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Research Article

A numerical method for solving singular Riccati equation with fractional order by modified fractional power series method

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Abstract

In this paper, we study the singular Riccati equation with fractional order. The modified fractional power series method (MFPS) is used to solve the proposed problem. The validity of the MFPS method is ascertained by presenting several examples. We prove the existence of the solution of the Riccati equation with fractional order. The convergence of the approximate solution using the proposed method is investigated. Theoretical and numerical results are presented.

Keywords: Singular Riccati equation, Caputo fractional derivative, Nonlinear initial value problem, Modified fractional power series method.

1 Introduction

Fractional differential equations (FDEs) appear as generalizations to existing models with integer derivative and they also present new models for some physical problems [1]. In recent years, great interests was devoted to the analytical and numerical treatments of fractional differential equations. In general, fractional differential equations don't have a closed form of the exact solution, and therefore, numerical methods such as the variational iteration [2], the homotopy analysis method [3], and the Adomian decomposition method [4, 5] have been implemented for several types of fractional differential equations. Also, the maximum principle and the method of lower and upper solutions have been extended to deal with FDEs and to obtain some analytical and numerical results [6]. The Pseudo-spectral method and the wavelet method based on the Legendre polynomials have been implemented for several types of FDEs [7].

Consider

$$a(x)\, D^{\alpha}u(x) + b(x)u(x) + c(x)u^2(x) = g(x), \quad x \in (0,1], \;\; 0 < \alpha \le 1,$$

subject to

$$u(0) = \theta$$

where $a, b, c, g \in C^1\,[0,1]$, θ is a constant, and $a(x) > 0$ on $[0,1]$ has been investigated by several researchers as a result of its importance in different applications. The numerical solution of the fractional Riccati differential equation was discussed by several researchers. Some of these numerical techniques are the polynomial least squares method [8],

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Adomian decomposition method [9], Bernstein polynomials [10], Legendre wavelet operational matrix method [11], and Bezier curves method [12]. Syam [13] solved the fractional Riccati differential equation by the fractional-order Legendre operational matrix of fractional integration. Moreover, Liao [14] used the homotopy analysis method. The homotopy perturbation method was implemented in [15, 16]. Additionally, a computational intelligence technique was presented for solution of nonlinear quadratic Riccati differential equations of fractional order based on artificial neural networks and sequential quadratic programming [17]. Moreover, Runge-Kutta methods and spectral methods were used to solve such problem [18]-[25].

In this paper, we discuss how to solve the following class of fractional singular Riccati differential equation of the form

$$a(x)\, D^{\alpha}u(x)+b(x)u(x)+c(x)u^2(x)=g(x), \quad x\in(0,1],\ 0<\alpha\leq 1, \tag{1.1}$$

subject to

$$u(0)=\theta \tag{1.2}$$

where $a,b,c,g\in C^1\ [0,1]$, θ is a constant, and $a(x)>0$ on $(0,1]$ with $a(0)=0$ and $D^{\alpha}a(0)\neq 0$. We use the modified fractional power series (MFPS) to solve it.

We organize this paper as follows. In Section 2, we present some preliminaries which we use in this paper. In Section 3, we present the MFPS method for solving Problem (1.1)-(1.2). Convergence and error estimate of the proposed method is presented in this section. Some numerical results are presented in Section 4 to show the efficiency of the presented method. Finally, in Section 5, we write some conclusions.

2 Preliminaries

In this section, We write the definition and some preliminary results of the Caputo fractional derivative as well as the definition of the fractional power series and one of its properties.

Definition 2.1. *A real function $f(t), t>0$, is said to be in the space $C_{\mu}, \mu\in\mathbb{R}$ if there exists a real number $p>\mu$, such that $f(t)=t^p f_1(t)$, where $f_1(t)\in C[0,\infty)$, and it is said to be in the space C_{μ}^m if $f^{(m)}\in C_{\mu}$, $m\in\mathbb{N}$.*

Definition 2.2. *For $\delta>0$, $m-1<\delta<m$, $m\in\mathbb{N}, t>0$, and $f\in C_{-1}^m$, the Caputo fractional derivative is defined by*

$$D^{\delta}f(t)=\begin{cases}\frac{1}{\Gamma(m-\delta)}\int_0^t(t-s)^{m-1-\delta}f^{(m)}(s)ds, & \delta>0,\\ f'(t), & \delta=0,\end{cases} \tag{2.3}$$

where Γ is the well-known Gamma function.

The Caputo fractional derivative satisfies the following properties for $\alpha>0$, see [26].

1. $D^{\alpha}c=0$, where c is constant,
2. $D^{\alpha}t^{\gamma}=\left\{\begin{array}{cc} 0, & \gamma<\alpha, \gamma\in\{0,1,2,...\}\\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha}, & \text{otherwise}\end{array}\right\}.$

Next, we write the definition and one of the properties of the fractional power series which are used in this paper. More details can be found in [27].

Definition 2.3. *A power series expansion of the form*

$$\sum_{m=0}^{\infty}c_m(t-t_0)^{m\alpha}=c_0+c_1(t-t_0)^{\alpha}+c_2(t-t_0)^{2\alpha}+...$$

is called fractional power series (FPS) about $t=t_0$.

Suppose that g has a FPS representation at $t = t_0$ of the form

$$g(t) = \sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha}, \quad t_0 \leq t < t_0 + \beta.$$

If $D^{m\alpha} g(t),\ \ m = 0,1,2,...,$ are continuous on $\mathbb{R}$, then $c_m = \frac{D^{m\alpha} g(t_0)}{\Gamma(1+m\alpha)}$.

3 MFPS method for solving fractional singular Riccati differential equation

In this section, we discuss how to solve the following class of fractional singular Riccati differential equation of the form

$$a(x)\ D^{\alpha} u(x) + b(x)u(x) + c(x)u^2(x) = g(x), \quad x \in (0,1],\ \ 0 < \alpha \leq 1, \tag{3.4}$$

subject to

$$u(0) = \theta \tag{3.5}$$

where $a,b,c,g \in C^1\ [0,1]$, θ is a constant, and $a(x) > 0$ on $(0,1]$ with $a(0) = 0$ and $D^{\alpha} a(0) \neq 0$. The MFPS method proposes the solution of the problem in the form of fractional power series as

$$u(x) = \sum_{n=0}^{\infty} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{3.6}$$

To obtain the approximate values of $f_n,\ n = 0,1,2,...,$ in Eq. (3.6), we consider the k-th truncated series $u_k(x)$ which has the form

$$u_k(x) = \sum_{n=0}^{k} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}. \tag{3.7}$$

Since $u(0) = f_0 = \theta$, we rewrite Eq. (3.7) as

$$u_k(x) = \theta + \sum_{n=1}^{k} f_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}, \quad k = 1,2,3,... \tag{3.8}$$

where $u_0(x) = f_0 = \theta$ is considered to be the 0^{th} RPS approximate solution of $u(x)$. To find the values of the RPS-coefficients $f_k,\ k = 1,2,3,...,$ we solve the fractional differential equation

$$D^{k\alpha} Res_k(0) = 0,\ k = 1,2,3,...$$

where $Res_k(x)$ is the k-th residual function and it is defined by

$$Res_k(x) = a(x)\ D^{\alpha} u_k(x) + b(x)u_k(x) + c(x)u_k^2(x) - g(x). \tag{3.9}$$

Suppose that

$$a(x) = \sum_{n=0}^{\infty} a_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}, b(x) = \sum_{n=0}^{\infty} b_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)},\ c(x) = \sum_{n=0}^{\infty} c_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)},\ g(x) = \sum_{n=0}^{\infty} g_n \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}$$

where

$$\begin{aligned} a_0 &= 0,\ a_n = D^{n\alpha} a(0),\ n = 1,2,3,..., \\ b_n &= D^{n\alpha} b(0),\ c_n = D^{n\alpha} c(0),\ g_n = D^{n\alpha} g(0),\ n = 0,1,2,.... \end{aligned}$$

Using the Cauchy product theorem, we get

$$\begin{aligned} b(x)u(x) &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{f_l\ b_{n-l}}{\Gamma(1+l\alpha)\Gamma(1+(n-l)\alpha)}\right)x^{n\alpha}, \\ a(x)\ D^{\alpha}u(x) &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{f_{l+1}a_{n+1-l}}{\Gamma(1+l\alpha)\Gamma(1+(n+1-l)\alpha)}\right)x^{(n+1)\alpha}, \end{aligned}$$

and

$$\begin{aligned} c(x)u^2(x) &= \left(\sum_{n=0}^{\infty}c_n\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\right)\left(\sum_{n=0}^{\infty}f_n\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\right)^2 \\ &= \left(\sum_{n=0}^{\infty}c_n\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}\right)\left(\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{f_l\ f_{n-l}}{\Gamma(1+l\alpha)\Gamma(1+(n-l)\alpha)}\right)x^{n\alpha}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{l=0}^{i}\frac{c_{n-i}\ f_l\ f_{i-l}}{\Gamma(1+(n-i)\alpha)\Gamma(1+l\alpha)\Gamma(1+(i-l)\alpha)}\right)x^{n\alpha}. \end{aligned}$$

Next, we consider the k-th truncated series of $a(x)\ D^{\alpha}u(x), b(x)u(x), c(x)u^2(x)$, and $g(x)$ which have the form

$$\begin{aligned} b(x)u(x) &\approx \sum_{n=0}^{k}\left(\sum_{l=0}^{n}\frac{f_l\ b_{n-l}}{\Gamma(1+l\alpha)\Gamma(1+(n-l)\alpha)}\right)x^{n\alpha}, \\ a(x)\ D^{\alpha}u(x) &\approx \sum_{n=0}^{k}\left(\sum_{l=0}^{n}\frac{f_{l+1}a_{n+1-l}}{\Gamma(1+l\alpha)\Gamma(1+(n+1-l)\alpha)}\right)x^{(n+1)\alpha}, \\ c(x)u^2(x) &\approx \sum_{n=0}^{k}\left(\sum_{i=0}^{n}\sum_{l=0}^{i}\frac{c_{n-i}\ f_l\ f_{i-l}}{\Gamma(1+(n-i)\alpha)\Gamma(1+l\alpha)\Gamma(1+(i-1)\alpha)}\right)x^{n\alpha}, \\ g(x) &\approx \sum_{n=0}^{k}g_n\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}. \end{aligned}$$

To determine the coefficient $f_k, k=1,2,3,...$, in Eq. (3.7), we substitute the k^{th} RPS approximate solution

$$u_k(x)=\theta+\sum_{n=1}^{k}f_n\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}$$

into Eq.(3.4) to get

$$\begin{aligned} Res_k(x) &= a(x)\ D^{\alpha}u(x)+b(x)u(x)+c(x)u^2(x)-g(x) \\ &= \sum_{n=0}^{k}\left(\sum_{l=0}^{n}\frac{f_{l+1}a_{n+1-l}}{\Gamma(1+l\alpha)\Gamma(1+(n+1-l)\alpha)}\right)x^{(n+1)\alpha} \\ &\quad+\sum_{n=0}^{k}\left(\sum_{l=0}^{n}\frac{f_l\ b_{n-l}}{\Gamma(1+l\alpha)\Gamma(1+(n-l)\alpha)}\right)x^{n\alpha} \\ &\quad+\sum_{n=0}^{k}\left(\sum_{i=0}^{n}\sum_{l=0}^{i}\frac{c_{n-i}\ f_l\ f_{i-l}}{\Gamma(1+(n-i)\alpha)\Gamma(1+l\alpha)\Gamma(1+(i-l)\alpha)}\right)x^{n\alpha} \\ &\quad-\sum_{n=0}^{k}g_n\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}. \end{aligned} \tag{3.10}$$

Then, we solve $D^{k\alpha}Res_k(0)=0$ to get

$$\begin{aligned} & \sum_{l=0}^{k-1}\frac{f_{l+1}a_{k-l}\Gamma(1+k\alpha)}{\Gamma(1+l\alpha)\Gamma(1+(k-l)\alpha)}+\sum_{l=0}^{k}\frac{f_l\, b_{k-l}\Gamma(1+k\alpha)}{\Gamma(1+l\alpha)\Gamma(1+(k-l)\alpha)} \\ & +\sum_{i=0}^{k}\sum_{l=0}^{i}\frac{c_{k-i}\, f_l\, f_{i-l}\Gamma(1+k\alpha)}{\Gamma(1+(k-i)\alpha)\Gamma(1+l\alpha)\Gamma(1+(i-l)\alpha)}-g_k \\ = \; & 0 \end{aligned} \tag{3.11}$$

for $k=1,2,3,\ldots$. Thus, the approximate solution is given by

$$u_k(x)=\theta+\sum_{n=1}^{k}f_n\frac{x^{n\alpha}}{\Gamma(1+n\alpha)},\quad k=2,3,\ldots. \tag{3.12}$$

In the next theorem, we study the convergence of the series (3.12) to the solution of problem (3.4)-(3.5).

Theorem 3.1. *Let $u(x)=\sum_{n=0}^{\infty}f_n\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}$ and $0<\alpha\le 1$. Then, the sequence $\{u_k(x)\}$ converges to the solution of problem (3.4)-(3.5).*

Proof. First, we want to prove that $\sum_{n=1}^{\infty}f_n\frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}$ converges to $D^{\alpha}u(x)$ when $x\in(0,1]$. For any $x\in(0,1]$,

$$\begin{aligned} D^{\alpha}u(x) &= \frac{1}{\Gamma(1-\alpha)}\int_0^x(x-s)^{-\alpha}u'(s)ds \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^x(x-s)^{-\alpha}\left(\sum_{n=0}^{\infty}f_n\frac{s^{n\alpha}}{\Gamma(1+n\alpha)}\right)'ds \\ &= \sum_{n=0}^{\infty}\frac{f_n}{\Gamma(1+n\alpha)}\frac{1}{\Gamma(1-\alpha)}\int_0^x(x-s)^{-\alpha}\,(s^{n\alpha})'ds \\ &= \sum_{n=0}^{\infty}\frac{f_n}{\Gamma(1+n\alpha)}D^{\alpha}(x^{n\alpha})=\sum_{n=1}^{\infty}\frac{f_n}{\Gamma(1+(n-1)\alpha)}x^{(n-1)\alpha}. \end{aligned}$$

Thus, $\sum_{n=1}^{\infty}f_n\frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}$ converges to $D^{\alpha}u(x)$ when $x\in(0,1]$.
Next, we want to prove the sequence $\{u_k(x)\}$ converges to the solution of problem (3.4)-(3.5). Let

$$\begin{aligned} & a(x)\,D^{\alpha}u(x)+b(x)u(x)+c(x)u^2(x)-g(x) \\ = \; & \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{f_{l+1}a_{n+1-l}}{\Gamma(1+l\alpha)\Gamma(1+(n+1-l)\alpha)}\right)x^{(n+1)\alpha} \\ & +\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{f_l\, b_{n-l}}{\Gamma(1+l\alpha)\Gamma(1+(n-1)\alpha)}\right)x^{n\alpha} \\ & +\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{l=0}^{i}\frac{c_{n-i}\, f_l\, f_{i-l}}{\Gamma(1+(n-i)\alpha)\Gamma(1+l\alpha)\Gamma(1+(i-1)\alpha)}\right)x^{n\alpha} \\ & -\sum_{n=0}^{\infty}g_n\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}=\sum_{n=0}^{\infty}\xi_n x^{n\alpha}. \end{aligned} \tag{3.13}$$

Since $D^{\alpha}u(x)=\sum_{n=1}^{\infty}f_n\frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}$ and $u(x)=\sum_{n=0}^{\infty}f_n\frac{x^{n\alpha}}{\Gamma(1+n\alpha)}$, then

$$\sum_{n=0}^{\infty}\xi_n x^{n\alpha}=0.$$

Let

$$S_k = \sum_{n=k}^{\infty} \xi_n x^{n\alpha}.$$

Then, the sequence $\{S_k\}$ converges to zero. From Eq. (3.9), we see that

$$Res_k(x) = S_k.$$

Thus,

$$\lim_{k\to\infty} Res_k(x) = \lim_{k\to\infty} S_k = 0.$$

Hence, the sequence $\{u_k(x)\}$ converges to the solution of problem (3.4)-(3.5). $\square$

Theorem 3.2. *Let $u_n(x) = \sum_{k=0}^{n} f_n \frac{x^{n\alpha}}{\Gamma(1+k\alpha)}$ be given as in Eq. (3.4). Let $u \in C^{n+1}[0,1]$ be a solution to problem (3.4)-(3.5) and $\left\| u^{(k)} \right\| \leq A$ for some positive real number A and $k = 0,1,...,n+1$. Then,*

$$\|u - u_n\| \leq \frac{1.13\,A}{(m-(n+1)\alpha)\,\Gamma((n+1)\alpha+1)}.$$

Proof. Let $x \in (0,1]$. Using the fractional Maclaurin series theorem,

$$R_n(x) = u(x) - u_n(x) = \frac{D^{(n+1)\alpha}u(c)}{\Gamma((n+1)\alpha+1)} x^{(n+1)\alpha},\ 0 \leq c \leq x,$$

which implies that

$$\|R_n(x)\| \leq \frac{\left|D^{(n+1)\alpha}u(c)\right|}{\Gamma((n+1)\alpha+1)}. \tag{3.14}$$

Let $m-1 < (n+1)\alpha < m$. Then,

$$\begin{aligned}
\left|D^{(n+1)\alpha}u(c)\right| &= \frac{1}{\Gamma(m-(n+1)\alpha)} \left| \int_0^c (c-s)^{m-1-(n+1)\alpha} u^m(s)ds \right| \\
&\leq \frac{\left\|u^{(m)}\right\|}{\Gamma(m-(n+1)\alpha)} \left| \int_0^c (c-s)^{m-1-(n+1)\alpha} ds \right| \\
&= \frac{\left\|u^{(m)}\right\|}{\Gamma(m-(n+1)\alpha)} \frac{c^{m-(n+1)\alpha}}{m-(n+1)\alpha} \\
&\leq \frac{A}{\Gamma(m-(n+1)\alpha)} \frac{c^{m-(n+1)\alpha}}{m-(n+1)\alpha}.
\end{aligned}$$

Since $0 < m-(n+1)\alpha < 1$ and $0 < c \leq 1$, then

$$0 < c^{m-(n+1)\alpha} \leq 1$$

and

$$\Gamma(m-(n+1)\alpha) \geq 0.885603.$$

Thus,

$$\left|D^{(n+1)\alpha}u(c)\right| \leq \frac{1.13\,A}{(m-(n+1)\alpha)}. \tag{3.15}$$

Moreover, from Eqs. (3.14)-(3.15), we get

$$\|u-u_n\| \leq \frac{1.13\,A}{(m-(n+1)\alpha)\,\Gamma((n+1)\alpha+1)} \to 0 \text{ as } n \to \infty.$$

□

4 Results and Discussion

In this section, we present some of our numerical results.

Example 4.1. *Consider the following singular fractional Riccati differential equation of the form*

$$x^{\frac{1}{2}} D^{\frac{1}{2}} u(x) + e^x u(x) + 2u^2(x) = g(x), \quad x \in (0,1] \tag{4.16}$$

subject to

$$u(0) = 1 \tag{4.17}$$

where

$$g(x) = \frac{16}{5\sqrt{\pi}} x^3 + e^x(x^3+1) + 2(x^3+1)^2.$$

The exact solution is $u(x) = x^3 + 1$. *Then,*

$$\begin{aligned}
a(x) &= x^{\frac{1}{2}}, \\
b(x) &= e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ..., \\
c(x) &= 2, \\
g(x) &= \frac{16}{5\sqrt{\pi}} x^3 + \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ...\right)(x^3+1) + 2(x^3+1)^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
f_0 &= 1,\ f_1 = 0,\ f_2 = 0,\ f_3 = 0, \\
f_4 &= 0,\ f_5 = 0,\ f_6 = 6, \\
f_n &= 0,\ n = 7, 8, ...
\end{aligned}$$

Thus,

$$u_6(x) = 1 + x^3$$

which is the exact solution.

Example 4.2. *Consider the following singular fractional Riccati differential equation of the form*

$$(x^{\frac{1}{4}} + x^{\frac{5}{4}}) D^{\frac{1}{4}} u(x) + 2x^2 u(x) + e^x u^2(x) = g(x), \quad x \in (0,1] \tag{4.18}$$

subject to

$$u(0) = 0 \tag{4.19}$$

where

$$\begin{aligned}
g(x) = {} & \left(\frac{x}{\Gamma(1.75)} - \frac{x^3}{\Gamma(3.75)} + \frac{x^5}{\Gamma(5.75)} - ...\right) + \left(\frac{x^2}{\Gamma(1.75)} - \frac{x^4}{\Gamma(3.75)} + \frac{x^6}{\Gamma(5.75)} - ...\right) \\
& + 2x^2\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\right) + \left(\frac{1}{0!} + \frac{x}{1!} + \frac{x^5}{2!} + ...\right)\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - ...\right)^2.
\end{aligned}$$

The exact solution is $u(x)=\sin x$. *Then,*

$$\begin{aligned} a(x) &= x^{\frac{1}{4}}+x^{\frac{5}{4}}, \\ b(x) &= 2x^2, \\ c(x) &= e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+..., \\ g(x) &= (\frac{x}{\Gamma(1.75)}-\frac{x^3}{\Gamma(3.75)}+\frac{x^5}{\Gamma(5.75)}-...)+(\frac{x^2}{\Gamma(1.75)}-\frac{x^4}{\Gamma(3.75)}+\frac{x^6}{\Gamma(5.75)}-...) \\ &\quad +2x^2(\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-...)+(\frac{1}{0!}+\frac{x}{1!}+\frac{x^5}{2!}+...)(\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-...)^2. \end{aligned}$$

We find that for $n=0,1,2,...,28$,

$$\begin{aligned} f_4 &= 1,\ f_{12}=-1,\ f_{20}=1,\ f_{28}=-1, \\ f_j &= 0,\ j\in\{0,1,2,...\}-\{4,12,20,28\}. \end{aligned}$$

Thus,

$$u_{28}(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}.$$

If $n\to\infty$,

$$u_n(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...=\sin x$$

which is the exact solution.

Example 4.3. *Consider the following singular fractional Riccati differential equation of the form*

$$x^{\frac{1}{3}}\ D^{\frac{1}{3}}u(x)+2xu(x)-(\cos x\ )u^2(x)=g(x), \quad x\in(0,1) \tag{4.20}$$

subject to

$$u(0)=0 \tag{4.21}$$

where

$$g(x)=\frac{2}{\Gamma(8/3)}x^2+2x^3-(\cos x)\ x^4.$$

The exact solution is $u(x)=x^2$. *Then,*

$$\begin{aligned} a(x) &= x^{\frac{1}{3}}, \\ b(x) &= 2x, \\ c(x) &= -\cos x = -\left(1-\frac{x^2}{2}+...\right), \\ g(x) &= \frac{2}{\Gamma(8/3)}x^2+2x^3-(\cos x)\ x^4 = \frac{2}{\Gamma(8/3)}x^2+2x^3-(1-\frac{x^2}{2}+...)\ x^4. \end{aligned}$$

Thus,

$$\begin{aligned} f_0 &= 0,\ f_1=0,\ f_2=0, \\ f_3 &= 0,\ f_4=2,\ \ f_5=0,\ f_6=2, \\ f_n &= 0,\ n=7,8,.... \end{aligned}$$

Thus,

$$u_6(x)=x^2$$

which is the exact solution.

Example 4.4. *Consider the following singular fractional Riccati differential equation of the form*

$$(x+x^{\frac{1}{2}})\,D^{\frac{1}{2}}u(x)+xu(x)-2u^2(x)=g(x), \quad x\in(0,1) \tag{4.22}$$

subject to

$$u(0)=0 \tag{4.23}$$

where

$$g(x)=\frac{8}{3\sqrt{\pi}}(x^{\frac{5}{2}}+x^2)+x^3-2x^4.$$

The exact solution is $u(x)=x^2$. Then,

$$\begin{aligned} a(x) &= x+x^{\frac{1}{2}}, \\ b(x) &= x, \\ c(x) &= -2, \\ g(x) &= \frac{8}{3\sqrt{\pi}}(x^{\frac{5}{2}}+x^2)+x^3-2x^4. \end{aligned}$$

Thus,

$$\begin{aligned} f_0 &= 0,\ f_1=0,\ f_2=0, \\ f_3 &= 0,\ f_4=2, \\ f_n &= 0,\ n=5,6,\ldots. \end{aligned}$$

Thus,

$$u_4(x)=x^2$$

which is the exact solution.

5 Conclusion

In this paper, we study the singular Riccati equation with fractional order. The modified fractional power series method is employed to compute an approximation to the proposed problem. The construction of the modified fractional power series method is presented. The validity of the MFPS method is ascertained by presenting four of our examples. We prove the existence of the solution of the singular Riccati equation with fractional order. The convergence of the approximate solution using the proposed method is investigated. From the previous section, we get the exact solutions in Examples (4.1)-(4.4) using the MFPS method. This gives us the numerical evidence that the MFPS method is an excellent tool due to the rapid convergence. In addition, the results in this paper confirm that the MFPS method is a powerful and an efficient method for solving nonlinear differential equations in different fields of sciences and engineering.

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