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Extension Operators for TU Games and the Lovász Extension

André Casajus

Prof. Dr. André Casajus is Research Professor at the Chair of Economics and Information Systems, HHL Leipzig Graduate School of Management, Leipzig, Germany.
Email: andre.casajus@hhl.de

Abstract
An extension operator assigns to any TU game its extension, a mapping that assigns a worth to any non-negative resource vector for the players. It satisfies three properties: linearity in the game, homogeneity of extensions, and the extension property. The latter requires the indicator vector of any coalition to be assigned the worth generated by this coalition in the underlying TU game. Algaba et al. (2004, Theor Decis 56, 229-238) advocate the Lovász extension (Lovász, 1983, Mathematical Programming: The State of the Art, Springer, 235-256) as a natural extension operator. We show that it is the unique extension operator that satisfies two desirable properties. Resources of players outside a carrier of the TU game do not affect the worth generated. For monotonic TU games, extensions are monotonic. Further, we discuss generalizations of the Lovász extension using CES production functions.

Extension operators for TU games and the Lovász extension☆

André Casajus[a,b]

[a] HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany
[b] Dr. Hops Craft Beer Bar, Eichendorffstr. 7, 04277 Leipzig, Germany

Abstract

An extension operator assigns to any TU game its extension, a mapping that assigns a worth to any non-negative resource vector for the players. It satisfies three properties: linearity in the game, homogeneity of extensions, and the extension property. The latter requires the indicator vector of any coalition to be assigned the worth generated by this coalition in the underlying TU game. Algaba et al. (2004, Theor Decis 56, 229–238) advocate the Lovász extension (Lovász, 1983, Mathematical Programming: The State of the Art, Springer, 235–256) as a natural extension operator. We show that it is the unique extension operator that satisfies two desirable properties. Resources of players outside a carrier of the game do not affect the worth generated. For monotonic games, extensions are monotonic. Further, we discuss generalizations of the Lovász extension using CES production functions.

Keywords: TU game, Lovász extension, CES production function
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1. Introduction

A cooperative game with transferable utility for a finite player set (TU game) is given by a coalition function that assigns a worth to any coalition (subset of the player set), where the empty coalition obtains zero. In a coalition, a player is either (fully) present for or (completely) absent from cooperation. A more general way to model the players' involvement in the generation of worth are resource vectors that assign to any player a non-negative amount of a resource she owns. In this setting, a coalition is represented by its indicator resource vector that assigns to any player inside this coalition an amount of one of her resource and an amount of zero to any player outside this coalition. The natural question now arises how to extend a TU game from coalitions to resource vectors.

A resource game is a mapping that assigns to any non-negative resource vector a real number, the worth generated by the resources. An extension operator is a mapping that

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Email address: mail@casajus.de (André Casajus)
URL: www.casajus.de (André Casajus)

assigns to any TU game a resource game, its extension, with the following properties (Algaba et al., 2004, Section 4).[1]

- **X** The worth generated by the indicator resource vector of a coalition in the resource game assigned to a TU game equals the worth generated by this coalition in the TU game.
- **L** Linearity in the game: the resource game assigned to the linear combination of two TU games equals the linear combination of the resource game assigned to these TU games.
- **HX** Homogeneity of resource games: the worth generated by a scaled resource vector in the resource game assigned to a TU game equals the scaled worth generated by the original resource vector in this resource game.

For a given TU game, its Lovász extension assigns a worth to a resource vector as follows (Lovász, 1983; Algaba et al., 2004). The worth of an active coalition is scaled by the uniform amount of their resource the players of this coalition make use of. The first active coalition comprises all players that own at least the smallest amount of the resource, i.e., all players, which also make use of this amount. The second active coalition comprises all players that own at least the second smallest amount, which make use of the difference between second smallest amount and the smallest amount. The third active coalition comprises all players that own at least the third smallest amount, which make use of the difference between third smallest amount and the second smallest amount. And so on. Finally, the scaled worths of all active coalitions are added up. Algaba et al. (2004, p. 233) advocate the Lovász extension (operator) as "a natural way of extending" TU games to resource vectors.

There exist a vast number of extension operators besides the Lovász extension, for example, the extension operators given by (11) or identified by Proposition 5 later on. However, these fail one of two economically desirable properties for resource games.

- **CX** Carrier property for resource games: resources outside a carrier of a TU game do not affect the worth generated in the resource game assigned to this TU game. Carrier: a coalition such that the worth of any coalition equals the worth of its intersection with the former coalition.[2]
- **MX** Monotonicity of resource games for monotonic games: In the resource game assigned to a monotonic TU game, the worth generated does not decrease when the resource vector increases. Monotonic TU game: all marginal contributions of players to coalitions not containing them are non-negative.

As our main result, we show that the Lovász extension is the unique extension operator that satisfies these two properties (Theorem 1). In view of this result, we discuss generalizations of the Lovász extension, a class of mappings based on CES (constant elasticity of substitution) production functions with uniform distribution parameter and homogeneity of degree 1, the CES mappings. Whereas all these CES mappings satisfy linearity in the game,

[1] The domain of Owen's (1972) multi-linear extension is the standard cube for the player set. The players' "ressources" stand for their (independent) probabilities of being present for cooperation.

[2] This property resembles the carrier property used by Shapley (1953) in the characterization of his value.

homogeneity of resource games, and the carrier property for resource games (Proposition 3), besides the Lovász extension, only very few of the CES mappings are extensions (one) or satisfy monotonicity of resource games for monotonic games (two) (Propositions 4 and 5).

This paper is organized as follows. In the second section, we provide basic definitions and notation. In the third section, we introduce and discuss extension operators and the Lovász extension. In the fourth section, we provide and discuss our main result. In the fifth section, we discuss the CES mappings. Some remarks conclude the paper. The Appendix contains all the proofs.

2. Basic definitions and notation

A TU game for a non-empty and finite player set N is given by a **coalition function** $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, where 2^N denotes the power set of N. Subsets of N are called **coalitions**; $v(S)$ is called the worth of coalition S. The set of all games for N is denoted by $\mathbb{V}$. A coalition $C \subseteq N$ is called a **carrier** of $v \in \mathbb{V}$, if $v(S) = v(S \cap C)$ for all $S \subseteq N$.

For $v, w \in \mathbb{V}$, and $\alpha \in \mathbb{R}$, the coalition functions $v + w \in \mathbb{V}$ and $\alpha \cdot v \in \mathbb{V}$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$. For $T \subseteq N$, $T \neq \emptyset$, the game $u_T \in \mathbb{V}$ given by $u_T = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise is called a **unanimity game**. Any $v \in \mathbb{V}$ can be uniquely represented by unanimity games. In particular, we have

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T, \tag{1}$$

where the coefficients $\lambda_T(v)$ are known as the Harsanyi dividends (Harsanyi, 1959) and can be determined recursively by

$$\lambda_T(v) := v(T) - \sum_{S \subsetneq T : S \neq \emptyset} \lambda_S(v). \tag{2}$$

A **rank order** for N is a bijection $\rho : N \to \{1, 2, \ldots, |N|\}$ with the interpretation that i is the $\rho(i)$th player in ρ; the set of rank orders of N is denoted by R. The set of players before i in ρ is denoted by $B_i(\rho) = \{\ell \in N : \rho(\ell) < \rho(i)\}$. The **marginal contribution** of i in ρ is denoted by

$$MC_i^v(\rho) := v(B_i(\rho) \cup \{i\}) - v(B_i(\rho)). \tag{3}$$

A game $v \in \mathbb{V}$ is called **monotonic** if $MC_i^v(\rho) \geq 0$ for all $\rho \in R$ and $i \in N$. Player $i \in N$ is called a **null player** in $v \in \mathbb{V}$, if $MC_i^v(\rho) = 0$ for all $\rho \in R$.

3. Extension operators and the Lovász extension

In a coalition $S \subseteq N$, a player i is either present with all her resources, $i \in N$, or not, $i \in N \setminus S$. A more general way to model the distribution of resources are **resource vectors** $s \in \mathbb{R}_+^N$, where a coalition $S \subseteq N$ is represented by its **indicating resource vector** $1_S \in \mathbb{R}_+^N$ given by

$$(1_S)_i := \begin{cases} 1, & i \in S, \\ 0, & i \in N \setminus S \end{cases} = 1 \qquad \text{for all } i \in N. \tag{4}$$

The question now arises how to extend a TU game from coalitions to resource vectors.

Let $\mathbb{E} := \left\{ f : \mathbb{R}_+^N \to \mathbb{R} \right\}$; the members of $\mathbb{E}$ are called **resource games** for N. This set is a linear space on the reals in the obvious sense. For all $f, g \in \mathbb{E}$ and $\alpha \in \mathbb{R}$, the resource games $f + g$ and $\alpha \cdot f$ are given by $(f + g)(s) = f(s) + g(s)$ and $(\alpha \cdot f)(s) = \alpha \cdot f(s)$ for all $s \in \mathbb{R}_+^N$, respectively. A mapping $E : \mathbb{V} \to \mathbb{E}$, $v \mapsto Ev$ is an **extension operator** if it satisfies the extension property, it is linear in the game, and the resource games assigned to TU games are homogenous (Algaba et al., 2004, Section 4); Ev is called the **extension** of v.

Extension (X). For all $v \in \mathbb{V}$ and $S \subseteq N$, we have $Ev(1_S) = v(S)$.

This property is at the heart of the very idea of an extension. The representative of a coalition in the associated resource game, the indicating resource vector, generates the same worth as the coalition in the underlying TU game.

Linearity in the game, L. For all $v, w \in \mathbb{V}$ and $\alpha \in \mathbb{R}$, we have $E(v + w) = Ev + Ew$ and $E(\alpha \cdot v) = \alpha \cdot Ev$.

Both the space of TU games $\mathbb{V}$ and the space of resource games $\mathbb{E}$ are linear spaces on the reals. Hence, it seems to be rather natural to require an extension operator to respect the linear structure of these spaces. Moreover, by definition, we have

$$(v + w)(S) = v(S) + w(S) \qquad \text{and} \qquad (\alpha \cdot v)(S) = \alpha \cdot v(S)$$

for all $v, w \in \mathbb{V}$ and $\alpha \in \mathbb{R}$. The extension property now implies

$$E(v + w)(1_S) = Ev(1_S) + Ew(1_S) \qquad \text{and} \qquad E(\alpha \cdot v)(1_S) = \alpha \cdot Ev(1_S),$$

that is, extension operators are linear in the game for the indicating resource vectors. Linearity in the game extends this to arbitrary resource vectors, which reflects that extensions should be determined by the underlying TU game.

Homogeneity of resource games, HX. For all $v \in \mathbb{V}$, $s \in \mathbb{R}_+^N$, and $\alpha \in \mathbb{R}_+$, we have

$$Ev(\alpha \cdot s) = \alpha \cdot Ev(s).$$

This property requires extensions to show no economies or diseconomies of scale. On the one hand, of course, this is a strong assumption on the technology embodied in extensions. On the other hand, in presence of the homogeneity-part of linearity in the game, homogeneity of resource games is equivalent to interchangeability of scaling of resource vectors and games. On its own, the latter may be viewed as less demanding in economic terms than the former.

Interchangeability of scaling, IS. For all $v \in \mathbb{V}$, $s \in \mathbb{R}_+^N$, and $\alpha \in \mathbb{R}_+$, we have $E(\alpha \cdot v)(s) = Ev(\alpha \cdot s)$.

The **Lovász extension (operator)** (Lovász, 1983; Algaba et al., 2004) $\mathrm{L} : \mathbb{V} \to \mathbb{E}$, $v \mapsto \mathrm{L}v$ for all $v \in \mathbb{V}$, is given as follows. For $s \in \mathbb{R}_+^N$, let $\#s := |\{s_i \mid i \in N\}|$ denote the number of different entries of s and let $\bar{s} \in \mathbb{R}_+^{\#s}$ be given by $\bar{s}_\ell = s_i$ for some $i \in N$ for all $\ell = 1, 2, \ldots, \#s$ and $\bar{s}_k < \bar{s}_\ell$ for all $k, \ell \in 1, 2, \ldots, \#s$ such that $k < \ell$. That is, $\bar{s}_\ell$ is the size of the ℓth smallest entry of s. Set $\bar{s}_0 := 0$ and let

$$P_\ell(s) := \{i \in N \mid s_i \geq \bar{s}_\ell\} \qquad \text{for all } \ell = 1, 2, \ldots, \#s, \tag{5}$$

i.e., $P_\ell(s)$ contains the players with an entry in s at least as great as the ℓth smallest entry of s. For $v \in \mathbb{V}$, its **Lovász extension** $\mathrm{L}v$ is given by

$$\mathrm{L}v(s) := \sum_{\ell=1}^{\#s} (\bar{s}_\ell - \bar{s}_{\ell-1}) \cdot v(P_\ell(s)) \qquad \text{for all } s \in \mathbb{R}_+^N. \tag{6}$$

Algaba et al. (2004, Theorem 5) express the Lovász extensions in terms of Harsanyi dividends. For all $v \in \mathbb{V}$, we have

$$\mathrm{L}v(s) := \sum_{T \subseteq N: T \neq \emptyset} \min_T(s) \cdot \lambda_T(v) \qquad \text{for all } s \in \mathbb{R}_+^N, \tag{7}$$

where

$$\min_T(s) := \min_{i \in T} s_i \qquad \text{for all } s \in \mathbb{R}_+^N. \tag{8}$$

Casajus and Wiese (2017, Equation 11) express the Lovász extensions in terms marginal contributions. For $s \in \mathbb{R}_+^N$, let

$$R(s) := \{\rho \in R \mid \rho(i) < \rho(j) \text{ for all } i, j \in N \text{ with } s_i > s_j\}, \tag{9}$$

i.e., $R(s)$ contains those rank orders for which players with a greater entry in s come before players with a smaller entry. For all $v \in \mathbb{V}$, we have

$$\mathrm{L}v(s) = \sum_{i \in N} s_i \cdot MC_i^v(\rho) \qquad \text{for all } s \in \mathbb{R}_+^N \text{ and } \rho \in R(s). \tag{10}$$

4. Uniqueness of the Lovász extension

In this section, we show that the Lovász extension is the unique extension that satisfies two economically desirable properties: a carrier property and a monotonicity property.

Carrier property for resource games, CX. For all $v \in \mathbb{V}$, $C \subseteq N$, and $s, r \in \mathbb{R}_+^N$ such that C is a carrier of v and $s_i = r_i$ for all $i \in C$, we have $Ev(s) = Ev(r)$.

Players outside a carrier are unproductive, i.e., null players. Hence, their resources should not affect the worth generated. Moreover, if player $i \in N$ is a null player in $v \in \mathbb{V}$, then $v(S \cup \{i\}) = v(S)$ for all $S \in N \setminus \{i\}$. The extension property now implies $Ev(1_{S \cup \{i\}}) = Ev(1_S)$, i.e., the resource of player i doesn't affect the generation of worth in the associated resource game. This property extends this to arbitrary resource vectors, which reflects that extensions should be determined by the underlying TU game.

Monotonicity of resource games for monotonic games, MX. For all $v \in \mathbb{V}$ and $s, r \in \mathbb{R}_+^N$ such that v is monotonic and $s_i \leq r_i$ for all $i \in N$, we have $Ev(s) \leq Ev(r)$.

In monotonic games, all players are productive, i.e., all marginal contributions are non-negative. Hence, whenever resources increase, the worth created should not decrease. Particularly, if the game $v \in \mathbb{V}$ is monotonic, then $v(S \cup \{i\}) \geq v(S)$ for all $i \in N$ and $S \subseteq N \setminus \{i\}$. The extension property now implies $Ev(1_{S \cup \{i\}}) \geq Ev(1_S)$, i.e., an increase of player i´s resource doesn't decrease the generation of worth in the extension. This property extends this to arbitrary resource vectors, which reflects that extensions should be determined by the underlying TU game.

Theorem 1. *The Lovász extension is the unique mapping $E : \mathbb{V} \to \mathbb{E}$ that satisfies linearity in the game (**L**), homogeneity of resource games (**HX**), the extension property (**X**), the carrier property for resource games (**CX**), and monotonicity of resource games for monotonic games (**MX**).*

Remark 2. *Careful inspection of the proof of Theorem 1 in Appendix A reveals that linearity in the game can be relaxed into additivity in the game: for all $v, w \in \mathbb{V}$, we have $E(v+w) = Ev + Ew$.*

On the one hand, the Leontief type technology in (7) underlying the Lovász extension is rather particular. On the other hand, the theorem now tells us that it is inevitable if an extension is supposed to satisfy economically desirable properties. In an informal sense, linearity in the game, homogeneity of resource games, the carrier property for resource games, and monotonicity of resource games for monotonic games seem to be less specific with respect to the technology of extensions than the extension property. Hence, one could attribute the Leontief type technology inherent in the Lovász extension to the latter.

We conclude this section by showing that the characterization of the Lovász extension in Theorem 1 is non-redundant for $|N| > 1$. To save space, we anticipate results from Section 5. By Theorem 1 and Propositions 3, 4, and 5, the mapping $\mathrm{LCES} : \mathbb{V} \to \mathbb{E}$ given by

$$\mathrm{LCES}v = \begin{cases} \mathrm{L}v, & v(N) \geq 0, \\ \mathrm{CES}^0 v, & 0 > v(N) \end{cases}$$

for all $v \in \mathbb{V}$ inherits all properties but linearity in the game from the Lovász extension and CES^0. The mapping $\mathrm{L}^2 : \mathbb{V} \to \mathbb{E}$ given by

$$\mathrm{L}^2\, v(s) := \mathrm{L}v\left(s^2\right) \qquad \text{and} \qquad \left(s^2\right)_i = s_i^2$$

for all $v \in \mathbb{V}$, $s \in \mathbb{R}_+^N$, and $i \in N$ inherits all properties from the Lovász extension but homogeneity of resource games. By Propositions 3, 4, and 5, the mapping CES^1 satisfies all properties but the extension property. The extension $\mathrm{L}^\star$ given by

$$\mathrm{L}^\star\, v(s) = \mathrm{L}v(s^\star) \qquad \text{and} \qquad s_i^\star = \sqrt{s_i \cdot \max_N(s)} \tag{11}$$

for all $v \in \mathbb{V}$, $s \in \mathbb{R}_+^N$, and $i \in N$ inherits all properties but the carrier property for resource games from the Lovász extension. By Propositions 3, 4, and 5, the mapping CES^0 satisfies all properties but monotonicity of resource games for monotonic games.

5. CES mappings

Formula (7) shows that a CES (constant elasticity of substitution) technology determines the Lovász extension, the minimum operators (8). In this section, we generalize the Lovász extension using CES production functions in (7) and explore their properties.

For any coalition $T \neq \emptyset$ and substitution parameter $\varrho \in \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, the CES production function $c_T^\varrho : \mathbb{R}_+^N \to \mathbb{R}$ with uniform distribution parameter and homogeneity of degree 1 is given by

$$c_T^\varrho(s) := \left[\sum_{\ell \in T} \frac{1}{|T|} \cdot s_\ell^\varrho\right]^{\frac{1}{\varrho}} \qquad \text{for all } s \in \mathbb{R}_+^N. \tag{12}$$

For $\varrho \in \{-\infty, 0, +\infty\}$, one consider the appropriate limits and obtains

$$c_T^{-\infty}(s) := \min_T(s), \qquad c_T^0(s) := \sqrt[|T|]{\prod_{\ell \in T} s_\ell}, \qquad c_T^{+\infty}(s) := \max_T(s) \tag{13}$$

for all $s \in \mathbb{R}_+^N$, where

$$\max_T(s) := \max_{i \in T} s_i \qquad \text{for all } s \in \mathbb{R}_+^N. \tag{14}$$

For $\varrho \in \bar{\mathbb{R}}$, the CES mapping $\mathrm{CES}^\varrho : \mathbb{V} \to \mathbb{E}$ is given by

$$\mathrm{CES}^\varrho v(s) := \sum_{T \subseteq N, T \neq \emptyset} c_T^\varrho(s) \cdot \lambda_T(v) \qquad \text{for all } v \in \mathbb{V} \text{ and } s \in \mathbb{R}_+^N. \tag{15}$$

By (7), (13), and (15), we have $\mathrm{CES}^{-\infty} = \mathrm{L}$, i.e., the CES mappings indeed generalize the Lovász extension. By construction, the CES mappings satisfy linearity in the game, homogeneity of resource games, and the carrier property for resource games.

Proposition 3. *The CES mappings* CES^ϱ, $\varrho \in \bar{\mathbb{R}}$ *satisfy the following properties: linearity in the game (**L**), homogeneity of resource games, and the carrier property for resource games (**CX**).*

In view of Theorem 1, all CES mappings except the Lovász extension must fail either the extension property or monotonicity of resource games for monotonic games. Indeed, only very few of the other CES mappings satisfy even one of these properties.

Proposition 4. *For* $|N| > 1$, *a CES mapping* CES^ϱ, $\varrho \in \bar{\mathbb{R}}$ *satisfies the extension property (**X**) if and only if* $\varrho \in \{-\infty, 0\}$.

Only for $\varrho \in \{-\infty, 0\}$, a player's resource cannot be completely substituted by the other players' resources in the CES production functions (12) and (13). This is what drives this result.

Proposition 5. *For* $|N| > 1$, *a CES mapping* CES^ϱ, $\varrho \in \bar{\mathbb{R}}$ *satisfies monotonicity of resource games for monotonic games (**MX**) if and only if* $\varrho \in \{-\infty, 1, +\infty\}$.

Whereas all CES production functions (12) and (13) are monotonic, only for $\varrho = 1$, they are additive in the resource vector. And only for $\varrho \in \{-\infty, +\infty\}$, their elasticity of substitution is zero. This is what seems to drive this result.

6. Concluding remarks

Casajus and Wiese (2017, Section 4) use the Lovász extension to construct the Lovász-Shapley value, a solution for TU games that are enriched by resource vectors. Later on, Casajus et al. (2020) use this value in order to construct replicator dynamics that are derived from TU games. Since their stability results for these dynamics crucially rely on the Leontief type technology inherent in the Lovász extension, they suggest to explore alternative extensions that are based on CES technologies. Propositions 4 and 5 indicate that one has to waive either the extension property or monotonicity of resource games for monotonic games.

Appendix A. Proof of Theorem 1

The Lovász extension indeed is an extension (Algaba et al., 2004, p. 233), i.e., it satisfies **X**, **L**, and **HX**. Players outside a carrier of a game are null players. If $i \in N$ is a null player in $v \in \mathbb{V}$, then $\lambda_T(v) = 0$ for all $T \subseteq N$ such that $i \in T$. In view of (7), the Lovász extension satisfies **CX**. Remains to show **MX**.

Since the minimum operators (8) are continuous and by (7), the Lovász extensions are continuous in the resource vector. Hence, we are also allowed to restrict attention to generic resource vectors s for which any two players own a different amount of resources. For such resource vectors, the set of induced rank orders $R(s)$ contains a single rank order ρ_s. Moreover, we are allowed to restrict attention to scenarios where only one player's resource increases (i) without changing this rank order or (ii) where only this player changes the position with her predecessor. In monotonic games, all marginal contributions are non-negative. By (10), the worth generated does not decrease in scenario (i). Let us turn to scenario (ii). Let $s, r \in \mathbb{R}_+^N$ and $i, j \in N$, $i \neq j$ be such that both s and r are generic, $s_\ell = r_\ell$ for all $\ell \in N \setminus \{i\}$ and $s_i < r_i$, $\rho_s(\ell) = \rho_r(\ell)$ for all $\ell \in N \setminus \{i, j\}$, $\rho_s(i) = \rho_r(j)$, $\rho_s(j) = \rho_r(i)$, and $\rho_r(i) = \rho_s(i) - 1$. This implies $s_i < s_j = r_j < r_i$ and therefore

$$\begin{aligned}
\mathrm{L}v(r) - \mathrm{L}v(s) &\overset{(10)}{=} r_i \cdot MC_i^v(\rho_r) + r_j \cdot MC_j^v(\rho_r) - s_i \cdot MC_i^v(\rho_s) + s_j \cdot MC_j^v(\rho_s) \\
&\geq r_j \cdot MC_i^v(\rho_r) + r_j \cdot MC_j^v(\rho_r) - r_j \cdot MC_i^v(\rho_r) + r_j \cdot MC_j^v(\rho_s) \\
&= 0,
\end{aligned}$$

where the last equation follows from

$$MC_i^v(\rho_r) + MC_j^v(\rho_r) = MC_i^v(\rho_s) + MC_j^v(\rho_s) \geq 0,$$

which drops from the fact that i and j are neighbors in ρ_r and ρ_s who just swap their positions and the game being monotonic. Hence, the Lovász extension satisfies **MX**.

Let now the extension E be as in the theorem. In view of **L**, it suffices to show $Eu_T = \mathrm{L}u_T$ for all $T \subseteq N$, $T \neq \emptyset$.

Claim 1, C1. For all $s \in \mathbb{R}_+^N$ such that $s_i = 0$ for some $i \in T$, we have $Eu_T(s) = 0$.

We obtain

$$0 \overset{\mathbf{X},\mathbf{HX}}{=} Eu_T\left(\min_{N\setminus\{i\}}(s)\cdot 1_{N\setminus\{i\}}\right) \overset{\mathbf{MX}}{\leq} Eu_T(s)$$
$$\overset{\mathbf{MX}}{\leq} Eu_T\left(\max_{N\setminus\{i\}}(s)\cdot 1_{N\setminus\{i\}}\right) \overset{\mathbf{X},\mathbf{HX}}{=} 0$$

and therefore $Eu_T(s) = 0$.

Claim 2, C2. For all $s \in \mathbb{R}^N_+$, we have $\min_T(s) \leq Eu_T(s) \leq \max_T(s)$.

Coalition T is a carrier of u_T, and we obtain

$$\min_T(s) \overset{\mathbf{X},\mathbf{HX}}{=} Eu_T(\min_T(s)\cdot 1_T) \overset{\mathbf{MX}}{\leq} Eu_T(s) \overset{\mathbf{CX},\mathbf{MX}}{\leq} Eu_T(\max_T(s)\cdot 1_T) \overset{\mathbf{X},\mathbf{HX}}{=} \max_T(s).$$

Claim 3, C3. For all $s \in \mathbb{R}^N_+$, we have $Eu^N_T(s) = \min_T(s)$.

If $|T| = 1$, the claim is immediate from **C2**. Let now $|T| > 1$. Further, let (*) $i \in T$ be such that $s_i = \min_T(s)$ and let $v \in \mathbb{V}$ and $s_{-i} \in \mathbb{R}^N_+$ be given by

$$v := \left(\sum_{\ell\in T} u_{\{\ell\}}\right) - u_T \tag{A.1}$$

and

$$\left(s_{-i}\right)_i := 0 \quad \text{and} \quad \left(s_{-i}\right)_\ell := s_\ell \quad \text{for all } \ell \in N\setminus\{i\}. \tag{A.2}$$

We obtain

$$\begin{aligned}\sum_{\ell\in T\setminus\{i\}} s_\ell &\overset{\mathbf{C1},\mathbf{C2},(A.2)}{=} \left(\sum_{\ell\in T} Eu_{\{\ell\}}(s_{-i})\right) - Eu_T(s_{-i}) \\ &\overset{\mathbf{L},(A.1)}{=} Ev(s_{-i}) \\ &\overset{\mathbf{MX},(A.2)}{\leq} Ev(s) \\ &\overset{\mathbf{L}}{=} \left(\sum_{\ell\in T} Eu_{\{\ell\}}(s)\right) - Eu_T(s) \\ &\overset{\mathbf{C2}}{=} \left(\sum_{\ell\in T} s_\ell\right) - Eu_T(s)\end{aligned}$$

and therefore

$$\min_T(s) \overset{\mathbf{C2}}{\leq} Eu_T(s) \leq s_i \overset{(*)}{=} \min_T(s).$$

By (1) and (7), this concludes the proof.

Appendix B. Proof of Proposition 4

For $\varrho \in \mathbb{R} \setminus \{0\}$, we have

$$\mathrm{CES}^{\varrho} u_N \left(1_{\{i\}}\right) = \left[\frac{1}{|N|}\right]^{\frac{1}{\varrho}} \neq 0 = u_N\left(\{i\}\right)$$

and $\mathrm{CES}^{+\infty} u_N \left(1_{\{i\}}\right) = 1 \neq 0 = u_N\left(\{i\}\right)$ for $i \in N$. That is, the mapping CES^{ϱ} fails **X** for $\varrho \in \bar{\mathbb{R}} \setminus \{-\infty, 0\}$. We already know that the Lovász extension $\mathrm{L} = \mathrm{CES}^{-\infty}$ satisfies **X**. For $S \subseteq N$, we obtain

$$\mathrm{CES}^{0} v\left(1_S\right) \overset{(13),(15)}{=} \sum_{T \subseteq N, T \neq \emptyset} \sqrt[|T|]{\prod_{\ell \in T} s_{\ell}} \cdot \lambda_T\left(v\right) \overset{(4)}{=} \sum_{T \subseteq S, T \neq \emptyset} \lambda_T\left(v\right) \overset{(2)}{=} v\left(S\right).$$

Hence, the mapping CES^{0} satisfies **X**.

Appendix C. Proof of Proposition 5

For $i, j \in N$, $i \neq j$, let the (monotonic) game $v \in \mathbb{V}$ be given by

$$v = u_{\{i\}} + u_{\{j\}} - u_{\{i,j\}}. \tag{C.1}$$

For $\alpha \in \mathbb{R}_+$ and $\mathbb{R} \setminus \{0\}$, we obtain

$$\mathrm{CES}^{\varrho} v\left(\alpha \cdot 1_{\{i\}}\right) \overset{(12),(15),(C.1)}{=} \alpha \cdot \left(1 - \left[\frac{1}{2}\right]^{\frac{1}{\varrho}}\right).$$

For $\varrho \in (-\infty, 0)$, we have

$$1 - \left[\frac{1}{2}\right]^{\frac{1}{\varrho}} < 0,$$

which implies that CES^{ϱ} fails **MX** for $\varrho \in (-\infty, 0)$. In view of Theorem 6 and Propositions 3 and 4, the extension CES^{0} must fail **MX**.

For $\alpha \in \mathbb{R}_+$ and $\varrho \in \mathbb{R} \setminus \{0\}$, we obtain

$$\mathrm{CES}^{\varrho} v\left(\alpha \cdot 1_{\{i\}} + 1_{\{j\}}\right) \overset{(12),(15),(C.1)}{=} \alpha + 1 - \left[\frac{1}{2}\right]^{\frac{1}{\varrho}} \cdot \left(\alpha^{\varrho} + 1\right)^{\frac{1}{\varrho}}$$

and

$$\frac{d}{d\alpha} \mathrm{CES}^{\varrho} v\left(\alpha \cdot 1_{\{i\}} + 1_{\{j\}}\right) = 1 - \left[\frac{1}{2}\right]^{\frac{1}{\varrho}} \cdot \left(\frac{\alpha}{\left(\alpha^{\varrho} + 1\right)^{\frac{1}{\varrho}}}\right)^{\varrho - 1}.$$

Hence, we have

$$\lim_{\varepsilon \downarrow 0} \frac{d}{d\alpha} \mathrm{CES}^{\varrho} v\left(\varepsilon^{\frac{1}{\varrho}} \cdot 1_{\{i\}} + 1_{\{j\}}\right) = -\infty \qquad \text{for } \varrho \in (0, 1)$$

and

$$\lim_{\varepsilon\to\infty}\frac{d}{d\alpha}\mathrm{CES}^{\varrho}v\left(\varepsilon^{\frac{1}{\varrho}}\cdot 1_{\{i\}}+1_{\{j\}}\right)=-\infty \qquad \text{for } \varrho\in(1,\infty)\,.$$

Therefore, CES^{ϱ} fails **MX** for $\varrho\in(0,1)$ and $\varrho\in(1,+\infty)$.

By Theorem 6, we already know that the Lovász extension $\mathrm{L}=\mathrm{CES}^{-\infty}$ satisfies **MX**. Remains to show that both CES^{1} and $\mathrm{CES}^{+\infty}$ satisfy **MX**. First, we show that

$$\mathrm{CES}^{1}v\,(s)=\sum_{i\in N}s_i\cdot\sum_{\rho\in R}\frac{1}{|R|}\cdot MC_i^{v}\,(\rho)\,. \tag{C.2}$$

and

$$\mathrm{CES}^{+\infty}v\,(s)=\sum_{i\in N}s_i\cdot MC_i^{v}\,(\rho) \qquad \text{for } \rho\in R^{*}\,(s) \tag{C.3}$$

for all $v\in\mathbb{V}$ and $s\in\mathbb{R}_{+}^{N}$, where

$$R^{*}\,(s):=\{\rho\in R\mid\rho\,(i)<\rho\,(j)\ \text{ for all } i,j\in N \text{ with } s_i<s_j\}\,. \tag{C.4}$$

Since both sides of (C.2) and (C.3) are linear in the game, we are allowed to restrict attention to unanimity games. Let $T\subseteq N$, $T\neq\emptyset$. For any $\rho\in R$, the marginal contribution of the last player from T is one for u_T, whereas the marginal contributions of all other players are zero. Since all rank orders have the same probability in the right-hand side of (C.2), any of the players in T has a probability of $1/|T|$ of being the last player from T. Hence, we have

$$\sum_{i\in N}s_i\cdot\sum_{\rho\in R}\frac{1}{|R|}\cdot MC_i^{u_T}\,(\rho)=\sum_{i\in T}\frac{1}{|T|}\cdot s_i\overset{(1),(12),(15)}{=}\mathrm{CES}^{1}u_T\,(s)\,.$$

For any $s\in\mathbb{R}_{+}^{N}$ and $\rho\in R^{*}\,(s)$, the marginal contribution of the last player from T is one for u_T, whereas the marginal contributions of all other players are zero. By (C.4), the amount of the resource of any of the last players from T is $\max_T(s)$. Hence, we have

$$\sum_{i\in N}s_i\cdot MC_i^{u_T}\,(\rho)=\max\nolimits_T(s)\overset{(1),(13),(15)}{=}\mathrm{CES}^{+\infty}v\,(s)\,.$$

In monotonic games, all marginal contribution are non-negative. By (C.2), it is therefore immediate that CES^{1} satisfies **MX**. Using arguments analogous to those showing that $\mathrm{L}=\mathrm{CES}^{-\infty}$ satisfies **MX** in the proof of Theorem 6, one shows that $\mathrm{CES}^{+\infty}$ satisfies **MX**.

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