Reyes' Topos of Reference and Modality from a fibrational Perspective

Master thesis by Joël Doat
Date of submission: 9 September, 2021

1. Review: Thomas Streicher
Darmstadt

TECHNISCHE UNIVERSITÄT DARMSTADT

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Darmstadt, 9 September, 2021

J. Doat

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1 Introduction

The basis of this thesis is given by Reyes' *A Topos-Theoretic Approach to Reference and Modality* providing semantics for modal higher-order theory language. Using **geometric morphisms** $\Delta \dashv \Gamma : \mathbb{E} \to \mathbb{S}$ between **topoi**, he defined the base topos $\mathbb{S}$ as the „universe of constant sets" which is best understood as the category of kinds, i.e. objects PERSON, DOG, CAT, etc. are sets representing count nouns containing „urelements" (e.g. references like proper names). With $\mathbb{E}$ as the „universe of variable sets" accounting for the different worlds in which the properties of kinds should be interpreted, the predicates on kinds (or constant sets) should thus be expressed in terms of morphisms (and objects) in $\mathbb{E}$. Therefore, predicates on kinds are identified with morphisms $\Delta(S) \to \Omega_{\mathbb{E}}$ in $\mathbb{E}$ for a constant set S, where Δ is given as the constant functor. In other words, these predicates can also be seen as variable subsets of $\Delta(S)$. As a consequence, it makes sense to also define the collection of objects $\Delta(S)$ for S in $\mathbb{S}$ as constant sets.

Nevertheless, as Reyes' theory is introduced in a rather unintuitive way using geometric morphisms, chapter 2 is concerned with bridging the geometric intuition with a logical one. Even though **fibered categories** were introduced by Grothendieck for geometric purposes (see [3]), Moens (see [9]) provided results linking these two sides using such categories: while the „geometric" reasoning is given by functors corresponding to inverse images and continuous maps, the „logical" properties can be explained in purely fibrational terms. This allows to motivate Reyes' use of geometric concepts to describe the interpretation of logical operations.
The chapter starts with some necessary preliminaries about topoi and fibered categories. **Elementary topoi** are categories satisfying certain axioms which allow to formalize all traditional mathematical structures and e.g. first-order logic without the use of set theory. In Section 2.1, we first briefly define topoi together with some important results that are needed for further developments and set-valued presheaves/sheaves as running example to demonstrate the connections throughout the different subsections including Chapter 3. Then, the aim of the subsequent sections is to view a topos (or more generally, a

category with pullbacks) *as fibered over itself* and how it relates to geometric morphisms between topoi. More concretely, the theory of **fibered categories**[1] allows to see objects of a category $\mathbb{C}$ intuitively as indexed sets. For a fibered category $P : \mathbb{X} \to \mathbb{C}$, an object X in $\mathbb{X}$ can thought of as a family indexed by $I = P(X)$. To obtain such a fibration over $\mathbb{C}$, we define for a functor $F : \mathbb{C} \to \mathbb{B}$, a glueing functor $F^*P_{\mathbb{B}} : \mathbb{B} \downarrow F \to \mathbb{C}$ where $P_{\mathbb{B}} : \mathbb{B}^2 \to \mathbb{B}$ is the codomain functor.
Further properties of fibrations are then discussed leading to Moens' central lemma offering an alternative characterization of such glueing functors for finite limit preserving functors. More importantly, as a consequence of Moens' lemma, geometric morphisms $F \dashv U : \mathbb{C} \to \mathbb{B}$ can be „lifted" to **fibered adjunctions** between $F^*P_{\mathbb{B}}$ and $P_{\mathbb{C}}$ which stand in one-to-one correspondence with the former, linking the geometric intuition with a purely fibrational perspective. These fibered adjunctions will useful to intrinsically define modal adjoint operators in Chapter 3 by restricting them to subobjects of the corresponding topos.
The chapter finishes with discussing further properties of geometric morphisms such as **local-connectedness**, **connectedness**, and **boundedness** together with the characterizations in terms of their corresponding fibered adjunction.

Chapter 3 introduces topos semantics as a mathematical formalization of kinds that can be used to describe the modal operators of **necessity** and **possibility**[2]. As it will be shown, further advantages are the implicit logic of quantification and exponentiability of kinds. In Subsections 3.1.1-3.1.4, the concept of reference in terms of constant and variable sets is again briefly introduced using geometric morphisms while also the connection to the fibrational results of the previous chapter is clarified to define necessary operations on subobjects (i.e. predicates on kind). Using these operations, the **modal adjoint operators** of necessity and possibility satisfying the axioms of S4 modal logic[3] are intrinsically definable when assuming local-connectedness on the geometric morphism. Furthermore, these modal operators are exemplified using again the set-valued presheaf/sheaf examples of the previous chapter.
In 3.1.5-3.1.6, further concepts inspired by the semantics of natural language are discussed: intensionality and extensionality as described originally by Montague (see [11]) and coincidence relations (i.e. subquotients of some $\Delta(S)$ representing objects in $\mathbb{E}$) modelling phenomena of opacity.
In Section 3.2, the chapter concludes with the relationship of the previous considerations

[1] This work follows the structure of Streicher in [14].
[2] They do not correspond exactly with modal operators in traditional modal logic.
[3] S4 is especially interesting due to its rather well-behavedness in classic modal logic.

to the language of modal higher-order theory. By identifying constant sets (i.e. $\Delta(S)$ for some S in $\mathbb{S}$) with **sorts**, they are used to form the „meaningful expressions“ of the language consisting of **terms** and **formulae**. Finally, the semantics of these expressions is given using our topos-theoretic constructions.

2 Category-Theoretic Preliminaries

2.1 Topoi

2.1.1 Motivation

In this subsection, we will explain the important parts of a topos in terms of (the category of) sets. The core part of such a topos is given by its *subobject classifier*.
First of all, for a set X, its subobjects are defined as all its subsets. These subsets can be expressed as functions rather than elements, i.e. the function $f : A \to X$ representing the image of a set A under f as a subset of X. Since then two different functions $f : A \to X$ and $f' : B \to X$ can represent the same subset in X, it would be helpful to assume that the domains are at least isomorphic to include a concept of equality for subobjects.

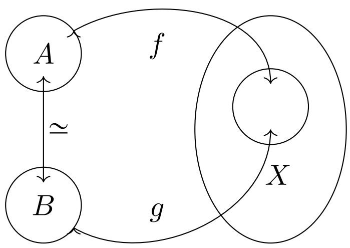
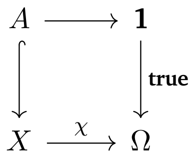


Figure 2.1: Isomorphism between subobjects in **Set**

Thus, we have to require functions that represent subsets to be injective. In an arbitrary category, this definition can then be replaced by monomorphism as we will see in Definition 2.1.1.

The second method to define subsets as functions are characteristic functions, i.e. a function $\chi : X \to \Omega$ with $\Omega = \{t, f\}$ such that t represents 'true', f represents 'false', and χ maps only those elements to t that are contained in the corresponding subset. Furthermore, to formally designate one element as 'true' in Ω, we define the function **true** : $\mathbf{1} \to \Omega$ with a singleton $\mathbf{1}$. The advantage of this definition is that we are able to talk about this classifying set Ω without talking about its elements. Now, we can combine the functions χ and **true** to create the corresponding subset of X: for a characteristic function $\chi : X \to \Omega$ there is a unique subset $A \subseteq X$ such that $A = \chi^{-1}(\mathbf{true}(*))$. This universal property that links subsets to the classifying set Ω is given by the pullback of **true** along χ:



The pullback generalizes the subset inclusion to arbitrary categories as we will see in Definition 2.1.2.

The two previous descriptions amount now to the relation between the representation of subsets with injective functions and characteristic functions. Let $\mathrm{Sub}(X)$ be the set of sets that have an injective map into X modulo isomorphism. Then the unique subset given by a characteristic function represents exactly the corresponding equivalence class, i.e.

$$\mathrm{Sub(X)} \simeq \mathrm{Hom}(X, \Omega). \tag{2.1}$$

In Proposition 2.1.3, we will see this correspondence for arbitrary categories.

2.1.2 Definition

As described in Subsection 2.1.1 to define subobjects, we generalize from injective maps in the category of sets to monomorphism in arbitrary categories. Moreover, the equality of subobjects can be defined similarly to the isomorphisms of sets, as in Fig. 2.1 with the corresponding collection of subobjects modulo equality.

Definition 2.1.1. [6]

(a) Let $\mathbb{E}$ be a category. For an object X in $\mathbb{E}$, we call a monomorphism into X a **subobject**.

(b) For subobjects $m : A \hookrightarrow X$ and $n : B \hookrightarrow X$, m **is contained in** n iff there is a morphism $f : A \to B$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle m}\big\downarrow & \swarrow{\scriptstyle n} & \\ X & & \end{array}$$

commutes.

(c) Subobjects m and n are **equal** (denoted by $m \sim n$) iff m is contained in n and vice versa.

(d) We define the collection of subobjects of X modulo $\sim$ as $\mathrm{Sub}_{\mathbb{E}}(X)$.

Remark. Let $\mathbb{E}$ be a category with pullbacks. Then $\mathrm{Sub}_{\mathbb{E}}$ extends to a functor $\mathrm{Sub}_{\mathbb{E}} : \mathbb{E}^{op} \to \mathbf{Set}$ by defining $\mathrm{Sub}_{\mathbb{E}}(f)([m]_{\sim}) = [f^*m]_{\sim}$ where f^*m is the pullback of m along f. Moreover, note that $\mathrm{Sub}_{\mathbb{E}}(X)$ defines a poset (category).

Definition 2.1.2. [6] Let $\mathbb{E}$ be some category with terminal object $\mathbf{1}$. An object Ω together with a map $t : \mathbf{1} \to \Omega$ is called **subobject classifier** if for any monomorphism $m : A \hookrightarrow X$ in $\mathbb{E}$, there is a unique map $\chi_m : X \to \Omega$ such that

$$\begin{array}{ccc} A & \xrightarrow{!} & \mathbf{1} \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle t} \\ X & \xrightarrow{\chi_m} & \Omega \end{array}$$

is a pullback square.

Concluding the foregoing considerations, the existence of a subobject classifier in a category $\mathbb{E}$ can be characterized by the functor $\mathrm{Sub}_{\mathbb{E}}$.

Proposition 2.1.3. [6] Let $\mathbb{E}$ be an arbitrary category with pullbacks. Then $\mathbb{E}$ has a subobject classifier iff $\mathrm{Sub}_{\mathbb{E}} : \mathbb{E}^{op} \to \mathbf{Set}$ is representable.

Definition 2.1.4. [6] A category $\mathbb{E}$ is a **topos** if

(a) $\mathbb{E}$ has finite limits,

(b) $\mathbb{E}$ is cartesian closed,

(c) $\mathbb{E}$ has a subobject classifier $t : \mathbf{1} \to \Omega_{\mathbb{E}}$ with terminal object $\mathbf{1}$.

Example 2.1.5. The category **Set** of sets the subobject classifier is given by a two-element set $\mathbf{2}$ together with $t : \mathbf{1} \to \mathbf{2}$ where $\mathbf{1}$ is a one-element set $\{*\}$.

Example 2.1.6. For a small category $\mathbb{C}$ the category $\mathbf{Set}^{\mathbb{C}^{op}}$ forms a topos. The subobject classifier $\Omega : \mathbb{C}^{op} \to \mathbf{Set}$ such that for objects X in $\mathbb{C}$, the set $\Omega(X)$ contains all sieves[1] on X and for morphisms $f : Y \to X$

$$\Omega(f)(\sigma) := \{g : Z \to Y \mid f \circ g \in \sigma\}.$$

Then the components for every object X in $\mathbb{C}$ of $t : \mathbf{1} \to \Omega$ are given by the set of morphisms with codomain X, i.e. the maximal sieve on X, where $\mathbf{1} : \mathbb{C}^{op} \to \mathbf{Set}$ is mapping constantly to a one-element set and its identity morphisms. Consider the following important sub-examples:

(i) Let $\mathbb{C} := I$ for some set I. For a two-element set $\mathbf{2}$, the subobject classifier of $\mathbf{Set}^I$ is the functor $\Omega : I \to \mathbf{Set}$ such that $\Omega(X) = \mathbf{2}$ together with $t : \mathbf{1} \to (\mathbf{2})_{i \in I}$ such that $t_i(*) = \top$ (assuming $\top$ represents „true“).

(ii) Let $\mathbb{C} := \mathbb{P} = (|\mathbb{P}|, \leqslant_{\mathbb{P}})$ be a preorder category. Its subobject classifier $\Omega : \mathbb{P}^{op} \to \mathbf{Set}$ is defined by

$$\Omega(U) = \{K \subseteq \downarrow_{\mathbb{P}} U \mid K \text{ is downward closed } \}$$

for objects U, where $\downarrow_{\mathbb{P}} U := \{R \in |\mathbb{P}| \mid R \leqslant_{\mathbb{P}} U\}$ and for the unique arrow $f : R \to U$ the image $\Omega(f)$ is the restriction of $\Omega(U)$ to $\Omega(R)$. Then the components for every object U in $\mathbb{P}$ of $t : \mathbf{1} \to \Omega$ are given by

$$t_U(*) = \downarrow_{\mathbb{P}} U.$$

[1] A sieve σ is a set of arrows with codomain X such that for all $f \in \sigma$, we have $f \circ g \in \sigma$ for suitable g.

Example 2.1.7. Let $\mathrm{Sh}(\mathcal{C})$ be the category of sheaves on a cite $\mathcal{C} = (\mathbb{C}, J)$. Recall that the **closure of a sieve** U (see [10]) is defined by

$$\mathrm{cl}(U) := \{C \in \mathrm{Ob}(\mathbb{C}) \mid \exists \{C_i \to C\}_{i \in I} \in J \forall i \in I : C_i \in U\}$$

and a sieve is **closed** if it corresponds to its closure. Then the subobject classifier $\Omega^{cl} : \mathbb{C}^{op} \to \mathbf{Set}$ is defined by

$$\Omega^{cl}(U) = \{K \mid K \text{ is a closed sieve on } U\}.$$

for objects U and for morphisms

$$\Omega^{cl}(f) = f^*,$$

where f^* is the pullback of sieves. Then $t : \mathbf{1} \to \Omega^{cl}$ is mapping to the maximal sieve on each object of $\mathbb{C}$.

Note that the previous examples can be generalized in the following theorem.

Theorem 2.1.8. [6] A category of presheaves is a topos.

Finally, we define the notion of homomorphisms between two topoi. Intuitively, these preserve the structure of a topos with respect to logic. Therefore, they will be especially useful later on to explain the „logical“ aspects of geometric morphisms.

Definition 2.1.9. [6] Let $F : \mathbb{S} \to \mathbb{E}$ be a functor between topoi. We call F **logical** if

(1) F preserve finite limits,

(2) for every object X in $\mathbb{S}$ the canonical morphism

$$\varphi_X : F(\Omega^X) \to \Omega^{F(X)}$$

is an isomorphism.

Remark. Note that with regard to the previous considerations, the exponential object Ω^X is just the internalization of $\mathrm{Hom}(X, \Omega)$.

2.2 Fibered Categories

2.2.1 Motivation

In this section, we will discuss the generalization of *presheaves over* $\mathbb{C}$

$$F : \mathbb{C}^{op} \to \textbf{Set}$$

for a category $\mathbb{C}$. Of course, it is possible to extend the codomain of such a presheaf to „set-based“ categories like e.g. the categories of groups, rings, etc. since they appear already as a special instance in the functor category $[\mathbb{C}^{op}, \textbf{Set}]$. In our more general context, it is useful to even consider pseudofunctorial „presheaves of categories“

$$\mathcal{H} : \mathbb{C}^{op} \to \textbf{Cat}.$$

Thanks to the **Grothendieck construction**, these functors can be translated to the notion of **fibered categories**.
But first, we discuss some examples of such „presheaves of categories“ that are interesting in the context of this work.

Example 2.2.1. For a category $\mathbb{C}$, consider the functor

$$\text{Fam}(\mathbb{C}) : \textbf{Set}^{op} \to \textbf{Cat}$$

such that for objects I we have $\text{Fam}(\mathbb{C})(I) = \mathbb{C}^I$ and for morphisms $u : J \to I$ we have $\text{Fam}(\mathbb{C})(u) = \mathbb{C}^I \to \mathbb{C}^J$.

Even though, this functor is an archetypical example in the theory of fibered categories, it might be interesting to replace **Set** with an arbitrary category. Especially, for our purposes topoi will be interesting but to generalize this example, even a category with pullbacks suffices.

Example 2.2.2. For a category $\mathbb{C}$ with pullbacks, consider the functor

$$\mathcal{H} : \mathbb{C}^{op} \to \textbf{Cat}$$

such that for objects I we have $\mathcal{H}(I) = \mathbb{C}/I$ and for morphisms $u : J \to I$, we obtain the pullback functor $\mathcal{H}(u) = u^* : \mathbb{C}/I \to \mathbb{C}/J$.

However, note that the maps $\mathcal{H}(u)$ require a choice of pullbacks and thus, the composition of such pullbacks holds only up to isomorphism, i.e.

$$\mathcal{H}(uv) \simeq \mathcal{H}(v) \circ \mathcal{H}(u).$$

Usually, such functors that only preserve composition up to isomorphism are called **pseudo-functors**. Nevertheless, thanks to the Grothendieck construction, we are able to replace such pseudo-functors $\mathcal{H} : \mathbb{C}^{op} \to \textbf{Cat}$ with regular functors $P : \mathbb{X} \to \mathbb{C}$. Before explaining the Grothendieck construction, we add a concluding example that involves the subobjects discussed in the last section, as a restriction of the previous example.

Example 2.2.3. For a category $\mathbb{C}$ with pullbacks, consider again the functor $\text{Sub}_{\mathbb{C}} : \mathbb{C}^{op} \to \textbf{Set}$ from the remark of Definition 2.1.1 which sends objects I to $\text{Sub}_{\mathbb{C}}(I)$ the set of monomorphisms to I modulo $\sim$ and for morphisms u defining $\text{Sub}_{\mathbb{E}}(u)([j]_{\sim}) = [u^*j]_{\sim}$ where u^*j is the pullback of j along u.
As we can see the collection of subobjects can be seen a poset category, the functor extends to the category **Cat**

$$\text{Sub}_{\mathbb{C}} : \mathbb{C}^{op} \to \textbf{Cat}$$

Now, after the introductory examples, we can introduce briefly the intuition behind the Grothendieck construction[2] that will help us to motivate the equivalence between „presheaves of categories“ and fibered categories:
for a category $\mathbb{C}$, consider the 2-functor

$$\int : [\mathbb{C}^{op}, \textbf{Cat}] \to \textbf{Cat}/\mathbb{C}.$$

Then the functor $\int \mathcal{H} : \mathbb{X} \to \mathbb{C}$ is defined as follows:

- The objects of $\mathbb{X}$ are tuples (I, X) with $I \in \mathbb{C}$ and $X \in \mathcal{H}(I)$.
- The morphisms from (J, Y) to (X, I) in $\mathbb{X}$ is a pair (u, α) with $u : J \to I$ in $\mathbb{C}$ and $\alpha : Y \to \mathcal{H}(u)(J)$ in $\mathcal{H}(J)$.
- For maps $(v, \beta) : (K, Z) \to (Y, J)$ and $(u, \alpha) : (J, Y) \to (X, I)$ in $\mathbb{X}$ the composition $(u, \alpha) \circ (v, \beta)$ is defined by $(u \circ v, \mathcal{H}(u)(\alpha) \circ v)$.
- The identities $\text{id}_{(I,X)}$ are defined by $(\text{id}_I, \text{id}_X)$
- The functor $\int \mathcal{H}$ is sending objects (X, I) to I and morphisms (u, α) to u.

[2]For a complete explanation of the Grothendieck construction see [6]

2.2.2 Definition

In the spirit of the Grothendieck construction, the intuition behind fibered categories can be motivated via an indexed family of sets:
Given some set S, let $q : I \to \mathcal{P}(S)$ be some family of subsets. Then we can define X as the set of tuples (i, s) such that $i \in I$ and $s \in q(i)$ together with a map $p : X \to I$ such that $p((i, s)) := i$. The map p can be seen as same family of subsets indexed over I where the set with index $i \in I$ is $p^{-1}(i) \subseteq X$.
With the help of the Grothendieck construction, this idea can be generalized to consider such morphisms $p : X \to I$ in arbitrary categories as families indexed over I even in categories where I is not a set. In this more general setting, this „inverse image" will be called **fibre**.

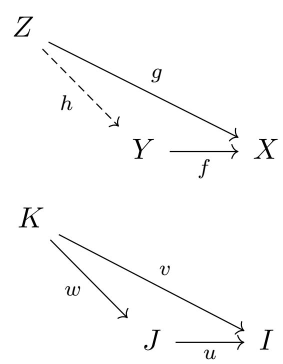
Definition 2.2.4. [14] Let $F : \mathbb{C} \to \mathbb{B}$ be a functor.

(a) Let X be an object in $\mathbb{C}$ and I in $\mathbb{B}$. Then X is **over** I if $F(X) = I$. Let $f : Y \to X$ be a morphism in $\mathbb{C}$ and $u : J \to I$ in $\mathbb{B}$. Then f is **over** u if $F(f) = u$.

(b) Let $u : J \to I$ be a morphism in $\mathbb{B}$, Y over J, and X over I. Then the collection of morphisms from Y to X over u is denoted by $\mathbb{C}_u(Y, X)$.

(c) Let J be an object in $\mathbb{B}$. Then the **fibre over** J is the subcategory $\mathbb{S}_J$ whose objects are the objects over J and whose morphisms are morphisms over id_J. The morphism in such a fibre are called **vertical**.

(d) Let $f : Y \to X$ be over $u : J \to I$. Then f is **cartesian** if for all $g : Z \to X$ over $v : K \to I$ and all $w : K \to J$ with $v = u \circ w$ there is a unique $h : Z \to Y$ over w with $g = f \circ h$. The morphism f is called a **cartesian lifting** of u.



(e) The functor F is a **fibration** or **fibered category** if for all objects Y over J and all morphisms $u : I \to J$ there is a cartesian lifting $f : X \to Y$.

Example 2.2.5. For a category $\mathbb{C}$, define the category $\mathrm{Fam}(\mathbb{C})$ such that

- the objects are set-indexed families $(X_i)_{i\in I}$ of objects in $\mathbb{C}$,
- a morphism from $(Y_j)_{j\in J}$ to $(X_i)_{i\in I}$ is a pair $(u, (f_j)_{j\in J})$ such that $u : J \to I$ is a map in **Set** and $(f_j)_{j\in J}$ is J-indexed family of morphisms $f_j : Y \to X_{u(j)}$ in $\mathbb{C}$,
- composition of maps $(w, (h_k)_{k\in K}) : (Z_k)_{k\in K} \to (Y_j)_{j\in J}$ and $(u, (f_j)_{j\in J}) : (Y_j)_{j\in J} \to (X_i)_{i\in I}$ is given by $(u \circ w, (f_{w(k)} \circ h_k)_{k\in K})$

Consider the functor

$$P : \mathrm{Fam}(\mathbb{C}) \to \mathbf{Set}$$

where $P((X_i)_{i\in I}) = I$ such that $P(u, (f_j)_{j\in J}) = u$. Then the fibre over $(X)_{i\in I} \in \mathrm{Fam}(\mathbb{C})$ is the index category $\mathbb{C}^I$ and a morphism $(u, (f_j)_{j\in J})$ is cartesian if every f_j is an isomorphism. Furthermore, the functor from Example 2.2.1 can be replaced by P via $\textstyle\int$.

Example 2.2.6. For a category $\mathbb{C}$, we define its *arrow category* $\mathbb{C}^2$ whose objects are the arrows in $\mathbb{C}$ and morphism the commuting squares. We introduce the *codomain functor*

$$\partial_1 : \mathbb{C}^2 \to \mathbb{C}$$

which sends a morphism to its codomain and a commuting square

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle x} \\ J & \xrightarrow[u]{} & I \end{array}$$

to u. As we will see in Lemma 2.2.7, the codomain functor is a fibration if and only if the category $\mathbb{C}$ has pullbacks. In this case, the pseudo-functor from Example 2.2.2 can be replaced by ∂_1 via $\textstyle\int$.

We will refer to the codomain functor $P_{\mathbb{C}} = \partial_1$ as the **fundamental fibration** of $\mathbb{C}$. The fundamental fibration will play an important role in this fibrational framework and therefore, it is worth stating explicitly its properties as a lemma.

Lemma 2.2.7. [14] For a category $\mathbb{C}$ and its fundamental fibration $P_{\mathbb{C}}$, we have

(a) the fibre over an object I in $\mathbb{C}$ is the slice $\mathbb{C}/I$,

(b) the cartesian morphisms in $\mathbb{C}^2$ are exactly the pullback squares,

(c) $P_{\mathbb{C}}$ is a fibration iff $\mathbb{C}$ has pullbacks.

Example 2.2.8. By restricting the domain of the fundamental fibration $P_{\mathbb{C}}$ to monomorphisms modulo $\sim$, we obtain the *subobject fibration*

$$P_{\mathsf{Sub}(\mathbb{C})} : \mathsf{Sub}(\mathbb{C}) \to \mathbb{C}$$

which is a fibration if and only if $\mathbb{C}$ has pullbacks along monomorphism (see Lemma 2.2.7) and the fibre over $I \in \mathbb{C}$ is exactly the poset category of subobjects of I. Furthermore, using $\int$, the functor from Example 2.2.3 can be replaced by the subobject fibration.

Lastly, we define functors between fibered categories.

Definition 2.2.9. [14] For fibrations $P : \mathbb{X} \to \mathbb{C}$ and $Q : \mathbb{Y} \to \mathbb{C}$, a **fibered functor** between P and Q is given by a functor $F : \mathbb{X} \to \mathbb{Y}$ such that

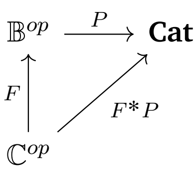
(1) $Q \circ F = P$,

(2) $F(f)$ is cartesian in Q if f is cartesian in P.

2.2.3 Properties

Change of Base

In terms of „presheaves of categories“, the intuition is given by the following diagram



where F^*P is the **change of base of (the pseudofunctor) P along F**.
This can be again translated into the theory of fibered categories: consider a fibration $P : \mathbb{X} \to \mathbb{B}$ and an arbitrary functor $F : \mathbb{C} \to \mathbb{B}$. Then the diagram

$$\begin{array}{ccc} \mathbb{Y} & \xrightarrow{K} & \mathbb{X} \\ {\scriptstyle F^*P}\downarrow & & \downarrow{\scriptstyle P} \\ \mathbb{C} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

forms a pullback in **Cat**. Here again, the functor F^*P is obtained by the **change of base of P along F**. Using the fundamental fibration from Example 2.2.6 in the last subsection, we obtain the following important instance.

Definition 2.2.10. [14] Let $\mathbb{B}$ and $\mathbb{C}$ be arbitrary categories, $F : \mathbb{C} \to \mathbb{B}$ a functor and $P_{\mathbb{B}}$ the fundamental fibration of $\mathbb{B}$. Then we call the pullback square

$$\begin{array}{ccc} \mathbb{B} \downarrow F & \xrightarrow{K} & \mathbb{B}^2 \\ {\scriptstyle P_F}\downarrow & & \downarrow{\scriptstyle P_{\mathbb{B}}} \\ \mathbb{C} & \xrightarrow[F]{} & \mathbb{B} \end{array}$$

the **Artin glueing**, where $P_F = F^*P_{\mathbb{B}}$.

Remark. One also writes $\mathrm{gl}_F(\mathbb{B})$ and $\mathrm{Gl}_F(\mathbb{B})$ for P_F and $\mathbb{B} \downarrow F$, respectively.

As we will later see, a useful application of this construction are geometric morphisms $F \dashv U : \mathbb{E} \to \mathbb{S}$ for topoi $\mathbb{E}$ and $\mathbb{S}$. Interestingly, in this case, $\mathbb{E} \downarrow F$ is again a topos and P_F preserves the topos structure.

Furthermore, the Artin glueing is an instance where the functor P_F is itself always a fibration again.

Theorem 2.2.11. [14] Let $F : \mathbb{C} \to \mathbb{B}$ be a functor. If $P : \mathbb{X} \to \mathbb{B}$ is a fibration then F^*P is also a fibration.

The following example of the Artin glueing describes more concretely the intuition behind fibered categories as indexed families.

Example 2.2.12. Let $\mathbb{C}$ be a category with finite limits and set-indexed coproducts such that these are universal and disjoint. Then we can define a coproduct functor

$$\begin{aligned} \Delta : \mathbf{Set} &\to \mathbb{C}, \\ I &\mapsto \coprod_{i \in I} \mathbf{1}. \end{aligned}$$

When observing the corresponding Artin glueing

$$\begin{array}{ccc} \mathbb{C} \downarrow \Delta & \longrightarrow & \mathbb{C}^2 \\ {\scriptstyle P_\Delta}\downarrow & & \downarrow{\scriptstyle P_{\mathbb{C}}} \\ \mathbf{Set} & \xrightarrow[\Delta]{} & \mathbb{C} \end{array}$$

the fibre of P_Δ over I is given by $\mathbb{C}/(\coprod_{i\in I} \mathbf{1})$. Then the equivalence $\mathbb{C}/(\coprod_{i\in I} \mathbf{1}) \simeq \mathbb{C}^I$ can be proven using functors

$$\mathbb{C}/(\textstyle\coprod_{i\in I} \mathbf{1}) \underset{G_I}{\overset{F_I}{\rightleftarrows}} \mathbb{C}^I$$

such that

$$\begin{aligned} F(Y \to \coprod_{i \in I} \mathbf{1}) &= (Y_i)_{i \in I}, \\ G((X_i)_{i \in I}) &= \coprod_{i\in I} X_i \to \coprod_{i \in I} \mathbf{1}. \end{aligned}$$

Thus, we obtain an equivalence between indexed families $(X_i)_{i\in I}$ and indexing $X \to \coprod_{i\in I} \mathbf{1}$ in the fibered sense. This gives rise to the Artin glueing

$$\begin{array}{ccc} \mathrm{Fam}(\mathbb{C}) & \longrightarrow & \mathbb{C}^2 \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P_{\mathbb{C}}} \\ \mathbf{Set} & \xrightarrow[\Delta]{} & \mathbb{C} \end{array}$$

where P is the family fibration from Example 2.2.5.

Internal sums

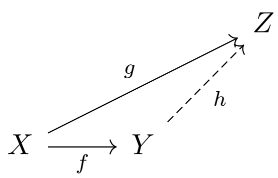
In Example 2.2.12, we already saw a special case where categories with sums might have interesting properties in the fibered context. Here, we will generalize the notion of categories with small sums to fibrations over a category with pullbacks. For this purpose, we need to introduce first **cocartesian arrows**. For this purpose, we can express a property of the family fibration $\mathrm{Fam}(\mathbb{C}) \to \mathbf{Set}$ from Example 2.2.5 that is equivalent to $\mathbb{C}$ having small sums.

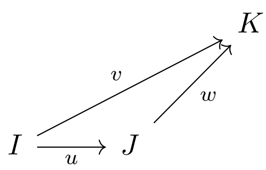
Example 2.2.13. Assume that the category $\mathbb{C}$ has small sums and recall the family fibration $P : \mathrm{Fam}(\mathbb{C}) \to \mathbf{Set}$ from Example 2.2.5. Let $X = (X_i)_{i \in I}$ and $Y = (\coprod_{i \in u^{-1}(j)} X_i)_{j \in J}$ and define the morphism $f := (u, \phi) : (I, X) \to (J, Y)$ over $u : I \to J$ such that

$$\phi_i = \mathsf{inj}_i : X_i \to (\coprod_{k \in u^{-1}(u(i))} X_k).$$

Then for all $g := (v, \psi) : X \to Z$ over $v : I \to K$ and $w : J \to K$ with $v = w \circ u$ there is a unique $h := (w, \theta) : Y \to Z$ over w such that $\theta_{u(i)} \circ \mathsf{inj}_i = \psi_i$. We call f **cocartesian**.

Definition 2.2.14. [14] For a category $\mathbb{C}$ with pullbacks, let $P : \mathbb{X} \to \mathbb{C}$ be a fibration. Let $f : X \to Y$ be over $u : I \to J$. Then f is **cocartesian** if for all $g : X \to Z$ over $v : I \to K$ and $w : J \to K$ with $v = w \circ u$ there is a unique $h : Y \to Z$ over w with $g = h \circ f$.



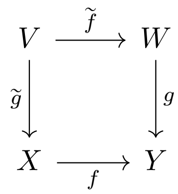


Furthermore, these cocartesian arrows also have to statisfy the so-called **Beck-Chevalley-Condition** (BCC).

Definition 2.2.15. [14] For a category $\mathbb{C}$ with pullbacks, let $P : \mathbb{X} \to \mathbb{C}$ be a fibration. P has **internal sums** if the following conditions are satisfied:

(i) for all $X \in \mathbb{X}_I$ and $u : I \to J$ in $\mathbb{C}$ there is a cocartesian morphism $\phi : X \to Y$ over u,

(ii) the **Beck-Chevalley Condition** holds, i.e. for all commuting squares



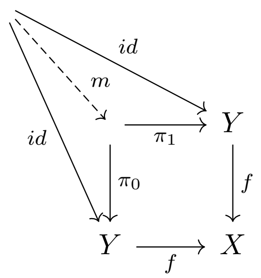
in $\mathbb{X}$ over a pullback, if f is cocartesian and and g, $\tilde{g}$ are cartesian then the morphism $\tilde{f}$ is cocartesian.

Using this definition of internal sums, we can now generalize the concepts of universality (also known as stability) and disjointness of sums to fibered categories.

Definition 2.2.16. Let $P : \mathbb{X} \to \mathbb{C}$ be a fibration between categories with finite limits.

(1) P has **stable internal sums** if cocartesian morphisms are stable under arbitrary pullbacks in $\mathbb{X}$.

(2) P has **disjoint internal sums** if for all cocartesian moprhisms $f : X \to Y$ the mediating morphism m in the pullback



is also cocartesian.

These definitions will be especially useful in the next subsection when we turn to geometric morphisms. The inverse image part is a finite limit preserving functor and thus, the Artin glueing preserves interesting properties.

Theorem 2.2.17. [14] For a category $\mathbb{C}$ with finite limits, let $F : \mathbb{C} \to \mathbb{B}$ preserve them. Then

(1) $\mathrm{Gl}_F(\mathbb{B})$ has finite limits and P_F preserves them,

(2) P_F has stable disjoint internal sums.

Moens' Lemma

Now, for such fibrations $P : \mathbb{X} \to \mathbb{C}$ with stable disjoint sums, we can show that a finite limit preserving functor D can be recovered for which $P \simeq P_D$. For every object X in $\mathbb{X}$ over I, we start by choosing a cocartesian arrow $\varphi_X : X \to \coprod_I X$ over the terminal projection I to $\mathbf{1}$. Then define $\Delta : \mathbb{X} \to \mathbb{X}_{\mathbf{1}}$ such that the image of an object X in $\mathbb{X}$ is the codomain of φ_X and the image of morphisms $f : X \to Y$ is given by the unique vertical morphism such that

$$\begin{array}{ccc} X & \xrightarrow{\varphi_X} & \Delta(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Delta(f)} \\ Y & \xrightarrow[\varphi_Y]{} & \Delta(Y) \end{array}$$

commutes. The functor Δ can be defined by the commuting triangle

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{D} & \mathbb{X}_{\nVdash} \\ {\scriptstyle 1}\downarrow & \nearrow_{\Delta} & \\ \mathbb{X} & & \end{array}$$

such that $1 : \mathbb{C} \to \mathbb{X}$ chooses fiberwise terminal objects, i.e. $1(I) := \mathbf{1}_I$ where $\mathbf{1}_I$ is the terminal object in the fibre over I.

Theorem 2.2.18. [14] For a category $\mathbb{C}$ with finite limits, let $P : \mathbb{X} \to \mathbb{C}$ be a fibration preserving them and having stable disjoint internal sums. Then $P \simeq P_D$ with

$$D : \mathbb{C} \to \mathbb{X}_{\mathbf{1}} : I \mapsto \coprod_I \mathbf{1}_I.$$

As a special case of Moens' Lemma, we obtain this equivalence also for the Artin glueing of finite limit preserving functors which is essential for the study of geometric morphisms in the next subsection and thus, worth mentioning as an additional result. Note that this is just a direct consequence of Theorem 2.2.17.

Corollary 2.2.19. For a category $\mathbb{C}$ with finite limits, let $F : \mathbb{C} \to \mathbb{B}$ preserve them. Then $P_F \simeq P_D$ with

$$D : \mathbb{C} \to (\mathrm{Gl}_F(\mathbb{B}))_{\mathbf{1}} : I \mapsto \coprod_I \mathbf{1}_I .$$

Fibered Topoi

Fibered topoi are interesting for the next section since they correspond to the Artin glueing of the inverse image of a geometric morphism. More specifically, fibered topoi over $\mathbb{S}$ that are cocomplete and locally small[3] are equivalent to geometric morphisms to topos $\mathbb{S}$.

Definition 2.2.20. [14] Let $P : \mathbb{X} \to \mathbb{C}$ be a fibration. Then

(1) for $u : J \to I$, we define a functor $u^* : \mathbb{X}_I \to \mathbb{X}_J$ such that (using axiom of choice) for an object X over I the function $u^*(X) \to X$ over u is cartesian and for $\alpha : X \to Y$ the morphism $u^*(\alpha)$ is the unique vertical arrow making the diagram

$$\begin{array}{ccc} u^*(X) & \longrightarrow & X \\ {\scriptstyle u^*(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ u^*(Y) & \longrightarrow & Y \end{array}$$

commute. We call u^* a **reindexing functor**.

(2) P is a **topos fibered over** $\mathbb{C}$ if all its reindexing functors are logical and $\mathbb{C}$ has finite limits.

Interestingly, as shown by Jibladze in [5], fibered topoi with internal sums have automatically stable and disjoint sums.

[3]See [14] for more details.

Theorem 2.2.21. [14] Let $P : \mathbb{X} \to \mathbb{C}$ be a fibered topos. If P has internal sums then these sums are stable and disjoint.

Therefore, we obtain by Moens' Lemma 2.2.18 an equivalence between topoi fibered over $\mathbb{C}$ with internal sums and fibrations of the form $P_D : \mathbb{B} \downarrow D \to \mathbb{C}$ for some finite limit preserving $D : \mathbb{C} \to \mathbb{B}$ and $\mathbb{B}$ having finite limits.

Corollary 2.2.22. Let $P : \mathbb{X} \to \mathbb{C}$ be a fibered topos with internal sums. Then $P \simeq P_D$ with

$$D : \mathbb{C} \to \mathbb{X}_\mathbf{1} : I \mapsto \coprod_I \mathbf{1}_I.$$

2.3 Geometric Morphism

2.3.1 Definition

Definition 2.3.1. [6] Let $\mathbb{E}$ and $\mathbb{S}$ be topoi. A **geometric morphism** is a pair of functors $U : \mathbb{E} \to \mathbb{S}$ and $F : \mathbb{S} \to \mathbb{E}$ with $F \dashv U$ such that F preserves finite limits.
The functors U and F are referred to as **direct image** and **inverse image**, respectively.

Remark. Let $f : F(X) \to Y$ and $g : X \to U(Y)$ be adjuncts of $F \dashv U$, i.e. correspond under the bijection

$$\mathbb{E}(F(X), Y) \simeq \mathbb{S}(X, U(Y)).$$

Then we denote $g = f^\sharp$ as the **(right) transpose** of f and $f = g^\flat$ as the **(left) transpose** of g.

A special case of geometric morphism that will play an important role in Chapter 3 is the copower functor from Example 2.2.12.

Proposition 2.3.2. [1] For a topos $\mathbb{E}$, the **copower functor** is given by

$$\begin{aligned} \Delta : \mathbf{Set} &\to \mathbb{E}, \\ I &\mapsto \coprod_{i \in I} \mathbf{1} \end{aligned}$$

such that for $u : I \to J$

$$\Delta(u) \circ \mathsf{inj}_i = \mathsf{inj}_{u(i)} : \mathbf{1} \to \coprod_{j \in J} \mathbf{1}.$$

Then the right adjoint of Δ is the **global section functor** given by the representable copresheaf

$$\Gamma : \mathbb{E} \to \mathbf{Set},$$
$$X \mapsto \hom_{\mathbb{E}}(\mathbf{1}, X)$$

such that for $f : X \to Y$ the morphism $\hom_{\mathbb{E}}(\mathbf{1}, f)$ maps every g to $f \circ g$.

In the next section, the following result is also useful. In the case where $\mathbb{E} = \mathsf{Set}^{\mathbb{I}}$ for some category $\mathbb{I}$, the copower can replaced by the (generalized) diagonal functor.

Corollary 2.3.3. For a category $\mathbb{I}$, the copower functor $\Delta : \mathbf{Set} \to \mathbf{Set}^{\mathbb{I}}$ is given by the (generalized) **diagonal functor** sending each object to the constant functor and arrows $u : I \to J$ to the natural transformation with components u.

Example 2.3.4. Consider from Example 2.1.6 the following two cases using limits:

(i) Consider for a set I the category $\mathbf{Set}^I$ of I-indexed families of sets. Then the copower is given by the diagonal functor Δ that sends sets S to $(S)_{i \in I}$.
The global section functor $\Gamma : \mathbf{Set}^I \to \mathbf{Set}$ is defined on an object $(X_i)_{i \in I}$ in $\mathbf{Set}^I$ by

$$\hom_{\mathbf{Set}^I}((\mathbf{1}), (X_i)_{i \in I}) \simeq \prod_{i \in I} X_i.$$

(ii) Consider for the preorder category $\mathbb{P} = (|\mathbb{P}|, \leqslant_{\mathbb{P}})$ the category $\mathbf{Set}^{\mathbb{P}^{op}}$. Then the copower is given by the diagonal functor Δ such that $\Delta(S)(U) = S$ for objects U in $\mathbb{P}$ and $\Delta(S)(f) = \mathsf{id}_S$ for morphisms f in $\mathbb{P}$.
The global section functor $\Gamma : \mathbf{Set}^{\mathbb{P}^{op}} \to \mathbf{Set}$ is defined on an object F in $\mathbf{Set}^{\mathbb{P}^{op}}$ by

$$\hom_{\mathbf{Set}^{\mathbb{P}^{op}}}(\mathbf{1}, F) \simeq \{(S_U)_{U \in |\mathbb{P}|} \in \prod_{U \in |\mathbb{P}|} F(U) \mid f : U \leqslant_{\mathbb{P}} U' \implies F(f)(S_{U'}) = S_U\}.$$

2.3.2 Fibered Adjunctions

First, consider Moens' D from Theorem 2.2.18 as a fibered functor: for a finite limit preserving functor $F : \mathbb{C} \to \mathbb{B}$, let $P_F : \mathbb{X} \to \mathbb{C}$ be the induced fibration. Using the equivalence $P_F \simeq P_D$, we define the fibered functor D_P from P_B to P_F that maps a morphism

$$\begin{array}{ccc} I_1 & \xrightarrow{v} & I_2 \\ \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_2} \\ J_1 & \xrightarrow{w} & J_2 \end{array}$$

to $D_P(w, v)$ in $\mathbb{X}$ over w such that the diagram

$$\begin{array}{ccc} \mathbf{1}_{I_1} & \longrightarrow & \mathbf{1}_{I_2} \\ \downarrow{\scriptstyle \varphi_{u_1}} & & \downarrow{\scriptstyle \varphi_{u_2}} \\ D_P(u_1) & \xrightarrow{D_P(w,v} & D_P(u_2) \end{array}$$

where φ_{u_1} and φ_{u_2} are cocartesian morphisms over u_1 and u_2, respectively. In other words, Moens' D can be seen as the restriction of D_P to one.

Together with a right adjoint G_P of D_P, the geometric morphisms $F \dashv U$ now be characterized in the fibered context using this fibered adjunction.

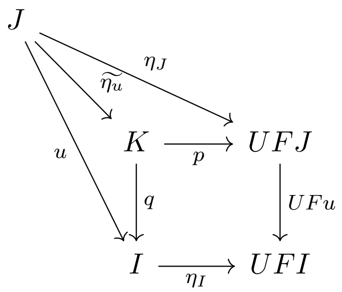
Definition 2.3.5. [14] For fibrations $P : \mathbb{X} \to \mathbb{C}$ and $Q : \mathbb{Y} \to \mathbb{C}$, a **fibered adjunction** consists of fibered functors $F : \mathbb{X} \to \mathbb{Y}$ and $Q : \mathbb{Y} \to \mathbb{X}$ such that $F \dashv U$ is an adjunction with vertical unit and counit.

Theorem 2.3.6. [14] Let $\mathbb{C}$ be a category with finite limits and $F : \mathbb{C} \to \mathbb{B}$ preserves them. Then F has a right adjoint iff D_P has a fibered right adjoint G_P.

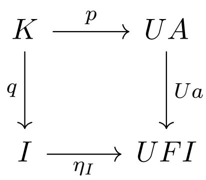
For later purposes, it is helpful to describe the fibered adjunction $D_P \dashv G_P$ more explicitly.

Theorem 2.3.7. [14] Let $F \dashv U : \mathbb{E} \to \mathbb{S}$ be a geometric morphism. Then the induced fibered adjunction $D_{P_F} \dashv G_{P_F}$ is given by the following:

- D_{P_F} applies F to objects and morphisms of $\mathbb{S}$,
- the fibre of G_{P_F} over I is given by $\eta_I^* \circ U/FI$,
- the unit $\widetilde{\eta}_u$ of $u : J \to I$ is given by the pullback



- the counit $\widetilde{\epsilon}_a$ is given by $\epsilon_A \circ F(p) : F(q) \to a$ with pullback



2.3.3 Properties

Recall Definition 2.1.9. As already mentioned, the fibrational perspective allows to infer „logical“ properties from geometric morphisms.

Proposition 2.3.8. [14] Let $F \dashv U : \mathbb{E} \to \mathbb{S}$ be a geometric morphism. Then

(1) the slice category $\mathrm{Gl}(F) = \mathbb{E} \downarrow F$ is a topos,

(2) the functor $\mathrm{gl}(F) = P_F : \mathrm{Gl}(F) \to \mathbb{S}$ is logical.

Further properties of geometric morphisms that will be interesting later on, are **local connectedness** and **connectedness** together with their characterization in terms of their fibered adjoints.

Definition 2.3.9. [14] A geometric morphism $F \dashv U : \mathbb{E} \to \mathbb{S}$ is called **locally connected** if one of the two equivalent conditions is satisfied:

(1)) F has a left adjoint L such that for transposes $\bar{y}$ and $\bar{x}$ of y and x, respectively, if

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle x} \\ F(J) & \xrightarrow[F(u)]{} & F(I) \end{array}$$

is a pullback square then

$$\begin{array}{ccc} L(Y) & \xrightarrow{L(f)} & L(X) \\ {\scriptstyle \bar{y}}\downarrow & & \downarrow{\scriptstyle \bar{x}} \\ J & \xrightarrow[u]{} & I \end{array}$$

is a pullback square.

(2) the fibered functor D_{P_F} has a fibered left adjoint S_{P_F}.

It is worth mentioning that in the case where I is the terminal object, the condition of local connectedness satisfies the **Frobenius reciprocity**, i.e.

$$L(F(S) \times R) \simeq S \times L(R), \tag{2.2}$$

which will be important in the next chapter.

The left adjoint of the copower functor Δ is given by the **connected component functor**. Consider the following examples.

Example 2.3.10. Expanding the Example 2.3.4, consider the following two cases with colimits:

(i) The connected component functor $\Pi_0 : \mathbf{Set}^I \to \mathbf{Set}$ is given by

$$\Pi_0((X_i)_{i\in I}) = \coprod_{i\in I} X_i.$$

(ii) The connected component functor $\Pi_0 : \mathbf{Set}^{\mathbb{P}^{op}} \to \mathbf{Set}$ is given by

$$\Pi_0(F) = \coprod_{U \in |\mathbb{P}|} F(U)/\sim$$

such that $\sim$ is the equivalence relation generated by

$$S_U \sim S_{U'} :\Leftrightarrow \exists f : U \leqslant U' \text{ with } F(f)(S_{U'}) = S_U.$$

for $S_U \in F(U)$, $S_{U'} \in F(U')$.

Theorem 2.3.11. [6] Let $F \dashv U : \mathbb{E} \to \mathbb{S}$ be a geometric morphism then the following are equivalent:

(1) $F \dashv U$ is locally connected,

(2) F preserves dependent products, i.e. for all morphisms $h : X \to Y$ in $\mathbb{S}$, the canonical natural transformation

$$(F/Y)^* \circ \prod_h \xrightarrow{\sim} \prod_{F(h)} \circ (F/X)^*$$

is an isomorphism,

(3) all slices of F preserve exponentials, i.e. for every object X in $\mathbb{S}$, the functor $F/X : \mathbb{S}/X \to \mathbb{E}/F(X)$ is cartesian closed.

The requirement on the slices of F preserving exponentials ensures exponentiability of objects of $\mathbb{S}$ in $\mathbb{E}$ which will be very useful to define the interpretation of „exponential kinds“.

Definition 2.3.12. [14] A geometric morphism $F \dashv U : \mathbb{E} \to \mathbb{S}$ is called **connected** if F is full and faithful.

Theorem 2.3.13. [14] c. Then the following are equivalent:

(1) $F \dashv U$ is connected,

(2) $UF \simeq \mathrm{Id}_{\mathbb{S}}$,

(3) the fibered functor G_{P_F} preserves cocartesian arrows.

As a consequence of adding connectedness to local connectedness, the further left adjoint has to preserve the terminal object.

Theorem 2.3.14. [6] Let $F \dashv U : \mathbb{E} \to \mathbb{S}$ be a locally connected geometric morphism with further left adjoint L. Then the following are equivalent:

(1) $F \dashv U : \mathbb{E} \to \mathbb{S}$ is connected,

(2) $LF \simeq \mathrm{Id}_{\mathbb{S}}$,

(3) $L(\mathbf{1}) = \mathbf{1}$.

Next, we introduce the characterization of specific geometric morphisms using subquotients, i.e. every object in a topos is given by quotients of subobjects. Just as quotients are in correspondence with equivalence relations, so are subquotients in correspondence with partial equivalence relations[4](PER), e.g. for a PER R on a set, the elements satisfying xRx restrict to a usual equivalence relation.

Definition 2.3.15. [14, 6] Let $F \dashv U : \mathbb{E} \to \mathbb{S}$ be a geometric morphism.

(a) It is called **bounded** iff there is an object X (called **bound**) in $\mathbb{E}$ such that for every object R in $\mathbb{E}$ there is an object I in $\mathbb{S}$ such that X appears as a subquotient of $X \times F(I)$, i.e. there is a subobject S such that

$$\begin{array}{ccc} S & \hookrightarrow & X \times F(I) \\ \downarrow\!\!\downarrow & & \\ R & & \end{array}$$

(b) It is called **localic** iff it is bounded by $\mathbf{1}$.

Theorem 2.3.16. [14, 6] Let $F \dashv U : \mathbb{E} \to \mathbb{S}$ be a geometric morphism. Then the following are equivalent:

(1) $F \dashv U$ is bounded,

(2) $\mathbb{E} \simeq \mathrm{Sh}_{\mathbb{E}}(\mathbb{C})$ for an internal site $\mathbb{C}$ in $\mathbb{S}$,

[4] symmetric and transitive binary relations

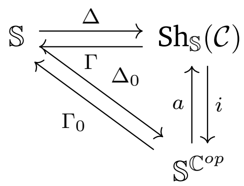
(3) P_F has a generating family, i.e. an object G in the fibre of I such that for all pair of distinct vertical morphisms $\alpha_1, \alpha_2 : X \to Y$ there is a cartesian morphism $\varphi : Z \to G$ and $\psi : Z \to X$ with $\alpha_1 \circ \psi \neq \alpha_2 \circ \psi$.

Furthermore, if $F \dashv U$ is bounded

(4) it is locally connected iff in (2) the constant presheaves are sheaves.

(5) it is localic iff in (2) $\mathbb{C} = (\mathbb{P}, J)$ for an internal preorder $\mathbb{P}$.

Example 2.3.17. Consider the topos of sheaves from Example 2.1.7. Let $\Delta \dashv \Gamma : \mathbb{E} \to \mathbb{S}$ be bounded. Then by Theorem 2.3.16, $\mathbb{E} \simeq \mathrm{Sh}_{\mathbb{S}}(\mathcal{C})$ for some internal site $\mathcal{C} = (\mathbb{C}, J)$ in $\mathbb{S}$ and we obtain the diagram



with the associated sheaf functor a, inclusion i (see [10]), Δ_0 diagonal, and $\Gamma_0 = \lim_{\mathbb{C}^{op}}$ (see Proposition 2.3.2) such that $\Delta = a\Delta_0$ and $\Gamma = \Gamma_0 i$.
If $\Delta \dashv \Gamma$ is also locally connected then the left adjoint Π_0 is defined analogously. Furthermore, from Theorem 2.3.16, the constant presheaves are sheaves and therefore, Δ is the diagonal functor.

3 Topoi of Reference and Modality

3.1 Topos-Theoretic Semantics

3.1.1 Constancy & Variation

Let $\mathbb{E}$ and $\mathbb{S}$ be topoi such that $\mathbb{E}$ is defined over $\mathbb{S}$, i.e. we have geometric morphism $\Delta \dashv \Gamma$

$$\mathbb{S} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \mathbb{E}$$

The functor Γ represents the global section functor whereas Δ represents the constant functor (see Proposition 2.3.2). Intuitively, the topos $\mathbb{S}$ is the „universe of constant sets" or the category of kinds, i.e. objects PERSON, DOG, CAT, etc. are sets representing count nouns containing „urelements" (e.g. references like proper names). With $\mathbb{E}$ as the „universe of variable sets" accounting for the different worlds in which the properties of kinds should be interpreted, the predicates on kinds (constant sets) should thus be expressed in terms of morphisms (and objects) in $\mathbb{E}$. Therefore, predicates on kinds are identified with morphisms $\Delta(S) \to \Omega_{\mathbb{E}}$ in $\mathbb{E}$ for a constant set S. Then predicates can be seen as variable sets. As a consequence, it makes sense to also define the subcategory of objects $\Delta(S)$ for S in $\mathbb{S}$ as constant sets which will be used to interpret the semantics of the modal language.

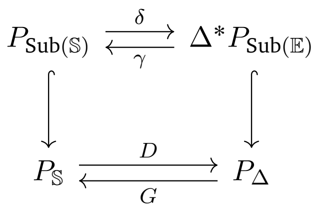
Definition 3.1.1. The **subcategory of constants sets** $\mathbb{C}$ in $\mathbb{E}$ is defined as the image of Δ.

Proposition 3.1.2. Let $\Delta \dashv \Gamma : \mathbb{E} \to \mathbb{S}$ be connected. Then the objects $\Delta(S)$ in $\mathbb{E}$ for some S in $\mathbb{S}$ form a full subcategory.

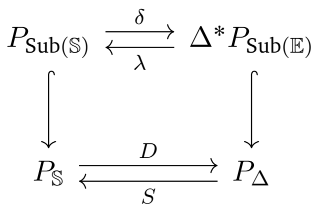
Proof. This follows directly from Definition 2.3.12. □

Even though having kinds as a full subcategory of $\mathbb{E}$ is an interesting property, we do not require connectedness in general as this would be to restrictive on the topos-theoretic models. Consider Example 3.1.3 for some details.

Next, the idea is to define the logical operations of necessity $\Box$ and possibility $\Diamond$ for predicates on kinds, i.e. for morphisms $\Delta(S) \to \Omega_{\mathbb{E}}$. For the functor Sub from Definition 2.1.1, we obtain $\text{Sub}_{\mathbb{E}}(\Delta(S))$ as the poset of predicates on $\Delta(S)$. Furthermore, note that $\text{Sub}_{\mathbb{E}}(\Delta(S))$ corresponds to the fibre of the Artin glueing $\Delta^* P_{\text{Sub}(\mathbb{E})}$ over I for the subobject fibration $P_{\text{Sub}(\mathbb{E})}$ from Definition 2.2.8 which is obtained by the Grothendieck construction of Sub. Due to the induced fibered adjunction $D \dashv G$ of $\Delta \dashv \Gamma$ from Theorem 2.3.7, we obtain the following diagram



such that $\delta \dashv \gamma$. Additionally, if $\Delta \dashv \Gamma$ is locally connected, Theorem 2.3.11 admits a further left adjoint for the fibered adjunction



such that $\lambda \dashv \delta$. Since this further fibered left adjoint is necessary for our topos-theoretic model, the geometric morphism $\Delta \dashv \Gamma$ is assumed to be locally connected if not stated otherwise.

Example 3.1.3. Consider locally connected and connected geometric morphisms for the two presheaves from Example 2.1.6:

(i) for the geometric morphism $\Pi_0 \dashv \Delta \dashv \Gamma : \mathbf{Set}^I \to \mathbf{Set}$

$$\Pi_0(\mathbf{1}) = \Pi_0(\{*\}_{i\in I}) = \coprod_{i\in I}\{*\} = I$$

and by Theorem 2.3.14, $I = \{*\}$. Thus, in this case only one possible world can exist, which is not very interesting from a modal logical perspective.

(ii) for for the geometric morphism $\Pi_0 \dashv \Delta \dashv \Gamma : \mathbf{Set}^{\mathbb{P}^{op}} \to \mathbf{Set}$

$$\Pi_0(\mathbf{1}) = \text{weakly connected components of } \mathbb{P}$$

and by Theorem 2.3.14, $\Pi_0(\mathbf{1}) = \mathbf{1}$ which is equivalent to the existence of a finite chain of elements

$$U \leqslant V_1 \geqslant V_2 \leqslant \cdots \geqslant V_n \leqslant U'$$

for all objects U, U' in $\mathbb{P}$.

3.1.2 Fibered Adjunctions on Subobjects

In the following, a more explicit construction for the restricted adjunction $\lambda \dashv \delta \dashv \gamma$ is given. Consider the functors

$$\mathrm{Sub}_{\mathbb{S}}(S) \underset{\gamma_S}{\overset{\delta_S}{\rightleftarrows}} \mathrm{Sub}_{\mathbb{E}}(\Delta(S)),$$

where for a subobject $P \hookrightarrow S$, we define

$$\delta_S(P \hookrightarrow S) := \Delta(P) \hookrightarrow \Delta(S)$$

and for a subobject $K \hookrightarrow \Delta(S)$, we apply Γ and form the pullback

$$\begin{array}{ccc} \gamma_S(K) & \longrightarrow & \Gamma(K) \\ \big\downarrow & & \big\downarrow \\ S & \xrightarrow[\eta_S]{} & \Gamma\Delta(S) \end{array} \tag{3.1}$$

with $\eta : \mathrm{Id} \to \Gamma\Delta$ the unit of the adjunction $\Delta \dashv \Gamma$.

Proposition 3.1.4. [13] For a geometric morphism $\Delta \dashv \Gamma$, we have

$$\gamma_S(K) = \bigvee\{P \hookrightarrow S \,|\, \Delta(P) \hookrightarrow K \hookrightarrow \Delta(S)\}.$$

Proof. As transpose of 3.1, we obtain for some subobject $P \hookrightarrow S$ the commutative diagram

$$\begin{array}{ccc} \Delta(P) & \longrightarrow & K \\ \downarrow & & \downarrow \\ \Delta(S) & = & \Delta(S) \end{array}$$

Since $\Delta(P) \hookrightarrow \Delta(S)$ and $K \hookrightarrow \Delta(S)$, the morphism $\Delta(P) \to K$ is monic. Finally, the maximality of P follows from the universality condition on the pullback square. $\square$

Note that

$$\mathrm{Sub}_{\mathbb{S}}(S) \simeq \mathrm{Hom}(S, \Omega_{\mathbb{S}}),$$

and

$$\mathrm{Sub}_{\mathbb{E}}(\Delta(S)) \simeq \mathrm{Hom}(\Delta(S), \Omega_{\mathbb{E}}) \simeq \mathrm{Hom}(S, \Gamma(\Omega_{\mathbb{E}})). \tag{3.2}$$

By using this identification, we can define maps $\delta : \Omega_{\mathbb{S}} \to \Gamma(\Omega_{\mathbb{E}})$ such that for all $P : S \to \Omega_S$

$$\delta_S(P) = \delta \circ P,$$

and $\gamma : \Gamma(\Omega_{\mathbb{E}}) \to \Omega_{\mathbb{S}}$ such that for all $K : S \to \Gamma(\Omega_{\mathbb{E}})$

$$\gamma_S(K) = \gamma \circ K$$

(with abuse of notation for K as the transpose of K).
In other words, we essentially obtain δ and γ by restricting Δ and Γ to subterminals, respectively.

Lemma 3.1.5. [12] The following holds for δ and γ as defined above:

(a) $\delta \dashv \gamma$,

(b) $\delta(\top) = \top$,

(c) $\delta(p \wedge q) = \delta(p) \wedge \delta(q)$.

Proof.

(a) The identity

$$\text{Hom}(\delta(P), K) \simeq \text{Hom}(P, \gamma(K))$$

is given by

$$\begin{aligned} \delta(P) \hookrightarrow K &\iff \Delta(P) \hookrightarrow K \\ &\iff P \in \{P \mid \Delta(P) \hookrightarrow K\} \\ &\iff P \hookrightarrow \bigvee\{P \mid \Delta(P) \hookrightarrow K\} \\ &\overset{3.1.4}{\iff} \delta \hookrightarrow \gamma(K). \end{aligned}$$

(b) Since Δ preserves finite limits, δ preserves the terminal object.

(c) Since Δ preserves finite limits, δ preserves finite meets.

$\square$

Assume the geometric morphism $\Delta \dashv \Gamma$ is locally connected, i.e. Δ has a left adjoint Π_0. Then we define

$$\text{Sub}_{\mathbb{S}} \xrightarrow{\lambda_S} \text{Sub}_{\mathbb{E}}(\Delta(S)),$$

such that for a subobject $K \hookrightarrow \Delta(S)$ we apply Π_0 and form the image factorization

$$\begin{array}{ccc} \Pi_0(K) & \hookrightarrow & \Pi_0\Delta(S) \\ \downarrow & & \downarrow{\scriptstyle \epsilon_s} \\ \lambda_S(K) & \hookrightarrow & S \end{array} \tag{3.3}$$

with $\epsilon : \Pi_0\Delta \to \text{Id}$ the counit of the adjunction $\Pi_0 \dashv \Delta$.

Proposition 3.1.6. [13] For a locally connected geometric morphism $\Delta \dashv \Gamma$, we have

$$\lambda_S(K) = \bigwedge\{P \hookrightarrow S \mid K \hookrightarrow \Delta(P) \hookrightarrow \Delta(S)\}.$$

Proof. As transpose of 3.3, we obtain by Definition 2.3.9 the commutative diagram for some subobject $P \hookrightarrow S$

$$\begin{array}{ccc} K & \hookrightarrow & \Delta(S) \\ \downarrow & & \| \\ \Delta(P) & \hookrightarrow & \Delta(S) \end{array}$$

Since $\Delta(P) \hookrightarrow \Delta(S)$ and $K \hookrightarrow \Delta(S)$, the morphism $K \to \Delta(P)$ is monic. Finally, the maximality of P follows from the universality condition on the image factorization. □

By using identification 3.2, we can define the map $\lambda : \Gamma(\Omega_{\mathbb{E}}) \to \Omega_{\mathbb{S}}$ such that for all $K : S \to \Gamma(\Omega_{\mathbb{E}})$

$$\lambda_S(K) = \lambda \circ K.$$

Here again, we obtain λ by restricting Π_0 to subterminals.

Lemma 3.1.7. [12] The following holds for γ and λ as defined above:

(a) $\lambda \dashv \delta$,

(b) $\lambda(p \wedge \delta(q)) = \lambda(p) \wedge q$.

Proof.

(a) The identity

$$\mathrm{Hom}(\lambda(K), P) \simeq \mathrm{Hom}(K, \delta(P))$$

is given by

$$\begin{aligned} \lambda(K) \hookrightarrow P &\overset{3.1.6}{\iff} \bigwedge\{P \mid K \hookrightarrow \Delta(P)\} \hookrightarrow P \\ &\iff P \in \{P \mid K \hookrightarrow \Delta(P)\} \\ &\iff K \hookrightarrow \Delta(P) \\ &\iff K \hookrightarrow \delta(P). \end{aligned}$$

(b) Since $\lambda \dashv \delta$ from (a), the assertion follows from the Frobenius reciprocity condition 2.2.

□

3.1.3 Necessity

Now, we are ready to define the box operator for necessity.

Definition 3.1.8. [12] Given δ and γ as defined above, we define

(a) the **box operator** $\Box = (\delta \circ \gamma) : \Gamma(\Omega_{\mathbb{E}}) \to \Gamma(\Omega_{\mathbb{E}})$,

(b) the **box operator** $\Box_S$ **on predicates of** $\Delta(S)$ for all predicates $\varphi : \Delta(S) \to \Omega_{\mathbb{E}}$ as the (left) transpose of $(\Box \circ \varphi^{\sharp}) : S \to \Gamma(\Omega_{\mathbb{E}})$.

Proposition 3.1.9. [12] The following holds for the box operator $\Box$:

(a) $\Box \leqslant \mathrm{Id}$,

(b) $\Box = \Box^2$,

(c) $T = \Box(T)$,

(d) $\Box(K_0 \wedge K_1) = \Box(K_0) \wedge \Box(K_1)$.

Proof. Given Proposition 3.1.4, note that for a $K : S \to \Gamma(\Omega_{\mathbb{E}})$

$$\Box(K) = \bigvee\{\Delta(P) \hookrightarrow \Delta(S) \mid \Delta(P) \hookrightarrow K\}.$$

Then,

(a) we get $\Box(K) \hookrightarrow K$,

(b) we get

$$\Box\Box(K) = \bigvee\{\Delta(P) \hookrightarrow \Delta(S) \mid \Delta(P) \hookrightarrow \Box(K)\} = \Box(K),$$

(c) for $T : \mathbf{1} \to \Gamma(\Omega_{\mathbb{E}})$, we get

$$\Box(T) = \bigvee\{P \hookrightarrow \mathbf{1} \mid \Delta(P) \hookrightarrow T\} = T,$$

(d) we get

$$\begin{aligned}\Box(K_0 \wedge K_1) &= \delta(\bigvee\{P \hookrightarrow S \mid \Delta(P) \hookrightarrow K_0 \wedge K_1\}) \\ &\overset{3.1.5}{=} \delta(\bigvee\{P \hookrightarrow S \mid \Delta(P) \hookrightarrow K_0\}) \wedge \delta(\bigvee\{P \hookrightarrow S \mid \Delta(P) \hookrightarrow K_1\}) \\ &= \bigvee\{\Delta(P) \hookrightarrow \Delta(S) \mid \Delta(P) \hookrightarrow K_0\} \wedge \bigvee\{\Delta(P) \hookrightarrow \Delta(S) \mid \Delta(P) \hookrightarrow K_1\}.\end{aligned}$$

□

Consider the following examples for the copower functor from Proposition 2.3.2.

Example 3.1.10. From Example 2.1.6, let $\mathbb{C}$ be a small category with $\mathbb{E} := \mathbf{Set}^{\mathbb{C}^{op}}$ and $\mathbb{S} := \mathbf{Set}$. Then the corresponding adjoint maps

$$\Omega_{\mathbb{S}} = \mathbf{2} \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \Gamma(\Omega_{\mathbb{E}})$$

are defined as follows in the following previously discussed cases:

(i) $\Gamma(\Omega_{\mathbb{E}}) = \mathbf{2}^I$ if $\mathbb{C} = I$ for a set I and the corresponding adjoint maps are defined by

$$\delta(p) = \begin{cases} I \text{ , if } p = \top, \\ \varnothing \text{ , if } p = \bot. \end{cases}$$

$$\gamma(K) = \begin{cases} \top \text{ , if } K = I, \\ \bot \text{ , if } K \neq I. \end{cases}$$

In other words,

$$\delta(p) = \{i \in I \mid p\},$$
$$\gamma(K) = ||\forall i \in I : i \in K||,$$

where $||\cdot||$ is mapping to the corresponding truth value.

Thus, the box operator $\Box_S$ on a predicate $\varphi : \Delta(S) \to \Omega_{\mathbb{E}}$ can be defined for all $s \in S$ as

$$i \vDash \Box_S \varphi[s] \qquad \text{iff} \qquad \forall j \in I : j \vDash \varphi[s].$$

(ii) $\Gamma(\Omega_{\mathbb{E}}) = \{K \subseteq P \mid K \text{ is downward closed}\}$ if $\mathbb{C} = \mathbb{P}$ is a preorder category $\mathbb{P} = (P, \leqslant)$ and the corresponding adjoint maps are given by

$$\delta(p) = \begin{cases} P \text{ , if } p = \top, \\ \varnothing \text{ , if } p = \bot. \end{cases}$$
$$\gamma(K) = \begin{cases} \top \text{ , if } K = P, \\ \bot \text{ , if } K \neq P. \end{cases}$$

In other words,

$$\delta(p) = \{V \in P \mid p\},$$
$$\gamma(K) = ||\forall U \in P : U \in K||.$$

Thus, the box operator $\Box_S$ on a predicate $\varphi : \Delta(S) \to \Omega_{\mathbb{E}}$ can be defined for all $s \in S$ as

$$U \vDash \Box_S \varphi[s] \qquad \text{iff} \qquad \forall V \in P : V \vDash \varphi[s].$$

Example 3.1.11. Assume that $\Delta \dashv \Gamma$ is localic. Then by Theorem 2.3.16, $\mathbb{E} \simeq \mathbf{Sh}_{\mathbb{S}}(\mathcal{C})$ where $\mathcal{C} = (\mathbb{P}, J)$ for some internal preorder $\mathbb{P}$ in $\mathbb{S}$. With Example 2.1.7, $\Gamma(\Omega_{\mathbb{E}})$ is the set of closed sieves. Then the corresponding adjoint maps

$$\Omega_{\mathbb{S}} = \mathbf{2} \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \Gamma(\Omega_{\mathbb{E}})$$

are given by

$$\delta(p) = \{V \in \mathsf{Ob}(\mathbb{C}) \mid p\}^1,$$
$$\gamma(K) = ||\forall U \in \mathsf{Ob}(\mathbb{C}) : U \in K||.$$

Thus, the box operator $\Box_S$ on a predicate $\varphi : \Delta(S) \to \Omega_{\mathbb{E}}$ can be defined for all $s \in S$ as

$$U \vDash \Box_S \varphi[s] \qquad \text{iff} \qquad \forall V \in \mathsf{Ob}(\mathbb{C}) : V \vDash \varphi[s]$$

where s is seen as the corresponding element in $i\Delta(S)(U)$.

3.1.4 Possibility

Similarly, we can define the diamond operator for possibility.

Definition 3.1.12. [12] Given δ and λ as defined above, we define

(a) the **diamond operator** $\Diamond = (\delta \circ \lambda) : \Gamma(\Omega_{\mathbb{E}}) \to \Gamma(\Omega_{\mathbb{E}})$,

(b) the **diamond operator** $\Diamond_S$ **on predicates of** $\Delta(S)$ for all predicates $\varphi : \Delta(S) \to \Omega_{\mathbb{E}}$ as the (left) transpose of $(\Diamond \circ \varphi^\flat) : S \to \Gamma(\Omega_{\mathbb{E}})$.

Proposition 3.1.13. [12] The following holds for the box operator $\Box$ and diamond operator $\Diamond$:

(a) $\Diamond \dashv \Box$,

(b) $\Box \leqslant \mathrm{Id} \leqslant \Diamond$,

(c) $\Box = \Box^2$, $\Diamond = \Diamond^2$,

(d) $\Diamond(K_0 \wedge \Box(K_1)) = \Diamond(K_0) \wedge \Box(K_1)$.

Proof. Given Proposition 3.1.6, note that for $K : S \to \Gamma(\Omega_{\mathbb{E}})$

$$\Diamond(K) = \bigwedge\{\Delta(P) \hookrightarrow \Delta(S) \mid K \hookrightarrow \Delta(P)\}.$$

Then

[1] Note that this should actually be the closure $\mathrm{cl}\{c \in C \mid p\}$ (see [10]). Nevertheless, the set is aleady closed.

(a) the identity

$$\mathrm{Hom}(\Diamond(K), P) \simeq \mathrm{Hom}(K, \Box(P))$$

is given by

$$\begin{aligned}
\Diamond(K) \hookrightarrow P &\iff \bigwedge\{\Delta(\bar{P}) \mid K \hookrightarrow \Delta(\bar{P})\} \hookrightarrow P \\
&\iff P \in \{\Delta(\bar{P}) \mid K \hookrightarrow \Delta(\bar{P})\} \\
&\iff K \hookrightarrow P \\
&\iff K \in \{\Delta(\bar{P}) \mid \Delta(\bar{P}) \hookrightarrow P\} \\
&\iff K \hookrightarrow \bigvee\{\Delta(\bar{P}) \mid \Delta(\bar{P}) \hookrightarrow P\} \\
&\iff K \hookrightarrow \Box(P),
\end{aligned}$$

(b) we get $K \hookrightarrow \Diamond(K)$,

(c) we get

$$\Diamond\Diamond(K) = \bigwedge\{\Delta(P) \hookrightarrow \Delta(S) \mid \Diamond(K) \hookrightarrow \Delta(P)\} = \Diamond(K),$$

(d) we get

$$\begin{aligned}
\Diamond(K_0 \wedge \Box K_1) &= \delta\lambda(K_0 \wedge \delta\gamma(K_1)) \\
&\overset{2.3.11}{=} \delta(\lambda(K_0) \wedge \gamma(K_1)) \\
&\overset{3.1.5}{=} \delta\lambda(K_0) \wedge \delta\lambda(K_1) \\
&= \Diamond(K_0) \wedge \Box(K_1).
\end{aligned}$$

□

The tuple $(\Diamond, \Box)$ satisfying the preceding conditions are also referred to as **modal adjoint operators couple** (MAO couple).

Consider again the following example for the copower functor from Proposition 2.3.2.

Example 3.1.14. From Example 2.1.6, let $\mathbb{C}$ be a small category with $\mathbb{E} := \mathbf{Set}^{\mathbb{C}^{op}}$ and $\mathbb{S} := \mathbf{Set}$. Then the corresponding adjoint map λ

$$\Omega_{\mathbb{S}} = \mathbf{2} \underset{\gamma}{\overset{\delta}{\rightleftarrows}} \Gamma(\Omega_{\mathbb{E}})$$

as follows in the following previously discussed cases:

(i)

$$\lambda(K) = \begin{cases} \top, \text{ if } K \neq \varnothing, \\ \bot, \text{ if } K = \varnothing. \end{cases}$$

In other words,

$$\lambda(K) = ||\exists i \in I : i \in K||.$$

Thus, the diamond operator $\Diamond_S$ on a predicate $\varphi : \Delta(S) \to \Omega_{\mathbb{E}}$ can be defined for all $s \in S$ as

$$i \vDash \Diamond_S \varphi[s] \qquad \text{iff} \qquad \exists j \in I : j \vDash \varphi[s].$$

In this case, we obtain the **possible world semantics**.

(ii)

$$\lambda(K) = \begin{cases} \top \text{ , if } K \neq \varnothing, \\ \bot \text{ , if } K = \varnothing. \end{cases}$$

In other words,

$$\lambda(K) = ||\exists U \in P : U \in K||.$$

Thus, the diamond operator $\Diamond_S$ on a predicate $\varphi : \Delta(S) \to \Omega_{\mathbb{E}}$ can be defined for all $s \in S$ as

$$U \vDash \Diamond_S \varphi[s] \qquad \text{iff} \qquad \exists V \in P : V \vDash \varphi[s].$$

In this case, we obtain the **possible situation semantics**.

Example 3.1.15. Assume that $\Delta \dashv \Gamma$ is localic and locally connected. Then by Theorem 2.3.16, $\mathbb{E} \simeq \mathrm{Sh}_{\mathbb{S}}(\mathcal{C})$ where $\mathcal{C} = (\mathbb{P}, J)$ for some internal preorder $\mathbb{P}$ in $\mathbb{S}$ such that the constant presheaves are sheaves. With Example, 2.1.7 $\Gamma(\Omega_{\mathbb{E}})$ is the set of closed sieves. Then the corresponding adjoint map λ

$$\Omega_{\mathbb{S}} = \mathbf{2} \underset{\lambda}{\overset{\delta}{\rightleftarrows}} \Gamma(\Omega_{\mathbb{E}})$$

is given by

$$\lambda(K) = ||\exists U \in \mathrm{Ob}(\mathbb{C}) : U \in K||.$$

Thus, the diamond operator $\Diamond_S$ on a predicate $\varphi : \Delta(S) \to \Omega_{\mathbb{E}}$ can be defined for all $s \in S$ as

$$U \vDash \Diamond_S \varphi[s] \qquad \text{iff} \qquad \exists V \in \mathrm{Ob}(\mathbb{C}) : V \vDash \varphi[s].$$

3.1.5 Intensionality & Extensionality

Montague's [11] formalization of intensionality and extensionality can also be represented in this context using the unit and counit of the adjunction. Since for his original work, one has to choose $\mathbb{E} = \mathbf{Set}^I$, consider the following example.

Example 3.1.16. From Example 3.1.10, consider again the geometric morphism

$$\mathbf{Set} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \mathbf{Set}^I$$

where the set I is meant to be the set of possible worlds. Recall that $\Delta(S) = (S)_{i\in I}$ and therefore, $\Gamma\Delta(S) \simeq \prod_{i\in I} S$. The unit $\eta_S : S \to \prod_{i\in I} S$ of $\Delta \dashv \Gamma$ is then defined by

$$s \mapsto (I \to S,\ i \mapsto s)$$

i.e. η_S is mapping an element s to the constant map with value s. This constant map $(I \to S,\ i \mapsto s)$ is also called the **intensions** of s. More generally, set-theoretical functions on possible worlds with denotations as values, i.e. objects of the form $(i \mapsto x_i) \in \prod_{i\in I} X_i$ correspond to the intensions in the sense of Montague. This operation of „intensionalization“[2] can be translated to the constant sets of $\mathbb{E}$ by composing the unit with Δ, i.e. $\Delta\eta_S$ maps elements of constant sets seen as $\Delta(S)$ to their intensions.

[2]Denoted by $[\hat{}S]$ in [11].

On the other hand, for an object $X = (X_i)_{i\in I}$ in $\mathbb{E}$, we have $\Gamma(X) = \prod_{i\in I} X_i$ and therefore, $\Delta\Gamma(X) = (\prod_{i\in I} X_i)_{i\in I}$. The counit $\epsilon_X : (\prod_{i\in I} X_i)_{i\in I} \to X$ of $\Delta \dashv \Gamma$ is then defined in each component $(\epsilon_X)_j : \prod_{i\in I} X_i \to X$ for $j \in I$ by

$$(i \mapsto x_i) \mapsto x_j.$$

This function value is also called the **extension** of $(i \mapsto x_i)$ with respect to j. In each component, the counit corresponds to the map sending an intensions to the denotation at a possible world j, in the sense of Montague. Again this „extensionalization"[3] can be applied to the constant sets $\Delta(S)$ of $\mathbb{E}$ by $\epsilon_{\Delta(S)}$.

3.1.6 Coincidence

Assume that $x = y$ for some entities x and y. Then a language is called **transparent** if $\varphi(x)$ implies $\varphi(y)$ for some property φ. It is called **opaque** if this implication does not hold in general. In [7], Keenan and Faltz already argued that opaque phenomena usually exists in natural language. In our context, multiple predicates on kinds may coincide in a given world whereas their members differ with respect to their individual properties. Thus, identity and coincidation are in general different concepts. Reyes modeled this phenomenon using **coincidence relations**.

Definition 3.1.17. [12] Let A be a complete Heyting algebra in $\mathbb{E}$. For the subcategory of constant set $\mathbb{C}$ (see Definition 3.1.1), the category $\mathbb{C}(A)$ of objects with **coincidence relation** is given by

- objects (X, E_X) with object X in $\mathbb{C}$ and $E : X \times X \to A$ such that

$$E(x, x') = E(x', x), \qquad\qquad E(x, x') \wedge E(x', x'') \leqslant E(x, x'').$$

- morphisms $f : (X, E) \to (Y, E')$ is a morphisms $F : X \to Y$ in $\mathbb{C}$ such that

$$E(x, x') \leqslant E'(f(x), f(x')).$$

Remark. Reyes himself defined the category of objects with coincidence relations for an arbitrary topos $\mathbb{E}$ with complete Heyting algebra A. Nevertheless, for reasons of simplicity, we focus on the category of constant sets $\mathbb{C}$ since this will be needed for the interpretation in the next section.

[3] Denoted by $[{}^{\smile}S]$ in [11]

At this point, it is useful to discern the following two cases:

(a) the binary relations E correspond to the usual equality $=$ and therefore, the categories $\mathbb{C}$ and $\mathbb{C}(\Omega_{\mathbb{E}})$ are isomorphic,

(b) the binary relations E correspond to Reyes' motivation behind coincidence relations and thus, must be treated as arbitrary.

Most of the modal language and thus, the semantics in the next section does not make use of an additional arbitrary relation. In fact, only one term ($t \asymp s$ denoting coincidence) needs it for its interpretation and hence, this differentiation makes sense to imagine the interpretation of all other cases inside the regular category of constant sets $\mathbb{C}$.

Furthermore, in case (b), note that the objects in $\mathbb{C}(A)$ do not correspond to A-valued sets (see e.g. [4]) due to the unusual definition of morphisms between them. Additionally, the category $\mathbb{C}(\Omega_{\mathbb{E}})$ is not a topos but a quasi-topos as shown by Reyes (see [12]). As a consequence, this category is at least locally cartesian closed which is essential for the exponents and products of kinds.
In the case of **Set**, it is worth mentioning that sets with coincidence relations essentially correspond to partial setoids which also permits a type-theoretic approach (see e.g. [2]). However, it is questionable if the exclusion of reflexivity in Reyes' definition is justified when modeling opacity of natural language. Including reflexivity to coincidence relations would result in a correspondence with setoids.

3.2 Relationship to the Language of Modal Higher-Order Theory

In this section, the language of modal higher-order theory given by Montague's intentional logic is introduced in the topos-theoretic approach discussed so far. In [8], Reyes himself summarized together with La Palme-Reyes the necessary considerations this approach is based on.

Given a locally connected geometric morphism $\Delta \dashv \Gamma : \mathbb{E} \to \mathbb{S}$, recall from Definition 3.1.1 the subcategory of constant sets $\mathbb{C}$ in $\mathbb{E}$ consisting of objects $\Delta(S)$ for S in $\mathbb{S}$. Together with additional binary predicates of subsection 3.1.6, the semantics of our modal higher-order language takes place in $\mathbb{C}(\Omega_{\mathbb{E}})$ consisting of objects $(\Delta(S), E_{\Delta(S)})$. Note that if these additional predicates E correspond to the usual equality on the corresponding object, the categories $\mathbb{C}$ and $\mathbb{C}(\Omega_{\mathbb{E}})$ are isomorphic. In the case where E is meant to be Reyes' coincidence relation, the predicate E has to be treated as an additional arbitrary binary relation and thus, differing from the usual equality.

3.2.1 Sorts & Terms

First, the syntactical units **sorts** are defined for the language. Since the language is then interpreted in $\mathbb{C}(\Omega_{\mathbb{E}})$, a correspondence between these sorts and constant sets has to be established. Therefore, basic sorts represent basic kinds S in $\mathbb{S}$ like PERSON, HUMAN, BOOK, etc. More complex sorts are then built by recursion.

Definition 3.2.1. [12] The collection of **sorts** is recursively defined only by

(a) basic sorts,

(b) 1,

(c) PROP,

(d) $X \times Y$ for sorts X, Y,

(e) Y^X for sorts X, Y.

Next, we define **terms** t of a given sort X denoted by $t : X$.

Definition 3.2.2. [12] The collection of **terms** is recursively defined only by

(a) if $c \in \text{Con}_X$ then $c : X$, where Con_X is the set of basic constant term of sort X, e.g. $* \in \text{Con}_1$, Reyes $\in \text{Con}_{\text{PERSON}}$, read $\in \text{Con}_{\text{PROP}^{\text{PERSON}}}$,

(b) if $x \in \text{Var}_X$ then $x : X$, where Var_X is an infinite set of variables of sort X,

(c) if $t : X$, $s : Y$ then $(t, s) : X \times Y$,

(d) if $t : Y^X$, $s : X$ then $t(s) : Y$,

(e) if $x \in \text{Var}_X$, $t : Y$ then $\lambda x t : Y^X$.

(f) $\top : \text{PROP}$, $\perp : \text{PROP}$,

(g) if $t : X$, $s : X$ then $t = s : \text{PROP}$ and $t \asymp s : \text{PROP}$,

(h) if $\varphi : \text{PROP}$, $\psi : \text{PROP}$ then $\varphi \bullet \psi : \text{PROP}$ for $\bullet \in \{\wedge, \vee, \rightarrow\}$,

(i) if φ : PROP and $x \in \text{Var}_X$ then $\exists x\varphi$: PROP and $\forall x\varphi$: PROP,

(j) if φ : PROP, ψ : PROP then $\Box\varphi$: PROP and $\Diamond\varphi$: PROP.

Terms of sort PROP are called **formulae**.

Remark. One might also refer to sorts as **types**. However, Reyes used these two concepts to distinguish the interpretation as constant sets and variable sets, respectively.

3.2.2 Interpretation

Now, given a locally connected geometric morphism $\Delta \dashv \Gamma : \mathbb{E} \to \mathbb{S}$, we interpret sorts and terms in $\mathbb{C}(\Omega_{\mathbb{E}})$. Sorts are interpreted as kinds, i.e. tuples containing constant sets and coincidence relations. In other words, sorts $\widetilde{S}$ are mapped to the interpretation $(\Delta(S), E_{\Delta(S)})$ for the corresponding reference object S in $\mathbb{S}$.

Definition 3.2.3. [12] The interpretation I of sorts as objects of $\mathbb{C}(\Omega_{\mathbb{E}})$ is defined recursively by

(a) $I(\hat{S}) = (\Delta(S), E_{\Delta(S)})$ for basic sorts $\widetilde{S}$,

(b) $I(1) = (\Delta(\mathbf{1}), E_\top)$ such $E_\top$ maps to the top element of $\Omega_{\mathbb{E}}$,

(c) $I(\text{PROP}) = (\Delta\Gamma(\Omega_{\mathbb{E}}), E_{\Delta\Gamma(\Omega_{\mathbb{E}})})$ with $E_{\Delta\Gamma(\Omega_{\mathbb{E}})} = \epsilon_{\Omega_{\mathbb{E}}} \circ \Delta\Gamma(\leftrightarrow)$

(d) $I(X \times Y) = I(X) \times I(Y)$,

(e) $I(Y^X) = I(Y)^{I(X)}$.

Remark. In the previous definition, due to the preservation of finite limits, the relations E in (b) and (c) correspond to the usual equality on the given object. Furthermore, for the same reason, the interpretation in (d) and (e) preserves the usual equality. Thus, as a consequence, the use of Reyes' coincidence relations in Subsection 3.1.6 only plays a role for the interpretation of basic sorts.

In the next step, we define the interpretation $||(-)||$ for terms. For notational simplicity, we define for sorts $||\widetilde{S}|| = \bigcup(I(\hat{S}))$ with the forgetful functor $\bigcup : \mathbb{C}(\Omega_{\mathbb{E}}) \to \mathbb{C}$, i.e. $||\widetilde{S}|| = \Delta(S)$.

Definition 3.2.4. [12] For each term, we choose distinct variables $\overrightarrow{x} = x_1, \ldots, x_n$ such that the free variables of the term are among them. The interpretation of terms

$$||\overrightarrow{x} : t|| : ||X_1|| \times \cdots \times ||X_n|| \to ||X||$$

is defined recursively by

(a) basic constant terms $c : X$ as global sections $||c|| : \mathbf{1} \to ||X||$ such that

$$||\overrightarrow{x} : c|| \longrightarrow \mathbf{1} \xrightarrow{||c||} ||X||,$$

(b) variables $x_i : X$ as

$$||\overrightarrow{x} : X_i|| = \pi_i : ||X_1|| \times \cdots \times ||X_n|| \longrightarrow ||X_i||,$$

(c) $(t, s) \in X \times Y$ as

$$||\overrightarrow{x} : (t, s)|| = (||\overrightarrow{x} : t||, ||\overrightarrow{x} : s||),$$

(d) $t(s) : Y$ as

$$||\overrightarrow{x} : t(s)|| = \text{ev} \circ (||\overrightarrow{x} : t||, ||\overrightarrow{x} : s||)$$

with the usual $\text{ev} : ||Y||^{||X||} \times ||X|| \to ||Y||$.

(e) $\lambda xt : Y^X$ as

$$||\overrightarrow{x} : \lambda xt|| : ||X_1|| \times \cdots \times ||X_n|| \longrightarrow ||Y||^{|}|X||$$

as the exponential transpose of

$$||\overrightarrow{x}x : t|| : ||X_1|| \times \cdots \times ||X_n|| \times X \longrightarrow ||Y||,$$

(f) $\top$: PROP and $\bot$: PROP as

$$||\overrightarrow{x}\top|| = ||X_1|| \times \cdots \times ||X_n|| \longrightarrow \mathbf{1},$$
$$||\overrightarrow{x}\bot|| = ||X_1|| \times \cdots \times ||X_n|| \longrightarrow \mathbf{1},$$

such that

$$\mathbf{1} \simeq \Delta(t) : \Delta(\mathbf{1}) \longrightarrow \Delta\Gamma(\Omega_{\mathbb{E}}),$$
$$\mathbf{1} \simeq \Delta(f) : \Delta(\mathbf{1}) \longrightarrow \Delta\Gamma(\Omega_{\mathbb{E}}),$$

respectively.

(g) $t = s : \mathrm{PROP}$ and $t \asymp s : \mathrm{PROP}$ as

$$||\overrightarrow{x} : t = s|| : ||X_1|| \times \cdots \times ||X_n|| \overset{(||\overrightarrow{x}:t)||,||\overrightarrow{x}:s||)}{\longrightarrow} ||X|| \times ||X|| \overset{\Delta((\mathrm{eq}_{||X||})^\sharp)}{\longrightarrow} \Delta\Gamma(\Omega_{\mathbb{E}}),$$

$$||\overrightarrow{x} : t \asymp s|| : ||X_1|| \times \cdots \times ||X_n|| \overset{(||\overrightarrow{x}:t)||,||\overrightarrow{x}:s||)}{\longrightarrow} ||X|| \times ||X|| \overset{\Delta((E_{||X||})^\sharp)}{\longrightarrow} \Delta\Gamma(\Omega_{\mathbb{E}}),$$

such that $\mathrm{eq}_{||X||}$ classifies the diagonal $||X|| \mapsto ||X|| \times ||X||$ and $I(X) = (||X||, E_{||X||})$.

(h) $\varphi \bullet \psi : \mathrm{PROP}$ as

$$||\overrightarrow{x} : \varphi \bullet \psi|| : ||X_1|| \times \cdots \times ||X_n|| \overset{(||\overrightarrow{x}:\varphi)||,||\overrightarrow{x}:\psi||)}{\longrightarrow} \Delta\Gamma(\Omega_{\mathbb{E}}) \times \Delta\Gamma(\Omega_{\mathbb{E}}) \overset{\Delta\Gamma(\bullet)}{\longrightarrow} \Delta\Gamma(\Omega_{\mathbb{E}})$$

such that $\bullet : \Omega_{\mathbb{E}} \times \Omega_{\mathbb{E}} \to \Omega_{\mathbb{E}}$ is given by the operations $\wedge, \vee, \to$ on $\Omega_{\mathbb{E}}$,

(i) $\exists x\varphi : \mathrm{PROP}$ and $\forall x\varphi : \mathrm{PROP}$ as

$$||\overrightarrow{x} : \exists x\varphi|| : ||X_1|| \times \cdots \times ||X_n|| \overset{\Delta((\exists_\pi(\epsilon_{\Omega_{\mathbb{E}}} \circ ||\overrightarrow{x}\, x:\varphi||))^\sharp)}{\longrightarrow} \Delta\Gamma(\Omega_{\mathbb{E}}),$$

$$||\overrightarrow{x} : \forall x\varphi|| : ||X_1|| \times \cdots \times ||X_n|| \overset{\Delta((\forall_\pi(\epsilon_{\Omega_{\mathbb{E}}} \circ ||\overrightarrow{x}\, x:\varphi||))^\sharp)}{\longrightarrow} \Delta\Gamma(\Omega_{\mathbb{E}}),$$

such that

$$\pi : ||X_1|| \times \cdots \times ||X_n|| \times ||X|| \longrightarrow ||X_1|| \times \cdots \times ||X_n||$$

is the canonical projection,

(j) $\Box\varphi : \mathrm{PROP}$ and $\Diamond\varphi : \mathrm{PROP}$ as

$$||\overrightarrow{x} : \Box\varphi|| : ||X_1|| \times \cdots \times ||X_n|| \overset{||\overrightarrow{x}:\varphi||}{\longrightarrow} \Delta\Gamma(\Omega_{\mathbb{E}}) \overset{\Delta\Box}{\longrightarrow} \Delta\Gamma(\Omega_{\mathbb{E}}),$$

$$||\overrightarrow{x} : \Diamond\varphi|| : ||X_1|| \times \cdots \times ||X_n|| \overset{||\overrightarrow{x}:\varphi||}{\longrightarrow} \Delta\Gamma(\Omega_{\mathbb{E}}) \overset{\Delta\Diamond}{\longrightarrow} \Delta\Gamma(\Omega_{\mathbb{E}}).$$

Remark. As one can easily see, the interpretation of $t \asymp s$ in (g) differs in the sense of Reyes' coincidence relation but correspond to the interpretation of $t = s$ if these relations E are the same as the usual equality.

As a final remark, it is worth pointing out the unusual use of $\Delta\Gamma(\Omega_{\mathbb{E}})$ for the interpretation of propositions (i.e. PROP). One would maybe expect to use $\Omega_{\mathbb{E}}$ instead. However, in this context, this is motivated by the reduction of the interpretation on $\mathbb{C}$ (or $\mathbb{C}(\Omega_{\mathbb{E}})$) instead of the whole of $\mathbb{E}$ (or $\mathbb{E}(\Omega_{\mathbb{E}})$). As a consequence, it is required to define the interpretation using only kinds/constant sets, i.e. objects $\Delta(S)$ for S in $\mathbb{S}$. Thus, $\Omega_{\mathbb{E}}$ has to be „transformed“ into $\Delta\Gamma(\Omega_{\mathbb{E}})$ to satisfy this requirement.
Nevertheless, this second interpretation can be obtained for some term φ by composing $||\overrightarrow{x} : \varphi||$ with the counit $\epsilon_{\Omega_{\mathbb{E}}} : \Delta\Gamma(\Omega_{\mathbb{E}}) \to \Omega_{\mathbb{E}}$.

4 Conclusion

In this work, we have seen an alternative characterization of geometric morphisms providing a different background for Reyes' topos of reference and modality (see [12]). Thanks to Moens' treatment of fibered categories (see [9]), for categories $\mathbb{C}$ with finite limits, we obtain an equivalence between finite limit preserving fibrations P_D such that

$$D : \mathbb{C} \to \mathbb{X}_{\mathbf{1}} : I \mapsto \coprod_I \mathbf{1}_I$$

and finite limit preserving fibrations with stable disjoint internal sums. This equivalence even holds for fibered topoi with internals sums, because these are automatically stable and disjoint, as shown by Jibladze (see [5]). Since for geometric morphisms $F \dashv U$ the inverse image F preserves finite limits, it holds that $P_F \simeq P_D$. This consequence permits to lift geometric morphisms to fibered adjunctions and thus, allow a purely fibrational characterization.

The fibrational perspective helps now to motivate Reyes' use of geometric morphisms in this context. Fibered adjunctions allow to intrinsically define operations on subobjects (or predicates on kinds) and thus, assuming the geometric morphism is locally connected, leading to modal adjoint operators satisfying the S4 axioms of modal logic exemplified on presheaves/sheaves. Nevertheless, note that these modal adjoint operators of necessity and possibility differ from the operators of classic modal logic as, e.g. they do not satisfy the law of De Morgan duality.

Further concepts like the intensionality and extensionality of Montague (see [11]) can be included into this topos-theoretic approach. For phenomena of opacity that can be found in natural language, we can define coincidence relations as subquotients of some constant set $\Delta(S)$. While differing from identity, they express that predicates on kinds might coincide in some world even though they are not equal.

Finally, the expressions (terms and formulae) of the language of modal higher-order theory can be formulated using sorts which in turn are identified with the constant sets of our topoi. Then, using these constant sets together with their coincidence relation, an interpretation of sorts and terms can be given with our topos semantics. However, note that the corresponding sequence calculus is not described in this thesis. Nevertheless, Reyes' himself (see [12]) elaborated such a formal system for which he proved its soundness.

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