Homogenization of a Ginzburg-Landau equation in thin multidomains

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Abstract
This paper deals with homogenization of the Ginzburg-Landau boundary value problem in a thin multidomain. More precisely we are interested in the minimization of the following Ginzburg-Landau energy

$E_{n,\varepsilon}(u) = \frac{1}{2} \int_{\Omega_n} |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_{\Omega_n} (1 - |u|^2)^2,$

over all maps $u : \Omega_n \to \mathbb{C}$, satisfying a partial boundary Dirichlet condition $u = g$ on a specific subset of $\partial \Omega_n$.

Here $\Omega_n \subset \mathbb{R}^2$ is a thin bounded multidomain, and $g : \mathbb{C} \to S^1$ is a given smooth map. The analysis is performed in the case where $\Omega_n$ is made of a thin horizontal plate of vanishing thickness with a forest of vertical cylinders periodically distributed on the top of it of vanishing width as $n \to \infty$. The main issue addressed here is to determine the limit energy and the behavior of minimizers as $n \to \infty$.

Keywords: Homogenization; Harmonic capacity; Extension operator; Ginzburg-Landau equations.


1 Introduction

The homogenization theory deals with the properties of heterogeneous materials, which are of critical importance for modern technology. It is a Mathematical Theory, or more precisely, an Asymptotic Analysis Theory that originates from Material Engineering, or more precisely, from understanding the way Constitutive Equation of composite material can be gotten from the Constitutive Equation of each component of the concerned material and from their topological and geometrical distributions.

Modeling of such materials raises fundamental mathematical questions, primarily in partial differential equations and Calculus of Variations.

For general references about homogenization, we refer to [1], [4], [7], [8], [9], [10], [12], [16].

Vortices of the minimizers of the Ginzburg-Landau energy functional capture essential features of superconductors and superfluids.

They have many common features with vortices in fluids, defects in liquid crystals, dislocations in solids, etc.

The Ginzburg-Landau energy has been used to describe many physical situations presenting topological defects known

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The aim of this work is to study the asymptotic behavior of the minimizing solutions of the Ginzburg-Landau energy.

The authors study the problem of minimizing the Ginzburg-Landau energy under a suitable boundary condition ensuring a topological obstruction in the limit \( \varepsilon \to 0 \), and a fortiori the formation of vortices.

The homogenization of the Ginzburg-Landau equation in a domain of \( \mathbb{R}^2 \) with oscillating boundary is studied in [15]. The limit behaviour of the Ginzburg-Landau equation in a perforated domain in \( \mathbb{R}^3 \) with holes along a plane is studied in [3].

For more details on the application of the homogenization theory on the Ginzburg-Landau equation we refer to [3], [4], [16], [21] and [15].

In this paper we consider the problem of minimizing the following Ginzburg-Landau energy

\[
E_{n,\varepsilon}(u) = \frac{1}{2} \int_{\Omega_n} |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_{\Omega_n} (1-|u|^2)^2,
\]

over the functional space

\[
V_n = \{ u \in H^1(\Omega_n, \mathbb{R}^2), \ u = g \text{ on } \Gamma_{n,1} \text{ and } \frac{\partial u}{\partial v} = 0 \text{ on } \Gamma_{n,2} \}.
\]

Where \( g : \Gamma_{n,1} \to \mathbb{C} \) is a prescribed smooth map with \( |g| = 1 \) on \( \Gamma_{n,1} \), \( \Omega_n \) is a thin multidomains of \( \mathbb{R}^2 \), constituted of the section of an horizontal plate \( \Omega^H_n = [-1,1] \times -h_n,0] \), with small thickness \( h_n \) and \( 2n-1 \) cylinders \( \Omega^i_n = \{|a^i_n - r_n, a^i_n + r_n| \times [0,1] \} \) with small radius \( r_n \), where \( a^i_n = (\frac{1}{n},0) \) \( i = -(n-1), ..., n-1 \) which means that \( \Omega^i_n \) are periodically distributed, set \( \Omega^m_n = \bigcup_{i=-(n-1)}^{n-1} \Omega^i_n \).

\( \Gamma_{n,1} \) is a subset of the boundary, \( \partial \Omega_{n} \), of the multidomain \( \Omega_n = \Omega^m_n \cup \Omega^b_n \), \( \Gamma_{n,1} = \bigcup_{i=-(n-1)}^{n-1} \{|a^i_n - r_n, a^i_n + r_n| \times [1] \} \cup \{-1,1\} \times -h_n,0 \} \) and \( \Gamma_{n,2} = \partial \Omega_n / \Gamma_{n,1} \).

Set \( \Omega^H_n = [-1,1] \times [0,1], \ \Omega^b_n = [-1,1] \times [1,0], \ \Omega = \Omega^H \cup \Omega^b \) and \( \Sigma = \Omega^H \cap \Omega^b \).

The aim of this work is to study the asymptotic behavior of the minimizing solutions of the Ginzburg-Landau energy.

Let \( \chi_{\Omega^m_n} \) be the characteristic function of \( \Omega^m_n = \bigcup_{i=-(n-1)}^{n-1} \Omega^i_n \), under the assumption that \( 2nr_n \) converges to some constant \( \theta \in [0,1] \), we have

\[
\chi_{\Omega^m_n} \rightharpoonup_{n \to +\infty} \theta \chi_{\Omega^H} \text{ weakly } \ast \text{ in } L^\infty(\mathbb{R}^2)
\]

The results in this paper are about the asymptotic behavior, as \( n \to +\infty \), of Problem (1.1) in the case where the number of cylinders is depending on \( n \), and we assume that

\[
\lim_{n \to +\infty} \frac{nh_n}{r_n} = q \in [0, +\infty].
\]

Let’s define \( T_n : [-1,1] \times [-1,1] \times -h_n,0] \)

then the function \( v_{n,\varepsilon} \) is a solution of the following minimization problem

\[
\min_{v \in V} \frac{h_n}{2} \int_{\Omega^H_n} \left| \frac{\partial v}{\partial x} \right|^2 + \frac{1}{h_n} \int_{\Omega^b_n} \left| \frac{\partial v}{\partial y} \right|^2 + \frac{1}{2\varepsilon^2} \left( 1 - |v|^2 \right)^2 + \frac{1}{2} \int_{\Omega^H_n} |\nabla u|^2 + \frac{1}{2\varepsilon^2} \left( 1 - |u|^2 \right)^2
\]

where

\[
V = \{ v = (v^H, v^b) : H^1(\Omega^b_n, \mathbb{C}) \times H^1(\Omega^H_n, \mathbb{C}); \ v^H = g \text{ on } \bigcup_{i=-(n-1)}^{n-1} \{|a^i_n - r_n, a^i_n + r_n| \times \{1\} \}, \ v^b = g \text{ on } \{\pm 1\} \times [-1,0], \text{ and } v^H = v^b \text{ on } \bigcup_{i=-(n-1)}^{n-1} \{|a^i_n - r_n, a^i_n + r_n| \times \{0\} \}
\]

In the following we present the main results.
Proposition 1.1. Let $u_{n,e}$ be a solution of Problem (1.1), $h_n$ and $r_n$ satisfying (1.2) and (1.3). Then, there exists a constant independent of $n$, such that
\[
\|v_{n,e}\|_{H^1(\Omega_n)} \leq c \tag{1.4}
\]

Proposition 1.2. Let $u_{n,e}$ be a solution of Problem (1.1), $h_n$ and $r_n$ satisfying (1.2) and (1.3). Then there exists an extension linear operator $P_n \in \mathcal{L}(H^1(\Omega_n); H^1(\Omega))$ and an increasing sequence of positive integer numbers \(\{n_j\} \in \mathbb{N}\) such that
\[
\begin{align*}
P_n v_{n_j,e} &\rightarrow v_e \quad \text{weakly in } H^1(\Omega^e) \\
P_{n_j,e} v_{n_j,e} &\rightarrow v_e \quad \text{weakly in } H^1(\Omega^b) \tag{1.5}
\end{align*}
\]

Theorem 1.1. Let $u_{n,e}$ be a solution of Problem (1.1), $h_n$ and $r_n$ satisfying (1.3). Then
\[
P_{n,e} v_{n,e}^\theta \rightarrow v_e^\theta, \quad \frac{\partial P_{n,e} v_{n,e}^\theta}{\partial y} \rightarrow \frac{\partial v_e^\theta}{\partial y} \quad \text{weakly in } L^2(\Omega^e) \tag{1.6}
\]
\[
\frac{\partial P_{n,e} v_{n,e}^\theta}{\partial x} \rightarrow 0 \quad \text{weakly in } L^2(\Omega^e) \tag{1.7}
\]
\[
v_{n,e}^b \rightarrow v_e^b \quad \text{weakly in } L^2(\Omega^b) \tag{1.8}
\]
\[
\frac{\partial v_e^b}{\partial y} \rightarrow 0 \quad \text{strongly in } L^2(\Omega^b) \tag{1.9}
\]
as $n \to +\infty$ where $(v_e^\theta, v_e^b)$ is the unique solution of the following problem
\[
\text{Min}\{\theta \int_{\Omega^e} \left| \frac{\partial v}{\partial y} \right|^2 + \frac{1}{\varepsilon^2} (1 - |v|^2)^2, \; v = (v^\theta, v^b) \in L^2(\Omega^e, H^1([0,1])) \times H^1(\Omega^b),
\]
\[
v^b(x,0) = v^\theta(x,0), \; \text{for almost every } x \in [-1,1] \quad \text{and } v^b(x,-1) = g(x,-1),
\]
\[
v^\theta(x,1) = g(x,1)\}
\]

Moreover the energy converges, that is
\[
\lim_{n \to +\infty} E_{n,e}(v_{n,e}) = \theta \int_{\Omega^e} \left| \frac{\partial v_e^\theta}{\partial y} \right|^2 + \frac{1}{\varepsilon^2} (1 - |v_e^\theta|^2)^2. \tag{1.11}
\]

As far as the asymptotic behavior, as $n \to +\infty$, of the solution $u_{n,e}$ of the problem (1.1) is concerned, Theorem 1.1 leads immediately to the following result ( $\Omega_n$ and $\Sigma$ denote the Lebesgue measure of $\Omega_n^e$ and $\Sigma$ in $\mathbb{R}^2$ and $\mathbb{R}$ respectively):

Corollary 1.1. Let $u_{n,e}$ be a solution of problem (1.1). Then under the assumptions of Theorem 1.1, it results
\[
P_{n,e} u_{n,e}^\theta \rightarrow v_e^\theta \quad \text{strongly in } L^2([-1,1],H^1([0,1])) \tag{1.12}
\]
\[
\frac{\partial u_{n,e}^\theta}{\partial x} \rightarrow 0 \quad \text{weakly in } L^2(\Omega^e) \tag{1.13}
\]
\[
\frac{\partial u_{n,e}^\theta}{\partial y} \rightarrow \theta \frac{\partial v_e^\theta}{\partial y} \quad \text{strongly in } L^2(\Omega^e) \tag{1.14}
\]

up to a subsequence,
\[
\lim_n \int_{\Omega_n^b} |u_{n,e}^b|^2 dx = 0 \tag{1.15}
\]
\[
\lim_n \frac{1}{|\Omega_n^e|} \int_{\Omega_n^e} u_{n,e}^\theta dx dy = \frac{1}{|\Sigma|} \int_{\Sigma} v dx \tag{1.16}
\]
where \((v_{\varepsilon}^a, v_{\varepsilon}^b)\) is the unique solution of the following problem

\[
\min \{ \theta \int_{\Omega^\varepsilon} \frac{1}{2} \frac{\partial v^a}{\partial y}^2 + \frac{1}{\varepsilon^2} (1 - |v^a|^2)^2, \quad v = (v^a, v^b) \in L^2(\Omega^\varepsilon, H^1(\Omega^\varepsilon)) \times H^1(\Omega^\varepsilon), \quad \}
\]

\[
 v^b(x, 0) = v^a(x, 0), \quad \text{for almost every } x \in [-1, 1], \quad \text{and } v^b(x, -1) = g(x, -1),
\]

\[
v^a(x, 1) = g(x, 1) \}
\]

Moreover the energy converges, that is

\[
\lim_{n \to +\infty} E_{n, \varepsilon}(u_{n, \varepsilon}) = \theta \int_{\Omega^\varepsilon} \frac{1}{2} \frac{\partial u^a}{\partial y}^2 + \frac{1}{\varepsilon^2} (1 - |u^a|^2)^2.
\]

2 A priori norm-estimates

This section is devoted to give some a priori estimates of the norm of \(v_{n, \varepsilon}\), in order to introduce to the proof of Theorem 1.1.

**Proposition 2.1.** Let \(v_{n, \varepsilon}\) be a solution of problem (1.1) under the assumptions of Theorem 1.1. Then, there exists a constant \(c\) such that

\[
\|v_{n, \varepsilon}\|_{H^1(\Omega^\varepsilon)} \leq c
\]

\[
\|\nabla v_{n, \varepsilon}\|_{(L^2(\Omega^\varepsilon))^2} \leq c
\]

\[
\|\sqrt{h_n} v_{n, \varepsilon}\|_{(L^2(\Omega^\varepsilon))^2} \leq c
\]

\[
\|\sqrt{h_n} (\bar{a}_{\varepsilon} v_{n, \varepsilon} - \bar{b}_{\varepsilon} v_{n, \varepsilon})\|_{(L^2(\Omega^\varepsilon))^2} \leq c
\]

for every \(n\).

**Proof.** The minimizer \(v_{n, \varepsilon}\) of the energy of Ginzburg-Landau is a solution of the following variational problem

\[
h_n \int_{\Omega^\varepsilon} \frac{\partial v_{n, \varepsilon}}{\partial x} \frac{\partial v}{\partial x} + \frac{1}{h_n^2} \frac{\partial v_{n, \varepsilon}}{\partial y} \frac{\partial v}{\partial y} - \frac{1}{\varepsilon^2} (1 - |v_{n, \varepsilon}|^2) v_{n, \varepsilon} v
\]

\[
+ \int_{\Omega^\varepsilon} \nabla v_{n, \varepsilon} \nabla v - \frac{1}{\varepsilon^2} (1 - |v_{n, \varepsilon}|^2) v_{n, \varepsilon} v = 0 \quad \text{for all } v \in H^1(\Omega^\varepsilon, \mathbb{C}).
\]

The a priori norm-estimates are obtained by using this variational formulation, in which we choose \(v = v_{n, \varepsilon}\) as test function and the Young inequality.

**Corollary 2.1.** Let \(v_{n, \varepsilon}\) be a solution of problem (1.1) under the assumptions of Theorem 1.1. Then, there exists a constant \(c\) such that

\[
\|v_{n, \varepsilon}\|_{L^2(\Omega^\varepsilon; \Sigma)} \leq c
\]

for every \(n\).

**Proof.** The estimate (2.24) follows from estimate (2.19), if one observes that there exists a constant \(c\) such that

\[
\|v_{n, \varepsilon}\|_{L^2(\Omega^\varepsilon; \Sigma)} \leq c(\|v_{n, \varepsilon}\|^2_{L^2(\Omega^\varepsilon)} + \|\frac{\partial v_{n, \varepsilon}}{\partial y}\|^2_{L^2(\Omega^\varepsilon)})
\]

for every \(n\).

**Proposition 2.2.** Let \(v_{n, \varepsilon}\) be a solution of problem (1.1) under the assumptions of Theorem 1.1. Then, there exists a constant \(c\) such that

\[
\|v_{n, \varepsilon}\|_{L^2(\Omega^\varepsilon)} \leq c
\]

for every \(n\).
Proof. For the sake of clarity, first we introduce some notations which we will use in what follows of this proof. Set $I_n = \{ k \in \mathbb{N}, (r_n - 1, 1 + \frac{1}{n}) \cap 1, 1 \neq \emptyset \}$, $C_n = \cup_{k \in I_n} (r_n - 1, 1 + \frac{1}{n}) \times 1, 1 - 1.0]$ and $C$ be a bounded open set of $\mathbb{R}^2$ such that $C_n \subseteq C$ for every $n$.

It is easy to prove the existence of a linear extension operator $Q \in L(H^1(\Omega^b), H^1(C))$ such that

$$
\|\partial_{x}Qv\|_{L^2(C)} \leq c\|\partial_{x}Qv\|_{L^2(\Omega^b)}, \quad \|\partial_{y}Qv\|_{L^2(C)} \leq c\|\partial_{y}Qv\|_{L^2(\Omega^b)} \quad \forall v \in H^1(\Omega^b),
$$

(2.26)

where $c$ is a constant independent of $v$ (see [10]). In particular, we have,

$$
\int_{\Omega^b}|u_{n,\varepsilon}|^2 \leq \int_{C_n}|qu_{n,\varepsilon}|^2, \quad \text{for every } n
$$

(2.27)

by making use of the change of variable $x = r_n + \frac{k}{n}$, it results

$$
\int_{C_n}|Qu_{n,\varepsilon}|^2 \, dx dy = r_n \sum_{k \in I_n} \int_{|x| = 1 + \frac{1}{n}}|Qu_{n,\varepsilon}(r_n + \frac{k}{n}, y)|^2 \, dx dy,
$$

we recall a helpful tool, the well known Friedrichs inequality: Let $G$ be an open bounded connected subset of $\mathbb{R}^N$ with Lipschitz boundary and let $\Gamma \subseteq \partial G$ be such that the $(N - 1)$-dimensional Hausdorff measure of $\Gamma$ is positive. Then, there exists a constant $c$ such that

$$
\int_G |v|^p \, dx \leq c \left( \int_G |v|^p \, d\sigma + \int_G |Dv|^p \, dx \right) \quad \forall v \in W^{1,p}(G).
$$

(2.28)

Making use of the Friedrichs inequality yields

$$
\int_{C_n}|u_{n,\varepsilon}|^2 \, dx dy \leq c \left( \int_{\Omega^b \cap \Sigma|} |u_{n,\varepsilon}|^2 \, dx + r_n^2 \int_{C_n}|\partial_{x}u_{n,\varepsilon}|^2 \, dx dy + \int_{C_n}|\partial_{y}u_{n,\varepsilon}|^2 \, dS dy \right),
$$

(2.29)

for every $n$, where $c$ is a constant independent of $n$. By combining (2.27) and (2.29) with (2.26), it follows that

$$
\int_{\Omega^b}|u_{n,\varepsilon}|^2 \, dx dy \leq c \left( \int_{\Omega^b \cap \Sigma|} |u_{n,\varepsilon}|^2 \, dx + r_n^2 \int_{\Omega^b}|\partial_{x}u_{n,\varepsilon}|^2 \, dx dy + \int_{\Omega^b}|\partial_{y}u_{n,\varepsilon}|^2 \, dS dy \right),
$$

for every $n$, where $c$ is a constant independent of $n$. Finally, by making use of (2.22), (2.24) and (1.3), we obtain (2.25).

Proposition 2.3. Let $u_{n,\varepsilon}$ be a solution of problem (2.23), let $\Theta$ be a constant given by (1.2). Then (1.9) holds true. Moreover, there exist a subsequence of $\{ u_{n,\varepsilon} \}$, still denoted by $\{ u_{n,\varepsilon} \}$, such that (1.6) hold true. Furthermore, for this subsequence, under assumption (1.3), convergence (1.8) holds true.

Before proving this proposition, the main properties of the two-scale convergence method introduced by Nguetseng in [21] and developed by Allaire in [1] are recalled. Here, $Y = [0, 1]$ $(N \geq 1)$, $1 < p < \infty$, $C_{per}$ denote the space of infinitely differentiable functions in $\mathbb{R}^d$ that are periodic of period $Y$ and $W^{1,p}_{per}(Y)$ is the completion of $C_{per}(Y)$ for the norm of $W^{1,p}(Y)$.

Definition 2.1. (II)

Let $A$ be an open subset of $\mathbb{R}^N$. A sequence $\{v_{\varepsilon}\} \subseteq L^p(A)$ is said to "two-scale converges" to a limit $v \in L^p(A \times Y)$ if, for any function $\psi \in D(A, C_{per}(Y))$, it results that

$$
(\ast) \lim_{\varepsilon \to 0} \int_A v_{\varepsilon}(x) \psi(x, \frac{x}{\varepsilon}) \, dx = \int_{A \times Y} v(x, y) \psi(x, y) \, dx dy
$$

Proposition 2.4. (II) Let $A$ be an open subset of $\mathbb{R}^N$. 

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\[ L(\{v_e\}_e \subset L^p(A) be a sequence converging to \( v \) strongly in \( L^p(A) \). Then, \( \{v_e\}_e \) two-scale converges to the same limit \( v \).

\[ \text{ii)} \ \text{Let } \{v_e\}_e \subset L^p(A) be a sequence two-scale converging to \( v \in L^p(A \times Y) \). Then, \( \{v_e\}_e \) converges to \( f \), \( v(.,y)dy \) weakly in \( L^p(A) \).

\[ \text{iii)} \ \text{Let } \{v_e\}_e \subset L^p(A) be a bounded sequence. Then, there exist a subsequence of \( \{e\}_e \), still denoted by \( \{e\}_e \), and a function \( v \in L^p(A \times Y) \) such that \( \{v_e\}_e \) two-scale converges to \( v \).

\[ \text{iv)} \ \text{Let } \{v_e\}_e \subset W^{1,p}(A) be a sequence such that \( \{v_e\}_e \) and \( \{\varepsilon Dv_e\}_e \) are bounded \( L^p(A) \) and \( (L^p(A))^N \), respectively. Then, there exist a subsequence of \( \{e\}_e \), still denoted by \( \{e\}_e \), and a function \( v \in L^p(A, W^{1,p}_p(Y)) \) such that \( \{v_e\}_e \) and \( \{\varepsilon Dv_e\}_e \) two-scale converge to \( v \) and \( Dv \), respectively.

\[ \text{Remark 2.1.} \ \text{Due to density properties, it is easily seen that if } \{v_e\}_e \subset L^p(A) \text{ two-scale converges to } v \in L^p(A \times Y), \text{ convergence } (*) \text{ holds true also for any function } \psi \text{ of the form } \psi(x,y) = \phi_1(x)\phi_2(y), \text{ where } \phi_1 \in C_0(A) \text{ and } \phi_2 \in L^\infty_p(Y).

\[ \text{proof of Proposition 1.2:} \ \text{Convergence (1.9) follows from (2.22). By virtue of (2.19), there exist a subsequence of } \{n\}, \text{ still denoted by } \{n\}, \text{ and } v_1 \in L^2(\Omega^b, H^1([0,1])) \text{ such that (1.6) holds true. Moreover, under assumption (1.2), estimate (2.25) holds true. Consequently, in view of (1.9), there exist a subsequence of the previous one, still denoted by } \{n\}, \text{ and } w \in L^2(\Omega^b), \text{ independent of } y, \text{ such that}

\[ v_{n,e} \rightharpoonup w \ \text{ weakly in } L^2(\Omega^b).

\[ \text{(2.30)}

\[ \text{In order to prove (1.8), we show that, up to a subsequence,}

\[ \{v_{n,e}\}_n \text{ two-scale converges to } w \text{ in } \Omega^b.

\[ \text{(2.31)}

\[ \text{To this aim, first observe that (2.22) and (1.3) provide that}

\[ \lim_{n \to +\infty} \frac{1}{n} Dv_{n,e} \| (L^2(\Omega^b))^2 = 0.

\[ \text{Consequently it follows that}

\[ \{\frac{1}{n} Dv_{n,e}\}_n \text{ two-scale converges to } 0 \text{ in } \Omega^b.

\[ \text{(2.32)}

\[ \text{Combining (2.25) with (2.32), one can see that we can extract again a subsequence and } f \in L^2(\Omega^b) \text{ such that}

\[ \{v_{n,e}\}_n \text{ two-scale converges to } f \text{ in } \Omega^b.

\[ \text{(2.33)}

\[ \text{Finally, since } f \text{ is independent of } x, (2.31) \text{ follows by comparing (2.30) with (2.33). By setting } \psi_n(x,y) = \chi_{\Omega^b/\Sigma}(x) \text{ for } (x,y) \in \Omega^b, \text{ convergence (2.31) provides that}

\[ \lim_n \int_{\Omega^b} v_{n,e} \psi_n \varphi dx dy = \theta \int_{\Omega^b} w \varphi dx dy \ \forall \varphi \in C_0(\Omega^b),

\[ \text{from which it follows that, since } \{v_{n,e}\}_n \text{ is bounded in } L^2(\Omega^b),

\[ v_{n,e} \rightharpoonup \theta w \ \text{ weakly in } L^2(\Omega^b).

\[ \text{(2.34)}

\[ \text{Moreover, equation (1.9) provides that}

\[ \lim_n \| \frac{\partial (v_{n,e})}{\partial y} \|_{L^2(\Omega^b)} = \lim_n \| \psi_n \frac{\partial v_{n,e}}{\partial y} \|_{L^2(\Omega^b)} \leq \lim_n \frac{\partial v_{n,e}}{\partial y} \|_{L^2(\Omega^b)} = 0.

\[ \text{(2.35)}

\[ \text{International Scientific Publications and Consulting Services} \]
Then, by making use of (2.34) and (2.35), and by the fact that $w$ is independent of $y$, it results that

$$
\lim_n \int_{\Sigma} P_{n,e} v_{n,e} \varphi dx = \lim_n \left( \int_{\Omega^b} \frac{\partial (v_{n,e} \Psi_n)}{\partial y} \varphi dxdy + \int_{\Omega^b} v_{n,e} \Psi_n \frac{\partial \varphi}{\partial y} dxdy \right) 
= \theta \int_{\Omega^b} w \frac{\partial \varphi}{\partial y} dxdy = \theta \int_{\Sigma} w \varphi dx \quad \forall \varphi \in C_0^\infty (\Omega^a \cup \Omega^b \cup \Sigma).
$$

(2.36)

On the other hand, from (1.6) it follows that

$$
\lim_n \int_{\Sigma} P_{n,e} v_{n,e} \varphi dx = - \lim_n \left( \int_{\Omega^a} \frac{\partial (P_{n,e} v_{n,e})}{\partial y} \varphi dxdy + \int_{\Omega^a} P_{n,e} v_{n,e} \frac{\partial \varphi}{\partial y} dxdy \right) 
= - \theta \int_{\Omega^a} \nu_e \frac{\partial \varphi}{\partial y} dxdy - \theta \int_{\Omega^a} \frac{\partial \nu_e}{\partial y} \varphi dxdy 
= \theta \int_{\Sigma} \nu_e \varphi dx \quad \forall \varphi \in C_0^\infty (\Omega^a \cup \Omega^b \cup \Sigma).
$$

(2.37)

By comparing (2.36) and (2.37), we obtain

$$
\int_{\Sigma} w \varphi dx = \int_{\Sigma} \nu_e \varphi dx \quad \forall \varphi \in C_0^\infty (\Sigma).
$$

Then, $w$ is the function independent of $y$ which is equal, on $\Sigma$, to the trace of $\nu_e$. Finally, convergence (1.8) follows from (2.30).

\hfill \square

### 2.1 A proof of Theorem 1

By virtue of estimations (2.19), (2.20) and Proposition 2.5, there exist $\nu_e \in L^2(\Omega^a \cap \{1\}, H^1(\Omega^a \cap [0, 1]))$ satisfying (1.6) and (1.7). Furthermore there exist $\eta = (\eta_1, \eta_2) \in (L^2(\Omega^a))^2$ such that

$$
P_{n,e} \nabla v_{n,e} \rightharpoonup \eta \quad \text{weakly in} \quad (L^2(\Omega^a))^2.
$$

(2.38)

Let us show that $\eta_1 = 0$ a.e in $\Omega^a$.

Set $(w_n)_{n \in \mathbb{N}}$ be a sequence in $W^{1,\infty}(\Omega^a)$ satisfying the following conditions

$$
w_n \to x \quad \text{strongly in} \quad L^\infty(\Omega^a) \quad \text{as} \quad n \to +\infty.
$$

(2.39)

$$
\nabla w_n = 0 \quad \text{a. e in} \quad \Omega^a_n \quad \forall \ n \in \mathbb{N}
$$

(2.40)

The existence of such sequences is proved in [9] Lemma 4.3.

By choosing $v = \varphi w_n$ and $v = \varphi x$ with $\varphi \in C_0^\infty (\Omega^a)$, as test function in (2.23) then by virtue of (2.40) it results

$$
\int_{\Omega^a} \nabla (P_{n,e} v_{n,e}) \nabla \varphi w_n - \frac{1}{\varepsilon^2} (1 - |v_{n,e}|^2) v_{n,e} \varphi w_n dxdy = 0 \quad \forall \varphi \in C_0^\infty (\Omega^a)
$$

(2.41)

$$
\int_{\Omega^a} \nabla (P_{n,e} v_{n,e}) \nabla (\varphi x) - \frac{1}{\varepsilon^2} (1 - |v_{n,e}|^2) v_{n,e} \varphi x dxdy = 0 \quad \forall \varphi \in C_0^\infty (\Omega^a)
$$

(2.42)

for every $n \in \mathbb{N}$. By passing to the limit, as $n$ diverges, in (2.41) and (2.42) we obtain

$$
\int_{\Omega^a} \eta \nabla \varphi x - \frac{1}{\varepsilon^2} (1 - |v_e|^2) v_e \varphi x dxdy = 0 \quad \forall \varphi \in C_0^\infty (\Omega^a)
$$

(2.43)

$$
\int_{\Omega^a} (\eta \nabla (\varphi x) - \frac{1}{\varepsilon^2} (1 - |v_e|^2) v_e \varphi x) dxdy = 0 \quad \forall \varphi \in C_0^\infty (\Omega^a).
$$

(2.44)
By subtracting (2.43) from (2.44) we obtain
\[
\int_{\Omega^e} \eta (\nabla \varphi - \nabla (\varphi x)) dx dy = 0 \quad \forall \varphi \in C_0^\infty(\Omega^e) \tag{2.45}
\]
which is equivalent to
\[
\int_{\Omega} \eta_1 \varphi dx dy = 0 \quad \forall \varphi \in C_0^\infty(\Omega^a).
\]
Which implies that
\[
\eta_1 = 0 \quad \text{a.e in } \Omega^a. \tag{2.46}
\]
Now, let us prove the convergence of the energies. For any \( v \in C^1(\Omega^e) \), we pass to the limit in (2.23), as \( n \) tends to \(+\infty\), with the test function
\[
w(x,y) = \begin{cases} v(x,y) & \text{if } (x,y) \in \Omega^a \\ v(x,0) & \text{if } (x,y) \in \Omega^b. \end{cases}
\]
Then, by virtue of (1.2), (2.21), (2.22), (2.38) and (2.46), one has the convergence of energies (1.11).

3 Case with two periods

We consider the same minimization problem on the space
\[
V_n = \{ u \in H^1(\Omega_n, \mathbb{R}^2), \ u = g \text{ on } \Gamma_1 \text{ and } \frac{\partial u}{\partial y} = 0 \text{ on } \Gamma_2 \}.
\]
Where \( g : \Gamma_{n,1} \to \mathbb{C} \) is a prescribed smooth map with \( |g| = 1 \) on \( \Gamma_{n,1} \), \( \Omega_n \) is a thin multidomain of \( \mathbb{R}^2 \), constituted of the section of an horizontal plate with small thickness \( h_n \) and \( n^2 + n \) cylinders \( \Omega_n = [a_n' - s_n, a_n' + s_n] \times [0,1] \) with small radius \( r_n \) and \( r'_n \), where \( a_n' = (\frac{1}{n},0) \), \( s_n = r_n \) for \( i = 0, \ldots, n-1 \) and \( a_n' = (0,0) \), \( s_n = r'_n \) for \( i = -(n-1), \ldots, -1 \) which means that \( \Omega_n \) are periodically distributed with two different periods moving in the two x directions.

\( \Gamma_{n,1} \) is a subset of the boundary, \( \partial \Omega_n \), of the multidomain \( \Omega_n \).
\( \Gamma_{n,1} = \bigcup_{i=-(n-1)}^{(n-1)} [a_n' - s_n, a_n' + s_n] \times \{1\} \cup \{1\} \times [i] - h_n, 0\} \) and \( \Gamma_{n,2} = \partial \Omega_n / \Gamma_{n,1} \).

Set \( \Omega^a = [-1,1] \times [0,1], \Omega^b = [-1,1] \times [-1,0], \Omega = \Omega^a \cup \Omega^b \) and \( \Sigma = \Omega^a \cap \Omega^b \).

Our aim is to study the asymptotic behavior of the minimizing solutions of the Ginzburg-Landau energy. Let \( \chi_{\Omega_n^e} \) be the characteristic function of \( \Omega_n^e \), we have
\[
\chi_{\Omega_n^e} \rightharpoonup_{n \to +\infty} \theta = \lim_{n \to +\infty} 2nr_n \chi_{[-1,0]}^2 + 4nr_n \chi_{[-1,0]} \quad \text{weakly * in } L^\infty(\Omega^a) \tag{3.47}
\]
To study the asymptotic behavior, as \( n \to +\infty \), of Problem (1.1) in the case where the number cylinders is not constant, assume that the volumes of \( \Omega_n^a \) and \( w_n \) tend to zero with same rate, that is
\[
\lim_{n \to +\infty} \frac{nh_n}{r_n} = q \in [0, +\infty]. \tag{3.48}
\]
Set
\[
T_n = [-1,1] \times [-1,0] - 1, 1] \times [-1,0] \]
defined by \( T_n(x,y) = (x, h_n y) \).

Let’s set
\[
v_n(x,y) = \begin{cases} u_n(x,y) & \text{for } (x,y) \in \Omega_n^a, \\
u_n(T_n(x,y)) & \text{for } (x,y) \in \Omega_n^b = [-1,1] \times [-1,0]. \end{cases}
\]
Then the function \( v_n(x,y) \) is a solution of the following minimization problem
\[
\min_{v \in V} \left( \frac{h_n}{2} \int_{\Omega^b} \frac{\partial v}{\partial x}^2 + \frac{1}{2} \frac{\partial v}{\partial y}^2 + \frac{1}{2} \frac{\partial v}{\partial x}^2 (1 - |v|^2)^2 + \frac{1}{2} \int_{\Omega^a} |\nabla v|^2 + \frac{1}{2} e^2 (1 - |v|^2)^2 \right)
\]
where
\[
V = \{ v = (v^a, v^b) \in H^1(\Omega_0^a, C) \times H^1(\Omega_0^b, C); \ v^a = g \text{ on } \bigcup_{k=-(n-1)}^{(n-1)} [a_n^k - s_n, a_n^k + s_n] \times \{1\}, \\
\ v^b = g \text{ on } \{\pm 1\} \times -1, 0, \ \text{and} \ v^a = v^b \text{ on } \bigcup_{k=-(n-1)}^{(n-1)} [a_n^k - s_n, a_n^k + s_n] \times \{0\}\}.
\]

In the following we present the main results

**Proposition 3.1.** Let \( u_{n,e} \) be a solution of Problem (1.1), \( h_n \) and \( r_n \) satisfying (3.47) and (3.48). Then, there exists a constant independent of \( n \), such that
\[
\|v_{n,e}\|_{H^1(\Omega_e)} \leq c \tag{3.49}
\]

**Proposition 3.2.** Let \( u_{n,e} \) be a solution of Problem (1.1), \( h_n \) and \( r_n \) satisfying (3.47) and (3.48). Then there exists an extension linear operator \( P_n \in \mathcal{L}(H^1(\Omega_n), H_{0e}^1(\Omega)) \) and an increasing sequence of positive integer numbers \( \{n_j\}_{j \in \mathbb{N}} \) and \( v_e \in L^2([-1, 1], H^1([0, 1])) \) \( H^1(\Omega^b) \) such that
\[
\begin{align*}
 &P_{n_j}v_{n_j,e} \to v_e^a \text{ weakly in } H^1(\Omega^a) \\
 &P_{n_j}v_{n_j,e} \to v_e^b \text{ weakly in } H^1(\Omega^b)
\end{align*}
\tag{3.50}
\]

**Theorem 3.1.** Let \( u_{n,e} \) be a solution of Problem (1.1), \( h_n \) and \( r_n \) satisfying (3.48). Then
\[
P_{n,e}v^a_{n,e} \to v_e^a, \quad \frac{\partial P_{n,e}v^a_{n,e}}{\partial y} \to \frac{\partial v_e^a}{\partial y} \text{ weakly in } L^2(\Omega^a) \tag{3.51}
\]
\[
\frac{\partial P_{n,e}v^a_{n,e}}{\partial x} \to 0 \text{ weakly in } L^2(\Omega^a) \tag{3.52}
\]
\[
v^b_{n,e} \to v_e^b \text{ weakly in } L^2(\Omega^b) \tag{3.53}
\]
\[
\frac{\partial v^b_{n,e}}{\partial y} \to 0 \text{ strongly in } L^2(\Omega^b) \tag{3.54}
\]
as \( n \to +\infty \) where \((v^a_e, v^b_e)\) is the unique solution of the following problem
\[
\begin{align*}
 &\text{Min} \{ \theta \int_{\Omega^a} \left| \frac{\partial v^a}{\partial y} \right|^2 + \frac{1}{\varepsilon^2} (1 - |v^a|^2)^2, \ v = (v^a, v^b) \in L^2(\Omega^a, H^1([0, 1])) \times H^1(\Omega^b); \\
 &v^b(x, 0) = v^a(x, 0), \text{ for almost every } x \in [-1, 1] \text{ and } v^b(x, -1) = g(x, -1), \ v^a(x, 1) = g(x, 1) \} \tag{3.55}
\end{align*}
\]
Moreover the energy converges, that is
\[
\lim_{n \to +\infty} E_{n,e}(v_{n,e}) = \theta \int_{\Omega^a} \left| \frac{\partial v^a_e}{\partial y} \right|^2 + \frac{1}{\varepsilon^2} (1 - |v^a_e|^2)^2. \tag{3.56}
\]
The proofs of these results are similar to those of the case of one period with simple modifications to adapt them to the case of two periods.

4 The general Case

We consider the same minimization problem over all functions in the space
\[
V_n = \{ u \in H^1(\Omega_n, \mathbb{R}^2), u = g \text{ on } \Gamma_{n,1} \text{ and } \frac{\partial u}{\partial v} = 0 \text{ on } \Gamma_{n,2} \}.
\]
Where \( g : \Gamma_{n,1} \to \mathbb{C} \) is a prescribed smooth map with \(|g| = 1\) on \( \Gamma_{n,1} \).
\( \Omega_n \) is a thin multidomain of \( \mathbb{R}^2 \), constituted of the section of an horizontal plane \( \Omega_n^a = (-1, 1] \times \Omega_n, 0[ \), with small
thickness \( h_n \) and \( n \) cylinders \( \Omega_n^i = [d_n^i - r_n, d_n^i + r_n] \times [0, 1] \) with small radiiuses \( r_n \), where \( 2r_n < |d_n^{i+1} - d_n^i| \) for all \( i = 0, \ldots, n \) which means that \( \Omega_n^i \) are disjoints. Set \( \Omega_n = \cup_{i=0}^{n-1} \Omega_n^i \). \( \Gamma_{n,1} \) is a subset of the boundary, \( \partial \Omega_n \), of the multidomain \( \Omega_n = \Omega_n^a \cup \Omega_n^b \). \( \Gamma_{n,1} = \cup_{i=0}^{n-1} \{d_n^i - r_n, d_n^i + r_n[\times \{1\}] \cup \{-1, 1\} \times -h_n, 0\} \) and \( \Gamma_{n,2} = \partial \Omega_n / \Gamma_{n,1} \). Set \( \Omega^a = [-1, 1[ \times [0, 1], \Omega^b = [-1, 1[ \times [-1, 0], \Omega = \Omega^a \cup \Omega^b \) and \( \sum = \Omega^a \cap \Omega^b \).

Denote by \( \chi_{\Omega_n^i} \) the characteristic function of \( \Omega_n^i \).

Our aim is to study the asymptotic behavior of the minimizing solutions of the Ginzburg-Landau system. To study the asymptotic behavior, as \( n \to +\infty \), of Problem (1.1) in the case where the number cylinders is not constant, assume that the volumes of \( \Omega_n^a \) and \( \Omega_n^b \) tend to zero and satisfy the following

\[
\lim_{n \to \infty} \frac{h_n}{r_n^n} = q \in [0, +\infty[. \tag{4.57}
\]

The homogenization result is obtained using a method of coordinates transformation, namely the periodic unfolding operator, and multiscale convergence.

Let us recall some definitions and properties that we will need in what follows of this section.

### 4.1 The periodic unfolding operator

The periodic unfolding operator was introduced by D. Cioranescu, A. Damlamian and G. Griso [8].

**Definition of the periodic unfolding operator and \( L^p \) properties:**

In \( \mathbb{R}^n \), let \( Y \) be a reference cell (ex. \( [0, 1]^n \), or more generally a set having the paving property with respect to a basis \( (b_1, \ldots, b_n) \) defining the periods). For \( y \in \mathbb{R}^n \), we denote \([y]_Y\) the unique integer combination \( \sum_{j=1}^n k_j b_j \) of the periods such that \( y - [y]_Y \) belongs to \( Y \) and we set \([y]_Y = y - [y]_Y \). \( \mathbb{Y} \).

**Definition 4.1.** Let \( \delta > 0 \), \( Y \) be a reference cell and \( u : \mathbb{Y}^N \to S \), where \( S \) is a set, the unfolding operator \( \tau_\delta^Y \) is defined as follows

\[
\tau_\delta^Y (u) : \mathbb{Y}^N \times \mathbb{Y}^N \to S
\]

\[
(x, y) \mapsto \tau_\delta^Y (u)(x, y) = u(\delta \frac{x}{|x|} + \delta y).
\]

One readily sees that

\[
\forall x \in \mathbb{Y}^N, \tau_\delta^Y (u)(x, \{\frac{x}{|x|}\}) = u(x).
\]

Moreover, \( \tau_\delta^Y (u) \) is invariant under the action on \( \mathbb{Z}^N \):

\[
\tau_\delta^Y (u)(x + \delta \xi, y - \xi) = \tau_\delta^Y (u)(x, y).
\]

If \( u : \mathbb{Y}^N \to S \) and \( f : S \to S' \), then

\[
\tau_\delta^Y (f \circ u) = f \circ \tau_\delta^Y (u).
\]

In particular if \( u : \mathbb{Y}^N \to S \) and \( v : \mathbb{Y}^N \to T \), the preceding property applied to the projections \( P : (u, v) \mapsto u \) and \( Q : (u, v) \mapsto v \) yields

\[
\tau_\delta^Y ((u, v)) = (\tau_\delta^Y (u), \tau_\delta^Y (v)).
\]

Therefore, if \( F : S \times T \to \mathbb{R} \),

\[
\tau_\delta^Y (F(u, v)) = F(\tau_\delta^Y (u), \tau_\delta^Y (v)). \tag{4.58}
\]

Useful particular cases are when \( S = \mathbb{R} \), \( T = \mathbb{R} \) and \( F : (s, t) \mapsto st \) and where \( S = \mathbb{Y}^N \), \( T = \mathbb{Y}^N \) and \( F \) is the dot product.

**Remark 4.1.** The previous statements allow to define the unfolded of an operator \( \tau_\delta^Y (u) \).
Proposition 4.1. [8]. For every $u \in L^1(\mathbb{R}^N)$, 
\[
\int_{\mathbb{R}^N} u(x)dx = \frac{1}{|Y|} \int_{\mathbb{R}^N \times Y} \tau_{\delta}^Y(u)(x,y)dxdy.
\]
In particular, if $1 \leq p < +\infty$ and $u \in L^p(\mathbb{R}^N)$, then $\tau_{\delta}^Y(u) \in L^p(\mathbb{R}^N \times Y)$, and
\[
\|\tau_{\delta}^Y(u)\|_{L^p(\mathbb{R}^N \times Y)} = |Y|^{-\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^N)}.
\]

If $\chi_A$ denotes the characteristic function of a measurable set $A$, the combination of Proposition 4.1 together with (4.58) yields

Proposition 4.2. [8]. Let $A \subset \mathbb{R}^N$ be a measurable. If $u \in L^1(A)$, then $\tau_{\delta}^Y(\chi_A)\tau_{\delta}^Y(u)$ is well-defined on $\mathbb{R}^N \times \mathbb{R}^N$, $\tau_{\delta}^Y(\chi_A)\tau_{\delta}^Y(u) \in L^1(\mathbb{R}^N \times Y)$, and
\[
\int_A u(x)dx = \frac{1}{|Y|} \int_{\mathbb{R}^N \times Y} \tau_{\delta}^Y(\chi_A)\tau_{\delta}^Y(u)(x,y)dxdy.
\]
In particular, if $1 \leq p < +\infty$ and $u \in L^p(A)$, then $\tau_{\delta}^Y(\chi_A)\tau_{\delta}^Y(u) \in L^p(\mathbb{R}^N \times \mathbb{R}^N)$, $\tau_{\delta}^Y(\chi_A)\tau_{\delta}^Y(u) \in L^p(\mathbb{R}^N \times Y)$, and
\[
\|\tau_{\delta}^Y(\chi_A)\tau_{\delta}^Y(u)\|_{L^p(\mathbb{R}^N \times Y)} = |Y|^{-\frac{1}{p}} \|u\|_{L^p(A)}.
\]

Remark 4.2. Proposition 4.2 shows that the natural domain of the unfolding $\tau_{\delta}^Y(u)$ of a function $u$ defined on $A$ is the unfolding of the set $A$. As we shall see later, $\tau_{\delta}^Y(\chi_A) \rightarrow \chi_A \times Y$ in $L^1_{loc}(\mathbb{R}^N \times Y)$ if $|\partial A| = 0$. The convergence is in $L^1(\mathbb{R}^N \times Y)$ if $A$ has a finite measure. If $\Omega$ is a bounded open set with a Lipschitz boundary, then there is a constant $C$ such that when $\delta$ is sufficiently small,
\[
\|\tau_{\delta}^Y(\chi_A) - \chi_{A \times Y}\|_{L^1(\mathbb{R}^N \times Y)} \leq C\delta.
\]

Unfolding locally summable functions
Since the unfolding operator has a local action, it is natural to examine its effect on locally summable functions.

Lemma 4.1. For every bounded open set $\Omega \subset \mathbb{R}^N \times \mathbb{R}^N$, and every $\delta_0 > 0$, there is $C \geq 1$ and a bounded open set $\Omega' \subset \mathbb{R}^N$, such that for every $u \in L^p_{\text{loc}}(\mathbb{R}^N)$, $1 \leq p < +\infty$, for every $\delta < \delta_0$,
\[
\|\tau_{\delta}^Y(u)\|_{L^p(\Omega')} \leq C|Y|^{-\frac{1}{p}} \|u\|_{L^p(\Omega)}. \tag{4.59}
\]

Proof. The case $p > 1$ follows from the case $p = 1$ applied to the function $|u|^p$. We can thus consider that $p = 1$. Assume first that $\Omega = \Omega_1 \times (\xi + Y)$. Define
\[
\Omega' = \{x \in \mathbb{R}^N \text{ s.t. } d(\Omega_1,x) \leq 2\delta_0(diam(Y) + |\xi|)\}. \tag{4.60}
\]
Note that if $x + \delta\xi \in \Omega_1$ and $y \in Y$, then $|x - \delta(\frac{1}{\delta_0})y - \delta_0\xi| = |\delta(\frac{1}{\delta_0})y - \delta_0\xi| < \delta$, and therefore $\delta(\frac{1}{\delta_0})y + \delta_0\xi \in \Omega$. This means that $\tau_{\delta}^Y(\chi_{\Omega'}) \geq \chi_{(\Omega_1 - \delta Y) \times Y}$. Therefore, by the group invariance of the unfolding and by Proposition 4.2
\[
\int_{\Omega_1 \times (Y + \xi)} |\tau_{\delta}^Y(u)|dxdy \leq \int_{(\Omega_1 - \delta Y) \times Y} |\tau_{\delta}^Y(u)|dxdy \\
\leq \int_{\mathbb{R}^N \times Y} \tau_{\delta}^Y(\chi_{\Omega})|\tau_{\delta}^Y(u)|dxdy = \int_{\Omega} |u|dxdy.
\]
In general, consider $\Omega = \Omega_1 \times \Omega_2$. Since $\Omega_2$ is bounded, there is a finite collection $(\xi_i)_{i=1}^m \in \mathbb{Z}$ such that $\Omega_2 \subset \bigcup_{i=1}^m (Y + \xi_i)$. 

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Indeed, in this case we can assume that \(|u_\delta| < 1\) (otherwise we consider the sequence \(\arctg(u_\delta)\)) and \(u_\delta \rightharpoonup \hat{u} \) *-weakly in \(\mathcal{M}(\mathbb{R}^N \times \mathbb{R}^N)\), therefore we can apply the previous lemma.

**Remark 4.3.** In particular if

\[ \tau^Y_\delta (u_\delta) \rightharpoonup \hat{u} \quad \text{a. e.,} \]

then \(\hat{u}\) is \(Y\)-periodic.
Unfolding operator and gradients:
In this paragraph we study the properties of the unfolding operator applied on the gradient of some functions. If \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \) then by Proposition 4.3, \( \tau^Y_\delta(u) \in L^p_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^N) \) and \( \tau^Y_\delta(\nabla u) \in L^p_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^N) \). Moreover, for every test function \( \varphi \in \mathcal{D}(\mathbb{R}^N \times \mathbb{R}^N) \)
\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} \nabla_x \varphi \, \tau^Y_\delta(u) \, dx \, dy = \int_{\mathbb{R}^N \times \mathbb{R}^N} \nabla_x \varphi(x,y) u(\delta \frac{x}{\delta Y} y + \delta y) \, dx \, dy
\]
\[
= - \int_{\mathbb{R}^N \times \mathbb{R}^N} \varphi(x,y) \delta \nabla u(\delta \frac{x}{\delta Y} y + \delta y) \, dx \, dy
\]
\[
= - \int_{\mathbb{R}^N \times \mathbb{R}^N} \varphi \delta \tau^Y_\delta(\nabla u) \, dx \, dy,
\]
therefore \( \tau^Y_\delta(u) \) is weakly differentiable with respect to \( Y \) and
\[
\delta \tau^Y_\delta(\nabla u) = \nabla_y (\tau^Y_\delta(u)).
\]

**Proposition 4.4.** [8]. Let \( (u_\delta)_\delta \) be sequence of \( W^{1,p}_{\text{loc}}(\mathbb{R}^N) \) and let \( \hat{u} \in L^p_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^N) \). If \( (u_\delta)_\delta \) is bounded in \( L^p_{\text{loc}}(\mathbb{R}^N) \), \( (\delta \nabla u_\delta)_\delta \) is bounded in \( L^p_{\text{loc}}(\mathbb{R}^N) \) and
\[
\tau^Y_\delta(u_\delta) \rightharpoonup \hat{u} \quad \text{weakly in } L^p_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^N) \quad \text{as } \delta \to 0,
\]
then
\[
\delta \tau^Y_\delta(\nabla u_\delta) \rightharpoonup \nabla_y \hat{u} \quad \text{weakly in } L^p_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^N) \quad \text{as } \delta \to 0.
\]
Moreover \( \hat{u} \) is \( Y \)-periodic in \( y \).

Now let us come back to our problem and see how to use the unfolding operator introduced below. Set
\[
T_n := [1,1[ x ] - 1,0[ \to 1,1] \times [-h_n,0[]
\]
defined by \( T_n(x_1,x_2) = (x_1,h_n x_2) \), and
\[
v_{n,e}(x,y) = \begin{cases} 
\tau^Y_{n,e}(u_{n,e})(x,y) & \text{for } (x,y) \in \Omega^n_{n,e} \times Y, \\
u_{n,e}(T_n(x)) & \text{for } x = (x_1,x_2) \in \Omega^n.
\end{cases}
\]
where \( Y \) is a reference cell in \( \mathbb{R} \) and \( \tau^Y_{n,e}(u_{n,e})(x,y) = u_{n,e}(r_n \frac{x_1}{r_n} y + r_n y) \).

Then the function \( v_{n,e} \) is a solution of the following minimization problem
\[
\min_{v \in V} \frac{h_n}{2} \int_{\Omega^n} \left| \frac{\partial v}{\partial x} \right|^2 + \frac{1}{h_n} \left| \frac{\partial v}{\partial y} \right|^2 + \frac{1}{2 \varepsilon^2} \left( 1 - |v|^2 \right)^2 + \frac{r_n^2}{2 |y|} \left| \int_{\Omega^n} \nabla^Y v(\nabla u)^2 + \frac{1}{2 \varepsilon^2} \left( 1 - |\tau^Y_{n,e}(u)|^2 \right)^2 \right|
\]
where
\[
V = \{ v = (v^a,v^b) \in H^1(\Omega^n_{n,e}, C) \times H^1(\Omega^n_{n,e}, C) ; \ v^a = g \ on \ \cup_{k=0}^{n} \{ a^k_n - r_n, a^k_n + r_n \} \times \{ 1 \} , \\
v^b = g \ on \ \{ \pm 1 \} \times [-1,0] , \ and \ v^0 = v^b \ on \ \cup_{k=0}^{n} \{ a^k_n - r_n, a^k_n + r_n \} \times \{ 0 \} \}.
\]

In order to establish our aim results we are in the necessity of an additional assumption on the characteristic function \( \chi_{\Omega^n} \), thus we suppose the existence of a subset \( \Omega' \) of \( \Omega^n \) such that the following convergence holds
\[
\tau^Y_{n,e}(\chi_{\Omega^n}) \rightharpoonup \chi_{\Omega' \times Y} \quad \text{almost everywhere.}
\]

In the following we present the main results

**Proposition 4.5.** Let \( u_{n,e} \) be a solution of Problem (1.1), \( h_n \) and \( r_n \) satisfying (4.58). Then there exists an increasing sequence of positive integer numbers \( \{ n_j \}_{j \in \mathbb{N}} \) and \( v_e \in H^1(\Omega'^n) \times H^1(\Omega'^b) \) such that
\[
v_{n_j,e} \rightharpoonup v_e \quad \text{weakly in } H^1(\Omega'^n) \times H^1(\Omega'^b)
\]
Proof. The function $v_{n,e}$ is a minimizer of the energy, which is finite and bounded. We conclude the existence of a constant $c$, constant independent of $n$, such that

$$\|v_{n,e}\|_{H^1(\Omega)} \leq c. \quad (4.63)$$

Then, the proposition is a simple conclusion of this inequality. \qed

Theorem 4.3. Let $u_{n,e}$ be a solution of Problem (1.1), $h_n$ and $r_n$ satisfying (4.58). Then

$$v_{n,e}^a \to v_e^a, \quad \frac{\partial v_{n,e}^a}{\partial x_2} \to \frac{\partial v_e^a}{\partial x_2} \text{ weakly in } L^2(\Omega^2)$$

$$\frac{\partial v_{n,e}^a}{\partial x_1} \to 0 \text{ weakly in } L^2(\Omega^2)$$

$$v_{n,e}^b \to v_e^b \text{ strongly in } L^2(\Omega^b)$$

$$\frac{\partial v_{n,e}^b}{\partial x_2} \to 0 \text{ strongly in } L^2(\Omega^b)$$

as $n \to +\infty$ where $(v_e^a, v_e^b)$ is the first term of the unique solution $(v_e, \hat{v})$ of the following problem

$$\min \{ q \int_{\Omega^a} \{ |\nabla v^a(x)|^2 + \frac{1}{\varepsilon^2} (1 - |v^a(x)|^2)^2 \} dx$$

$$+ \int_{\Omega^a \times Y} \{ |\nabla_x v^a(x) + \nabla_y \hat{v}(x,y)|^2 + \frac{1}{\varepsilon^2} (1 - |v^a(x) + \hat{v}(x,y)|^2)^2 \} dxdy,$$

$v = (v^a, v^b) \in H^1(\Omega^a) \times H^1(\Omega^b)$; $v^a(x_1,0) = v^a(x_1,0)$, for almost every $(x_1,0) \in \Omega^a$; $v^b(x_1,-1) = g(x_1,-1)$, $v^b(x_1,1) = g(x_1,1)$

and $\hat{v} \in L^2(\Omega; H^1_{per}(Y))$ with $\int_Y \hat{v}(x,y)dy = 0.$

Moreover the energy converges, that is

$$\lim_{n \to +\infty} E_{n,e}(v_{n,e}) = \frac{q}{2} \int_{\Omega^a} \{ |\nabla v^a(x)|^2 + \frac{1}{\varepsilon^2} (1 - |v^a(x)|^2)^2 \} dx$$

$$+ \frac{1}{2|Y|} \int_{\Omega^a \times Y} \{ |\nabla_x v^a(x) + \nabla_y \hat{v}(x,y)|^2 + \frac{1}{\varepsilon^2} (1 - |v^a(x) + \hat{v}(x,y)|^2)^2 \} dxdy. \quad (4.69)$$

Proof. Let us give some a priori estimates of the norm of $v_{n,e}$, in order to introduce to the proof of Theorem 4.3. \qed

Proposition 4.6. Let $v_{e,n}$ be a solution of problem (1.1) under the assumptions of Theorem 4.3. Then, there exists a constant $c$ such that

$$\|v_{n,e}\|_{H^1(\Omega^a)} \leq c \quad (4.70)$$

$$\|\nabla_x v_{n,e}\|_{L^2(\Omega^a)}^2 \leq c \quad (4.71)$$

$$\|v_{n,e}\|_{L^2(\Omega^b)}^2 \leq c \quad (4.72)$$

$$\|\partial_1 v_{n,e} - \frac{1}{h_n} \partial_2 v_{n,e}\|_{L^2(\Omega^b)}^2 \leq c \quad (4.73)$$

for every $n$. 
Theorem 4.2 and Proposition 4.2, there is $\hat{u}$.

First we see that

we have by Proposition 4.2 $

\tau$

We are now going to prove that the sequence

$\phi$

From the previous statement and the density of the test functions

$\phi$

For

If

We are now ready to obtain a first homogenized equation. For the test function

$\tau$

such that

$\eta$

Similarly, the sequence $\tau_n^\nu(\nabla u_{n,e})$ is bounded in $L^2(\Omega \times Y)$. Up to a subsequence, there is $\eta \in L^2(\Omega \times Y)$ such that

$\tau_n^\nu(\nabla u_{n,e}) \rightarrow \eta$ weakly in $L^2(\Omega \times Y)$.

We are now ready to obtain a first homogenized equation. For the test function $\varphi_{n,e} \in H_0^1(\Omega_n)$, that we extend to $\Omega$, we have by Proposition 4.2

$$
\frac{1}{|Y|} \int_{\Omega^e \times Y} \tau_n^\nu(\chi_{\Omega_n}) \tau_n^\nu(\nabla u_{n,e}) \tau_n^\nu(\nabla \varphi_{n,e}) dxdy = \frac{1}{2\varepsilon^2} \int_{\Omega^e} (1 - |u_{n,e}|^2) u_{n,e} \varphi_{n,e} d\Omega.
$$

If $\varphi_{n,e} = \Psi \in \mathcal{D}(\Omega^e)$, and if $n \rightarrow +\infty$, since by Theorem 4.1, $\tau_n^\nu(\nabla \Psi) \rightarrow \nabla \Psi \otimes 1$ strongly in $L^2(\Omega^e \times Y)$, we obtain

$$
\frac{1}{|Y|} \int_{\Omega^e \times Y} \eta(x,y) \nabla \Psi dxdy = \frac{1}{2\varepsilon^2} \int_{\Omega^e} (1 - |u_{n,e}|^2) u_{n,e} \Psi d\Omega.
$$

For $\varphi \in \mathcal{D}(\Omega^e)$ and $\Psi \in H_{per}^1(Y)$, consider $\varphi_{n,e}(x) = \frac{1}{n} \varphi(x) \psi(nx)$. The sequence $\varphi_{n,e}$ converges weakly to 0 in $H^1(\Omega^e)$ while the unfolded sequence $\tau_n^\nu(\nabla \varphi_{n,e})$ converges strongly to $\varphi(x) \nabla \Psi(y)$ in $L^2(\Omega; H_{per}^1(Y))$, hence

$$
\frac{1}{|Y|} \int_{\Omega^e \times Y} \eta(x,y) \nabla \varphi(x,y) dxdy = 0.
$$

From the previous statement and the density of the test functions $\varphi_{n,e}$ in $L^2(\Omega; H_{per}^1(Y))$, we deduce that for all $\varphi \in L^2(\Omega; H_{per}^1(Y))$

$$
\frac{1}{|Y|} \int_{\Omega^e \times Y} \eta(x,y) \nabla \varphi(x,y) dxdy = 0.
$$

We are now going to prove that the sequence $\tau_n^\nu(\nabla \varphi_{n,e})$ converges in fact strongly.

First we see that

$$
\int_{\Omega^e \times Y} \tau_n^\nu(\chi_{\Omega_n}) \nabla \varphi_{n,e} \nabla \varphi_{n,e} + \nabla \varphi_{n,e} \nabla \varphi_{n,e} \tau_n^\nu(\nabla \varphi_{n,e})^2 dxdy

\begin{align*}
&\leq \left( \int_{\Omega^e \times Y} \tau_n^\nu(\chi_{\Omega_n}) (1 + |\nabla \varphi_{n,e}| + |\tau_n^\nu(\nabla \varphi_{n,e})|)^2 dxdy \right)^{1 - \frac{2}{p}}
\end{align*}

\frac{1}{|Y|} \int_{\Omega^e \times Y} \eta(x,y) \nabla \varphi(x,y) dxdy = 0.
$$
Finally we prove that up to a subsequence, in $L^2(\Omega^d \times Y)$, thanks to the fact that $(\tau_n^Y(\chi_{\Omega^d}))_n$ is bounded in $L^\infty(\Omega^d \times Y)$ and converges almost everywhere, the first factor in the right-hand side is bounded as $n \to +\infty$, and we have

$$
\int_{\Omega^d \times Y} \tau_n^Y(\chi_{\Omega^d})|\nabla v_{n,e} + \nabla_y \hat{u} - \tau_n^Y(\nabla v_{n,e})|^2 \, dxdy \\
\leq c \int_{\Omega^d \times Y} \tau_n^Y(\chi_{\Omega^d})|\nabla v_{n,e} + \nabla_y \hat{u} - \tau_n^Y(\nabla v_{n,e})|^2 \, dxdy.
$$

We need to prove that the right-hand side goes to 0 as $n \to +\infty$. First by Proposition 4.2,

$$\frac{1}{|Y|} \int_{\Omega^d \times Y} \tau_n^Y(\chi_{\Omega^d})|\tau_n^Y(\nabla v_{n,e})|^2 \, dxdy = \frac{1}{2|\varepsilon|} \int_{\Omega^d} (1 - |v_{n,e}|^2) v_{n,e} \nabla v_{n,e} \, dx \to \frac{1}{2|\varepsilon|} \int_{\Omega^d} (1 - |v|)^2 v \nabla v \, dx.
$$

While the weak convergence of $\tau_n^Y(\chi_{\Omega^d})\tau_n^Y(\nabla v_{n,e})$ to $\eta$ previously established yields

$$\frac{1}{|Y|} \int_{\Omega^d \times Y} \tau_n^Y(\chi_{\Omega^d}) \tau_n^Y(\nabla v_{n,e})(\nabla_x v_{n,e} + \nabla_y \hat{u}) \, dxdy \to \frac{1}{|Y|} \int_{\Omega^d \times Y} \eta(x,y)(\nabla_x v_{e} + \nabla_y \hat{u}) \, dxdy = \frac{1}{2|\varepsilon|} \int_{\Omega^d} (1 - |v|^2) v \nabla v \, dx.
$$

Now, it remains to prove that

$$\int_{\Omega^d \times Y} \tau_n^Y(\chi_{\Omega^d})(\nabla_x v_{e} + \nabla_y \hat{u})(\nabla_x v_{e} + \nabla_y \hat{u} - \tau_n^Y(\nabla v_{n,e})) \, dxdy \to 0.
$$

Since

$$\nabla_x v_{e} + \nabla_y \hat{u} - \tau_n^Y(\nabla v_{n,e}) \to 0
$$

weakly in $L^2(\Omega^d \times Y)$, and by the Lebesgue dominated convergence Theorem we prove that

$$\tau_n^Y(\chi_{\Omega^d})\tau_n^Y(\nabla v_{n,e}) \to \chi_{\Omega^d \times Y}(\nabla_x v_{e} + \nabla_y \hat{u})
$$

strongly in $L^2(\Omega^d \times Y)$ as $n \to +\infty$.

Finally we prove that

$$\eta(x,y) = \nabla_x v_{e} + \nabla_y \hat{u},
$$

this will follow from the strong convergence

$$\tau_n^Y(\chi_{\Omega^d})\tau_n^Y(\nabla v_{n,e}) \to \chi_{\Omega^d \times Y}(\nabla_x v_{e} + \nabla_y \hat{u})
$$

in $L^2(\Omega^d \times Y)$.

Up to a subsequence,

$$\tau_n^Y(\chi_{\Omega^d})\tau_n^Y(\nabla v_{n,e}) \to \chi_{\Omega^d \times Y}(\nabla_x v_{e} + \nabla_y \hat{u}) \quad \text{almost everywhere}.
$$

References

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