, dt = \sum_{(\alpha, \beta, \gamma, \delta, \rho, \mu, \nu) \in \mathbb{C}: \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0, \mathbb{R}(\gamma) > 0, \mathbb{R}(\delta) > 0, \mathbb{R}(\rho), \mathbb{R}(\mu) > 0, \mathbb{R}(\nu) > 0, \mathbb{R}(\lambda) > 0, \rho, \nu > 0 \text{ and } q < \gamma(\rho + \mu + 1) + \arg \left( \frac{\exp(z\lambda^n)}{\lambda^n} \right) < \pi, \text{ then}$$

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ct+d)^{\gamma-1} \exp \left( \frac{-A}{t-a}(b-t) \right) J_{\nu,\gamma,\delta}^{\mu,q,p}(z\lambda^n) \, dt$$

$$= (ac + d)^\gamma \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-A)^r}{r!} \frac{(\gamma)_n}{\Gamma(\mu n + \nu + 1)(\delta)^n} B(\alpha - r, \beta + fn - r)(b-a)^{\alpha + \beta + fn - 2r - 1}$$

$$\times _{\alpha - r, -\lambda; \alpha + \beta + fn - 2r; -\frac{(b-a)c}{ca + d}} F_1 \left[ \frac{a}{c(a + d)} \gamma(\rho + \mu + 1) + \arg \left( \frac{\exp(z\lambda^n)}{\lambda^n} \right) < \pi \right] \right].$$

Theorem 2.2. If $\alpha, \beta, \gamma, \delta, \rho, \mu, \nu \in \mathbb{C}$, $\mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0, \mathbb{R}(\gamma) > 0, \mathbb{R}(\delta) > 0, \mathbb{R}(\rho), \mathbb{R}(\mu) > 0, \mathbb{R}(\nu) > 0, \mathbb{R}(\lambda) > 0, \rho, \nu > 0$ and $q < \gamma(\rho + \mu + 1) + \arg \left( \frac{\exp(z\lambda^n)}{\lambda^n} \right) < \pi, \text{ then}$

$$\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ct+d)^{\gamma-1} \exp \left( \frac{-A}{t-a}(b-t) \right) J_{\nu,\gamma,\delta}^{\mu,q,p}(z\lambda^n) \, dt$$

$$= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-A)^r}{r!} \frac{(\gamma)_n}{\Gamma(\mu n + \nu + 1)(\delta)^n} \int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ct+d)^{\gamma-1} \exp \left( \frac{-A}{t-a}(b-t) \right) \, dt.$$ 

By using the integral (1.16), we obtain at the required result (2.19).
Corollary 2.2. For $A = 0$ in Theorem 2.2, we get
\[
\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ct+d)^{\lambda} J_{\nu,\gamma,\delta}^{\mu,q,p} (z(b-t)^{\frac{\beta}{\gamma}})dt
\]
\[
= (ac + d)^{\lambda} \sum_{n=0}^{\infty} \frac{(\gamma)_m(-z)^{n}}{\Gamma(m + n + 1)(\delta)_{pm}} B(\alpha, \beta + fn - m)
\times \left[ \alpha - m, -\lambda; \alpha + \beta + fn - 2m \right].
\]  

(2.20)

Corollary 2.3. For $a = 0, b = 1$ in Theorem 2.2, we get
\[
\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}(ct+d)^{\lambda} \exp \left( \frac{-A}{t(1-t)} \right) J_{\nu,\gamma,\delta}^{\mu,q,p} (z(1-t)^{\frac{\beta}{\gamma}})dt
\]
\[
= d^{\lambda} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\gamma_q m(-z)^{n}}{\Gamma(m + n + 1)(\delta)_{pm}m!} B(\alpha - m, \beta + fn - m)
\times \left[ \alpha - m, -\lambda; \alpha + \beta + fn - 2m \right].
\]  

(2.21)

Theorem 2.3. If $\eta, \gamma, \delta, p, \mu, v, \lambda, \sigma \in \mathbb{C}$, $R(\eta) > 0, R(\gamma) > 0, R(\sigma) > 0, R(\delta) > 0, R(v) > 0, R(\sigma) > 0, R(\lambda) > 0, R(\mu) > 0, R(\eta) > 0, p, q > 0$ and $q < R(\alpha) + p$; then
\[
\int_0^1 t^{\lambda-1}(1-t)^{\eta-\lambda-1}(1-ut^{p}(1-t)^{\sigma})^{-q} \exp \left( \frac{-A}{t(1-t)} \right) J_{\nu,\gamma,\delta}^{\mu,q,p} (z^{\alpha}t^{\lambda})dt
\]
\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\gamma)_n(z)^{n}}{\Gamma(m + n + 1)(\delta)_{pm}n!} \int_0^1 t^{\lambda-\eta+\rho r-\alpha-1}(1-t)^{\eta-\lambda-1} \exp \left( \frac{-A}{t(1-t)} \right) dt
\]  

(2.22)

Proof. On taking L.H.S. of Theorem 2.3, using the definition of generalized Bessel-Maitland function (1.3), and then by changing the order of summation and integration, we get
\[
\int_0^1 t^{\lambda-1}(1-t)^{\eta-\lambda-1}(1-ut^{p}(1-t)^{\sigma})^{-q} \exp \left( \frac{-A}{t(1-t)} \right) J_{\nu,\gamma,\delta}^{\mu,q,p} (z^{\alpha}t^{\lambda})dt
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\gamma)_n(z)^{n}}{\Gamma(m + n + 1)(\delta)_{pm}n!} \int_0^1 t^{\lambda+\rho r-\alpha-1}(1-t)^{\eta+\sigma-\lambda-1} \exp \left( \frac{-A}{t(1-t)} \right) dt
\]  

which further on using the integral (1.15), gives the required result (2.22).

Corollary 2.4. For $a = 0$ in Theorem 2.3 reduces to the following result as:
\[
\int_0^1 t^{\lambda-1}(1-t)^{\mu-\lambda-1} \exp \left( \frac{-A}{t(1-t)} \right) J_{\nu,\gamma,\delta}^{\mu,q,p} (z^{\alpha}t^{\lambda})dt
\]
\[
= \sum_{n=0}^{\infty} \frac{(\gamma)_n(z)^{n}}{\Gamma(m + n + 1)(\delta)_{pm}} B(\lambda + \alpha n, \eta - \lambda; A).
\]  

(2.23)
Corollary 2.5. For $A = 0$ in Theorem 2.3 to get:

$$
\int_0^1 t^{\lambda-1}(1-t)^{\eta-1}\{-\textstyle{a} \frac{d^p}{dt^p} \left( t^{\alpha} \right) \} V_{\nu, \gamma, \delta} dt
= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-z)^n}{n!} \frac{\Gamma(n+\nu+1)(\delta)_p}{\Gamma(\mu+\nu+1)(\delta)_p} B(\lambda+n\alpha+\nu r, \mu-\lambda+r\sigma).
$$

(2.24)

Corollary 2.6. Setting $\nu = q = 0$, $A = 0$ in Theorem 2.3, we immediately deduce to the following result:

$$
\frac{\Gamma(\nu)}{\Gamma(\eta-\lambda)} \frac{\Gamma(\eta-\lambda-\delta)}{\Gamma(\eta-\lambda-\delta-1)} \int_0^1 t^{\nu-1}(1-t)^{\eta-1} \left\{ -\textstyle{a} \frac{d^p}{dt^p} \left( t^{\alpha} \right) \right\} V_{\nu, \gamma, \delta} dt
= 3 \Psi_3 \left[ \begin{array}{c} (\gamma,q),(\lambda,\alpha),(1,1); \\ (\mu,\nu+1),(\delta,p),(\eta,\alpha); -z \end{array} \right].
$$

(2.25)

3 Special cases

1. On setting $\gamma = q = 1$ in Theorem 2.1, we get

$$
\frac{\Gamma(\delta)}{\Gamma(\eta-\lambda)} \int_0^1 t^{\nu-1}(1-t)^{\eta-1} \left\{ -\frac{A}{t(1-t)} \right\} \left[ (1,1); -zt^\alpha \right] dt
= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \frac{\Gamma(n+\nu+1)(\delta)_p}{\Gamma(\mu+\nu+1)(\delta)_p} B(n\alpha+x,y,A).
$$

(3.26)

2. On setting $\gamma = \delta = q = p = 1$ in Theorem 2.1, we find

$$
\frac{\Gamma(\delta)}{\Gamma(\eta-\lambda)} \int_0^1 t^{\nu-1}(1-t)^{\eta-1} \left\{ -\frac{A}{t(1-t)} \right\} \left[ -; -zt^\alpha \right] dt
= \sum_{n=0}^{\infty} \frac{(-z)^n}{n^2} \frac{\Gamma(n+\nu+1)(\delta)_p}{\Gamma(\mu+\nu+1)(\delta)_p} B(n\alpha+x,y,A).
$$

(3.27)

3. On setting $\gamma = p = 1$ in Theorem 2.1, we obtain

$$
\frac{1}{\Gamma(\delta)} \int_0^1 t^{\nu-1}(1-t)^{\eta-1} \left\{ -\frac{A}{t(1-t)} \right\} \left[ (1,1); -zt^\alpha \right] dt
= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \frac{\Gamma(n+\nu+1)(\delta)_p}{\Gamma(\mu+\nu+1)(\delta)_p} B(n\alpha+x,y,A).
$$

(3.28)

4. Replacing $\nu$ by $\nu-1$ in Theorem 2.1 and then by using relation (1.4), we acquire

$$
\frac{\Gamma(\nu)}{\Gamma(\delta)} \int_0^1 t^{\nu-1}(1-t)^{\eta-1} \left\{ -\frac{A}{t(1-t)} \right\} \left[ (\gamma,q),(1,1); zt^\alpha \right] dt
= \sum_{n=0}^{\infty} \frac{\gamma^n(t)^n}{n!} \frac{\Gamma(n+\nu+1)(\delta)_p}{\Gamma(\mu+\nu+1)(\delta)_p} B(n\alpha+x,y,A).
$$

(3.29)
5. On setting $\delta = p = 1$ and replacing $v$ by $v - 1$ in Theorem 2.1 and then by using relation (1.5), we obtain
\[
\frac{1}{\Gamma(\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{\gamma-1} \exp\left(\frac{-A}{t(1-t)}\right) I_1 \psi_1 \left[ (\gamma, q); \mu, v \right] z^\alpha dt = \sum_{n=0}^\infty \frac{(\gamma)_n z^n}{\Gamma(\mu+n+v)n!} B(\mu+n+v, y; A).
\]

6. On setting $\delta = p = q = 1$ and replacing $v$ by $v - 1$ in Theorem 2.1, and then by using the relation (1.6), we find
\[
\frac{1}{\Gamma(\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{\gamma-1} \exp\left(\frac{-A}{t(1-t)}\right) I_1 \psi_1 \left[ (1, 1); (\mu, v) \right] z^\alpha dt = \sum_{n=0}^\infty \frac{(\gamma)_n z^n}{\Gamma(\mu+n+v)n!} B(n\alpha + x, y; A).
\]

7. On setting $\delta = p = q = \gamma = 1$ and replacing $v$ by $v - 1$ in Theorem 2.1, and then by using the relation (1.7), we get
\[
\frac{1}{\Gamma(\gamma)} \int_0^1 t^{\gamma-1}(1-t)^{\gamma-1} \exp\left(\frac{-A}{t(1-t)}\right) I_1 \psi_1 \left[ (1, 1); (\mu, v) \right] z^\alpha dt = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\mu+n+v) n!} B(n\alpha + x, y; A).
\]

8. On setting $p = q = \delta = \gamma = 1$ and $v = 0$ in Theorem 2.1, and then by using the relation (1.8), we obtain
\[
\frac{1}{\Gamma(\gamma)} \int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ct+d)^{\lambda} \exp\left(\frac{-A}{(t-a)(b-t)}\right) I_1 \psi_1 \left[ (1, 1); (\mu, v+1) \right] -z(b-t)^{\gamma} dt = (ac+d)^{\lambda} \sum_{r=0}^{\infty} \sum_{r=0}^\infty \frac{(-A)^r(-z)^n}{\Gamma(\mu+n+\gamma)(\delta)_r n!} B(\alpha-m, \beta+nf-m)(b-a)^{\alpha+\beta+nf-2m-1}
\times \mathcal{F}_1 \left[ \alpha-m, -\lambda, \alpha+\beta+nf-2m; \frac{-(b-a)c}{ac+d} \right].
\]

9. On setting $\gamma = q = 1$ in Theorem 2.2, we get
\[
\frac{1}{\Gamma(\gamma)} \int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ct+d)^{\lambda} \exp\left(\frac{-A}{(t-a)(b-t)}\right) I_1 \psi_1 \left[ (1, 1); (\mu, v+1) \right] -z(b-t)^{\gamma} dt = (ac+d)^{\lambda} \sum_{r=0}^{\infty} \sum_{r=0}^\infty \frac{(-A)^r(-z)^n}{\Gamma(\mu+n+\gamma)(\delta)_r n!} B(\alpha-m, \beta+nf-m)(b-a)^{\alpha+\beta+nf-2m-1}
\times \mathcal{F}_1 \left[ \alpha-m, -\lambda, \alpha+\beta+nf-2m; \frac{-(b-a)c}{ac+d} \right].
\]

10. On setting $\gamma = \delta = p = q = 1$ in Theorem 2.2, we acquire
\[
\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}(ct+d)^{\lambda} \exp\left(\frac{-A}{(t-a)(b-t)}\right) I_1 \psi_1 \left[ (1, 1); (\mu, v+1) \right] -z(b-t)^{\gamma} dt = (ac+d)^{\lambda} \sum_{r=0}^{\infty} \sum_{r=0}^\infty \frac{(-A)^r(-z)^n}{\Gamma(\mu+n+\gamma)(\delta)_r n!} B(\alpha-m, \beta+nf-m)(b-a)^{\alpha+\beta+nf-2m-1}
\times \mathcal{F}_1 \left[ \alpha-m, -\lambda, \alpha+\beta+nf-2m; \frac{-(b-a)c}{ac+d} \right].
\]
On setting $v$ by $v - 1$ in Theorem 2.2 and then by using the relation (1.4), we get

$$
\frac{\Gamma(\gamma)}{\Gamma(\delta)} \int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} (ct+d)^k \exp \left( \frac{-A}{(t-a)(b-t)} \right) z(1,1) d\Psi_2 \left[ (\mu,\nu), (\delta, \rho) \right] dt
$$

$$
= (ac+d)^k \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-A)^n \Gamma(\nu)(\nu+\rho)^r}{\Gamma(\mu+n+\nu+1)(\delta)^{n-r}} B(\alpha-m, \beta+n-1, \gamma) \left( -\frac{b-a}{ac+d} \right)
$$

$$
\times \frac{\Gamma(\gamma)}{\Gamma(\delta)} \int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} (ct+d)^k \exp \left( \frac{-A}{(t-a)(b-t)} \right) z(1,1) d\Psi_2 \left[ (\mu,\nu), (\delta, \rho) \right] dt
$$

$$
= (ac+d)^k \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-A)^n \Gamma(\nu)(\nu+\rho)^r}{\Gamma(\mu+n+\nu+1)(\delta)^{n-r}} B(\alpha-m, \beta+n-1, \gamma) \left( -\frac{b-a}{ac+d} \right)
$$

$$
\times \frac{\Gamma(\gamma)}{\Gamma(\delta)} \int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} (ct+d)^k \exp \left( \frac{-A}{(t-a)(b-t)} \right) z(1,1) d\Psi_2 \left[ (\mu,\nu), (\delta, \rho) \right] dt
$$

12. On setting $p = \delta = 1$ and replacing $v$ by $v - 1$ in Theorem 2.2 and then by using the relation (1.5), we find

$$
\frac{\Gamma(\gamma)}{\Gamma(\delta)} \int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} (ct+d)^k \exp \left( \frac{-A}{(t-a)(b-t)} \right) z(1,1) d\Psi_2 \left[ (\mu,\nu), (\delta, \rho) \right] dt
$$

$$
= (ac+d)^k \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-A)^n \Gamma(\nu)(\nu+\rho)^r}{\Gamma(\mu+n+\nu+1)(\delta)^{n-r}} B(\alpha-m, \beta+n-1, \gamma) \left( -\frac{b-a}{ac+d} \right)
$$

13. On setting $p = q = \delta = 1$ and replacing $v$ by $v - 1$ in Theorem 2.2 and then by using the relation (1.6), we obtain

$$
\frac{\Gamma(\gamma)}{\Gamma(\delta)} \int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1} (ct+d)^k \exp \left( \frac{-A}{(t-a)(b-t)} \right) z(1,1) d\Psi_2 \left[ (\mu,\nu), (\delta, \rho) \right] dt
$$

$$
= (ac+d)^k \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-A)^n \Gamma(\nu)(\nu+\rho)^r}{\Gamma(\mu+n+\nu+1)(\delta)^{n-r}} B(\alpha-m, \beta+n-1, \gamma) \left( -\frac{b-a}{ac+d} \right)
$$

14. On setting $\gamma = q = 1$ in Theorem 2.3, we get

$$
\frac{1}{\Gamma(\delta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} (1-ut)^{\rho-1} (1-t)^{\sigma-1} \exp \left( \frac{-A}{t(1-t)} \right) z(1,1) d\Psi_2 \left[ (\mu,\nu+1), (\delta, \rho) \right] dt
$$

$$
= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-A)^n \Gamma(\nu)(\nu+\rho)^r}{\Gamma(\mu+n+\nu+1)(\delta)^{n-r}} B(\alpha-m, \beta+n-1, \gamma) \left( -\frac{b-a}{ac+d} \right)
$$

15. On setting $\gamma = \delta = p = q = 1$ in Theorem 2.3, we find

$$
\frac{1}{\Gamma(\delta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} (1-ut)^{\rho-1} (1-t)^{\sigma-1} \exp \left( \frac{-A}{t(1-t)} \right) z(1,1) d\Psi_2 \left[ (\mu,\nu+1), (\delta, \rho) \right] dt
$$

$$
= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-A)^n \Gamma(\nu)(\nu+\rho)^r}{\Gamma(\mu+n+\nu+1)(\delta)^{n-r}} B(\alpha-m, \beta+n-1, \gamma) \left( -\frac{b-a}{ac+d} \right)
$$
16. Replacing \( v \) by \( v - 1 \) in Theorem 2.3 and then by using the relation (1.4), we get

\[
\frac{\Gamma(\gamma)}{\Gamma(\delta)} \int_0^1 t^{\lambda-1}(1-t)^{\eta-\lambda-1} (1 - wt^\rho (1-t)^\sigma)^{-a} \exp \left( \frac{-A}{t(1-t)} \right) \right) \left[ \begin{array}{c} (\gamma, q) ; (1, 1) \\ (\mu, v) ; (\delta, p) \\ \end{array} \right] dt
\]

\[
= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{t^r}}{\Gamma(\mu n + v) n! r!} B(\lambda + n\alpha + \rho r, \eta - \lambda + \sigma r; \lambda).
\]

(3.41)

17. On setting \( \delta = p = 1 \) and replacing \( v \) by \( v - 1 \) in Theorem 2.3 and then by using the relation (1.5), we acquire

\[
\frac{\Gamma(\gamma)}{\Gamma(\delta)} \int_0^1 t^{\lambda-1}(1-t)^{\eta-\lambda-1} (1 - wt^\rho (1-t)^\sigma)^{-a} \exp \left( \frac{-A}{t(1-t)} \right) \right) \left[ \begin{array}{c} (\gamma, q) ; (1, 1) \\ (\mu, v) ; (\delta, p) \\ \end{array} \right] dt
\]

\[
= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{t^r}(\gamma)_r}{\Gamma(\mu n + v) n! r!} B(\lambda + n\alpha + \rho r, \eta - \lambda + \sigma r; \lambda).
\]

(3.42)

18. On setting \( \delta = p = q = 1 \) and replacing \( v \) by \( v - 1 \) in Theorem 2.3 and then by using the relation (1.6), we get

\[
\frac{\Gamma(\gamma)}{\Gamma(\delta)} \int_0^1 t^{\lambda-1}(1-t)^{\eta-\lambda-1} (1 - wt^\rho (1-t)^\sigma)^{-a} \exp \left( \frac{-A}{t(1-t)} \right) \right) \left[ \begin{array}{c} (\gamma, 1) ; (1, 1) \\ (\mu, v) ; (\delta, p) \\ \end{array} \right] dt
\]

\[
= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{t^r}(\gamma)_r}{\Gamma(\mu n + v) n! r!} B(\lambda + n\alpha + \rho r, \eta - \lambda + \sigma r; \lambda).\]

(3.43)

19. On setting \( \delta = \gamma = p = q = 1 \) and replacing \( v \) by \( v - 1 \) in Theorem 2.3 and then by using the relation (1.7), we obtain

\[
\int_0^1 t^{\lambda-1}(1-t)^{\eta-\lambda-1} (1 - wt^\rho (1-t)^\sigma)^{-a} \exp \left( \frac{-A}{t(1-t)} \right) \right) \left[ \begin{array}{c} (1, 1) ; (1, 1) \\ (\mu, v) ; (\delta, p) \\ \end{array} \right] dt
\]

\[
= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{t^r}}{\Gamma(\mu n + v) n! r!} B(\lambda + n\alpha + \rho r, \eta - \lambda + \sigma r; \lambda).
\]

(3.44)

20. On setting \( p = q = \delta = \gamma = 1 \) and \( v = 0 \) in Theorem 2.3 and then by using the relation (1.8), we find

\[
\int_0^1 t^{\lambda-1}(1-t)^{\eta-\lambda-1} (1 - wt^\rho (1-t)^\sigma)^{-a} \exp \left( \frac{-A}{t(1-t)} \right) \right) \left[ \begin{array}{c} (1, 1) ; (1, 1) \\ (\mu, 1) ; (\delta, p) \\ \end{array} \right] dt
\]

\[
= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{t^r}}{\Gamma(\mu n + 1) n! r!} B(\lambda + n\alpha + \rho r, \eta - \lambda + \sigma r; \lambda).
\]

(3.45)

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