Solving the Intuitionistic fuzzy fractional equation by means of the homotopy analysis method

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Abstract
In this work, we use the homotopy analysis method in order to solve a fractional differential equations with intuitionistic fuzzy initial data under generalized fuzzy Caputo derivative.

Keywords: The homotopy analysis method, Generalized intuitionistic fuzzy derivative, Caputo fractional derivative, Hukuhara difference.

1 Introduction
In this work, we will used the homotopy analysis in order to solve the following problem
\[
\begin{align*}
\text{gH}D^\gamma_t u(x,t) &= \frac{\partial^2}{\partial x^2} u(x,t), \quad 0 \leq x, t < 1 \\
u(x,0) &= \phi(x) \\
\frac{\partial}{\partial x} u(x,0) &= \psi(x)
\end{align*}
\]
(1.1)

where \(1 \leq \gamma < 2\), the operator \(\text{gH}D^\gamma_t\) denote the Caputo fractional generalized derivative of order \(\gamma\), and \(\phi, \psi : [0, 1] \rightarrow \mathbb{F}_1(\mathbb{R})\).

The concept of intuitionistic fuzzy sets is introduced by K. Atanassov [1]. The authors in [2] built the concept of intuitionistic fuzzy metric space and intuitionistic fuzzy numbers. In [3] S. Melliani introduce the extension of Hukuhara difference in the intuitionistic fuzzy case. The authors in [5] introduce the concept of intuitionistic fuzzy Laplace’s transform. Muhammet Kurulay in [4] solve the fractional nonlinear Klein-Gordon equation by means of the homotopy analysis method. From this end idea we introduce in this paper the concept of generalized intuitionistic fuzzy Caputo derivative, and we give a solution of an intuitionistic fuzzy fractional equation by mean the homotopy analysis method.

This paper is organized as follows. In section 2 we recall some concept concerning the intuitionistic fuzzy numbers. The concept of generalized intuitionistic fuzzy derivative and generalized intuitionistic fuzzy Caputo derivative, takes place in section 3. In section 4 we present the intuitionistic fuzzy Laplace’s transform. The homotopy analysis method is presented in section 5. Finally in section 6 we give the solution of the problem 1.1.

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2 preliminaries

Definition 2.1. [2] The set of all intuitionistic fuzzy numbers is given by

\[ \mathcal{IF}_1 = \mathcal{IF}_1(\mathbb{R}) = \left\{ <u,v>: \mathbb{R} \rightarrow [0,1]^2, 0 \leq u + v \leq 1 \right\} \]

with the following conditions:

1. For Each \( <u,v> \in \mathcal{IF}_1 \) is normal, i.e., \( \exists x_0, x_1 \in \mathbb{R} \), such that \( u(x_0) = 1 \) and \( v(x_1) = 1 \).
2. For Each \( <u,v> \in \mathcal{IF}_1 \) is a convex intuitionistic set, i.e, \( u \) is fuzzy convex and \( v \) is fuzzy concave.
3. For Each \( <u,v> \in \mathcal{IF}_1 \), \( u \) is a lower continuous and \( v \) is appear continuous.
4. cl \( \{ x \in \mathbb{R}, v(x) \leq \alpha \} \) is bounded.

Definition 2.2. [2] For \( \alpha \in [0,1] \), we define the appear and lower \( \alpha \)-cut by

\[ [<u,v>]_\alpha = \{ x \in \mathbb{R}, u(x) \geq \alpha \} \]
\[ [<u,v>]^\alpha = \{ x \in \mathbb{R}, v(x) \leq 1 - \alpha \} \]

Definition 2.3. The intuitionistic fuzzy zero is intuitionistic fuzzy set defined by

\[ \tilde{0}(x) = \begin{cases} (1,0) & x = 0 \\ (0,1) & x \neq 0 \end{cases} \]

Proposition 2.1. [2] We can write

\[ [<u,v>]_\alpha = [<[u,v]>]^\alpha (\alpha), [<[u,v]>]_{1-\alpha} (\alpha) \]
\[ [<u,v>]^\alpha = [<[u,v]>]^\alpha (\alpha), [<[u,v]>]_{1-\alpha} (\alpha) \]

Remark 2.1. We can write \( [<u,v>]_\alpha = [u]^\alpha \) and \( [<u,v>]^\alpha = [1-v]^\alpha \), in the fuzzy case

Proposition 2.2. [2] For all \( <u,v>, <u',v'> \in \mathcal{IF}_1 \), we have

\[ <u,v> = <u',v'> \iff [<u,v>]_\alpha \quad \text{and} \quad [<u,v>]^\alpha, \forall \alpha \in [0,1] \]

We define two operations on \( \mathcal{IF}_1 \) by

\[ <u,v> \oplus <u',v'> = <u \lor v, u' \land v'>, \forall <u,v>, <u',v'> \in \mathcal{IF}_1 \]

\[ \lambda <u,v> = \lambda u, \lambda v, \forall \lambda \in \mathbb{R}, \forall <u,v> \in \mathcal{IF}_1 \]

According to Zadeh extension, we have

\[ [<u,v> \oplus <u',v'>]_\alpha = [<u,v>]_\alpha + [<u',v'>]_\alpha \]
\[ [<u,v> \oplus <u',v'>]^\alpha = [<u,v>]^\alpha + [<u',v'>]^\alpha \]
\[ [\lambda <u,v>]_\alpha = \lambda [<u,v>]_\alpha \]
\[ [\lambda <u,v>]^\alpha = \lambda [<u,v>]^\alpha \]

Theorem 2.1. [2] Let \( \mathcal{M} = \{ M_\alpha, M^\alpha, \alpha \in [0,1] \} \) be a family of subsets in \( \mathbb{R} \) stisfying the following conditions

1. \( \alpha \leq s \Rightarrow M_s \subseteq M_\alpha \) and \( M^\alpha \subseteq M^s \), for each \( \alpha, s \in [0,1] \).
2. \( M_\alpha \) and \( M_s \) are nonempty compact convex sets in \( \mathbb{R} \) for each \( \alpha \in [0,1] \).
3. for any nondecreasing sequence $\alpha_i \rightarrow \alpha$ on $[0, 1]$, we have $M_\alpha \subseteq [0, 1] = \bigcap_\alpha M_\alpha$ and $M^\alpha = \bigcap_\alpha M^\alpha$.

We define $u$ and $v$ by

$$
\begin{align*}
  u(x) &= \begin{cases} 
    0, & x \notin M_0 \\
    \sup_{\alpha \in [0, 1]} M_\alpha & x \in M_0
  \end{cases} \\
  v(x) &= \begin{cases} 
    1, & x \notin M^0 \\
    1 - \sup_{\alpha \in [0, 1]} M_\alpha & x \in M^0
  \end{cases}
\end{align*}
$$

Then

$$< u, v > \in I^F_1$$

with $M_\alpha = [< u, v >]_\alpha$ and $M^\alpha = [< u, v >]^\alpha$.

Remark 2.2. [2]

1. The family $\{ [< u, v >]_\alpha, [< u, v >]^\alpha, \alpha \in [0, 1] \}$ satisfying (1) -- (3) of the previous theorem.

2. For all $\alpha \in [0, 1],
\quad [< u, v >]_\alpha \subseteq [< u, v >]^\alpha$

Theorem 2.2. [2] On $I^F_1$ we can define the metric

$$
\begin{align*}
  d_\alpha ((u, v), (z, w)) &= \frac{1}{4} \sup_{0 < \alpha \leq 1} \left| \left[(u, v)^+\right]_r (\alpha) - \left[(z, w)^+\right]_r (\alpha) \right| \\
  &\quad + \frac{1}{4} \sup_{0 < \alpha \leq 1} \left| \left[(u, v)^-_l\right]_f (\alpha) - \left[(z, w)^-_l\right]_f (\alpha) \right| \\
  &\quad + \frac{1}{4} \sup_{0 < \alpha \leq 1} \left| \left[(u, v)^+_l\right]_f (\alpha) - \left[(z, w)^+\right]_r (\alpha) \right| \\
  &\quad + \frac{1}{4} \sup_{0 < \alpha \leq 1} \left| \left[(u, v)^-_r\right]_f (\alpha) - \left[(z, w)^-_l\right]_f (\alpha) \right|
\end{align*}
$$

and

$$
\begin{align*}
  d_p (< u, v >, < u', v' >) &= \left( \frac{1}{4} \int_0^1 \left| [< u, v >]_l^+ (\alpha) - [< u', v' >]_l^+ (\alpha) \right|^p dt \right. \\
  &\quad + \frac{1}{4} \int_0^1 \left| [< u, v >]_l^- (\alpha) - [< u', v' >]_l^- (\alpha) \right|^p dt \\
  &\quad + \frac{1}{4} \int_0^1 \left| [< u, v >]_r^- (\alpha) - [< u', v' >]_r^- (\alpha) \right|^p dt \\
  &\quad + \left. \frac{1}{4} \int_0^1 \left| [< u, v >]_r^+ (\alpha) - [< u', v' >]_r^+ (\alpha) \right|^p dt \right)^{\frac{1}{p}}
\end{align*}
$$

For $p \in [1, \infty)$, we have $(I^F_1, d_p)$ is a complete metric space.

Definition 2.4. [4] The mittag-leffler function is given by

$$
E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + ak)} \quad \alpha \in \mathbb{C}, \Re(\alpha) > 0, z \in \mathbb{C} \quad (2.2)
$$

and its general form

$$
E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + ak)} \quad \alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, z \in \mathbb{C} \quad (2.3)
$$
3 The generalized Hukuhara derivative of an intuitionistic fuzzy-valued function

The concept of intuitionistic fuzzy Hukuhara difference is introduced by the authors in [3], in this paper we will give the definition of generalized Hukuhara difference between two intuitionistic fuzzy number.

**Definition 3.1.** The generalized Hukuhara difference of two fuzzy number \(<u,v>, <u',v'>\in IF_1\) is defined as follows

\(<u,v>-gH<u',v'>=<z,w>\iff\begin{cases} <u,v>=<u',v'>+<z,w> \\
\text{or} \\
<u',v'>=<u,v>+(−1)<z,w>
\end{cases}

Note that the \((α,β)\)-level representation of fuzzy-valued function \(f: [0,T] \rightarrow IF_1\) expressed by \([f]_{α} = [f_{α,l}, f_{α,r}]\) and \([f]^{α} = [f^{α,l}, f^{α,r}]\)

**Definition 3.2.** The generalized Hukuhara derivative of a intuitionistic fuzzy-valued function \(f: [0,T] \rightarrow IF_1\) at \(t_0\) is defined as

\[ f'_g(t_0) = \lim_{t \to t_0} \frac{f(t) - g_{H}(t_0) f(t_0)}{t - t_0} \]

if \(f'_g(t_0) \in IF_1\), we say that \(f\) is generalized Hukuhara differentiable at \(t_0\).

Also we say that \(f\) is \([(i) - gH]\)-differentiable at \(t_0\) if

\[
\begin{cases}
(f'_{gH})^α &= (f_{α,l})', (f_{α,r})' \\
(f'_{gH})^β &= (f_{β,l})', (f_{β,r})'
\end{cases}
\]

And that \(f\) is \([(ii) - gH]\)-differentiable at \(t_0\) if

\[
\begin{cases}
(f'_{gH})^α &= (f_{α,r})', (f_{α,l})' \\
(f'_{gH})^β &= (f_{β,r})', (f_{β,l})'
\end{cases}
\]

**Remark 3.1.** We can defined the generalized derivative of higher order by

\[
\begin{cases}
f^{(0)} = f \\
f^{(n)}_{gH} = (f^{(n-1)})'_{gH}, \quad \forall n \in \mathbb{N}
\end{cases}
\]

**Theorem 3.1.** Let \(f(t)\) and \(f'(t)\) are two differentiable intuitionistic fuzzy-valued functions. We set \([f(t)]_α = [f_{α,l}(t), f_{α,r}(t)]\) and \([F(t)]^β = [f^β(t), f^β(t)]\), where \(0 \leq α + β \leq 1\)

- Let \(f(t)\) and \(f'(t)\) be (i)-differentiable, or, let \(f(t)\) and \(f'(t)\) be (ii)-differentiable; then: \(f_{α,l}(t), f_{α,r}(t), f^β(t)\) and \(g^β(t)\) have first-order and second-order derivatives and

\[
\{[f''(t)]_α = [f''_{α,l}(t), f''_{α,r}(t)]\}_{f''(t)}^β = [f''^β(t), g''^β(t)]
\]

- Let \(f(t)\) be (i)-differentiable and \(f'(t)\) be (ii)-differentiable, or, let \(f(t)\) be (ii)-differentiable and \(f'(t)\) be (ii)-differentiable; then: \(f_{α,l}, g_{α}, f^β\) and \(g^β\) have first-order and second-order derivatives and

\[
\{[f''(t)]_α = [g''_{α,l}(t), f''_{α,r}(t)]\}_{f''(t)}^β = [g''^β(t), f''^β(t)]
\]
Proof. Just use the proof of theorem 3.1 in [6] for \( [f(t)]_\alpha \) and \( [f(t)]_\beta \).

Definition 3.3. Let \( f : (0, T) \rightarrow \mathbb{F}_1 \). We say that \( f \) of classe \( \mathcal{C}^m \), \( m \in \mathbb{N} \), if \( f_{gh}^{(m)} \) exists and continues, by respect to metric \( d_\infty \).

Now if the \( \alpha \)-levels of \( f : (0, T) \rightarrow \mathbb{F}_1 \), are given by \( [f]_\alpha = [f_{\alpha,l}, f_{\alpha,r}] \) and \( [f]_\beta = [f_{\beta,l}, f_{\beta,r}] \) and \( f_{\alpha,l}, f_{\alpha,r}, f_{\beta,l}, f_{\beta,r} \) are Riemann integrable on \([0, T]\). Since the family

\[
\left\{ [f_{\alpha,l}, f_{\alpha,r}], [f_{\beta,l}, f_{\beta,r}] \right\}
\]

built an intuitionistic element and the integrals preserve the monotony then by the Theorem 2.1 the family

\[
\left\{ \left[ \int_{[0,T]} f_{\alpha,l}, \int_{[0,T]} f_{\alpha,r} \right], \left[ \int_{[0,T]} f_{\beta,l}, \int_{[0,T]} f_{\beta,r} \right] \right\}
\]

define an intuitionistic fuzzy element, which is the integral of \( f \) on \([0, T]\), we denote \( \int_{[0,T]} f \).

Definition 3.4. Let \( f : [0, T] \rightarrow \mathbb{F}_1 \) be a intuitionistic fuzzy-valued function, we say that \( f \) is integrable on \([0, T]\) if \( f_{\alpha,l}, f_{\alpha,r}, f_{\beta,l}, f_{\beta,r} \) defined in the previous are integrable on \([0, T]\).

4 Intuitionistic fuzzy generalized caputo-derivative

Let \( f : [0, T] \rightarrow \mathbb{F}_1 \) be a intuitionistic fuzzy-valued integrable function on \([0, T]\), and \( \delta \in (m-1, m] \) and \( m \in \mathbb{N}^* \) it’s \((\alpha, \beta)\)-levels are defined by \( [f]_\alpha = [f_{\alpha,l}, f_{\alpha,r}] \) and \( [f]_\beta = [f_{\beta,l}, f_{\beta,r}] \)
where \( f_{\alpha,l}, f_{\alpha,r}, f_{\beta,l}, f_{\beta,r} \in \mathcal{C}^m([0, T]) \).

We set

\[
M_\alpha = \left[ \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-m-1} \left( f_{\alpha,l} \right)^{(m)} (s) \cdot \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-m-1} \left( f_{\alpha,r} \right)^{(m)} (s) \right]
\]

and

\[
M_\beta = \left[ \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-m-1} \left( f_{\beta,l} \right)^{(m)} (s) \cdot \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-m-1} \left( f_{\beta,r} \right)^{(m)} (s) \right]
\]

Proposition 4.1. The the family \( \{M_\alpha, M_\beta, \alpha, \beta \in [0, 1]\} \) defined an intuitionistic fuzzy element. Proof. Just use the Theorem 2.1

Definition 4.1. The intuitionistic fuzzy preceding item is called the generalized caputo derivative of \( f \), we denote \( D^\alpha f \).

we say that \( f \) is \( c^f[(i) - gH]\)-differentiable at to if

\[
\left[ \varepsilon H D^\delta f \right]_\alpha = \left[ D^\delta f_{\alpha,l}, D^\delta f_{\alpha,r} \right]
\]

and that \( f \) is \( c^f[(ii) - gH]\)-differentiable at \( t_0 \) if

\[
\left[ \varepsilon H D^\delta f \right]_\alpha = \left[ D^\delta f_{\alpha,r}, D^\delta f_{\alpha,l} \right]
\]

\[
\left[ \varepsilon H D^\delta f \right]_\beta = \left[ D^\delta f_{\beta,r}, D^\delta f_{\beta,l} \right]
\]
As in the previous definition we will give the definition of intuitionistic fuzzy fractional Riemann-Liouville integral. If the \((\alpha, \beta)\)-levels of \(f : (0, T) \rightarrow \mathbb{F}_1\), are given by \([f]_\alpha = [f_{a, l}, f_{a, r}]\) and \([f]_\beta = [f^{\beta, l}_1, f^{\beta, r}_1]\) and \(f_{a, l}, f_{a, r}, f^{\beta, l}_1, f^{\beta, r}_1\) are Riemann integrable on \((0, T)\). Since the family
\[
\left\{ [f_{a, l}, f_{a, r}], \; [f^{\beta, l}_1, f^{\beta, r}_1] \right\}
\]
built an intuitionistic element and the integrals preserve the monotony then by Theorem 2.1 the family
\[
\left\{ \mathcal{L}_\alpha, \mathcal{L}_\beta : \alpha + \beta \in [0, 1] \right\}
\]
where
\[
\mathcal{L}_\alpha = \left\{ \frac{1}{\Gamma(\delta)} \int_{(0, t]} (t-s)^{\delta-1} f_{a, l}(s) \frac{1}{\Gamma(\delta)} \int_{(0, t]} (t-s)^{\delta-1} f_{a, r}(s) \right\}
\]
and
\[
\mathcal{L}_\beta = \left\{ \frac{1}{\Gamma(\delta)} \int_{(0, t]} (t-s)^{\delta-1} f^{\beta, l}_1(s) \frac{1}{\Gamma(\delta)} \int_{(0, t]} (t-s)^{\delta-1} f^{\beta, r}_1(s) \right\}
\]
define an intuitionistic fuzzy element, which is the Riemann-Liouville fractional integral of \(f\) on \((0, T)\), we denote
\[
\frac{1}{\Gamma(\delta)} \int_{(0, t]} (t-s)^{\delta-1} f(s) ds
\]

**Definition 4.2.** The Riemann-Liouville fractional integral of \(f\) on \((0, T)\), defined as
\[
I^\delta f(t) = \frac{1}{\Gamma(\delta)} \int_{(0, t]} (t-s)^{\delta-1} f(s) ds
\]
where \(\delta \in (m-1, m)\)

### 5 The intuitionistic fuzzy Laplace transform

In this section \(f : [a, \infty) \rightarrow \mathbb{F}_1\) is an intuitionistic fuzzy-valued function where \(a > 0\). We set \([f(t)]_\alpha = [f_{a, l}(t), f_{a, r}(t)]\) and \([f(t)]_\beta = [f^{\beta, l}_1(t), f^{\beta, r}_1(t)]\), where \(0 \leq \alpha + \beta \leq 1\). assume that these four function are Riemann-integrable on \([a, b]\], and assume there are four positive function \(M_1(\alpha), M_2(\alpha), N_1(\beta)\) and \(N_2(\beta)\) such that
\[
\int_a^b f_{a, l}(t) dt \leq M_1(\alpha)
\]
\[
\int_a^b f^{\beta, l}_1(t) dt \leq N_1(\beta)
\]
\[
\int_a^b f_{a, r}(t) dt \leq M_2(\alpha)
\]
\[
\int_a^b f^{\beta, r}_1(t) dt \leq N_2(\beta)
\]
For every \(b \geq a\).

We define
\[
\mathcal{A}_\alpha = \mathcal{L}(f_{a, l}(t)) = \int_0^a e^{-rt} f_{a, l}(t) dt
\]
\[
\mathcal{B}_\beta = \mathcal{L}(f^{\beta, l}_1(t)) = \int_0^a e^{-rt} f^{\beta, l}_1(t) dt
\]
\[
\mathcal{X}_\alpha = \mathcal{L}(f_{a, r}(t)) = \int_0^a e^{-rt} f_{a, r}(t) dt
\]
\[
\mathcal{B}_\beta = \mathcal{L}(f^{\beta, r}_1(t)) = \int_0^a e^{-rt} f^{\beta, r}_1(t) dt
\]

By theorem 1 we get the following definition
Definition 5.1. The Laplace’s transform is defined as follows
\[ \mathcal{L}(f(t)) = \int_0^\infty e^{-st} \circ \overline{\mathcal{B}}(t) \, dt \]
\[ = \left\{ [A_\alpha, \overline{A}_\alpha]; [B_\beta, \overline{B}_\beta] \right\} \]

Theorem 5.1. [5] Let \( f, g : [a, b] \rightarrow \mathbb{F}_1 \) are continuous intuitionistic fuzzy valued function and \( c_1, c_2 \) are constants. Then
\[ \mathcal{L} \left( c_1 \circ f(t) \oplus c_2 \circ g(t) \right) = c_1 \circ \mathcal{L}(f(t)) \oplus c_2 \circ \mathcal{L}(g(t)) \]

Theorem 5.2. [5] Let \( f \) is continuous intuitionistic fuzzy valued function on \([0, \infty)\) and \( \lambda \in \mathbb{R} \). Then \( \mathcal{L}[\lambda \circ f(t)] = \lambda \circ \mathcal{L}[f(t)] \)

Theorem 5.3. [5] Let \( f \) is continuous intuitionistic fuzzy valued function and \( \mathcal{L}[f(t)] = F(p) \). Then
\[ \mathcal{L}[e^{pt} \circ f(t)] = F(p-a) \]
where \( e^{pt} \) is real valued function and \( p - a > 0 \).

Theorem 5.4. [5] Let \( f'(t) \) be an integrable fuzzy valued function, and \( f(t) \) is the primitive of \( f'(t) \) on \([0, \infty)\). Then \( \mathcal{L}[f'(t)] = p \circ \mathcal{L}(f(x)) - \mathcal{gH}(f(0)) \) i.e. \( \mathcal{L}[f'(t)] = p \circ \mathcal{L}(f(x)) - f(0) \), when \( f \) is \((i)\)-differentiable and \( \mathcal{L}[f'(t)] = \mathcal{gH}(f(x)) \), when \( f \) is \((ii)\)-differentiable

Theorem 5.5. Let \( f(t) \) and \( f'(t) \) are two differentiable intuitionistic fuzzy-valued functions. We set \([f(t)]_\alpha = [\int_0^t (s^\alpha t^\beta, \overline{s^\alpha t^\beta}) dt, \overline{\int_0^t (s^\alpha t^\beta, \overline{s^\alpha t^\beta}) dt}] \)
and \([f(t)]^\beta = [\int_0^t (s^\alpha t^\beta, \overline{s^\alpha t^\beta}) dt, \overline{\int_0^t (s^\alpha t^\beta, \overline{s^\alpha t^\beta}) dt}] \), where \( 0 \leq \alpha + \beta \leq 1 \)

- If \( f' \) and \( D^\gamma f \) are \((i)\)-differentiable then
  \[ \mathcal{L}(D^\gamma f(s)) = s^{2-\gamma} \mathcal{L}(f(s)) - h^{s^{1-\gamma} f'(0)} \]
- If \( D^\gamma f \) is \((i)\)-differentiable and \( f' \) is \((ii)\)-differentiable then
  \[ \mathcal{L}(D^\gamma f(s)) = \left(s^{1-\gamma} f'(0) - h (s^{2-\gamma} \mathcal{L}(f(s))) \right) - h s^{\gamma} f(0) \]
- If \( D^\gamma f \) is \((ii)\)-differentiable and \( f' \) is \((i)\)-differentiable then
  \[ \mathcal{L}(D^\gamma f(s)) = s^{-\gamma} f(0) - h \left(s^{-2-\gamma} \mathcal{L}(f(s)) - h s^{1-\gamma} f'(0) \right) \]
- If \( f' \) and \( D^\gamma f \) are \((ii)\)-differentiable then
  \[ \mathcal{L}(D^\gamma f(s)) = -s^{-\gamma} f(0) - h \left(-s^{1-\gamma} f'(0) - h (-s^{2-\gamma} \mathcal{L}(f(s))) \right) \]

\[ \text{Proof.} \text{ Observe that } D^\gamma = D^{\gamma-1} D \text{ and use the theorem 5.4} \]

6 The homotopy method
We apply the homotopy analysis method to the discussed problem. Let us consider the fractional differential equation,
\[ FDu(x, t) = 0 \]
Based on the constructed zero-order deformation equation by Liao (2003), we give the following zero-order deformation equation in the similar way:
\[ (1 - q)L(U(x, t; q)) - u_0(x, t) = qhFD(U(x, t; q)), \quad q \in [0, 1], \quad h \neq 0 \]
$L$ is an auxiliary linear integer order operator and it possesses the property $L(C) = 0$. $U$ is an unknown function. Expanding $U$ in Taylor series with respect to $q$, one has

$$U(x, t; q) = u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t)q^m$$

where

$$u_m(x, t) = \left. \frac{\partial U(x, t; q)}{\partial q^m} \right|_{q=0}$$

Differentiating the equation $m$ times with respect to the embedding parameter $q$, then setting $q = 0$, and finally dividing them by $m!$, we have the $m$th-order deformation equation

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = hR_m (\vec{u}_{m-1}(x, t))$$

where

$$R_m (\vec{u}_{m-1}(x, t)) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} FD(U(x, t; q))}{\partial q^{m-1}} \right|_{q=0}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}$$

7 Solution of the problem (1.1)

The solution is given by two cases.

- Case i): $u$ and $u'$ are $(i)$ differentiable by rapport to $x$ and $D^\gamma u$ is $(i)$-differentiable by rapport to $t$.

$$\begin{cases} D^\gamma f_{\alpha}(x, t) = \frac{\partial^2 f_{\alpha}}{\partial x^2}(x, t), & 0 \leq x, t < 1 \\ f_{\alpha}(x, 0) = \varphi_{\alpha}(x) \\ \frac{\partial}{\partial x}(x, 0) = \phi_{\alpha}(x) \end{cases} \quad (7.5)$$

In this case we set $\varphi_{\alpha}(x) = \alpha(1 + \sin(x))$ and $\phi_{\alpha}(x) = 0$.

We now successively obtain

$$f_{\gamma, \alpha}(x, t) = \frac{\alpha t^{\gamma}}{\Gamma(\gamma + 1)} h \left( 3 \sin(x) + 2 - \cos^2(x) \right)$$

$$f_{2, \alpha}(x, t) = -\frac{\alpha t^{\gamma+2}}{\Gamma(\gamma + 3)} h \sin(x) \left( -13 + 2 \cos^2(x) \right)$$

$$-3 \frac{\alpha t^{\gamma}}{\Gamma(\gamma)} h^2 \sin(x)$$

$$-2 \frac{\alpha t^{\gamma}}{\Gamma(\gamma)} h + 12 \frac{\alpha t^{\gamma+2}}{\Gamma(\gamma)} h^2 \cos^2(x)$$

$$+ \ldots$$

For the special case $h = -1$, we obtain from

$$f_{\alpha}(x, t) = \alpha + \alpha(x - \frac{x^3}{3!} + \ldots) + \alpha \frac{t^{\gamma}}{\Gamma(\gamma + 1)} (-1 - 3x - x^2 + \ldots) + \alpha \frac{t^{\gamma+2}}{\Gamma(\gamma + 3)} (11x + 12x^2 + \ldots)$$
In this case we set $\phi_{\beta}(x) = \beta (1 + \sin(x))$, and $\phi_{\beta}(x) = 0$.

We now successively obtain

$$f_{1, \beta}(x, t) = \frac{\beta t^\gamma}{\Gamma(\gamma + 1)} h \left( 3 \sin(x) + 2 - \cos^2(x) \right)$$

$$f_{2, \beta}(x, t) = -\frac{\beta t^{\gamma+2}}{\Gamma(\gamma+3)} h \sin(x) \left( -13 + 2 \cos^2(x) \right)$$

$$- 3 \frac{\beta t^\gamma}{\Gamma(\gamma)} h^2 \sin(x)$$

$$- 2 \frac{\beta t^\gamma}{\Gamma(\gamma)} h + 12 \frac{\beta t^{\gamma+2}}{\Gamma(\gamma)} h^2 \cos^2(x)$$

$$+ ...$$

For the special case $h = -1$, we obtain from

$$f_{\beta}(x, t) = \beta + \beta \left( x - \frac{x^3}{3!} + ... \right) + \beta \frac{t^\gamma}{\Gamma(\gamma + 1)} (-1 - 3x - x^2 + ...) + \beta \frac{t^{\gamma+2}}{\Gamma(\gamma+3)} (11x + 12x^2 + ...)$$

Figure 1: The figures show the 3rd-order approximation solution of $f_{\beta}(x, t)$ and $f_{\alpha}(x, t)$ for $\gamma = 1.5$, $\beta = 0.3$ and $\alpha = 0.5$.
For the special case $h = -1$, we obtain from
\[
\mathcal{J}_\beta(x,t) = -\alpha - \alpha(x - \frac{x^3}{3!} + ...) - \alpha \frac{t^\gamma}{\Gamma(\gamma + 1)} (-1 - 3x - x^2 + ...) - \alpha \frac{t^{\gamma+2}}{\Gamma(\gamma + 3)} (11x + 12x^2 + ...)
\]
\[
\left\{
\begin{aligned}
(D)^2 \mathcal{J}_\beta(x,t) &= \frac{\partial^3 \mathcal{J}_\beta(x,t)}{\partial x^3}(x,t), \quad 0 \leq x, t < 1 \\
\mathcal{J}_\beta(x,0) &= \varphi_\beta(x) \\
\frac{\partial \mathcal{J}_\beta(x,0)}{\partial x} &= \varphi_\beta(x)
\end{aligned}
\right.
\]
(7.8)

with $\varphi_\beta(x) = -\beta (1 + \sin(x))$, $\varphi_\beta(x) = 0$

\[
\mathcal{J}_{1,\beta}(x,t) = -\frac{\beta t^\gamma}{\Gamma(\gamma + 1)} h (3 \sin(x) + 2 - \cos^2(x))
\]
\[
\mathcal{J}_{2,\beta}(x,t) = \frac{\beta t^{\gamma+2}}{\Gamma(\gamma + 3)} h \sin(x) (-13 + 2\cos^2(x))
\]

For the special case $h = -1$, we obtain from
\[
\mathcal{J}_\beta(x,t) = -\beta - \beta(x - \frac{x^3}{3!} + ...) - \beta \frac{t^\gamma}{\Gamma(\gamma + 1)} (-1 - 3x - x^2 + ...) - \beta \frac{t^{\gamma+2}}{\Gamma(\gamma + 3)} (11x + 12x^2 + ...)
\]

Figure 2: The figures show the 3rd-order approximation solution of $\mathcal{J}_\beta(x,t)$ and $\mathcal{J}_\alpha(x,t)$ for $\gamma = 1.5$, $\beta = 0.3$ and $\alpha = 0.5$

8 Conclusion

In this study, the homotopy analysis method has been employed to obtain an approximate analytical solution of intuitionistic fuzzy fractional equations given in (1.1).

It is quite important to notice that a higher number of iteration and higher order of $p$ are needed to gain more accuracy. The basic ideas of this approach are expected to be further employed to solve other intuitionistic fuzzy problems in intuitionistic fuzzy fractional calculus.

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