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Stable Distributions

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SFB 649 ECONOMIC RISK BERLIN

1 Stable distributions

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1.1 Introduction

Many of the concepts in theoretical and empirical finance developed over the past decades – including the classical portfolio theory, the Black-Scholes-Merton option pricing model and the RiskMetrics variance-covariance approach to Value at Risk (VaR) – rest upon the assumption that asset returns follow a normal distribution. However, it has been long known that asset returns are not normally distributed. Rather, the empirical observations exhibit fat tails. This heavy tailed or leptokurtic character of the distribution of price changes has been repeatedly observed in various markets and may be quantitatively measured by the kurtosis in excess of 3, a value obtained for the normal distribution (Bouchaud and Potters, 2000; Carr et al., 2002; Guillaume et al., 1997; Mantegna and Stanley, 1995; Rachev, 2003; Weron, 2004).

It is often argued that financial asset returns are the cumulative outcome of a vast number of pieces of information and individual decisions arriving almost continuously in time (McCulloch, 1996; Rachev and Mittnik, 2000). As such, since the pioneering work of Louis Bachelier in 1900, they have been modeled by the Gaussian distribution. The strongest statistical argument for it is based on the Central Limit Theorem, which states that the sum of a large number of independent, identically distributed variables from a finite-variance distribution will tend to be normally distributed. However, as we have already mentioned, financial asset returns usually have heavier tails.

In response to the empirical evidence Mandelbrot (1963) and Fama (1965) proposed the stable distribution as an alternative model. Although there are other heavy-tailed alternatives to the Gaussian law – like Student's t , hyperbolic, normal inverse Gaussian, or truncated stable – there is at least one good reason

for modeling financial variables using stable distributions. Namely, they are supported by the generalized Central Limit Theorem, which states that stable laws are the only possible limit distributions for properly normalized and centered sums of independent, identically distributed random variables.

Since stable distributions can accommodate the fat tails and asymmetry, they often give a very good fit to empirical data. In particular, they are valuable models for data sets covering extreme events, like market crashes or natural catastrophes. Even though they are not universal, they are a useful tool in the hands of an analyst working in finance or insurance. Hence, we devote this chapter to a thorough presentation of the computational aspects related to stable laws. In Section 1.2 we review the analytical concepts and basic characteristics. In the following two sections we discuss practical simulation and estimation approaches. Finally, in Section 1.5 we present financial applications of stable laws.

1.2 Definitions and basic characteristics

Stable laws – also called α -stable, stable Paretian or Lévy stable – were introduced by Levy (1925) during his investigations of the behavior of sums of independent random variables. A sum of two independent random variables having an α -stable distribution with index α is again α -stable with the same index α . This invariance property, however, does not hold for different α 's.

The α -stable distribution requires four parameters for complete description: an index of stability $\alpha \in (0, 2]$ also called the tail index, tail exponent or characteristic exponent, a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\sigma > 0$ and a location parameter $\mu \in \mathbb{R}$. The tail exponent α determines the rate at which the tails of the distribution taper off, see the left panel in Figure 1.1. When $\alpha = 2$, the Gaussian distribution results. When $\alpha < 2$, the variance is infinite and the tails are asymptotically equivalent to a Pareto law, i.e. they exhibit a power-law behavior. More precisely, using a central limit theorem type argument it can be shown that (Janicki and Weron, 1994; Samorodnitsky and Taqqu, 1994):

$$\begin{cases} \lim_{x \rightarrow \infty} x^\alpha \mathbb{P}(X > x) = C_\alpha(1 + \beta)\sigma^\alpha, \\ \lim_{x \rightarrow \infty} x^\alpha \mathbb{P}(X < -x) = C_\alpha(1 - \beta)\sigma^\alpha, \end{cases} \quad (1.1)$$

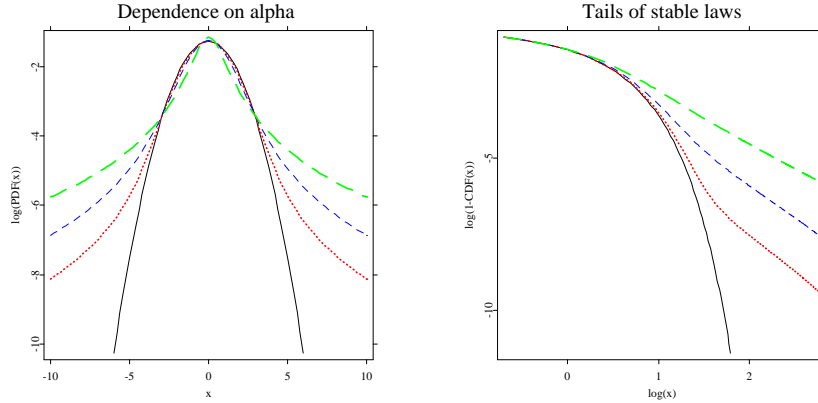


Figure 1.1: *Left panel:* A semilog plot of symmetric ($\beta = \mu = 0$) α -stable probability density functions (pdfs) for $\alpha = 2$ (black solid line), 1.8 (red dotted line), 1.5 (blue dashed line) and 1 (green long-dashed line). The Gaussian ($\alpha = 2$) density forms a parabola and is the only α -stable density with exponential tails. *Right panel:* Right tails of symmetric α -stable cumulative distribution functions (cdfs) for $\alpha = 2$ (black solid line), 1.95 (red dotted line), 1.8 (blue dashed line) and 1.5 (green long-dashed line) on a double logarithmic paper. For $\alpha < 2$ the tails form straight lines with slope $-\alpha$.

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where:

$$C_\alpha = \left(2 \int_0^\infty x^{-\alpha} \sin(x) dx \right)^{-1} = \frac{1}{\pi} \Gamma(\alpha) \sin \frac{\pi\alpha}{2}.$$

The convergence to a power-law tail varies for different α 's and, as can be seen in the right panel of Figure 1.1, is slower for larger values of the tail index. Moreover, the tails of α -stable distribution functions exhibit a crossover from an approximate power decay with exponent $\alpha > 2$ to the true tail with exponent α . This phenomenon is more visible for large α 's (Weron, 2001).

When $\alpha > 1$, the mean of the distribution exists and is equal to μ . In general, the p th moment of a stable random variable is finite if and only if $p < \alpha$. When the skewness parameter β is positive, the distribution is skewed to the right,

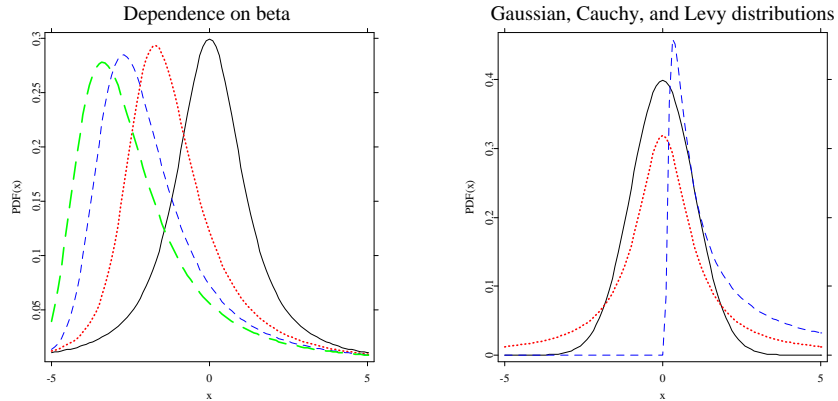


Figure 1.2: *Left panel*: Stable pdfs for $\alpha = 1.2$ and $\beta = 0$ (black solid line), 0.5 (red dotted line), 0.8 (blue dashed line) and 1 (green long-dashed line). *Right panel*: Closed form formulas for densities are known only for three distributions – Gaussian ($\alpha = 2$; black solid line), Cauchy ($\alpha = 1$; red dotted line) and Levy ($\alpha = 0.5, \beta = 1$; blue dashed line). The latter is a totally skewed distribution, i.e. its support is \mathbb{R}_+ . In general, for $\alpha < 1$ and $\beta = 1$ (-1) the distribution is totally skewed to the right (left).

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i.e. the right tail is thicker, see the left panel of Figure 1.2. When it is negative, it is skewed to the left. When $\beta = 0$, the distribution is symmetric about μ . As α approaches 2, β loses its effect and the distribution approaches the Gaussian distribution regardless of β . The last two parameters, σ and μ , are the usual scale and location parameters, i.e. σ determines the width and μ the shift of the mode (the peak) of the density. For $\sigma = 1$ and $\mu = 0$ the distribution is called standard stable.

1.2.1 Characteristic function representation

Due to the lack of closed form formulas for densities for all but three distributions (see the right panel in Figure 1.2), the α -stable law can be most

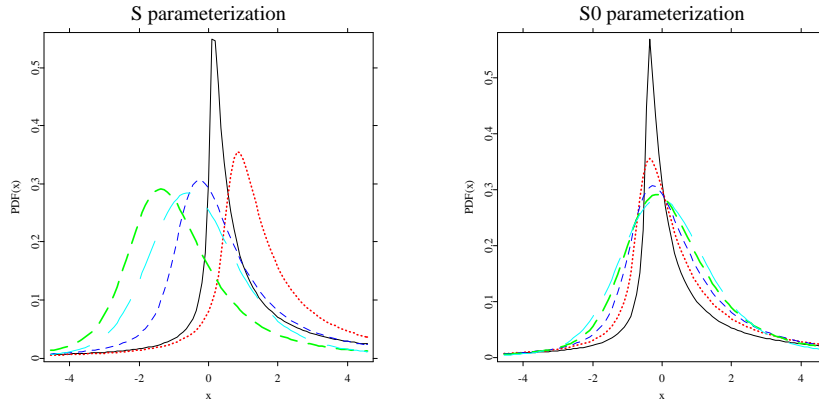



Figure 1.3: Comparison of S and S^0 parameterizations: α -stable pdfs for $\beta = 0.5$ and $\alpha = 0.5$ (black solid line), 0.75 (red dotted line), 1 (blue short-dashed line), 1.25 (green dashed line) and 1.5 (cyan long-dashed line).

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conveniently described by its characteristic function $\phi(t)$ – the inverse Fourier transform of the probability density function. However, there are multiple parameterizations for α -stable laws and much confusion has been caused by these different representations, see Figure 1.3. The variety of formulas is caused by a combination of historical evolution and the numerous problems that have been analyzed using specialized forms of the stable distributions. The most popular parameterization of the characteristic function of $X \sim S_\alpha(\sigma, \beta, \mu)$, i.e. an α -stable random variable with parameters α , σ , β , and μ , is given by (Samorodnitsky and Taqqu, 1994; Weron, 2004):

$$\ln \phi(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 - i\beta \text{sign}(t) \tan \frac{\pi\alpha}{2}\} + i\mu t, & \alpha \neq 1, \\ -\sigma |t| \{1 + i\beta \text{sign}(t) \frac{2}{\pi} \ln |t|\} + i\mu t, & \alpha = 1. \end{cases} \quad (1.2)$$

For numerical purposes, it is often advisable to use Nolan's (1997) parameterization:

$$\ln \phi_0(t) = \begin{cases} -\sigma^\alpha |t|^\alpha \{1 + i\beta \text{sign}(t) \tan \frac{\pi\alpha}{2} [(\sigma|t|)^{1-\alpha} - 1]\} + i\mu_0 t, & \alpha \neq 1, \\ -\sigma |t| \{1 + i\beta \text{sign}(t) \frac{2}{\pi} \ln(\sigma|t|)\} + i\mu_0 t, & \alpha = 1. \end{cases} \quad (1.3)$$

The $S_\alpha^0(\sigma, \beta, \mu_0)$ parameterization is a variant of Zolotariev's (M)-parameterization (Zolotarev, 1986), with the characteristic function and hence the density and the distribution function jointly continuous in all four parameters, see the right panel in Figure 1.3. In particular, percentiles and convergence to the power-law tail vary in a continuous way as α and β vary. The location parameters of the two representations are related by $\mu = \mu_0 - \beta\sigma \tan \frac{\pi\alpha}{2}$ for $\alpha \neq 1$ and $\mu = \mu_0 - \beta\sigma \frac{2}{\pi} \ln \sigma$ for $\alpha = 1$. Note also, that the traditional scale parameter σ_G of the Gaussian distribution defined by:

$$f_G(x) = \frac{1}{\sqrt{2\pi}\sigma_G} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma_G^2} \right\}, \quad (1.4)$$

is not the same as σ in formulas (1.2) or (1.3). Namely, $\sigma_G = \sqrt{2}\sigma$.

1.2.2 Stable density and distribution functions

The lack of closed form formulas for most stable densities and distribution functions has negative consequences. For example, during maximum likelihood estimation computationally burdensome numerical approximations have to be used. There generally are two approaches to this problem. Either the fast Fourier transform (FFT) has to be applied to the characteristic function (Mittnik, Doganoglu, and Chenyao, 1999) or direct numerical integration has to be utilized (Nolan, 1997, 1999).

For data points falling between the equally spaced FFT grid nodes an interpolation technique has to be used. Taking a larger number of grid points increases accuracy, however, at the expense of higher computational burden. The FFT based approach is faster for large samples, whereas the direct integration method favors small data sets since it can be computed at any arbitrarily chosen point. Mittnik, Doganoglu, and Chenyao (1999) report that for $N = 2^{13}$ the FFT based method is faster for samples exceeding 100 observations and slower for smaller data sets. Moreover, the FFT based approach is less universal – it is efficient only for large α 's and only for pdf calculations. When

computing the cdf the density must be numerically integrated. In contrast, in the direct integration method Zolotarev's (1986) formulas either for the density or the distribution function are numerically integrated.

Set $\zeta = -\beta \tan \frac{\pi\alpha}{2}$. Then the density $f(x; \alpha, \beta)$ of a standard α -stable random variable in representation S^0 , i.e. $X \sim S_\alpha^0(1, \beta, 0)$, can be expressed as (note, that Zolotarev (1986, Section 2.2) used yet another parametrization):

- when $\alpha \neq 1$ and $x > \zeta$:

$$f(x; \alpha, \beta) = \frac{\alpha(x - \zeta)^{\frac{1}{\alpha-1}}}{\pi |\alpha - 1|} \int_{-\xi}^{\frac{\pi}{2}} V(\theta; \alpha, \beta) \exp \left\{ -(x - \zeta)^{\frac{\alpha}{\alpha-1}} V(\theta; \alpha, \beta) \right\} d\theta, \quad (1.5)$$

- when $\alpha \neq 1$ and $x = \zeta$:

$$f(x; \alpha, \beta) = \frac{\Gamma(1 + \frac{1}{\alpha}) \cos(\xi)}{\pi(1 + \zeta^2)^{\frac{1}{2\alpha}}},$$

- when $\alpha \neq 1$ and $x < \zeta$:

$$f(x; \alpha, \beta) = f(-x; \alpha, -\beta),$$

- when $\alpha = 1$:

$$f(x; 1, \beta) = \begin{cases} \frac{1}{2|\beta|} e^{-\frac{\pi x}{2\beta}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(\theta; 1, \beta) \exp \left\{ -e^{-\frac{\pi x}{2\beta}} V(\theta; 1, \beta) \right\} d\theta, & \beta \neq 0, \\ \frac{1}{\pi(1+x^2)}, & \beta = 0, \end{cases}$$

where

$$\xi = \begin{cases} \frac{1}{\alpha} \arctan(-\zeta), & \alpha \neq 1, \\ \frac{\pi}{2}, & \alpha = 1, \end{cases}$$

and

$$V(\theta; \alpha, \beta) = \begin{cases} (\cos \alpha \xi)^{\frac{1}{\alpha-1}} \left(\frac{\cos \theta}{\sin \alpha(\xi + \theta)} \right)^{\frac{\alpha}{\alpha-1}} \frac{\cos\{\alpha\xi + (\alpha-1)\theta\}}{\cos \theta}, & \alpha \neq 1, \\ \frac{2}{\pi} \left(\frac{\frac{\pi}{2} + \beta\theta}{\cos \theta} \right) \exp \left\{ \frac{1}{\beta} \left(\frac{\pi}{2} + \beta\theta \right) \tan \theta \right\}, & \alpha = 1, \beta \neq 0. \end{cases}$$

The distribution $F(x; \alpha, \beta)$ of a standard α -stable random variable in representation S^0 can be expressed as:

- when $\alpha \neq 1$ and $x > \zeta$:

$$F(x; \alpha, \beta) = c_1(\alpha, \beta) + \frac{\text{sign}(1 - \alpha)}{\pi} \int_{-\xi}^{\frac{\pi}{2}} \exp\left\{-(x - \zeta)^{\frac{\alpha}{\alpha-1}} V(\theta; \alpha, \beta)\right\} d\theta,$$

where

$$c_1(\alpha, \beta) = \begin{cases} \frac{1}{\pi} \left(\frac{\pi}{2} - \xi\right), & \alpha < 1, \\ 1, & \alpha > 1, \end{cases}$$

- when $\alpha \neq 1$ and $x = \zeta$:

$$F(x; \alpha, \beta) = \frac{1}{\pi} \left(\frac{\pi}{2} - \xi\right),$$

- when $\alpha \neq 1$ and $x < \zeta$:

$$F(x; \alpha, \beta) = 1 - F(-x; \alpha, -\beta),$$

- when $\alpha = 1$:

$$F(x; 1, \beta) = \begin{cases} \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp\left\{-e^{-\frac{\pi x}{2\beta}} V(\theta; 1, \beta)\right\} d\theta, & \beta > 0, \\ \frac{1}{2} + \frac{1}{\pi} \arctan x, & \beta = 0, \\ 1 - F(x, 1, -\beta), & \beta < 0. \end{cases}$$

Formula (1.5) requires numerical integration of the function $g(\cdot) \exp\{-g(\cdot)\}$, where $g(\theta; x, \alpha, \beta) = (x - \zeta)^{\frac{\alpha}{\alpha-1}} V(\theta; \alpha, \beta)$. The integrand is 0 at $-\xi$, increases monotonically to a maximum of $\frac{1}{e}$ at point θ^* for which $g(\theta^*; x, \alpha, \beta) = 1$, and then decreases monotonically to 0 at $\frac{\pi}{2}$ (Nolan, 1997). However, in some cases the integrand becomes very peaked and numerical algorithms can miss the spike and underestimate the integral. To avoid this problem we need to find the argument θ^* of the peak numerically and compute the integral as a sum of two integrals: one from $-\xi$ to θ^* and the other from θ^* to $\frac{\pi}{2}$.

1.3 Simulation of α -stable variables

The complexity of the problem of simulating sequences of α -stable random variables results from the fact that there are no analytic expressions for the

inverse F^{-1} of the cumulative distribution function. The first breakthrough was made by Kanter (1975), who gave a direct method for simulating $S_\alpha(1, 1, 0)$ random variables, for $\alpha < 1$. It turned out that this method could be easily adapted to the general case. Chambers, Mallows, and Stuck (1976) were the first to give the formulas.

The algorithm for constructing a standard stable random variable $X \sim S_\alpha(1, \beta, 0)$, in representation (1.2), is the following (Weron, 1996):

- generate a random variable V uniformly distributed on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and an independent exponential random variable W with mean 1;
- for $\alpha \neq 1$ compute:

$$X = S_{\alpha,\beta} \cdot \frac{\sin\{\alpha(V + B_{\alpha,\beta})\}}{\{\cos(V)\}^{1/\alpha}} \cdot \left[\frac{\cos\{V - \alpha(V + B_{\alpha,\beta})\}}{W} \right]^{(1-\alpha)/\alpha}, \quad (1.6)$$

where

$$B_{\alpha,\beta} = \frac{\arctan(\beta \tan \frac{\pi\alpha}{2})}{\alpha},$$

$$S_{\alpha,\beta} = \left\{ 1 + \beta^2 \tan^2 \left(\frac{\pi\alpha}{2} \right) \right\}^{1/(2\alpha)};$$

- for $\alpha = 1$ compute:

$$X = \frac{2}{\pi} \left\{ \left(\frac{\pi}{2} + \beta V \right) \tan V - \beta \ln \left(\frac{\frac{\pi}{2} W \cos V}{\frac{\pi}{2} + \beta V} \right) \right\}. \quad (1.7)$$

Given the formulas for simulation of a standard α -stable random variable, we can easily simulate a stable random variable for all admissible values of the parameters α , σ , β and μ using the following property: if $X \sim S_\alpha(1, \beta, 0)$ then

$$Y = \begin{cases} \sigma X + \mu, & \alpha \neq 1, \\ \sigma X + \frac{2}{\pi} \beta \sigma \ln \sigma + \mu, & \alpha = 1, \end{cases} \quad (1.8)$$

is $S_\alpha(\sigma, \beta, \mu)$. It is interesting to note that for $\alpha = 2$ (and $\beta = 0$) the Chambers-Mallows-Stuck method reduces to the well known Box-Muller algorithm for generating Gaussian random variables (Janicki and Weron, 1994). Although many other approaches have been proposed in the literature, this method is regarded as the fastest and the most accurate (Weron, 2004).

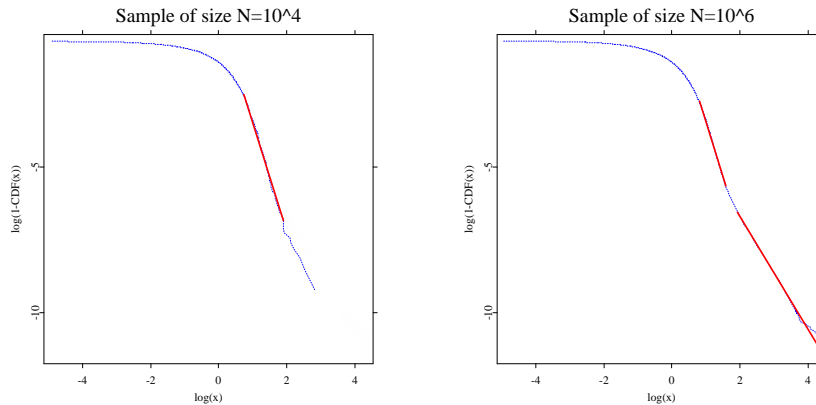


Figure 1.4: A double logarithmic plot of the right tail of an empirical symmetric 1.9-stable distribution function for a sample of size $N = 10^4$ (*left panel*) and $N = 10^6$ (*right panel*). Thick red lines represent the linear regression fit. The tail index estimate ($\hat{\alpha} = 3.7320$) obtained for the smaller sample is close to the initial power-law like decay of the larger sample ($\hat{\alpha} = 3.7881$). The far tail estimate $\hat{\alpha} = 1.9309$ is close to the true value of α .

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1.4 Estimation of parameters

Like simulation, the estimation of stable law parameters is in general severely hampered by the lack of known closed-form density functions for all but a few members of the stable family. Either the pdf has to be numerically integrated (see the previous section) or the estimation technique has to be based on a different characteristic of stable laws.

All presented methods work quite well assuming that the sample under consideration is indeed α -stable. However, if the data comes from a different distribution, these procedures may mislead more than the Hill and direct tail estimation methods. Since the formal tests for assessing α -stability of a sample are very time consuming we suggest to first apply the “visual inspection” tests to see whether the empirical densities resemble those of α -stable laws.

1.4.1 Tail exponent estimation

The simplest and most straightforward method of estimating the tail index is to plot the right tail of the empirical cdf on a double logarithmic paper. The slope of the linear regression for large values of x yields the estimate of the tail index α , through the relation $\alpha = -\text{slope}$.

This method is very sensitive to the size of the sample and the choice of the number of observations used in the regression. For example, the slope of about -3.7 may indicate a non- α -stable power-law decay in the tails or the contrary – an α -stable distribution with $\alpha \approx 1.9$. This is illustrated in Figure 1.4. In the left panel a power-law fit to the tail of a sample of $N = 10^4$ standard symmetric ($\beta = \mu = 0$, $\sigma = 1$) α -stable distributed variables with $\alpha = 1.9$ yields an estimate of $\hat{\alpha} = 3.732$. However, when the sample size is increased to $N = 10^6$ the power-law fit to the extreme tail observations yields $\hat{\alpha} = 1.9309$, which is fairly close to the original value of α .

The true tail behavior (1.1) is observed only for very large (also for very small, i.e. the negative tail) observations, after a crossover from a temporary power-like decay (which surprisingly indicates $\alpha \approx 3.7$). Moreover, the obtained estimates still have a slight positive bias, which suggests that perhaps even larger samples than 10^6 observations should be used. In Figure 1.4 we used only the upper 0.15% of the records to estimate the true tail exponent. In general, the choice of the observations used in the regression is subjective and can yield large estimation errors, a fact which is often neglected in the literature.

A well known method for estimating the tail index that does not assume a parametric form for the entire distribution function, but focuses only on the tail behavior was proposed by Hill (1975). The Hill estimator is used to estimate the tail index α , when the upper (or lower) tail of the distribution is of the form: $1 - F(x) = Cx^{-\alpha}$, see Figure 1.5. Like the log-log regression method, the Hill estimator tends to overestimate the tail exponent of the stable distribution if α is close to two and the sample size is not very large. For a review of the extreme value theory and the Hill estimator see Härdle, Klinke, and Müller (2000, Chapter 13) or Embrechts, Klüppelberg, and Mikosch (1997).

These examples clearly illustrate that the true tail behavior of α -stable laws is visible only for extremely large data sets. In practice, this means that in order to estimate α we must use high-frequency data and restrict ourselves to the most “outlying” observations. Otherwise, inference of the tail index may be strongly misleading and rejection of the α -stable regime unfounded.

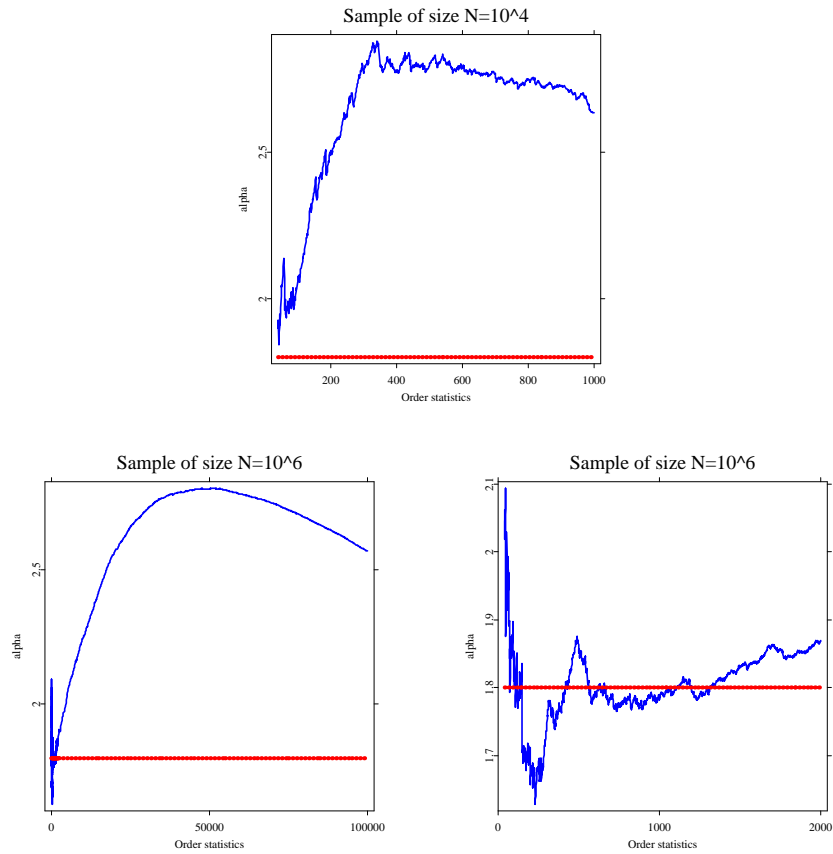


Figure 1.5: Plots of the Hill statistics $\hat{\alpha}_{n,k}$ vs. the maximum order statistic k for 1.8-stable samples of size $N = 10^4$ (*top panel*) and $N = 10^6$ (*left and right panels*). Red horizontal lines represent the true value of α . For better exposition, the right panel is a magnification of the left panel for small k . A close estimate is obtained only for $k = 500, \dots, 1300$ (i.e. for $k < 0.13\%$ of sample size).

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We now turn to the problem of parameter estimation. We start the discussion with the simplest, fastest and ... least accurate quantile methods, then develop the slower, yet much more accurate sample characteristic function methods and, finally, conclude with the slowest but most accurate maximum likelihood approach. Given a sample x_1, \dots, x_n of independent and identically distributed $S_\alpha(\sigma, \beta, \mu)$ observations, in what follows, we provide estimates $\hat{\alpha}$, $\hat{\sigma}$, $\hat{\beta}$, and $\hat{\mu}$ of all four stable law parameters.

1.4.2 Quantile estimation

Already in 1971 Fama and Roll provided very simple estimates for parameters of symmetric ($\beta = 0, \mu = 0$) stable laws when $\alpha > 1$. McCulloch (1986) generalized and improved their method. He analyzed stable law quantiles and provided consistent estimators of all four stable parameters, with the restriction $\alpha \geq 0.6$, while retaining the computational simplicity of Fama and Roll's method. After McCulloch define:

$$v_\alpha = \frac{x_{0.95} - x_{0.05}}{x_{0.75} - x_{0.25}}, \quad (1.9)$$

which is independent of both σ and μ . In the above formula x_f denotes the f -th population quantile, so that $S_\alpha(\sigma, \beta, \mu)(x_f) = f$. Let \hat{v}_α be the corresponding sample value. It is a consistent estimator of v_α . Now, define:

$$v_\beta = \frac{x_{0.95} + x_{0.05} - 2x_{0.50}}{x_{0.95} - x_{0.05}}, \quad (1.10)$$

and let \hat{v}_β be the corresponding sample value. v_β is also independent of both σ and μ . As a function of α and β it is strictly increasing in β for each α . The statistic \hat{v}_β is a consistent estimator of v_β .

Statistics v_α and v_β are functions of α and β . This relationship may be inverted and the parameters α and β may be viewed as functions of v_α and v_β :

$$\alpha = \psi_1(v_\alpha, v_\beta), \quad \beta = \psi_2(v_\alpha, v_\beta). \quad (1.11)$$

Substituting v_α and v_β by their sample values and applying linear interpolation between values found in tables provided by McCulloch (1986) yields estimators $\hat{\alpha}$ and $\hat{\beta}$.

Scale and location parameters, σ and μ , can be estimated in a similar way. However, due to the discontinuity of the characteristic function for $\alpha = 1$ and $\beta \neq 0$ in representation (1.2), this procedure is much more complicated. We refer the interested reader to the original work of McCulloch (1986).

1.4.3 Characteristic function approaches

Given a sample x_1, \dots, x_n of independent and identically distributed (i.i.d.) random variables, define the sample characteristic function by

$$\hat{\phi}(t) = \frac{1}{n} \sum_{j=1}^n e^{itx_j}. \quad (1.12)$$

Since $|\hat{\phi}(t)|$ is bounded by unity all moments of $\hat{\phi}(t)$ are finite and, for any fixed t , it is the sample average of i.i.d. random variables $\exp(itx_j)$. Hence, by the law of large numbers, $\hat{\phi}(t)$ is a consistent estimator of the characteristic function $\phi(t)$.

Press (1972) proposed a simple estimation method, called the method of moments, based on transformations of the characteristic function. The obtained estimators are consistent since they are based upon estimators of $\phi(t)$, $\text{Im}\{\phi(t)\}$ and $\text{Re}\{\phi(t)\}$, which are known to be consistent. However, convergence to the population values depends on a choice of four points at which the above functions are evaluated. The optimal selection of these values is problematic and still is an open question. The obtained estimates are of poor quality and the method is not recommended for more than preliminary estimation.

Koutrouvelis (1980) presented a regression-type method which starts with an initial estimate of the parameters and proceeds iteratively until some prespecified convergence criterion is satisfied. Each iteration consists of two weighted regression runs. The number of points to be used in these regressions depends on the sample size and starting values of α . Typically no more than two or three iterations are needed. The speed of the convergence, however, depends on the initial estimates and the convergence criterion.

The regression method is based on the following observations concerning the characteristic function $\phi(t)$. First, from (1.2) we can easily derive:

$$\ln(-\ln |\phi(t)|^2) = \ln(2\sigma^\alpha) + \alpha \ln |t|. \quad (1.13)$$

The real and imaginary parts of $\phi(t)$ are for $\alpha \neq 1$ given by

$$\Re\{\phi(t)\} = \exp(-|\sigma t|^\alpha) \cos \left[\mu t + |\sigma t|^\alpha \beta \text{sign}(t) \tan \frac{\pi\alpha}{2} \right],$$

and

$$\Im\{\phi(t)\} = \exp(-|\sigma t|^\alpha) \sin \left[\mu t + |\sigma t|^\alpha \beta \text{sign}(t) \tan \frac{\pi\alpha}{2} \right].$$

The last two equations lead, apart from considerations of principal values, to

$$\arctan\left(\frac{\Im\{\phi(t)\}}{\Re\{\phi(t)\}}\right) = \mu t + \beta\sigma^\alpha \tan\frac{\pi\alpha}{2}\text{sign}(t)|t|^\alpha. \quad (1.14)$$

Equation (1.13) depends only on α and σ and suggests that we estimate these parameters by regressing $y = \ln(-\ln|\phi_n(t)|^2)$ on $w = \ln|t|$ in the model

$$y_k = m + \alpha w_k + \epsilon_k, \quad k = 1, 2, \dots, K, \quad (1.15)$$

where t_k is an appropriate set of real numbers, $m = \ln(2\sigma^\alpha)$, and ϵ_k denotes an error term. Koutrouvelis (1980) proposed to use $t_k = \frac{\pi k}{25}$, $k = 1, 2, \dots, K$; with K ranging between 9 and 134 for different estimates of α and sample sizes.

Once $\hat{\alpha}$ and $\hat{\sigma}$ have been obtained and α and σ have been fixed at these values, estimates of β and μ can be obtained using (1.14). Next, the regressions are repeated with $\hat{\alpha}$, $\hat{\sigma}$, $\hat{\beta}$ and $\hat{\mu}$ as the initial parameters. The iterations continue until a prespecified convergence criterion is satisfied.

Kogon and Williams (1998) eliminated this iteration procedure and simplified the regression method. For initial estimation they applied McCulloch's (1986) method, worked with the continuous representation (1.3) of the characteristic function instead of the classical one (1.2) and used a fixed set of only 10 equally spaced frequency points t_k . In terms of computational speed their method compares favorably to the original method of Koutrouvelis (1980). It has a significantly better performance near $\alpha = 1$ and $\beta \neq 0$ due to the elimination of discontinuity of the characteristic function. However, it returns slightly worse results for very small α .

1.4.4 Maximum likelihood method

The maximum likelihood (ML) estimation scheme for α -stable distributions does not differ from that for other laws, at least as far as the theory is concerned. For a vector of observations $x = (x_1, \dots, x_n)$, the ML estimate of the parameter vector $\theta = (\alpha, \sigma, \beta, \mu)$ is obtained by maximizing the log-likelihood function:

$$L_\theta(x) = \sum_{i=1}^n \ln \tilde{f}(x_i; \theta), \quad (1.16)$$

where $\tilde{f}(\cdot; \theta)$ is the stable pdf. The tilde denotes the fact that, in general, we do not know the explicit form of the density and have to approximate it

numerically. The ML methods proposed in the literature differ in the choice of the approximating algorithm. However, all of them have an appealing common feature – under certain regularity conditions the maximum likelihood estimator is asymptotically normal.

Modern ML estimation techniques either utilize the FFT approach for approximating the stable pdf (Mittnik et al., 1999) or use the direct integration method (Nolan, 2001). Both approaches are comparable in terms of efficiency. The differences in performance are the result of different approximation algorithms, see Section 1.2.2.

Simulation studies suggest that out of the five described techniques the method of moments yields the worst estimates, well outside any admissible error range (Stoyanov and Racheva-Iotova, 2004; Weron, 2004). McCulloch’s method comes in next with acceptable results and computational time significantly lower than the regression approaches. On the other hand, both the Koutrouvelis and the Kogon-Williams implementations yield good estimators with the latter performing considerably faster, but slightly less accurate. Finally, the ML estimates are almost always the most accurate, in particular, with respect to the skewness parameter. However, as we have already said, maximum likelihood estimation techniques are certainly the slowest of all the discussed methods. For example, ML estimation for a sample of a few thousand observations using a gradient search routine which utilizes the direct integration method is slower by 4 orders of magnitude than the Kogon-Williams algorithm, i.e. a few minutes compared to a few hundredths of a second on a fast PC! Clearly, the higher accuracy does not justify the application of ML estimation in many real life problems, especially when calculations are to be performed on-line.

1.5 Financial applications of stable laws

Many techniques in modern finance rely heavily on the assumption that the random variables under investigation follow a Gaussian distribution. However, time series observed in finance – but also in other applications – often deviate from the Gaussian model, in that their marginal distributions are heavy-tailed and, possibly, asymmetric. In such situations, the appropriateness of the commonly adopted normal assumption is highly questionable.

It is often argued that financial asset returns are the cumulative outcome of a vast number of pieces of information and individual decisions arriving almost continuously in time. Hence, in the presence of heavy-tails it is natural

Table 1.1: Fits to 2000 Dow Jones Industrial Average (DJIA) index returns from the period February 2, 1987 – December 29, 1994. Test statistics and the corresponding p -values based on 1000 simulated samples (in parentheses) are also given.

Parameters:	α	σ	β	μ
α -stable fit	1.6411	0.0050	-0.0126	0.0005
Gaussian fit		0.0111		0.0003
Tests:	Anderson-Darling		Kolmogorov	
α -stable fit	0.6441 (0.020)		0.5583 (0.500)	
Gaussian fit	+ ∞ (<0.005)		4.6353 (<0.005)	

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to assume that they are approximately governed by a stable non-Gaussian distribution. Other leptokurtic distributions, including Student's t , Weibull, and hyperbolic, lack the attractive central limit property.

Stable distributions have been successfully fit to stock returns, excess bond returns, foreign exchange rates, commodity price returns and real estate returns (McCulloch, 1996; Rachev and Mittnik, 2000). In recent years, however, several studies have found, what appears to be strong evidence against the stable model (Gopikrishnan et al., 1999; McCulloch, 1997). These studies have estimated the tail exponent directly from the tail observations and commonly have found α that appears to be significantly greater than 2, well outside the stable domain. Recall, however, that in Section 1.4.1 we have shown that estimating α only from the tail observations may be strongly misleading and for samples of typical size the rejection of the α -stable regime unfounded. Let us see ourselves how well the stable law describes financial asset returns.

In this section we want to apply the discussed techniques to financial data. Due to limited space we chose only one estimation method – the regression approach of Koutrouvelis (1980), as it offers high accuracy at moderate computational time. We start the empirical analysis with the most prominent example – the Dow Jones Industrial Average (DJIA) index, see Table 1.1. The data set

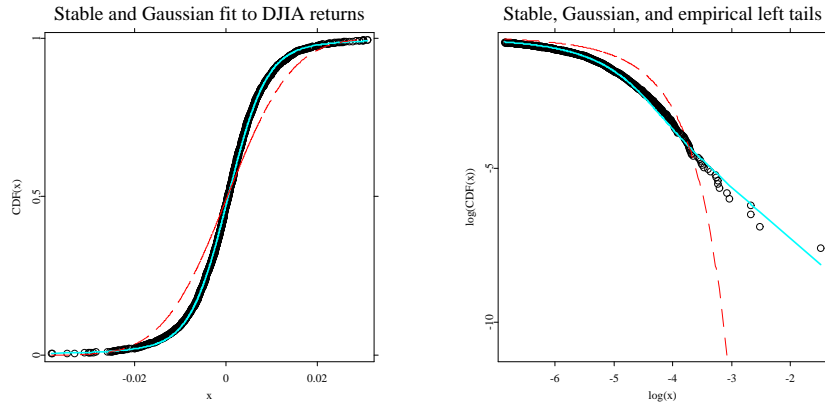


Figure 1.6: Stable (cyan) and Gaussian (dashed red) fits to the DJIA returns (black circles) empirical cdf from the period February 2, 1987 – December 29, 1994. Right panel is a magnification of the left tail fit on a double logarithmic scale clearly showing the superiority of the 1.64-stable law.

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covers the period February 2, 1987 – December 29, 1994 and comprises 2000 daily returns. Recall, that it includes the largest crash in Wall Street history – the Black Monday of October 19, 1987. Clearly the 1.64-stable law offers a much better fit to the DJIA returns than the Gaussian distribution. Its superiority, especially in the tails of the distribution, is even better visible in Figure 1.6.

To make our statistical analysis more sound, we also compare both fits through Anderson-Darling and Kolmogorov test statistics (D'Agostino and Stephens, 1986). The former may be treated as a weighted Kolmogorov statistics which puts more weight to the differences in the tails of the distributions. Although no asymptotic results are known for the stable laws, approximate p -values for these goodness-of-fit tests can be obtained via the Monte Carlo technique. First the parameter vector is estimated for a given sample of size n , yielding $\hat{\theta}$, and the test statistics is calculated assuming that the sample is distributed according to $F(x; \hat{\theta})$, returning a value of d . Next, a sample of size n of $F(x; \hat{\theta})$ -

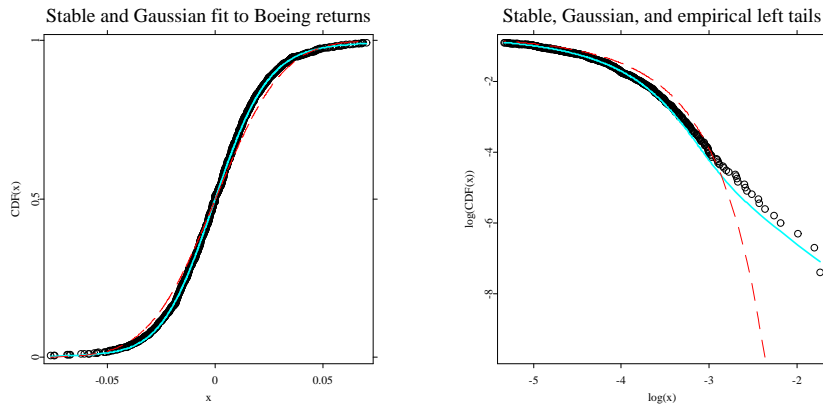


Figure 1.7: Stable (cyan) and Gaussian (dashed red) fits to the Boeing stock returns (black circles) empirical cdf from the period July 1, 1997 – December 31, 2003. Right panel is a magnification of the left tail fit on a double logarithmic scale clearly showing the superiority of the 1.78-stable law.

 STFstab07.xpl

distributed variates is generated. The parameter vector is estimated for this simulated sample, yielding $\hat{\theta}_1$, and the test statistics is calculated assuming that the sample is distributed according to $F(x; \hat{\theta}_1)$. The simulation is repeated as many times as required to achieve a certain level of accuracy. The estimate of the p -value is obtained as the proportion of times that the test quantity is at least as large as d .

For the α -stable fit of the DJIA returns the values of the Anderson-Darling and Kolmogorov statistics are 0.6441 and 0.5583, respectively. The corresponding approximate p -values based on 1000 simulated samples are 0.02 and 0.5 allowing us to accept the α -stable law as a model of DJIA returns. The values of the test statistics for the Gaussian fit yield p -values of less than 0.005 forcing us to reject the Gaussian law, see Table 1.1.

Next, we apply the same technique to 1635 daily returns of Boeing stock prices from the period July 1, 1997 – December 31, 2003. The 1.78-stable distribution fits the data very well, yielding small values of the Anderson-Darling (0.3756)

Table 1.2: Fits to 1635 Boeing stock price returns from the period July 1, 1997 – December 31, 2003. Test statistics and the corresponding p -values based on 1000 simulated samples (in parentheses) are also given.

Parameters:	α	σ	β	μ
α -stable fit	1.7811	0.0141	0.2834	0.0009
Gaussian fit		0.0244		0.0001
Tests:	Anderson-Darling		Kolmogorov	
α -stable fit	0.3756 (0.18)		0.4522 (0.80)	
Gaussian fit	9.6606 (<0.005)		2.1361 (<0.005)	

 STFstab07.xpl

and Kolmogorov (0.4522) test statistics, see Figure 1.7 and Table 1.2. The approximate p -values based, as in the previous example, on 1000 simulated samples are 0.18 and 0.8, respectively, allowing us to accept the α -stable law as a model of Boeing returns. On the other hand, the values of the test statistics for the Gaussian fit yield p -values of less than 0.005 forcing us to reject the Gaussian distribution.

The stable law seems to be tailor-cut for the DJIA index and Boeing stock price returns. But does it fit other asset returns as well? Unfortunately, not. Although, for most asset returns it does provide a better fit than the Gaussian law, in many cases the test statistics and p -values suggest that the fit is not as good as for these two data sets. This can be seen in Figure 1.8 and Table 1.3, where the calibration results for 4444 daily returns of the Japanese yen against the US dollar (JPY/USD) exchange rate from December 1, 1978 to January 31, 1991 are presented. The empirical distribution does not exhibit power-law tails and the extreme tails are largely overestimated by the stable distribution. For a risk manager who likes to play safe this may not be a bad idea, as the stable law overestimates the risks and thus provides an upper limit of losses. However, from a calibration perspective other distributions, like the hyperbolic or truncated stable, may be more appropriate for many data sets (Weron, 2004).

Table 1.3: Fits to 4444 JPY/USD exchange rate returns from the period December 1, 1978 – January 31, 1991. Test statistics and the corresponding p -values (in parentheses) are also given.

Parameters:	α	σ	β	μ
α -stable fit	1.3274	0.0020	-0.1393	-0.0003
Gaussian fit		0.0049		-0.0001
Tests:	Anderson-Darling		Kolmogorov	
α -stable fit	4.7833		1.4520	
	(<0.005)		(<0.005)	
Gaussian fit	91.7226		6.7574	
	(<0.005)		(<0.005)	

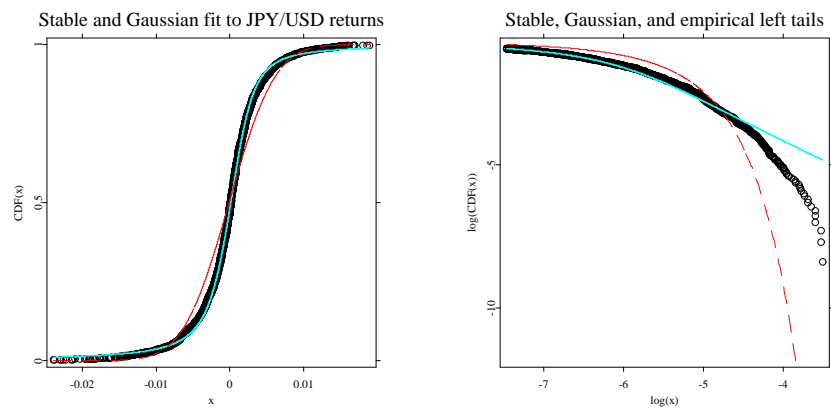


Figure 1.8: Stable (cyan) and Gaussian (dashed red) fits to the JPY/USD exchange rate returns (black circles) empirical cdf from the period December 1, 1978 – January 31, 1991. Right panel is a magnification of the left tail fit on a double logarithmic scale. The extreme returns are largely overestimated by the stable law.

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Index

- α -stable variable, 8
- algorithm
 - Box-Muller, 9
- asset return, 16
- Black Monday, 18
- Chambers-Mallows-Stuck method, 9
- characteristic function, 4
- distribution
 - Lévy stable, 2
 - stable,
 - stable distribution
 - stable Paretian, 2
- Dow Jones Industrial Average (DJIA), 18
- extreme event, 2
- Fourier transform, 5
- function
 - characteristic,
 - characteristic function
- goodness-of-fit, 18
- Hill estimator, 11
- Monte Carlo
 - method, 18
- power-law tail, 3
- stable distribution, 1
 - characteristic exponent, 2
 - density function, 6
 - direct integration, 6, 16
 - distribution function, 6
 - FFT-based approach, 6, 16
 - index of stability, 2
 - maximum likelihood method, 15
 - method of moments, 14
 - quantile estimation, 13
 - regression-type method, 14, 15
 - simulation, 8
 - skewness parameter, 2
 - tail exponent, 2
 - estimation, 11
 - tail index, 2
- stable laws, 16
 - asset returns, 1
- tail exponent, 2, 11
 - estimation, 11
 - log-log regression, 11
- test statistic
 - Anderson-Darling, 18
 - Kolmogorov, 18

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