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Lagrangian mean curvature flow

Jason D. Lotay[1]

Lagrangian mean curvature flow is a powerful tool in modern mathematics with connections to topics in analysis, geometry, topology and mathematical physics. I will describe some of the key aspects of Lagrangian mean curvature flow, some recent progress, and some major open problems.

1 Shortest curves

A famous mathematical question which goes back to antiquity is the *isoperimetric problem*. A simple version of the problem asks: given an area A, what is the shortest curve in the plane that encloses the given area A? The answer is well-known: the shortest curve is a circle as in Figure 1.

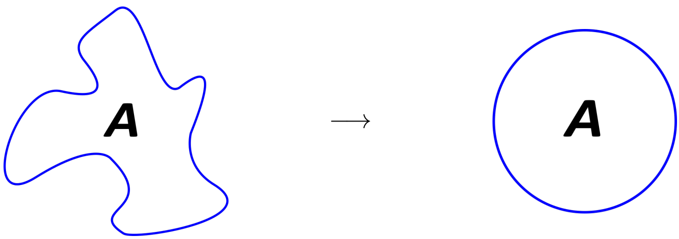

Figure 1: The isoperimetric problem

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1.1 Curves on surfaces

Another way to think about the isoperimetric problem is to imagine that the plane is really the surface of a sphere with a point (say, the North pole) removed: this is the familiar *stereographic projection* (see Figure 2).

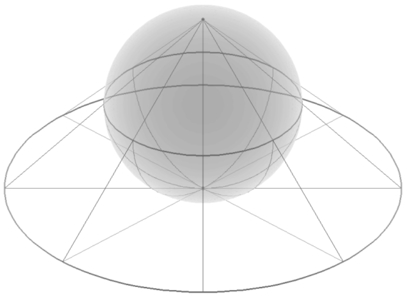

Figure 2: Stereographic projection

When we "wrap up" the plane to get a sphere, a given curve in the plane will become a curve in the sphere, as in Figure 3, and the area A it encloses in the plane will become the area below the curve on the sphere. By rescaling our stereographic projection, we can always ensure that the curve divides the sphere into regions of equal area (so A becomes half the area of the sphere). The isoperimetric problem then becomes: what is the shortest loop which divides the sphere into regions of equal area? The answer, of course, is the equator or, more accurately, any *great circle*, which means the intersection of the sphere with a plane passing through the centre of the sphere (as shown on the right-hand side of Figure 3).

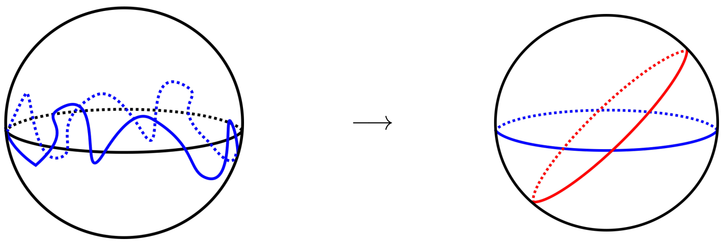

Figure 3: Curves dividing the sphere into regions of equal area

We can now generalize our isoperimetric problem to other surfaces, such as the torus (as in Figure 4), and ask: given a class of curves, what is the shortest curve (or curves) representing that class? In particular, how can we find these shortest curves?

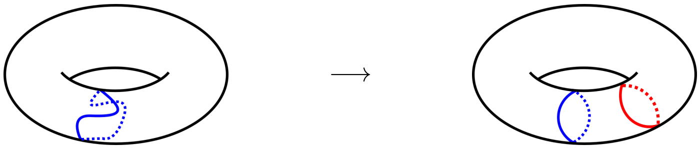

Figure 4: Curves on the torus

1.2 Curve shortening flow

Given an initial curve γ_0, one way to approach the problem of finding the shortest curve(s) in the same class as γ_0 is to consider a family of curves $\gamma(t)$, where $t \geq 0$ represents "time", with the properties that $\gamma(0) = \gamma_0$ and, as time increases, the length of $\gamma(t)$ decreases as quickly as possible. Explicitly, the family of curves γ must satisfy the following equation, which informally says that the "velocity" of the family of curves $\gamma(t)$ is the "curvature" $\kappa(t)$ of the curve $\gamma(t)$:

$$\frac{\partial \gamma}{\partial t} = \kappa. \tag{1}$$

(Note here that the time derivative appears as a partial derivative because the function $\gamma(t)$ depends on two variables: the position on the curve as well as time.) Informally, the curvature is how "curved" the curve is at a point: for example, in the plane, a straight line has $\kappa = 0$, whereas a circle of radius r has curvature $\kappa = 1/r^2$ (so a circle is "more curved" if it is smaller).

Equation (1) is called the *curve-shortening flow*, since it shortens the curve as quickly as possible[2]. The curve-shortening flow is an example of what is called a "geometric flow". These are currently the subject of much research, and are, in a sense, nonlinear versions of the heat equation, which was first developed by Joseph Fourier in 1822 to describe how heat dissipates. In fact, the curve shortening flow is the simplest example of "Lagrangian mean curvature flow", which we will introduce in Section 2.

[2] The notion of a *flow* comes from physics and formalises the idea of the motion of a particle in a fluid. It is informally viewed as a continuous motion of points in time.

1.3 Examples

If we start with a circle in the plane, then under the curve shortening flow (1) it will always remain a circle, but it will get smaller and smaller until it shrinks to a point in a finite amount of time (as in Figure 5).

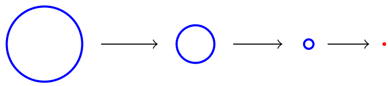

Figure 5: Circle shrinking to a point under curve shortening flow

If we instead start with a curve on the torus as in the left part of Figure 4, it will converge (that is, get closer and closer) to a loop like one in the right part of Figure 4 as t goes to infinity (in other words, as t gets larger and larger).

One particularly important example is that if we start with any curve on the sphere as on the left part of Figure 3 which divides the sphere into regions of equal area, then the curve shortening flow will exist for all $t > 0$ and the flow will converge to a great circle as on the right part of Figure 3 as t tends to infinity.

2 Lagrangians

Mathematicians are very interested in a class of geometric objects called *Lagrangians*, which are named after Joseph-Louis Lagrange (1738–1813) and naturally arise in classical mechanics, encoding some of the key properties of a mechanical system, though they are now a fundamental part of modern mathematics. Lagrangians can have any dimension, but always live in a space of twice their dimension: the simplest examples are curves (of dimension 1) on surfaces (of dimension 2), as we saw before. To give an intuitive sense of these objects is somewhat complicated, but they are important because they connect geometry, topology and physics.

2.1 Mean curvature flow

Just as for curves, given a Lagrangian L_0 we can try to decrease its "volume" (remember now it can have any dimension) as fast as possible with a family $L(t)$ for $t \geq 0$ with $L(0) = L_0$. This again gives a geometric flow called the *mean curvature flow*:

$$\frac{\partial L}{\partial t} = H, \tag{2}$$

where H is known as the *mean curvature* of $L(t)$, which is a sort of "average" curvature of $L(t)$ at each point. When L_0 is a curve in a surface then the mean curvature H is equal to the curvature κ of the curve, and so (2) becomes (1).

2.2 Finding smallest Lagrangians

Using the mean curvature flow (2) we can try to find the smallest Lagrangian representing a given class of Lagrangians, just like for curves on surfaces. However, in general, starting with a Lagrangian $L(0)$ does not mean that $L(t)$ is Lagrangian for all later times $t > 0$. That is, the mean curvature flow (2) does not preserve the Lagrangian condition. An important observation by Smoczyk [7] is that the Lagrangian condition is preserved by the mean curvature flow if the space in which the Lagrangian lives has the special property known as *Kähler–Einstein*, named after Erich Kähler (1906–2000) and Albert Einstein (1879–1955) (because of their relation to Einstein's equations from General Relativity). The Kähler-Einstein condition is informally a natural extension of the notion of surfaces with constant curvature, such as the round sphere, to higher even dimensions. In these particular spaces, we can then solve (2) so that $L(t)$ is Lagrangian for all t: this gives rise to the *Lagrangian mean curvature flow*.

Lagrangian mean curvature flow is extremely important mathematically, as well as having links to String Theory in theoretical physics. The great challenge is to understand when the flow will exist for all $t > 0$ and converge to a minimal Lagrangian (one for which $H = 0$) as t tends to infinity.

3 Examples

Picturing Lagrangians which are not curves is a little difficult: for example, Lagrangian surfaces must live in 4 dimensions. However, we can visualize some Lagrangian surfaces as follows.

3.1 Lagrangian tori

Suppose we have a curve in the plane, such as the ellipse in Figure 6, which does not go through the origin. Then, we can define a Lagrangian so that for every point in the curve we have a circle in the Lagrangian, where the size of the circle is equal to the distance from the origin at that point of the curve. This will mean that the Lagrangian defined by an ellipse will be a torus.

If we take $a = b$ in Figure 6 then we get a circle and the corresponding Lagrangian is known as the *Clifford torus*, named after William Clifford (1845-1879). In this case, since $a = b$, the circles at each point are all the same size,

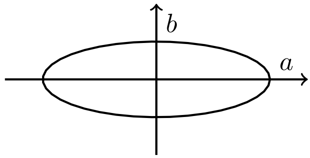

Figure 6: An ellipse defining a Lagrangian torus

and so it can be viewed just like the torus in Figure 4. The Clifford torus is a fascinating object which has been the subject of mathematical research for more than 100 years, and has recently been the focus of great interest. In particular, the Clifford torus is the natural higher-dimensional version of the usual circle in the plane, or the equator in the sphere.

If $a = b = 1$, say, then the area enclosed by the circle will be π. If we want the area of an ellipse as in Figure 6 to also enclose area π, then we need the condition $ab = 1$. If we have two different ellipses with $ab = 1$, so that they enclose the same area, then the Lagrangian tori they define are called *Hamiltonian isotopic*[3]. This is a higher-dimensional version of saying that two curves in the plane contain the same area, or that two curves in the sphere both split the sphere into regions of equal area, as we considered earlier.

3.2 Lagrangian spheres

If instead we have a curve in the plane which passes through the origin, such as the "figure eight" curve in Figure 7, then we define a Lagrangian in the same way, except that for the origin in the curve we have just a point in the Lagrangian (since the corresponding circle in the Lagrangian has "zero length").

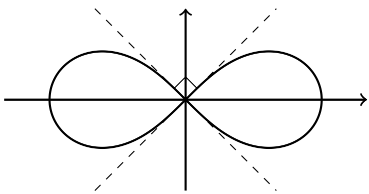

Figure 7: A figure eight defining a Lagrangian sphere

[3] The name comes from William Hamilton (1805–1865) and his work on classical mechanics.

This Lagrangian is a little more complicated to understand, but one way to see what Lagrangian you get is described in Figure 8, where we imagine starting with an ellipse (given a Lagrangian torus) and deforming it into a figure eight.

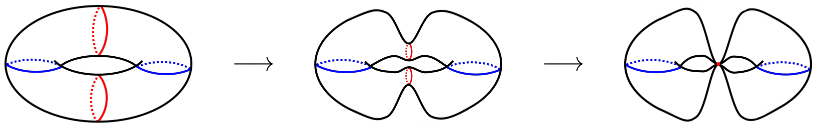

Figure 8: A torus collapsing into a sphere

The resulting object we get after collapsing the torus is *topologically* a sphere, meaning that if you imagine it was made of plasticine, then you can smoothly deform it into a round sphere. To convince yourself that this is the case, imagine pushing the blue circle around the surface until it shrinks to a point at the red dot. This is exactly what happens if you take the equator and push it up to the North pole in the sphere. The Lagrangian sphere we get from Figure 7 is known as the *Whitney sphere*, named after Hassler Whitney (1907–1989).

3.3 Clifford torus

If we start the Lagrangian mean curvature flow with L_0 being the Clifford torus described above, then the corresponding Lagrangian $L(t)$ (solving the Lagrangian mean curvature flow) will always be given by a circle in the plane at each t (and so a Lagrangian torus) until it shrinks to a point, as in Figure 5.

However, if we start the Lagrangian mean curvature flow at a torus defined by an ellipse with b much smaller than a (but with $ab = 1$ so that the torus is Hamiltonian isotopic to the Clifford torus), it is shown in [2, 5] that the ellipse will collapse into a figure eight under the flow, so the torus will collapse into a sphere as in Figure 8.

In [1] I showed recently, in collaboration with Evans and Schulze, that you can take b arbitrarily close to a, with $ab = 1$, and still the Lagrangian mean curvature flow starting at that torus cannot shrink to a point. We believe that the torus will again collapse as in Figure 8 above. This result is very surprising because it says that the Clifford torus is *unstable* for Lagrangian mean curvature flow under arbitrarily small Hamiltonian perturbations, meaning that you can change the initial Clifford torus a very small amount to one that is Hamiltonian isotopic and yet get very different behaviour for the flow.

One key open problem is this: does the Clifford torus have the least volume amongst all Lagrangians which are Hamiltonian isotopic to it? The result in [1] shows that tackling this problem using Lagrangian mean curvature flow, which is a natural approach, will unfortunately be very difficult.

3.4 Whitney sphere

One can also ask: what happens if we start the Lagrangian mean curvature flow at the Whitney sphere? This has recently been answered, under certain assumptions, in [6, 9]: the sphere will collapse to a point, and the corresponding figure eight will not stay the same shape but deform as in Figure 9, "squashing" vertically faster than it does horizontally.

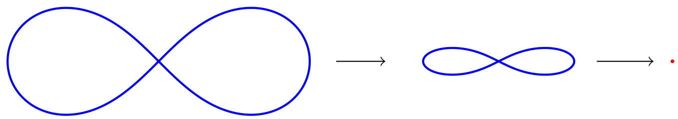

Figure 9: A figure eight collapsing under the flow

4 Thomas–Yau Conjecture

We have seen various examples in the previous section where we know what the Lagrangian mean curvature flow looks like. However, in general, we do not know what the flow does, even in flat space. A particularly important class of spaces with the Kähler–Einstein property (recall that this was mentioned in Section 2.2 above), where we want to understand the Lagrangian mean curvature flow are called *Calabi–Yau manifolds*[4]: these are of central importance in many parts of mathematics, as well as being relevant to String Theory in theoretical physics. In Calabi–Yau manifolds, the minimal Lagrangians (which the Lagrangian mean curvature flow is supposed to find) are basically the same as "special Lagrangians", which are the subject of a great deal of research.

In 2002, Thomas and Yau [8] made a conjecture that the Lagrangian mean curvature flow in Calabi–Yau manifolds should exist for all time and converge to a special Lagrangian if and only if the initial Lagrangian is stable, in a certain sense. Arguably the largest open problem in Lagrangian mean curvature flow is to try to prove (or disprove) the Thomas–Yau Conjecture, or more sophisticated, updated versions of the conjecture due to Joyce [3].

Recently, Oliveira and I showed in [4] that a version of the Thomas–Yau Conjecture is true for a large class of 2-dimensional Lagrangians. However, the general conjectures by Thomas–Yau and Joyce remain out of reach, and resolving these conjectures will be invaluable for Lagrangian mean curvature flow, which will be of significant importance for both mathematics and theoretical physics.

[4] A manifold is a space that can be curved, but if we zoom sufficiently close to any point, it locally looks as flat as a plane (like the sphere and torus we have already seen).

Image credits

Figure 2 Author: CheChe. Licensed under Creative Commons Attribution-Share Alike 4.0 International via Wikimedia Commons, https://commons.wikimedia.org/wiki/File:Stereographic_projection_in_3D.svg, visited on August 5, 2020.

All other images created by the author.

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Jason D. Lotay *is a professor of pure mathematics at the University of Oxford.*

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Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9 – 11
77709 Oberwolfach
Germany

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