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Masterarbeit

On the Number of Delaunay Triangles occurring in all Contiguous Subsequences

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Abstract

Given an ordered sequence of points $P = \{p_1, p_2, \ldots, p_n\}$, we consider all contiguous subsequences $P_{i,j} := \{p_i, \ldots, p_j\}$ of P and the set T of distinct Delaunay triangles within their Delaunay triangulations. For arbitrary point sets and orderings, we give an $O(n^2)$ bound on $|T|$. Furthermore, for arbitrary point sets in uniformly random order, we give two proofs of a $\Theta(n \log n)$ bound on $E[|T|]$.

Kurzfassung

Wir betrachten für eine geordnete Punktreihe $P = \{p_1, p_2, \ldots, p_n\}$ alle zusammenhängenden Teilreihen $P_{i,j} := \{p_i, \ldots, p_j\}$ und die Menge T aller verschiedenen Delaunay-Dreiecke in ihren Delaunay-Triangulierungen. Wir beweisen für beliebige Punktmengen und Reihenfolgen eine $O(n^2)$-Schranke für $|T|$. Weiterhin präsentieren wir für beliebige Punktmengen in zufällig gleichverteilter Reihenfolge zwei Beweise für eine $\Theta(n \log n)$-Schranke für $E[|T|]$.

Contents

1 Introduction and Related Work

Given an ordered sequence of points $P = \{p_1, p_2, \ldots, p_n\}$, we consider all contiguous subsequences $P_{i,j} := \{p_i, \ldots, p_j\}$ of P and their Delaunay triangulations $T_{i,j} := DT(P_{i,j})$. We are interested in the set T of distinct Delaunay triangles occurring in such triangulations, that is $T := \bigcup_{i<j} \{t | t \text{ triangle of } T_{i,j}\}$. [GKS92] gives an example where the Delaunay triangulations of $O(n)$ subsequences already contain $\Omega(n^2)$ distinct Delaunay triangles. We will prove that this is in fact a worst-case example, i.e. we will show that $O(n^2)$ is also an upper bound on $|T|$. More importantly, we show that for arbitrary point sets in uniformly random order, $E[|T|] \in \Theta(n \log n)$.

Our results give an optimistic outlook on the possibility of precomputing all Delaunay triangles of contiguous subsequences and indexing them in a clever way to allow quick retrieval of Delaunay triangulations of contiguous subsequences. Subcomplexes of the Delaunay triangulation, like α-shapes ([EKS83]) or the β-skeleton ([KR85]), could also be computed in this manner. In [BNH19], the authors already give one way of computing such a retrieval data structure for α-shapes, which they use to visualize storm events within the United States in certain time frames.

In an approach related to ours, Kaplan et al. in [KRS11] consider the randomized incremental construction of minimization diagrams and investigate the complexity of the overlay of all features, including removed ones, constructed during a run. The Delaunay triangulation can also be formulated as a minimization diagram, but Kaplan et al. only consider prefixes while we consider all contiguous subsequences. [GKS92] gives a randomized analysis of the incremental construction algorithm for the Delaunay triangulation and shows that when considering only prefixes, the number of Delaunay triangles is $\Theta(n)$ in expectation.

The following chapters are organized as follows. Chapter 2 gives a basic introduction to Delaunay triangulations. Chapters 3 and 4 give two ways of proving that for arbitrary point sets P in a uniformly random ordering, $E[|T|]$ is in $O(n \log n)$. Chapter 5 tightens the bound by describing two ways to prove an $\Omega(n \log n)$ bound on $E[|T|]$. Finally, chapter 6 summarizes this work and gives an outlook on future work.

Some of the proofs in this thesis have previously been presented digitally at the 36th European Workshop on Computational Geometry (EuroCG 2020) ([FW20]).

2 Preliminaries

In this chapter we explain some necessary preliminaries concerning Delaunay triangulations. Interested readers will find a much deeper introduction to the topic in [DVOC08].

2.1 The Delaunay triangulation

A Delaunay triangulation ([Del33], [Del+34]) of some point set $P \subseteq \mathbb{R}^2$ is a planar triangulation which maximizes the minimum internal angle. This has the useful effect of avoiding skinny triangles, where possible. Such a triangulation can be used to model geography, objects, or other spatial data, for example. When constructing height maps from a discrete set of height measurements, avoiding skinny triangles means we tend to avoid height interpolations from far away points when closer points are available. Figure 2.1 shows two ways to triangulate a point set of 4 points, one with two skinny triangles and one with two more pleasantly shaped triangles. Assuming the points represent height measurements, interpolating the height measurements at point p in the middle of the quadrilateral will interpolate between the two closer points if we avoid the skinny triangle situation on the left and instead choose the triangulation on the right. Note that requiring the triangulation to be planar means we have to allow the outside face to not be triangular if the convex hull of P isn't defined by exactly three points. However, all other faces will always be triangles.

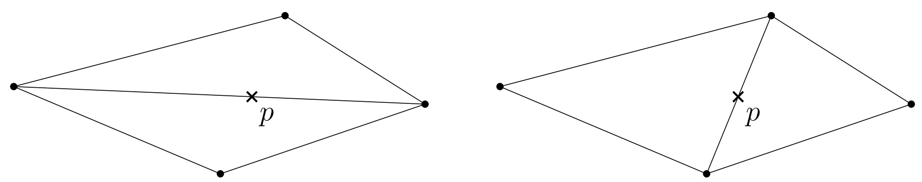

Figure 2.1: Two ways to triangulate a quadrilateral. **Left:** Skinny triangles, interpolating at p based on far away points. **Right:** The Delaunay triangulation, allowing interpolation at p based on closer points.

As usual, we will assume P to be non-degenerate in this thesis, i.e. we assume there are no four co-circular or three co-linear points. For such P, the Delaunay triangulation is unique. There are multiple equivalent ways to define the Delaunay triangulation, such as:

1. The triangulation maximizing the minimal internal angle
2. The triangulation where all triangles are *Delaunay*
3. The triangulation where all edges are *Delaunay*

We say a triangle $t = p_a p_b p_c$ is *Delaunay* if its circumcircle $cc(t)$ – the circle that goes through its three vertices – contains only p_a, p_b, and p_c on its boundary and no points of P in its interior. Similarly, we say an edge $e = \{p_a, p_b\}$ is *Delaunay* if there exists an open disk which contains only p_a and p_b on its boundary and no points of P in its interior. Figure 2.2 shows the Delaunay triangulation of a small point set with an overlay of the resulting triangles' empty circumcircles.

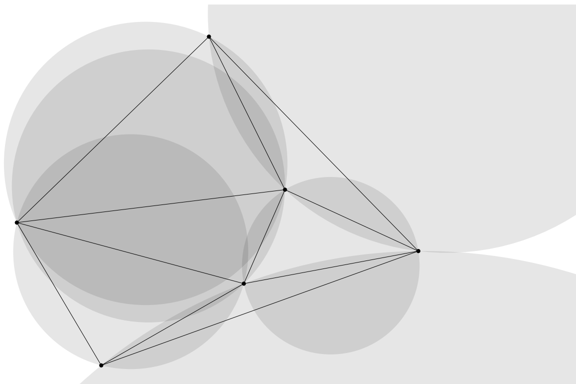

Figure 2.2: Delaunay triangulation of a few points and the triangles' empty circumcircles.

2.2 The Delaunay Flipping algorithm

This section describes a construction algorithm for the Delaunay triangulation. As such, this section is not strictly necessary to understand this thesis, but it provides some background for the original context of Lemma 5.

Another definition of the Delaunay triangulation can be given using the *local Delaunay* property: A triangulation is Delaunay if all its edges are locally Delaunay; An edge is said to be *locally Delaunay* if it is part of the convex hull of P, or if its two adjacent triangles don't contain each others' third vertices in their circumcircles. This definition provides a more efficient way to check whether some triangulation is Delaunay. It also leads to the *Delaunay flipping algorithm* which turns any triangulation into the Delaunay triangulation by repeatedly flipping edges that don't satisfy the local Delaunay property, *illegal edges*. To flip an edge means to replace it with the edge connecting its two adjacent triangles' third points. It can be shown that flipping an illegal edge produces a locally Delaunay edge, and that this process terminates and computes the Delaunay triangulation. Figure 2.1 actually shows such an edge flip which legalizes the edge in the center.

The Flipping algorithm can also be used to compute the Delaunay triangulation incrementally, by repeatedly inserting a new point, connecting it with all points visible from its position, and running the flipping algorithm based on the edges connecting these visible points. When inserting points in a uniformly random order, one can show an expected runtime of $O(n \log n)$ ([GKS92]).

3 Upper bound: Proof by edge counting

In this version of the proof we first show that the expected number of Delaunay edges in all contiguous subsequences of P is $O(n \log n)$. We then apply that bound to Delaunay triangles by showing that the number of Delaunay edges is equal to the number of Delaunay triangles, up to a constant factor.

3.1 Bounding the number of edges

We consider the set of Delaunay edges occurring in the Delaunay triangulations of contiguous subsequences of P, assuming $n > 2$:

$$E_T := \{e \mid \exists\, t \in T : e \text{ edge of } t\}$$

The following Lemma allows us to focus on smaller subsequences when considering specific potential Delaunay edges $\{p_i, p_j\}$.

▶ **Lemma 1** *Any edge $e = \{p_i, p_j\} \in E_T$ (w.l.o.g. $i < j$) appears in $T_{i,j}$.*

Proof. There exists some $T_{a,b}$ with $a \le i, b \ge j$ in which e is a Delaunay edge, so there must exist a disk with p_i, p_j on its boundary and no points from $P_{a,b}$ in its interior. As $P_{i,j} \subseteq P_{a,b}$, the disk's interior is also free of points from $P_{i,j}$, hence $e \in T_{i,j}$. ☐

Lemma 1 states that it suffices to consider the minimal contiguous subsequence containing p_i and p_j to argue about the probability of the edge $\{p_i, p_j\}$ being present in E_T. We can thus turn our attention towards $T_{i,j}$ to bound the probability that an edge $e = \{p_i, p_j\}$ is a Delaunay edge.

▶ **Lemma 2** *For a potential Delaunay edge $e = \{p_i, p_j\}$, $i < j$, we have $Pr[e \in T_{i,j}] < \frac{6}{j-i}$.*

Proof. For $j = i + 1$, the claim holds because $Pr[e \in T_{i,j}] \le 1 < \frac{6}{j-i}$. For $j > i + 1$, observe that when considering the point set $P_{i,j}$, clearly $T_{i,j}$ will be the same regardless of how the points in $P_{i,j}$ are labelled by their ordering. $T_{i,j}$ is a planar graph with $j - i + 1 > 2$ nodes, and hence per Euler's formula contains at most $3(j - i + 1) - 6$ edges. All points in $P_{i,j}$ are equally likely to be p_i, or p_j. $Pr[e \in T_{i,j}]$ is thus bounded by the probability of two randomly chosen nodes in a graph with $j - i + 1$ nodes and $\le 3(j - i + 1) - 6$ edges to be connected with an edge. By randomly choosing two nodes, we randomly choose one edge amongst all potential $\binom{j-i+1}{2}$ edges. The probability of that edge to be one of the $\le 3(j - i + 1) - 6$ edges of $T_{i,j}$ is $\le \frac{3(j-i+1)-6}{\binom{j-i+1}{2}} < \frac{6}{j-i}$. ☐

Armed with the knowledge that $Pr[e \in E_T] = Pr[e \in T_{i,j}]$ from Lemma 1, we can use Lemma 2 to bound the expected size of E_T.

▶ **Lemma 3** *The expected size of E_T is $O(n \log n)$.*

Proof. By linearity of expectation we can simply sum over all potential $\binom{n}{2}$ edges to obtain a bound on $E[|E_T|]$, using Lemma 2 to bound edges' probability of existence.

$$\begin{aligned} E[|E_T|] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr[\{p_i, p_j\} \in E_T] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr[\{p_i, p_j\} \in T_{i,j}] \\ &< \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{6}{j-i} \\ &= 6 \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \frac{1}{j} \\ &\leq 6 \sum_{i=1}^{n-1} H_n \end{aligned}$$

And thus $E[|E_T|] \in O(n \log n)$. □

3.2 Applying the edge bound to triangles

Note that generally, Delaunay edges may be used by many Delaunay triangles across all the $T_{i,j}$, some may even be used by $\Omega(n)$ different Delaunay triangles, so it is not obvious that our bound on edges also applies to T. The following Lemma shows why it does.

▶ **Lemma 4** $|T| \in \Theta(|E_T|)$ *for arbitrary point sets and arbitrary orderings.*

Proof. Consider some Delaunay triangle $t = p_a p_b p_c \in T$ (w.l.o.g. $a < b < c$). Due to Lemma 1, we have $t \in T_{a,c}$. Apart from t, there can exist at most one other triangle $t' \in T_{a,c}$ adjacent to the edge $\{p_a, p_c\}$. This way, we can charge every Delaunay triangle of T to some Delaunay edge of E_T, charging at most 2 triangles to any edge. So the overall number of Delaunay triangles is at most twice the overall number of Delaunay edges, hence $|T| \in O(|E_T|)$. Also observe that for any Delaunay triangle $t \in T$, there are at most 3 Delaunay edges $e \in E_T$, so $|E_T| \in O(|T|)$. Together, we get $|T| \in \Theta(|E_T|)$. □

▶ **Corollary 1** $|T| \in O(n^2)$ *for arbitrary point sets and orderings.*

Proof. There are only $\binom{n}{2}$ potential Delaunay edges, so $|E_T|, |T|$ are always $O(n^2)$. □

We can now state our main theorem.

▶ **Theorem 1** *The expected number of distinct Delaunay triangles occurring in all contiguous subsequences of a uniformly randomly ordered point set of size n is $O(n \log n)$.*

Proof. Follows from Lemmas 3 and 4. □

4 Upper bound: Proof by backwards analysis

In this version of the proof we use backwards analysis to bound the expected number of conflicts between triangles created by the insertion of some point p_j into $T_{i,j-1}$, and all points of P. A triangle t is in conflict with a point p if and only if p is contained in t's circumcircle. We introduce $k(t) := |cc(t) \cap P|$ to denote the number of points from P inside a triangle's circumcircle. We then use that conflict bound to show that $E[|T|] \in O(n \log n)$.

4.1 Bounding the number of conflicts

Disclaimer: This section (4.1) follows the proof from [SF19], starting after the proof of Lemma 4.12 and ending shortly before section 4.9. Adjustments were made mainly to adapt the proof to infixes of P, opposed to prefixes of P as in the original work. This section is included here so the proof for Theorem 1 can be understood without consulting further literature for a proof of Lemma 5.

▶ **Lemma 5** $E\left[\sum_{t \in T_{i,j} \setminus T_{i,j-1}} k(t)\right] \in O(\frac{n-j+i-1}{j-i+1})$.

Proof. $T_{i,j} \setminus T_{i,j-1}$ are exactly those triangles in $T_{i,j}$ that have p_j as a vertex. Since p_j is a uniformly random point of $T_{i,j}$, and each triangle in $T_{i,j}$ has three vertices, we have

$$E\left[\sum_{t \in T_{i,j} \setminus T_{i,j-1}} k(t)\right] = E\left[\sum_{\text{t has } p_j \text{ as a vertex}} k(t)\right] = \frac{3}{j-i+1} E\left[\sum_{t \in T_{i,j}} k(t)\right]$$

On the other hand, $T_{i,j} \setminus T_{i,j+1}$ are exactly those triangles in $T_{i,j}$ that have p_{j+1} in their circumcircle. Since p_{j+1} is a uniformly random point of $P \setminus P_{i,j}$, we have

$$E[|T_{i,j} \setminus T_{i,j+1}|] = \frac{1}{n-j+i-1} E\left[\sum_{t \in T_{i,j}} k(t)\right]$$

Combining these two equations results in

$$E\left[\sum_{t \in T_{i,j} \setminus T_{i,j-1}} k(t)\right] = \frac{3(n-j+i-1)}{j-i+1} E[|T_{i,j} \setminus T_{i,j+1}|]$$

In any step, the number of destroyed triangles is exactly two less than the number of created triangles. The latter is O(1) in expectation, hence $E\left[\sum_{t \in T_{i,j} \setminus T_{i,j-1}} k(t)\right] \leq \frac{3c(n-j+i-1)}{j-i+1}$ for some appropriate c. ◻

4.2 Bounding the number of triangles

We define an order of triangulations $T_{i,j}$, allowing us to associate each triangle t of T with the $T_{i,j}$ it *first* appears in.

$$T_{a,b} < T_{c,d} \iff (a < c \vee (a = c \wedge b < d))$$

We can now count Delaunay triangles:

$$E[|T|] = E\left[\sum_{i=1}^{n}\sum_{j=i}^{n} |\{t \in T | t \text{ first appears in } T_{i,j}\}|\right]$$

Observe that for $i = 1$, we count exactly the triangles one encounters when executing the traditional randomized incremental construction algorithm. [GKS92] gives an $O(n)$ bound with a constant of $3e^2$ on the expected number of such triangles, so we focus on the cases where $i > 1$. We also disregard $T_{i,j}$ with $j - i < 2$ as these $T_{i,j}$ don't contain any triangles.

$$E[|T|] \leq 3e^2 n + E\left[\sum_{i=2}^{n-2}\sum_{j=i+2}^{n} |\{t \in T | t \text{ first appears in } T_{i,j}\}|\right]$$

We restate the second term using the fact that any t which first appears in some $T_{i,j}$ surely didn't appear in $T_{i,j-1}$:

$$E[|T|] \leq 3e^2 n + \sum_{i=2}^{n-2}\sum_{j=i+2}^{n}\sum_{t \in T_{i,j} \setminus T_{i,j-1}} Pr[t \text{ first appears in } T_{i,j}]$$

To bound $Pr[t \text{ first appears in } T_{i,j}]$ we consider a necessary condition of t being new in $T_{i,j}$:

▶ **Lemma 6** *A triangle t which first appears in $T_{i,j}$, $i > 1$, contains p_{i-1} in its circumcircle.*
Proof. If $p_{i-1} \notin cc(t)$, t would be Delaunay in $T_{i-1,j}$, but t only appears later in $T_{i,j}$. ☐

Using Lemma 6 and the fact that p_{i-1} is a uniformly random point of $P \setminus P_{i,j}$, we have for $i > 1$:

$$Pr[t \text{ first appears in } T_{i,j}] \leq Pr[p_{i-1} \in cc(t)] = \frac{k(t)}{n - j + i - 1}$$

Plugging this bound and Lemma 5 into our $E[|T|]$ bound, we can prove our main theorem.

▶ **Theorem 1** *The expected number of distinct Delaunay triangles occurring in all contiguous subsequences of a uniformly randomly ordered point set of size n is $O(n \log n)$.*

Proof.

$$\begin{aligned} E[|T|] &\leq 3e^2 n + \sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \sum_{t \in T_{i,j} \setminus T_{i,j-1}} \frac{k(t)}{n-j+i-1} \\ &\leq 3e^2 n + \sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \frac{1}{n-j+i-1} \frac{3c(n-j+i-1)}{j-i+1} \\ &= 3e^2 n + 3c \sum_{i=2}^{n-2} \sum_{j=i+2}^{n} \frac{1}{j-i+1} \\ &\leq 3e^2 n + 3c \sum_{i=2}^{n-2} \sum_{j=1}^{n-i-1} \frac{1}{j} \\ &\leq 3e^2 n + 3c \sum_{i=2}^{n-2} H_n \end{aligned}$$

And thus $E[|T|] \in O(n \log n)$. □

5 Lower bound: Two simple proofs

We give two simple ways to prove the $\Omega(n \log n)$ lower bound on $E[|T|]$, one using nearest neighbor graphs, and one using graph connectivity.

▶ **Theorem 2** *The expected number of distinct Delaunay triangles occurring in all contiguous subsequences of a uniformly randomly ordered point set of size n is $\Omega(n \log n)$.*

Proof. The nearest neighbor graph of a point set P is a subgraph of its Delaunay triangulation, and if P is in uniformly random order, p_1's nearest neighbor changes $\Theta(\log n)$ times within $P_{1,1}, \ldots, P_{1,n}$, in expectation. This gives us an expected $\Theta(\log n)$ nearest neighbor edges incident to p_1 within $T_{1,1}, \ldots, T_{1,n}$. The same argument applies to p_2 and the sequence $T_{2,2}, \ldots, T_{2,n}$, and so on. Aggregated over all n points, we get $\Theta(n \log n)$ nearest neighbor edges in expectation, giving us an $\Omega(n \log n)$ lower bound on Delaunay edges. ◻

Proof. Any Delaunay triangulation of n points is a connected graph, so it must have at least $n-1$ edges. Applying the argument from Lemma 2, we get $Pr[e \in T_{i,j}] \geq \frac{j-i}{\binom{j-i+1}{2}} = \frac{2}{j-i+1}$. We can now once again use linearity of expectation to sum over all potential edges and obtain a bound on $E[|E_T|]$, which per Lemma 4 also applies to $E[|T|]$.

$$\begin{aligned}
E[|E_T|] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr[\{p_i, p_j\} \in E_T] \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} Pr[\{p_i, p_j\} \in T_{i,j}] \\
&\geq \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{j=2}^{n-i+1} \frac{2}{j} \\
&\geq \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \frac{1}{j} \\
&\geq \sum_{i=1}^{\lfloor n/2 \rfloor} H_{\lceil n/2 \rceil}
\end{aligned}$$

And thus $E[|E_T|], E[|T|] \in \Omega(n \log n)$. ◻

6 Conclusion and Outlook

We have shown that the expected number of distinct Delaunay triangles occurring in all contiguous subsequences of a uniformly randomly ordered point set of size n is $\Theta(n \log n)$. We did so by giving two proofs each for both the $\Omega(n \log n)$ bound, and the $O(n \log n)$ bound. We also gave an $O(n^2)$ bound on $|T|$ which is tight in the worst case.

A natural next step will be to apply our results and observations on Delaunay triangulations of contiguous subsequences to create an algorithm to actually compute T. It would also be interesting to see how our complexity results generalize to higher dimensions, such as $\mathbb{R}^3$.

Bibliography

[BNH19] A. Bonerath, B. Niedermann, J. Haunert. “Retrieving α-Shapes and Schematic Polygonal Approximations for Sets of Points within Queried Temporal Ranges”. In: *SIGSPATIAL/GIS*. ACM, 2019, pp. 249–258 (cit. on p. 7).

[Del+34] B. Delaunay et al. “Sur la sphere vide”. In: *Izv. Akad. Nauk SSSR, Otdelenie Matematicheskii i Estestvennyka Nauk* 7.793-800 (1934), pp. 1–2 (cit. on p. 9).

[Del33] B. Delaunay. “Neue Darstellung der geometrischen Kristallographie”. In: *Zeitschrift für Kristallographie-Crystalline Materials* 84.1-6 (1933), pp. 109–149 (cit. on p. 9).

[DVOC08] M. De Berg, M. Van Kreveld, M. Overmars, O. Cheong. *Computational Geometry*. Springer, 2008 (cit. on p. 9).

[EKS83] H. Edelsbrunner, D. G. Kirkpatrick, R. Seidel. “On the shape of a set of points in the plane”. In: *IEEE Trans. Information Theory* 29.4 (1983), pp. 551–558 (cit. on p. 7).

[FW20] S. Funke, F. Weitbrecht. “On the Number of Delaunay Triangles occurring in all Contiguous Subsequences”. In: *EuroCG*. 2020, pp. 263–266 (cit. on p. 7).

[GKS92] L. J. Guibas, D. E. Knuth, M. Sharir. “Randomized incremental construction of Delaunay and Voronoi diagrams”. In: *Algorithmica* 7.1 (June 1992), pp. 381–413. ISSN: 1432-0541 (cit. on pp. 7, 10, 14).

[KR85] D. G. Kirkpatrick, J. D. Radke. “A Framework for Computational Morphology”. In: *Computational Geometry*. Ed. by G. T. Toussaint. Vol. 2. Machine Intelligence and Pattern Recognition. North-Holland, 1985, pp. 217–248 (cit. on p. 7).

[KRS11] H. Kaplan, E. Ramos, M. Sharir. “The overlay of minimization diagrams in a randomized incremental construction”. In: *Discrete & Computational Geometry* 45.3 (2011), pp. 371–382 (cit. on p. 7).

[SF19] S. Funke. *Lecture notes for Computational Geometry*. (only accessible from within university networks). 2019. URL: https://fmi.uni-stuttgart.de/files/alg/teaching/s19/compgeo/scribe_notes_19.pdf (visited on 04/17/2020) (cit. on p. 13).

Declaration

I hereby declare that the work presented in this thesis is entirely my own and that I did not use any other sources and references than the listed ones. I have marked all direct or indirect statements from other sources contained therein as quotations. Neither this work nor significant parts of it were part of another examination procedure. I have not published this work in whole or in part before. The electronic copy is consistent with all submitted copies.

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