Common coupled fixed point for generalized rational type contractions in $b$-metric spaces

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Abstract
In this paper, we establish the existence and uniqueness of a common coupled fixed point for generalized rational type contractions in $b$-metric spaces which generalizes some existing results in the literature.

Keywords: Common fixed point, Coupled fixed point, Rational type contractive mappings, $b$-metric spaces.

1 Introduction and preliminaries

Fixed Point Theory is an attractive and fascinating subject with an enormous number of applications in various fields of mathematics and other branches of science. One of the fundamental tool in fixed point theory is the Banach contraction theorem established by Banach in 1922. The Banach fixed point theorem for contraction mappings has been generalized and extended in various directions by many researchers.

In 1993, Czerwik [5] proposed the concept of $b$-metric spaces as a generalization of metric spaces and also proved the Banach contraction mapping principle in this framework. Since then many authors have studied fixed point theorems for single-valued and multi-valued mappings in $b$-metric spaces (see [[4]-[12]] and references therein). A $b$-metric space was also called a metric-type space in [7].

In 2006, Bhaskar and Lakshmikantham [6] introduced the concept of coupled fixed points for a given partially ordered set $X$ and studied the existence and uniqueness of a coupled fixed point in partially ordered metric spaces using the concept of mixed monotone property. The study of common coupled fixed points of mappings satisfying certain contractive conditions has been at the centre of vigorous research activity. In 2015 [10], we studied the existence and uniqueness of common coupled fixed points for a pair of mappings in view of diverse contractive conditions in $b$-metric spaces. Recently, Sarwar et al. [13] generalized and extended these results in $b$-metric spaces. In this paper, we further generalize and extend the existing results to present common coupled fixed point theorems for generalized rational type contractions in the setting of $b$-metric spaces.

Definition 1.1. [2] Let $X$ be a (nonempty) set and $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}^+$ (nonnegative real numbers) is called a $b$-metric provided that, for all $x, y, z \in X$, the following conditions are satisfied:

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair $(X, d)$ is called a $b$-metric space with parameter $s$. 

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We now give some examples of b-metric spaces.

**Example 1.1.** [4] The space \( L_p(0 < p < 1) \), \( L_p = \{(x_n) \in \mathbb{R} : \sum |x_n|^p < \infty \} \), together with the function \( d : L_p \times L_p \to \mathbb{R} \)
\[
d(x,y) = \left( \sum |x_n - y_n|^p \right)^{\frac{1}{p}}
\]
where \( x = (x_n) ; y = (y_n) \in L_p \) is a b-metric space with \( s = 2^{\frac{1}{p}} \).

**Example 1.2.** [4] The space \( L_p(0 < p < 1) \) of all real functions \( x(t) \), \( t \in [0,1] \) such that \( \int_0^1 |x(t)|^p dt < \infty \), is a b-metric space if we take
\[
d(x,y) = \left( \int_0^1 |x(t) - y(t)|^p dt \right)^{\frac{1}{p}}, \text{ for each } x, y \in L_p.
\]

**Remark 1.1.** We note that a metric space is evidently a b-metric space for \( s = 1 \). However, in general, a b-metric on \( X \) need not be a metric on \( X \) as shown in the following example:

**Example 1.3.** [2] Let \( X = \{0,1,2\} \) and \( d(2,0) = d(0,2) = m \geq 2 \), \( d(0,1) = d(1,2) = d(1,0) = d(2,1) = 1 \) and
\[
d(0,0) = d(1,1) = d(2,2) = 0.
\]
Then \( d(x,y) \leq \frac{m}{2} [d(x,z) + d(z,y)] \) for all \( x,y,z \in X \). If \( m > 2 \), the ordinary triangle inequality does not hold.

**Definition 1.2.** [4] Let \((X,d)\) be a b-metric space. Then a sequence \( \{x_n\} \) in \( X \) is called a Cauchy sequence if for every \( \varepsilon > 0 \), there exists \( K(\varepsilon) \in \mathbb{N} \), such that \( d(x_n,x_m) < \varepsilon \) for all \( n,m \geq K(\varepsilon) \).

**Definition 1.3.** [4] Let \((X,d)\) be a b-metric space. Then a sequence \( \{x_n\} \) in \( X \) is said to converge to \( x \in X \) if for every \( \varepsilon > 0 \), there exists \( K(\varepsilon) \in \mathbb{N} \), such that \( d(x_n,x) < \varepsilon \) for all \( n \geq K(\varepsilon) \). In this case, we write \( \lim_{n \to \infty} x_n = x \).

**Definition 1.4.** [4] The b-metric space \((X,d)\) is complete if every Cauchy sequence in \( X \) converges in \( X \).

**Remark 1.2.** In a b-metric space \((X,d)\) the following assertions hold:

(i) A convergent sequence has a unique limit.

(ii) Every convergent sequence is Cauchy.

**Definition 1.5.** [6] An element \((x,y) \in X \times X\) is called a coupled fixed point of \( T : X \times X \to X \) if \( x = T(x,y) \), \( y = T(y,x) \).

**Definition 1.6.** An element \((x,y) \in X \times X\) is called a common coupled fixed point of \( S,T : X \times X \to X \) if \( x = S(x,y) = T(x,y) \), \( y = S(y,x) = T(y,x) \).

**Example 1.4.** Let \( X = \mathbb{R} \) and \( S,T : X \times X \to X \) defined as \( S(x,y) = xy \) and \( T(x,y) = x + (y-x)^2 \) for all \( x,y \in X \). Then \((0,0)\) and \((1,1)\) are common coupled fixed points of \( S \) and \( T \).

### 2 Main results

**Theorem 2.1.** Let \((X,d)\) be a complete b-metric space with parameter \( s \geq 1 \) and let the mappings \( S,T : X \times X \to X \) satisfy
\[
d(S(x,y),T(u,v)) \leq \alpha_1 \frac{d(x,u) + d(y,v)}{2} + \alpha_2 \frac{d(x,S(x,y))d(u,T(u,v))}{1 + d(x,u) + d(y,v) + d(u,S(x,y))}
\]
\[
+ \alpha_1 \frac{d(x,T(u,v))d(u,v)}{1 + d(x,u) + d(y,v) + d(u,T(u,v))}
\]
\[
+ \alpha_2 \frac{d(S(x,y),T(u,v))d(x,u)}{1 + d(x,u) + d(y,v) + d(u,S(x,y))}
\]
\[
+ \alpha_5 \frac{d(S(x,y),T(u,v))d(y,v)}{1 + d(x,u) + d(y,v) + d(u,T(u,v))}
\]
\[
+ \alpha_6 \frac{d(u,T(u,v))d(x,u)}{1 + d(x,u) + d(y,v) + d(u,T(u,v))}
\]
\[
+ \alpha_7 \frac{d(u,S(x,y))d(x,u)}{1 + d(x,u) + d(y,v) + d(u,S(x,y))}
\]
\[
+ \alpha_8 \frac{d(u,S(x,y))d(y,v)}{1 + d(x,u) + d(y,v) + d(u,S(x,y))}
\]
\[
+ \alpha_9 \max \{d(u,S(x,y)),d(S(x,y),T(u,v))\}
\]
for all \(x, y, u, v \in X\) and \(\alpha_i \geq 0\), \(i = 1, 2, \ldots, 9\) with \(s \alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_9 < 1\) and \(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8 + \alpha_9 < 1\). Then \(S\) and \(T\) have a unique common coupled fixed point in \(X\).

**Proof.** Let \(x_0\) and \(y_0\) be arbitrary points. Define \(x_{2k+1} = S(x_{2k}, y_{2k}), y_{2k+1} = S(y_{2k}, x_{2k})\) and \(x_{2k+2} = T(x_{2k+1}, y_{2k+1}), y_{2k+2} = T(y_{2k+1}, x_{2k+1})\) for \(k = 0, 1, 2, \ldots\) Then,

\[
d(x_{2k+1}, x_{2k+2}) = d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1}))
\]

\[
\leq \alpha_1 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \alpha_2 \frac{d(x_{2k}, S(x_{2k}, y_{2k}))d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, y_{2k+1})}
\]

\[
+ \alpha_3 \frac{d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1}))d(x_{2k}, x_{2k+1})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, y_{2k+1})}
\]

\[
+ \alpha_4 \frac{d(x_{2k}, S(x_{2k}, y_{2k}))d(y_{2k}, y_{2k+1})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, y_{2k+1})}
\]

\[
+ \alpha_5 \frac{d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1}))d(y_{2k}, y_{2k+1})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, y_{2k+1})}
\]

\[
+ \alpha_6 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, y_{2k+1})}{2} + \alpha_7 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, y_{2k+1})}{2} + \alpha_8 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, y_{2k+1})}{2} + \alpha_9 \max\{d(x_{2k+1}, S(x_{2k}, y_{2k})), d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1}))\}
\]

\[
\leq \alpha_1 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \alpha_2 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \alpha_3 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \alpha_4 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \alpha_5 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \alpha_6 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \alpha_7 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \alpha_8 \frac{d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})}{2} + \alpha_9 \max\{d(x_{2k+1}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\}
\]

\[
\Rightarrow d(x_{2k+1}, x_{2k+2}) \leq \frac{\alpha_1}{2(1-(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_9))}d(x_{2k}, x_{2k+1}) + \frac{\alpha_8}{2(1-(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_9))}d(y_{2k}, y_{2k+1}).
\]
Similarly,
\[ d(y_{2k+1}, y_{2k+2}) \leq \frac{a_1}{2(1-(a_2+a_4+a_5+a_6+a_9))} d(y_{2k}, y_{2k+1}) + \frac{a_1}{2(1-(a_2+a_4+a_5+a_6+a_9))} d(x_{2k}, x_{2k+1}). \]

Adding the above two inequalities, we get
\[ [d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})] \leq \frac{a_1}{2[1-(a_2+a_4+a_5+a_6+a_9)]} [d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] \]
\[ = h[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] \text{ where } 0 \leq h = \frac{a_1}{1-(a_2+a_4+a_5+a_6+a_9)} < 1. \]

Also, \( d(x_{2k+2}, x_{2k+3}) \leq \frac{a_1}{2[1-(a_2+a_4+a_5+a_6+a_9)]} d(x_{2k+1}, x_{2k+2}) + \frac{a_1}{2[1-(a_2+a_4+a_5+a_6+a_9)]} d(y_{2k+1}, y_{2k+2}) \)
and
\[ d(y_{2k+2}, y_{2k+3}) \leq \frac{a_1}{2[1-(a_2+a_4+a_5+a_6+a_9)]} d(y_{2k+1}, y_{2k+2}) + \frac{a_1}{2[1-(a_2+a_4+a_5+a_6+a_9)]} d(x_{2k+1}, x_{2k+2}). \]

Adding the above two inequalities, we get
\[ [d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3})] \leq \frac{a_1}{2[1-(a_2+a_4+a_5+a_6+a_9)]} [d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})] \]
\[ = h[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})]. \]

Continuing like this, we have
\[ (d(x_n, x_{n+1}) + d(y_n, y_{n+1})) \leq h(d(x_{n-1}, x_n) + d(y_{n-1}, y_n)) \leq \ldots \leq h^n(d(x_0, x_1) + d(y_0, y_1)). \]

Now, if \( d(x_n, x_{n+1}) + d(y_n, y_{n+1}) = \delta_n \), then \( \delta_n \leq h \delta_{n-1} \leq \ldots \leq h^n \delta_0. \)

For \( m > n, m, n \in \mathbb{N} \), we have
\[ (d(x_n, x_m) + d(y_n, y_m)) \leq s(d(x_n, x_{n+1}) + d(y_n, y_{n+1})) + s^{m-n} (d(x_{m-1}, x_m) + d(y_{m-1}, y_m)) \]
\[ \leq sh^n \delta_0 + s^2 h^{n+1} \delta_0 + \ldots + s^{m-n} h^{m-1} \delta_0 \]
\[ < sh^n \delta_0 + (sh)^2 \delta_0 + \ldots \]
\[ = \frac{sh^n}{1-sh} \delta_0 \to 0 \text{ as } n \to \infty. \]

This shows that \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequences in \( X \). Since \( X \) is a complete \( b \)-metric space, there exist \( x, y \in X \) such that \( x_n \to x \) and \( y_n \to y \) as \( n \to \infty \).

Now we show that \( x = S(x,y) \) and \( y = S(y,x) \).
We suppose on the contrary that \( x \neq S(x,y) \) and \( y \neq S(y,x) \) so that
\[ d(x, S(x,y)) = l_1 > 0 \text{ and } d(y, S(y,x)) = l_2 > 0. \]

Consider \( l_1 = d(x, S(x,y)) \leq s(d(x, x_{2k+2}) + d(x_{2k+2}, S(x,y)))] \)
Similarly, one can prove that
\[ y = S(y, x) \]
That is,
\[ S(x, y) = x \]
So we have proved that \( (x, y) \) is a common coupled fixed point of \( S \) and \( T \).

By taking limit as \( k \to \infty \), we get
\[ (1 - \alpha_0)I_1 \leq 0. \]
\[ \Rightarrow I_1 \leq 0, \] which is a contradiction.

Therefore, \( d(x, S(x, y)) = 0 \).
That is, \( x = S(x, y) \).
Similarly, one can prove that \( y = S(y, x) \).
It follows similarly that \( x = T(y, x) \) and \( y = T(y, x) \).
So we have proved that \( (x, y) \) is a common coupled fixed point of \( S \) and \( T \).
Similarly, one can easily prove that

\[ d(x, x') = d(S(x, y), T(y', x')) \leq \alpha_1 \frac{d(x, x') + d(y, y')}{2} + \alpha_2 \frac{d(x, x') + d(y, y')}{2} + \alpha_3 d(x, x') + d(y, y') \]

\[ + \alpha_4 \frac{d(x, x') + d(y, y')}{2} + \alpha_5 d(x, x') + d(y, y') \]

\[ + \alpha_6 \frac{d(x, x') + d(y, y')}{2} + \alpha_7 d(x, x') + d(y, y') \]

\[ + \alpha_8 \frac{d(x, x') + d(y, y')}{2} + \alpha_9 \max\{d(x', S(x, y)), d(S(x, y), T(y', x'))\} , \]

\[ \Rightarrow [1 - (\frac{\alpha_0}{2} + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8 + \alpha_9)]d(x, x') \leq \frac{\alpha_0}{2}d(y, y'). \]

\[ d(x, x') \leq \frac{\alpha_0}{2 - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_7 + 2\alpha_8 + 2\alpha_9)}d(y, y'). \]

Similarly, one can easily prove that

\[ d(y, y') \leq \frac{\alpha_0}{2 - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_7 + 2\alpha_8 + 2\alpha_9)}d(x, x'). \]

Adding the above two inequalities, we get

\[ d(x, x') + d(y, y') \leq \frac{\alpha_0}{2 - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_7 + 2\alpha_8 + 2\alpha_9)}(d(x, x') + d(y, y')). \]

\[ \Rightarrow [1 - \frac{\alpha_0}{2 - (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_7 + 2\alpha_8 + 2\alpha_9)](d(x, x') + d(y, y')) \leq 0. \]
\[ \Rightarrow [1 - (\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8 + \alpha_9)](d(x, x') + d(y, y')) \leq 0. \]

Since \( \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8 + \alpha_9 < 1 \), \( d(x, x') + d(y, y') \leq 0. \)

\[ \Rightarrow d(x, x') + d(y, y') = 0. \]

\[ \Rightarrow x = x' \text{ and } y = y'. \]

Thus, \( S \) and \( T \) have a unique common fixed point in \( X \).

**Corollary 2.1.** Let \( (X, d) \) be a complete \( b \)-metric space with parameter \( s \geq 1 \) and let the mapping \( T : X \times X \to X \) satisfy

\[
d(T(x, y), T(u, v)) \leq \alpha_1 \frac{d(x, u) + d(y, v)}{2} + \alpha_2 \frac{d(x, T(x, y))d(u, T(u, v))}{1 + d(x, y) + d(u, T(x, y))} + \alpha_3 \frac{d(x, T(x, y))d(x, T(u, v))}{1 + d(x, y) + d(u, T(x, y))} + \alpha_4 \frac{d(x, u) + d(y, v) + d(u, T(x, y))}{1 + d(x, y) + d(u, T(x, y))} + \alpha_5 \frac{d(T(x, y), T(u, v))d(y, v)}{1 + d(x, y) + d(u, T(x, y))} + \alpha_6 \frac{d(T(x, y), T(u, v))d(y, v)}{1 + d(x, y) + d(u, T(x, y))} + \alpha_7 \frac{d(u, T(x, y))d(u, T(y, v))}{1 + d(x, y) + d(u, T(x, y))} + \alpha_8 \frac{d(u, T(x, y))d(u, T(y, v))}{1 + d(x, y) + d(u, T(x, y))} + \alpha_9 \frac{d(u, T(x, y))d(T(y, v), T(u, v))}{1 + d(x, y) + d(u, T(x, y))} + \alpha_0 \max\{d(u, T(x, y)), d(T(x, y), T(u, v))\}
\]

for all \( x, y, u, v \in X \) and \( \alpha_i \geq 0, i = 1, 2, \ldots, 9 \) with \( s\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + s\alpha_9 < 1 \) and \( \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8 + \alpha_9 < 1 \). Then \( T \) has a unique coupled fixed point in \( X \).

**Proof.** Take \( T = S \) in Theorem 2.1. Then we get the desired result.

**Theorem 2.2.** Let \( (X, d) \) be a complete \( b \)-metric space with parameter \( s \geq 1 \) and let the mappings \( S, T : X \times X \to X \) satisfy

\[
d(S(x, y), T(u, v)) \leq \alpha \frac{d(x, u) + d(y, v)}{2} + \beta \frac{d(x, S(x, y))d(S(x, y), T(u, v))}{1 + s(d(x, u) + d(y, v) + d(u, S(x, y)) + d(x, T(u, v))} + \gamma \max\{d(u, S(x, y)), d(S(x, y), T(u, v))\}
\]

for all \( x, y, u, v \in X \) and \( \alpha, \beta, \gamma \geq 0 \) with \( s(\alpha + \beta + \gamma) < 1 \). Then \( S \) and \( T \) have a unique common coupled fixed point in \( X \).

**Proof.** Let \( x_0 \) and \( y_0 \in X \) be arbitrary points.

Define \( x_{2k+1} = S(x_{2k}, y_{2k}), y_{2k+1} = S(y_{2k}, x_{2k}) \) and \( x_{2k+2} = T(x_{2k+1}, y_{2k+1}), y_{2k+2} = T(y_{2k+1}, x_{2k+1}) \) for \( k = 0, 1, 2, \ldots \).
Then
\[
d(x_{2k+1}, x_{2k+2}) = d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1}))
\]
\[
\leq \alpha d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})
\]
\[
+ \beta \left[ 1 + s(d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, T(x_{2k+1}, y_{2k+1})) \right]
\]
\[
+ \gamma \max \{d(x_{2k+1}, S(x_{2k}, y_{2k})), d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \}
\]
\[
= \alpha d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})
\]
\[
+ \beta \left[ 1 + s(d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) + d(x_{2k+1}, T(x_{2k+1}, y_{2k+1})) \right]
\]
\[
+ \gamma \max \{d(x_{2k+1}, x_{2k+1}), d(x_{2k+1}, x_{2k+2}) \}
\]
\[
\leq \alpha \frac{d(x_{2k}, x_{2k+1})}{2} + \alpha \frac{d(y_{2k}, y_{2k+1})}{2} + (\beta + \gamma)d(x_{2k+1}, x_{2k+2}).
\]

\[\Rightarrow d(x_{2k+1}, x_{2k+2}) \leq \frac{\alpha}{1-\beta-\gamma} d(x_{2k}, x_{2k+1}) + \frac{\alpha}{1-\beta-\gamma} d(y_{2k}, y_{2k+1}).\]

Similarly
\[d(y_{2k+1}, y_{2k+2}) \leq \frac{\alpha}{1-\beta-\gamma} d(y_{2k}, y_{2k+1}) + \frac{\alpha}{1-\beta-\gamma} d(x_{2k}, x_{2k+1}).\]

Adding the above two inequalities, we get
\[|d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})| \leq \frac{\alpha}{1-\beta-\gamma} [d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})]\]
\[= r[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) \text{ where } 0 \leq r = \frac{\alpha}{1-\beta-\gamma} < 1.\]

Also, \[d(x_{2k+2}, x_{2k+3}) \leq \frac{\alpha}{2(1-\beta-\gamma)} d(x_{2k+1}, x_{2k+2}) + \frac{\alpha}{2(1-\beta-\gamma)} d(y_{2k+1}, y_{2k+2})\]

and
\[d(y_{2k+2}, y_{2k+3}) \leq \frac{\alpha}{2(1-\beta-\gamma)} d(y_{2k+1}, y_{2k+2}) + \frac{\alpha}{2(1-\beta-\gamma)} d(x_{2k+1}, x_{2k+2}).\]

Adding the above two inequalities, we get
\[|d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3})| \leq \frac{\alpha}{1-\beta-\gamma} [d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})]\]
\[= r[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})].\]

Therefore,
\[\rho(d(x_0, x_n) + d(y_0, y_n)) \leq r^\rho(d(x_{n-1}, x_n) + d(y_{n-1}, y_n)) \leq \cdots \leq r^{n-1}d(x_0, x_1) + d(y_0, y_1).\]

Now, if \(d(x_n, x_{n+1}) + d(y_n, y_{n+1}) = t_n\), then \(t_n \leq h t_{n-1} \leq \cdots \leq r^{n-1}t_0.\)

For \(m, n \in \mathbb{N}, m > n\), we have
\[d(x_m, x_n) + d(y_m, y_n) \leq d(x_m, x_{n+1}) + d(y_m, y_{n+1}) + \cdots + d(x_{m-n}, x_m) + d(y_{m-n}, y_m)\]
\[\leq s^m t_0 + s^{m-1}t_0 + \cdots + s^0 t_0\]
\[= s^m t_0 + (s^m + (s^2) + \cdots + t_0)
\[= \frac{s^m}{1-s^m} t_0 \to 0 \text{ as } n \to \infty.\]
This shows that \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequences in \( X \). Since \( X \) is a complete \( b \)-metric space, there exist \( x, y \in X \) such that \( x_n \to x \) and \( y_n \to y \) as \( n \to \infty \).

Now we show that \( x = S(x, y) \) and \( y = S(y, x) \).

We suppose on the contrary that \( x \neq S(x, y) \) and \( y \neq S(y, x) \) so that

\[
d(x, S(x, y)) = l_1 > 0 \quad \text{and} \quad d(y, S(y, x)) = l_2 > 0.
\]

Consider

\[
l_1 = d(x, S(x, y)) \leq s[d(x, x_{2k+2}) + d(x_{2k+2}, S(x, y))] \leq sd(x, x_{2k+2}) + s \frac{\alpha y_{2k+1} + d(y_{2k+1}, y)}{2} + s \beta \frac{d(x, S(x, y))d(S(x, y), T(y, x_{2k+1}, y_{2k+1}))}{1 + s(d(x_{2k+1}, x) + d(y_{2k+1}, y) + d(x_{2k+1}, S(x, y)) + d(T(y, x_{2k+1}, y_{2k+1}))} + s \gamma \max \{d(x_{2k+1}, S(x, y))d(S(x, y), T(y, x_{2k+1}, y_{2k+1}))\} \leq sd(x, x_{2k+2}) + s \alpha y_{2k+1} + d(y_{2k+1}, y) + s \beta d(x, S(x, y))d(S(x, y), T(y, x_{2k+1}, y_{2k+1})) + s \beta d(x, S(x, y))d(S(x, y), x_{2k+2}) + s \alpha y_{2k+1} + d(y_{2k+1}, y) + s \gamma \max \{d(x_{2k+1}, S(x, y))d(S(x, y), T(y, x_{2k+1}, y_{2k+1}))\}.
\]

By taking limit as \( k \to \infty \), we get \( (1 - s \beta - s \gamma)l_1 \leq 0 \).

\( \Rightarrow l_1 \leq 0 \), which is a contradiction.

Therefore, \( d(x, S(x, y)) = 0 \).

That is, \( x = S(x, y) \).

Similarly, one can prove that \( y = S(y, x) \).

It follows similarly that \( x = T(x, y) \) and \( y = T(y, x) \).

So we have proved that \( (x, y) \) is a common coupled fixed point of \( S \) and \( T \).

We now show that \( S \) and \( T \) have a unique common coupled fixed point.

Uniqueness: Let \( (x^*, y^*) \in X \times X \) be another common coupled fixed point of \( S \) and \( T \).

Then,

\[
d(x, x^*) = d(S(x, y), T(x^*, y^*)) \leq \frac{\alpha}{2} d(x, x^*) + d(y, y^*) + \frac{\beta}{2} d(x, S(x, y))d(S(x, y), T(x^*, y^*)) + \frac{\gamma}{2} \max \{d(x, x^*), d(y, y^*)\}.
\]

\[
= \frac{\alpha d(x, x^*) + d(y, y^*)}{2} + \frac{\beta}{2} \frac{d(x, x^*)}{1 + s d(x, x^*) + d(y, y^*)} + \frac{\gamma}{2} \max \{d(x, x^*), d(y, y^*)\}.
\]

\( \Rightarrow d(x, x^*) \leq \frac{\alpha}{\beta + 2 \gamma} d(x, x^*). \)

Similarly, one can easily prove that

\( d(y, y^*) \leq \frac{\alpha}{\beta + 2 \gamma} d(y, y^*). \)

Adding the above two inequalities, we get

\[
d(x, x^*) + d(y, y^*) \leq \frac{\alpha}{\beta + 2 \gamma} [d(x, x^*) + d(y, y^*)].
\]

\( \Rightarrow (2 - 2 \alpha - 2 \gamma)(d(x, x^*) + d(y, y^*)) \leq 0. \)
⇒ \( d(x, x^*) + d(y, y^*) = 0. \)
⇒ \( x = x^* \) and \( y = y^*. \)
⇒ \( (x, y) = (x^*, y^*). \)

Thus \( S \) and \( T \) have a unique common coupled fixed point in \( X \).

**Corollary 2.2.** Let \((X, d)\) be a complete \( b \)-metric space with parameter \( s \geq 1 \) and let the mapping \( T : X \times X \to X \) satisfy

\[
d(T(x, y), T(u, v)) \leq \alpha \frac{d(x, u) + d(y, v)}{2} + \frac{\beta}{1 + s(d(x, u) + d(y, v) + d(u, T(x, y)) + d(x, T(u, v)))} \]

\[
+ \gamma \max\{d(u, T(x, y)), d(T(x, y), T(u, v))\}
\]

for all \( x, y, u, v \in X \) and \( \alpha, \beta, \gamma \geq 0 \) with \( s(\alpha + \beta + \gamma) < 1 \). Then \( T \) has a unique coupled fixed point in \( X \).

**Proof.** Take \( T = S \) in Theorem 2.2. Then we get the desired result.

**References**


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