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Null Players, Outside Options, and Stability

The Conditional Shapley Value

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Abstract
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Null players, outside options, and stability: the conditional Shapley value☆

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Abstract

We suggest a new component efficient solution for monotonic TU games with a coalition structure, the conditional Shapley value. Other than other such solutions, it satisfies the null player property. Nevertheless, it accounts for the players' outside options in productive components of coalition structures. For all monotonic games, there exist coalition structures that are stable under the conditional Shapley value. For voting games, the stability of coalition structures under the conditional Shapley value supports Gamson's theory of coalition formation (Gamson, Am Sociol Rev 26, 1961, 373–382).

Keywords: TU game, Shapley value, outside options, null player, stability, Gamson's law
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1. Introduction

The Shapley value (Shapley, 1953) probably is the most eminent one-point solution concept for cooperative games with transferable utility (TU games). Since the Shapley value is efficient. i.e., the worth generated by the grand coalition is distributed among the players, the implicit underlying assumption is that the grand coalition is the productive unit, i.e., all players cooperate in order to generate worth. In view of Young's (1985) characterization, the Shapley value can be viewed as *the* efficient solution that reflects the players' individual productivities in TU games.

A simple way to model more general production arrangements are coalition structures, i.e., partitions of the player set where its components are interpreted as the productive units. Solutions for TU games that are enriched by a coalition structure (CS games, CS solutions) reflect this interpretation when they are component efficient, i.e., the worth generated by a

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component of the coalition structure is distributed among its members.[1]

The component efficient CS solution suggested by Aumann and Drèze (1974), determines the players' payoffs by applying the Shapley value to the subgames induced by the coalition structure. This way, only the players' productivity within their components is taken into account but not their joint productivity with players outside their components. Outside options, however, might be important:

> Any particular alliance describes only one particular consideration which enters the minds of the participants when they plan their behavior. Even if a particular alliance is ultimately formed, the division of the proceeds between the allies will be decisively influenced by the other alliances which each one might alternatively have entered. [...] Even if [...] one particular alliance is actually formed, the others are present in "virtual" existence: Although they have not materialized, they have contributed essentially to shaping and determining the actual reality. (von Neumann and Morgenstern, 1944, p. 36)

> During the course of negotiations there comes a moment when a certain coalition structure is "crystallized". The players will no longer listen to "outsiders", yet each coalition has still to adjust the final share of its proceeds. (This decision may depend on options outside the coalition, even though the chances of defection are slim). (Maschler, 1992, pp. 595)

Wiese (2007) and Casajus (2009) suggest component efficient CS solutions that take into account outside options, the outside option value and the χ-value. As the Aumann-Drèze value, these solutions coincide with the Shapley value for the trivial coalition structure containing the grand coalition as the only component. In this sense, these CS solutions generalize the Shapley value. In contrast to the Shapley value and the Aumann-Drèze value, however, the solutions due to Wiese and Casajus fail the null player property, i.e., totally unproductive players may obtain a negative or positive payoff. Indeed, Casajus (2009, p. 52) demonstrates that a component efficient CS solution satisfying the null player property necessarily must neglect some of the outside options of players in CS games.

In some situations, negative or positive payoffs for null players are not that plausible or interpretable. In particular, this seems to be the case for voting games, i.e., simple superadditive games. In such games, coalitions create either a worth of zero, indicating a losing coalition, or a worth of one, indicating a winning coalition. Moreover, adding players never turns a winning coalition into a losing coalition. Further, the complement of a winning coalition is losing. A winning coalition in a coalition structure can be viewed as the government coalition in a parliament, for example, where the players stand for political parties. The payoffs of the parties in a government coalition then may reflect their relative

[1] Alternatively, the components can be interpreted as bargaining units formed during the process of bargaining on the distribution of the worth generated by the grand coalition as the productive unit. CS solutions that fit this interpretation are efficient, for example, the CS solutions suggested by Owen (1977) and Kamijo (2009).

power or influence within the government, which is expressed by the (relative) number of ministries a party can claim for itself, for example.

In this paper, we suggest a CS solution for monotonic games that satisfies the null player property, the *conditional Shapley value*. It is characterized by five properties, four standard properties, component efficiency, symmetry within components, the null player property, additivity restricted to the trivial partition containing only the grand coalition, and a new property, the relative splitting property (Theorem 3). Component efficiency: the worth generated by a component is distributed among its members. Symmetry within components: equally productive players who inhabit the same component obtain the same payoff. Null player property: unproductive players obtain a zero payoff. Additivity: the payoffs for the sum of two games equal the sum of the payoffs for the single games. The relative splitting property is a relative version of the (absolute) splitting property (Casajus, 2009). Splitting: whenever components split, players who stay together gain or lose the same amount of payoff; put differently, the difference of their payoffs doesn't change. *Relative splitting*: whenever components split, players who stay together gain or lose proportionally by the same factor; put differently, the ratio of their payoffs doesn't change.

For all monotonic games, there exist coalition structures that are stable under the conditional Shapley value in the following sense (Theorem 4). There exists no coalition that can deviate from that coalition structure and make all its members better off no matter how the other players organize into groups. The conditional Shapley value this feature with the χ-value (Casajus, 2009, Theorem 6.1). In contrast, such stable coalition structures may not exist under the CS solutions due to Aumann and Drèze (1974) and Wiese (2007) for games with more than three players (Tutic, 2010).

In voting games, a player's payoff/power equals the probability that she is the pivot in a rank order, i.e., the coalition of her predecessors is losing and turns into winning coalition when she enters. In Section 4.1, we demonstrate that a player's power in a winning component/coalition is the probability that she is the pivot in a rank order in which one of the players of her coalition is the pivot. Under the conditional Shapley value, a partition is stable for a voting game if it contains a component that comprises a minimal winning coalition with minimal total power (Theorem 7). This supports Gamson's (1961) theory of coalition formation.

> It is clear, then, that the prediction from our model will be simply the *cheapest winning coalition* in the applicable situations. (Gamson, 1961, p. 377)

This paper is organized as follows. In the second section, we provide basic definitions and notation. In the third section, we introduce our new CS solution, the conditional Shapley value, and study the stability of coalition structures under this solution. In the fourth section, we apply the conditional Shapley value to voting games. So far, no remarks conclude the paper.

2. Basic definitions and notation

A TU game for a finite player set N is given by a **coalition function** $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, where 2^N denotes the power set of N. Subsets of N are called **coalitions**; $v(S)$ is called the worth of coalition S. The set of all games is denoted by $\mathbb{V}(N)$. Frequently, we address $v \in \mathbb{V}$ as a game. For $S \subseteq N$, $S \neq \emptyset$, the **subgame** $v|_S \in \mathbb{V}(S)$ of $v \in \mathbb{V}(N)$ with respect to S is given by $v|_S(T) = v(T)$ for all $T \subseteq S$.

For $v, w \in \mathbb{V}$, $\alpha \in \mathbb{R}$, and $T \subseteq N$, the coalition functions $v+w \in \mathbb{V}(N)$ and $\alpha \cdot v \in \mathbb{V}(N)$ are given by $(v+w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$. The game $\mathbf{0} \in \mathbb{V}(N)$ given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$ is called the **null game**. For $T \subseteq N$, $T \neq \emptyset$, the game $u_T \in \mathbb{V}(N)$, $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, is called a **unanimity game**. Any $v \in \mathbb{V}$ can be uniquely represented by unanimity games. In particular, we have

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T, \tag{1}$$

where the coefficients $\lambda_T(v)$ are known as the Harsanyi dividends (Harsanyi, 1959) and can be determined recursively by

$$\lambda_T(v) := v(T) - \sum_{S \subsetneq T : S \neq \emptyset} \lambda_S(v). \tag{2}$$

A game $v \in \mathbb{V}(N)$ is called **monotonic**, if $v(S) \leq v(T)$ for all $S, T \subseteq N$ such that $S \subseteq T$; a game $v \in \mathbb{V}(N)$ is called **superadditive**, if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ such that $S \cap T \neq \emptyset$; a game $v \in \mathbb{V}(N)$ is called **simple**, if $v(S) \in \{0, 1\}$ for all $S \subseteq N$. The set of all monotonic games is denoted by $\mathbb{M}(N)$; the set of all simple superadditive games is denoted by $\mathbb{S}(N)$.

A rank order for N is a bijection $\rho : N \to \{1, 2, \ldots, |N|\}$; the set of all rank orders is denoted by $R(N)$. For $v \in \mathbb{V}(N)$, $i \in N$, and $\rho \in R(N)$, let $P_i(\rho)$ denote the set of players before i in ρ, i.e.,

$$P_i(\rho) := \{j \in N \mid \rho(j) < \rho(i)\},$$

and let $MC_i^v(\rho)$ denote the marginal contribution of player i in v for ρ, i.e.,

$$MC_i^v(\rho) = v(P_i(\rho) \cup \{i\}) - v(P_i(\rho)).$$

A **solution** for a subset $\mathcal{V}$ of $\mathbb{V}(N)$ is a mapping $\varphi : \mathcal{V} \to \mathbb{R}^N$, which assigns a payoff $\varphi_i(v)$ to any player $i \in N$ for any game $v \in \mathcal{V}$. The **Shapley value** (Shapley, 1953), Sh, is given by

$$\mathrm{Sh}_i(v) := \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{|T|} = \sum_{\rho \in R(N)} \frac{MC_i^v(\rho)}{|R(N)|}, \qquad \text{for all } v \in \mathbb{V}(N),\ i \in N. \tag{3}$$

It is characterized by four properties, efficiency, additivity, symmetry, and the null player property.

Efficiency, E. For all $v \in \mathbb{V}(N)$, we have $\sum_{\ell \in N} \varphi_\ell(v) = v(N)$.

Additivity, A. For all $v, w \in \mathbb{V}(N)$, we have $\varphi(v+w) = \varphi(v) + (w)$.

Symmetry, S. For all $v \in \mathbb{V}(N)$ and $i, j \in N$ such that i and j are symmetric in v, we have $\varphi_i(v) = \varphi_j(v)$.

Null player, N. For all $v \in \mathbb{V}(N)$ and $i \in N$ such that i is a null player in v, we have $\varphi_i(v) = 0$.

A **coalition structure** for N is a partition $\mathcal{P}$ of N; let $\mathfrak{P}(N)$ denote the set of all coalition structures for N. The component of player $i \in N$ in $\mathcal{P} \in \mathfrak{P}(N)$ is denoted by $\mathcal{P}(i) \in \mathcal{P}$. A coalition structure $\mathcal{P} \in \mathbb{P}(N)$ is **finer** than a coalition structure $\mathcal{Q} \in \mathfrak{P}(N)$, if $\mathcal{P}(i) \subseteq \mathcal{Q}(i)$ for all $i \in N$. The **restriction** $\mathcal{P}|_S \in \mathfrak{P}(S)$ of $\mathcal{P} \in \mathfrak{P}(N)$ to $S \subseteq N$ is given by $\mathcal{P}|_S(i) = \mathcal{P}(i) \cap S$ for all $i \in S$.

A pair $(v, \mathcal{P}) \in \mathbb{V}(N) \times \mathfrak{P}(N)$ is called a **CS game**. A **CS solution** for $\mathcal{V} \subseteq \mathbb{V}$ is a mapping $\mathcal{V} \times \mathfrak{P}(N) \to \mathbb{R}^N$, which assigns a payoff $\varphi_i(v, \mathcal{P})$ to any player $i \in N$ for any CS game $(v, \mathcal{P}) \in \mathcal{V} \times \mathfrak{P}(N)$. The Aumann-Drèze value (Aumann and Drèze, 1974), AD, is given by

$$\mathrm{AD}_i(v, \mathcal{P}) = \mathrm{Sh}_i\left(v|_{\mathcal{P}(i)}\right) \qquad \text{for all } (v, \mathcal{P}) \in \mathbb{V}(N) \times \mathfrak{P}(N) \text{ and } i \in N. \tag{4}$$

The χ-value (Casajus, 2009), χ, is given by

$$\chi_i(v, \mathcal{P}) = \mathrm{Sh}_i(v) + \frac{v(\mathcal{P}(i)) - \sum_{\ell \in \mathcal{P}(i)} \mathrm{Sh}_\ell(v)}{|\mathcal{P}(i)|} \qquad \text{for all } (v, \mathcal{P}) \in \mathbb{V}(N) \times \mathfrak{P}(N) \text{ and } i \in N. \tag{5}$$

3. The conditional Shapley value for monotonic games

In this section, we introduce a new component efficient extension of the Shapley value to monotonic games with a coalition structure, i.e., a CS solution that coincides with the Shapley value for monotonic CS games with the trivial coalition structure containing only the grand coalition.

3.1. Motivation and definition

The Aumann-Drèze value for the full class of CS games is characterized by four properties: component efficiency, additivity, symmetry within components, and the null player property (Aumann and Drèze, 1974, Theorem 3). One easily checks that this characterization also works within the class of monotonic CS games.

Component efficiency, CE. For all $v \in \mathbb{V}(N)$, $\mathcal{P} \in \mathfrak{P}(N)$, and $P \in \mathcal{P}$, we have $\sum_{\ell \in P} \varphi_\ell(v, \mathcal{P}) = v(P)$.

Additivity, A. For all $v, w \in \mathbb{V}(N)$, $\mathcal{P} \in \mathfrak{P}(N)$, and $i \in N$, we have $\varphi_i(v+w, \mathcal{P}) = \varphi_i(v, \mathcal{P}) + \varphi_i(w, \mathcal{P})$.

Symmetry within components, CS. For all $v \in \mathbb{V}(N)$, $\mathcal{P} \in \mathfrak{P}(N)$, and $i, j \in N$ such that i and j are symmetric in v and $\mathcal{P}(i) = \mathcal{P}(j)$, we have $\varphi_i(v, \mathcal{P}) = \varphi_j(v, \mathcal{P})$.

Null player, N. For all $v \in \mathbb{V}(N)$, $\mathcal{P} \in \mathfrak{P}(N)$, and $i \in N$ such that i is a null player in v, we have $\varphi_i(v, \mathcal{P}) = 0$.

In view of its definition it is clear that Aumann-Drèze value cannot recognize the players' outside options. Since component efficiency is at the heart of the interpretation of the CS solutions under consideration and since symmetry within components is a natural relaxation of symmetry for CS solutions, one has to relax either additivity or the null player property in order to allow a CS solution to take into account outside options. In his characterization of the χ-value, Casajus (2009, Theorem 4.1) relaxes the null player property by restricting its applicability to the trivial coalition structure, $\{N\}$, where there are no outside options.

Grand coalition null player, GN. For all $v \in \mathbb{V}(N)$ and $i \in N$ such that i is a null player in v, we have $\varphi_i(v, \{N\}) = 0$.

There exist many CS solutions that satisfy component efficiency, additivity, symmetry within components, and the grand coalition null player property. In order to single out one particular of them, Casajus (2009) introduces and imposes the (absolute) splitting property.

Splitting, SP. For all $v \in \mathbb{V}(N)$, $\mathcal{P}, \mathcal{Q} \in \mathfrak{P}(N)$, and $i, j \in N$ such that $\mathcal{P}$ is finer than $\mathcal{Q}$ and $\mathcal{P}(i) = \mathcal{P}(j)$, we have

$$\varphi_i(v, \mathcal{P}) - \varphi_i(v, \mathcal{Q}) = \varphi_j(v, \mathcal{P}) - \varphi_j(v, \mathcal{Q}).$$

Whenever components split, this property requires that players who stay together to gain or lose the same amount of payoff, i.e., to gain or lose equally in *absolute* terms. One easily checks that the characterization below also works within the class of monotonic CS games.

Theorem 1 (Casajus, 2009). *The χ-value is the unique CS solution that satisfies component efficiency (**CE**), additivity (**A**), symmetry within components (**CS**), the grand coalition null player property (**GN**), and the splitting property (**SP**).*

The χ-value fails the null player property. Indeed, Casajus (2009, p. 52) demonstrates that a component efficient CS solution satisfying the null player property necessarily must neglect some of the outside options of players in CS games. So, if one wants a CS solution to obey the null player property one necessarily has to neglect some of the outside options. Moreover, in view of the discussion above, additivity has to be relaxed. We consider the following relaxation of additivity, which relaxes additivity in the same vein as the grand coalition null player property relaxes the null player property.

Grand coalition additivity, GA. For all $v, w \in \mathbb{V}(N)$, we have $\varphi(v + w, \{N\}) = \varphi(v, \{N\}) + \varphi(w, \{N\})$.

For games with more than two players, the splitting property is incompatible with component efficiency, grand coalition additivity, symmetry within components, the null player property.

Proposition 2. *For $|N| > 2$, there exists no CS solution for $\mathbb{V}(N)$ or $\mathbb{M}(N)$ that satisfies component efficiency (**CE**), grand coalition additivity (**GA**), symmetry within components (**CS**), the null player property (**N**), and the splitting property (**SP**).*

Proof. Let $|N| > 2$ and the CS solution φ for $\mathbb{V}(N)$ be as in the theorem. For fixed $\mathcal{P} = \{N\}$, the properties **CE**, **GA**, **CS**, and **N** become **E**, **A**, **S**, and **N** for TU games. Since the latter characterize the Shapley value both for $\mathbb{V}(N)$ and for $\mathbb{M}(N)$, we have (*) $\varphi(v, \{N\}) = \mathrm{Sh}(v)$ for all $v \in \mathbb{V}(N)$. W.l.o.g., let $1, 2, 3 \in N$. Set $v = u_{\{1,2\}}$ and $\mathcal{P} = \{\{1\}, \{2, 3\}\}$. We obtain

$$\begin{aligned}
\frac{1}{2} - \varphi_2(v, \mathcal{P}) &\overset{(3)}{=} \mathrm{Sh}_2(v) - \varphi_2(v, \mathcal{P}) \\
&\overset{(*)}{=} \varphi_2(v, \{N\}) - \varphi_2(v, \mathcal{P}) \\
&\overset{\mathbf{SP}}{=} \varphi_3(v, \{N\}) - \varphi_3(v, \mathcal{P}) \\
&\overset{(*)}{=} \mathrm{Sh}_3(v, \{N\}) - \varphi_3(v, \mathcal{P}) \overset{(3),\mathbf{N}}{=} 0 - 0,
\end{aligned}$$

i.e., $\varphi_2(v, \mathcal{P}) = \frac{1}{2}$. By **N**, we have $\varphi_3(v, \mathcal{P}) = 0$. Hence, $\varphi_2(v, \mathcal{P}) + \varphi_3(v, \mathcal{P}) = \frac{1}{2} \neq 0 = v(\{2, 3\})$, which contradicts **CE**. □

There exist many CS solutions that satisfy component efficiency, grand coalition additivity, symmetry within components, and the null player property. In order to single out one particular of them, we introduce and impose the relative splitting property.

Relative splitting, RSP. For all $v \in \mathbb{V}(N)$, $\mathcal{P}, \mathcal{Q} \in \mathfrak{P}(N)$, and $i, j \in N$ such that $\mathcal{P}$ is finer than $\mathcal{Q}$ and $\mathcal{P}(i) = \mathcal{P}(j)$, we have

$$\varphi_i(v, \mathcal{P}) \cdot \varphi_j(v, \mathcal{Q}) = \varphi_i(v, \mathcal{Q}) \cdot \varphi_j(v, \mathcal{P}). \tag{6}$$

If all expressions are defined, then (6) can be written as

$$\frac{\varphi_i(v, \mathcal{P})}{\varphi_i(v, \mathcal{Q})} = \frac{\varphi_j(v, \mathcal{P})}{\varphi_j(v, \mathcal{Q})}.$$

That is, whenever components split, players who stay together are required to gain or lose proportionally by the same factor, i.e., they gain or lose equally in *relative* terms. Expressed differently, the ratio of their payoffs doesn't change. In contrast, the splitting property requires the differences of the players' payoffs not to change.

For games with three and more players, there exist no solutions on the full domain of games component efficiency, grand coalition additivity, symmetry within components,, the null player property, the relative splitting property. In contrast, on the domain of monotonic games, a unique such a solution exists.

Theorem 3. *(i) For $|N| > 2$, there exists no CS solution for $\mathbb{V}(N)$ that satisfies component efficiency (**CE**), grand coalition additivity (**GA**), symmetry within components (**CS**), the null player property (**N**), and the relative splitting property (**RSP**).*

*(ii) There exists a unique CS solution for $\mathbb{M}(N)$ that satisfies component efficiency (**CE**), grand coalition additivity (**GA**), symmetry within components (**CS**), the null player property (**N**), and the relative splitting property (**RSP**).*

Proof. (i) Let $|N| > 2$ and the CS solution φ for $\mathbb{V}(N)$ be as in the theorem. For fixed $\mathcal{P} = \{N\}$, the properties **CE**, **GA**, **CS**, and **N** become **E**, **A**, **S**, and **N** for TU games. Since the latter characterize the Shapley value, we have

$$\varphi(v, \{N\}) = \mathrm{Sh}(v) \qquad \text{for all } v \in \mathbb{V}(N). \tag{7}$$

For $\mathcal{P} \in \mathfrak{P}(N)$ and $P \in \mathcal{P}$ such that $|P| > 1$ and $i \in P$, we have

$$\begin{aligned} \varphi_i(v, \mathcal{P}) \cdot \mathrm{Sh}_j(v) &\overset{(7)}{=} \varphi_i(v, \mathcal{P}) \cdot \varphi_j(v, \{N\}) \\ &\overset{\mathbf{ASP}}{=} \varphi_j(v, \mathcal{P}) \cdot \varphi_i(v, \{N\}) \\ &\overset{(7)}{=} \varphi_j(v, \mathcal{P}) \cdot \mathrm{Sh}_i(v) \end{aligned} \tag{8}$$

for all $j \in P$. Summing up (8) over $j \in P$ gives

$$\varphi_i(v, \mathcal{P}) \cdot \sum_{j \in P} \mathrm{Sh}_j(v) = \mathrm{Sh}_i(v) \cdot \sum_{j \in P} \varphi_j(v, \mathcal{P}) \overset{\mathbf{CE}}{=} \mathrm{Sh}_i(v) \cdot v(P). \tag{9}$$

W.l.o.g., let $1, 2, 3 \in N$. Set $v = u_{\{1\}} + u_{\{1,2\}} - 3 \cdot u_{\{1,2,3\}}$ and $\mathcal{P} = \{\{1,2\}, \{3\}\}$. We obtain

$$0 = \varphi_1(v, \mathcal{P}) \cdot (\mathrm{Sh}_1(v) + \mathrm{Sh}_2(v)) \overset{(9)}{=} \mathrm{Sh}_1(v) \cdot v(\{1,2\}) \overset{\mathbf{CE}}{=} \frac{1}{2} \cdot 2$$

a contradiction.

(ii) Let the CS solution φ for $\mathbb{M}(N)$ be as in the theorem. Since **E**, **A**, **S**, and **N** also characterize the Shapley value on $\mathbb{M}(N)$, again, we obtain (9). In monotonic games, the Shapley payoffs are non-negative and zero only for null players. Hence, for all $v \in \mathbb{M}(N)$, $\mathcal{P} \in \mathfrak{P}(N)$, and $P \in \mathcal{P}$, (*) $v(P) > 0$ implies $\sum_{\ell \in P} \mathrm{Sh}_\ell(v) > 0$ and (**) $\sum_{\ell \in P} \mathrm{Sh}_\ell(v) = 0$ implies that all $j \in P$ are null players in v. By **N** and (9), we obtain

$$\varphi_i(v, \mathcal{P}) = \begin{cases} \dfrac{\mathrm{Sh}_\ell(v) \cdot v(\mathcal{P}(i))}{\sum_{\ell \in \mathcal{P}(i)} \mathrm{Sh}_\ell(v)}, & v(\mathcal{P}(i)) > 0, \\ 0, & v(\mathcal{P}(i)) = 0 \end{cases} \tag{10}$$

for all $v \in \mathbb{M}(N)$, $\mathcal{P} \in \mathfrak{P}(N)$, and $i \in N$. One easily checks that the CS solution for $\mathbb{M}(N)$ defined by (10) satisfies all the properties in the theorem. $\square$

The CS solution for $\mathbb{M}(N)$ defined by (10) can be interpreted as conditional version the Shapley value. We therefore call it the **conditional Shapley value**, cSh. For $v \in \mathbb{M}(N)$ with $v(N) > 0$, i.e., $v \neq \mathbf{0}$, the relative Shapley payoffs

$$p_i(v) := \frac{\mathrm{Sh}_i(v)}{\sum_{\ell \in N} \mathrm{Sh}_\ell(v)}, \qquad i \in N$$

can be interpreted as the players' probabilities of obtaining $v(N)$. Then, the Shapley payoffs are the expected payoffs for this interpretation. For components $C \in \mathcal{P}$ with $v(C) > 0$, one can calculate the conditional probabilities of getting $v(C)$,

$$p_i(v|C) = \frac{p_i(v)}{\sum_{\ell \in C} p_i(v)} = \frac{\mathrm{Sh}_i(v)}{\sum_{\ell \in C} \mathrm{Sh}_\ell(v)}, \qquad i \in C.$$

Now, the conditional Shapley payoffs are the expected payoffs for this interpretation.

Casajus (2009, p. 52) demonstrates that a component efficient CS solution satisfying the null player property necessarily must neglect some of the outside options of players in CS games. Nevertheless, the conditional Shapley value doesn't fare too bad when it comes to outside options. In view of Young's (1985) characterization, the Shapley value can be viewed of reflecting the players' productivities within the grand coalition. Hence, with respect to a coalition structure, it reflects the players' productivities within their components and their outside options. Whenever the worth generated by a component is greater than zero, a player with a greater total productivity obtains a greater payoff than a player in her component with a lower total productivity. That is, the conditional Shapley value recognizes outside options in productive components.

3.2. Stability of coalition structures

In this subsection, we explore the stability of coalition structures under the conditional Shapley value. A coalition structure $\mathcal{P} \in \mathfrak{P}(N)$ is called cSh-**stable** for $v \in \mathbb{M}(N)$ if there is no coalition $S \subseteq N$, $S \neq \emptyset$ such that

$$\mathrm{cSh}_i(v, \mathcal{Q}) > \mathrm{cSh}_i(v, \mathcal{P}) \qquad \text{for all } \mathcal{Q} \in \mathfrak{P}(N) \text{ such that } S \in \mathcal{Q} \text{ and } i \in S$$

That is, no coalition can deviate from this coalition structure and make all their members strictly better off no matter how the other players organize. As the Wiese value (Wiese, 2007) and the χ-value, the conditional Shapley value is component independent, i.e., a player's payoff does not depend on how the players outside her component are organized.

Component independence, CI. For all $v \in \mathbb{M}(N)$, $\mathcal{P}, \mathcal{Q} \in \mathfrak{P}(N)$, and $i \in N$ such that $\mathcal{P}(i) = \mathcal{Q}(i)$, we have

$$\varphi_i(v, \mathcal{P}) = \varphi_i(v, \mathcal{Q}).$$

Hence, our notion of stability is analogous to the one used by Wiese (2007) and Casajus (2009).

Theorem 4. *For any $v \in \mathbb{M}(N)$, there exists a* cSh*-stable coalition structure.*

Proof. Fix $v \in \mathbb{M}(N)$ and set

$$\Pi(S) := \begin{cases} \dfrac{v(S)}{\sum_{\ell \in S} \mathrm{Sh}_\ell(v)}, & \sum_{\ell \in S} \mathrm{Sh}_\ell(v) > 0, \\ 0, & \sum_{\ell \in S} \mathrm{Sh}_\ell(v) = 0 \end{cases} \qquad \text{for } S \subseteq N,\ S \neq \emptyset. \tag{11}$$

Construct a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_t, \ldots, P_k\} \in \mathfrak{P}(N)$, $k \in \mathbb{N}$ by iteration on t as follows:

(i) Set $Q_1 = \emptyset$.

(ii)) Chose

$$P_t \in \operatorname*{argmax}_{S \subseteq N \setminus Q_t} \Pi(S). \tag{12}$$

(iii) Set $Q_{t+1} = Q_t \cup P_t$.

Repeat steps (i)–(iii) until $Q_{t+1} = N$.

Suppose, $\mathcal{P}$ were not $\dot{\chi}$-stable. By (12) and **CI**, there were some coalition $S \notin \mathcal{P}$ such that $\mathrm{cSh}_i(v, \{S, N \setminus S\}) > \mathrm{cSh}_i(v, \mathcal{P})$ for all $i \in S$. The only reason for S not being in $\mathcal{P}$ is that there is some $P \in \mathcal{P}$ such that $S \cap P \neq \emptyset$ and $\Pi(S) \leq \Pi(P)$. By (10) and (11), we had

$$\mathrm{cSh}_i(v, \{S, N \setminus S\}) = \Pi(S) \cdot \mathrm{Sh}_i(v) \leq \Pi(P) \cdot \mathrm{Sh}_i(v) = \mathrm{cSh}_i(v, \mathcal{P})$$

for all $i \in S \cap P$, a contradiction. □

Remark 5. *The proof of Theorem 4 is constructive. One easily checks that all* cSh*-stable coalition structures can be constructed as in this proof.*

Remark 6. *From proof of Theorem 4 it is also clear, that a coalition structure remains $\dot{\chi}$-stable when a null players are "moved around". More precisely, if $\mathcal{P} \in \mathfrak{P}(N)$ is* cSh*-stable for $v \in \mathbb{M}(N)$ and $\mathcal{Q} \in \mathfrak{P}(N)$ is such that*

$$\mathcal{Q}|_{N \setminus N_0(v)} = \mathcal{P}|_{N \setminus N_0(v)},$$

where $N_0(v)$ denotes the set of null players in v, then $\mathcal{Q}$ also is cSh*-stable for v.*

4. Coalition formation in voting games

In this section, we apply the conditional Shapley value to simple superadditive games, i.e., to **voting games**, and study coalition formation. In a voting game, a coalition is called **losing** if it generates a worth of zero and it is called **winning** if it generates a worth of one. A winning coalition is called **minimal** if it does not contain a winning coalition as a proper subcoalition.

4.1. The conditional Shapley value for voting games

In voting games, all marginal contributions are either zero or one. In non-trivial voting games, exactly one player's marginal contribution is one in any rank order. Such a player is called a pivot for a rank order and this rank order is called a swing for this player. Hence, the Shapley value indicates a player's probability of being a pivot in a rank order, where all rank orders are equally probable, i.e.,

$$\mathrm{Sh}_i(v) = \frac{|\{\rho \in R(N) \mid MC_i^v(\rho) = 1\}|}{|R(N)|} \qquad \text{for all } v \in \mathbb{S}(N) \text{ and } i \in N. \tag{13}$$

When the players are organized into groups represented by a partition, one can employ the conditional Shapley value in order to determine players' power. In this context, the phrase "conditional" can be motivated by the following observation. While $\mathrm{cSh}_i(v \mid S) = 0$ for all $i \in S$ whenever S is losing, we have

$$\mathrm{cSh}_i(v \mid S) = \frac{\mathrm{Sh}_i(v)}{\sum_{j \in S} \mathrm{Sh}_j(v)} = \frac{|\{\rho \in R(N) \mid MC_i^v(\rho) = 1\}|}{\left|\left\{\rho \in R(N) \mid MC_j^v(\rho) = 1 \text{ for some } j \in S\right\}\right|} \tag{14}$$

if S is winning. That is, a player's power according to the conditional Shapley value is the probability of a player being a pivot for a rank order *conditional* on one of the players in the coalition under consideration to be a pivot in a rank order.

Since the conditional Shapley value is component independent, one can use it to define conditional payoffs without reference to a coalition structure. For all $v \in \mathbb{S}(N)$, $S \subseteq N$, and $i \in S$, we define the power of player i in the game v given that coalition S has formed as

$$\mathrm{cSh}_i(v \mid S) := \mathrm{cSh}_i(v, \mathcal{P}) \qquad \text{for some/all } \mathcal{P} \in \mathfrak{P}(N) \text{ such that } S \in \mathcal{P}. \tag{15}$$

4.2. Stability in voting games and Gamson's law

Using the "partition-free" version of the conditional Shapley value in (15), the stability of coalition structures for voting games can be expressed as follows. A partition $\mathcal{P} \in \mathfrak{P}(N)$ is cSh**-stable** for a voting game $v \in \mathbb{S}(N)$ there is no coalition $S \subseteq N$ such that

$$\mathrm{cSh}_i(v \mid S) > \mathrm{cSh}_i(v \mid \mathcal{P}(i)) \qquad \text{for all } i \in S.$$

In voting games, any partition contains at most one winning coalition. By Remark 5, the algorithm in the proof of Theorem 4 implies that a partition is cSh-stable if and only if it contains one winning coalition for which the sum of its members' powers according to the Shapley value is minimal, while the other players can be organized arbitrarily. In voting games, all players have non-negative power, where only null players have zero power. Hence, this winning coalition consists of a minimal winning coalition and plus an arbitrary number of null players. *Cum grano salis*, a partition is cSh-stable for a voting game if it contains one of the minimal winning coalitions with minimal total power.

Theorem 7. *A partition $\mathcal{P} \in \mathfrak{P}(N)$ is* cSh*-stable for a voting game $v \in \mathbb{S}(N)$ if and only if it contains a component $P \in \mathcal{P}$ that contains a minimal winning coalition $S \subseteq N$ in v such that*

$$\sum_{i \in S} \mathrm{Sh}_i(v) \leq \sum_{i \in T} \mathrm{Sh}_i(v) \qquad \textit{for all minimal winning coalitions } T \subseteq N \textit{ in } v$$

and that all players in $P \setminus S$ are null players in v.

If one interprets a player's power in a voting game as her contribution to a coalition formed, then the conditional Shapley value fits the empirical hypothesis of Gamson (1961) underlying his theory of coalition formation.

> Any participant will expect others to demand from a coalition a share of the payoff proportional to the amount of resources which they contribute to a coalition. (Gamson, 1961, p. 376)

Moreover, the fact that any cSh-stable coalition structure contains one of the minimal winning coalitions with minimal total power also fits his prediction on the outcome of coalition formation.

> It is clear, then, that the prediction from our model will be simply the *cheapest winning coalition* in the applicable situations. (Gamson, 1961, p. 377)

Hence, stability under the conditional Shapley value provides a cooperative foundation of Gamson's (1961) theory of coalition formation.[2]

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[2] Fréchette et al. (2005), for example, provide a non-cooperative foundation of Gamson's (1961) theory of coalition formation.

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