Dissertation

STOCHASTIC HYPODISSIPATIVE HYDRODYNAMIC EQUATIONS: WELL-POSEDNESS, STATIONARY SOLUTIONS AND ERGODICITY

zur Erlangung des akademischen Grades
Doktor der Mathematik (Dr. math)
Eingereicht von

Siyu Liang

Fakultät für Mathematik
Universität Bielefeld
September 2021

Acknowledgement

I would like to thank my supervisor, Prof. Dr. Michael Röckner, for his great effort of providing me help and ideas. Every time during our discussion, he can always quickly find the problems in my writing and provide me with suggestions and solutions. He offers me much help not only in Mathematics, but also in English writing.

I would like to thank Prof. Dr. Zhiming Ma in Chinese Academy of Sciences, for leading me to the area of probability theory and recommending me to Bielefeld University for PhD studies. During his last visit in Bielefeld University two years ago, he gave us a lot of encouragement.

I would like to thank Prof. Dr. Rongchan Zhu for infinity help in my problems and thesis. Although due to the Covid-19 pandemic, she cannot go to Bielefeld University, but by our discussion online, she has offered me thousands of ideas and checked my proof very carefully.

I would like to thank Prof. Dr. Ping Zhang in Chinese Academy of Sciences for offering me much help in partial differential equations.

I am grateful for the financial support from Deutsche Forschungsgemeinschaft(DFG) for my PhD studies. I would like to thank the whole IRTG 2235 group, especially Nadine Brehme, Dr. Claudia Köhler and Prof. Dr. Moritz Kaßmann for offering me help in everything, both studies and life.

I am grateful to many of my colleagues in IRTG group. I want to thank Marco Rehmeier, we shard an office for more than two years and we often discuss Mathematics problems in the office. Moreover, as a foreigner, when I have some questions or problems in the life in Bielefeld, he always helps me as much as possible.

I want to thank Marco Rehmeier and Dr. Chengcheng Ling for reading part of my thesis and pointing out some typos in it.

In the end, I want to thank my family, especially my parents for their always supporting me.

Preface

In my thesis we discuss the well-posedness and long time behaviours of stochastic equations from fluid dynamics. More precisely, we consider the following topics:

1. We study the well-posedness of both deterministic and anisotropic $2D$ Navier–Stokes equations. For the deterministic case, we prove the global well-posedness of the system with initial data in the anisotropic Sobolev space $\tilde{H}^{0,1}$. For the stochastic case, we obtain existence of martingale solutions and pathwise uniqueness of the solutions, which imply existence of the probabilistically strong solution to this system by the Yamada–Watanabe Theorem.

2. We also study stationary solutions and ergodicity of both anisotropic Navier–Stokes equations and Euler equations with positive damping term on both the torus $\mathbb{T}^2$ and the whole space $\mathbb{R}^2$. We first show the existence of (H^1-valued) martingale solutions for both the $2D$ anisotropic Navier–Stokes equations and Euler equations. Then we prove the existence of H^1-valued stationary martingale solutions for the equations with positive damping term on both $\mathbb{T}^2$ and $\mathbb{R}^2$, while previous work in the literature focused on the stochastic Euler equations on $\mathbb{T}^2$.

   Finally, for the case of anisotropic Navier–Stokes equations with positive damping term, we prove uniqueness of the invariant measure when the noise term is small enough with respect to the damping term by the coupling method. Moreover, the convergence to the (unique) invariant measure is proved to be exponentially fast.

3. We show that there exists a white noise stationary solution of the modified Surface Quasi-Geostrophic equations equations on $\mathbb{R}^2$, i.e. there exists a stationary solution of the modified Surface Quasi-Geostrophic equations with the marginal distribution equal to space white noise in $\mathbb{R}^2$.

4. We investigate the stochastic 3D Navier–Stokes equations perturbed by Gaussian noise of convolution type by transformation to a Random PDE. Instead of obtaining global well-posedness result when the initial data are small enough, we focus on the mild solution in some time-space Sobolev space. We prove that for $\mathbb{P}$-a.e. path ω, there exists a mild solution in the time interval $[0, T_*(u_0, \omega)]$, where u_0 is the initial condition.

Contents

Chapter 0

Introduction

0.1 Background of Incompressible Fluid System

As an important part of fluid dynamics, the incompressible Navier–Stokes equations is a topic studied by many mathematicians since the early last century.

Let us first briefly recall the physics background of (incompressible) Navier–Stokes equations. Navier–Stokes systems consist of a momentum conservation equation and a mass conservation equation. Recall the process of derivation from Landau and Lifshitz's book [63]. We only show a simple version of this derivation. More details can be found in books such as [63] and Temam's book [87].
We first introduce some notations.

- D: an open region of $\mathbb{R}^d$.
- ρ: the density of the fluid. For simplicity we assume it is a constant.
- t: time, which satisfies $t \geq 0$.
- $u(t, x) : \mathbb{R}^+ \times D \to \mathbb{R}^d$: the velocity vector at time t and position $x \in D$.
- $X(t, x) : \mathbb{R}^+ \times D \to D$: the position at time t of the fluid element which was at position x when $t = 0$.
- $V(0)$: any volume of the fluid (at time 0).
- $V(t)$: the new position of $V(0)$ after time t.
- $d\tau$: the volume element.
- dS: the surface element.
- $\boldsymbol{n}$: the outer normal vector field of the surface.

- n_i: the ith component of $\boldsymbol{n}$.
- $\boldsymbol{F}(t,x) : \mathbb{R}^+ \times D \to \mathbb{R}^d$: the force vector at time t and position x.

  There are two kinds of force: one is due to the pressure, another is due to the viscosity. We assume that there is no other external force.
- $F_i(t,x) : \mathbb{R}^+ \times D \to \mathbb{R}$: the force in the ith direction at time t and position x.

Mass conservation equation: By conservation of mass, one has

$$\int_{V(t)} \rho d\tau = constant.$$

By Reynolds transport theorem, one has

$$0 = \frac{d}{dt}\int_{V(t)} \rho \; d\tau = \int_{V(t)} \partial_t \rho \; d\tau + \int_{\partial V(t)} \rho u \cdot \boldsymbol{n} dS.$$

By the Gauss-Green formula,

$$\int_{\partial V(t)} \rho u \cdot \boldsymbol{n} dS = \int_{V(t)} \operatorname{div}(\rho u) \; d\tau.$$

Due to the reason that ρ is a constant, we have

$$0 = \frac{d}{dt}\int_{V(t)} \rho \; d\tau = \rho \int_{V(t)} \operatorname{div} u \; d\tau.$$

Since $V(t)$ is arbitrary, we finally obtain

$$\operatorname{div} u = 0. \tag{0.1}$$

Momentum equations: By the definition of X and u, we immediately know

$$\begin{cases} \dot{X}(t,x) = u(t, X(t,x)), \\ X(0,x) = x. \end{cases}$$

The rate of the change of momentum at time t is as follows:

$$\frac{d}{dt}\Big(\rho \int_{V(0)} u(t, X(t,\tau)) d\tau\Big),$$

Since ρ is a constant, by the chain rule we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\rho\int_{V(0)} u(t,X(t,\tau))d\tau\Big) \\
&= \rho\int_{V(0)} \partial_t u(t,X(t,\tau)) + \dot{X}(t,\tau)\cdot\nabla u(t,X(t,\tau))\ d\tau \\
&= \rho\int_{V(0)} \partial_t u(t,X(t,\tau)) + u(t,X(t,\tau))\cdot\nabla u(t,X(t,\tau))\ d\tau \\
&= \rho\int_{V(t)} \partial_t u(t,\tau) + u(t,\tau)\cdot\nabla u(t,\tau)\ d\tau,
\end{aligned} \tag{0.2}$$

where the second equality is due to the fact that $\dot{X}(t,\tau) = u(t,X(t,\tau))$.
Now we consider the force. We can write the force F in the following form:

$$F_i = \sum_{j=1}^{d}\int_{\partial V} \sigma_{ij} n_j ds.$$

$(\sigma_{ij})_{1\le i,j\le d}$ is a $d\times d$ matrix with the following elements:

- $\sigma_{ii} = -q$, where q is the function which denotes the pressure.
- $\sigma_{ij} = \frac{\mu}{2}(\partial_j u_i + \partial_i u_j)$, for some constant μ, when $i \ne j$, which is the term caused by the viscosity and depends on the 'difference' of the speed between neighbouring layers.

Thus the *ith* component of the total force on $V(t)$ at time t is:

$$\begin{aligned}
\sum_{j=1}^{d}\int_{\partial V(t)} \sigma_{ij} n_j ds &= \sum_{j=1}^{d}\int_{V(t)} \partial_j \sigma_{ij} d\tau \\
&= \sum_{j=1}^{d}\int_{V(t)} \partial_j[-q\delta_{i,j} + \mu(\partial_j u_i + \partial_i u_j)\ d\tau \\
&= \int_{V(t)} -\partial_i q + \mu\Delta u_i d\tau,
\end{aligned} \tag{0.3}$$

where the first inequality is due to the Gauss-Green formula, and

$$\delta_{i,j}(x) = \begin{cases} 0 & \text{if } i\ne j, \\ 1 & \text{if } i = j. \end{cases}$$

Combining (0.2) and (0.3), we obtain

$$\rho\partial_t u + u\cdot\nabla u = -\nabla q + \mu\Delta u.$$

Note that ρ is a constant, thus we can write it in the following way:

$$\partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p. \tag{0.4}$$

Combining (0.1) and (0.4), one obtains the celebrated Navier–Stokessystem for incompressible fluids (initial value problem):

$$(NS_\nu) \quad \begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p, \\ \operatorname{div} u = 0, \\ u \mid_{t=0} = u_0, \end{cases}$$

where $\nu > 0$ is the viscosity of the fluid, u and p denote the velocity and the pressure of the fluid respectively. Note that for the inviscid cases (Euler system), we have a similar derivation except for $\nu = 0$. In 1934, J. Leray proved global existence of finite energy weak solutions to (NS_ν) in the whole space $\mathbb{R}^d$, for $d = 2, 3$, in the seminar paper [66]. When $d = 2$, global weak solutions to (NS_ν) are regular and unique. However, when $d = 3$, the regularity and uniqueness of Leray solutions to (NS_ν) are still widely open in the field of mathematical fluid mechanics except for the case when the norm of the initial data are small compared to the viscosity ν.

Considering strong solutions, Ladyzhenskaya [62] has proved the existence of a global strong solution in 2 dimensions. A similar result is also obtained in Temam [87]. In 3 dimensions, Fujita and Kato [42] proved well-posedness for initial values small enough in the Sobolev space $\dot{H}^{\frac{1}{2}}$ which is a scaling invariant space for the 3D Navier–Stokes equations. There are some further results in other scaling invariant spaces such as in [57](L^3), [79], [14]$(B_{p,\infty}^{-1+\frac{3}{p}}, p > 3)$ and [60](BMO^{-1}). The existence of global strong solutions in 3 dimensions for any smooth initial data remains to be an open problem.

0.2 Anisotropic Viscosity

Besides the classical Naver-Stokes equations, we may also consider the incompressible Navier–Stokes equations with partial dissipation. Systems of this type appear in geophysical fluids (see for instance [21, 78]). In fact, instead of putting the classical viscosity $-\nu\Delta$ in (NS_ν), meteorologists often model turbulent diffusion by putting a viscosity of the form: $-\nu_h \Delta_h - \nu_3 \partial_{x_3}^2$, where ν_h and ν_3 are empirical constants, and ν_3 is usually much smaller than ν_h. We refer to the book of J. Pedlovsky [78], Chapter 4 for a more complete discussion, and in the particular case of the so-called Ekman layers [36, 48] for rotating fluids, where $\nu_3 = \epsilon \nu_h$ and ϵ is a very small parameter. In [20, 23, 76], the authors consider the global well-posedness of such a system with small initial data in some anisotropic Besov type spaces. However, for this 3D anisotropic Navier–Stokes equation, there is not any result concerning the global existence of weak solutions.

The aim of Chapter 1 is to investigate both the following deterministic systems on $\mathbb{R}^2$ or on the two-dimensional torus $\mathbb{T}^2$

$$\begin{cases} \partial_t u + u \cdot \nabla u - \partial_1^2 u = -\nabla p, \\ \operatorname{div} u = 0, \\ u \mid_{t=0} = u_0, \end{cases} \tag{0.5}$$

and the following stochastic system on $\mathbb{T}^2$

$$\begin{cases} du + (u \cdot \nabla u - \partial_1^2 u)dt = \sigma(t,u)dW - \nabla p dt, \\ \operatorname{div} u = 0, \\ u \mid_{t=0} = u_0, \end{cases} \tag{0.6}$$

where σ is the random external force and W is an ℓ^2- cylindrical Wiener process, the definition of which will be introduced in Chapter 1. It is well-known that there exists a unique global solution in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+, H^s(\mathbb{R}^2))$ for the 2D incompressible Euler system with initial data in an s-order Sobolev space $H^s(\mathbb{R}^2)$ for $s > 2$ (see [1] for instance). As (0.5) is an intermediate equation between 2D Euler equations and 2D Navier Stokes equations, we also have similar global well-posedness for (0.5) with initial data in $H^s(\mathbb{R}^2)$ for $s > 2$. Since (0.5) has more dissipation than the Euler system, we are going to prove its global well-posedness with initial data in $\tilde{H}^{0,1}$ (see Section 1.1 for the definition of $\tilde{H}^{0,1}$). For the stochastic $2D$ Euler equation, we can only deal with it when driven by additive or linear multiplicative noise (See [46]). But for the anisotropic system (0.5), we can solve the martingale problem with general multiplicative noise. The main novelty is an $H^{0,1}$-uniform estimate, the proof of which depends crucially on the divergence free condition (see (1.12) in Chapter 1). In Section 1.1, we introduce some notations and recall some preliminaries. In the following two sections, we first study the deterministic equation (0.5) and then we consider (0.6) with a stochastic external force which may depend on the velocity u.

Main results for the deterministic part: We prove the existence and uniqueness of weak solutions in the space $L^\infty_{loc}(\mathbb{R}^+; H^{0,1}) \cap L^2_{loc}(\mathbb{R}^+; \dot{H}^{1,1})$ for the deterministic equation (0.5) (see Theorem 1.3 below). (The definition of $H^{0,1}$ and $\dot{H}^{1,1}$ are given in Section 1.1.) In order to prove existence results, we need both the L^2 as well as the $H^{0,1}$ uniform estimate for appropriate approximating solutions to (0.5). Note that the L^2-uniform estimate alone does not provide the compactness of the approximating solutions due to the lack of an estimate for $\|\partial_2 u\|_{L^2}$. To obtain the uniform $H^{0,1}$ estimate, we have to use the divergence free condition of the velocity field in a crucial way. The uniqueness of solutions is proved by estimating the difference between any two solutions, $w = u - v$, in the space L^2.

Main results for the stochastic part: We prove the existence of martingale solutions to (0.6) (see Theorem 1.19 below). A similar result for $3D$ equations with a

Brickman-Forchheimer term, $|u|^{2\alpha}u$, has been obtained in [6]. In this section we prove the existence and uniqueness of probabilistically strong solutions to (0.6) in $2D$ without the Brickman–Forchheimer term. In order to do so, we first use Galerkin approximations to project (0.6) to a finite-dimensional space. Then we use Itô's formula to obtain the uniform estimates of u_n in both L^2 and $H^{0,1}$. Similar to the deterministic case, the proof depends heavily on the divergence free condition of the velocity field. However, since we have to take the expectation, we cannot use Itô's formula to estimate $\|u_n\|^2_{H^{0,1}}$ directly, instead we shall multiply it by an exponential term $e^{-2c\int_0^t \|\partial_1 u(s)\|^2_{L^2} ds}$ for some proper c. Then by tightness methods (Skorokhod Theorem), we obtain the existence of martingale solutions. Here we emphasize that we heavily rely on the divergence free condition and we could not use similar methods as in [6], since we do not have the Brickman–Forchheimer term, (which helps to obtain a better regularity estimate for the solution). Moreover, we can prove the pathwise uniqueness of solutions in L^2 space. Finally, by the Yamada–Watanabe theorem, we obtain the existence and uniqueness of a (probabilistically) strong solution to (0.6).

The results of Chapter 1 have been published in [68], which is a joint paper with Prof. Dr. Ping Zhang and Prof. Dr. Rongchan Zhu. I am grateful for their supervising me in the deterministic part and stochastic parts, respectively.

0.3 Long Time Behaviour of Anisotropic Navier–Stokes and Euler equations

In Chapter 2 we mainly study the long time behaviour of some equations.

In the first part of Chapter 2, (from Section 2.1 to Section 2.4) we study the long time behaviour of the following stochastic (damped) anisotropic Navier–Stokes and Euler system on $\mathbb{R}^2$ and $\mathbb{T}^2$ with a linear damping term λu for $\lambda > 0$, $\lambda_1 \geq 0$

$$\begin{cases} du + (u \cdot \nabla u - \lambda_1 \partial_1^2 u + \lambda u)dt = \sigma(u)dW - \nabla p dt, \\ \operatorname{div} u = 0, \\ u\,|_{t=0} = u_0. \end{cases} \tag{0.7}$$

In the first part of Chapter 2 (from Section 2.1 to Section 2.4) we always assume that $u_0 \in H^1$ and $\operatorname{div} u_0 = 0$, σ is a Hilbert–Schmidt operator from ℓ^2 to H^1, where H^1 is the first order Sobolev space (see Section 1.1 for the definition) and W is an ℓ^2-cylindrical Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The reason we need more smoothness of the initial data is that in this chapter we have vanishing dissipative term. In other words, in the first part of the chapter we consider analytically strong solutions.

For equations (0.7) with $\lambda = 0$ and $\lambda_1 = 1$, we have obtained the existence of martingale solutions and pathwise uniqueness with an initial value in the anisotropic Sobolev space $H^{0,1}$, which imply the existence of (probabilistically) strong solutions. Now we are interested in the long time behaviour of the solution to (0.7). Many researchers study the long time behaviour of the stochastic Navier–Stokes equations (i.e. replacing $\partial_1^2 u$ in (0.7) by Δu and setting $\lambda = 0$) on bounded domains such as the torus or on some unbounded Poincare domains. For the classical stochastic 2D Navier–Stokes equations, the existence of a stationary solution has been studied in [38] on open bounded domains with smooth boundary and multiplicative noise. For unbounded Poincare domains, the existence of invariant measures is proved in [7] for additive noise and in [10] for multiplicative noise. For general unbounded domains, there is not any result on existence of invariant measures. Concerning the uniqueness of invariant measures, there are a lot of results for 2D Navier–Stokes equations on open bounded domains with regular boundary or on the torus, (such as [40] and [50]). These results rely on the dissipative term Δu and the finite volume of the domain heavily.

In the first part of Chapter 2 we study the long time behaviour of the solutions to (0.7) in the 2 dimensional case. We investigate the systems not only on the torus, but also on the whole space $\mathbb{R}^2$. We concentrate on the existence and uniqueness of the invariant measures of (0.7). Unfortunately we do not have the whole Laplacian, but only one direction $\partial_1^2 u$, which makes the system not dissipative enough. As a result, we need an extra damping term λu to improve this system.

Main results for the well-posedness: on the whole plane $\mathbb{R}^2$ and the torus $\mathbb{T}^2$, we prove the global well-posedness (i.e., the existence and uniqueness of the strong solutions, both analytically and probabilistically) for the damped stochastic Euler equations and damped anisotropic Navier–Stokes equations for multiplicative noise by approximating them by damped stochastic Navier–Stokes equations. Moreover, we show that the solution to the stochastic anisotropic Navier-Stoke equations forms a Markov process.

Main results for the existence of stationary solutions for damped Euler/ anisotropic Navier–Stokes equations: we obtain the existence of stationary solutions/ invariant measures for stochastic damped Euler/ anisotropic Navier–Stokes equations on $\mathbb{R}^2$ and $\mathbb{T}^2$ driven by multiplicative noise.

Main results for the uniqueness of stationary solutions to damped anisotropic Navier–Stokes equations: by modifying a classical coupling method in [75], we obtain the ergodicity and exponentially mixing result when the noise is small enough with respect to the damping term (see Theorem 2.17).

In related results in the literature **Section 6 of [45]** the authors construct a sequence of stationary solutions to stochastic damped Navier–Stokes equations on the torus such that when the viscosity goes to 0 a subsequence thereof converges to the stationary martingale solution of damped Euler equations.

It is obtained in [4] that there exists an invariant measure in L^∞ for vorticity equations of type

$$\partial_t w + u \cdot \nabla w + \lambda w = \nabla^\perp \cdot W$$

on smooth bounded domains with strictly positive damping term λw for $\lambda > 0$, where $\nabla^{\perp} = (-\partial_2, \partial_1)$.

It is also proved in [61] that a sequence of stationary solutions to stochastic Navier–Stokes equations when the viscosity and the noise tend to 0 together (noise tends to 0 at the rate $\sqrt{\nu}$) contain a subsequence which converges to the stationary solution of (deterministic) Euler equations. Roughly speaking, (damped) Euler equations can be approximated by 2D (damped) Navier–Stokes equations for both $\mathbb{T}^2$ and $\mathbb{R}^2$.

0.4 White Noise Stationary Solutions of mSQG Equations

In the second part of Chapter 2 we shift to the vorticity form of the deterministic equations. We first recall the definition of the space white noise on $\mathbb{R}^2$ and $\mathbb{T}^2$. Then we recall the previous results about space white noise solutions, i.e., solutions with the marginal distribution equal to space white noise, which satisfy the equation in the weak sense to the following vorticity form of the Euler equations on $\mathbb{T}^2$:

$$\partial_t \omega + \mathcal{K}\omega \cdot \nabla \omega = 0,$$

where $\mathcal{K}$ is the Biot-Savart operator which we shall define later in Section 0.4.1. However, for $\mathbb{R}^2$, there is still not any result of white noise stationary solutions due to the bad integrability of the Biot-Savart kernel at infinity. Instead, we prove the existence of white noise stationary solutions to the following modified Surface Quasi-Geostrophic equations (we will call them mSQG equations for simplicity from now on) on $\mathbb{R}^2$:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ \qquad u = \nabla^{\perp}(-\Delta)^{-(1+\epsilon)/2}\omega. \end{cases}$$

based on the previous results of [71] on $\mathbb{T}^2$ and letting the radius of the torus go to infinity. Different from the first part of Chapter 2 which considers solutions in function space, in the second part of the Section 2, we consider solutions which are only tempered distributions.

0.4.1 Euler Cases: Previous Results

First we consider the vorticity form of the Euler equations:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ \qquad u = \nabla^{\perp}(-\Delta)^{-1}\omega, \end{cases} \tag{0.8}$$

where $\nabla^{\perp} = (-\partial_2, \partial_1)$ and the operator $\nabla^{\perp}(-\Delta)^{-1} =: \mathcal{K}$. If we write $\mathcal{K}$ in terms of a convolution with some kernel $K(x)$, the convolution kernel is called the Biot-Savart kernel. In the whole space the kernel has the form

$$K(x) = \frac{1}{2\pi}\frac{x^{\perp}}{|x|^2}.$$

On the torus $\mathbb{T}^2 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$, the operator is defined through Fourier coefficients by $(\hat{\mathcal{K}\omega})_k = \frac{k^{\perp}}{|k|^2}\omega_k$ for $k \neq 0$, where $\hat{f}_k$ is the kth Fourier coefficients of f. For $k = 0$, we always require $(\mathcal{K}\omega)_k = \omega_k = 0$.

In Flandoli's paper [37], it is proved that there exists a white noise solution with trajectories of class $C\left([0,T]\,; H^{-1-}\left(\mathbb{T}^2\right)\right)$ to the following weak vorticity form of the Euler equations:

$$\langle \omega_t, \phi\rangle = \langle \omega_0, \phi\rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi\rangle\, ds, \tag{0.9}$$

where

$$H_\phi\left(x,y\right) := \frac{1}{2}K\left(x-y\right)\left(\nabla\phi\left(x\right) - \nabla\phi\left(y\right)\right)$$

and $K\left(x\right)$ is Biot-Savart kernel on the torus $\mathbb{T}^2$. The form of the Biot-Savart kernel on the torus $\mathbb{T}^2$ is a little more complicated, but it still satisfies the following two properties:

1. $K(-x) = -K(x)$.

2. For some constant C,
$$|K\left(x\right)| \leq C\frac{1}{|x|},$$

which are the essential properties of the kernel that are used for the proof in the paper [37].

Note that there is some (formal, but not rigorous) equivalence between the equation (0.9) and the equation (0.8). Indeed, if ω_t satisfies (0.8) in the weak sense, formally we can write for $\phi \in C^{\infty}(\mathbb{T}^2)$,

$$\langle \omega_t, \phi\rangle = \langle \omega_0, \phi\rangle + \int_0^t \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K\left(x-y\right)\nabla\phi\left(x\right)\omega_s\left(dx\right)\omega_s\left(dy\right)ds.$$

Since $K\left(x-y\right) = -K(y-x)$, we have by commutating x and y,

$$\langle \omega_t, \phi\rangle = \langle \omega_0, \phi\rangle - \int_0^t \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K\left(x-y\right)\nabla\phi\left(y\right)\omega_s\left(dx\right)\omega_s\left(dy\right)ds.$$

By adding these two equations, we obtain the equation (0.9).

Note that the integration of the space white noise over $\mathbb{T}^2$ is not 0, (although the expectation is 0), which is different from the classical vorticity on the torus. By the definition of the vorticity, $\omega = \partial_1 u_2 - \partial_2 u_1$, where u_1 and u_2 are the two coordinates of u. Hence we immediately obtain that the integration of the vorticity over the torus is equal to 0.

Inspired by Flandoli's paper [37], we may want to know whether there exists a similar result for the whole space $\mathbb{R}^2$, i.e. whether we can construct a white noise stationary solution of the Euler equations on the whole space $\mathbb{R}^2$. Unfortunately, it seems not easy. We will explain the reason later in the main chapter, but roughly speaking, the difficulty comes from the bad integrability of the Biot-Savart kernel at infinity. Therefore, we consider a similar problem for mSQG equations which has a kernel with better integrability.

0.4.2 mSQG Equations

In the second part of Chapter 2, we study the following mSQG equations.

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ \qquad\qquad u = \nabla^{\perp}(-\Delta)^{-(1+\epsilon)/2}\omega, \end{cases}$$

where $0 < \epsilon < 1$. Note that when $\epsilon = 1$ it is the Euler equation and when $\epsilon = 0$ it is the Surface Quasi-Geostrophic equation (SQG) equation. The SQG equations (sometimes also called quasi-geostrophic approximations) are approximations to the shallow water equations with a small Rossby number (which goes to 0 in the limit), where a small Rossby number means the system is mainly determined by the Coriolis force which is caused by the rotation of the earth. This is also called "(nearly) in geostrophic balance".

The SQG equation is obtained from the $3D$ Quasi-Geostrophic equation by assuming the potential vorticity is identically equal to 0. SQG($\epsilon = 0$) equations are introduced in the paper [25], where a striking mathematical and physical analogy is developed between the structure and formation of singular solutions of SQG equations and the potential formation of finite time singular solutions for the $3D$ Euler equations. For more physical background of Quasi-Geostrophic equations and the formulation of SQG equations we refer to [51, 64, 35, 78] and [90].

Many results have been obtained for the solutions of SQG equations since they have some similarities to Euler equations. The global existence of weak solutions to SQG equations is known in the spaces $L^p(\mathbb{R}^2)$, for $p \in (4/3, \infty)$, see [81, 72]. In [13] the nonuniqueness of weak solutions is proved in a certain class by using the convex integration method.

When $0 < \epsilon < 1$, such equations are called modified Surface Quasi-Geostrophic (mSQG) equations, while in some papers such as [53, 54, 17, 44] they are also called generalized Surface Quasi-Geostrophic (gSQG) equations. In a recent preprint paper [16], the authors construct nontrivial global (classical) solutions of the mSQG equations.

Similarly as Euler equations, there are also some results of using point-vortex model to study mSQG equations, such as [41], [70],[71], [43](for more general models), [44] and [85].

The reason we consider the mSQG equations on $\mathbb{R}^2$ is that the kernel corresponding to the mSQG equations (which we denote by K_ϵ) is dominated by $C\frac{1}{|x|^{2-\epsilon}}$, which has a better integrability at infinity compared to $2D$ Biot-Savart kernel. By the symmetric property of K_ϵ, when $0 < \epsilon < 1$, we can again transfer the equations to the following form: for $\phi \in C_c^\infty(\mathbb{R}^2)$,

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_{\phi,\epsilon} \rangle \, ds, \tag{0.10}$$

where

$$H_{\phi,\epsilon}(x, y) := \frac{1}{2} K_\epsilon (x - y) (\nabla \phi (x) - \nabla \phi (y)) .$$

When $0 < \epsilon < 1$, K_ϵ and $H_{\phi,\epsilon}$ have better integrability at infinity compared to H_ϕ. We will show $H_{\phi,\epsilon}$ is in the space $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$. When $\epsilon = 1$, it becomes even better at infinity, but it becomes worse around the origin.

If we return to the torus, there are some previous results on white noise solutions. In [41], the authors show in Theorem 1 a similar result to Flandoli's paper [37], i.e. there exists a white noise stationary solution on the torus which satisfies (0.10). In the further paper [71], the authors generalize the results to the stochastic cases as well as obtain results on scaling limit. We generalize their results of white noise stationary solutions to the whole space $\mathbb{R}^2$ by letting the length of the torus go to infinity. We will define the nonlinear term and show the connection between the kernel on the torus and the kernel on the whole space.

Main results on mSQG equations (Theorem 2.41): We prove the existence of white noise stationary solutions of the mSQG equations on $\mathbb{R}^2$.

0.5 3D Stochastic Navier–Stokes with Convolution Type Noise

In Chapter 3 we are back to $3D$ cases and the classical Navier–Stokes equations with full Laplacian. In this chapter we study the stochastic Navier–Stokes equation with the noise of convolution type.

Consider the following stochastic $3D$ Navier–Stokes equation

$$\begin{cases} du + (u \cdot \nabla u - \Delta u)dt = \sum_{i=1}^{n} (B_i(u) + \lambda_i u) d\beta_i(t) - \nabla p dt, \\ \operatorname{div} u = 0, \\ u \mid_{t=0} = u_0, \end{cases} \tag{0.11}$$

on the whole space $\mathbb{R}^3$, where $\beta_i(t)$, $i = 1, ..., n$ are one dimensional independent Brownian motions on a given probability space $(\Omega, \mathcal{F}, P)$, λ_i, $i = 1, ..., n$, are non-zero constants and B_i , $i = 1, ..., n$ are the convolution operators

$$B_i(u)(\xi) = \int_{\mathbb{R}^3} h_i(\xi - \bar{\xi}) u(\bar{\xi}) d\bar{\xi} = (h_i * u)(\xi), \ \xi \in \mathbb{R}^3,$$

where $h_i \in L^1(\mathbb{R}^3)$, $i = 1, 2, ..., n$, and Δ is the (weak) Laplacian on $(L^2(\mathbb{R}^3))^3$. The vorticity form of this system has been investigated in [2] by Barbu and Röckner , where the authors prove the existence and uniqueness in $(L^p(\mathbb{R}^3))^3$, $\frac{3}{2} < p < 2$, of a global mild solution to random vorticity equations associated to stochastic $3D$ Navier–Stokes equations for sufficiently small initial vorticity. In their paper the smallness of the initial values depends on the whole Brownian path, hence the solutions obtained are not adapted. In the paper [86] by Röckner, R. C. Zhu and X. C. Zhu, the authors prove the solution satisfies the vorticity equation with the stochastic integration being understood in the sense of the integration of controlled rough paths. In the further paper [74] , the authors generalize this result for gradient-type noise in 2 or 3 dimensions by a different type of transformation which is adapted to their gradient-type noise. Then they also obtain the existence of a solution adapted to the Brownian filtration up to some stopping time.

In Chapter 3 we consider the original equations (0.11) instead of the vorticity form. We do not assume the initial values are small. Instead we only assume initial data are smooth. In other words, for any fixed path ω, the initial data are in Sobolev spaces H^N for any $N \in \mathbb{N}$. We focus on the existence of solutions in the following space: Let $\mathcal{Z}_T^\gamma$ be the subspace of $L^1([0,T], L^1_{loc}(\mathbb{R}^3))$ with the norm

$$\|u\|^2_{\mathcal{Z}_T^\gamma} := \|u\|^2_{L^\infty([0,T];\dot{H}^{\frac{1}{2}+\gamma})} + \int_0^T \|\nabla u(t)\|^2_{\dot{H}^{\frac{1}{2}+\gamma}} dt$$

finite.

Main results of Chapter 3: We prove in Theorem 3.6 the well-posedness (i.e. the existence and uniqueness of the mild solution in the time interval $[0, T_*(u_0)]$ which belongs to the space $\mathcal{Z}^\gamma_{T_*(u_0)}$), where

$$T_*(u_0, \omega) = c_\gamma \big(\sup_{t \geq 0} \eta_t\big)^{-1} \|u_0\|^{-\frac{2}{\gamma}}_{\dot{H}^{\frac{1}{2}+\gamma}},$$

and the strict positive number c_γ depends only on γ.

For the deterministic classical $3D$ Navier–Stokes equations, by Fujita-Kato's fixed point procedure, global well-posedness results have been obtained in certain scaling invariant spaces (one also calls them critical spaces). For general initial values, the solution will blow up (in the sense of strong solutions) after some time. For some spaces which are above the critical spaces, a fixed point procedure (Picard's contraction principle) can also be applied to study the blow-up time of the solution. We refer to Chapter 15 in the book [65] for such results for the Sobolev spaces $\dot{H}^s$ and L^p, for $s \geq \frac{1}{2}$ and $p \geq 3$. Also in Poulon's paper [80], the author introduces the notion of the minimal blow-up Navier–Stokes solutions. The authors also show that the set of such solutions is not only nonempty but also compact in a certain sense. Based on this, a lower bound estimate of the maximal time up to which the solution remains regular for initial values in the space $\dot{H}^s$, $\frac{1}{2} < s < \frac{3}{2}$, is obtained in Proposition 1.1 of a later paper [22] by J. -Y. Chemin and I. Gallagher. For deterministic anisotropic $3D$ Navier–Stokes equations, in [67], we also study the maximal time up to which the solutions remain regular.

The idea of the proof is that we first apply the transformation in [2] to transform the equation to a random PDE and write the solution in the form of mild solutions. Then we apply the Littlewood-Paley theory, thanks to the commutativity of the convolution (this is the reason why we have to limit our noise to convolution type multiplicative noise), our convolution operator B_i and transformation operator Γ operate on any Besov space. Afterwards, due to the contraction property of the semigroup $e^{t\Delta}$, we obtain the estimates necessary for the fixed point argument. This method has also been used to calculate a lower bound of the time T up to which the regular solution exists for deterministic $3D$ Navier–Stokes equations.

In my thesis we use C to denote the constant which can be different from line to line. And we use the notation '$A \lesssim B$' to mean that there is some constant C such that $A \leq CB$.

Chapter 1

Well-posedness of 2D Navier–Stokes Equations with Anisotropic Viscosity

1.1 Preliminaries and Notations

In this chapter we prove the well-posedness of two dimensional Navier–Stokes equations with anisotropic viscosity for both deterministic and stochastic cases. In Section 1.1, we introduce some notations. In Section 1.2 we prove the results for deterministic cases, i.e. we study the equation

$$\begin{cases} \partial_t u + u\cdot\nabla u - \partial_1^2 u = -\nabla p, \\ \operatorname{div} u = 0, \\ u\mid_{t=0} = u_0. \end{cases} \tag{1.1}$$

In Section 1.3 we prove the stochastic cases, i.e. we consider the following equation

$$\begin{cases} du + (u\cdot\nabla u - \partial_1^2 u)dt = \sigma(t,u)dW - \nabla p dt, \\ \operatorname{div} u = 0, \\ u\mid_{t=0} = u_0. \end{cases} \tag{1.2}$$

1.1.1 Function and Distribution Spaces on $\mathbb{R}^2$

Denote by $\mathcal{S}(\mathbb{R}^2)$ the Schwartz space and $\mathcal{S}'(\mathbb{R}^2)$ its dual space. On $\mathbb{R}^2$, we recall the classical (non-homogeneous) Sobolev spaces:

$$H^s(\mathbb{R}^2) := \left\{ u \in \mathcal{S}'(\mathbb{R}^2); \| u \|^2_{H^s(\mathbb{R}^2)} := \int_{\mathbb{R}^2} (1+ \mid \xi \mid^2)^s \mid \hat{u}(\xi) \mid^2 d\xi < \infty \right\}, \tag{1.3}$$

where $s \in \mathbb{R}$, and

$$\hat{u}(\xi) = \mathcal{F}u(\xi) := \int_{\mathbb{R}^2} f(x) e^{-ix\cdot\xi} dx,$$

denotes the Fourier transform of u on $\mathbb{R}^2$.
$H^s(\mathbb{R}^2)$ is a Hilbert space with $H^{-s}(\mathbb{R}^2)$ as its dual space. Since in later sections of this chapter, we will consider anisotropic equations, we also introduce (non-homogeneous) anisotropic Sobolev spaces

$$H^{s,s'}(\mathbb{R}^2) := \Big\{ u \in \mathcal{S}'(\mathbb{R}^2); \| u \|^2_{H^{s,s'}(\mathbb{R}^2)} := \int_{\mathbb{R}^2} (1+ \mid \xi_1 \mid^2)^s (1+ \mid \xi_2 \mid^2)^{s'} \mid \hat{u}(\xi) \mid^2 d\xi < \infty \Big\},$$

where $s, s' \in \mathbb{R}$ and $\xi = (\xi_1, \xi_2)$.
We remark that the space $H^{s,s'}(\mathbb{R}^2)$ endowed with the norm $\| \cdot \|_{H^{s,s'}(\mathbb{R}^2)}$ is a Hilbert space-with the dual space $H^{-s,-s'}(\mathbb{R}^2)$. We also recall the horizontally homogeneous anisotropic Sobolev norm and the space:

$$\dot{H}^{s,s'}(\mathbb{R}^2) := \Big\{ u \in \mathcal{S}'(\mathbb{R}^2), \hat{u} \in L^1_{loc}; \|u\|^2_{\dot{H}^{s,s'}(\mathbb{R}^2)} := \int_{\mathbb{R}^2} \mid \xi_1 \mid^{2s} (1+ \mid \xi_2 \mid^2)^{s'} \mid \hat{u}(\xi) \mid^2 d\xi < \infty \Big\},$$

where $s, s' \geq 0$. In what follows, we shall use 'h' to denote the horizonal variable x_1, and 'v' to denote the vertical direction x_2.

Let $\mathbb{R}^2 = (\mathbb{R}_\mathrm{h}, \mathbb{R}_\mathrm{v})$. For exponents $p, q \in [1, \infty)$, we denote the space $L^p(\mathbb{R}_\mathrm{h}, L^q(\mathbb{R}_\mathrm{v}))$ by $L^p_\mathrm{h}(L^q_\mathrm{v})(\mathbb{R}^2)$, which is endowed with the norm

$$\| u \|_{L^p_\mathrm{h}(L^q_\mathrm{v})(\mathbb{R}^2)} := \Big\{ \int_{\mathbb{R}_\mathrm{h}} \big(\int_{\mathbb{R}_\mathrm{v}} \mid u(x_1, x_2) \mid^q dx_2 \big)^{\frac{p}{q}} dx_1 \Big\}^{\frac{1}{p}}.$$

Similar notation for $L^p_\mathrm{v}(L^q_\mathrm{h})(\mathbb{R}^2)$. Then it follows from Minkowski inequality that

$$\| u \|_{L^p_\mathrm{h}(L^q_\mathrm{v})(\mathbb{R}^2)} \leq \| u \|_{L^q_\mathrm{v}(L^p_\mathrm{h})(\mathbb{R}^2)} \text{ when } 1 \leq q \leq p \leq \infty,$$
$$\| u \|_{L^q_\mathrm{v}(L^p_\mathrm{h})(\mathbb{R}^2)} \leq \| u \|_{L^p_\mathrm{h}(L^q_\mathrm{v})(\mathbb{R}^2)} \text{ when } 1 \leq p \leq q \leq \infty.$$

The proof of the above inequalities can be found in Proposition 1.3 of [1].

We define the vector fields $L^p_\mathrm{h}(L^q_\mathrm{v})(\mathbb{R}^2; \mathbb{R}^2)$, $L^q_\mathrm{v}(L^p_\mathrm{h})(\mathbb{R}^2; \mathbb{R}^2)$, $H^s(\mathbb{R}^2; \mathbb{R}^2)$, $H^{s,s'}(\mathbb{R}^2; \mathbb{R}^2)$ and $\dot{H}^{s,s'}(\mathbb{R}^2; \mathbb{R}^2)$ to be the sets of the vectors with both components in $L^p_\mathrm{h}(L^q_\mathrm{v})(\mathbb{R}^2)$, $L^q_\mathrm{v}(L^p_\mathrm{h})(\mathbb{R}^2)$, $H^s(\mathbb{R}^2)$, $H^{s,s'}(\mathbb{R}^2)$ and $\dot{H}^{s,s'}(\mathbb{R}^2)$ respectively.
For simplicity, from now on, in this chapter, we will use the same notations of vector fields and function spaces when there is no confusion.

1.1.2 Function Spaces on $\mathbb{T}^2$

Now we recall some function spaces on two dimensional torus $\mathbb{T}^2$. Let $\mathbb{T}^2 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} = (\mathbb{T}_\mathrm{h}, \mathbb{T}_\mathrm{v})$. Similar to the whole space $\mathbb{R}^2$, we recall the anisotropic L^p spaces:

$$\| u \|_{L^p_\mathrm{h}(L^q_\mathrm{v})(\mathbb{T}^2)} := \Big\{ \int_{\mathbb{T}_\mathrm{h}} \big(\int_{\mathbb{T}_\mathrm{v}} \mid u(x_1, x_2) \mid^q dx_2 \big)^{\frac{p}{q}} dx_1 \Big\}^{\frac{1}{p}}.$$

Similar to $\mathbb{R}^2$ case, we also have:

$$\begin{aligned}
\| u \|_{L^p_{\mathrm{h}}(L^q_{\mathrm{v}})(\mathbb{T}^2)} &\leq \| u \|_{L^q_{\mathrm{v}}(L^p_{\mathrm{h}})(\mathbb{T}^2)} \quad \text{when } 1 \leq q \leq p \leq \infty, \\
\| u \|_{L^q_{\mathrm{v}}(L^p_{\mathrm{h}})(\mathbb{T}^2)} &\leq \| u \|_{L^p_{\mathrm{h}}(L^q_{\mathrm{v}})(\mathbb{T}^2)} \quad \text{when } 1 \leq p \leq q \leq \infty.
\end{aligned}$$

Noting that $\{\frac{1}{2\pi}e^{ik\cdot x}\}_{k\in\mathbb{Z}^2}$ is the orthonormal basis of $L^2(\mathbb{T}^2, \mathbb{C})$, for $u \in L^2(\mathbb{T}^2)$, we consider the Fourier expansion of u:

$$u(x) = \sum_{k\in\mathbb{Z}^2} \hat{u}_k \frac{1}{2\pi} e^{ik\cdot x} \quad \text{with} \quad \hat{u}_k = \overline{\hat{u}_{-k}},$$

where $\hat{u}_k := \frac{1}{2\pi}\int_{\mathbb{T}^2} u(x)e^{-ik\cdot x}dx$ denotes the kth Fourier coefficient of u on $\mathbb{T}^2$. It follows from Fourier-Plancherel equality that the series is convergent in $L^2(\mathbb{T}^2)$.

Define the Sobolev norm :

$$\| u \|^2_{H^s(\mathbb{T}^2)} := \sum_{k\in\mathbb{Z}^2} (1+ | k |^2)^s | \hat{u}_k |^2, \tag{1.4}$$

and the anisotropic Sobolev norms:

$$\begin{aligned}
\| u \|^2_{H^{s,s'}(\mathbb{T}^2)} &= \sum_{k\in\mathbb{Z}^2} (1+ | k_1 |^2)^s (1+ | k_2 |^2)^{s'} | \hat{u}_k |^2, \\
\| u \|^2_{\dot{H}^{s,s'}(\mathbb{T}^2)} &= \sum_{k\in\mathbb{Z}^2} | k_1 |^{2s} (1+ | k_2 |^2)^{s'} | \hat{u}_k |^2,
\end{aligned}$$

where $k = (k_1, k_2)$.
We remark that the nonhomogeneous spaces $H^s(\mathbb{T}^2)$ and $H^{s,s'}(\mathbb{T}^2)$ are Hilbert spaces with dual spaces $H^{-s}(\mathbb{T}^2)$ and $H^{-s,-s'}(\mathbb{T}^2)$, respectively.

We define the Sobolev spaces $H^s(\mathbb{T}^2)$, $H^{s,s'}(\mathbb{T}^2)$ and $\dot{H}^{s,s'}(\mathbb{T}^2)$ for $s, s' \geq 0$ as the completion of $C^\infty(\mathbb{T}^2)$ with the norms $\| \cdot \|_{H^s(\mathbb{T}^2)}$, $\| \cdot \|_{H^{s,s'}(\mathbb{T}^2)}$ and $\| \cdot \|_{\dot{H}^{s,s'}(\mathbb{T}^2)}$ respectively. We define the vector fields $L^p_{\mathrm{h}}(L^q_{\mathrm{v}})(\mathbb{T}^2;\mathbb{R}^2)$, $L^q_{\mathrm{v}}(L^p_{\mathrm{h}})(\mathbb{T}^2;\ \mathbb{R}^2)$, $H^s(\mathbb{T}^2;\mathbb{R}^2)$, $H^{s,s'}(\mathbb{T}^2;\mathbb{R}^2)$ and $\dot{H}^{s,s'}(\mathbb{T}^2;\mathbb{R}^2)$ to be the sets of the vectors with both components in $L^p_{\mathrm{h}}(L^q_{\mathrm{v}})(\mathbb{T}^2)$, $L^q_{\mathrm{v}}(L^p_{\mathrm{h}})(\mathbb{T}^2)$, $H^s(\mathbb{T}^2)$, $H^{s,s'}(\mathbb{T}^2)$ and $\dot{H}^{s,s'}(\mathbb{T}^2)$ respectively.

1.1.3 Preliminaries of Topology and Construction of Some Other Topological Spaces

Metric Spaces Local in Time

For any Banach space X and $1 \leq p \leq \infty$, define the metric space $L^p_{loc}(\mathbb{R}^+; X)$ to be the space of all those functions $u : \mathbb{R}^+ \to X$, such that

$$\|u\|_{L^p([0,T];X)} < \infty \text{ for any } T > 0,$$

with the following distance:

$$\rho_{L^p_{loc}(\mathbb{R}^+;X)}(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|x-y\|_{L^p([0,j];X)}}{1+\|x-y\|_{L^p([0,j];X)}}.$$

Then $L^p_{loc}(\mathbb{R}^+;X)$ is a Fréchet Space generated by the seminorms $\{L^p([0,n];X)\}_{n\geq 1}$. Moreover, note that the convergence in $L^p_{loc}(\mathbb{R}^+;X)$ is equivalent to the convergence in each $L^p([0,T];X)$ for $T>0$.

Similarly, define the metric space $C(\mathbb{R}^+;X)$ to be the space of all those functions $u : \mathbb{R}^+ \to X$, such that

$$\|u\|_{C([0,T];X)} < \infty \text{ for any } T>0,$$

with the metric generated by the seminorms $\{C([0,n];X)\}_{n\geq 1}$, i.e.

$$\rho_{C(\mathbb{R}^+;X)}(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|x-y\|_{C([0,j];X)}}{1+\|x-y\|_{C([0,j];X)}}.$$

Observe that this metric which we defined coincides with the compact-open topology, which is usually equipped in the space of continuous functions or maps.

Metric Spaces Local in Space

Define the space $L^p_{loc}(\mathbb{R}^d)$ to be the metric space of the functions which are L^p integrable in any compact set with the following metric:

$$\rho_{L^p}(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|x-y\|_{L^p(B_j)}}{1+\|x-y\|_{L^p(B_j)}},$$

where B_j is the ball of radius j (open or close).

Weak and Weak Star topology

For $1<p<\infty$ and $T>0$, let $L^p_w([0,T];X)$ be the space $L^p([0,T];X)$ with weak topology. Let $L^\infty_{w*}([0,T];X)$ be the space $L^\infty([0,T];X)$ with weak star topology. We know that in the reflexive Banach spaces, weak and weak star topology coincide. However, for L^∞ space, they are different.

Projection Maps and Spaces $\bar{\Omega}$, Ω^T

Let X be any topological space and $\pi_{[0,t]}$ be the projection map from $C(\mathbb{R}^+;X)$ to $C([0,t];X)$ which is the restriction of the function in $C(\mathbb{R}^+;X)$ to the interval $[0,t]$. Similarly, for $t_1 \geq t_2$, we define the projection map π_{t_1,t_2} to be the map which restricts any $x \in C([0,t_1];X)$ to the interval $[0,t_2]$.

Finally , we define $\bar{\Omega} := C(\mathbb{R}^+; H^{-1})$ and $\Omega^T := C([0,T]; H^{-1})$, which are the spaces we use in the martingale problems.

1.1.4 Some Other Notations and Definitions

We use D to denote the domain $\mathbb{R}^2$ or $\mathbb{T}^2$. Let us denote the following vector spaces and we omit the domain again if there is no confusion:

$$\begin{aligned} H &:= \left\{ u \in L^2(D,\mathbb{R}^2);\ \operatorname{div} u = 0 \right\}, \\ V &:= \left\{ u \in H^1(D,\mathbb{R}^2);\ \operatorname{div} u = 0 \right\}, \\ \tilde{H}^{s,s'} &:= \left\{ u \in H^{s,s'}(D,\mathbb{R}^2);\ \operatorname{div} u = 0 \right\}. \end{aligned}$$

Moreover, we use $(\cdot,\cdot)$ or $(\cdot \mid \cdot)$ to denote the scalar product

$$(u,v) = (u \mid v) = (u,v)_{L^2} = \sum_{j=1}^{2} \int_D u_j(x) v_j(x) dx.$$

We use $(\cdot,\cdot)_{H^{0,1}}$ or $(\cdot,\cdot)_{0,1}$ to denote the inner product

$$(u,v)_{H^{0,1}} = \sum_{j=1}^{2} \int_D \big(u_j(x) v_j(x) + \partial_2 u_j(x) \partial_2 v_j(x) \big)\, dx.$$

Let

$\mathbf{P} : L^2(D;\mathbb{R}^2) \to H$ be the Leray projection operator to divergence free space.

By applying $\mathbf{P}$ to (1.1), we write

$$\partial_t u = \mathbf{P}(\partial_1^2 u - u \cdot \nabla u).$$

As usual, when $u, v, w \in H^1$, we denote

$$\begin{aligned} B(u,v) &:= u \cdot \nabla v, \\ B(u) &:= u \cdot \nabla u, \\ b(u,v,w) &:= (u \cdot \nabla v, w). \end{aligned}$$

Then we have $b(u,v,w) = -b(u,w,v)$ for $u,v,w \in V$. In particular, $b(u,v,v) = 0$.

Let us end this section by the definition of weak solution to (1.1)

Definition 1.1 (Weak Solution)**.** *We call $u(t,x) : \mathbb{R}^+ \times D \to \mathbb{R}^2$ a global weak solution of* (1.1) *with the initial data u_0 if u satisfies:*

(i) $u \in L^\infty([0,T]; \tilde{H}^{0,1}) \cap L^2([0,T]; \dot{H}^{1,1})$ *for any* $T > 0$ *;*

(ii) for any $T > 0$ *and* $\varphi \in C_c^\infty([0,T] \times D)$ *with* $\operatorname{div} \varphi(t) = 0$*,* $t > 0$*,*

$$\int_0^t \big\{-(u, \partial_t \varphi) + (\partial_1 u, \partial_1 \varphi) + (u \cdot \nabla u, \varphi)\big\}\, ds = (u_0, \varphi(0)) - (u(t), \varphi(t)), \tag{1.5}$$

(iii) $u \in C(\mathbb{R}^+; L^2)$ *with* $u(0) = u_0$*.*

Remark 1.2. *From (i) and (ii) we obtain (iii) by the same method of 2D Navier–Stokes equations. Indeed, for any $T > 0$, we deduce from (i), Sobolev interpolation inequality and Hölder's inequality that*

$$\int_0^T \|u\|_{H^{\frac{1}{2}}}^4 dt \lesssim \int_0^T \|u\|_{L^2}^2 \|u\|_{H^1}^2 dt \lesssim \|u\|_{L^\infty([0,T];L^2)}^2 \|u\|_{L^2([0,T];H^1)}^2,$$

which together with the Sobolev embedding

$$H^{\frac{1}{2}} \hookrightarrow L^4$$

ensure that $u \in L^4([0,T]; L^4)$, and thus $u \cdot \nabla u \in L^2([0,T]; H^{-1})$. Let

$$Gu := -u \cdot \nabla u + \partial_1^2 u - \nabla \Delta^{-1} \partial_i \partial_j \left((u)^i (u)^j\right) \in L^2([0,T]; H^{-1}).$$

Then it follows from (ii) that

$$\int_0^t \big\{(Gu, \varphi) + (u, \partial_t \varphi)\big\}\, ds = (u(t), \varphi(t)) - (u_0, \varphi(0))$$

for any $T > 0$ and $\varphi \in C_c^\infty([0,T] \times D)$ with $\operatorname{div} \varphi(t) = 0$, $t > 0$. Thus Gu is the weak derivative of u with respect to time t, which we denote by $\partial_t u$ from now on. Since $u \in L^2([0,T]; H^1)$ for any $T > 0$, we have for any $t > 0$,

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + 2 \int_0^t (\partial_t u, u) ds.$$

Therefore, $u \in C(\mathbb{R}^+; L^2)$. Finally $u(0) = u_0$ comes from (ii) by setting $t = 0$.

1.2 The Deterministic Cases

In this section we consider the deterministic equation (1.1). For simplicity, we always omit the domain D in this section when there is no confusion. We have the following theorem of well-posedness for both $\mathbb{R}^2$ and $\mathbb{T}^2$.

Theorem 1.3. *Given solenoidal vector field u_0 in $\tilde{H}^{0,1}$, (1.1) has a unique global weak solution in the sense of Definition 1.1.*

The a priori estimates are similar on the whole space $\mathbb{R}^2$ and the torus $\mathbb{T}^2$. We first state and prove some estimates which hold in both cases.

1.2.1 A Priori Estimates

Lemma 1.4. *Let u be a global smooth enough solution to (1.1). Then one has*

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\partial_1 u(s)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2. \tag{1.6}$$

Proof. Indeed by taking the L^2 inner product of the momentum equation of (1.1) with u and using $\operatorname{div} u = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\partial_1 u\|_{L^2}^2 = 0.$$

Integrating the above inequality over $[0, t]$ leads to (1.6). □

Lemma 1.5. *Under the same assumption of Lemma 1.4, one has the following estimates:*

$$\begin{aligned} \left| \big(\partial_2(u\cdot\nabla u)|\partial_2 u\big) \right| \lesssim \big(&\|\partial_2 u^1\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\partial_1 u^2\|_{L^2_{\rm h}(L^\infty_{\rm v})} \\ &+ \|\partial_1 u^1\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\partial_2 u^2\|_{L^\infty_{\rm h}(L^2_{\rm v})}\big)\|\partial_2 u^2\|_{L^2}. \end{aligned} \tag{1.7}$$

Proof. It is easy to observe that

$$(\partial_2(u\cdot\nabla u) \mid \partial_2 u) = (\partial_2(u\cdot\nabla u^1) \mid \partial_2 u^1) + (\partial_2(u\cdot\nabla u^2) \mid \partial_2 u^2), \tag{1.8}$$

where $u = (u^1, u^2)$.
For the first term on the right-hand side of (1.8), we have

$$\begin{aligned} (\partial_2(u\cdot\nabla u^1) \mid \partial_2 u^1) =& (\partial_2(u^1\partial_1 u^1 + u^2\partial_2 u^1) \mid \partial_2 u^1) \\ =& (\partial_2 u^1 \partial_1 u^1 \mid \partial_2 u^1) + (u^1\partial_2\partial_1 u^1 \mid \partial_2 u^1) \\ &+ (\partial_2 u^2\partial_2 u^1 \mid \partial_2 u^1) + (u^2\partial_2^2 u^1 \mid \partial_2 u^1). \end{aligned}$$

Yet due to $\operatorname{div} u = 0$, we achieve

$$(\partial_2 u^1\partial_1 u^1 \mid \partial_2 u^1) + (\partial_2 u^2\partial_2 u^1 \mid \partial_2 u^1) = 0,$$

and

$$\begin{aligned}(u^1\partial_2\partial_1 u^1 \mid \partial_2 u^1)+(u^2\partial_2^2 u^1 \mid \partial_2 u^1) &= \big(u\cdot\nabla\partial_2 u^1|\partial_2 u^1\big)\\ &= \frac{1}{2}\int_D u\cdot\nabla|\partial_2 u^1|^2\,dx = -\frac{1}{2}\int_D \operatorname{div} u|\partial_2 u^1|^2\,dx = 0.\end{aligned}$$

This leads to

$$(\partial_2(u\cdot\nabla u^1) \mid \partial_2 u^1) = 0. \tag{1.9}$$

For the second term on the right-hand side of (1.8), again due to $\operatorname{div} u = 0$, we have:

$$\begin{aligned}(\partial_2(u\cdot\nabla u^2) \mid \partial_2 u^2) =&(\partial_2(u^1\partial_1 u^2) \mid \partial_2 u^2) + (\partial_2(u^2\partial_2 u^2) \mid \partial_2 u^2)\\ =&\big(\partial_2 u\cdot\nabla u^2|\partial_2 u^2\big).\end{aligned}$$

The second equality is due to

$$(u^1\partial_1\partial_2 u^2 \mid \partial_2 u^2) + (u^2\partial_2^2 u^2 \mid \partial_2 u^2) = -\frac{1}{2}(\partial_1 u^1\partial_2 u^2 \mid \partial_2 u^2) - \frac{1}{2}(\partial_2 u^2\partial_2 u^2 \mid \partial_2 u^2) = 0.$$

We finish the proof of the lemma when we notice

$$\begin{aligned}&\mid \big(\partial_2 u\cdot\nabla u^2|\partial_2 u^2\big) \mid\\ =&\mid (\partial_2 u^1\partial_1 u^2 \mid \partial_2 u^2) + (\partial_2 u^2\partial_2 u^2 \mid \partial_2 u^2) \mid\\ \le& \big(\|\partial_2 u^1\|_{L^\infty_{\rm h}(L^2_{\rm v})}\|\partial_1 u^2\|_{L^2_{\rm h}(L^\infty_{\rm v})} + \|\partial_1 u^1\|_{L^2_{\rm h}(L^\infty_{\rm v})}\|\partial_2 u^2\|_{L^\infty_{\rm h}(L^2_{\rm v})}\big)\|\partial_2 u^2\|_{L^2}.\end{aligned}$$

□

The following estimates are different between $\mathbb{R}^2$ and $\mathbb{T}^2$. First we consider the whole space $\mathbb{R}^2$, which is easier.

Lemma 1.6. *Let f be a smooth function on $\mathbb{R}^2$, we have*

$$\|f\|^2_{L^2_{\rm v}(L^\infty_{\rm h})} \le 2\|f\|_{L^2}\|\partial_1 f\|_{L^2}.$$

$$\|f\|^2_{L^2_{\rm h}(L^\infty_{\rm v})} \le 2\|f\|_{L^2}\|\partial_2 f\|_{L^2}.$$

Proof. We only present the proof to the first inequality, the second one follows along the same line. Indeed observing that

$$\begin{aligned}f^2(x_1,x_2) = \int_{-\infty}^{x_1}\partial_1 f^2(y,x_2)\,dy =&2\int_{-\infty}^{x_1} f(y,x_2)\partial_1 f(y,x_2)\,dy\\ \le&2\|f(\cdot,x_2)\|_{L^2_{\rm h}}\|\partial_1 f(\cdot,x_2)\|_{L^2_{\rm h}},\end{aligned}$$

which implies

$$\|f(\cdot,x_2)\|^2_{L^\infty_{\rm h}} \le 2\|f(\cdot,x_2)\|_{L^2_{\rm h}}\|\partial_1 f(\cdot,x_2)\|_{L^2_{\rm h}}.$$

Applying Hölder's inequality in the x_2 variable gives

$$\|f\|^2_{L^2_{\rm v}(L^\infty_{\rm h})} \leq 2\|f\|_{L^2}\|\partial_1 f\|_{L^2}.$$

This completes the proof of the lemma. □

Now we consider the torus, which is more complicated because we do not assume the average value of the function on the torus is 0.

Lemma 1.7. *Let f be a smooth function on the torus $\mathbb{T}^2$, there hold*

$$\|f\|^2_{L^2_{\rm v}(L^\infty_{\rm h})} \lesssim \|f\|_{L^2}\|\partial_1 f\|_{L^2} + \|f\|^2_{L^2},$$

and

$$\|f\|^2_{L^2_{\rm h}(L^\infty_{\rm v})} \lesssim \|f\|_{L^2}\|\partial_2 f\|_{L^2} + \|f\|^2_{L^2}.$$

Proof. We only prove the first inequality, the second one follows similarly.
Indeed for any fixed $x_2 \in [0, 2\pi]$; we define $\bar{f}(x_2) = \frac{1}{2\pi}\int_{\mathbb{T}_{\rm h}} f(r, x_2)dr$ to be the average of f on $\mathbb{T}_{\rm h}$. By mean value theorem, there exists some $a(x_2) \in [0, 2\pi]$, such that $f(a(x_2), x_2) = \bar{f}(x_2)$. As a consequence, we deduce for any $x_1 \in [0, 2\pi]$ that

$$\begin{aligned}\big(f(x_1, x_2) - \bar{f}(x_2)\big)^2 &= \big(f(x_1, x_2) - f(a(x_2), x_2)\big)^2\\ &\leq \int_{\mathbb{T}_{\rm h}} |\partial_1(f(r, x_2) - \bar{f}(x_2))^2|\, dr\\ &= 2\int_{\mathbb{T}_{\rm h}} |f(r, x_2) - \bar{f}(x_2)| \cdot |\partial_1 f(r, x_2)|\, dr,\end{aligned}$$

which implies

$$\|f(x_1, x_2)\|^2_{L^\infty_{\rm h}} \lesssim \bar{f}(x_2)^2 + \int_{\mathbb{T}_{\rm h}} |f(r, x_2)| \cdot |\partial_1 f(r, x_2)|dr + |\bar{f}(x_2)| \int_{\mathbb{T}_{\rm h}} \cdot|\partial_1 f(r, x_2)|dr. \quad (1.10)$$

Claim that

$$\|\bar{f}(x_2)\|^2_{L^2_{\rm v}} \leq \|f\|^2_{L^2}.$$

Indeed, by Hölder's inequality,

$$\|\bar{f}(x_2)\|^2_{L^2_{\rm v}} = \int_{\mathbb{T}_{\rm v}} |\frac{1}{2\pi}\int_{\mathbb{T}_{\rm h}} f(r, x_2)dr|^2 dx_2 \leq \int_{\mathbb{T}_{\rm v}}\int_{\mathbb{T}_{\rm h}} |f(r, x_2)|^2 dr dx_2.$$

Then we finish the proof by integrating (1.10) over $\mathbb{T}_{\rm v}$ and using the claim. □

Therefore, combining the above two lemmas, we obtain the following result.

Lemma 1.8. *Under the same assumption of Lemma 1.4, the following estimate holds on both $\mathbb{R}^2$ and $\mathbb{T}^2$:*

$$\|\partial_2 u(t)\|^2_{L^2} + \int_0^t \|\partial_1\partial_2 u(s)\|^2_{L^2} ds \leq e^{2C\left(\|u_0\|^2_{L^2} + t^{\frac{1}{2}}\|u_0\|_{L^2}\right)}\|\partial_2 u_0\|^2_{L^2}.$$

Proof. By taking ∂_2 to the momentum equation of (1.1) and then taking L^2 inner product of the resulting equation with $\partial_2 u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_2 u(t)\|_{L^2}^2 + \|\partial_1\partial_2 u\|_{L^2}^2 \le -(\partial_2(u\cdot\nabla u) \mid \partial_2 u). \tag{1.11}$$

While it follows from Lemmas 1.5 , 1.6 and 1.7 that

$$\begin{aligned}
&|\ \big(\partial_2(u\cdot\nabla u)|\partial_2 u\big)\ | \\
&\lesssim \big(\|\partial_2 u\|_{L^2} + \|\partial_2 u\|_{L^2}^{\frac12}\|\partial_1\partial_2 u\|_{L^2}^{\frac12}\big)\big(\|\partial_1 u\|_{L^2} + \|\partial_1 u\|_{L^2}^{\frac12}\|\partial_1\partial_2 u\|_{L^2}^{\frac12}\big)\|\partial_2 u^2\|_{L^2} \\
&\lesssim \|\partial_1\partial_2 u\|_{L^2}\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2} \\
&+ \|\partial_1\partial_2 u\|_{L^2}^{\frac12}\big(\|\partial_2 u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2} + \|\partial_1 u\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2}\big)\|\partial_2 u^2\|_{L^2} \\
&+ \|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2}\|\partial_2 u^2\|_{L^2},
\end{aligned} \tag{1.12}$$

where the second inequality is due to the divergence free condition of u so that $\|\partial_2 u^2\|_{L^2} = \|\partial_1 u^1\|_{L^2}$.
By Young's Inequality, we have

$$\|\partial_1\partial_2 u\|_{L^2}\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2} \le \frac{1}{6}\|\partial_1\partial_2 u\|_{L^2}^2 + C\|\partial_1 u\|_{L^2}^2\|\partial_2 u\|_{L^2}^2,$$

and

$$\begin{aligned}
&\|\partial_1\partial_2 u\|_{L^2}^{\frac12}(\|\partial_2 u\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2} + \|\partial_1 u\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2})\|\partial_2 u^2\|_{L^2} \\
&\le \frac{1}{6}\|\partial_1\partial_2 u\|_{L^2}^2 + (\|\partial_2 u\|_{L^2}^{\frac23}\|\partial_1 u\|_{L^2}^{\frac43} + \|\partial_1 u\|_{L^2}^{\frac23}\|\partial_2 u\|_{L^2}^{\frac43})\|\partial_2 u^2\|_{L^2}^{\frac43} \\
&\le \frac{1}{6}\|\partial_1\partial_2 u\|_{L^2}^2 + C\|\partial_1 u\|_{L^2}^{\frac43}\|\partial_2 u\|_{L^2}^2,
\end{aligned}$$

where in the second inequality we used again the divergence free condition of u.

As a result, it comes out

$$\begin{aligned}
&|\ \big(\partial_2(u\cdot\nabla u)|\partial_2 u\big)\ | \\
&\le \frac{1}{2}\|\partial_1\partial_2 u\|_{L^2}^2 + C\|\partial_1 u\|_{L^2}^2\|\partial_2 u\|_{L^2}^2 + C\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2}^2 + C\|\partial_1 u\|_{L^2}^{\frac43}\|\partial_2 u\|_{L^2}^2 \\
&\le \frac{1}{2}\|\partial_1\partial_2 u\|_{L^2}^2 + C(\|\partial_1 u\|_{L^2}^2 + \|\partial_1 u\|_{L^2})\|\partial_2 u\|_{L^2}^2,
\end{aligned} \tag{1.13}$$

where in the last inequality, we used $\|\partial_1 u\|_{L^2}^{\frac43} \lesssim \|\partial_1 u\|_{L^2} + \|\partial_1 u\|_{L^2}^2$.
By inserting the estimate (1.13) into (1.11) and applying Gronwall's inequality and using (1.6), we obtain

$$\begin{aligned}
\|\partial_2 u(t)\|_{L^2}^2 + \int_0^t \|\partial_1\partial_2 u(s)\|_{L^2}^2 ds &\le e^{2C\int_0^t \|\partial_1 u\|_{L^2}^2 + \|\partial_1 u\|_{L^2} ds}\|\partial_2 u_0\|_{L^2}^2 \\
&\le e^{2C(\|u_0\|_{L^2}^2 + t^{\frac12}\|u_0\|_{L^2})}\|\partial_2 u_0\|_{L^2}^2.
\end{aligned}$$

This completes the proof of Lemma 1.8. ∎

Remark 1.9. *In case that $D = \mathbb{R}^2$, the proof of Lemma 1.8 is much easier and tight. In fact, by Lemmas 1.5 and 1.6, we have*

$$| (\partial_2(u \cdot \nabla u) \mid \partial_2 u) | \lesssim \|\partial_1\partial_2 u\|_{L^2}\|\partial_1 u\|_{L^2}\|\partial_2 u\|_{L^2}.$$

Young's inequality yields

$$| (\partial_2(u \cdot \nabla u) \mid \partial_2 u) | \leq \frac{1}{2}\|\partial_1\partial_2 u\|_{L^2}^2 + C\|\partial_1 u\|_{L^2}^2\|\partial_2 u\|_{L^2}^2.$$

Inserting the above inequality into (1.11) gives rise to

$$\frac{d}{dt}\|\partial_2 u(t)\|_{L^2}^2 + \|\partial_1\partial_2 u\|_{L^2}^2 \leq 2C\|\partial_1 u\|_{L^2}^2\|\partial_2 u\|_{L^2}^2.$$

Applying Gronwall's inequality and using (1.6), we obtain

$$\begin{aligned}\|\partial_2 u(t)\|_{L^2}^2 + \int_0^t \|\partial_1\partial_2 u(s)\|_{L^2}^2 ds &\leq e^{2C\int_0^t \|\partial_1 u\|_{L^2}^2 ds}\|\partial_2 u_0\|_{L^2}^2 \\ &\leq e^{2C\|u_0\|_{L^2}^2}\|\partial_2 u_0\|_{L^2}^2.\end{aligned}$$

1.2.2 Proof of Theorem 1.3

We divide the proof of this theorem into the following two parts:

(1) **Existence part.**

It is standard that the first step to prove the existence of weak solutions to some nonlinear partial differential equations is to construct appropriate approximate solutions. Here we consider the following approximating equations:

$$\begin{cases}\partial_t u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon - \partial_1^2 u^\epsilon - \epsilon^2\partial_2^2 u^\epsilon = -\nabla p^\epsilon, \\ \operatorname{div} u^\epsilon = 0, \\ u^\epsilon(0) = u_0 * j_\epsilon,\end{cases}, \tag{1.14}$$

where j is a smooth function on $\mathbb{R}^2$ with

$$j(x) = 1, \quad | x | \leq 1; \; j(x) = 0, \quad | x | \geq 2,$$

and

$$j_\epsilon(x) = \frac{1}{\epsilon^2} j(\frac{x}{\epsilon}).$$

It follows from classical theory on Navier–Stokes system that (1.14) has a unique global smooth solution (u^ϵ, p^ϵ) for any fixed ϵ. (See, for example, Thm. 6.1 of [87] and Theorem

7.1 of [82]). Furthermore, along the same line to the proof of Lemmas 1.4 and 1.8, we have

$$\|u^\epsilon(t)\|_{L^2}^2 + 2\int_0^t \|\partial_1 u^\epsilon\|_{L^2}^2 ds + \epsilon^2 \int_0^t \|\partial_2 u^\epsilon\|_{L^2}^2 ds \le \|u_0\|_{L^2}^2,$$
$$\|\partial_2 u^\epsilon(t)\|_{L^2}^2 + \int_0^t \|\partial_1\partial_2 u^\epsilon\|_{L^2}^2 ds + \epsilon^2 \int_0^t \|\partial_2^2 u^\epsilon\|_{L^2}^2 ds \le \|\partial_2 u_0\|_{L^2}^2 e^{2C(\|u_0\|_{L^2}^2 + t^{\frac{1}{2}}\|u_0\|_{L^2})}.$$

It is obvious that for $\varphi \in C_c^\infty([0,T]\times D)$ with $\operatorname{div}\varphi(t) = 0$, u^ϵ satisfies the following equation:

$$\int_0^t \big(-(u^\epsilon, \partial_t\varphi) + (u^\epsilon\cdot\nabla u^\epsilon, \varphi) + (\partial_1 u^\epsilon, \partial_1\varphi) + \epsilon^2(\partial_2 u^\epsilon, \partial_2\varphi)\big)\, ds = (u^\epsilon(0), \varphi(0)) - (u^\epsilon(t), \varphi(t)). \tag{1.15}$$

Then for any fixed $T>0$, $\{u^\epsilon\}_{\epsilon>0}$ is uniformly bounded in $L^\infty([0,T];H^{0,1})\cap L^2([0,T];H^{1,1})$. By interpolation, $\{u^\epsilon\}_{\epsilon>0}$ is uniformly bounded in $L^4([0,T];H^{\frac{1}{2}})$. Sobolev imbedding theorem implies that $\{u^\epsilon\}_{\epsilon>0}$ is bounded in $L^4([0,T];L^4)$. Hence the nonlinear term in (1.14) is bounded in $L^2([0,T];H^{-1})$. Moreover, $\nabla p^\epsilon = \nabla\Delta^{-1}\partial_i\partial_j\left((u^\epsilon)^i(u^\epsilon)^j\right)$ is uniformly bounded in $L^2([0,T];H^{-1})$. As a result, it comes out that

$$\{\partial_t u^\epsilon\}_{\epsilon>0} \quad \text{is uniformly bounded in } L^2([0,T];H^{-1}). \tag{1.16}$$

At this stage, we need to use the following Aubin-Lions lemma (see, for example, Theorem 4.12 of [82]).

Lemma 1.10 (Aubin-Lions). *Let K be the torus or a smooth bounded (open) domain. If the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded sequence in $L^q([0,T];H^1(K))$ for $q\in(1,\infty)$, and $(\partial_t u_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $L^p([0,T];H^{-1}(K))$ for some $p\in(1,\infty)$, then there exist $u\in L^q([0,T];H^1(K))$ and a subsequence of $(u_{n_j})_{j\in\mathbb{N}}$ so that $(u_{n_j})_{j\in\mathbb{N}}$ converges strongly to u in $L^q([0,T];L^2(K))$.*

Let us now take $\epsilon = \frac{1}{n}$ in (1.14). Set $u_n = u^{\frac{1}{n}}$.

For the case that $D=\mathbb{R}^2$, note that $\bigcup_{i=1}^\infty B_i = \mathbb{R}^2$, where B_i is the open ball of radius i. By a classical diagonal methods, we can choose a subsequence of $(u_n)_{n\in\mathbb{N}}$ (which we still denote by $(u_n)_{n\in\mathbb{N}}$ for simplicity) so that

$$u_n \to u \text{ strongly in } L^2([0,T];L^2(B_i)) \text{ for any } i.$$

Since the test function φ in (1.5) satisfies $\varphi\in C_c^\infty(D)$, it must be supported in some B_i. Then we can take $n\to\infty$ in (1.15) to obtain (1.5).
Finally note that since u^ϵ is uniformly bounded in $L^\infty(\mathbb{R}^+;H^{0,1})\cap L^2(\mathbb{R}^+;\dot H^{1,1})$, we can choose a subsequence of u_n (which we denote by u_n again) and some $\tilde u$, such that

- $u_n \to \tilde{u}$ weakly in $L^2([0,T]; H^{1,1})$ for each $T > 0$,
- $u_n \to \tilde{u}$ weakly $*$ in $L^\infty([0,T]; H^{0,1})$ for each $T > 0$.

By the uniqueness of the limits of weak convergence, u and $\tilde{u}$ coincide.

The case for $D = \mathbb{T}^2$, the compactness argument is a direct application of Aubin-Lions' Lemma, we omit the details here.

Remark 1.11. *Indeed on $\mathbb{R}^2$, we deduce from Remark 1.9 that $u \in L^\infty(\mathbb{R}^+; \tilde{H}^{0,1}) \cap L^2(\mathbb{R}^+; \dot{H}^{1,1})$.*

(2) **Uniqueness part.**

Let $u, v \in L^\infty([0,T]; H^{0,1}) \cap L^2([0,T]; \dot{H}^{1,1})$ be two weak solutions of (1.1). We denote $w := u - v$. Then we have

$$\partial_t w + w \cdot \nabla v + u \cdot \nabla w - \partial_1^2 w = -\nabla p,$$

where $\nabla p = \nabla p_1 - \nabla p_2$, and p_1, p_2 are the pressures corresponding to the solution u and v, respectively. Taking L^2 inner product of the above equation with w gives

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \|\partial_1 w\|_{L^2}^2 \leq \mid (w \cdot \nabla v \mid w) \mid . \tag{1.17}$$

Observing that

$$\begin{aligned} \mid (w \cdot \nabla v \mid w) \mid &= \mid (w^1 \partial_1 v + w^2 \partial_2 v \mid w) \mid \\ &\leq \big(\|w^1\|_{L^\infty_{\rm h}(L^2_{\rm v})} \|\partial_1 v\|_{L^2_{\rm h}(L^\infty_{\rm v})} + \|w^2\|_{L^2_{\rm h}(L^\infty_{\rm v})} \|\partial_2 v\|_{L^\infty_{\rm h}(L^2_{\rm v})}\big) \|w\|_{L^2}, \end{aligned} \tag{1.18}$$

where $w = (w^1, w^2)$, from which and Lemmas 1.6 and 1.7, we deduce that

$$\begin{aligned} \mid (w \cdot \nabla v \mid w) \mid &\lesssim \big(\|w\|_{L^2}^{\frac{1}{2}} \|\partial_1 w\|_{L^2}^{\frac{1}{2}} + \|w\|_{L^2}\big) H(v) \|w\|_{L^2} \\ &\lesssim \|w\|_{L^2}^{\frac{3}{2}} \|\partial_1 w\|_{L^2}^{\frac{1}{2}} H(v) + \|w\|_{L^2}^2 H(v), \end{aligned} \tag{1.19}$$

where

$$H(v) := \|\partial_2 v\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 v\|_{L^2}^{\frac{1}{2}} + \|\partial_1 v\|_{L^2} + \|\partial_1 v\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 v\|_{L^2}^{\frac{1}{2}} + \|\partial_2 v\|_{L^2}.$$

Then applying Young's Inequality to the first term on the right of (1.19) yields that for some constant C,

$$\mid (w \cdot \nabla v \mid w) \mid \leq \frac{1}{2}\|\partial_1 w\|_{L^2}^2 + C H(v)^{\frac{4}{3}} \|w\|_{L^2}^2 + C H(v) \|w\|_{L^2}^2. \tag{1.20}$$

By Young's inequality for any smooth $f_1, f_2 \in L^2$,

$$\|f_1\|_{L^2}^{\frac{1}{2}} \|f_2\|_{L^2}^{\frac{1}{2}} \lesssim \|f_1\|_{L^2} + \|f_2\|_{L^2}.$$

By inserting the estimate (1.20) into (1.17) and applying Gronwall's Inequality, we obtain for some constant C_0 that

$$| (w \cdot \nabla v \mid w) | \leq \frac{1}{2}\|\partial_1 w\|_{L^2}^2 + C_0 r(v,t)\|w\|_{L^2}^2, \tag{1.21}$$

and

$$\|w(t)\|_{L^2}^2 \leq \|w_0\|_{L^2}^2 e^{2C_0 \int_0^t r(v,s)ds}.$$

Here for $v \in H^{1,1}$, define

$$r(v,t) := \|\partial_1 v(t)\|_{L^2}^{\frac{4}{3}} + \|\partial_2 v(t)\|_{L^2}^{\frac{4}{3}} + \|\partial_1 v(t)\|_{L^2} + \|\partial_2 v(t)\|_{L^2} + \|\partial_1\partial_2 v(t)\|_{L^2} + \|\partial_1\partial_2 v(t)\|_{L^2}^{\frac{4}{3}} \in L^1([0,T]),$$

which implies the uniqueness.

1.3 The Stochastic Cases

For the stochastic case, we consider the equation (1.2) on $\mathbb{T}^2$, and again for simplicity of the notation, we always omit the domain $\mathbb{T}^2$ in what follows.

1.3.1 Preliminaries and Notations

Let $(e_k, k \geq 1)$ be an orthonormal basis of H whose elements belong to H^2 and orthogonal in $\tilde{H}^{0,1}$ and $\tilde{H}^{1,0}$ (hence also $\tilde{H}^{1,1}$). For integers $k, l \geq 1$ with $k \neq l$, we deduce that for $i = 1, 2$:

$$(\partial_i^2 e_k, e_l) = -(\partial_i e_k, \partial_i e_l) = 0.$$

Therefore, $\partial_i^2 e_k$ is a constant multiple of e_k.

Remark 1.12. *Note that, such basis exists, for example, let* $\mathbb{Z}^2 \setminus \{(0,0)\} = \mathbb{Z}_+^2 \cup \mathbb{Z}_-^2$, *where*

$$\begin{aligned} \mathbb{Z}_+^2 &= \left\{(k_1,k_2) \in \mathbb{Z}^2 \,|\, k_2 > 0\right\} \cup \left\{(k_1,0) \in \mathbb{Z}^2 \,|\, k_1 > 0\right\}, \\ \mathbb{Z}_-^2 &= \left\{(k_1,k_2) \in \mathbb{Z}^2 \,|\, -k \in \mathbb{Z}_+^2\right\}. \end{aligned}$$

For $k \in \mathbb{Z}^2 \backslash \{(0,0)\}$, *set*

$$e_k(x) = \begin{cases} \frac{k^\perp}{|k|}\sin(k \cdot x) & \text{if } k \in \mathbb{Z}_+^2, \\ \frac{k^\perp}{|k|}\cos(k \cdot x) & \text{if } k \in \mathbb{Z}_-^2. \end{cases}$$

where $k \cdot x = k_1 x_1 + k_2 x_2$ *and* $k^\perp = (-k_2, k_1)$. *Then* $\{e_k\}_{k \in \mathbb{Z}^2 \backslash \{(0,0)\}}$ *together with the vector* $(1,0)$ *and* $(0,1)$ *is orthonormal basis of* H *which satisfies our requirements.*

Let $\mathcal{H}_n = span(e_1, ..., e_n)$ and let P_n(resp. $\tilde{P}_n$, $\bar{P}_n$) denote the orthogonal projection from H (resp. $\tilde{H}^{0,1}$, $\tilde{H}^1$) to $\mathcal{H}_n$. We deduce that

$$P_n u = \tilde{P}_n u, \text{ for } u \in \tilde{H}^{0,1}.$$

Indeed, for $v \in \mathcal{H}_n$, we have $\partial_2^2 v \in \mathcal{H}_n$ and for any $u \in \tilde{H}^{0,1}$:

$$(P_n u, v) = (u, v) \text{ and } (\partial_2 P_n u, \partial_2 v) = -(P_n u, \partial_2^2 v) = -(u, \partial_2^2 v) = (\partial_2 u, \partial_2 v).$$

Hence given $u \in \tilde{H}^{0,1}$, we have

$$(P_n u, v)_{H^{0,1}} = (u, v)_{H^{0,1}}, \text{ for any } v \in \mathcal{H}_n.$$

This proves that P_n and $\tilde{P}_n$ coincide on $\tilde{H}^{0,1}$. Similarly, we can prove P_n, $\tilde{P}_n$ and $\bar{P}_n$ coincide on $\tilde{H}^1$.

Let $(W(t), t \geq 0)$ be an ℓ^2-cylindrical Wiener process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $W_n(t) = \sum_{j=1}^{n} \psi_j \beta_j(t) := \Pi_n W(t)$, where $\{\beta_j(t)\}$ is a sequence of independent Brownian Motions on $(\Omega, \mathcal{F}, \mathbb{P})$ and ψ_j is an orthonormal basis of ℓ^2.
Let $L_2(\ell^2, \mathbb{U})$ denote the space of Hilbert–Schmidt operators from ℓ^2 to $\mathbb{U}$ for any Hilbert space $\mathbb{U}$.
For a topological vector space $\mathbb{V}$, let $\mathcal{B}(\mathbb{V})$ denote its Borel σ-algebra and $\mathcal{P}(\mathbb{V})$ denote all the probability measures on $(\mathbb{V}, \mathcal{B}(\mathbb{V}))$.
Let σ be a measurable mapping from $\big(\mathbb{R}^+ \times \tilde{H}^{1,1}, \mathcal{B}(\mathbb{R}^+ \times \tilde{H}^{1,1})\big)$ to $\big(L_2(\ell^2, \tilde{H}^{1,1}), \mathcal{B}(L_2(\ell^2, \tilde{H}^{1,1}))\big)$.
After the above preliminaries, we introduce probabilistically weak, strong solutions and martingale solutions. Set $F : H^1 \to H^{-1}$ with

$$F(u) := -B(u) + \partial_1^2 u.$$

Definition 1.13 ((Probabilistically) weak solution)**.** *We say that a pair (u, W) is a (probabilistically) weak solution to (1.2) if there exists a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that $u = (u(t))_{t\geq 0}$ is an $(\mathcal{F}_t)$- adapted process and W is an ℓ^2-cylindrical Wiener process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and the following holds:*

(i) $u \in L^\infty_{loc}(\mathbb{R}^+; \tilde{H}^{0,1}) \cap L^2_{loc}(\mathbb{R}^+; \tilde{H}^{1,1}) \cap C(\mathbb{R}^+; H^{-1})$ $\mathbb{P}$-*a.s.* ;

(ii) $\int_0^T \|F(u(s))\|_{H^{-1}} ds + \int_0^T \|\sigma(s, u(s))\|^2_{L_2(\ell^2, H)} ds < +\infty$ $\mathbb{P}$-*a.s., for any* $T > 0$;

(iii) For every $l \in C^1(\mathbb{T}^2)$ *with* $\operatorname{div} l = 0$, $\mathbb{P}$-*a.s.*

$$u(0) = u_0,$$

$$\langle u(t), l\rangle = \langle u_0, l\rangle + \int_0^t \langle -u \cdot \nabla u + \partial_1^2 u, l\rangle \, ds + \int_0^t \langle \sigma(s, u(s)) dW(s), l\rangle.$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality bracket. Note that $\langle u, v\rangle$ and (u, v) coincide when $u, v \in L^2$.

Now we define the (probabilistically) strong solution of (1.2) and we fix a probability space$(\Omega, \mathcal{F}, \mathbb{P})$ and an ℓ^2-cylindrical Wiener process W on it.

Definition 1.14 ((Probabilistically) strong solution)**.** *We say that u is a (probabilistically) strong solution to the equation (1.2) on the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the fixed cylindrical Wiener process W, if it satisfies:*

(i) $u : \Omega\times\mathbb{R}^+ \to L^\infty_{loc}(\mathbb{R}^+; \tilde{H}^{0,1})\cap L^2_{loc}(\mathbb{R}^+; \tilde{H}^{1,1})\cap C(\mathbb{R}^+; H^{-1})$ is adapted to $\hat{\mathcal{F}}_t$, which is the normal filtration generated by W_s. (See [69, Definition 2.1.12] for the definition of normal filtration.)

(ii) u satisfies (i),(ii) and (iii) of Definition 1.13.

Finally we define the martingale solutions.

For $t \geq 0$, define the filtration:

$$\mathcal{F}_t = \sigma\{\pi_r : 0 \leq r \leq t\},$$

where the projection map $\pi_r(x) := x(r)$, for $x \in \bar{\Omega}$, where $\bar{\Omega}$ is defined in Section 1.1.3.

Definition 1.15 ((Global) Martingale solution)**.** *We say that a probability measure $P \in \mathcal{P}(\bar{\Omega})$ ($\mathcal{P}(\bar{\Omega})$ is defined before Definition 1.13) is called a martingale solution of the equation (1.2) with initial value u_0 if*

(M1) $P\big(x \in \bar{\Omega}; x(0) = u_0\big) = 1$, and for any $T > 0$,

$$P(x \in \bar{\Omega}; \sup_{t\in[0,T]} \|\pi_t(x)\|_{\tilde{H}^{0,1}} < +\infty, \int_0^T \|\pi_t(x)\|^2_{\tilde{H}^{1,1}} dt < +\infty\big) = 1.$$

$$P\{x \in \bar{\Omega}; \int_0^T \|F(\pi_t(x))\|_{H^{-1}} dt + \int_0^T \|\sigma(t, \pi_t(x))\|^2_{L_2(\ell^2,H)} dt < +\infty\} = 1;$$

(M2) For every $l \in C^\infty(\mathbb{T}^2)$ with $\operatorname{div} l = 0$, the process

$$M_l(t, \cdot) = \langle \pi_t(\cdot), l\rangle - \int_0^t \langle F\pi_s(\cdot), l\rangle ds,$$

is a continuous square integrable $\mathcal{F}_t-$martingale with respect to P, whose quadratic variation process is $\int_0^t \|\sigma^*(s, u(s))(l)\|^2_{\ell^2} ds$, where $\sigma^*(s, \cdot)$ is the adjoint operator of $\sigma(s, \cdot)$;

(M3) We have for any $T > 0$,

$$\mathbb{E}^P\Big(\sup_{t\in[0,T]} \|\pi_t(\cdot)\|^2_{L^2} + \int_0^T \|\pi_t(\cdot)\|^2_{H^{1,0}} dt\Big) \leq C_T(1 + \|u_0\|^2_{L^2}).$$

Remark 1.16.
By the above definition, we know immediately that if u is a (probabilistically) strong solution with respect to the fixed cylindrical Wiener process W, (u, W) is a (probabilistically) weak solution. Moreover, let P denote the law of u in $\bar{\Omega}$, then P is a martingale solution. Note that by martingale representation theorem, (see for example Theorem 8.2 of [34] and Theorem 3.3.6 of [56]) the existence of martingale solution leads to the existence of (probabilistically) weak solution. The law of the weak solution gives a martingale solution P.

Hypothesis 1.17. *The diffusion coefficient σ is a measurable mapping from $\big(\mathbb{R}^+ \times \tilde{H}^{1,1}, \mathcal{B}(\mathbb{R}^+ \times \tilde{H}^{1,1})\big)$ to $\big(L_2(\ell^2, \tilde{H}^{1,1}), \mathcal{B}(L_2(\ell^2, \tilde{H}^{1,1}))\big)$ such that :*

(i) ***Growth condition***
There exist nonnegative constants K_i', K_i and $\tilde{K}_i$ $(i = 0, 1, 2)$ such that for every $t \geq 0$ and $u \in \tilde{H}^{1,1}$:

$$
\begin{aligned}
\|\sigma(t,u)\|^2_{L_2(\ell^2,H^{-1})} &\leq K_0' + K_1'\|u\|^2_{L^2};\\
\|\sigma(t,u)\|^2_{L_2(\ell^2,H)} &\leq K_0 + K_1\|u\|^2_{L^2} + K_2\|\partial_1 u\|^2_{L^2};\\
\|\sigma(t,u)\|^2_{L_2(\ell^2,H^{0,1})} &\leq \tilde{K}_0 + \tilde{K}_1\|u\|^2_{0,1} + \tilde{K}_2(\|\partial_1 u\|^2_{L^2} + \|\partial_2\partial_1 u\|^2_{L^2}).
\end{aligned}
$$

(ii) ***Lipschitz condition***
There exist constants L_1 and L_2 such that for $t \geq 0$ and $u, v \in \tilde{H}^{1,1}$:

$$
\|\sigma(t,u) - \sigma(t,v)\|^2_{L_2(\ell^2,H)} \leq L_1\|u - v\|^2_{L^2} + L_2\|\partial_1(u - v)\|^2_{L^2}.
$$

Remark 1.18. *A typical example of σ satisfying Hypothesis 1.17 is the following: First we recall the Hölder space $C^{k+\tau}$ (k is an nonnegative integer and $0 \leq \tau < 1$) as: $u \in C^{k+\tau}$ if and only if it has kth continuous derivatives and*

$$
\|u\|_{C^{k+\tau}} := \sum_{|\alpha|\leq k} \|D^\alpha u\| + \sum_{|\alpha|=k} \sup_{x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\tau} < \infty.
$$

For $u \in H^{1,1}$ and $y \in \ell^2$, let

$$
\sigma(t,u)y = \sum_{k=1}^{\infty}(a_k u + c_k\partial_1 u + b_k g \circ u)\langle y, \psi_k\rangle_{\ell^2},
$$

where

1. *ψ_k, is the orthonormal basis of ℓ^2;*

2. *$c_k \in C^\rho$, $\sum_{k=1}^{\infty} \|c_k\|^2_{C^\rho} \leq M_1$ for some $\rho > 2$;*

3. *$a_k, b_k \in L^\infty(\mathbb{T}^2)$, $\partial_2 a_k \in L^\infty(\mathbb{T}^2)$, $\partial_2 b_k \in L^\infty(\mathbb{T}^2)$.*
   $\sum_{k=1}^{\infty}(\|a_k\|_{L^\infty}^2 + \|b_k\|_{L^\infty}^2) \le M_2$, and $\sum_{k=1}^{\infty}(\|\partial_2 a_k\|_{L^\infty}^2 + \|\partial_2 b_k\|_{L^\infty}^2) \le M_2$;

4. *g is a continuous differentiable function from $\mathbb{R}^2$ to $\mathbb{R}^2$ and $\|g\|_{C^1} \le C(g)$. Here C^ρ and C^1 are the Hölder norms.*

5. *Suppose also that $\operatorname{div}(a_k u + c_k \partial_1 u + b_k g \circ u) = 0$ for any $u \in H^{1,1}$.*

Then we have

1.
$$\begin{aligned}\|\sigma(t,u)\|_{L_2(\ell^2,H^{-1})}^2 &= \sum_k \|a_k u + c_k \partial_1 u + b_k g \circ u\|_{H^{-1}}^2 \\ &\lesssim \sum_k \|a_k u\|_{H^{-1}}^2 + \sum_k \|c_k \partial_1 u\|_{H^{-1}}^2 + \sum_k \|b_k g \circ u\|_{H^{-1}}^2.\end{aligned}$$

   Note that

$$\begin{aligned}\|c_k \partial_1 u\|_{H^{-1}} &= \sup_{\|v\|_{H^1} \le 1} |\langle c_k \partial_1 u, v\rangle| \\ &\le \sup_{\|v\|_{H^1} \le 1} [|\langle \partial_1 c_k u, v\rangle| + |\langle \partial_1 c_k u, v\rangle|] \\ &\le \|(\partial_1 c_k) u\|_{L^2} + \|c_k u\|_{L^2} \\ &\le \|c_k\|_{C^\rho}\|u\|_{L^2}.\end{aligned}$$

$$\|a_k u\|_{H^{-1}} \le \|a_k u\|_{L^2} \le \|a_k\|_{L^\infty}\|u\|_{L^2}.$$

$$\begin{aligned}\|b_k g \circ u\|_{H^{-1}} \le \|b_k g \circ u\|_{L^2} &\le \|b_k\|_{L^\infty}\|g \circ u\|_{L^2} \\ &\lesssim \|b_k\|_{L^\infty}\|g \circ u\|_{L^\infty} \\ &\le \|b_k\|_{L^\infty}\|g\|_{L^\infty} \\ &\le \|b_k\|_{L^\infty} C(g),\end{aligned}$$

   where the second inequality is due to Holder's inequality. Therefore,

$$\|\sigma(t,u)\|_{L_2(\ell^2,H^{-1})} \lesssim (\sqrt{M_1} + \sqrt{M_2})\|u\|_{L^2} + \sqrt{M_2}C(g).$$

2. *Similarly,*

$$\begin{aligned}\|\sigma(t,u)\|^2_{L_2(\ell^2,H)} &= \sum_k \|a_k u + c_k\partial_1 u + b_k g\circ u\|^2_{H^{-1}} \\ &\lesssim \sum_k \|a_k u\|^2_{L^2} + \sum_k \|c_k\partial_1 u\|^2_{L^2} + \sum_k \|b_k g\circ u\|^2_{L^2}.\end{aligned}$$

$$\|c_k\partial_1 u\|_{L^2} \le \|c_k\|_{L^\infty}\|\partial_1 u\|_{L^2}.$$

Hence

$$\|\sigma(t,u)\|_{L_2(\ell^2,H)} \lesssim \sqrt{M_1}\|\partial_1 u\|_{L^2} + \sqrt{M_2}\|u\|_{L^2} + \sqrt{M_2}C(g).$$

3.

$$\begin{aligned}\|\sigma(t,u)\|_{L_2(\ell^2,H^{0,1})} \lesssim&\sqrt{M_1}\|\partial_1 u\|_{L^2} + \sqrt{M_2}\|u\|_{L^2} + \sqrt{M_2}C(g) \\ &+ (\sum_{k=1}^\infty \|\partial_2(a_k u)\|^2_{L^2})^{\frac12} + (\sum_{k=1}^\infty \|\partial_2(c_k\partial_1 u)\|^2_{L^2})^{\frac12} + (\sum_{k=1}^\infty \|\partial_2(b_k g\circ u)\|^2_{L^2})^{\frac12} \\ \lesssim&\sqrt{M_1}\|\partial_1 u\|_{L^2} + \sqrt{M_2}\|u\|_{L^2} + \sqrt{M_2}C(g) \\ &+ \sqrt{M_2}(\|u\|_{L^2} + \|\partial_2 u\|_{L^2}) \\ &+ \sqrt{M_1}(\|\partial_1 u\|_{L^2} + \|\partial_1\partial_2 u\|_{L^2}) \\ &+ \sqrt{M_2}C(g)(1 + \|u\|_{L^2} + \|\partial_2 u\|_{L^2}),\end{aligned}$$

where the last inequality is due to the chain rule.

4.

$$\|\sigma(t,u)-\sigma(t,v)\|_{L_2(\ell^2,H)} \lesssim\sqrt{M_2}\|u-v\|_{L^2} + \sqrt{M_1}\|\partial_1(u-v)\|_{L^2} + \sqrt{M_2}C(g)\|u-v\|_{L^2}.$$

1.3.2 Main Theorems for Stochastic Cases

In this section we state two theorems about the well-posedness of equation (1.2), which will be proved in the following sections.

Theorem 1.19. *Assume that $u_0 \in \tilde{H}^{0,1}$ and suppose that σ satisfies Hypothesis 1.17 with $K_2 < \frac{2}{21}$ and $\tilde{K}_2 < \frac15$. Then (1.2) has a global martingale solution on $[0,\infty)$ in the sense of Definition 1.15.*

Theorem 1.20 (Pathwise uniqueness)**.** *Assume that $u_0 \in \tilde{H}^{0,1}$. Suppose that σ satisfies Hypothesis 1.17 with $K_2 < \frac{2}{21}$, $\tilde{K}_2 < \frac15$ and $L_2 < \frac15$. If u, v are two weak solutions in $[0,\infty)$ on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$ and with the same ℓ^2-Wiener process W. Then we have $u = v$ $\mathbb{P}-a.s.$*

Remark 1.21. *By the Yamada–Watanabe theorem, (cf. [69]) the existence of (probabilistically) weak solution in* $[0,\infty)$ *and pathwise uniqueness lead to the existence of the (probabilistically) strong solution on* $[0,\infty)$*.*

1.3.3 Galerkin Approximation and A Priori Estimates

Fix $n \geq 1$ and consider the following stochastic ordinary differential equations on $\mathcal{H}_n$:

$$u_n(0) = P_n u_0,$$

and for $t \geq 0$, $v \in \mathcal{H}_n$:

$$d(u_n(t), v) = (P_n F(u_n(t)), v)dt + (P_n \sigma(t, u_n(t))dW_n(t), v). \tag{1.22}$$

Then for $k = 1, ..., n$ we have for $t \geq 0$:

$$d(u_n(t), e_k) = (P_n F(u_n(t)), e_k)dt + \sum_{j=1}^{n} (P_n \sigma(t, u_n(t))\psi_j, e_k)d\beta_j(t).$$

Now we use [69, Theorem 3.1.1], which is about the existence and uniqueness of solutions to stochastic differential equations. Note that since it is in finite dimensions, there exists some constant $C(n)$ such that $\|v\|_{H^2} \leq C(n)\|v\|_{L^2}$ for $v \in \mathcal{H}_n$.
Let $\varphi, \psi, v \in \mathcal{H}_n$. Integration by parts implies that

$$| \langle \partial_1^2 \varphi - \partial_1^2 \psi, v \rangle | \leq \|\varphi - \psi\|_{H^{1,0}} \|v\|_{H^{1,0}} \leq C(n)^2 \|\varphi - \psi\|_{L^2} \|v\|_{L^2}.$$

Moreover, we have

$$\begin{aligned} | \langle B(\varphi) - B(\psi), v \rangle | &= | -\langle B(\varphi - \psi, v), \varphi \rangle - \langle B(\psi, v), \varphi - \psi \rangle | \\ &\leq C C(n)^3 \|\varphi - \psi\|_{L^2} (\|\varphi\|_{L^2} + \|\psi\|_{L^2}) \|v\|_{L^2}. \end{aligned}$$

Hence we know that for $u, v \in \mathcal{H}_n$, and $\|u\|_{L^2}, \|v\|_{L^2} \leq R$,

$$| \langle F(u) - F(v), u - v \rangle | \leq 2C(R)C(n)^3 \|u - v\|_{L^2}^2,$$

The Hypothesis 1.17 implies that for $u, v \in \mathcal{H}_n$, and $\|u\|_{L^2}, \|v\|_{L^2} \leq R$,

$$\begin{aligned} \|P_n(\sigma(t,u) - \sigma(t,v))\|_{L_2(\ell^2,H)}^2 &\leq \|\sigma(t,u) - \sigma(t,v)\|_{L_2(\ell^2,H)}^2 \\ &\leq L_1 \|u - v\|_{L^2}^2 + L_2 \|\partial_1(u - v)\|_{L^2}^2 \\ &\leq C(n)^2 (L_1 + L_2) \|u - v\|_{L^2}^2. \end{aligned}$$

So it satisfies local weak monotonicity. Moreover, for $u \in \mathcal{H}_n$,

$$\begin{aligned} 2\langle u, P_n F(u) \rangle + \|P_n \sigma(t,u)\|_{L_2(\ell^2,H)}^2 &\leq \|u\|_{H^{1,0}}^2 + \|\sigma\|_{L_2(\ell^2,H)}^2 \\ &\leq C(n)^2 \|u\|_{L^2}^2 + K_0 + K_1 \|u\|_{L^2}^2 + K_2 \|\partial_1 u\|_{L^2}^2 \\ &\leq K_0 + \big(C(n)^2 + K_1 + K_2 C(n)^2\big) \|u\|_{L^2}^2. \end{aligned}$$

Thus it satisfies weak coercivity.
Hence by [69, Theorem 3.1.1], there exists a unique global strong solution $u_n(t)$ to (1.22). Moreover, $u \in C(\mathbb{R}^+, \mathcal{H}_n), \mathbb{P} - a.s.$

1.3.4 The L^2 Energy Estimates

In this section, we give the following a priori estimates.

Lemma 1.22. *For each $T > 0$, we have the following energy estimates under the hypothesis of Thm 1.19:*

$$\mathbb{E}(\sup_{t\in[0,T]} \|u_n(t)\|_{L^2}^2) + \mathbb{E}\int_0^T \|u_n(t)\|_{H^{1,0}}^2 dt \leq C(1 + \|u_0\|_{L^2}^2),$$

where C is a constant depending on K_0, K_1, K_2, T but independent of n.

Proof. Let $u_n(t)$ be the solution to (1.22) described above. By Itô's formula, we have:

$$\begin{aligned}\|u_n(t)\|_{L^2}^2 =& \|P_n u_0\|_{L^2}^2 + 2\int_0^t (\sigma(s, u_n(s))dW_n(s), u_n(s)) \\ &- 2\int_0^t \|\partial_1 u_n(s)\|_{L^2}^2 ds + \int_0^t \|P_n\sigma(s, u_n(s))\Pi_n\|_{L_2(\ell^2,H)}^2 ds.\end{aligned} \tag{1.23}$$

The growth condition implies that

$$\begin{aligned}&\int_0^t \|P_n\sigma(s, u_n(s))\Pi_n\|_{L_2(\ell^2,H)}^2 ds \\ &\leq \int_0^t [K_0 + K_1\|u_n(s)\|_{L^2}^2 + K_2\|\partial_1 u_n(s)\|_{L^2}^2] ds.\end{aligned} \tag{1.24}$$

The Burkholder–Davis–Gundy inequality(see Thm. 3.14 of [34]) and the Young inequality as well as the growth condition imply that:

$$\begin{aligned}&\mathbb{E}\Big(\sup_{s\leq t} \mid 2\int_0^s (P_n\sigma(r, u_n(r))dW_n(r), u_n(r)) \mid\Big) \\ &\leq 6\mathbb{E}\Big\{\int_0^t \|P_n\sigma(r, u_n)(r)\Pi_n\|_{L^2(\ell^2,H)}^2 \|u_n(r)\|_{L^2}^2 dr\Big\}^{\frac{1}{2}} \\ &\leq \beta\mathbb{E}(\sup_{s\leq t} \|u_n(s)\|_{L^2}^2) + \frac{9}{\beta}\mathbb{E}\int_0^t [K_0 + K_1\|u_n(s)\|_{L^2}^2 + K_2\|\partial_1 u_n(t)\|_{L^2}^2] ds.\end{aligned} \tag{1.25}$$

Since $K_2 < \frac{2}{21}$, we can choose $0 < \beta < 1$ such that $(\frac{9}{\beta} + 1)K_2 - 2 < 0$.

By (1.23)-(1.25) and dropping the negative terms, we deduce:

$$(1-\beta)\mathbb{E}\sup_{s\in[0,t]}\|u_n(s)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 + CK_0T + C\mathbb{E}\int_0^t K_1\|u_n(s)\|_{L^2}^2 ds.$$

Gronwall's lemma implies that

$$\mathbb{E}(\sup_{t\in[0,T]}\|u_n(t)\|_{L^2}^2) \leq C, \tag{1.26}$$

where C is a constant depending on K_0, K_1, K_2, T but not n.
Inserting (1.26) back to (1.23)-(1.25) yields

$$\mathbb{E}(\sup_{t\in[0,T]}\|u_n(t)\|_{L^2}^2) + \mathbb{E}\int_0^t \|u_n(t)\|_{H^{1,0}}^2 ds \leq C(1+\|u_0\|_{L^2}^2),$$

where C is a constant depending on K_0, K_1, K_2, T but not n.
This completes the proof. □

However, it is not enough that we only have $L^2(\Omega)$ estimates. We also need an $L^4(\Omega)$ uniform estimates of u_n.

Lemma 1.23. *We have the following uniform estimates for each $T > 0$ under the hypothesis of Thm. 1.19:*

$$\mathbb{E}(\sup_{t\in[0,T]}\|u_n(t)\|_{L^2}^4) + \mathbb{E}\int_0^T \|u_n(t)\|_{L^2}^2\|u_n(t)\|_{H^{1,0}}^2 dt \leq C(1+\|u_0\|_{L^2}^4),$$

where C is a constant depending on K_0, K_1, K_2, T but independent of n.

Proof. Applying once more the Itô's formula to the square of $\|\cdot\|_{L^2}^2$, we obtain:

$$\|u_n(t)\|_{L^2}^4 = \|P_nu_0\|_{L^2}^4 - 4\int_0^t \|\partial_1 u_n(s)\|_{L^2}^2\|u_n(s)\|_{L^2}^2 ds + I_1 + I_2 + I_3, \tag{1.27}$$

where

$$\begin{aligned}
I_1 =& 4\int_0^t (\sigma(s,u_n(s))dW_n(s), u_n(s))\|u_n(s)\|_{L^2}^2 \ , \\
I_2 =& 2\int_0^t \|P_n\sigma(s,u_n(s))\Pi_n\|_{L_2(\ell^2,H)}^2\|u_n(s)\|_{L^2}^2\, ds, \\
I_3 =& 4\int_0^t \|(P_n\sigma(s,u_n(s))\Pi_n)^*(u_n)\|_{l^2}^2 ds.
\end{aligned}$$

The growth condition implies that

$$I_2(t) + I_3(t) \leq 6 \int_0^t (K_0 + K_1 \|u_n(s)\|_{L^2}^2 + K_2 \|\partial_1 u_n(t)\|_{L^2}^2) \|u_n(s)\|_{L^2}^2 ds. \tag{1.28}$$

The Burkholder–Davis–Gundy inequality, the growth condition and the Young inequality imply that:

$$\begin{aligned} \mathbb{E}(\sup_{s\leq t} I_1(s)) \leq & 12\mathbb{E}\Big\{\int_0^t \|\sigma(r, u_n(r))\|_{L_2(\ell^2,H)}^2 \|u_n(r)\|_{L^2}^6 dr\Big\}^{\frac{1}{2}} \\ \leq & \gamma \mathbb{E}(\sup_{s\leq t} \|u_n(s)\|_{L^2}^4) \\ & + \frac{36}{\gamma}\mathbb{E}\int_0^t (K_0 + K_1 \|u_n(s)\|_{L^2}^2 + K_2 \|\partial_1 u_n(t)\|_{L^2}^2) \|u_n(s)\|_{L^2}^2 ds. \end{aligned} \tag{1.29}$$

Since $K_2 < \frac{2}{21}$, we can choose $0 < \gamma < 1$, such that $6K_2 + \frac{36}{\gamma}K_2 - 4 < 0$.
Thus combining (1.27)-(1.29) and dropping some negative terms on the right of the inequality, we have:

$$(1-\gamma)\mathbb{E}(\sup_{t\in[0,T]} \|u_n(t)\|_{L^2}^4) \leq \|u_0\|_{L^2}^4 + \mathbb{E}\int_0^t C_1 \|u_n(s)\|_{L^2}^4 + C_2 \|u_n(s)\|_{L^2}^2 ds.$$

Since we have obtained $\mathbb{E}(\sup_{t\in[0,T]} \|u_n(t)\|_{L^2}^2) \leq C$, the Gronwall inequality yields

$$\mathbb{E}(\sup_{t\in[0,T]} \|u_n(t)\|_{L^2}^4) \leq C,$$

where C is a constant which does not depend on n. Similar as in the proof of Lemma 1.22, we complete the proof. □

1.3.5 Tightness and the Skorokhod Theorem

In this section we use the classical tightness methods. Similar to the deterministic cases, L^2-estimates are not enough to obtain strong convergence. Instead we will introduce a new space $\mathcal{X}_T$ as follows:
Define

$$\mathcal{X}_T := C([0,T]; H^{-1}) \cap L^2([0,T]; H) \cap L_w^2([0,T]; H^{1,1}) \cap L_{w^*}^\infty([0,T]; H^{0,1}),$$

where $L_w^2([0,T]; H^{1,1})$ denotes $L^2([0,T]; H^{1,1})$ with the weak topology and $L_{w^*}^\infty([0,T]; H^{0,1})$ denotes $L^\infty([0,T]; H^{0,1})$ with the weak star topology.

Remark 1.24.

Here the intersection space $\mathcal{X}_T$ takes the following intersection topology: the class of open sets of $\mathcal{X}_T$ are generated by the sets of the form $\mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}_3 \cap \mathcal{O}_4$, where $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ are the open sets in the above four spaces, respectively.

Let $\hat{P}_n^T$ be the law of $\pi_{[0,T]}u_n$ (i.e., the restriction of u_n on $[0,T]$) on $\mathcal{X}_T$.

Lemma 1.25. *Under the hypothesis of Theorem 1.19, for any fixed $T > 0$, the family of measures $\{\hat{P}_n^T, n \in \mathbb{N}\}$ on $\mathcal{X}_T$ is tight.*

Proof of Lemma (1.25)
Firstly, since $\tilde{K}_2 < \frac{1}{5}$, we can choose $\tilde{\alpha}, \tilde{\beta} \in (0,1)$, such that:

$$\tilde{K}_2 + 2\tilde{\alpha} + \frac{9}{\tilde{\beta}}\tilde{K}_2 < 2.$$

From the calculation (1.13) in Lemma 1.8, by the Young inequality, we deduce that:

$$|\ (\partial_2(u \cdot \nabla u) \mid \partial_2 u)\ | \leq \tilde{\alpha}\|\partial_1\partial_2 u\|_{L^2}^2 + C(\tilde{\alpha})(\|\partial_1 u\|_{L^2}^2 + \|\partial_1 u\|_{L^2})\|\partial_2 u\|_{L^2}^2. \tag{1.30}$$

Set

$$A(u,T) = A_1(u,T) + A_2(u,T),$$

where

$$A_1(u,T) := \sup_{0\leq t\leq T} \|u(t)\|_{L^2}^2 + \int_0^T \|u(t)\|_{H^{1,0}}^2 dt + \|u\|_{C^{\frac{1}{8}}([0,T];H^{-1})},$$

and

$$A_2(u,T) := \sup_{0\leq t\leq T} [e^{-2C(\tilde{\alpha})\int_0^t \|\partial_1 u(s)\|_{L^2}^2 + \|\partial_1 u(s)\|_{L^2} ds}\|u(t)\|_{H^{0,1}}^2] + \int_0^T e^{-2C(\tilde{\alpha})\int_0^t \|\partial_1 u(s)\|_{L^2}^2 + \|\partial_1 u(s)\|_{L^2} ds}\|u(t)\|_{H^{1,1}}^2 dt.$$

$$K_R^T := \Big\{ u \in \mathcal{X}_T; A(u,T) \leq R \Big\}.$$

Now we want to show that

(i) For any $R > 0$, K_R^T is relatively compact in $\mathcal{X}_T$;

(ii) For any $\epsilon > 0$, there exists some $R > 0$ (which depends on T), such that $\hat{P}_n^T(K_R^T) > 1 - \epsilon$ for any n.

By definition and Banach-Alaoglu Theorem, K_R^T is bounded hence compact in $L_w^2([0,T];H^{1,1})$ and $L_{w^*}^\infty([0,T];H^{0,1})$.

Note that K_R^T is metrizable with the induced topology of $\mathcal{X}_T$, hence compactness is equivalent to sequentially compactness. By the definition of K_R^T , K_R^T is uniformly bounded in L^2, since L^2 could be embedded compactly in H^{-1} and K_R^T is equicontinuous in $C([0,T];H^{-1})$, the existence of subsequence which converges in $C([0,T];H^{-1})$ can be obtained by Arzela-Ascoli Lemma. Let us denote this subsequence by $\{x_n\}_{n\geq 1}$ as well for simplicity. By the definition of K_R^T, it is obvious that $\{x_n\}_{n\geq 1}$ is bounded in $L^2([0,T];H^{1,1})$.
Moreover, we have:

$$\begin{aligned}\int_0^T \|x_n-x_m\|_{L^2}^2 dt &\leq \int_0^T \|x_n-x_m\|_{H^1}\|x_n-x_m\|_{H^{-1}}dt\\ &\leq \Big(\int_0^T \|x_n-x_m\|_{H^1}^2\, dt\Big)^{\frac{1}{2}}\Big(\int_0^T \|x_n-x_m\|_{H^{-1}}^2\, dt\Big)^{\frac{1}{2}}\\ &\leq C_{R,T}\sup_{t\in[0,T]}\|x_n-x_m\|_{H^{-1}}^2\\ &\to 0,\end{aligned}$$

as $n,m\to\infty$, which finishes the proof of (i).
Proof of (ii):
By Lemma 1.22 as well as Chebyshev's inequality, we can choose $R_1(T)$ large enough such that:

$$P\Big(\sup_{t\in[0,T]}\|u_n(t)\|_{L^2}^2+\int_0^T \|u_n(t)\|_{H^{1,0}}^2+\|u_n(t)\|_{H^{1,0}}\,dt>\frac{R_1(T)}{4}\Big)<\frac{\epsilon}{4}. \tag{1.31}$$

Set for $u\in\mathcal{X}_T$, $h_u(t):=2C(\tilde{\alpha})\int_0^t(\|\partial_1 u\|_{L^2}^2+\|\partial_1 u\|_{L^2})ds$.
Now we need another estimate as following:

$$\begin{aligned}&\mathbb{E}(\sup_{t\in[0,T]}(e^{-h_{u_n}(t)}\|u_n(t)\|_{H^{0,1}}^2))+\mathbb{E}\int_0^T e^{-h_{u_n}(t)}\|u_n(t)\|_{H^{1,1}}^2dt\\ \leq &C(\tilde{K}_0,\tilde{K}_1,\tilde{K}_2,T)(1+\|u_0\|_{H^{0,1}}^2),\end{aligned} \tag{1.32}$$

the proof of which is postponed later to Lemma 1.26.
By (1.32) and Chebyshev's Inequality, we can choose $R_2(T)>R_1(T)$ large enough such that: $\forall n\in\mathbb{N}$,

$$\hat{P}_n^T\Big(u\in\mathcal{X}_T;\sup_{t\in[0,T]}e^{-h_u(t)}\|u(t)\|_{H^{0,1}}^2+\int_0^T e^{-h_u(t)}\|u\|_{H^{1,1}}^2dt>\frac{R_2(T)}{4}\Big)<\frac{\epsilon}{4}. \tag{1.33}$$

Now we fix $R_2(T)$ and set

$$\hat{K}_{R_2(T)}:=\{u\in\mathcal{X}_T;A_1(T)\leq\frac{R_2(T)}{4}\text{ and }A_2(T)\leq\frac{R_2(T)}{4}\}.$$

Then we know $\hat{P}_n^T(\bar{\Omega} \setminus \hat{K}_{R_2(T)}) < \frac{\epsilon}{2}$. Now we only consider $u \in \hat{K}_{R_2(T)}$. By Hölder's inequality, we have:

$$\begin{aligned} &\sup_{s\neq t\in[0,T]} \Big(\frac{\| \int_s^t -\partial_1^2 u(r) + \operatorname{div}(u\otimes u)dr\|_{H^{-1}}^2}{\mid t-s\mid} \Big) 1_{u\in \hat{K}_{R_2(T)}} \\ &\leq \int_0^T \| -\partial_1^2 u(r) + \operatorname{div}(u\otimes u)\|_{H^{-1}}^2 \, dr 1_{u\in\hat{K}_{R_2(T)}}. \end{aligned} \tag{1.34}$$

The boundedness of u in $L^2([0,T];H^{1,1})$ leads to the boundedness of $\partial_1^2 u$ in $L^2([0,T];H^{-1})$. By the definition of $\hat{K}_{R_2(T)}$, u is also bounded in $L^\infty([0,T];H^{0,1})$. By interpolation, u is bounded in $L^4([0,T];H^{\frac{1}{2}})$. By Sobolev imbedding, u is bounded in $L^4([0,T];L^4)$. Thus we obtain for $u \in \hat{K}_{R_2(T)}$,

$$\int_0^T \| -\partial_1^2 u(r) + \operatorname{div}(u\otimes u)\|_{H^{-1}}^2 \, dr \leq C, \tag{1.35}$$

where C is a constant independent of n.
By (1.34) and (1.35), we have for $u \in \hat{K}_{R_2(T)}$,

$$\sup_{s\neq t\in[0,T]} \Big(\frac{\| \int_s^t -\partial_1^2 u(r) + \operatorname{div}(u\otimes u)dr\|_{H^{-1}}^2}{\mid t-s\mid} \Big) \leq C. \tag{1.36}$$

Moreover, for any $T \geq t > s \geq 0$ and any $p \in \mathbb{N}$ we have

$$\begin{aligned} \mathbb{E}^{\hat{P}_n^T} \| \int_s^t P_n\sigma(r,\pi_r(\cdot))dW_n(r)\|_{H^{-1}}^{2p} &\leq C_p \mathbb{E}^{\hat{P}_n^T} \Big(\int_s^t \|\sigma(r,\pi_r(\cdot))\|_{L_2(\ell^2,H^{-1})}^2 \, dr\Big)^p \\ &\leq C_p \mid t-s\mid^{p-1} \int_s^t \mathbb{E}^{\hat{P}_n^T} \|\sigma(r,\pi_r(\cdot))\|_{L_2(\ell^2,H^{-1})}^{2p} dr \\ &\leq C_p \mid t-s\mid^{p-1} \int_s^t \mathbb{E}^{\hat{P}_n^T} (\|\pi_r(\cdot)\|_{L^2}^{2p} + 1) dr \\ &\leq C_{p,T} \mid t-s\mid^p \big(1 + \mathbb{E}(\sup_{t\in[0,T]} \|u_n(t)\|_{L^2}^{2p})\big). \end{aligned}$$

By Kolmogorov's criterion, for any $\alpha \in (0, \frac{p-1}{2p})$, we have:

$$\mathbb{E}^{\hat{P}_n^T} \Big(\sup_{s\neq t\in[0,T]} \frac{\| \int_s^t P_n\sigma(r,\pi_r(\cdot))dW_n(r)\|_{H^{-1}}^{2p}}{\mid t-s\mid^{p\alpha}} \Big) \leq C_{p,T}\big(1 + \mathbb{E}(\sup_{t\in[0,T]} \|u_n(t)\|_{L^2}^{2p})\big). \tag{1.37}$$

Choose $p = 2$. By (1.36) and (1.37) , we get for $\alpha = \frac{1}{8}$:

$$\sup_n \mathbb{E}^{\hat{P}_n^T} \Big(\sup_{s\neq t\in[0,T]} \frac{\|\pi_t(\cdot) - \pi_s(\cdot)\|_{H^{-1}}}{\mid t-s\mid^\alpha} 1_{u\in\hat{K}_{R_2(T)}} \Big) \leq C.$$

By Chebyshev's inequality, we choose $R_3(T) > R_2(T)$ large enough and obtain:

$$\hat{P}_n^T(u \in \mathcal{X}_T; \|u\|_{C^{\frac{1}{8}}([0,T];H^{-1})} > \frac{R_3(T)}{4} \text{ and } u \in \hat{K}_{R_2(T)}) < \frac{\epsilon}{4}. \tag{1.38}$$

Combining (1.31),(1.33) with (1.38) we complete the proof when we let R be any constant larger than $R_3(T)$. □

Lemma 1.26. *Under the hypothesis of Thm. 1.19 the uniform estimates* (1.32) *holds.*

Proof. Using again the Itô's Formula to $e^{-h_{u_n(t)}}\|u_n(t)\|^2_{H^{0,1}}$, we obtain:

$$\begin{aligned} e^{-h_{u_n}(t)}\|u_n(t)\|^2_{H^{0,1}} = \|P_n u_0\|^2_{H^{0,1}} + \sum_{j=1}^{3} T_j(t) - \int_0^t e^{-h_{u_n}(s)} h'_{u_n}(s)\|u_n(s)\|^2_{H^{0,1}} ds \\ + \int_0^t e^{-h_{u_n}(s)}[-2\|\partial_1 u_n(s)\|^2_{L^2} - 2\|\partial_1\partial_2 u_n(s)\|^2_{L^2}]ds, \end{aligned} \tag{1.39}$$

where

$$\begin{aligned} T_1(t) =& -2\int_0^t e^{-h_{u_n}(s)}(\partial_2(u_n \cdot \nabla u_n), \partial_2 u_n(s)), \\ T_2(t) =& 2\int_0^t e^{-h_{u_n}(s)}(\sigma(s, u_n(s))dW_n(s), u_n(s))_{H^{0,1}}, \\ T_3(t) =& \int_0^t e^{-h_{u_n}(s)}\|P_n\sigma(s, u_n(s))\Pi_n\|^2_{L_2(\ell^2, H^{0,1})}\, ds. \end{aligned}$$

The growth condition implies that

$$T_3(t) \le \int_0^t e^{-h_{u_n}(s)}\big[\tilde{K_0} + \tilde{K_1}\|u_n(s)\|^2_{H^{0,1}} + \tilde{K_2}(\|\partial_1 u_n(s)\|^2_{L^2} + \|\partial_1\partial_2 u_n(s)\|^2_{L^2})\big]\, ds. \tag{1.40}$$

For $T_1(t)$, we use (1.30) to have $\tilde{\alpha}$ as in Lemma 1.25.

$$| T_1(t) | \le \int_0^t e^{-h_{u_n}(s)}\big[2\tilde{\alpha}\|\partial_1\partial_2 u_n\|^2_{L^2} + 2C(\tilde{\alpha})(\|\partial_1 u_n\|^2_{L^2} + \|\partial_1 u_n\|_{L^2})\|\partial_2 u_n\|^2_{L^2}\big]\, ds. \tag{1.41}$$

Similar to (1.29), we have

$$\begin{aligned} &\mathbb{E}\Big(\sup_{s\le t} | 2\int_0^s e^{-h_{u_n}(r)}\big(\sigma(r, u_n(r))dW_n(r), u_n(r)\big)_{H^{0,1}} |\Big) \\ &\le 4\mathbb{E}\Big\{\int_0^t e^{-h_{u_n}(r)}\|P_n\sigma(r, u_n(r))\Pi_n\|^2_{L^2(\ell^2, H^{0,1})} e^{-h_{u_n}(r)}\|u_n(r)\|^2_{H^{0,1}} dr\Big\}^{\frac{1}{2}} \\ &\le \tilde{\beta}\mathbb{E}\big(\sup_{s\le t}(e^{-h_{u_n}(s)}\|u_n(s)\|^2_{H^{0,1}})\big) \\ &+ \frac{4}{\tilde{\beta}}\mathbb{E}\int_0^t e^{-h_{u_n}(s)}\big[\tilde{K_0} + \tilde{K_1}\|u_n(s)\|^2_{H^{0,1}} + \tilde{K_2}(\|\partial_1 u_n(s)\|^2_{L^2} + \|\partial_1\partial_2 u_n(s)\|^2_{L^2})\big]\, ds. \end{aligned} \tag{1.42}$$

Combining (1.39)-(1.42) and dropping the negative terms, we have:

$$\begin{aligned}&(1-\tilde{\beta})\mathbb{E}(\sup_{t\in[0,T]} e^{-h_{u_n}(t)}\|u_n(t)\|^2_{H^{0,1}})\\ &\leq \mathbb{E}\|P_n u_0\|^2_{H^{0,1}} + \mathbb{E}\int_0^T e^{-h_{u_n}(s)}(1+\frac{9}{\tilde{\beta}})(\tilde{K_0}+\tilde{K_1}\|u_n(s)\|^2_{H^{0,1}})ds.\end{aligned}$$

By Gronwall's inequality,

$$\mathbb{E}(\sup_{t\in[0,T]}(e^{-h_{u_n}(t)}\|u_n(t)\|^2_{H^{0,1}})) \leq C(T,\tilde{K_0},\tilde{K_1},\tilde{K_2})\|u_0\|^2_{H^{0,1}} \tag{1.43}$$

Combining (1.39)-(1.42) again with the estimate (1.43) we obtain:

$$\begin{aligned}\mathbb{E}(\sup_{t\in[0,T]}(e^{-h_{u_n}(t)}\|u_n(t)\|^2_{H^{0,1}}))+\mathbb{E}\int_0^T e^{-h_{u_n}(t)}\|u_n(t)\|^2_{H^{1,1}}dt\\ \leq C(\tilde{K_0},\tilde{K_1},\tilde{K_2},T)(1+\|u_0\|^2_{H^{0,1}}).\end{aligned}$$

□

The classical Skorokhod Theorem can only be used in metric space. We will use the following Jakubowski's version of the Skorokhod Theorem in the form given by Brzeźniak and Ondreját [11] Thm. A.1 and it was proved by A. Jakubowski in [55].

Theorem 1.27. *Let $\mathcal{Y}$ be a topological space such that there exists a sequence f_m of continuous functions $f_m:\mathcal{Y}\to\mathbb{R}$ that separates points of $\mathcal{Y}$. Let us denote by $\mathcal{S}$ the σ-algebra generated by the maps f_m. Then*

- *(j1) every compact subset of $\mathcal{Y}$ is metrizable;*
- *(j2) if (μ_m) is tight sequence of probability measures on $(\mathcal{Y},\mathcal{S})$, then there exists a subsequence (m_k), a probability space $(\Omega,\mathcal{F},\mathbb{P})$ (which, by [55], can be chosen to be $([0,1],\mathscr{B}([0,1]),\mathcal{L}))$ with $\mathcal{Y}$-valued Borel measurable variables ξ_k, ξ such that μ_{m_k} is the law of ξ_k and ξ_k converges to ξ almost surely on Ω. Moreover, the law of ξ is a Radon measure.*

Now we check the $\mathcal{X}_T$ defined in Lemma 1.25 satisfies the above condition. It is sufficient to prove that on each space appearing in the definition of $\mathcal{X}_T$ there exists a countable set of continuous real-valued functions separating points:
Since $C([0,T];H^{-1})$ and $L^2([0,T];H)$ are separable Banach spaces, it is easy to see the condition in Theorem 1.27 is satisfied.
For the space $L^2_w([0,T];H^{1,1})$ it is sufficient to put

$$f_m(u) := \int_0^T (u(t),v_m(t))_{H^{1,1}}dt \in \mathbb{R}, u\in L^2_w([0,T];H^{1,1}), m\in\mathbb{N},$$

where $\{v_m\}_{m\geq 1}$ is a dense subset of $L^2([0,T];H^{1,1})$. Since $\{v_m\}_{m\geq 1}$ is dense in $L^2([0,T];H^{1,1})$, it separates points of $L^2([0,T];H^{1,1})$.

Similarly for the space $L^\infty_{w^*}([0,T];H^{0,1})$ it is sufficient to put

$$\tilde{f}_m(u) := \int_0^T (u(t),\tilde{v}_m(t))_{H^{0,1}}dt \in \mathbb{R}, u \in L^\infty_{w^*}([0,T];H^{0,1}), m \in \mathbb{N},$$

where $\{\tilde{v}_m\}_{m\geq 1}$ is a dense subset of $L^1([0,T];H^{0,1})$. Since $\{\tilde{v}_m\}_{m\geq 1}$ is dense in $L^1([0,T];H^{0,1})$, it separates points of $L^\infty([0,T];H^{0,1})$.

Now all the conditions of the above Skorokhod theorem are satisfied.

Moreover, the σ- algebra generated by the sequence of the above continuous functions which separate the points in $\mathcal{X}_T$ is exactly $\mathscr{B}(\mathcal{X}_T)$, the proof of which we put in the appendix. (See Corollary B.9).

By Theorem 1.27, there exists a subsequence $\{\hat{P}^T_{n_k^T}\}_{k\geq 1}$ as well as $\mathcal{X}_T-$valued random variables $\tilde{u}^T_{n_k}$ in the space $([0,1],\mathscr{B}([0,1]),\mathcal{L})$, such that

(i) $\tilde{u}^T_{n_k^T}$ has the law $\hat{P}^T_{n_k^T}$;

(ii) $\hat{P}^T_{n_k^T}$ converges weakly to some $\hat{P}^T$;

(iii) $\tilde{u}^T_{n_k^T} \to \tilde{u}^T$ in $\mathcal{X}_T$ $\mathcal{L}-a.s.$ and $\tilde{u}^T$ has the law $\hat{P}^T \in \mathcal{P}(\Omega^T)$.

Selection by the Diagonal Method: Without loss of generality we can assume that n_k^{N+1} is a further subsequence of n_k^N for any positive integer N. By a diagonal method we select the subsequence $m_k := n_k^k$. Hence $\hat{P}^N_{m_k}$ converges weakly to $\hat{P}^N$ when k goes to infinity.

To sum up, we obtain the following lemma:

Lemma 1.28. *There exists a subsequence $\{m_k\}_{k\geq 1}$ such that*

- *For any positive integer N, $\hat{P}^N_{m_k}$ converges weakly to $\hat{P}^N$ when k goes to infinity.*
- *For any positive integer N, there exists a sequence of $\mathcal{X}_N-$valued random variables $\{\tilde{u}^N_{m_k}\}_{k\geq 1}$ in the space $([0,1],\mathscr{B}([0,1]),\mathcal{L})$, such that the law of $\tilde{u}^N_{m_k}$ is $\hat{P}^N_{m_k}$. Moreover, $\tilde{u}^N_{m_k}$ converges to $\tilde{u}^N$ $\mathcal{L}-a.s.$ and $\tilde{u}^N$ has the law $\hat{P}^N$.*

Definition 1.29 (Probability measure $\hat{P}$ on $\bar{\Omega}$)**.** *Define the probability measure $\hat{P}$ on $\bar{\Omega}$ as follows:*
For any positive interger N, let $\hat{P}(\pi^{-1}_{[0,N]}B) := \hat{P}^N(B)$ for $B \in \mathscr{B}(\Omega^N) \subset \mathscr{B}(\mathcal{X}_N)$.

Remark 1.30.

We need to show it is well defined. For any $N_1 < N_2$*, note that* $\pi_{N_2,N_1}\pi_{[0,N_2]} = \pi_{[0,N_1]}$.

Let m_k *be the sequence defined by Lemma 1.28 with* $m_k \geq k$*. Then* $\hat{P}^{N_2}_{m_k}$ *converges weakly to* $\hat{P}^{N_2}$*, hence* $\hat{P}^{N_2}_{m_k} \circ \pi^{-1}_{N_2,N_1}$ *converges weakly to* $\hat{P}^{N_2} \circ \pi^{-1}_{N_2,N_1}$.

On the other hand, since $\hat{P}^{N_2}_{m_k} \circ \pi^{-1}_{N_2,N_1} = \hat{P}^{N_1}_{m_k}$*, it converges also to* $\hat{P}^{N_1}$*. Therefore,* $\hat{P}^{N_1} = \hat{P}^{N_2} \circ \pi^{-1}_{N_2,N_1}$.

It is obvious that $\{\pi^{-1}_{[0,N]}B\}_{N\in\mathbb{N},B\in\mathscr{B}(\Omega^N)}$ *is an algebra on which* $\hat{P}$ *is* σ*-additive. By Carathéodory's extension theorem, there exists a unique extension of* $\hat{P}$ *to* $\mathscr{B}(\bar{\Omega})$.

Remark 1.31.

1. *Since for any positive integer* N*,* $\hat{P}^N$ *and* $\hat{P}\circ\pi^{-1}_{[0,N]}$ *coincide in* $\mathscr{B}(\Omega^N)$*. By Theorem B.8 and Theorem B.3 we know that* $\mathscr{B}(\mathcal{X}_N) = \mathscr{B}(\Omega^N) \cap \mathcal{X}_N$*. Hence* $\hat{P}\circ\pi^{-1}_{[0,N]}$ *is also a probability measure on* $\big(\mathcal{X}_N, \mathscr{B}(\mathcal{X}_N)\big)$ *(i.e.* $\hat{P}\circ\pi^{-1}_{[0,N]}$ *is supported in* $\mathcal{X}_N$*) and* $\hat{P}\circ\pi^{-1}_{[0,N]} = \hat{P}^N$ *on* $\big(\mathcal{X}_N, \mathscr{B}(\mathcal{X}_N)\big)$ *.) Therefore, we immediately have:*

$$\mathbb{E}^{\hat{P}}\big(\sup_{t\in[0,N]}\|\pi_t(\cdot)\|^2_{L^2}\big) + \mathbb{E}^{\hat{P}}\int_0^N \|\pi_t(\cdot)\|^2_{H^{1,0}}dt \leq C(N)(1+\|u_0\|^2_{L^2}).$$

Since the above inequality holds for any positive integer N*, we immediately have for any* $T > 0$,

$$\mathbb{E}^{\hat{P}}\big(\sup_{t\in[0,T]}\|\pi_t(\cdot)\|^2_{L^2}\big) + \mathbb{E}^{\hat{P}}\int_0^T \|\pi_t(\cdot)\|^2_{H^{1,0}}dt \leq C(T)(1+\|u_0\|^2_{L^2}). \tag{1.44}$$

Similarly for $L^4(\Omega)$ *estimates, we have:*

$$\mathbb{E}^{\hat{P}}\big(\sup_{t\in[0,T]}\|\pi_t(\cdot)\|^4_{L^2}\big) + \mathbb{E}^{\hat{P}}\int_0^T \|\pi_t(\cdot)\|^2_{L^2}\|\pi_t(\cdot)\|^2_{H^{1,0}}dt \leq C(T)(1+\|u_0\|^4_{L^2}). \tag{1.45}$$

1.3.6 Passing to the Limit and the Proof of Main Theorems

In this section we pass the limit as $n \to \infty$. We use a similar method as in [47].

Proof of Theorem 1.19.
Let us prove $\hat{P}$ satisfies (M1),(M2) and (M3) of Definition 1.15 .
(M3) is satisfied by (1.44).
For (M1), first we choose some positive integer $N > T$. Since $\hat{P}\circ\pi^{-1}_{[0,N]}$ is supported in $\mathcal{X}_N$, by the definition of $\mathcal{X}_N$ we immediately obtain

$$\hat{P}\Big(x\in\bar{\Omega};\ \sup_{t\in[0,T]}\|\pi_t(x)\|_{\tilde{H}^{0,1}} < +\infty, \int_0^T \|\pi_t(x)\|^2_{H^{1,1}}dt < +\infty\Big) = 1.$$

Recall that $F(u) := -B(u) + \partial_1^2 u$, hence

$$\int_0^T \|F(x(t))\|_{H^{-1}} dt \leq \int_0^T \|x(t) \otimes x(t)\|_{L^2} dt + \int_0^T \|\partial_1^2 x(t)\|_{H^{-1}} dt$$
$$\lesssim \int_0^T \|x(t)\|^2_{H^{\frac{1}{2}}} dt + \int_0^T \|x(t)\|_{H^1} dt$$
$$\leq C(T) \int_0^T \|x(t)\|^2_{H^1} dt,$$

where the second inequality is due to the Sobolev embedding $H^{\frac{1}{2}}$ into L^4 and the last inequality is due to the Holder's inequality. Together with the growth condition of Hypothesis 1.17 and the fact that $\hat{P} \circ \pi^{-1}_{[0,N]}$ is supported in $\mathcal{X}_N$, we obtain

$$\hat{P}\{x \in \bar{\Omega}; \int_0^T \|F(\pi_t(x))\|_{H^{-1}} dt + \int_0^T \|\sigma(t, \pi_t(x))\|^2_{L_2(\ell^2, H)} dt < +\infty\} = 1.$$

Moreover, noting that $\tilde{u}^N_{m_k}(0) = u_{m_k}(0) = P_{m_k} u_0$, we have:

$$\hat{P}\{x \in \bar{\Omega}; x(0) = u_0\} = \hat{P}^N(x \in \Omega^N; x(0) = u_0) = \lim_{k \to \infty} \mathcal{L}(\tilde{u}^N_{m_k}(0) = P_{m_k} u_0) = 1,$$

which finishes the proof of (M1).
For (M2), note that since $\hat{P} \circ \pi^{-1}_{[0,N]} = \hat{P}^N$, it is equivalent to show that for any positive integer N, $\hat{P}^N$ satisfies the following:

(M2') For every $l \in C^\infty(\mathbb{T}^2)$ with $\operatorname{div} l = 0$, and $0 \leq t \leq N$, the process

$$M_l(t, \cdot) = \langle \pi_t(\cdot), l \rangle - \int_0^t \langle F\pi_s(\cdot), l \rangle ds,$$

is a continuous square integrable $\mathcal{F}_t - martingale$ with respect to $\hat{P}^N$, whose quadratic variation process is $\int_0^t \|\sigma^*(s, \pi_s(\cdot))(l)\|^2_{\ell^2} ds$, where the asterisk denotes the adjoint operator.

Now we prove (M2'):
Fix $l \in C^\infty(\mathbb{T}^2)$ with $\operatorname{div} l = 0$. Set

$$M_l^{(m_k)}(t, u) := \langle u(t), l \rangle - \int_0^t \langle F(u(s), P_{m_k} l \rangle ds.$$

Since $\tilde{u}^N_{m_k} \to \tilde{u}^N$ in $C([0, N]; H^{-1})$ as k goes to infinity , we have for $\mathcal{L} - a.s.$

$$\|\langle \tilde{u}^N_{m_k}(t), l \rangle - \langle \tilde{u}^N(t), l \rangle\|_{L^\infty([0,N])} \to 0$$

Moreover, since $\tilde{u}^N_{m_k}$ is bounded in $L^4([0,1];L^\infty([0,N];L^2))$, $\langle \tilde{u}^N_{m_k}(t), l\rangle$ is bounded in $L^4([0,1])$ for any t, (note that the notation $[0,1]$ here and later is the probability space $([0,1],\mathscr{B}([0,1]),\mathcal{L})$ that $\tilde{u}^N_{m_k}$ is defined to be on when we apply Skorokhod Theorem.) we obtain

$$\lim_{k\to\infty}\mathbb{E}^{\mathcal{L}} \mid \langle \tilde{u}^N_{m_k}(t), l\rangle - \langle \tilde{u}^N(t), l\rangle \mid = 0.$$

Since

$$\tilde{u}^N_{m_k} \to \tilde{u}^N \text{ in } L^2([0,N];H)\ \mathcal{L}-a.s.,$$

we have

$$\int_0^t \langle F(\tilde{u}^N_{m_k}(s)), P_{m_k} l\rangle\, ds \to \int_0^t \langle F(\tilde{u}^N(s)), l\rangle\, ds \quad \mathcal{L}-a.s.$$

Moreover, $\{\tilde{u}^N_{m_k}\}_{k\geq 1}$ is bounded in $L^4([0,1];L^\infty([0,N];H))\cap L^2([0,1];L^2([0,N];H^1))$.

Therefore, we have

$$\sup_k \mathbb{E}^{\mathcal{L}}\Big(\int_0^N \mid \langle F(\tilde{u}^N_{m_k}(s)), P_{m_k} l\rangle \mid ds\Big)^2 \leq C.$$

Therefore, we have

$$\lim_{k\to\infty}\mathbb{E}^{\mathcal{L}} \mid \langle F(\tilde{u}^N_{m_k}(s)), P_{m_k} l\rangle - \langle F(\tilde{u}^N(s)), l\rangle \mid = 0.$$

By the definition of M_l in (M2'), we have

$$\lim_{k\to\infty}\mathbb{E}^{\mathcal{L}} \mid M_l^{(m_k)}(t, \tilde{u}^N_{m_k}) - M_l(t, \tilde{u}^N) \mid = 0. \tag{1.46}$$

Let $t > s$ and g be any bounded and real-valued $\mathcal{F}_s$-measurable continuous function on $\mathcal{X}_N$. Using (1.46) we have:

$$\begin{aligned}\mathbb{E}^{\hat{P}^N}\Big((M_l(t,\cdot) - M_l(s,\cdot))g(\cdot)\Big) &= \mathbb{E}^{\mathcal{L}}\Big(\big(M_l(t,\tilde{u}^N) - M_l(s,\tilde{u}^N)\big)g(\tilde{u}^N)\Big)\\ &= \lim_{k\to\infty}\mathbb{E}^{\mathcal{L}}\Big(\big(M_l^{(n)}(t,\tilde{u}^N_{m_k}) - M_l^{(n)}(s,\tilde{u}^N_{m_k})\big)g(\tilde{u}^N_{m_k})\Big)\\ &= 0,\end{aligned}$$

where the last step is due to (M2) for $\hat{P}^N_{m_k}$ since $\hat{P}^N_{m_k}$ is a martingale solution to (1.22). Then we have

$$\mathbb{E}^{\hat{P}^N}(M_l(t,u) \mid \mathcal{F}_s) = M_l(s,u). \tag{1.47}$$

On the other hand, by Burkholder–Davis–Gundy's inequality, growth condition of σ, Lemma 1.22 and Lemma 1.23 we have:

$$\begin{aligned}\sup_k \mathbb{E}^{\mathcal{L}} \mid M_l^{(m_k)}(t,\tilde{u}^N_{m_k}) \mid^4 &\le C\sup_k \mathbb{E}^{\mathcal{L}}(\int_0^t \|\sigma^*(\tilde{u}^N_{m_k}(s))(P_{m_k}l)\|^2_{\ell^2}\, ds)^2\\ &\le C\sup_k \int_0^t \mathbb{E}^{\mathcal{L}}(\|\sigma^*(\tilde{u}^N_{m_k}(s))(P_{m_k}l)\|^4_{\ell^2})\, ds\\ &\le C\sup_k \mathbb{E}^{\mathcal{L}}\int_0^t \|\sigma^*(s,\tilde{u}^N_{m_k})\|^4_{L_2(H^1,\ell^2)}ds\|l\|^4_{H^1}\\ &\le C\sup_k \mathbb{E}^{\mathcal{L}}\int_0^t \|\sigma(s,\tilde{u}^N_{m_k})\|^4_{L_2(\ell^2,H^{-1})}ds\|l\|^4_{H^1}\\ &\le C\sup_k \mathbb{E}^{\mathcal{L}}\int_0^t (K_0'+K_1'\|\tilde{u}^N_{m_k}\|^2_{L^2})^2 ds\\ &< +\infty,\end{aligned}$$

where the third inequality is due to the reason that the normal norm of the operator is smaller than the Hilbert–Schmidt norm of the operator and the fact that P_n is also a projection on $\tilde{H}^1$, the fourth inequality is a result of $\|\sigma\|_{L_2(\ell^2,H^{-1})} = \|\sigma^*\|_{L_2(H^1,\ell^2)}$.
Then by (1.46) we obtain

$$\lim_{k\to\infty} \mathbb{E}^{\mathcal{L}} \mid M_l^{(m_k)}(t,\tilde{u}^N_{m_k}) - M_l(t,\tilde{u}^N) \mid^2 = 0.$$

On the other hand, by Lipchitz condition of σ,

$$\lim_{k\to\infty} \mathbb{E}^{\mathcal{L}}\int_0^t \|\sigma^*(s,\tilde{u}^N_{m_k}(s))(l) - \sigma^*(s,\tilde{u}^N(s))(l)\|^2_{\ell^2} ds = 0.$$

Thus, using the same method used for proving $\mathbb{E}^{\hat{P}^N}(M_l(t,u)\mid \mathcal{F}_s) = M_l(s,u)$, we obtain

$$\mathbb{E}^{\hat{P}^N}(M_l^2(t,\cdot) - \int_0^t \|\sigma^*(s,\pi_s(\cdot))(l)\|^2_{l^2}ds \mid \mathcal{F}_s) = M_l^2(s,\cdot) - \int_0^s \|\sigma^*(r,\pi_r(\cdot))(l)\|^2_{l^2}dr.$$

(M2) holds.
The results follow. □

Finally let us turn to the proof of the pathwise uniqueness.

Proof of Theorem 1.20. Set

$$\tilde{w} := u - v.$$

Then we have

$$\langle u(t), e_k\rangle = \langle u(0), e_k\rangle + \int_0^t \langle -u\cdot\nabla u + \partial_1^2 u, e_k\rangle ds + \int_0^t \langle \sigma(s,u(s))dW(s), e_k\rangle, \qquad (1.48)$$

and

$$\langle v(t), e_k\rangle = \langle v(0), e_k\rangle + \int_0^t \langle -v\cdot\nabla v + \partial_1^2 v, e_k\rangle ds + \int_0^t \langle \sigma(s, v(s))dW(s), e_k\rangle. \tag{1.49}$$

(1.48)-(1.49) ensures that

$$\langle \tilde{w}(t), e_k\rangle = \int_0^t \langle -\tilde{w}\cdot\nabla u + v\cdot\nabla\tilde{w} + \partial_1^2\tilde{w}, e_k\rangle ds + \int_0^t \langle \sigma(s,u(s)) - \sigma(s,v(s))dW(s), e_k\rangle. \tag{1.50}$$

with $\tilde{w}(0) = 0$.
Set $\varphi_k(s) := \langle \tilde{w}(s), e_k\rangle$. Itô's formula and (1.50) yield:

$$\begin{aligned} d\varphi_k^2 &= 2\varphi_k d\varphi_k + \|(\sigma(s,u(s)) - \sigma(s,v(s)))^* e_k\|_{l^2}^2 ds \\ &= 2\langle \tilde{w}(s), e_k\rangle\langle -\tilde{w}\cdot\nabla u + v\cdot\nabla\tilde{w} + \partial_1^2\tilde{w}, e_k\rangle ds \\ &+ 2\langle \tilde{w}(s), e_k\rangle\langle \big(\sigma(s,u(s)) - \sigma(s,v(s))\big)dW(s), e_k\rangle \\ &+ \|\big(\sigma(s,u(s)) - \sigma(s,v(s))\big)^* e_k\|_{l^2}^2 ds. \end{aligned}$$

From the similar calculation of (1.21), we know that for any $\hat{\alpha} > 0$, there exists $C(\hat{\alpha})$ such that for any $s > 0$,

$$\begin{aligned} &|\ (\tilde{w}(s)\cdot\nabla u(s)\ |\ \tilde{w}(s))\ | \\ &\le \hat{\alpha}\|\partial_1\tilde{w}(s)\|_{L^2}^2 + C(\hat{\alpha})r(u,s)\|\tilde{w}(s)\|_{L^2}^2. \end{aligned} \tag{1.51}$$

Set $q(t) := \int_0^t 2C(\hat{\alpha})r(u,s)ds$.
By Itô's formula:

$$\begin{aligned} e^{-q(t)}\varphi_k^2(t) =& 2\int_0^t e^{-q(s)}\langle \tilde{w}(s), e_k\rangle\langle -\tilde{w}\cdot\nabla u + v\cdot\nabla\tilde{w} + \partial_1^2\tilde{w}, e_k\rangle ds \\ &+ 2\int_0^t e^{-q(s)}\langle \tilde{w}(s), e_k\rangle\langle \big(\sigma(s,u(s)) - \sigma(s,v(s))\big)dW(s), e_k\rangle \\ &+ \int_0^t e^{-q(s)}\|(\sigma(s,u(s)) - \sigma(s,v(s)))^* e_k\|_{l^2}^2 ds - \int_0^t q'(s)e^{-q(s)}\varphi_k^2 ds. \end{aligned} \tag{1.52}$$

Note that

$$\sum_{k=1}^{\infty}\varphi_k^2(t) = \|\tilde{w}(t)\|_{L^2}^2. \tag{1.53}$$

The dominated convergence theorem imply when $N\to\infty$:

$$\begin{aligned} &2\ |\sum_{k\le N}\int_0^t e^{-q(s)}\langle \tilde{w}(s), e_k\rangle\langle -\tilde{w}\cdot\nabla u, e_k\rangle ds\ | \\ &\longrightarrow 2\ |\int_0^t e^{-q(s)}\langle \tilde{w}\cdot\nabla\tilde{w}, u\rangle ds\ |, \end{aligned}$$

since by (1.51), and

$$\langle \tilde{w}\cdot\nabla\tilde{w}, u\rangle = -\langle \tilde{w}\cdot\nabla u, \tilde{w}\rangle,$$

$$\begin{aligned}
&2\int_0^t e^{-q(s)}\|\tilde{w}\cdot\nabla u\|_{L^2}\|\tilde{w}\|_{L^2}ds \\
&\leq \int_0^t 2\hat{\alpha}e^{-q(s)}\|\partial_1\tilde{w}\|_{L^2}^2 + 2C(\hat{\alpha})e^{-q(s)}r(u)(s)ds \\
&= \int_0^t [2\hat{\alpha}e^{-q(s)}\|\partial_1\tilde{w}\|_{L^2}^2 + e^{-q(s)}q'(s)\|\tilde{w}\|_{L^2}^2]ds.
\end{aligned} \tag{1.54}$$

Note that

$$\|(\sum_{k\leq N}\langle \tilde{w}(s), e_k\rangle e_k)\|_{L^2} \leq \|\tilde{w}(s)\|_{L^2}.$$

Now we follow the same calculation of (1.19):

$$\begin{aligned}
&|\ e^{-q(s)}\langle v\cdot\nabla\tilde{w}, \sum_{k\leq N}\langle\tilde{w}(s), e_k\rangle e_k\rangle\ | \\
&\lesssim e^{-q(s)}\big(\|v\|_{L^2}^{\frac{1}{2}}\|\partial_1 v\|_{L^2}^{\frac{1}{2}} + \|v\|_{L^2}\big)H(w)\|w\|_{L^2}.
\end{aligned}$$

Since the latter belongs to $L^1([0,T])$ for any $T>0$, we use the dominated convergence theorem again and obtain:

$$2\sum_{k\leq N}\int_0^t e^{-q(s)}\langle\tilde{w}(s), e_k\rangle\langle v\cdot\nabla\tilde{w}, e_k\rangle ds \longrightarrow 0 \text{ as } N\to\infty\ . \tag{1.55}$$

Similarly, by the dominated convergence theorem, we have

$$2\sum_{k\leq N}\int_0^t e^{-q(s)}\langle\tilde{w}(s), e_k\rangle\langle\partial_1^2\tilde{w}, e_k\rangle ds \longrightarrow -2\int_0^t e^{-q(s)}\|\partial_1\tilde{w}\|_{L^2}^2 ds \text{ as } N\to\infty\ , \tag{1.56}$$

and

$$\begin{aligned}
&\sum_{k\leq N}\int_0^t e^{-q(s)}\|(\sigma(s,u(s)) - \sigma(s,v(s)))^* e_k\|_{l^2}^2 ds \\
&\longrightarrow \int_0^t e^{-q(s)}\|\sigma(s,u(s)) - \sigma(s,v(s))\|_{L^2(\ell^2,H)}^2 ds \\
&\leq \int_0^t e^{-q(s)}(L_1\|\tilde{w}\|_{L^2}^2 + L_2\|\partial_1\tilde{w}\|_{L^2}^2)ds.
\end{aligned} \tag{1.57}$$

Since $L_2 < \frac{1}{5}$, we can choose $0<\hat{\alpha}<1$ and $0<\hat{\beta}<1$, such that $L_2 + 2\hat{\alpha} + \frac{9}{\hat{\beta}}L_2 < 2$. By Burkholder–Davis–Gundy's inequality as well as the dominated convergence theorem, we deduce

$$\begin{aligned}
&2\mid \mathbb{E}(\sup_{0\le s\le t}\int_0^s \sum_{k\le N} e^{-q(r)}\langle \tilde{w}(r), e_k\rangle\langle(\sigma(r,u(r))-\sigma(r,v(r)))dW(r), e_k\rangle)\mid \\
&\le 6\mathbb{E}\Big(\int_0^t e^{-2q(s)}\|\sigma(s,u(s))-\sigma(s,v(s))\|^2_{L^2(\ell^2,H)}\|\tilde{w}\|^2_{L^2}\Big)^{\frac{1}{2}} \\
&\le \hat{\beta}\mathbb{E}(\sup_{s\le t}(e^{-q(s)}\|\tilde{w}(s)\|^2_{L^2})) \\
&+\frac{9}{\hat{\beta}}\mathbb{E}\int_0^t e^{-q(s)}(L_1\|\tilde{w}(s)\|^2_{L^2}+L_2\|\partial_1\tilde{w}(s)\|^2_{L^2})ds.
\end{aligned} \tag{1.58}$$

Combining (1.52)-(1.58) and dropping the negative terms, we obtain:

$$(1-\hat{\beta})\mathbb{E}(\sup_{0\le s\le t} e^{-q(s)}\|\tilde{w}\|^2_{L^2}) \le \mathbb{E}\int_0^t (1+\frac{9}{\hat{\beta}})L_1 e^{-q(s)}\|\tilde{w}(s)\|^2_{L^2}ds.$$

By Gronwall's inequality we obtain $\tilde{w}=0$ $\mathbb{P}-a.s.$ in the end. □

Chapter 2

Stationary Solutions and Ergodicity Results

In the first part of this chapter we discuss the existence and the uniqueness of the stationary solutions of the equations

$$\begin{cases} du + (u \cdot \nabla u - \lambda_1 \partial_1^2 u + \lambda u)dt = \sigma(u)dW - \nabla p dt, \\ \operatorname{div} u = 0, \\ u \mid_{t=0} = u_0, \end{cases} \tag{2.1}$$

on $\mathbb{R}^2$ and $\mathbb{T}^2$ with a linear damping term λu for $\lambda \geq 0$, $\lambda_1 \geq 0$, where $W(t), t \geq 0$ is an ℓ^2-cylindrical Wiener process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and σ is a continuous map from $\tilde{H}^1$ to $L_2(\ell^2, \tilde{H}^1)$.
In Section 2.1, we introduce some notations. In Section 2.2, we prove the strong well-posedness results for stochastic $2D$ anisotropic Navier–Stokes equations. In Section 2.3, we use Krylov–Bogoliubov methods to prove the existence of stationary solutions for damped equations. In Section 2.4, we use the coupling methods to prove ergodicity results.

In the second part of this chapter, in Section 2.5 we introduce the notation of space white noise and the previous results about space white noise solution of Euler equations on the torus. In Section 2.6 we prove the existence of white noise solutions for mSQG equations on $\mathbb{R}^2$.

2.1 Notations and Preliminaries

See Section 1.1 for the basic notations and preliminaries.

2.1.1 Some Other Notations and Definitions for Vector Fields

As usual, we use D to denote the domain $\mathbb{R}^2$ or $\mathbb{T}^2$. Let $C_c^\infty(\mathbb{R}^2)$ be the space of smooth functions in $\mathbb{R}^2$ with compact support. Recall that $H(D)$, $\tilde{H}^s(D)$ and $C_{sol}^\infty(D)$ are the vector spaces $L^2(D;\mathbb{R}^2)$,$H^s(D;\mathbb{R}^2)$ and $C_c^\infty(D,\mathbb{R}^2)$ with divergence free, respectively.

Moreover, recall that on $\mathbb{T}^2$ and $\mathbb{R}^2$, for $u, v \in (L^2)^2$, we use $(\cdot,\cdot)$ or $(\cdot \mid \cdot)$ to denote the inner product, i.e.

$$(u, v) = (u \mid v) = (u, v)_{L^2(D)} = \sum_{j=1}^{2} \int_D u_j(x) v_j(x) dx.$$

Here $u = (u_1, u_2)$ and $v = (v_1, v_2)$.
We also define for $\mathbb{T}^2$ and $\mathbb{R}^2$, and $u, v \in (H^1)^2$,

$$(u, v)_{H^1(D)} = \big((I - \Delta)^{\frac{1}{2}} u, (I - \Delta)^{\frac{1}{2}} v\big).$$

In particular,

$$\|u\|^2_{H^1(D)} = \big((I - \Delta)u, u\big) = \|u\|^2_{L^2(D)} + \|\nabla u\|^2_{L^2(D)},$$

for $u \in L^2$.

As usual, when $u, v, w \in (H^1)^2(D)$, we write

$$\begin{aligned} B(u, v) :=& u \cdot \nabla v, \\ B(u) :=& u \cdot \nabla u, \\ b(u, v, w) :=& (u \cdot \nabla v, w). \end{aligned}$$

Then we have $b(u, v, w) = -b(u, w, v)$ for $u, v, w \in \tilde{H}^1(D)$. In particular, $b(u, v, v) = 0$. Again, from now on in this chapter we omit the notation of the domain if no confusion will occur.

2.1.2 The Noise Term σ

As before, we assume that W is an ℓ^2-cylindrical Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Similar as in Chapter 1, for a Hilbert space X, we use the notation $L_2(\ell^2, X)$ to denote the Hilbert–Schmidt operators from ℓ^2 to X with the usual associated Hilbert–Schmidt norm.
We suppose $\sigma : \tilde{H}^1 \to L_2(\ell^2, \tilde{H}^1)$ to be a continuous operator. We introduce the following hypothesis for the noise term:

Hypothesis 2.1. ***H^1-Lipschitz condition***
There exists a constant L such that for $t \geq 0$ and $u, v \in \tilde{H}^1$,

$$\|\sigma(u) - \sigma(v)\|^2_{L_2(\ell^2, H^1)} \leq L\|u - v\|^2_{H^1}. \qquad (H^1\text{-Lip})$$

2.2 Well-posedness Results

Before we come to the main results of the long time behaviour of solutions to (2.1), we prove the existence and uniqueness of solutions. We will mainly present the proof of the case of $D = \mathbb{R}^2$, since there are only minor differences between the cases $D = \mathbb{R}^2$ and $D = \mathbb{T}^2$. We will point out these differences along the way.

2.2.1 Existence of Martingale Solutions

Consider the following approximating damped Navier–Stokes equations of (2.1)

$$\begin{cases} du + (u \cdot \nabla u - A_\nu \partial_1^2 u - \nu \partial_2^2 u + \lambda u)dt = \sigma(u)dW - \nabla p dt, \\ \operatorname{div} u = 0, \\ u \mid_{t=0} = u_0, \end{cases} \tag{2.2}$$

with $A_\nu = \nu + \lambda_1$. Note that when $\nu = 0$, (2.2) becomes (2.1).

First recall the definition of (H^1-valued) martingale solutions, (probabilistically) weak solutions and (probabilistically) strong solutions of (2.2).

Denote by $\mathcal{P}(C(\mathbb{R}^+; U'))$ the space of Borel probability measures on $C(\mathbb{R}^+; U')$, where U is another separable Hilbert space which is dense in H^1 and compactly embedded in H^1, thus H^{-1} is compactly and densely embedded in U'. (see [9] Appendix Lemma C.1 for the proof of the existence of such U). The reason we introduce such U is that in the case of $D = \mathbb{R}^2$, Sobolev embeddings are not compact. In the case of $D = \mathbb{T}^2$, we can use the space H^{-1} instead of U'.
For $t \geq 0$, define

$$\mathcal{B}_t = \sigma\{\pi_r(\cdot) : 0 \leq r \leq t\},$$

where π_r is the projection map $\pi_r x := x(r)$ for $x \in C(\mathbb{R}^+; U')$.
Let $G_\nu : H^1 \to H^{-1}$ be given by:

$$G_\nu(u) = u \cdot \nabla u - A_\nu \partial_1^2 u - \nu \partial_2^2 u + \lambda u. \tag{2.3}$$

Definition 2.2 ((H^1-valued) Martingale solution)**.** *Given $u_0 \in \tilde{H}^1$, a probability measure $P \in \mathcal{P}(C(\mathbb{R}^+; U'))$ is called a (H^1-valued) global martingale solution of (2.2) with initial value u_0 if for any $T > 0$,*

(M1) $P\big(x \in C(\mathbb{R}^+; U'); \pi_{[0,T]}x \in L^2([0,T]; \tilde{H}^1) \cap C([0,T]; H^1_w), x(0) = u_0\big) = 1$, *where $\pi_{[0,T]}x$ is the restriction of x to the interval $[0,T]$, and*

$$\int_0^T \|G_\nu(\pi_s(\cdot))\|_{H^{-1}} ds + \int_0^T \|\sigma(\pi_s(\cdot))\|^2_{L_2(\ell^2, H^1)} ds < +\infty, P - a.s.$$

(M2) *For every* $l \in C^{\infty}_{sol}(D)$, *the process*

$$M_l(t, \cdot) = \langle \pi_t(\cdot), l\rangle - \int_0^t \langle G_\nu \pi_s(\cdot), l\rangle ds,$$

$t \in [0, T]$ *is a continuous square integrable* $\mathcal{B}_t-$*martingale with respect to* P, *whose quadratic variation process is* $\int_0^t \|\sigma^*(\pi_s(\cdot))(l)\|^2_{\ell^2} ds$, $t \in [0, T]$, *where* σ^* *denotes the adjoint operator of* σ.

(M3) *For any* $p \geq 1$ *we have*

$$\mathbb{E}^P \Big(\sup_{t\in[0,T]} \|\pi_t(\cdot)\|^{2p}_{H^1} \Big) \leq C_T(1 + \|u_0\|^{2p}_{H^1}),$$

where $\mathbb{E}^P$ *denotes the expectation with respect to* P.

Remark 2.3. *Note that* H^1_w *is the space* H^1 *with weak topology. Let* $u : [0, T] \to H^1$. *Then* $u \in C([0, T], H^1_w)$ *if and only if for any* $\phi \in H^1$, $(u, \phi)_{H^1} \in C([0, T], \mathbb{R})$.
The topology of $C([0, T], H^1_w)$ *is the weakest topology which makes the map*

$$C([0, T], H^1_w) \to C([0, T], \mathbb{R})$$
$$u \mapsto (u, \phi)_{H^1}$$

continuous for any $\phi \in H^1$, *where* $(u, \phi)_{H^1}$ *is the inner product in the Hilbert space* H^1.

Similarly, we define H^1-valued (probabilistically) weak solutions as follows.

Definition 2.4 ((H^1-valued) (Probabilistically) Weak solution)**.** *We say that a pair* (u, W) *is a* (H^1*-valued) (probabilistically) weak solution to* (2.2) *if there exists a stochastic basis* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ *such that* $u = (u(t))_{t\geq 0}$ *is an* $(\mathcal{F}_t)$*- adapted process and* W *is an* ℓ^2*-cylindrical Wiener process on* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. *Moreover, the following holds:*

(i) $u \in L^2([0, T]; \tilde{H}^1) \cap C([0, T]; H^1_w) \cap C([0, T]; U')$ *for any* $T > 0$ *;*
$\int_0^T \|G_\nu(u(s))\|_{H^{-1}} ds + \int_0^T \|\sigma(u(s))\|^2_{L_2(\ell^2,H^1)} ds < +\infty$ $\mathbb{P} - a.s.$, *for any* $T > 0$*;*

(ii) *the identity*

$$du + G_\nu(u)dt = \sigma(u)dW - \nabla p dt$$

holds $\mathbb{P} - a.s.$*;*

(iii) *For any* $p \geq 1$ *and any* $T > 0$, *we have*

$$\mathbb{E}^{\mathbb{P}} \Big(\sup_{t\in[0,T]} \|u(t)\|^{2p}_{H^1} \Big) \leq C_T(1 + \|u_0\|^{2p}_{H^1}).$$

By the martingale representation theorem (see for example Theorem 8.2 in [34] and Theorem 3.3.6 in [56]), the above two definitions are equivalent.

Definition 2.5 ((H^1-valued) (Probabilistically) Strong solution)**.** *We say that u is an H^1-valued (probabilistically) strong solution to the equation* (2.2) *on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a fixed ℓ^2-cylindrical Wiener process W, if it satisfies:*

(i) u is adapted to the normal filtration $\hat{\mathcal{F}}_t$, which is the normal filtration associated to W (see [69, Definition 2.1.12] for the definition of normal filtration).

(ii) (u, W) with the stochastic basis $(\Omega, \mathcal{F}, \hat{\mathcal{F}}_t, \mathbb{P})$ satisfies Definition 2.4 .

Then we have the following result concerning the existence of the martingale solutions:

Theorem 2.6. *Under Hypothesis 2.1, for $\lambda \geq 0$, $\lambda_1 \geq 0$, $\nu \geq 0$, and the initial value $u_0 \in \tilde{H}^1$, there exists a (H^1-valued) martingale solution to* (2.2) *in the sense of Definition 2.2. In particular, there exists a (H^1-valued) martingale solution to* (2.1)

Remark 2.7. *Note that only for the results of the existence of the martingale solutions, we temporarily allow the damping term to vanish, i.e. $\lambda = 0$.*

To prove Theorem 2.6, we use the classical Galerkin approximations in H^1 and tightness methods.

Let $\{e_n\}_{n\geq 1}$ be an orthonormal basis of $\tilde{H}^1$ consisting of smooth elements. Note that this is different from Chapter 1 , where we defined e_n to be the orthonormal basis of H. Since in this chapter we need H^1 uniform estimates, we will use the Galerkin approximation methods in H^1. For $x \in (H^{-s})^2$, $s > 0$ (see (1.3) in Section 1.1.1 for the definition of H^{-s}), define the finite dimensional projection

$$P_n x := \sum_{i=1}^{n} \langle x, e_i \rangle e_i,$$

and let

$$B_n(u, u) = P_n B(u, u).$$

Similar as in Chapter 1, let $W_n(t) = \sum_{j=1}^{n} \psi_j \beta_j(t) := \Pi_n W(t)$, where $\{\beta_j(t)\}_{j\geq 1}$ is a sequence of independent Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\psi_j\}_{j\geq 1}$ is an orthonormal basis of ℓ^2. Consider the following approximating equations in the finite dimensional spaces $\mathcal{H}_n := span\{e_1, e_2, ... e_n\}$:

$$\begin{cases} du^{(\nu,n)} + [B_n(u^{(\nu,n)}, u^{(\nu,n)}) - A_\nu \partial_1^2 u^{(\nu,n)} - \nu \partial_2^2 u^{(\nu,n)}]dt = P_n \sigma(u^{(\nu,n)}) dW_n, \\ u^{(\nu,n)}(0) = P_n u_0. \end{cases} \tag{2.4}$$

Lemma 2.8. *Let $u_0 \in \tilde{H}^1$, and assume σ satisfies Hypothesis 2.1. Given a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ and an ℓ^2-cylindrical Wiener process W on it, then, for any $\nu > 0$ and n, there exists a unique global strong solution $u^{(\nu,n)} \in C(\mathbb{R}^+; \mathcal{H}_n)$ to (2.4).*

Proof. The result is standard, see, for example, Theorem 3.1.1 in [69], which implies that for each $T > 0$, there exists a unique strong solution $u^{(\nu,n)}$ in $C(\mathbb{R}^+; \mathcal{H}_n)$. □

From Hypothesis 2.1, we immediately obtain that there exists a constant L_0, such that

$$\|\sigma(u)\|^2_{L_2(\ell^2,H^1)} \leq L_0 + L\|u\|^2_{H^1},$$

where L is the Lipschitz constant in Hypothesis 2.1.

Lemma 2.9. *For any $T > 0$, we have the following uniform estimates for $u^{(\nu,n)}$:*

$$\mathbb{E} \sup_{0\leq t\leq T} \|u^{(\nu,n)}(t)\|^2_{H^1} + \nu\mathbb{E}\int_0^T \|\Delta u^{(\nu,n)}(t)\|^2_{L^2} dt \leq C(L_0, L, T)(1 + \|u_0\|^2_{H^1}).$$

Proof. Itô's formula applying to $\|u^{(\nu,n)}\|^2_{H^1} = \|(I-\Delta)^{\frac{1}{2}} u^{(\nu,n)}\|^2_{L^2}$ implies

$$\|u^{(\nu,n)}(t)\|^2_{H^1} \leq \|u_0\|^2_{H^1} - 2\int_0^t (u^{(\nu,n)}\cdot\nabla u^{(\nu,n)} - \nu\Delta u^{(\nu,n)}, u^{(\nu,n)})_{H^1} ds + M_t + \int_0^t \|P_n\sigma\|^2_{L_2(\ell^2,H^1)} ds, \tag{2.5}$$

where

$$M_t := 2\int_0^t (\sigma dW_n, u^{(\nu,n)}(r))_{H^1}.$$

By assumption

$$\int_0^t \|P_n\sigma\|^2_{L_2(\ell^2,H^1)} ds \leq \int_0^t L_0 + L\|u^{(\nu,n)}(s)\|^2_{H^1} ds. \tag{2.6}$$

By Burkholder–Davis–Gundy, Hölder and Young's inequalities, we have:

$$\begin{aligned}
&\mathbb{E}[\sup_{0\leq s\leq t} M_s] \\
&= 2\mathbb{E}\sup_{0\leq s\leq t}\int_0^s (\sigma dW_n(r), u^{(\nu,n)})_{H^1} \\
&\leq 6\mathbb{E}\Big(\int_0^t \|\sigma^*(u^{(\nu,n)})\|^2_{\ell^2} ds\Big)^{\frac{1}{2}} \\
&\leq 6\mathbb{E}\Big(\int_0^t \|u^{(\nu,n)}(s)\|^2_{H^1}\|\sigma\|^2_{L_2(\ell^2,H^1)} ds\Big)^{\frac{1}{2}} \\
&\leq 6\mathbb{E}\Big(\int_0^t (L_0 + L\|u^{(\nu,n)}\|^2_{H^1})\|u^{(\nu,n)}(s)\|^2_{H^1} ds\Big)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\mathbb{E}\sup_{0\leq s\leq t}\|u^{(\nu,n)}(s)\|^2_{H^1} + C(L_0, L)\mathbb{E}\int_0^t (1 + \|u^{(\nu,n)}\|^2_{H^1}) ds.
\end{aligned} \tag{2.7}$$

Moreover,

$$
\begin{aligned}
&-2\int_0^t (u^{(\nu,n)}\cdot\nabla u^{(\nu,n)}-\nu\Delta u^{(\nu,n)},u^{(\nu,n)})_{H^1}ds\\
&=-2\int_0^t (u^{(\nu,n)}\cdot\nabla u^{(\nu,n)}-\nu\Delta u^{(\nu,n)},(I-\Delta)u^{(\nu,n)})ds\\
&=-2\int_0^t (\nu\|\nabla u^{(\nu,n)}\|_{L^2}^2+\nu\|\Delta u^{(\nu,n)}\|_{L^2}^2)ds
\end{aligned}
\tag{2.8}
$$

Combining (2.5) to (2.8), using Gronwall's inequality, we obtain

$$
\mathbb{E}\sup_{0\le t\le T}\|u^{(\nu,n)}(t)\|_{H^1}^2\le C(L_0,L,T)(1+\|u_0\|_{H^1}^2). \tag{2.9}
$$

By (2.6), (2.9) and (2.5), we deduce

$$
\mathbb{E}\sup_{0\le t\le T}\|u^{(\nu,n)}(t)\|_{H^1}^2+\nu\int_0^T\|\Delta u^{(\nu,n)}(t)\|_{L^2}^2ds\le C(L_0,L,T)(1+\|u_0\|_{H^1}^2).
$$

□

In order to pass to the limit in (2.4), we need the following $L^p(\Omega)$-estimate.

Lemma 2.10. *For any $T>0$, we have for $p\ge 1$,*

$$
\mathbb{E}\sup_{0\le t\le T}\|u^{(\nu,n)}(t)\|_{H^1}^p\le C(p,L_0,L,T)(1+\|u_0\|_{H^1}^p).
$$

Proof. Applying Itô's formula to $\|u^{(\nu,n)}(t)\|_{H^1}^p$ implies

$$
\begin{aligned}
&d\|u^{(\nu,n)}(t)\|_{H^1}^p\\
\le&(p\|u^{(\nu,n)}(t)\|_{H^1}^{p-2}u^{(\nu,n)},du^{(\nu,n)})_{H^1}+\frac{1}{2}p(p-1)\|u^{(\nu,n)}(t)\|_{H^1}^{p-2}\|\sigma\|_{L_2(\ell^2,H^1)}^2dt\\
=&p\|u^{(\nu,n)}(t)\|_{H^1}^{p-2}(\sigma dW_n(t),u^{(\nu,n)})_{H^1}\\
&-p\|u^{(\nu,n)}(t)\|_{H^1}^{p-2}(u^{(\nu,n)}\cdot\nabla u^{(\nu,n)},u^{(\nu,n)})_{H^1}dt\\
&+p\|u^{(\nu,n)}(t)\|_{H^1}^{p-2}(A_\nu\partial_1^2u^{(\nu,n)}+\nu\partial_2^2u^{(\nu,n)},u^{(\nu,n)})_{H^1}dt\\
&+\frac{1}{2}p(p-1)\|u^{(\nu,n)}(t)\|_{H^1}^{p-2}\|\sigma\|_{L_2(\ell^2,H^1)}^2dt.
\end{aligned}
\tag{2.10}
$$

Note that

$$
(u^{(\nu,n)}\cdot\nabla u^{(\nu,n)},u^{(\nu,n)})_{H^1}=0,
$$

and

$$
\begin{aligned}
&\|u^{(\nu,n)}(t)\|_{H^1}^{p-2}(A_\nu\partial_1^2u^{(\nu,n)}+\nu\partial_2^2u^{(\nu,n)},u^{(\nu,n)})_{H^1}dt\\
&=-\|u^{(\nu,n)}(t)\|_{H^1}^{p-2}(A_\nu\|\partial_1u^{(\nu,n)}\|_{H^1}^2+\nu\|\partial_2u^{(\nu,n)}\|_{H^1}^2)\le 0.
\end{aligned}
$$

And by assumption of σ we obtain:

$$\begin{aligned} &d\|u^{(\nu,n)}(t)\|_{H^1}^p \\ \leq &p\|u^{(\nu,n)}(t)\|_{H^1}^{p-2}(\sigma dW_n, u^{(\nu,n)})_{H^1} \\ &+\frac{1}{2}p(p-1)(L_0+L\|u^{(\nu,n)}(t)\|_{H^1}^2)\|u^{(\nu,n)}(t)\|_{H^1}^{p-2}dt. \end{aligned} \tag{2.11}$$

Similarly to (2.7), by Burkholder–Davis–Gundy, Hölder and Young's inequalities, we have:

$$\begin{aligned} &p\mathbb{E}\sup_{0\leq s\leq t}|\int_0^s(\sigma dW_n(r), \|u^{(\nu,n)}(t)\|_{H^1}^{p-2}u^{(\nu,n)})_{H^1}| \\ &\leq \frac{1}{2}\mathbb{E}\sup_{0\leq s\leq t}\|u^{(\nu,n)}(s)\|_{H^1}^p + C(p,L_0,L)\int_0^t\mathbb{E}\|u^{(\nu,n)}(s)\|_{H^1}^p ds + C(p,L_0,L,T) \end{aligned} \tag{2.12}$$

Moreover, by Hölder and Young's inequalities, we also obtain:

$$\begin{aligned} &\frac{1}{2}p(p-1)\mathbb{E}\int_0^t(K_0+K_1\|u^{(\nu,n)}(s)\|_{H^1}^2)\|u^{(\nu,n)}(t)\|_{H^1}^{p-2}ds \\ &\leq C(p,L_0,L,T)(1+\int_0^t\mathbb{E}\|u^{(\nu,n)}(s)\|_{H^1}^p ds). \end{aligned} \tag{2.13}$$

Combining (2.10)-(2.13), and using Gronwall's inequality, we finally have:

$$\mathbb{E}\sup_{0\leq t\leq T}\|u^{(\nu,n)}(t)\|_{H^1}^p \leq C(p,L_0,L,T)(1+\|u_0\|_{H^1}^p).$$

□

Define the function space

$$\mathcal{Z}_T := C([0,T];U')\cap L^2_w([0,T];\tilde{H}^1)\cap L^2([0,T];H_{loc})\cap C([0,T];H^1_w).$$

Remark 2.11.

Here the intersection space $\mathcal{Z}_T$ takes the following intersection topology: the class of open sets of $\mathcal{Z}_T$ are generated by the sets of the form $\mathcal{O}_1\cap\mathcal{O}_2\cap\mathcal{O}_3\cap\mathcal{O}_4$, where $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ are the open sets in the above four spaces, respectively.

Let $\mu^{(\nu,n,T)}$ be the distribution of $u^{(\nu,n)}$ in $\mathcal{Z}_T$.

Lemma 2.12. *$\{\mu^{(\nu,n,T)}\}_{0<\nu<1,n\geq 1}$ is tight in $\mathcal{Z}_T$.*

Proof. Define $K_R^T = \{u\in\mathcal{Z}_T;\ \sup_{0\leq t\leq T}\|u(t)\|_{H^1}+\|u\|_{C^{\frac{1}{8}}([0,T];H^{-1})}\leq R\}$.

We need to show that K_R is relatively compact in $\mathcal{Z}_T$. We use Lemma 3.1 in [9], which is stated in Appendix A for the convenience of the readers.

That is, we need to show,
(i) $\sup\limits_{u\in K_R}\int_0^T \|u(t)\|_{H^1}^2 dt < \infty$;
(ii) $\lim\limits_{\delta\to 0}\sup\limits_{u\in K_R}\sup\limits_{t,s\in[0,T],|t-s|\le\delta}\|u(t)-u(s)\|_{U'} = 0.$

Both follow immediately from the definition of K_R^T, since by the definition of U, H^{-1} can be embedded compactly in U'. Note that the following equality holds

$$u^{(\nu,n)}(t) = P_n u_0 + \int_0^t P_n(A_\nu \partial_1^2 u^{(\nu,n)} + \nu\partial_2^2 u^{(\nu,n)} - u^{(\nu,n)}\cdot\nabla u^{(\nu,n)})ds + M^{\nu,n}(t),$$

where $M^{\nu,n}(t) = \int_0^t P_n\sigma dW_n$.
By Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}^{\mu^{(\nu,n,T)}}\Big[\sup_{s\neq t\in[0,T]}\frac{\|\int_s^t P_n(A_\nu\partial_1^2 u + \nu\partial_2^2 u - u\cdot\nabla u)dr\|_{H^{-1}}^2}{|t-s|}\Big]\\
&= \mathbb{E}^{\mathbb{P}}\Big[\sup_{s\neq t\in[0,T]}\frac{\|\int_s^t P_n(A_\nu\partial_1^2 u^{(\nu,n)} + \nu\partial_2^2 u^{(\nu,n)} - u^{(\nu,n)}\cdot\nabla u^{(\nu,n)})dr\|_{H^{-1}}^2}{|t-s|}\Big]\\
&\le \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \|A_\nu\partial_1^2 u^{(\nu,n)} + \nu\partial_2^2 u^{(\nu,n)} - u^{(\nu,n)}\cdot\nabla u^{(\nu,n)}\|_{H^{-1}}^2 dr\Big]\\
&\le \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \|u^{(\nu,n)}\|_{H^1}^2 + \|u^{(\nu,n)}\otimes u^{(\nu,n)}\|_{L^2}^2 dr\Big]\\
&\le \mathbb{E}^{\mathbb{P}}\Big[\int_0^T \|u^{(\nu,n)}\|_{H^1}^2 + \|u^{(\nu,n)}\|_{L^4}^4 dr\Big]\\
&\le C\mathbb{E}^{\mathbb{P}}\Big[\int_0^T \|u^{(\nu,n)}\|_{H^1}^2 + \|u^{(\nu,n)}\|_{H^1}^4 dr\Big]\\
&\le C_2,
\end{aligned} \tag{2.14}$$

where we use the Sobolev embedding from H^1 into L^4 and the last inequality is due to Lemma 2.10. Moreover, for any $p \ge 1$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\|M^{\nu,n}(t,\cdot) - M^{\nu,n}(s,\cdot)\|_{L^2}^{2p}\\
&\le C\mathbb{E}^{\mu^{(\nu,n,T)}}\Big(\int_s^t \|\sigma\|_{L_2(\ell^2,L^2)}^2 dr\Big)^p\\
&\le C|t-s|^{p-1}\mathbb{E}^{\mu^{(\nu,n,T)}}\int_s^t \|\sigma\|_{L_2(\ell^2,L^2)}^{2p} dr\\
&\le C|t-s|^p\mathbb{E}^{\mu^{(\nu,n,T)}}\sup_{0\le t\le T}(K_0 + K_1\|u\|_{H^1}^2)^p\\
&\le C(K_0,K_1,T,p)|t-s|^p(1+\|u_0\|_{H^1}^{2p}).
\end{aligned}$$

By Kolmogorov's criterion, for any $\alpha \in (0, \frac{p-1}{2p})$, we get

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\Big(\sup_{s\neq t\in[0,T]} \frac{\|M^{\nu,n}(t,\cdot)-M^{\nu,n}(s,\cdot)\|_{L^2}^{2p}}{|t-s|^{p\alpha}}\Big) \\ &\leq C(K_0,K_1,T). \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15) and taking $p=2$, we get

$$\sup_{0<\nu<1, n\in\mathbb{N}} \mathbb{E}^{\mu^{(\nu,n,T)}}\Big(\sup_{s\neq t\in[0,T]} \frac{\|\pi_t(\cdot)-\pi_s(\cdot)\|_{H^{-1}}}{|t-s|^{\frac{1}{8}}}\Big) < \infty.$$

Hence, together with Lemma 2.10 we have proved: for any $\epsilon > 0$, there exists a sufficiently large constant $R > 0$, such that $P((K_R^T)^c) < \epsilon$. It only remains to show that these sets are relatively compact in $C([0,T]; H_w^1)$, which is proved in Lemma 2.1 and Theorem 2.2 in [8]. □

Proof of Theorem 2.6

First, we prove the case $\nu = 0$. Now again we apply the Theorem 1.27 (Skorokhod Theorem). Similarly to the previous case of $\mathcal{X}_T$, we now find a sequence of continuous functions from $\mathcal{Z}_T$ to $\mathbb{R}$ which separate points in $\mathcal{Z}_T$. Recall that

$$\mathcal{Z}_T := C([0,T]; U') \cap L_w^2([0,T]; \tilde{H}^1) \cap L^2([0,T]; H_{loc}) \cap C([0,T]; H_w^1).$$

It suffices to prove that for each intersected space, there exits a sequence of continuous functions which separate the points in it.

Since $C([0,T]; U')$ is separable Banach space, it is easy to see there exists a sequence of continuous functions which separate the points in it.

For $L_w^2([0,T]; \tilde{H}^1)$, it is sufficient to put

$$f_m(u) := \int_0^T (u(t), v_m(t))_{H^1} dt \in \mathbb{R}, u \in L_w^2([0,T]; H^1), m \in \mathbb{N},$$

where $\{v_m\}_{m\geq 1}$ is a dense subset of $L^2([0,T]; H^1)$. Since $\{v_m\}_{m\geq 1}$ is dense in $L^2([0,T]; H^1)$, it separates points of $L^2([0,T]; H^1)$.

Similarly for $L^2([0,T]; H_{loc})$, let $\{\bar{v}_m\}_{m\geq 1}$ be a countable dense subset of $L^2([0,T]; C_c)$, where C_c is the space of continuous on $\mathbb{R}^2$ with compact support. Then $\{\bar{v}_m\}_{m\geq 1}$ is also dense in $L^2([0,T]; L^2)$, hence it can separates points in $L^2([0,T]; H_{loc})$.

Finally for the space $C([0,T]; H_w^1)$. Let $\{\tilde{v}_m\}_{m\geq 1}$ be a countable dense subset of $C([0,T]; H^1)$. Then due to the denseness of $\{\tilde{v}_m\}_{m\geq 1}$, it can separates points in $C([0,T]; H_w^1)$. Therefore, the points in $\mathcal{Z}_T$ can be separated by a sequence of continuous functions.

The σ-algebra generated by a sequence of continuous functions which separate the points in $\mathcal{Z}_T$ is exactly $\mathscr{B}(\mathcal{Z}_T)$, which we prove in the appendix (see Corollary B.9). By Theorem 1.27, there exists a subsequence $\{\mu^{(\nu_k,n_k,T)}\}$ ($\nu_k \to 0, n_k \to +\infty$), as well as random variables $\tilde{u}^{\nu_k,n_k}$ in the space $([0,1], \mathscr{B}([0,1]), \mathcal{L})$, such that

(i) $\tilde{u}^{(\nu_k,n_k,T)}$ has law $\mu^{(\nu_k,n_k,T)}$;

(ii) $\mu^{(\nu_k,n_k,T)}$ converges weakly to some $\tilde{\mu}^T$;

(iii) $\tilde{u}^{(\nu_k,n_k,T)} \to \tilde{u}^T$ in $\mathcal{Z}_T$ $\mathcal{L} - a.s.$, and $\tilde{u}^T$ has the law $\tilde{\mu}^T \in \mathcal{P}(C([0,T];U'))$.

By a similar method as in Lemma 1.28, Definition 1.29 and Remark 1.30 we can construct $\hat{\mu} \in C(\mathbb{R}^+;U')$. The proof of $\hat{\mu}$ satisfying the condition of the martingale solutions (Definition 2.2) is the same as in Section 1.3.6, by which we finish the proof of the existence of the martingale solutions for $\nu = 0$.

For $\nu > 0$, the proof works the same way, since we can fix ν and apply the tightness methods with respect to n. Hence we finish the proof of Theorem 2.6.

□

Just as we have mentioned before, by the martingale representation theorem (see for example [34, Theorem 8.2] and [56, Theorem 3.3.6]), we obtain the existence of (H^1-valued) (Probabilistically) weak solutions.

2.2.2 Discussion of Pathwise Uniqueness

For pathwise uniqueness of the equation, we note that the initial values in H^1 are not smooth enough to obtain the uniqueness results for (deterministic) Euler equations. In this case pathwise uniqueness does not hold for Euler equations. Instead, pathwise uniqueness holds only for anisotropic Navier–Stokes equations, both damped and not damped. By the same argument as in Chapter 1, we have the following similar conclusion:

Theorem 2.13 (Pathwise uniqueness)**.** *Assume that the noise term $\sigma(u)$ satisfies Hypothesis 2.1, and, in addition, the noise term satisfies the Hypothesis 1.17 (ii) for $L_2 < \frac{1}{5}$. Then for $\lambda_1 = 1$ and $\lambda \geq 0$, we have pathwise uniqueness for the solutions of (2.1), i.e. : if u, v are two (H^1-valued) (probabilistically) weak solutions on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, with the same ℓ^2-Wiener process W and common initial value $u_0 \in \tilde{H}^1$, then we have $u = v$ $\mathbb{P} - a.s.$*

By the Yamada–Watanabe Theorem (see [69] Appendix E), we obtain the existence of probabilistically strong solutions.

For Markov property, note that if we have pathwise uniqueness to (2.1), by the Yamada–Watanabe Theorem, (see [69] Appendix E) we obtain the existence of probabilistically strong solutions. Note that if u is a (probabilistically strong) solution to (2.1) on $(\Omega, \mathcal{F}, \hat{\mathcal{F}}_t, \mathbb{P})$, we obtain the Markov property on H^1 by [69, Proposition 4.3.3].

2.3 Existence of Stationary Solutions

In this section, we prove the existence of stationary solutions to (2.2) as well as to (2.1). The key method is the classical Krylov–Bogoliubov methods and compactness methods.

In [26], a similar result was proved for the vorticity form of the deterministic Navier–Stokes equations. We use a similar method as in [38]. The difference is that instead of using Krylov–Bogoliubov method in L^2, we use it in H^1.

First, we recall the definition of stationary martingale solutions.

Definition 2.14 (Stationary martingale solutions)**.** *A stationary (H^1-valued) martingale solution of (2.2) on $[0,\infty)$ consists of a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, an ℓ^2-cylindrical Wiener process W and a progressively measurable process*

$$u : [0,\infty) \times \Omega \to \tilde{H}^1,$$

with $\mathbb{P}-a.s.$ paths in $L^2([0,T]; H_{loc}) \cap C([0,T]; U') \cap C([0,T]; H^1_w)$ for any $T > 0$. Moreover, u is a stationary process in H^1, such that $\mathbb{P} - a.s.$ the identity

$$du + G_\nu(u)dt = \sigma(u)dW - \nabla p dt \tag{2.16}$$

holds, where $G_\nu(u)$ is defined as in (2.3).

Theorem 2.15. *Assume that $\lambda > 0$ and $\lambda_1 \geq 0$ and that σ satisfies Hypothesis 2.1 (H^1-Lip hypothesis) with $L < 2\lambda$. Then, for any $\nu \geq 0$, (2.2) has a stationary (H^1-valued) martingale solution u^ν.*

The proof is an application of the method in [38].

Proof. We only prove the result for $\nu = \lambda_1 = 0$, since the proof is analogous for strictly positive ν.

First, we apply the Krylov–Bogoliubov methods. Consider the following approximating equations with initial value 0:

$$\begin{cases} du_n^0 + \lambda u_n^0 + B_n(u_n^0, u_n^0) = P_n \sigma(u_n^0) dW_n, \\ u_n^0(0) = 0. \end{cases} \tag{2.17}$$

By the same argument as in Lemma 2.9, we apply Itô's formula to

$$\|u_n^0\|_{H^1}^2 = \|(I-\Delta)^{\frac{1}{2}} u_n^0\|_{L^2}^2$$

to obtain

$$\|u_n^0(t)\|_{H^1}^2 \leq -2\int_0^t (u_n^0 \cdot \nabla u_n^0 - \nu\Delta u_n^0 + \lambda u_n^0, u_n^0)_{H^1} ds + M_t^{n,0} + \int_0^t \|P_n\sigma\|_{L_2(\ell^2, H^1)}^2 ds,$$

where

$$M_t^{n,0} := 2\int_0^t (\sigma dW_n, u_n^0)_{H^1}.$$

By taking the expectation and dropping the negative term $2\int_0^t (\nu\Delta u_n^0, u_n^0)_{H^1} ds$, since $(u_n^0 \cdot \nabla u_n^0, u_n^0)_{H^1} = 0$, we obtain

$$\frac{d}{dt}\mathbb{E}\|u_n^0(t)\|_{H^1}^2 + 2\lambda\|\nabla u_n^0(t)\|_{H^1}^2 \leq \mathbb{E}\|P_n\sigma\|_{L_2(\ell^2,H^1)}^2.$$

Note that

$$\|P_n\sigma\|_{L_2(\ell^2,H^1)}^2 \leq \|\sigma\|_{L_2(\ell^2,H^1)}^2 \leq L_0 + L\|u\|_{H^1}^2.$$

Therefore,

$$\frac{d}{dt}\mathbb{E}\|u_n^0(t)\|_{H^1}^2 \leq -(2\lambda - L)\mathbb{E}\|u_n^0(t)\|_{H^1}^2 + L_0.$$

By Gronwall's inequality, we deduce

$$\mathbb{E}\|u_n^0(t)\|_{H^1}^2 \leq e^{-(2\lambda-L)t}L_0 t \leq \frac{L_0}{2\lambda - L}. \tag{2.18}$$

Since $L < 2\lambda$, the terms $\mathbb{E}\|u_n^0(t)\|_{H^1}^2$ are uniformly bounded with respect to n and ν. Similar as in the proof of Theorem 4.1 of [38], from (2.18), by applying Krylov–Bogoliubov methods, we deduce the existence of an invariant measure μ_n of (2.17) for any (fixed) n. Let $u_n \in C(\mathbb{R}^+, P_n H)$ be the corresponding stationary solutions obtained by the Krylov–Bogoliubov methods on some probability space, which for simplicity, we still denote by $(\Omega, \mathcal{F}, \mathbb{P})$. Then we immediately obtain

$$\mathbb{E}\|u_n(t)\|_{H^1}^2 \leq \frac{L_0}{2\lambda - L}.$$

In particular,

$$\mathbb{E}\|u_n(0)\|_{H^1}^2 \leq \frac{L_0}{2\lambda - L}.$$

Note that the similar estimates as in Lemma 2.9 and Lemma 2.10 still hold. In other word, we still have

$$\mathbb{E}\sup_{0\leq t\leq T}\|u_n(t)\|_{H^1}^2 \leq C(L_0, L, T)(1 + \mathbb{E}\|u_n(0)\|_{H^1}^2).$$

Thus

$$\mathbb{E}\sup_{0\leq t\leq T}\|u_n(t)\|_{H^1}^2 \leq C(L_0, L, T)(1 + \frac{L_0}{2\lambda - L}).$$

With the above estimates, the compactness methods work nearly identical to the proof of Theorem 4.1 of [38] as follows: endow $L^2_{loc}(\mathbb{R}^+; H_{loc})$ with the following distance

$$\rho_{L^2_{loc}(\mathbb{R}^+;H_{loc})} = \sum_{R=1}^{\infty} \frac{1}{2^R} \frac{\rho_{L^2_{loc}(\mathbb{R}^+;L^2(B_R))}}{1+\rho_{L^2_{loc}(\mathbb{R}^+;L^2(B_R))}},$$

where the definition of $\rho_{L^2_{loc}(\mathbb{R}^+;L^2(B_R))}$ is given in Section 1.1.3. Note that the law of u_n is tight in the metric space

$$L^2_{loc}(\mathbb{R}^+; H_{loc}) \cap C(\mathbb{R}^+; U'),$$

since the tightness of the law of u_n in

$$L^2_{loc}(\mathbb{R}^+; H_{loc}) \cap C(\mathbb{R}^+; U'),$$

is equivalent to the tightness of the law of $\pi_{[0,T]}u_n$ in each

$$L^2([0,T]; H_{loc}) \cap C([0,T]; U')$$

for each $T > 0$, the proof of which is almost the same as in the proof of the Lemma 2.12. Hence, we can use the Skorokhod theorem. We know that there exists another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and another sequence $\tilde{u}_n$, such that $\tilde{u}_n$ has the same law as u_n in $L^2_{loc}(\mathbb{R}^+; H_{loc}) \cap C(\mathbb{R}^+; U')$, and $\tilde{u}_n$ converges almost surely to some limit $\tilde{u}$ in $L^2_{loc}(\mathbb{R}^+; H_{loc}) \cap C(\mathbb{R}^+; U')$. Since u_n and $\tilde{u}_n$ are stationary in U', by the almost surely convergence in $C(\mathbb{R}^+; U')$, $\tilde{u}$ is stationary in U'. Moreover,

$$\mathbb{E}^{\tilde{\mathbb{P}}} \sup_{0\le t\le T} \|\tilde{u}_n(t)\|^2_{H^1} \le C(L_0, L, T)(1 + \frac{L_0}{2\lambda - L}),$$

which is due to the measurability (in fact, lower semi-continuous) property of the map

$$C([0,T]; U') \to \mathbb{R}^+ \cup \{\infty\}$$
$$u \mapsto \sup_{0\le t\le T} \|u(t)\|^2_{H^1},$$

where $\|u(t)\|_{H^1} = +\infty$, for any $u(t) \in U' \backslash H^1$.
Again by the lower semi-continuity property, we have

$$\mathbb{E}^{\tilde{\mathbb{P}}} \sup_{0\le t\le T} \|\tilde{u}(t)\|^2_{H^1} \le C(L_0, L, T)(1 + \frac{L_0}{2\lambda - L}).$$

Therefore, for each $t > 0$, $\tilde{u}(t)$ is also a H^1-valued random variables. Hence $\tilde{u}$ is also stationary in H^1. Moreover,

$$\tilde{u}_n(t) - \tilde{u}_n(0) - \int_0^t [\lambda \tilde{u}_n + B_n(\tilde{u}_n, \tilde{u}_n)] ds$$

is a martingale with quadratic variation $\int_0^t [P_n\sigma(\tilde{u}_n)]^* P_n\sigma(\tilde{u}_n) ds$. Passing to the limit is similar to Section 1.3.6, which finishes our proof.

□

2.4 Ergodicity and Exponential Mixing

In this section we consider (2.1) on both $\mathbb{T}^2$ and $\mathbb{R}^2$. We prove the uniqueness of invariant measures (i.e., ergodicity property) and the exponential mixing.

Hypothesis 2.16. *We assume that*

1. $\lambda > 0$ *and* $\lambda_1 = 1$,
   while the cases of other positive λ_1 *are similar.*

2. $\sigma(\cdot)$ *is a continuous map from* H^1 *to* $L_2(\ell^2, H^1)$.
   Then the equations (2.1) *become*

$$\begin{cases} du + (u \cdot \nabla u - \partial_1^2 u + \lambda u)dt = \sigma(u)dW - \nabla p dt, \\ \operatorname{div} u = 0, \\ u \mid_{t=0} = u_0. \end{cases} \tag{2.19}$$

3. $\sigma(\cdot)$ *satisfies Hypothesis 1.17 (ii) and Hypothesis 2.1 for* $L < 2\lambda$, $L_1 < \lambda$ *and* $L_2 < \frac{1}{5}$.

4. $\sigma(u)$ *is bounded in the* $L_2(\ell^2, H^1)$ *norm for any* $u \in H^1$, *i.e. there exists a constant* $K_2 > 0$, *such that*
$$\|\sigma(u)\|^2_{L_2(\ell^2,H^1)} \le K_2.$$

From 2 and 3 we obtain the existence of the probabilistically strong solutions for any $u_0 \in \tilde{H}^1$, which was shown in Section 2.2. Moreover, in Section 2.3 we proved the existence of the stationary solutions of the equation (2.19).

Claim that the process related to the solution is a Markov process.

Indeed we set
$$p_t(x, dy) := \mathbb{P} \circ (u(t,x))^{-1}(dy),$$
where $u(\cdot, x)$ denotes the unique solution to (2.1) with initial value $x \in H^1$.
We define the Markov semi-group P_t as follows.
Let $\mathcal{B}(\tilde{H}^1)$ be the Borel σ-algebra of $\tilde{H}^1$. Set for a bounded $\mathcal{B}(\tilde{H}^1)$-measurable function F,
$$P_t F(x) = \int F(y) p_t(x, dy).$$

Here we state the main theorem of this section:

Theorem 2.17. *If* $\sigma(u)$ *satisfies Hypothesis 2.16, then there exists a constant* $C_0^{(2)}$, *which is a constant only dependent on Sobolev embedding, such that if*

$$K_2 < C_0^{(2)} \frac{\lambda^3}{1 + \lambda^2},$$

there exists only one invariant measure μ to (2.1).
Moreover, we have for any $\Lambda \in \mathcal{B}(\tilde{H}^1)$,

$$\|P_t^*\Lambda - \mu\|_{**} \leq Ce^{-\gamma' t}\big(1 + \int_{\tilde{H}^1} \|x\|_{H^1}^2 d\Lambda(x)\big),$$

where $\|\cdot\|_{**}$ will be defined later in (2.29).

The proof of ergodicity is based on the coupling method and it is a modification of [75, Theorem 2.1], which we put in the next section for completeness.

2.4.1 [75, Theorem 2.1]

In this section we recall [75, Theorem 2.1].

Let U be a Hilbert space and W be a cylindrical Wiener process on U. Let u be a Markov process which takes value in a Polish space H. This Markov process u is assumed to be a non anticipative (=adapted) measurable map of W.

Denote the dependence of u with respect to (t, W, u_0) as follows:

$$u(t) = u(t, W, u_0).$$

Define the following assumptions.

A0 There exists a continuous process $\tilde{u}$ taking value in H and that is a non-anticipative measurable map of W. Moreover $(u, \tilde{u})$ is a homogeneous weak Markov process and its law $\mathcal{D}(u, \tilde{u})$ is measurably depending of its initial condition $(u_0, \tilde{u}_0)$.

Let $\mathcal{H} : H \to \mathbb{R}$ be a positive functional which plays the role of Lyapunov functional. We assume that there exist $\gamma, C_1, C > 0$ and a mapping $h : H^2 \to U$ such that the following hold.

A1 There exists a family $(C'_\alpha)_{\alpha\in(0,\infty)}$, such that for any $u_0 \in H$, any $t \geq 0$, any $\alpha > 0$ and any stopping time $\tau > 0$,

$$\begin{cases} E(\mathcal{H}(u(t, W, u_0)) \leq e^{-\gamma t}\mathcal{H}(u_0) + C_1 \\ E(e^{-\alpha\tau}\mathcal{H}(u(\tau, W, u_0))1_{\tau<\infty}) \leq \mathcal{H}(u_0) + C'_\alpha. \end{cases}$$

A2 For any $(u_0^1, u_0^2) \in H^2$, for any couple (W_1, W_2) of cylindrical Wiener processes of U and for any $t \geq 0$, we have

$$P(d_H(u_1(t), u_2(t)) \geq Ce^{-\gamma t} \text{ and } \tilde{u} = u_2 \text{ on } [0, t]) \leq Ce^{-\gamma t},$$

where

$$\begin{cases} u_i(t) = u(t, W_i, u_0^i) \text{ for } i = 1, 2, \\ \tilde{u}(t) = \tilde{u}(t, W_1, u_0^1, u_0^2), \\ 2C_1 \geq \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2). \end{cases}$$

A3 For any $(t, u_0^1, u_0^2) \in (0, \infty) \times H^2$, we have almost surely

$$\tilde{u}(t, W, u_0^1, u_0^2) = u\Big(t, W + \int_0^\cdot h\big(u(s, W, u_0^1), \tilde{u}(s, W, u_0^1, u_0^2)\big) ds, u_0^2\Big).$$

A4 For any couple (W_1, W_2) of cylindrical Wiener processes of U and for any $(t, u_0^1, u_0^2) \in [0, \infty) \times H^2$

$$P\Big(\int_{t_0}^{\tau} |h(t)|_U^2 dt \geq Ce^{-\gamma t_0} \text{ and } \tilde{u} = u_2 \text{ on } [s, \tau]\Big) \leq Ce^{-\gamma t_0},$$

where $(\tilde{u}, u_2)$ are defined in A2, where $\tau \geq t_0$ is any stopping time and where

$$\begin{cases} h(t) = h\big(u(t, W_1, u_0^1), \tilde{u}(t, W, u_0^1, u_0^2)\big), \\ 2C_1 \geq \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2). \end{cases}$$

A5 There exists $p_1 > 0$ such that for any $(u_0^1, u_0^2) \in H^2$, we have

$$P\Big(\int_0^\infty |h(t)|_U^2 dt \leq C\Big) \geq p_1,$$

where

$$\begin{cases} h(t) = h\big(u(t, W_1, u_0^1), \tilde{u}(t, W, u_0^1, u_0^2)\big), \\ 2C_1 \geq \mathcal{H}(u_0^1) + \mathcal{H}(u_0^2). \end{cases}$$

Now we state Theorem 2.1 in [75].

Theorem 2.18. *Denote by $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ the Markov transition semi-group associated to the Markov family $\big(u(\cdot, W, u_0)\big)_{u_0 \in H}$. Under the above assumptions, there exists a unique stationary probability measure μ of $(\mathcal{P}_t)_{t\in\mathbb{R}^+}$ on H. Moreover, μ satisfies*

$$\int_H \mathcal{H}(u) d\mu(u) < \infty,$$

and there exists $C, \gamma' > 0$ such that for any $\lambda \in \mathcal{P}(H)$

$$\|\mathcal{P}_t^* \lambda - \mu\|_* \leq Ce^{-\gamma' t}\big(1 + \int_H \mathcal{H}(u) d\lambda(u)\big).$$

2.4.2 Application of Theorem 2.1 in [75]

But we have to do some modification to Theorem 2.1 in [75] since we do not have A2 satisfied. In fact, we only have the exponential decay in L^2, which is a space larger than the state space H^1.

First just like [75] we define the coupling $(u, \tilde{u})$ as follows:

$$\big(u, \tilde{u}\big) = \big(u(t, W, u_0), \tilde{u}(t, W, u_0, \tilde{u}_0)\big) := \big(u(t, W, u_0), u(t, W, \tilde{u}_0)\big).$$

Here $u(t, W, u_0)$, $u(t, W, \tilde{u}_0)$ denote the strong solutions of (2.1) with the initial value u_0 and $\tilde{u}_0$, respectively. Note that here $\tilde{u}(t, W, u_0, \tilde{u}_0) = u(t, W, \tilde{u}_0)$ actually does not depend on u_0.

Take the function $h = 0$. Take the functional

$$\mathcal{H} : \tilde{H}^1 \to \mathbb{R}^+$$

$$u \mapsto \|u\|^2_{H^1}$$

as the role of Lyapunov functional.
Then by definition A3, A4 and A5 are satisfied immediately.

2.4.3 Proof of A1

For A1 we use Itô's Formula ([69, Theorem 4.2.5]). Applying Itô's Formula to $\|u(t)\|^2_{L^2}$ and noting that $\langle u \cdot \nabla u, u\rangle = 0$, we obtain:

$$\begin{aligned} &\|u(t)\|^2_{L^2} + 2\int_0^t \|\partial_1 u(s)\|^2_{L^2} ds + 2\lambda \int_0^t \|u(s)\|^2_{L^2} ds \\ &= \|u_0\|^2_{L^2} + 2\int_0^t \langle \sigma dW(s), u\rangle + \int_0^t \|\sigma\|^2_{L_2(\ell^2, H)} ds. \end{aligned} \tag{2.20}$$

Then by taking the expectation and noting that

$$\|\sigma(u)\|^2_{L_2(\ell^2,H)} \le K_2,$$

we have

$$\frac{d}{dt}\mathbb{E}\|u(t)\|^2_{L^2} \le -2\lambda \mathbb{E}\|u(t)\|^2_{L^2} + K_2.$$

By Gronwall's inequality we have

$$\mathbb{E}\|u(t)\|^2_{L^2} \le e^{-2\lambda t}\|u_0\|^2_{L^2} + \frac{K_2}{2\lambda}.$$

Similarly, apply the Itô's Formula to $\|\nabla u(t)\|_{L^2}^2$, note that $\langle u\cdot\nabla u, \Delta u\rangle = 0$ and we obtain:

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2 + 2\int_0^t \|\partial_1 \nabla u(s)\|_{L^2}^2 ds + 2\lambda \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \\ &= \|\nabla u_0\|_{L^2}^2 - 2\int_0^t \langle \sigma dW(s), \Delta u\rangle + \int_0^t \|\nabla \sigma\|_{L_2(\ell^2, L^2)}^2 ds.\end{aligned} \tag{2.21}$$

Then by taking the expectation and noting that for any $u \in H^1$,

$$\|\sigma(u)\|_{L_2(\ell^2, H^1)}^2 \leq K_2,$$

we have

$$\frac{d}{dt}\mathbb{E}\|\nabla u(t)\|_{L^2}^2 \leq -2\lambda \mathbb{E}\|\nabla u(t)\|_{L^2}^2 + K_2.$$

By Gronwall's inequality we have

$$\mathbb{E}\|\nabla u(t)\|_{L^2}^2 \leq e^{-2\lambda t}\|\nabla u_0\|_{L^2}^2 + \frac{K_2}{2\lambda}.$$

Thus we have

$$\mathbb{E}\|u(t)\|_{H^1}^2 \leq e^{-2\lambda t}\|u_0\|_{H^1}^2 + \frac{K_2}{\lambda}.$$

Take $C_1 = \frac{K_2}{\lambda}$.
For $\alpha > 0$, applying the Itô's Formula to $e^{-\alpha t}\|u(t)\|_{L^2}^2$, we deduce

$$\begin{aligned}&de^{-\alpha s}\|u(s)\|_{L^2}^2 + 2e^{-\alpha s}\|\partial_1 u(s)\|_{L^2}^2 ds + (2\lambda + \alpha)e^{-\alpha s}\|u(s)\|_{L^2}^2 ds \\ &= 2e^{-\alpha s}\langle \sigma(u)dW(s), u(s)\rangle + e^{-\alpha s}\|\sigma(u)\|_{L_2(\ell^2, L^2)}^2 ds.\end{aligned}$$

Let τ be a stopping time and $n \in \mathbb{N}$. Then we have

$$\mathbb{E}\big(e^{-\alpha(\tau\wedge n)}\|u(\tau\wedge n)\|_{L^2}^2\big) \leq \|u_0\|_{L^2}^2 + C_\alpha^{(1)}.$$

Similarly, applying the Itô's formula to $e^{-\alpha t}\|\nabla u(t)\|_{L^2}^2$, we have

$$\mathbb{E}\big(e^{-\alpha(\tau\wedge n)}\|\nabla u(\tau\wedge n)\|_{L^2}^2\big) \leq \|\nabla u_0\|_{L^2}^2 + C_\alpha^{(2)}.$$

Thus taking $C'_\alpha = C_\alpha^{(1)} + C_\alpha^{(2)}$, we have

$$\mathbb{E}\big(e^{-\alpha(\tau\wedge n)}\|u(\tau\wedge n)\|_{H^1}^2\big) \leq \|u_0\|_{H^1}^2 + C'_\alpha.$$

Let $n \to \infty$, and we finish the proof of A1.

2.4.4 Proof of A2'

Instead of A2, we can only prove A2' as follows. Given $(u_0^1, u_0^2) \in \tilde{H}^1 \times \tilde{H}^1$ and couple (W_1, W_2) of ℓ^2-cylindrical Wiener process, and for any $t \geq 0$, we have

$$P(\|u_1(t) - u_2(t)\|_{L^2} \geq Ce^{-\gamma t} \text{ and } \tilde{u} = u_2 \text{ on } [0,t]) \leq Ce^{-\gamma t},$$

where

$$\begin{cases} u_i(t) = u(t, W_i, u_0^i) \text{ for } i = 1, 2, \\ \tilde{u}(t) = \tilde{u}(t, W_1, u_0^1, u_0^2) = u(t, W_1, u_0^2), \\ 2C_1 \geq \|u_0^1\|_{H^1}^2 + \|u_0^2\|_{H^1}^2. \end{cases}$$

That means we can only get the estimates of distance of u_1, u_2 in the space L^2, but not H^1. (Note that by our construction $\tilde{u}(t) = u_2(t)$ immediately holds.) Therefore, we need to do some modifications to the proof of [75]. We prove A2' by a similar method of A2 in [75].

Define

$$E_u(t) = \|u(t)\|_{L^2}^2 + 2\int_0^t \|\partial_1 u(s)\|_{L^2}^2 ds + \lambda \int_0^t \|u(s)\|_{L^2}^2 ds.$$

$$\tilde{E}_u(t) = \|\nabla u(t)\|_{L^2}^2 + 2\int_0^t \|\partial_1 \nabla u(s)\|_{L^2}^2 ds + \lambda \int_0^t \|\nabla u(s)\|_{L^2}^2 ds.$$

Lemma 2.19. *Given $u_0 \in \tilde{H}^1$ and W, for the solution u, we have the following estimates for $\gamma_0 = \frac{\lambda}{4K_2}$:*

$$\mathbb{E}\Big(\exp\big(\gamma_0 \sup_{t\geq 0}(E_u(t) - K_2 t)\big)\Big) \leq 2e^{\gamma_0 \|u_0\|_{L^2}^2},$$

Proof. Use the method of [75]. Define

$$M(t) = 2\int_0^t (\sigma dW(s), u(s)).$$

$$M_\gamma(t) = M(t) - \frac{\gamma}{2}\langle M(t)\rangle.$$

Here

$$\begin{aligned} \langle M(t)\rangle =& 4\int_0^t \|\sigma^* u\|_{\ell^2}^2 ds \\ \leq& 4\int_0^t \|\sigma\|_{L_2(\ell^2, H)}^2 \|u\|_{L^2}^2 ds \\ \leq& 4K_2 \int_0^t \|u\|_{L^2}^2 ds, \end{aligned}$$

since σ^*, as the dual operator of σ, is bounded from L^2 to ℓ^2 with

$$\|\sigma^*\|_{L(L^2,\ell^2)} = \|\sigma\|_{L(\ell^2,H)} \leq \|\sigma\|_{L_2(\ell^2,H)},$$

where $L(L^2, \ell^2)$ and $L(\ell^2, H)$ are the norms of the operators.

From (2.20) we know that

$$E_u(t) \leq M_{\gamma_1}(t) + \|u_0\|_{L^2}^2 + K_2 t. \tag{2.22}$$

Since $e^{\gamma M_\gamma}$ is a positive supermartingale with value 1 at time 0. Maximal supermartingale inequality implies for any $\rho > 0$:

$$P\big(\sup_{t\geq 0} M_{\gamma_1}(t) \geq \rho\big) \leq e^{-\gamma_1 \rho}. \tag{2.23}$$

Let $\gamma_0 = \frac{1}{2}\gamma_1$. Then we have

$$\mathbb{E}(e^{\gamma_0 \sup M_{\gamma_1}}) = 1 + \gamma_0 \int_0^\infty e^{\gamma_0 x} P(\sup M_{\gamma_1} \geq x) dx,$$

which yields, by (2.23),

$$\mathbb{E}(e^{\gamma_0 \sup M_{\gamma_1}}) \leq 2. \tag{2.24}$$

Combining (2.22) and (2.24), it follows

$$\mathbb{E}\Big(\exp\big(\gamma_0 \sup_{t\geq 0}(E_u(t) - K_2 t)\big)\Big) \leq 2e^{\gamma_0 \|u_0\|_{L^2}^2},$$

which finishes the proof. □

Lemma 2.20. *Given $u_0 \in H^1$ and W, for the solution u, we also have the following estimates for $\gamma_0 = \frac{\lambda}{4K_2}$:*

$$\mathbb{E}\Big(\exp\big(\gamma_0 \sup_{t\geq 0}(\tilde{E}_u(t) - K_2 t)\big)\Big) \leq 2e^{\gamma_0 \|\nabla u_0\|_{L^2}^2},$$

Proof. The proof is similar as in Lemma 2.19. Define

$$\tilde{M}_t = -2\int_0^t \langle \sigma dW(s), \Delta u(s)\rangle.$$

$$\tilde{M}_\gamma(t) = \tilde{M}(t) - \frac{\gamma}{2}\langle \tilde{M}(t)\rangle.$$

Here

$$\langle \tilde{M}(t)\rangle = 4\int_0^t \|\sigma^*(\Delta u)\|_{\ell^2}^2 ds,$$

where σ^*, as the dual operator of σ, is bounded from H^{-1} to ℓ^2 with

$$\|\sigma^*\|_{L(H^{-1},\ell^2)} = \|\sigma\|_{L(\ell^2,H^1)} \le \|\sigma\|_{L_2(\ell^2,H^1)}.$$

Noting that

$$\|\sigma^*(\Delta u)\|_{\ell^2}^2 \le \|\Delta u\|_{H^{-1}}^2 \|\sigma\|_{L_2(\ell^2,H^1)}^2 \le \|\nabla u\|_{L^2}^2 \|\sigma\|_{L_2(\ell^2,H^1)}^2,$$

we have

$$\langle \tilde{M}(t)\rangle \le 4\int_0^t \|\nabla u\|_{L^2}^2 \|\sigma\|_{L_2(\ell^2,H^1)}^2 ds \le 4K_2 \int_0^t \|\nabla u\|_{L^2}^2 ds.$$

Again, set $\gamma_1 = \frac{\lambda}{2K_2}$. From (2.21) we know that

$$\tilde{E}_u(t) \le \tilde{M}_{\gamma_1}(t) + \|\nabla u_0\|_{L^2}^2 + K_2 t. \tag{2.25}$$

Since $e^{\gamma \tilde{M}_\gamma}$ is a positive supermartingale with value 1 at time 0. Maximal supermartingale inequality implies for any $\rho > 0$:

$$P\big(\sup_{t\ge 0} \tilde{M}_{\gamma_1}(t) \ge \rho\big) \le e^{-\gamma_1 \rho}. \tag{2.26}$$

Let $\gamma_0 = \frac{1}{2}\gamma_1$. Then we have

$$\mathbb{E}(e^{\gamma_0 \sup \tilde{M}_{\gamma_1}}) = 1 + \gamma_0 \int_0^\infty e^{\gamma_0 x} P(\sup \tilde{M}_{\gamma_1} \ge x) dx,$$

which yields, by (2.26),

$$\mathbb{E}(e^{\gamma_0 \sup \tilde{M}_{\gamma_1}}) \le 2. \tag{2.27}$$

Combining (2.25) and (2.27), it follows

$$\mathbb{E}\Big(\exp\big(\gamma_0 \sup_{t\ge 0}(\tilde{E}_u(t) - K_2 t)\big)\Big) \le 2e^{\gamma_0 \|\nabla u_0\|_{L^2}^2},$$

which finishes the proof. □

Take $\delta = u_1 - u_2$ and $q = p^{u_1} - p^{u_2}$.
We obtain that δ satisfies the following equation:

$$\begin{cases} d\delta - \partial_1^2 \delta dt + \lambda \delta dt = (-\nabla q - u_2\cdot\nabla\delta - \delta\cdot\nabla u_1)dt + (\sigma(u_1) - \sigma(u_2))dW_t, \\ \operatorname{div}\delta = 0, \\ \delta\mid_{t=0} = u_0 - \tilde{u}_0, \end{cases} \tag{2.28}$$

Lemma 2.21.

$$\mid (\delta\cdot\nabla u_1 \mid \delta) \mid \le \frac{1}{2}\|\partial_1 \delta\|_{L^2}^2 + C_0 r(u_1,t)\|\delta\|_{L^2}^2.$$

Here

$$r(u_1,t) = \|\partial_1 u_1(t)\|_{L^2}^{\frac{4}{3}} + \|\partial_2 u_1(t)\|_{L^2}^{\frac{4}{3}} + \|\partial_1 u_1(t)\|_{L^2} + \|\partial_2 u_1(t)\|_{L^2} + \|\partial_1\partial_2 u_1(t)\|_{L^2} + \|\partial_1\partial_2 u_1(t)\|_{L^2}^{\frac{4}{3}}.$$

Proof. See the inequality (1.21) in Chapter 1. □

Remark 2.22. *Note that C_0 is a constant which is omly related to Sobolev embeddings and some interpolation inequalities. From now on we always use the notation $C_0^{(1)}$, $C_0^{(2)}$, ... to denote different constants from line to line which are only related to C_0 (thus only related to Sobolev embeddings).*

Lemma 2.23. *We have the following estimates for δ:*

$$\mathbb{E}\|\delta(t)\|_{L^2}^2 \leq \exp\big\{-\lambda t + 2\int_0^t C_0 r(u_1,s)ds\big\}\|\delta_0\|_{L^2}^2.$$

Proof. By Itô's formula, we have:

$$d\|\delta(t)\|_{L^2}^2 + 2\|\partial_1\delta\|_{L^2}^2 + 2\lambda\|\delta\|_{L^2}^2 = -2(\delta\cdot\nabla u_1|\delta) + 2\langle(\sigma(u_1)-\sigma(u_2))dW_t,\delta\rangle + \|\sigma(u_1)-\sigma(u_2)\|_{L^2}^2 dt.$$

Thus by taking the expectation

$$\frac{d}{dt}\mathbb{E}\|\delta\|_{L^2}^2 + 2\mathbb{E}\|\partial_1\delta\|_{L^2}^2 + 2\lambda\mathbb{E}\|\delta\|_{L^2}^2 \leq 2\mathbb{E}|(\delta\cdot\nabla u_1|\delta)_{L^2}| + \mathbb{E}\|\sigma(u_1)-\sigma(u_2)\|_{L^2}^2.$$

From Lemma 2.21, since $L_1 < \lambda$ and $L_2 < \frac{1}{5}$, we deduce

$$\frac{d}{dt}\mathbb{E}\|\delta(t)\|_{L^2}^2 + \mathbb{E}\|\partial_1\delta(t)\|_{L^2}^2 + \lambda\mathbb{E}\|\delta(t)\|_{L^2}^2 \leq 2C_0 r(u_1,t)\mathbb{E}\|\delta(t)\|_{L^2}^2.$$

By Gronwall inequality,

$$\mathbb{E}\|\delta(t)\|_{L^2}^2 \leq \exp\big\{-\lambda t + 2\int_0^t C_0 r(u_1,s)ds\big\}\|\delta_0\|_{L^2}^2.$$

□

Proof of $A2'$

By Hölder's and Young's inequality,

$$\begin{aligned}&2\int_0^t C_0 r(u_1,s)ds\\ &\leq \frac{\lambda}{2}t + \Big(\frac{C_0^{(1)}}{\lambda} + C_0^{(1)}\Big)\int_0^t \|\nabla u_1\|_{L^2}^2 ds + \Big(\frac{C_0^{(1)}}{\lambda} + C_0^{(1)}\Big)\int_0^t \|\partial_1\partial_2 u_1\|_{L^2}^2 ds.\end{aligned}$$

Thus

$$\begin{aligned}&\mathbb{E}\|\delta(t)\|_{L^2}^2\\ &\leq \|\delta_0\|_{L^2}^2 \exp\Big\{-\frac{\lambda}{2}t + \Big(\frac{C_0^{(1)}}{\lambda} + C_0^{(1)}\Big)\int_0^t \|\nabla u_1\|_{L^2}^2 ds + \Big(\frac{C_0^{(1)}}{\lambda} + C_0^{(1)}\Big)\int_0^t \|\partial_1\partial_2 u_1\|_{L^2}^2 ds\Big\}.\end{aligned}$$

Recall that we have defined

$$\tilde{E}_u(t) = \|\nabla u(t)\|_{L^2}^2 + 2\int_0^t \|\partial_1 \nabla u(s)\|_{L^2}^2 ds + \lambda \int_0^t \|\nabla u(s)\|_{L^2}^2 ds.$$

Thus we have

$$\mathbb{E}\|\delta(t)\|_{L^2}^2 \leq \|\delta_0\|_{L^2}^2 \exp\big\{ -\frac{\lambda}{2}t + C_0^{(1)}(1+\frac{1}{\lambda}+\frac{1}{\lambda^2})\tilde{E}_{u_1}(t)\big\}.$$

Thus if we have $\tilde{E}_{u_1}(t) \leq \frac{\lambda^3}{4C_0^{(1)}(1+\lambda+\lambda^2)}t$, we will have

$$\|\delta(t)\|_{L^2}^2 \leq \|\delta_0\|_{L^2}^2 \exp\{-\frac{\lambda}{4}t\}.$$

From Lemma 2.20 we deduce:

$$P(\tilde{E}_{u_1}(t) > \frac{\lambda^3}{4C_0^{(1)}(1+\lambda+\lambda^2)}t) \leq \frac{2e^{\gamma_0\|\nabla u_0^1\|_{L^2}^2}}{\exp\{\gamma_0(\frac{\lambda^3}{4C_0^{(1)}(1+\lambda+\lambda^2)} - K_2)t\}}.$$

By assumption $\|\nabla u_0^1\|_{L^2}^2 \leq 2C_1$. Thus if we have $\frac{\lambda^3}{4C_0^{(1)}(1+\lambda+\lambda^2)} > K_2$, we have

$$P\big(\|\delta(t)\|_{L^2}^2 \geq \|\delta_0\|_{L^2}^2 \exp\{-\frac{\lambda}{4}t\}\big) \leq 2e^{2\gamma_0 C_1 - \gamma_0(\frac{\lambda^3}{4C_0^{(1)}(1+\lambda+\lambda^2)} - K_2)t}.$$

Take $\gamma < \min\{\frac{\lambda}{4}, \gamma_0(\frac{\lambda^3}{4C_0^{(1)}(1+\lambda+\lambda^2)} - K_2)\}$.
Then we finish the proof of A2'.

2.4.5 Proof of A0

A0 is equivalent to the following lemma:

Lemma 2.24. *Denote by $\mathcal{P}(\tilde{H}^1)$ the space of all the Borel probability measures on $\tilde{H}^1$ with the metric of the total variation distance. The law of $u(t, W, u_0)$ in $\tilde{H}^1$ (which we denote by $\mathcal{D}\big(u(t, W, u_0)\big) \in \mathcal{P}(\tilde{H}^1)$) measurably depends on the initial value $u_0 \in \tilde{H}^1$.*

Let us first briefly recall the definition of the total variation distance. Denote by $\mathcal{M}(\tilde{H}^1)$ the space of all the finite signed measures on $(\tilde{H}^1, \mathscr{B}(\tilde{H}^1))$ with the total variation norm. $\mathcal{P}(\tilde{H}^1)$, as a subset of $\mathcal{M}(\tilde{H}^1)$, is endowed with the following total variation distance:

$$\|\mu - \nu\|_{var} = \frac{1}{2} \sup_{f:\tilde{H}^1 \to \mathbb{R} \text{ measurable}, \|f\|_{L^\infty} \leq 1} \Big|\int_{\tilde{H}^1} f d\mu - \int_{\tilde{H}^1} f d\nu\Big|,$$

where $\|f\|_{L^\infty}$, as we all know, is the essential supremum of $|f|$.
On the other hand, the space $\mathcal{M}(\tilde{H}^1)$ can also be equipped with the following norm generate by bounded Lipschitz functions on L^2:

$$\|\mu\|_{**} = \sup_{f\in Lip(L^2), \|f\|_{Lip(L^2)}\le 1} |\int_{\tilde{H}^1} f d\mu|, \tag{2.29}$$

where $Lip(L^2)$ is the space of bounded Lipschitz functions from L^2 to $\mathbb{R}$ with the norm

$$\|f\|_{Lip(L^2)} := \|f\|_{L^\infty} + \sup_{u,v\in L^2, u\neq v} \frac{|f(u)-f(v)|}{\|u-v\|_{L^2}}.$$

Note that, the integral in (2.29) is meaningful because $\mathscr{B}(\tilde{H}^1) = \mathscr{B}(L^2)\cap \tilde{H}^1$, which is due to the fact that $\tilde{H}^1$ and L^2 are both standard Borel spaces. (See Appendix Definition B.2 for the definition of the standard Borel space.)

Remark 2.25. *$\|\cdot\|_{**}$ is really a norm, i.e. $\|\mu\|_{**} = 0$ if and only if $\mu = 0$. Note that, $\mu \in \mathcal{M}(\tilde{H}^1)$ could be also viewed as a finite signed measure μ' on $(L^2, \mathscr{B}(L^2))$. Therefore, if $\|\mu\|_{**} = 0$, then for any bounded Lipschitz function $\tilde{f}$ from L^2 to $\mathbb{R}$,*

$$\int_{L^2} \tilde{f} d\mu' = 0.$$

Hence $\mu' = 0$. Thus we know $\mu = 0$.

Now we show the following important proposition:

Proposition 2.26. *The Borel σ-algebras of $(\mathcal{P}(\tilde{H}^1), \|\cdot\|_{var})$ and $(\mathcal{P}(\tilde{H}^1), \|\cdot\|_{**})$ are the same.*

Before we prove the Proposition 2.26, we would like to point out that if we can prove the proposition, then together with Lemma 2.23, we immediately obtain Lemma 2.24.

Proof of Proposition 2.26

Denote the Borel σ-algebras of $(\mathcal{P}(\tilde{H}^1), \|\cdot\|_{var})$ and $(\mathcal{P}(\tilde{H}^1), \|\cdot\|_{**})$ by $\mathscr{B}(\|\cdot\|_{var})$ and $\mathscr{B}(\|\cdot\|_{**})$ respectively.

By Theorem 17.23 and 17.24 of [58] we know that $(\mathcal{P}(\tilde{H}^1), \mathscr{B}(\|\cdot\|_{var}))$ is a standard Borel space. (See Appendix Definition B.2 for the definition of the standard Borel space.)

By the proof of Remark 2.25, we claim that $\mathscr{B}(\|\cdot\|_{**})$ is countably generated. (See Appendix Definition B.1 for the definition of the countably generated.) Indeed let $\{\tilde{f}_n\}_{n\ge 1}$ be the countable dense subset of bounded Lipschitz functions from L^2 to $\mathbb{R}$ and denote by f_n the restriction of $\tilde{f}_n$ to $\tilde{H}^1$. Then $\{f_n\}_{n\ge 1}$ generates $(\mathcal{P}(\tilde{H}^1), \mathscr{B}\|\cdot\|_{**})$. Hence $(\mathcal{P}(\tilde{H}^1), \mathscr{B}\|\cdot\|_{**})$ is countably generated.

By Theorem B.3 we finish the proof.

□

2.4.6 Conclusion

Since we have verified all the conditions (except for some modifications to A2), we can go through the proof of Theorem 2.1 in [75]. Following the method of construction, we can still obtain (2.22) in Section 2.9 of [75], that is, there exist $\gamma' > 0$ and $C > 0$, such that

$$\mathbb{E}(\|u_1(t) - u_2(t)\|_{L^2} \wedge 1) \leq Ce^{-\gamma' t}(\|u_0^1\|_{H^1}^2 + \|u_0^2\|_{H^1}^2).$$

Then for any $f \in Lip(L^2)$, and any $u_0^1, u_0^2 \in \tilde{H}^1$,

$$\mathbb{E}(|f(u_1(t)) - f(u_2(t))|) \leq Ce^{-\gamma' t}\|f\|_{Lip(L^2)}(\|u_0^1\|_{H^1}^2 + \|u_0^2\|_{H^1}^2 + 1).$$

Since we have $P_t f(u_0^i) = \mathbb{E}f(u_i(t))$ for $i = 1, 2$, we obtain

$$|P_t f(u_0^1) - P_t f(u_0^2)| \leq Ce^{-\gamma' t}\|f\|_{Lip(L^2)}(\|u_0^1\|_{H^1}^2 + \|u_0^2\|_{H^1}^2 + 1).$$

Denote the invariant measure we obtained in the Theorem 2.15 by μ and by the proof of the existence of invariant measures, we have

$$\int_{H^1} \|u\|_{H^1}^2 d\mu < \infty.$$

Now for any measure Λ in $(\tilde{H}^1, \mathcal{B}(\tilde{H}^1))$, and any $f \in Lip(L^2)$, we have

$$\begin{aligned}
&|P_t^* \Lambda f - \mu(f)| \\
&= |\int P_t f d\Lambda - \int f d\mu| \\
&= |\int P_t f d\Lambda - \int P_t f d\mu| \\
&\leq \int_{\tilde{H}^1}\int_{\tilde{H}^1} |\mathbb{E}[f(u_1(t,x))] - \mathbb{E}[f(u_2(t,y))]| d\Lambda(x) d\mu(y) \\
&\leq e^{-\gamma' t}\|f\|_{Lip(L^2)} \int_{\tilde{H}^1}\int_{\tilde{H}^1} (1 + \|x\|_{\tilde{H}^1}^2 + \|y\|_{H^1}^2) d\Lambda(x) d\mu(y) \\
&\leq Ce^{-\gamma' t}\|f\|_{Lip(L^2)}\big(1 + \int_{\tilde{H}^1} \|x\|_{\tilde{H}^1}^2 d\Lambda(x)\big),
\end{aligned}$$

where $u_1(t,x)$, $u_2(t,y)$ are the solutions to the equations (2.19) with the initial value x and y, respectively.

2.5 Stationary Solutions of Vorticity Equations of Non-damping Euler Cases: Introduction of Space White Noise Distribution and Previous Results on the Torus

From now on in this chapter we always consider the equations without a damping term. In this section we recall the previous results that space white noise is a stationary solution of the vorticity form of (deterministic) Euler equations. First let us recall the deterministic incompressible Euler equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p, \\ \operatorname{div} u = 0, \end{cases}$$

and the vorticity form

$$\partial_t \omega + \mathcal{K}\omega \cdot \nabla w = 0, \tag{2.30}$$

where ω is the vorticity and $\mathcal{K}$ is the Biot-Savart operator with the following form:

$$u_t(x) = \mathcal{K}\omega_t(x) = \int_D K(x-y)\omega_t(y)dy,$$

where D, is the domain of $\mathbb{R}^2$ or $\mathbb{T}^2$ and K is the Biot-Savart Kernel.

2.5.1 Introduction of Space White Noise on $\mathbb{T}^2$

First we recall the definition and the construction of the space white noise distribution on the torus $\mathbb{T}^2 = \mathbb{T}^2_{2\pi} = (\mathbb{Z}/2\pi\mathbb{Z})^2$ (see, for example, [37]). The following definition and construction mainly come from [37], which we write here for completeness.

A **space white noise (variable)** ω on $\mathbb{T}^2$ is a Gaussian distributional valued random variable mapping from some probability space $(\Xi, \mathcal{F}, \mathbb{P})$ to $C^\infty(\mathbb{T}^2)'$ such that

- For any $\phi \in C^\infty(\mathbb{T}^2)$, $\langle \omega, \phi \rangle$ is a real valued Gaussian random variable.
- For any $\phi, \psi \in C^\infty(\mathbb{T}^2)$,

$$\mathbb{E}\langle \omega, \phi \rangle \langle \omega, \psi \rangle = \langle \phi, \psi \rangle_{L^2(\mathbb{T}^2)}.$$

We call the distribution of a space white noise on $C^\infty(\mathbb{T}^2)'$ the **space white noise distribution** (on $C^\infty(\mathbb{T}^2)'$).

Now we show the existence of the space white noise variable by constructing it.
Define

$$\omega = \sum_{n\in\mathbb{Z}^2} G_n(\theta)\frac{1}{2\pi}e^{inx},$$

where $\theta \in \Xi$, $G_n = \overline{G_{-n}}$, and G_n, $n \in \mathbb{Z}^2_+ \cup \{0\}$ are independent random variables with standard Gaussian distributions. Thus we have

$$\mathbb{E}[G_n G_m] = \delta_{mn}.$$

for $m, n \in \mathbb{Z}^2_+$.
Hence it is easy to verify that w is a space white noise, the details of which can be found in [37].

Remark 2.27.

- *We know that $\omega \in H^{-1-\epsilon}$ $\mathbb{P}$-a.s. for any $\epsilon > 0$, the proof of which can be found in [37, Section 2.1]. Therefore, the space white noise distribution is supported in H^{-1-}.*

- *By the definition of the space white noise on $\mathbb{T}^2$, any random variable with space white noise distribution on $\mathbb{T}^2$ in some probability space could be expanded by the series $\omega = \sum_{n\in\mathbb{Z}^2} G_n(\theta)\frac{1}{2\pi}e^{inx}$, where G_n, $n \in \mathbb{Z}^2_+ \cup \{0\}$ are independent random variables with standard Gaussian distributions in the same probability space.*

- *From now on we do not distinguish a space white noise (variable) and the space white noise distribution when no confusion occurs.*

2.5.2 Previous Results

It is proved in [37] that there exists a white noise solution which satisfies the weak formulation of the Euler equations as follows:

$$\langle \omega_t, \phi\rangle = \langle \omega_0, \phi\rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_\phi\rangle\, ds, \tag{2.31}$$

where

$$H_\phi(x, y) := \frac{1}{2}K(x-y)\left(\nabla\phi(x) - \nabla\phi(y)\right),$$

and $\langle \omega_s \otimes \omega_s, H_\phi\rangle$ is defined as the $L^2(\Xi)$ limit of the approximating sequences $\left\langle \omega\otimes\omega, H_\phi^n\right\rangle$.
Here $H_\phi^n \in H^{2+}(\mathbb{T}^2 \times \mathbb{T}^2))$, where

$$H^{2+}(\mathbb{T}^2 \times \mathbb{T}^2)) = \bigcup_{s>2} H^s(\mathbb{T}^2),$$

and $H^s(\mathbb{T}^2)$ are the non-homogeneous Sobolev spaces on $\mathbb{T}^2$ defined in (1.4).

Moreover, $\{H^n_\phi\}_{n\geq 1}$ are assumed to be symmetric and approximate H_ϕ in the following sense:

$$\lim_{n\to\infty}\int\int \left(H^n_\phi - H_\phi\right)^2(x,y)\,dxdy = 0$$

$$\lim_{n\to\infty}\int H^n_\phi(x,x)\,dx = 0.$$

We refer to [37] for more details.

Remark 2.28.

1. *(2.31) is formally equivalent to the equation (2.30) if we note that the Biot-Savart kernel satisfies*
$$K(x-y) = -K(y-x).$$

2. *However, there is no existence of the white noise stationary solutions for Euler equations on the whole space $\mathbb{R}^2$ due to the non-integrability of the Biot-Savart kernel. In fact, we have difficulty to define the nonlinear term $\langle\omega_s\otimes\omega_s, H_\phi\rangle$. We will explain it in more detail in the next section.*

2.5.3 Introduction of Space White Noise on $\mathbb{R}^2$

When it comes to the cases of the whole space, first we recall the definition of the space white noise distribution ω on $\mathbb{R}^2$ as a Gaussian distributional valued stochastic process mapping from Ξ to $C^\infty_c(\mathbb{R}^2)'$ such that

- For any $\phi\in C^\infty_c(\mathbb{R}^2)$, $\langle\omega,\phi\rangle$ is a real valued Gaussian random variable.
- For any $\phi,\psi\in C^\infty_c(\mathbb{R}^2)$,
$$\mathbb{E}\langle\omega,\phi\rangle\langle\omega,\psi\rangle = \langle\phi,\psi\rangle_{L^2(\mathbb{R}^2)}.$$

Similar to the case of the torus, we will give a construction of a random variable with this distribution by taking the limit of space white noise on the torus and letting the volume of the torus go to infinity. For fixed M, denote $\mathbb{T}^2_M = (\mathbb{Z}/M\mathbb{Z})^2$ to be the torus of length M.
Define

$$\{e^M_n\}_{n\in\mathbb{Z}^2} = \{\frac{1}{M}e^{2\pi i n\cdot x/M}\}_{n\in\mathbb{Z}^2} \tag{2.32}$$

as the orthonormal basis of $L^2(\mathbb{T}^2_M,\mathbb{C})$.

For $u \in C^\infty(\mathbb{T}_M^2)$, we consider the following Fourier expansion of u on the torus:

$$u(x) = \sum_{k\in\mathbb{Z}^2} \hat{u}_k^M e_k^M \quad \text{with} \quad \hat{u}_k^M = \overline{\hat{u}_{-k}^M},$$

where $\hat{u}_k^M := \frac{1}{M}\int_{\mathbb{T}_M^2} u(x)e^{-2\pi i k\cdot x/M}dx$ denotes the kth Fourier coefficient of u on $\mathbb{T}_M^2$. Define the Sobolev norm on $\mathbb{T}_M^2$ for $s \in \mathbb{R}$:

$$\| u \|^2_{H^s(\mathbb{T}_M^2)} := \sum_{k\in\mathbb{Z}^2} \Big(1 + (\frac{2\pi|k|}{M})^2\Big)^s |\hat{u}_k^M|^{2s}.$$

Define the space $H^s(\mathbb{T}_M^2)$ as the completion of $C^\infty(\mathbb{T}_M^2)$ under the norm $\| \cdot \|_{H^s(\mathbb{T}_M^2)}$. Let

$$\omega^M = \sum_{n\in\mathbb{Z}^2} G_n^M(\theta)e_n^M, \tag{2.33}$$

where $e_n^M = \frac{1}{M}e^{2\pi i n\cdot x/M}$, $G_n^M = \overline{G_{-n}^M}$, and G_n^M, $n \in \mathbb{Z}_+^2 \cup \{0\}$ are independent random variables with standard Gaussian distributions. Thus we have

$$\mathbb{E}[G_n^M G_m^M] = \delta_{mn}$$

for $m, n \in \mathbb{Z}_+^2$.

We extend (2.33) periodically to a distribution $\bar{\omega}^M$ on $\mathbb{R}^2$. (With the Fourier series we can extend it directly by viewing it as the series on $\mathbb{R}^2$.) That is,

$$\bar{\omega}^M = \sum_{n\in\mathbb{Z}^2} G_n^M(\theta)e_n^M \text{ on } \mathbb{R}^2.$$

However, $\bar{\omega}^M$ is not uniformly bounded with respect to M in $H^{-1-}(\mathbb{R}^2)$, thus it is not weak convergent. Therefore, we need to introduce a weighted function and weighted distribution space.

Definition 2.29 (Weighted Sobolev norms and spaces)**.** *Let $\rho \in L^1_{loc}(\mathbb{R}^d)$ and $\rho(x) \geq 0$. Define the weighted Sobolev norms $\| \cdot \|_{H^s(\mathbb{R}^d,\rho)}$ by*

$$\| \cdot \|_{H^s(\mathbb{R}^d,\rho)} := \|\rho \cdot \|_{H^s(\mathbb{R}^d)}.$$

Define the weighted Sobolev spaces $H^s(\mathbb{R}^d, \rho)$ which is the subset of $\mathcal{S}'(\mathbb{R}^d)$ with $H^s(\mathbb{R}^d, \rho)$ norm finite.

Since in this chapter we consider the $2D$ cases, from now on for simplicity we use the notation $H^s(\rho)$ instead of $H^s(\mathbb{R}^2,\rho)$ if there is no confusion. Moreover, we define the space $H^{-1-}(\rho)$ as the space $\bigcap_{\epsilon>0} H^{-1-\epsilon}(\rho)$ with the following Frechet metric:

$$d(u,v)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|u-v\|_{H^{-1-\frac{1}{n}}(\rho)}}{1+\|u-v\|_{H^{-1-\frac{1}{n}}(\rho)}}.$$

Similarly, $H^{-1-}(\rho)$ is also a complete metric space. Convergence in $H^{-1-}(\rho)$ is equivalent to convergence in every $H^{-1-\epsilon}(\rho)$ for each $\epsilon>0$.
Let $\rho(x)=\frac{1}{\langle x\rangle^{\sigma}}$ and $\rho'(x)=\frac{1}{\langle x\rangle^{\sigma'}}$, where $\langle x\rangle=(1+|x|^2)^{\frac{1}{2}}$.
The following lemma is shown in [89, Theorem 6.31]:

Lemma 2.30. *For $0<\sigma'<\sigma$, and $s'>s$, the distributional space $H^{s'}(\rho')$ is compactly embedded in $H^s(\rho)$.*

Therefore, we have the following lemma.

Lemma 2.31. *For any $\epsilon>0$ and $\sigma>2$, the distribution of $\{\bar{\omega}^M\}_{M>0}$ is tight in the weighted Sobolev space $H^{-1-\epsilon}(\rho)$. Hence $\{\bar{\omega}^M\}_{M>0}$ is tight in the metric space $H^{-1-}(\rho)$. Moreover, denote by $\bar{\mu}_M$ the distribution of $\{\bar{\omega}^M\}$ in $H^{-1-}(\rho)$. Then we have $\bar{\mu}_M$ converges weakly to the space white noise distribution on $\mathbb{R}^2$ in $H^{-1-}(\rho)$ as $M\to\infty$.*

Proof. By the definition of the weighted Sobolev norm,

$$\begin{aligned}
&\mathbb{E}\|\bar{\omega}^M\|^2_{H^{-1-\epsilon}(\rho)}\\
&=\mathbb{E}\|\rho\bar{\omega}^M\|^2_{H^{-1-\epsilon}(\mathbb{R}^2)}\\
&=\mathbb{E}\|\sum_{n\in\mathbb{Z}^2}\rho e_n^M G_n^M\|^2_{H^{-1-\epsilon}(\mathbb{R}^2)}\\
&=\int_{\mathbb{R}^2}\mathbb{E}|\int_{\mathbb{R}^2}\frac{\frac{1}{M}\sum_{n\in\mathbb{Z}^2}G_n^M e^{2\pi i n\cdot x/M}}{\langle x\rangle^{\sigma}}e^{-i\xi\cdot x}dx|^2(1+|\xi|^2)^{-1-\epsilon}d\xi\\
&\le\frac{1}{M^2}\int_{\mathbb{R}^2}\mathbb{E}|\int_{\mathbb{R}^2}\frac{|\sum_{n\in\mathbb{Z}^2}G_n^M e^{2\pi i n\cdot x/M}|}{\langle x\rangle^{\sigma}}dx|^2(1+|\xi|^2)^{-1-\epsilon}d\xi\\
&\lesssim\frac{1}{M^2}\int_{\mathbb{R}^2}\mathbb{E}\int_{\mathbb{R}^2}\frac{(\sum_{n\in\mathbb{Z}^2_+}G_n^M e^{2\pi i n\cdot x/M})^2}{\langle x\rangle^{\sigma}}dx(1+|\xi|^2)^{-1-\epsilon}d\xi,
\end{aligned}$$

where the last inequality is due to the Holder's inequality.

Since $\mathbb{E}[G_n^M G_m^M] = \delta_{mn}$ for $m, n \in \mathbb{Z}_+^2$ and $\sum_{n\in\mathbb{Z}^2} e^{4\pi i n\cdot x/M} \lesssim M^2$, we obtain

$$\mathbb{E}\|\bar{\omega}^M\|^2_{H^{-1-\epsilon}(\rho)} \lesssim \int_{\mathbb{R}^2} |\int_{\mathbb{R}^2} \frac{1}{\langle x\rangle^\sigma} dx|^2 (1+|\xi|^2)^{-1-\epsilon} d\xi$$
$$\lesssim 1.$$

Note that $H^{-1-\epsilon}(\rho)$ can be compactly embedded into $H^{-1-2\epsilon}(\rho')$. Since $\sigma > 2$ and $\epsilon > 0$, we obtain the conclusion.

Therefore, by the Prokhorov's theorem and Skorokhod's representation theorem there exists a subsequence M_k such that μ_{M_k} converges weakly to the limit $\bar{\mu}$. Moreover, there exists a sequence of random variables $\bar{\omega}'^{M_k}$ on another probability space $(\Xi', \mathcal{F}', \mathbb{P}')$, which have distributions μ_{M_k}, such that $\bar{\omega}'^{M_k}$ converges weakly to $\bar{\omega}$ $P' - a.e.$, with $\bar{\omega}$ has the distribution $\bar{\mu}$.

We claim that $\bar{\mu}$ is the space white noise distribution on $\mathbb{R}^2$.

Indeed, we know that for $\phi \in C_c^\infty(\mathbb{R}^2)$, there exists some k_0 such that for $k \geq k_0$, ϕ is supported on the ball with the radius smaller than $\frac{M_k}{2}$, then we can view ϕ as a function ϕ_M on the torus $\mathbb{T}_M$, thus for $k \geq k_0$,

$$\langle \bar{\omega}^{M_k}, \phi\rangle = \langle \omega^{M_k}, \phi_{M_k}\rangle$$

is centred Gaussian, therefore, $\langle \bar{\omega}, \phi\rangle$ is centred Gaussian.

Moreover, similarly, from the argument of the explanation of Definition 4 of [73], if we fix $\phi, \psi \in C_c^\infty$, when k is large enough, (ϕ and ψ are supported on the ball with the radius smaller than $\frac{M_k}{2}$), we have

$$\mathbb{E}\langle \bar{\omega}^{M_k}, \phi\rangle \langle \bar{\omega}^{M_k}, \psi\rangle = \langle \phi, \psi\rangle_{L^2(\mathbb{R}^2)},$$

thus we have

$$\mathbb{E}\langle \bar{\omega}, \phi\rangle \langle \bar{\omega}, \psi\rangle = \langle \phi, \psi\rangle_{L^2(\mathbb{R}^2)},$$

which finishes the proof of our claim.

□

2.6 Stationary Solutions on $\mathbb{R}^2$: Turning to mSQG Equations

Just as mentioned in Section 2.5, we cannot show the existence of the white noise stationary solution on the whole space. Instead we show a similar result for mSQG equations.

In [71] the authors studied the mSQG equation on the 2D torus $\mathbb{T}^2$, perturbed by multiplicative transport noise, and proved that the white noise measure on $\mathbb{T}^2$ is the invariant measure. We will show a similar result to the deterministic mSQG equations on the whole space $\mathbb{R}^2$.

2.6.1 Introduction of mSQG equations

First recall the following (deterministic) mSQG equations on both $\mathbb{T}^2_M$ and $\mathbb{R}^2$:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ \qquad\qquad u = \nabla^\perp (-\Delta)^{-(1+\epsilon)/2} \omega, \end{cases} \tag{2.34}$$

where

$$\nabla^\perp = (-\partial_2, \partial_1).$$

Note that when $\epsilon = 1$ it is Euler equation and when $\epsilon = 0$ it is Surface Quasi-Geostrophic equation (SQG) equation.

Remark 2.32.

- *On the whole space, we know that the operator $\nabla^\perp(-\Delta)^{-(1+\epsilon)/2}$ is defined by the Fourier multiplier $\xi^\perp|\xi|^{-(1+\epsilon)}$. If we write it in the form of the convolution, it is equivalent to the convolution with*

$$K_\epsilon := \mathcal{F}^{-1}\big(\xi^\perp|\xi|^{-(1+\epsilon)}\big).$$

Recall that on $\mathbb{R}^2$, the Fourier transform and Fourier inverse transform are defined as follows:

$$\hat{f}(\xi) = \mathcal{F}f(\xi) := \int_{\mathbb{R}^2} f(x) e^{-ix\cdot\xi} dx,$$

and

$$\mathcal{F}^{-1} f(\xi) := \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(x) e^{ix\cdot\xi} dx.$$

Thus we know that K_ϵ is dominated by $C_\epsilon \frac{1}{|x|^{2-\epsilon}}$ for some constant C_ϵ. Obviously the kernel is singular at the point $(0,0)$.

- *On the torus $\mathbb{T}^2_M$, the operator $\nabla^\perp(-\Delta)^{-(1+\epsilon)/2}$ is defined by Fourier transform. Let f be a distribution in some Sobolev space $H^{-N}(\mathbb{T}^2_M)$, for $N > 0$ with the Fourier expansion*

$$f(x) = \sum_{k\in\mathbb{Z}^2\setminus\{(0,0)\}} \hat{f}^M_k e^M_k \quad \text{with} \quad \hat{f}^M_k = \overline{\hat{f}^M_{-k}},$$

where $\hat{f}_k^M := \frac{1}{M}\int_{\mathbb{T}_M^2} f(x)e^{-2\pi ik\cdot x/M}dx$ *denotes the* k*th Fourier coefficient of* f *on* $\mathbb{T}_M^2$ *and recall that we have defined* $e_k^M = \frac{1}{M}e^{2\pi ik\cdot x/M}$ *before. Then the operator* $\nabla^\perp(-\Delta)^{-(1+\epsilon)/2}$ *is:*

$$\nabla^\perp(-\Delta)^{-(1+\epsilon)/2}f = \sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}} \big(\frac{M}{2\pi}\big)^\epsilon \frac{k^\perp}{|k|^{1+\epsilon}}\hat{f}_k^M e_k^M.$$

If we write it in the form of convolution,

$$\begin{aligned}\nabla^\perp(-\Delta)^{-(1+\epsilon)/2}f &= \sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}} \big(\frac{M}{2\pi}\big)^\epsilon \frac{k^\perp}{|k|^{1+\epsilon}}\frac{1}{M^2}\int_{\mathbb{T}_M^2} f(\xi)e^{-2\pi ik\cdot\xi/M}d\xi\ e^{2\pi ik\cdot x/M}\\ &= \sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}} \big(\frac{M}{2\pi}\big)^\epsilon \frac{k^\perp}{|k|^{1+\epsilon}}\frac{1}{M^2}e^{2\pi ik\cdot\cdot/M} * f\\ &=: K_\epsilon^M * f,\end{aligned} \tag{2.35}$$

where the convolution is defined on the torus $\mathbb{T}_M^2 = [-\frac{M}{2},\frac{M}{2}]^2$.

Hence for $x\in\mathbb{T}_M^2$,

$$|x|^{2-\epsilon}K_\epsilon^M(x) = \sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}} \big(\frac{1}{2\pi}\big)^\epsilon \frac{k^\perp}{|k|^{1+\epsilon}}\big(\frac{|x|}{M}\big)^{2-\epsilon}e^{2\pi ik\cdot x/M}.$$

Let $\eta = \frac{x}{M} \in [-\frac{1}{2},\frac{1}{2}]^2\backslash\{(0,0)\}$, *then*

$$|x|^{2-\epsilon}K_\epsilon^M(x) = \sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}} \big(\frac{1}{2\pi}\big)^\epsilon \frac{k^\perp}{|k|^{1+\epsilon}}|\eta|^{2-\epsilon}e^{2\pi ik\cdot\eta}.$$

We discuss this following two situations:

1. η *is small:*

   Note that when η *goes to* 0, *since*

$$\sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}} \big(\frac{1}{2\pi}\big)^\epsilon \frac{k^\perp}{|k|^{1+\epsilon}}|\eta|^{2-\epsilon}e^{2\pi ik\cdot\eta} = \frac{1}{4\pi^2}\sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}} \frac{2\pi\eta k^\perp}{\big(2\pi|\eta k|\big)^{1+\epsilon}}\big(2\pi\eta\big)^2 e^{2\pi ik\cdot\eta},$$

   it would converge to the integration

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\xi^\perp|\xi|^{-(1+\epsilon)}e^{i\xi}d\xi,$$

   which is a constant. Thus there exists a small δ , *such that when* $|\eta|<\delta$, $K_\epsilon^M(x)\le\frac{C}{|x|^{2-\epsilon}}$, *where* C *is a constant only depending on* ϵ.

2. *For* $\delta < |\eta| \le \frac{1}{2}$*, Since* $|\sum_{|k|\le N} e^{2\pi i k\cdot\eta}|$ *has uniformly bound* $\left(\frac{1}{1-\cos 2\pi\delta}\right)^2$*, the series* $\sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}} \left(\frac{1}{2\pi}\right)^{\epsilon} \frac{k^{\perp}}{|k|^{1+\epsilon}} |\eta|^{2-\epsilon} e^{2\pi i k\cdot\eta}$ *converges and has uniform bound not depending on* η.

Thus to conclude, we have proved the following lemma:

Lemma 2.33. *Let*

$$K_\epsilon = \mathcal{F}^{-1}\big(\xi^{\perp}|\xi|^{-(1+\epsilon)}\big),$$

$$K_\epsilon^M = \sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}} \big(\frac{M}{2\pi}\big)^{\epsilon} \frac{k^{\perp}}{|k|^{1+\epsilon}} \frac{1}{M^2} e^{2\pi i k\cdot\cdot/M}$$

be the kernel corresponding to the operator $\nabla^{\perp}(-\Delta)^{-(1+\epsilon)/2}$ *on* $\mathbb{R}^2$ *and* $\mathbb{T}_M^2$*, respectively. Then there exists a common constant* C_ϵ *which does not depend on* M*, such that*

$$K_\epsilon(x) \le \frac{C_\epsilon}{|x|^{2-\epsilon}},$$

and

$$K_\epsilon^M(x) \le \frac{C_\epsilon}{|x|^{2-\epsilon}},$$

for any $x \in \mathbb{R}^2$, $x \in \mathbb{T}_M^2$*, respectively.*

- *Moreover, note that if we fix some* $x \ne 0$ *and let* M *goes to infinity, the sum*

$$\sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}} \big(\frac{M}{2\pi}\big)^{\epsilon} \frac{k^{\perp}}{|k|^{1+\epsilon}} \frac{1}{M^2} e^{2\pi i k\cdot x/M} = \frac{1}{4\pi^2} \sum_{k\in\mathbb{Z}^2\backslash\{(0,0)\}} \frac{\frac{2\pi}{M}k^{\perp}}{\big(\frac{2\pi}{M}|k|\big)^{1+\epsilon}} \big(\frac{2\pi}{M}\big)^2 e^{2\pi i k\cdot x/M}$$

converges to the integration

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^2} \xi^{\perp}|\xi|^{-(1+\epsilon)} e^{ix\cdot\xi} d\xi,$$

which is exactly the Fourier inverse transform of $\xi^{\perp}|\xi|^{-(1+\epsilon)}$.

In other word, we have the following lemma

Lemma 2.34. *For any* $x \in \mathbb{R}^2\backslash\{0\}$, $K_\epsilon^M(x)$ *converges pointwisely to* K_ϵ *as* M *goes to infinity.*

2.6.2 Definition of the Nonlinear Term and the White Noise Solutions on $\mathbb{R}^2$

Finally we come to our main result. First we introduce the definition of the white noise stationary solution of (2.34).

Definition 2.35. *Fix any $T > 0$. We say that a measurable map $\omega_\cdot : \Xi \times [0,T] \to C_c^\infty(\mathbb{R}^2)'$ with trajectories of class $C([0,T]; C_c^2(\mathbb{R}^2)')$ is a white noise stationary (weak) solution of (2.34), if it satisfies the following:*

1. *For fixed t, w_t is the space white noise distribution on $\mathbb{R}^2$.*

2. *For any $\phi \in C_c^\infty(\mathbb{R}^2)$,*

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \langle \omega_s \otimes \omega_s, H_{\phi,\epsilon} \rangle \, ds,$$

where

$$H_{\phi,\epsilon}(x,y) := \frac{1}{2} K_\epsilon (x-y) (\nabla \phi(x) - \nabla \phi(y)),$$

and we will introduce the definition of the term $\langle \omega_s \otimes \omega_s, H_{\phi,\epsilon} \rangle$ later.

Similarly to the paper [37] we also need to define the nonlinear term. Let $\rho(x) = \frac{1}{\langle x \rangle^\sigma}, \sigma > 2$. We may think that since $\bar{\omega} \in H^{-1-}(\rho)$, $\bar{\omega} \otimes \bar{\omega}$ should be in the space $H^{-2-}(\mathbb{R}^4, \rho \times \rho)$, where $H^{-2-}(\mathbb{R}^4, \rho \times \rho)$ is the intersection of all the weighted Sobolev spaces $H^{-2-\epsilon}(\mathbb{R}^4, \rho \times \rho)$ for all $\epsilon > 0$. Therefore, we can define $\langle \bar{\omega} \otimes \bar{\omega}, f \rangle$ when $f \in H^{2+}(\mathbb{R}^4, \rho^{-1} \times \rho^{-1})$, where $H^{2+}(\mathbb{R}^4, \rho^{-1} \times \rho^{-1})$ is the union of the spaces $H^{2+\epsilon}(\mathbb{R}^4, \rho^{-1} \times \rho^{-1})$ for all $\epsilon > 0$, even we do not use the property of the space white noise. In particular, it could be defined on $C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$.

However, $H_{\phi,\epsilon}$ does not belong to the space $H^{2+}(\mathbb{R}^4, \rho^{-1} \times \rho^{-1})$. Thus similar to [37], we will introduce the definition of the nonlinear term by constructing an approximating sequence.

First of all, the following lemma gives for smooth and compactly supported function ϕ, $H_{\phi,\epsilon} \in L^2(\mathbb{R}^2 \times \mathbb{R}^2)$.

Lemma 2.36. *For $\phi \in C_c^2(\mathbb{R}^2)$, $H_{\phi,\epsilon} \in L^2(\mathbb{R}^2 \times \mathbb{R}^2)$.*

Proof. We prove directly by calculation. Since

$$H_{\phi,\epsilon}(x,y) := \frac{1}{2} K_\epsilon (x-y) (\nabla \phi(x) - \nabla \phi(y)),$$

and $K_\epsilon(x-y) \le \frac{1}{|x-y|^{2-\epsilon}}$, $0<\epsilon<1$, assuming that ϕ is supported in the ball of radius R, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |H_{\phi,\epsilon}(x,y)|^2 dxdy \\
&\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{|\nabla\phi(x)-\nabla\phi(y)|^2}{|x-y|^{4-2\epsilon}} dxdy \\
&\le \int_{|x|\le 2R}\int_{|y|\le 2R} \frac{\|D^2\phi\|_{L^\infty}^2}{|x-y|^{2-2\epsilon}} dxdy + 2\int_{|x|\le R}\int_{|y|\ge 2R} \frac{|\nabla\phi(x)-0|^2}{|x-y|^{4-2\epsilon}} dxdy \\
&\le C(R)\|D^2\phi\|_{L^\infty}^2 + 2\|\nabla\phi\|_{L^\infty}^2 \pi R^2 \int_{|y|\ge 2R} \frac{1}{(|y|-R)^{4-2\epsilon}} dy \\
&\le C(R)(\|D^2\phi\|_{L^\infty}^2 + \|\nabla\phi\|_{L^\infty}^2),
\end{aligned}$$

where $C(R)$ is a constant only depends on R and the second inequality is due to the symmetric property of $H_{\phi,\epsilon}(x,y)$. □

Since we only used the property $K_\epsilon(x-y) \le \frac{1}{|x-y|^{2-\epsilon}}$, by Lemma 2.33 we immediately have the following corollary:

Corollary 2.37. *For any $\phi \in C_c^2(\mathbb{R}^2)$, we also assume that ϕ is supported in $[-\frac{M_0}{2}, \frac{M_0}{2}]^2$. Thus for any $M > M_0$, ϕ can be viewed as a function in $C^2(\mathbb{T}_M^2)$. Then for any $M > M_0$,*

$$H_{\phi,\epsilon}^M(x,y) := \frac{1}{2}K_\epsilon^M(x-y)(\nabla\phi(x)-\nabla\phi(y)),$$

is in $L^2(\mathbb{T}_M^2 \times \mathbb{T}_M^2)$. Moreover, there exists a constant C_ϕ which does not depend on M, such that

$$\|H_{\phi,\epsilon}^M(x,y)\|_{L^2(\mathbb{T}_M^2\times\mathbb{T}_M^2)} \le C_\phi.$$

Similar as Theorem 8 of [37], we will prove the following theorem which gives the approximating sequence.

Theorem 2.38. *Fix $\phi \in C_c^2(\mathbb{R}^2\times\mathbb{R}^2)$. Assume that $f_n \in C_c^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ are symmetric and approximate $H_{\phi,\epsilon}$ in the following sense:*

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(f_n - H_{\phi,\epsilon})^2(x,y)\,dxdy = 0$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^2} f_n(x,x)\,dx = 0.$$

Then the sequence of r.v.'s $\langle\bar\omega\otimes\bar\omega, f_n\rangle$ is a Cauchy sequence in mean square. We denote by

$$\langle\bar\omega\otimes\bar\omega, H_{\phi,\epsilon}\rangle$$

its limit. Moreover, the limit is the same if f_n is replaced by $\widetilde{f}_n$ with the same properties and such that $\lim_{n\to\infty}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(\widetilde{f}_n - f_n)^2(x,y)\,dxdy = 0$.

Proof. Without loss of generality we assume that for each n, f_n is supported in $[-\frac{n}{4},\frac{n}{4}]^4$.

In the following we will use the relation between white noise on the torus and the whole space, the details of which can be found in [73].
Since f_n is supported in $[-\frac{n}{4},\frac{n}{4}]^4$, it could also be viewed as a smooth function on $\mathbb{T}_M^2\times\mathbb{T}_M^2$ when $M\geq n$. By the explanation of Definition 4 in [73], we know

$$\langle\bar{\omega}\otimes\bar{\omega}, f_n\rangle = \langle\omega^n\otimes\omega^n, f_n\rangle = \langle\omega^m\otimes\omega^m, f_n\rangle, \tag{2.36}$$

for $m\geq n$, where f_n is understood as a function on $\mathbb{R}^2$, $\mathbb{T}_n^2$ and $\mathbb{T}_m^2$ respectively.
Therefore, now what we need to prove is the following:

$\langle\omega^n\otimes\omega^n, f_n\rangle$ converges in $L^2(\Xi)$ as n goes to $+\infty$.

To prove the convergence it suffices to prove it is a Cauchy sequence.
Since $\lim_{n\to\infty}\int_{\mathbb{R}^2} f_n(x,x)\,dx = 0$, it is equivalent to show that $\langle\omega^n\otimes\omega^n, f_n\rangle - \int f_n(x,x)\,dx$ is a Cauchy sequence in mean square. We have for $m\geq n$,

$$\begin{aligned}
&\mathbb{E}\left[\left|\langle\omega^n\otimes\omega^n, f_n\rangle - \int_{\mathbb{R}^2} f_n(x,x)\,dx - \langle\omega^m\otimes\omega^m, f_m\rangle + \int_{\mathbb{R}^2} f_m(x,x)\,dx\right|^2\right]\\
&= \mathbb{E}\left[\left|\langle\omega^n\otimes\omega^n, f_n\rangle - \int_{\mathbb{R}^2} f_n(x,x)\,dx - \langle\omega^m\otimes\omega^m, f_m\rangle + \int_{\mathbb{R}^2} f_m(x,x)\,dx\right|^2\right] \qquad (2.37)\\
&= \mathbb{E}\left[\left|\langle\omega^m\otimes\omega^m, (f_n - f_m)\rangle - \int_{\mathbb{R}^2}(f_n - f_m)(x,x)\,dx\right|^2\right].
\end{aligned}$$

where the second equality is due to (2.36).
By (ii) and (iii) of the Corollary 6 in [37], (for completeness we attach the corollary later in Corollary 2.39) we know that (2.37) equals

$$2\int_{\mathbb{T}_m^2}\int_{\mathbb{T}_m^2}(f_n - f_m)^2(x,y)\,dxdy = 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(f_n - f_m)^2(x,y)\,dxdy,$$

which implies the Cauchy property of $\langle\bar{\omega}\otimes\bar{\omega}, f_n\rangle$ in mean square. Hence $\langle\bar{\omega}\otimes\bar{\omega}, H_{\phi,\epsilon}\rangle$ is well defined.

Moreover, by a similar way we prove that if we replace f_n by $\tilde{f}_n$ with the same properties and such that $\lim_{n\to\infty}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(\widetilde{f}_n - f_n)^2(x,y)\,dxdy = 0$, $\langle\bar{\omega}\otimes\bar{\omega}, \tilde{f}_n\rangle$ also converges in mean square to $\langle\bar{\omega}\otimes\bar{\omega}, H_{\phi,\epsilon}\rangle$. □

Corollary 2.39. *See [37, Corollary 6]*
i) If $\omega^M : \Xi \to C^\infty(\mathbb{T}_M^2)'$ *is a white noise and* $f\in H^{2+}(\mathbb{T}_M^2\times\mathbb{T}_M^2)$*, then for every* $p\geq 1$ *there is a constant* $C_{p,M}>0$ *such that*

$$\mathbb{E}\left[\left|\langle\omega^M\otimes\omega^M, f\rangle\right|^p\right] \leq C_{p,M}\,\|f\|_{L^\infty}^p .$$

ii) We have $\mathbb{E}\left[\left\langle \omega^M \otimes \omega^M, f\right\rangle\right] = \int_{\mathbb{T}_M^2} f(x,x)\,dx$.
iii) If f *is symmetric, then*

$$\mathbb{E}\left[\left|\left\langle \omega^M \otimes \omega^M, f\right\rangle - \mathbb{E}\left[\left\langle \omega^M \otimes \omega^M, f\right\rangle\right]\right|^2\right] = 2\int_{\mathbb{T}_M^2}\int_{\mathbb{T}_M^2} f(x,y)^2\,dxdy.$$

We now give an example of the approximating sequence $\{f_n\}_{n\geq 1}$.

Constructuion of the approximating sequence $\{f_n\}_{n\geq 1}$

We have proved in Lemma 2.36 that for fixed $\phi \in C_c^2(\mathbb{R}^2)$, $H_{\phi,\epsilon} \in L^2(\mathbb{R}^2\times\mathbb{R}^2)$, thus there exists a sequence of function $g_n \in C_c^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ which converge to $H_{\phi,\epsilon}$ in $L^2(\mathbb{R}^2\times\mathbb{R}^2)$. We can also assume that g_n is symmetric (otherwise, we let $\tilde{g}_n = \frac{1}{2}\big(g_n(x,y)+g_n(y,x)\big)$.) Without loss of generality we assume that for each n, g_n is supported in $[-\frac{n}{4},\frac{n}{4}]^4$. Let $f_n = r_n g_n$, where $r_n(x,y)$ is defined as follows:

$$r_n \begin{cases} = 1, & |x-y| \geq \frac{1}{n^6}; \\ = 0, & |x-y| \leq \frac{1}{2n^6}; \\ \in [0,1] \text{ such that } r_n \text{ smooth}, & \frac{1}{2n^6} \leq |x-y| \leq \frac{1}{n^6}. \end{cases}$$

Thus f_n is also smooth and supported in $[-\frac{n}{4},\frac{n}{4}]^4$. Moreover,

$$\begin{aligned} \|f_n - H_{\phi,\epsilon}\|_{L^2} &\leq \|g_n - H_{\phi,\epsilon}\|_{L^2} + \|(f_n-g_n)1_{|x-y|\leq\frac{1}{n^6}}\|_{L^2} \\ &\leq \|g_n - H_{\phi,\epsilon}\|_{L^2} + \|g_n 1_{|x-y|\leq\frac{1}{n^6}}\|_{L^2} \\ &\leq 2\|g_n - H_{\phi,\epsilon}\|_{L^2} + \|H_{\phi,\epsilon}1_{(x,y)\in[-\frac{n}{4},\frac{n}{4}]^4,|x-y|\leq\frac{1}{n^6}}\|_{L^2} \\ &\to 0, \end{aligned}$$

where the last line is due to the fact that g_n converges to $H_{\phi,\epsilon}$ in L^2 and the Lebesgue measure of the set $\{(x,y)\in[-\frac{n}{4},\frac{n}{4}]^4; |x-y|\leq\frac{1}{n^6}\}$ goes to 0.

Remark 2.40.

1. *Obviously all the* f_n *and* g_n *that we defined above relies on* ϕ *and* ϵ*, but for simplicity of the notation we skip them in our notation.*

2. *Note that the rate of convergence of* $\langle\bar{\omega}\otimes\bar{\omega}, f_n\rangle$ *to* $\langle\bar{\omega}\otimes\bar{\omega}, H_{\phi,\epsilon}\rangle$ *in* $L^2(\Xi)$ *only depends on the rate of the convergence of* f_n *to* $H_{\phi,\epsilon}$ *in* $L^2(\mathbb{R}^2\times\mathbb{R}^2)$*, but does not depend on* $\bar{\omega}$ *as long as it is a space white noise.*

3. *From our construction we can require that* $f_n(x,x) = 0$ *for any* n.

Therefore, after we define the nonlinear term, we manage to define the white noise solution of mSQG equations on $\mathbb{R}^2$.

Now we introduce the main theorem of this section.

Theorem 2.41. *There exists a white noise stationary solution of* (2.34) *according to the Definition 2.35.*

In other word, we prove a similar result of [41] by letting the volume of the torus go to infinity, which is in the next section.

2.6.3 Proof of the Theorem 2.41

First we recall the similar result on the torus. Recall the Theorem 1 of [41] and Theorem 1.1 of [71] by letting $\theta = 0$, the following theorem was proved

Theorem 2.42 (Existence)**.** *Let $\varepsilon \in (0,1]$. There exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a stationary process $\xi : \Omega \to C\big([0,T]; H^{-1-}\big)$ such that, for all $t \in [0,T]$, ξ_t is a white noise on $\mathbb{T}^2$; and for all $\phi \in C^\infty(\mathbb{T}^2)$, $\mathbb{P}$-a.s. for all $t \in [0,T]$, one has*

$$\big\langle \xi_t, \phi \big\rangle = \big\langle \xi_0, \phi \big\rangle + \int_0^t \big\langle \xi_s \otimes \xi_s, H_{\phi,\epsilon} \big\rangle \, ds. \tag{2.38}$$

Note that for $\epsilon = 1$ it is Euler equation, the result is also proved in [37].

Remark 2.43.

1. *In [37], [41] and [71], it is obvious that the zero set depends on ϕ. Indeed since the non-linear term is defined in mean square sense, for any ϕ we can change the value of $\big\langle \xi_s \otimes \xi_s, H_{\phi,\epsilon} \big\rangle$ in any probability zero set N_ϕ.*

2. *In [37], [41] and [71], the proofs only use the boundedness of second derivatives of test function ϕ. In other word, the above theorem also holds when $\phi \in C^2(\mathbb{T}^2)$.*

It is obvious that Theorem 2.42 also holds when we change the length of the torus. Let ω_t^M be the solution in the above theorem on the torus $\mathbb{T}_M^2$ on some probability space $(\Omega^M, \mathcal{F}^M, \mathbb{P}^M)$.

From now on for simplicity we replace the notation $C_c^2(\mathbb{R}^2)$ by C_c^2 and $C_c^\infty(\mathbb{R}^2)$ by C_c^∞ when there is no confusion.

First we fix $\phi \in C_c^2$. Assume that ϕ is supported in $[-\frac{A}{4}, \frac{A}{4}]^2$. Thus for $M > A$, ϕ could also be viewed as a function on the torus $\mathbb{T}_M^2$.

By Theorem 2.42 we have for $M > A$,

$$\big\langle \omega_t^M, \phi \big\rangle = \big\langle \omega_0^M, \phi \big\rangle + \int_0^t \big\langle \omega_s^M \otimes \omega_s^M, H_{\phi,\epsilon}^M \big\rangle \, ds,$$

where $H_{\phi,\epsilon}^M = \frac{1}{2} K_\epsilon^M(x-y)(\nabla\phi(x) - \nabla\phi(y))$ and $K_\epsilon^M(x)$ is the convolution kernel on the torus.

Similar as before, let $\bar{\omega}_t^M$ be its periopdic extension on $\mathbb{R}^2$. Thus we have for any $t \geq 0$,

$$\langle \bar{\omega}_t^M, \phi \rangle = \langle \omega_t^M, \phi_M \rangle,$$

where the left hand is defined on $\mathbb{R}^2$ and the right hand side is defined on the torus . Thus

$$\langle \bar{\omega}_t^M, \phi \rangle = \langle \bar{\omega}_0^M, \phi \rangle + \int_0^t \langle \omega_s^M \otimes \omega_s^M, H_{\phi_M,\epsilon}^M \rangle \, ds, \tag{2.39}$$

where $\langle \bar{\omega}_t^M, \phi \rangle$ and $\langle \bar{\omega}_0^M, \phi \rangle$ are duality products on $\mathbb{R}^2$ but $\langle \omega_s^M \otimes \omega_s^M, H_{\phi_M,\epsilon}^M \rangle$ is the duality product on the torus.

Before we prove the tightness we do some preparations. We begin with a lemma.

Lemma 2.44. *The metric space C_c^2 with the C^2 Hölder norm is separable.*

Proof.
Step 1: Prove C_c^2 approximated by $\mathcal{S}(\mathbb{R}^2)$
We fix a family of smooth functions which converge to the Dirac function, for example,

$$\gamma_R(x) = \begin{cases} C_R \exp\{\frac{1}{R|x|^2-1}\}, & |x|^2 \leq \frac{1}{R}; \\ 0, & |x|^2 \geq \frac{1}{R}, \end{cases}$$

where C_R is a constant such that

$$\int_{\mathbb{R}^2} \gamma_R(x) \, dx = 1.$$

For any function $f \in C_c^2$ and any index $|\alpha| \leq 2$,

$$D^\alpha(\gamma_R * f - f) = C_R \int_{\mathbb{R}^2} D^\alpha(f(x-y) - f(x))\gamma_R(y) \, dy.$$

Since $f \in C_c^2$, $D^\alpha(f(x-y) - f(x))$ goes to 0 uniformly as y tends to 0, hence $\gamma_R * f$ converges to f in C^2. It is obvious that $\gamma_R * f$ is smooth and has compact support. Therefore, $\gamma_R * f \in \mathcal{S}(\mathbb{R}^2)$.
Step 2: Find a Countable Dense Subset of C_c^2 .
Since $\mathcal{S}(\mathbb{R}^2)$ is separable, let $\{f_i\}_{i\geq 1}$ be its countable dense subset. Since C_c^∞ is dense in $\mathcal{S}(\mathbb{R}^2)$, for each f_i, we can find a sequence $f_{ij} \in C_c^\infty$, such that f_{ij} converges to f_i in $\mathcal{S}(\mathbb{R}^2)$ (hence in C^2) as j goes to infinity. Therefore, from above arguments we know $\{f_{ij}\}_{i\geq 1, j\geq 1}$ is a dense subset of C_c^2 .
Thus we have proved C_c^2 is a separable metric space. □

Definition 2.45. *We define the following function spaces:*

1. *Define $(C_c^2)'$ to be the space which contains all the continuous linear functional from C_c^2 to $\mathbb{R}$ with weak $*$ topology.*

2. *Define the time Sobolev space* $W^{1,2}([0,T];(C_c^2)')$ *be the space of all* $u \in C([0,T];(C_c^2)')$ *such that* $u(\phi) \in L^2([0,T];\mathbb{R})$ *and* $\partial_t u(\phi) \in L^2([0,T];\mathbb{R})$ *for any* $\phi \in C_c^2$. *The topology of* $W^{1,2}([0,T];(C_c^2)')$ *is defined to be the weakest topology on* $W^{1,2}([0,T];(C_c^2)')$ *such that for any* $\psi \in L^2([0,T];C_c^2)$, *the maps*

$$u \mapsto \langle u, \psi \rangle$$

and

$$u \mapsto \partial_t \langle u, \psi \rangle$$

are continuous from $W^{1,2}([0,T];(C_c^2)')$ *to* $\mathbb{R}$.

Remark 2.46.

1. C_c^2 *is not complete. Denote by* C_0^2 *its closure with respect to the* C^2 *norm. Then by Banach–Steinhaus theorem the space* $(C_c^2)'$ *is the same to the space* $(C_0^2)'$. *It is obvious that* C_0^2 *is also separable with the same countable dense subset of* C_c^2.

2. *Since* C_c^2 *is separable, the closed unit ball of* $(C_c^2)'$ *is a compact metric space by the Banach-Alaoglu Theorem, hence also separable. Therefore,* $(C_c^2)'$ *is also separable.*

We have the following tightness results.

Lemma 2.47. *Let* $\{\mathcal{D}(\bar{\omega}_t^M)\}_{M=1}^{\infty}$ *be the distribution of* $\bar{\omega}_t^M$ *in* $W^{1,2}([0,T];(C_c^2)')$. *Then for any* $T>0$, $\{\mathcal{D}(\bar{\omega}_t^M)\}_{M=1}^{\infty}$ *is tight in* $W^{1,2}([0,T];(C_c^2)')$.

Proof. By definition of the topology of $(C_c^2)'$, it suffices to prove that for any $\phi \in C_c^2$,

$$\mathbb{E}|\big\langle \bar{\omega}_t^M, \phi \big\rangle|^2 \le C \tag{2.40}$$

and

$$\mathbb{E}|\partial_t \big\langle \bar{\omega}_t^M, \phi \big\rangle|^2 \le C, \tag{2.41}$$

where C is a constant which depends on ϕ but not M and t. (2.40) is immediately obtained by Lemma 2.31.

To obtain (2.41), we note from (2.39) that for any $t>0$,

$$\partial_t \big\langle \bar{\omega}_t^M, \phi \big\rangle = \big\langle \omega_s^M \otimes \omega_s^M, H_{\phi_M,\epsilon}^M \big\rangle.$$

Assume that ϕ is supported in $[-\frac{A}{4}, \frac{A}{4}]^2$. Then for $M>A$, $\phi_M = \phi$.
Hence we have for $M>A$,

$$\mathbb{E}|\big\langle \omega_s^M \otimes \omega_s^M, H_{\phi_M,\epsilon}^M \big\rangle|^2 = \mathbb{E}|\big\langle \omega_s^M \otimes \omega_s^M, H_{\phi,\epsilon}^M \big\rangle|^2 \tag{2.42}$$

By Corollary 6 of [41], we deduce

$$\mathbb{E}|\big\langle \omega_s^M \otimes \omega_s^M, H_{\phi,\epsilon}^M \big\rangle|^2 = 2\int_{\mathbb{T}_M^2}\int_{\mathbb{T}_M^2} H_{\phi,\epsilon}^M(x,y)^2\, dxdy.$$

Recall that $H^M_{\phi,\epsilon}(x,y) = \frac{1}{2}K^M_\epsilon(x-y)(\nabla\phi(x) - \nabla\phi(y))$, where $|K^M_\epsilon(x)| \leq C_\epsilon \frac{1}{|x|^{2-\epsilon}}$ and C_ϵ is a uniform constant does not depend on M. Moreover, by Corollary 2.37, $H^M_{\phi,\epsilon}$ is uniformly bounded in L^2.

By Lemma 2.33 and the proof of Lemma 2.36 , we know that $\int_{\mathbb{T}^2_M}\int_{\mathbb{T}^2_M} H^M_{\phi,\epsilon}(x,y)^2\,dxdy$ is uniformly bounded with respect to M, since $\mathbb{E}|\langle\omega^M_s \otimes \omega^M_s, H^M_{\phi,\epsilon}\rangle|^2$ is uniformly bounded. Since there are only finite positive integers which are smaller than A, we come to the conclusion of (2.41). □

Again we will apply the Skorokhod Theorem 1.27. Since $L^2([0,T];C^2_c)$ is separable, there exists a sequence of continuous functions which separate the points in $W^{1,2}([0,T];(C^2_c)')$. Similar as before, we will show that the σ- algebra generated by the above sequence of continuous functions is exactly the Borel σ- algebra of $W^{1,2}([0,T];(C^2_c)')$. By Theorem B.4 it suffices to prove that $W^{1,2}([0,T];(C^2_c)')$ is a standard Borel space. (See Appendix B.2 for the definition of the standard Borel space.)

Lemma 2.48. *$W^{1,2}([0,T];(C^2_c)')$ is a standard Borel space.*

Proof. Let $X_1 := W^{1,2}([0,T];(C^2_c)')$ and $X_2 := W^{1,2}([0,T];L^2)$, where X_2 consists of all the functions w such that $w \in L^2([0,T]\times\mathbb{R}^2)$ and $\partial_t w \in L^2([0,T]\times\mathbb{R}^2)$ with the norm

$$\|w\|_{W^{1,2}([0,T];L^2)} := \|w\|_{L^2([0,T]\times\mathbb{R}^2)} + \|\partial_t w\|_{L^2([0,T]\times\mathbb{R}^2)}.$$

It is obvious that X_2 is a Polish space and it is continuously embedded in X_1. We need to prove

$$\mathscr{B}(X_2) = \mathscr{B}(X_1) \cap X_2.$$

Obviously $\mathscr{B}(X_1) \cap X_2 \subset \mathscr{B}(X_2)$.

It suffices to show that any open set of X_2 is in $\mathscr{B}(X_1) \cap X_2$.

Note that $\{B(x_m, r_n)\}_{m,n\geq 1}$ is a countable topology basis of X_2, where $\{x_m\}_{m\geq 1}$ is a countable dense subset of X_2 and $\{r_n\}_{n\geq 1}$ is the sequence of all the positive rational numbers. Therefore, we only need to prove

$$B(x_m, r_n) \in \mathscr{B}(X_1) \cap X_2.$$

Without loss of generality we only prove it for $x_m = 0$.

Note that

$$\begin{aligned} B(0,r_n) = &\quad \{x \in X_2; \|x\|_{L^2([0,T]\times\mathbb{R}^2)} + \|\partial_t x\|_{L^2([0,T]\times\mathbb{R}^2)} < r_n\} \\ = &\quad \bigcup_{j\geq 1}\bigcap_{k,l\geq 1}\{x \in X_2; |\langle x,\psi_k\rangle| + |\langle\partial_t x,\psi_l\rangle| < r_n - \tfrac{1}{j}\} \\ = &\quad X_2 \cap [\bigcup_{j\geq 1}\bigcap_{k,l\geq 1}\{x \in X_1; |\langle x,\psi_k\rangle| + |\langle\partial_t x,\psi_l\rangle| < r_n - \tfrac{1}{j}\}], \end{aligned}$$

where $\{\psi_k\}_{k\geq 1}$ is set to be a countable dense subset of the unit ball of $L^2([0,T]\times\mathbb{R}^2)$ such that $\{\psi_k\}_{k\geq 1}$ is also a subset of $L^2([0,T];C^2_c)$. Then

$$\{x \in X_1; |\langle x, \psi_k\rangle| + |\langle \partial_t x, \psi_l\rangle| < r_n - \frac{1}{j}\}$$

is an open set of X_1. Hence

$$B(0, r_n) \in \mathscr{B}(X_1) \cap X_2,$$

which finishes our proof. □

Therefore, there exists another probability space, which we still use the notation $(\Omega, \mathcal{F}, \mathbb{P})$ for simplicity, and a sequence of random variables $\bar{\bar{\omega}}_t^{M_k}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, such that

- $\bar{\bar{\omega}}_t^{M_k}$ has the same distribution to $\bar{\omega}_t^{M_k}$ in $W^{1,2}([0,T];(C_c^2)')$; (we also assume that M_k is increasing to infinity and $M_k \geq k$)
- $\bar{\bar{\omega}}_t^{M_k}$ converge $\mathbb{P}$-almost surely to some limit $\bar{\bar{\omega}}_t$ in $W^{1,2}([0,T];(C_c^2)')$.

Hence by the same argument of Lemma 2.31, we obtain that for any fixed $t \in [0,T]$, $\bar{\bar{\omega}}_t$ is a space white noise distribution on $\mathbb{R}^2$.

By the definition of the solution on the torus, $\bar{\omega}_t^M$ has the following form

$$\bar{\omega}_t^M = \sum_{n\in\mathbb{Z}^2} \bar{G}_n^M(t, \theta_M) e_n^M \text{ on } \mathbb{R}^2,$$

where $\theta_M \in \Omega^M$, $\bar{G}_n^M(\cdot, \theta_M) \in W^{1,2}([0,T];\mathbb{R})$ and for each t, $\bar{G}_n^M(t,\cdot)$, $n \in \mathbb{Z}_+^2 \cup \{0\}$ are independent random variables with standard Gaussian distributions on $(\Omega^M, \mathcal{F}^M, \mathbb{P}^M)$. Note that for fixed M , if $a_n(t) \in W^{1,2}([0,T];\mathbb{R})$ and $\sum_{n\in\mathbb{Z}^2} a_n(t) e_n^M \in W^{1,2}([0,T];(C_c^2)')$, the map

$$\sum_{n\in\mathbb{Z}^2} a_n(t) e_n^M \mapsto \big(a_{n_1}(t), a_{n_2}(t), ..., a_{n_k}(t)\big)$$

is continuous from $W^{1,2}([0,T];(C_c^2)')$ to $\big(W^{1,2}([0,T];\mathbb{R})\big)^k$ for any k and $n_1, n_2, ...n_k \in \mathbb{Z}^2$.

Therefore, $\bar{\bar{\omega}}_t^M$ also has the form

$$\bar{\bar{\omega}}_t^M = \sum_{n\in\mathbb{Z}^2} \bar{\bar{G}}_n^M(t,\theta) e_n^M \text{ on } \mathbb{R}^2$$

on $(\Omega, \mathcal{F}, \mathbb{P})$, where $\big(\bar{G}_{n_1}^M(t,\cdot), \bar{G}_{n_2}^M(t,\cdot), ...\bar{G}_{n_k}^M(t,\cdot)\big)$ and $\big(\bar{\bar{G}}_{n_1}^M(t,\cdot), \bar{\bar{G}}_{n_2}^M(t,\cdot), ..., \bar{\bar{G}}_{n_k}^M(t,\cdot)\big)$ have the same joint distributions on $\big(W^{1,2}([0,T];\mathbb{R})\big)^k$. Define

$$\hat{\omega}_t^M = \sum_{n\in\mathbb{Z}^2} \bar{\bar{G}}_n^M(t,\theta) e_n^M \text{ on } \mathbb{T}_M^2,$$

i.e. $\bar{\bar{\omega}}_t^M$ is an extension of $\hat{\omega}_t^M$ on $\mathbb{R}^2$. Moreover, $\hat{\omega}_t^M$ has the same distribution as ω_t^M, hence it also satisfies the equation (2.38).

Thus it satisfies the same equation as (2.39) for any $\phi \in C_c^2(\mathbb{R}^2)$ $\mathbb{P}$-a.s.:

$$\left\langle \bar{\bar{\omega}}_t^{M_k}, \phi \right\rangle = \left\langle \bar{\bar{\omega}}_0^{M_k}, \phi \right\rangle + \int_0^t \left\langle \hat{\omega}_s^{M_k} \otimes \hat{\omega}_s^{M_k}, H^{M_k}_{\phi_{M_k},\epsilon} \right\rangle ds. \tag{2.43}$$

Same as usual ϕ could also be viewed as a function on $\mathbb{T}^2_{M_k}$ when we fix ϕ and let M_k large enough. It suffices to prove for any fixed $\phi \in C_c^2(\mathbb{R}^2)$, we have $\mathbb{P}$-a.s.

$$\lim_{k\to\infty} \int_0^t \left\langle \hat{\omega}_s^{M_k} \otimes \hat{\omega}_s^{M_k}, H^{M_k}_{\phi,\epsilon} \right\rangle ds = \int_0^t \left\langle \bar{\bar{\omega}}_s \otimes \bar{\bar{\omega}}_s, H_{\phi,\epsilon} \right\rangle ds, \tag{2.44}$$

where on the left hand side, $\left\langle \omega_s^{M_k} \otimes \omega_s^{M_k}, H^{M_k}_{\phi,\epsilon} \right\rangle$ is the duality product on the torus and on the right hand side $\left\langle \bar{\bar{\omega}}_s \otimes \bar{\bar{\omega}}_s, H_{\phi,\epsilon} \right\rangle$ is the duality product on $\mathbb{R}^2$.

Proof of (2.44)

Step 1

Fix $\eta > 0$. Recall from Theorem 2.38, for a space white noise distribution $\bar{\omega}$ on $\mathbb{R}^2$ in some probability space, we define $\langle \bar{\omega} \otimes \bar{\omega}, H_{\phi,\epsilon} \rangle$ as the mean square limit of $\langle \bar{\omega} \otimes \bar{\omega}, f_n \rangle$, where $f_n \in C_c^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ are symmetric and approximate $H_{\phi,\epsilon}$ $\mathbb{P}$-a.s. in the following sense:

$$\lim_{n\to\infty} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(f_n - H_{\phi,\epsilon}\right)^2(x,y)\,dxdy = 0$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} f_n(x,x)\,dx = 0.$$

Moreover, without loss of generality we assume that for each n, f_n is supported in $[-\frac{n}{4}, \frac{n}{4}]^4$. By 3 of Remark 2.40, we can require $f_n(x,x) = 0$. And by 2 of Remark 2.40, we know the approximation is uniformly with respect to the time t. Hence we know $\left\langle \bar{\bar{\omega}}_s \otimes \bar{\bar{\omega}}_s, H_{\phi,\epsilon} \right\rangle$ is the $L^2(\Omega; L^2([0,T])) = L^2([0,T]; L^2(\Omega))$ limit of $\left\langle \bar{\bar{\omega}}_s \otimes \bar{\bar{\omega}}_s, f_n \right\rangle$. Thus we can find an n_0, such that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(f_{n_0} - H_{\phi,\epsilon}\right)^2(x,y)\,dxdy < \frac{\eta}{T}, \tag{2.45}$$

thus

$$\mathbb{E}\int_0^T \left|\left\langle \bar{\bar{\omega}}_s \otimes \bar{\bar{\omega}}_s, f_{n_0} \right\rangle - \left\langle \bar{\bar{\omega}}_s \otimes \bar{\bar{\omega}}_s, H_{\phi,\epsilon} \right\rangle\right|^2 dt < \eta.$$

Step 2

Fix n_0, since $\bar{\bar{\omega}}_t^{M_k}$ converge $\mathbb{P}$-a.s. to $\bar{\bar{\omega}}_t$ in $W^{1,2}([0,T]; (C_c^2)')$, (hence in $C([0,T]; (C_c^2)')$) $\left\langle \bar{\bar{\omega}}_s^{M_k} \otimes \bar{\bar{\omega}}_s^{M_k}, f_{n_0} \right\rangle$ converges to $\left\langle \bar{\bar{\omega}}_s \otimes \bar{\bar{\omega}}_s, f_{n_0} \right\rangle$ in $C([0,T]; \mathbb{R})$ $\mathbb{P}$-almost surely as k goes to infinity. Moreover, since when $k \geq n_0$, $\left\langle \bar{\bar{\omega}}_s^{M_k} \otimes \bar{\bar{\omega}}_s^{M_k}, f_{n_0} \right\rangle = \left\langle \hat{\omega}_s^{M_k} \otimes \hat{\omega}_s^{M_k}, f_{n_0} \right\rangle$ where f_{n_0} can be viewed as the product on the torus $\mathbb{T}^2_M$ when $k \geq n_0$, just as we have shown during the proof of the Theorem 2.38. By Corollary 2.39 it is uniformly integrable. Therefore,

$\langle \hat{\omega}_s^{M_k} \otimes \hat{\omega}_s^{M_k}, f_{n_0} \rangle$ converges to $\langle \bar{\bar{\omega}}_s \otimes \bar{\bar{\omega}}_s, f_{n_0} \rangle$ as $k \to \infty$ in $L^2(\Omega; L^2([0,T]))$.

Step 3

By step 1 and step 2, we know that there exists some $k_0 \geq n_0$ such that when $k \geq k_0$,

$$\mathbb{E} \int_0^T |\langle \hat{\omega}_s^{M_k} \otimes \hat{\omega}_s^{M_k}, f_{n_0} \rangle - \langle \bar{\bar{\omega}}_s \otimes \bar{\bar{\omega}}_s, H_{\phi,\epsilon} \rangle|^2 ds < 2\eta \tag{2.46}$$

Step 4

Just as we have mentioned in step 2, when $k \geq k_0 \geq n_0$, $\langle \bar{\bar{\omega}}_s^{M_k} \otimes \bar{\bar{\omega}}_s^{M_k}, f_{n_0} \rangle$ is the duality product on the torus $\mathbb{T}^2_{M_k}$.

By ii) iii) of Corollary 2.39 and the definition of $\langle \hat{\omega}_s^{M_k} \otimes \hat{\omega}_s^{M_k}, H_{\phi,\epsilon}^{M_k} \rangle$,

$$\mathbb{E} \int_0^T |\langle \hat{\omega}_s^{M_k} \otimes \hat{\omega}_s^{M_k}, f_{n_0} \rangle - \langle \hat{\omega}_s^{M_k} \otimes \hat{\omega}_s^{M_k}, H_{\phi,\epsilon}^{M_k} \rangle|^2 ds \leq T \int_{\mathbb{T}^2_{M_k}} \int_{\mathbb{T}^2_{M_k}} \left(f_{n_0} - H_{\phi,\epsilon}^{M_k} \right)^2 (x,y)\, dxdy.$$

If we view $H_{\phi,\epsilon}^M(x,y)$ as measurable functions on $\mathbb{R}^2$ which are 0 valued outside $[-\frac{M}{2}, \frac{M}{2}]^4$, we can view $\int_{\mathbb{T}^2_{M_k}} \int_{\mathbb{T}^2_{M_k}} \left(f_{n_0} - H_{\phi,\epsilon}^{M_k} \right)^2 (x,y)\, dxdy$ as $\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(f_{n_0} - H_{\phi,\epsilon}^{M_k} \right)^2 (x,y)\, dxdy$. Since for any x, $K_\epsilon^M(x)$ goes to $K_\epsilon(x)$ as M goes to infinity, $H_{\phi,\epsilon}^M$ converges pointwisely to $H_{\phi,\epsilon}$. Moreover, $\{H_{\phi,\epsilon}^M\}_{M>0}$, are all dominated by the $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$ integrable function $\frac{C(\nabla\phi(x) - \nabla\phi(y))}{|x-y|^{2-\epsilon}}$ for some constant C not depending on M, thus the convergence of $H_{\phi,\epsilon}^M$ to $H_{\phi,\epsilon}$ also holds in $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$. Thus combining with (2.45), we can find some $k_1 \geq k_0$, such that when $k \geq k_1$, $\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(f_{n_0} - H_{\phi,\epsilon}^{M_k} \right)^2 (x,y)\, dxdy < \frac{2\eta}{T}$, hence for any $k \geq k_1$,

$$\mathbb{E} \int_0^T |\langle \hat{\omega}_s^{M_k} \otimes \hat{\omega}_s^{M_k}, f_{n_0} \rangle - \langle \hat{\omega}_s^{M_k} \otimes \hat{\omega}_s^{M_k}, H_{\phi,\epsilon}^{M_k} \rangle|^2 ds < 2\eta. \tag{2.47}$$

Hence by (2.46) and (2.47), we obtain that (2.44) holds in $L^2(\Omega)$.

Since for any $0 \leq t \leq T$, $\langle \bar{\bar{\omega}}_t^{M_k}, \phi \rangle$ converges to $\langle \bar{\bar{\omega}}_t, \phi \rangle$ $\mathbb{P}$-a.s., the convergence of (2.44) also holds $\mathbb{P}$-a.s.

Chapter 3

Three Dimensional Navier–Stokes equations with Noise of Convolution Type

3.1 Introduction

In this chapter we consider the following stochastic $3D$ Navier–Stokes equations

$$\begin{cases} du + (u \cdot \nabla u - \Delta u)dt = \sum_{i=1}^{n} (B_i(u) + \lambda_i u) d\beta_i(t) - \nabla p dt, \\ \operatorname{div} u = 0, \\ u \mid_{t=0} = u_0, \end{cases} \tag{3.1}$$

on the whole space $\mathbb{R}^3$, where $\beta_i(t)$, $i = 1, ..., n$ are one dimensional independent Brownian motions on given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, λ_i, $i = 1, ..., n$ are non-zero constants and B_i , $i = 1, ..., n$ are the convolution operators

$$B_i(u)(\xi) = \int_{\mathbb{R}^3} h_i(\xi - \bar{\xi}) u(\bar{\xi}) d\bar{\xi} = (h_i * u)(\xi), \ \xi \in \mathbb{R}^3,$$

where $h_i \in L^1(\mathbb{R}^3)$, $i = 1, 2, ..., n$, and Δ is the (weak) Laplacian, hence could be defined on the tempered distribution space $(\mathcal{S}')^3$. Note that the convolution operator makes sense in any Sobolev space $H^s(\mathbb{R}^3)$, where $s \in \mathbb{R}$, the proof of which is later shown in Lemma 3.3. The vorticity form of this system has been investigated in [2], where the authors prove the existence and uniqueness in $(L^p(\mathbb{R}^3))^3$, $\frac{3}{2} < p < 2$, of a global mild solution to random vorticity equations associated with stochastic $3D$ Navier–Stokes equations for sufficiently small initial vorticity. In their paper the smallness of the initial values depends on the whole Brownian path, hence the solutions obtained are not adapted.

In this chapter we consider the original equations instead of the vorticity form. We do not assume the initial values are small. Instead we only assume initial data are smooth. In other words, for any fixed path ω, the initial data are in any Sobolev spaces H^N. We are focused on the local solution in a certain time-space Sobolev space for each fixed path ω.

The idea of the proof is that we first apply the transformation in [2] to transform the equation to random PDEs and write the solution in the form of mild solutions. Then thanks to the commutative property of the convolution (this is the reason why we have to limit our noise to the convolution type multiplicative noise) and the contraction property of the semigroup $e^{t\Delta}$, we obtain the estimates for fixed point argument.

3.2 Transform to Random PDEs

We use the same transform in [2], but instead of vorticity equation, we apply it to the original equation. For $t \geq 0$, we consider the transformation

$$u(t) = \Gamma(t)y(t),$$

where $\Gamma(t) : (L^2(\mathbb{R}^3))^3 \to (L^2(\mathbb{R}^3))^3$ is the linear continuous operator defined by the equations

$$d\Gamma(t) = \sum_{i=1}^{n}(B_i + \lambda_i I)\Gamma(t)d\beta_i(t),\ t \geq 0,$$

and $\Gamma(0) = I$. In other word, $\Gamma(t)$ is defined in the sense that, for every $z_0 \in (L^2(\mathbb{R}^3))^3$, the continuous $(\mathcal{F}_t)$–adapted $(L^2(\mathbb{R}^3))^3$–valued process $z(t) := \Gamma(t)z_0$, $t \geq 0$, solves the following SDE on $(L^2(\mathbb{R}^3))^3$,

$$dz(t) = \sum_{i=1}^{n} \tilde{B}_i z(t)d\beta_i(t), \quad z(0) = z_0.$$

Similar as in [2], we also set

$$\tilde{B}_i = B_i + \lambda_i I,\ i = 1, ..., n,$$

where I is the identity operator. Then we can also write Γ as the exponential form:

$$\Gamma(t) = \prod_{i=1}^{N} \exp\left(\beta_i(t)\tilde{B}_i - \frac{t}{2}\,\tilde{B}_i^2\right),\ t \geq 0.$$

Moreover, we immediately have all of $\Gamma(t)$,$\Gamma^{-1}(t)$ and $\tilde{B}_i$ commutate with (weak) derivatives. Note that

$$\Gamma^{-1}(t) = \prod_{i=1}^{N} \exp\left(-\beta_i(t)\tilde{B}_i + \frac{t}{2}\,\tilde{B}_i^2\right),\ t \geq 0.$$

Moreover, $\tilde{B}_i\tilde{B}_j = \tilde{B}_j\tilde{B}_i$.

Remark 3.1. *It is obvious that the operator B_i, $\Gamma(t)$ and $\Gamma^{-1}(t)$ can be defined (as a continuous operator) in any $(L^p(\mathbb{R}^3))^3$ for any $p \geq 1$ since the convolution with an L^1 function makes sense in any L^p space with the L^1 norm as the uniform bound of the operator (Young's inequality). Moreover, we have the following lemma as is proved in [2].*

Lemma 3.2. *We have*

$$\|\Gamma(t)z\|_{L^q} + \|\Gamma^{-1}(t)z\|_{L^q} \leq C_t\|z\|_{L^q},\ t \in [0,\infty),\ \forall z \in L^q(\mathbb{R}^3,\ \forall q \in [1,\infty),$$

and

$$\|\nabla(\Gamma(t)z)\|_{L^q} \leq \|\Gamma(t)\|_{L(L^q,L^q)}\|\nabla z\|_{L^q},\ \text{for all } z \text{ which satisfies } z, \nabla z \in L^q(\mathbb{R}^3).$$

Proof. See Lemma 2.1 of [2]. □

Moreover, the above lemma also holds when $q = \infty$. Since the proof is a direct result of Young's inequality

$$\|h_i * u\|_{L^q} \lesssim \|h\|_{L^1}\|u\|_{L^q},$$

which also holds as $q = \infty$, and the second inequality is due to the fact that Γ commutates with derivatives.

From Lemma 3.2 we can view Γ as a linear continuous operator from $(L^p(\mathbb{R}^3))^3$ to $(L^p(\mathbb{R}^3))^3$ for $1 \leq p \leq \infty$. Moreover, there is a common upper bound of $\|\Gamma(t)\|_{L(L^p,L^p)}$ which does not depend on p, but only depends on h_i,λ_i, t and, of course, the path ω.

The next lemma tells us for any $s \in \mathbb{R}$, for any $h \in L^1$, the convolution with h is a continuous operator map from any Sobolev space H^s to itself. Therefore, we can also extend the definition of operator B_i, $\Gamma(t)$ and $\Gamma^{-1}(t)$ to continuous operators from any Sobolev space H^s to itself.

Lemma 3.3. *For any $h \in L^1$ and $s \in \mathbb{R}$, the convolution with h is a continuous operator mapping from any Sobolev space H^s to itself.*

Proof. It suffices to prove that for any $s > 0$, and $a \in H^s$, we have $a * h \in H^s$.

$$\begin{aligned}
\|a * h\|_{H^s}^2 &= \int_{\mathbb{R}^3} |\mathcal{F}(a * h)|^2(\xi)(1 + |\xi|^2)^s d\xi \\
&= \int_{\mathbb{R}^3} |\hat{a}(\xi)|^2|\hat{h}(\xi)|^2(1 + |\xi|^2)^s d\xi \\
&\leq \|a\|_{H^s}^2\|\hat{h}\|_{L^\infty}^2 \\
&\leq \|a\|_{H^s}^2\|h\|_{L^1}^2.
\end{aligned}$$

Therefore, we have proved the result.

□

Thus we transform (3.1) to the following random PDEs of y:

$$dy + \Gamma^{-1}(\Gamma y \cdot \nabla \Gamma y)dt - \Delta y dt = -\Gamma^{-1}\nabla p dt, \qquad y(0) = u_0. \tag{3.2}$$

Note that

$$\nabla p = \nabla(-\Delta)^{-1} \operatorname{div}\operatorname{div}(\Gamma y \otimes \Gamma y) = \sum_{1\le i,j\le 3} \nabla(-\Delta)^{-1}\big(\partial_i\partial_j(\Gamma y^i \Gamma y^j)\big).$$

Let Q be the following bilinear operator from $(L^2(\mathbb{R}^3))^3 \times (L^2(\mathbb{R}^3))^3$ to $(\mathcal{S}'(\mathbb{R}^3))^3$:

$$Q(x,y) := \operatorname{div}(x\otimes y) + \sum_{1\le i,j\le 3} \nabla(-\Delta)^{-1}(t)\big(\partial_i\partial_j(x^i y^j)\big) = Q(y,x).$$

Thus we can rewrite the equation in the following form of mild solution

$$y(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\Gamma^{-1}Q(\Gamma y(s), \Gamma y(s))ds. \tag{3.3}$$

For simplicity from now on in this section we skip the dimension notation $\mathbb{R}^3$ if it is 3 dimension and there is no confusion.

Remark 3.4.

1. *Δ is an operator mapping from tempered distribution space $\mathcal{S}'$ to $\mathcal{S}'$.*

2. *We will show that Q, ΓQ and $\Gamma^{-1}Q$ are well defined and continuous from $(L^p)^3 \times (L^p)^3$ to (at least) the Sobolev space $(H^{-2})^3 \subset (\mathcal{S}')^3$ for any $2 \le p < \infty$.*
   Case of** $2 < p < \infty$**.
   It immediately follows from the $L^{\frac{p}{2}}$ boundedness of the Riesz transform.

   Case of** $p = 2$**.
   Recall that the inverse of the (minus) Laplacian $(-\Delta)^{-1}$ can be defined by Fourier multipliers: for any u which is in the range of the Δ ,

   $$(-\Delta)^{-1}u = \mathcal{F}^{-1}(|\xi|^{-2}\hat{u}(\xi)).$$

   Claim that for any $f \in L^1$, $\partial_i\partial_j f$ is in the domain of $(-\Delta)^{-1}$ (i.e. the range of the Δ).
   Indeed since $f \in L^1$, we have $\hat{f} \in L^\infty$, and $\xi_i\xi_j|\xi|^{-2}\hat{f}(\xi) \in L^\infty \subset \mathcal{S}'$. Therefore, the term of Fourier inverse transform $\mathcal{F}^{-1}(\xi_i\xi_j|\xi|^{-2}\hat{f}(\xi))$ is at least meaningful in $\mathcal{S}'$.
   We have

   $$\Delta\mathcal{F}^{-1}(\xi_i\xi_j|\xi|^{-2}\hat{f}(\xi)) = \partial_i\partial_j f$$

and

$$(-\Delta)^{-1}\big(\partial_i\partial_j f\big) = -\mathcal{F}^{-1}(\xi_i\xi_j|\xi|^{-2}\hat{f}(\xi)).$$

Moreover, since $\xi_i\xi_j|\xi|^{-2}\hat{f}(\xi) \in L^\infty$*,*

$$\|\mathcal{F}^{-1}(\xi_i\xi_j|\xi|^{-2}\hat{f}(\xi))\|^2_{H^{-2}} = \int_{\mathbb{R}^3} |\xi_i\xi_j|\xi|^{-2}\hat{f}(\xi)|^2(1+|\xi|^2)^{-2}d\xi \leq \|f\|_{L^1} < \infty.$$

Therefore, Q *maps* $(L^p)^3 \times (L^p)^3$ *continuously to* $(H^{-2})^3$*. Therefore, by Lemma 3.3 we know that* ΓQ *and* $\Gamma^{-1}Q$ *also map* $(L^p)^3 \times (L^p)^3$ *continuously to* $(H^{-2})^3$*.*

3. *Therefore, by Sobolev embedding* Q *and* ΓQ *can be well defined as continous maps from Sobolev space* $(\dot{H}^s)^3 \times (\dot{H}^s)^3$ *for* $0 < s < \frac{3}{2}$ *to* $(H^{-2})^3$ *.*

4. *From our definition of* $Q(x,y)$*, we immediately have* $\operatorname{div} Q(x,y) = 0$

5. *Let* X *be* L^p*,* $p \geq 2$ *or any Sobolev space* $\dot{H}^s$ *for* $0 < s < \frac{3}{2}$*. We have shown that* Q *is a continuous map from* $X^3 \times X^3$ *to* $(H^{-2})^3$*, thus if* u *and* v *are* $\mathcal{L}([0,t])/\mathscr{B}(X^3)$*-measurable, where* $\mathcal{L}$ *is the Lebesgue measure,* $Q(u,v)$ *is* $\mathcal{L}([0,t])/\mathscr{B}((H^{-2})^3)$*-measurable. Since* X *is dense in* $(H^{-2})^3$*, and both* X *and* $(H^{-2})^3$ *are Banach spaces, we have* $X \in \mathscr{B}((H^{-2})^3)$ *and* $\mathscr{B}(X) = \mathscr{B}((H^{-2})^3) \cap X$*.*
   Therefore, $Q(u,v)1_{Q(u,v)\in X}$ *is also* $\mathcal{L}([0,t])/\mathscr{B}(X)$*-measurable.*

6. *Due to 5, and the fact that* Γ *maps from any Sobolev or Lebesgue space to itself continuously, (which we would prove later in Remark 3.11,) the integral in (3.3) is meaningful in any Sobolev space if we can show the integration of their corresponding Sobolev norms are finite on the interval* $[0,T]$ *.*

Lemma 3.5. *Assume that* λ_i, h_i *satisfy*

$$|\lambda_i| > (\sqrt{12}+3)\|h_i\|_{L^1}, \quad \forall i = 1,2,...,N. \tag{3.4}$$

Let

$$\eta(t) = \|\Gamma(t)\|^2_{L(L^2,L^2)}\|\Gamma^{-1}(t)\|_{L(L^2,L^2)}, \ t \geq 0, \tag{3.5}$$

where for $q \in (1,\infty)$*,* $\|\cdot\|_{L(L^q,L^q)}$ *is the norm of the space* $L(L^q,L^q)$ *of linear continuous operators on* L^q*. Then we have*

$$\sup_{t\geq 0}\eta(t) < \infty, \mathbb{P}\text{-a.e.}$$

Proof. The proof is the same to Remark 1.2 of [2] if we note that the following still holds by calculation directly,

$$\eta(t) \leq \prod_{i=1}^N \exp\big(3|\beta_i(t)|(\|h_i\|_{L^1} + |\lambda_i|) - t\alpha_i\big), \ t \in [0,\infty),$$

where $\alpha_i := \frac{1}{2}\lambda_i^2 - \frac{3}{2}(\|h_i\|^2_{L^1} + 2|\lambda_i|\,\|h_i\|_{L^1})$ is strictly positive followed by (3.4). $\square$

3.3 The Main Theorem

On $\mathbb{R}^3$, we recall the homogeneous Sobolev spaces:

$$\dot{H}^s := \Big\{ u \in \mathcal{S}', \hat{u} \in L^1_{loc}; \| u \|^2_{\dot{H}^s} := \int_{\mathbb{R}^3} | \xi |^{2s} | \hat{u}(\xi) |^2 \, d\xi < \infty \Big\},$$

where $\hat{u}$ denotes the Fourier transform of u. 1Define the function space $\mathcal{Z}^\gamma_T$ to be functions from $[0,T] \times \mathbb{R}^3$ to $\mathbb{R}^d$ $(d \geq 1)$ with the corresponding norm

$$\|u\|^2_{\mathcal{Z}^\gamma_T} := \|u\|^2_{L^\infty([0,T];\dot{H}^{\frac{1}{2}+\gamma})} + \int_0^T \|\nabla u(t)\|^2_{\dot{H}^{\frac{1}{2}+\gamma}} dt$$

finite.
The completeness of $\mathcal{Z}^\gamma_T$ when $0 < \gamma < 1$ is proved in the appendix.

Theorem 3.6. *Let $0 < \gamma < 1$. Assume that (3.4) holds. Given vector field u_0 such that $u_0 \in \tilde{H}^N$ for any positive integer N. Then for $\mathbb{P}$-a.e. path, there exists a mild solution of (3.2) in the time interval $[0, T_*(u_0)]$ which belongs to the space $\mathcal{Z}^\gamma_{T_*(u_0)}$, where*

$$T_*(u_0, \omega) = c_\gamma \big(\sup_{t\geq 0} \eta(t)\big)^{-1} \|u_0\|^{-\frac{2}{\gamma}}_{\dot{H}^{\frac{1}{2}+\gamma}}, \tag{3.6}$$

and the strict positive number c_γ depends only on γ.

We put the proof in Section 3.3.2.

Remark 3.7.

1. *In particular, in (3.6), if there is no noise, we do not need any transformation, thus both Γ and Γ^{-1} are identity and $\eta_t = 1$ for any t, then we obtain the result which is consistent with the deterministic cases.*

2. *Similar to [2], the solution we obtain is not adapted.*

The proof of the main theorem relies heavily on the fixed point theorem. There are two key steps in the following proof: one is to show that Γ and Γ^{-1} could expand as operators in Besov spaces (see the next section for the definition of the Besov spaces), the proof of which relies on the commutative property of Γ and Littlewood-Paley operators (We will introduce the tool of Littlewood-Paley theory in the next section). In other words, it relies on the commutative property of convolutions since Littlewood-Paley operators are actually convolution operators. This is the reason why we need the noise to be also convolution types. Another step is that just like what we do in deterministic equations, we write the solution in the form of a mild solution and use the contraction property of the semigroup $e^{t\Delta}$ in order to obtain the estimates we need for the fixed point theorem.

3.3.1 Littlewood-Paley Theory

Let us first recall the (homogeneous) Littlewood-Paley decomposition in the book [1]. We will give a brief introduction to the Littlewood-Paley Theory, the details of which could be found in he book [1]. For $a \in \mathcal{S}'$, as usual, denote $\mathcal{F}a$ and $\widehat{a}$ the Fourier transform of the distribution a.

Definition 3.8 (the Space $\mathcal{S}'_h$, see Definition 1.26 of [1])**.** *Let $\mathcal{S}'_h$ be the space of tempered distributions u such that*

$$\lim_{\lambda\to\infty} \|\theta(\lambda D)u\|_{L^\infty} = 0 \tag{3.7}$$

for any $\theta \in C_c^\infty$, where $\theta(\lambda D)u$ is the Fourier multiplier defined as follows

$$\theta(\lambda D)u := \mathcal{F}^{-1}(\theta(\lambda\cdot)\hat{u}).$$

The next remark comes from Remark 1.27 of [1] and the examples afterwards.

Remark 3.9.

1. *Whether or not a tempered distribution u belongs to $\mathcal{S}'_h$ depends only on low frequencies. u belongs to $\mathcal{S}'_h$ if and only if one can find some smooth compactly supported function θ such that $\theta(0) \neq 0$ and (3.7) holds.*

2. *Directly by the definition we immediately know that the space $\mathcal{S}'_h$ contains all the tempered distributions whose Fourier transforms are locally integrable around 0. In particular, all the Sobolev spaces (homogeneous and non-homogeneous) are subsets of $\mathcal{S}'_h$.*

For $a \in \mathcal{S}'_h$, we set

$$\dot{\Delta}_k a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\widehat{a}), \tag{3.8}$$

where $\varphi(\tau)$ is a smooth function value in $[0,1]$ such that

$$\text{Supp } \varphi \subset \left\{\tau \in \mathbb{R};\ \frac{3}{4} \le |\tau| \le \frac{8}{3}\right\} \quad \text{and} \quad \forall \tau > 0\,,\ \sum_{k\in\mathbb{Z}} \varphi(2^{-k}\tau) = 1,$$

Then we have

$$\forall \xi \in \mathbb{R}^3 \backslash \{0\}, \frac{1}{2} \le \sum_{j\in\mathbb{Z}} \varphi^2(2^{-j}\xi) \le 1.$$

Moreover, we have the following equality for $a \in \mathcal{S}'_h$

$$a = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j a. \tag{3.9}$$

For $s \in \mathbb{R}$ and $(p, r) \in [1, \infty]^2$, define the homogeneous Besov spaces $\dot{B}^s_{p,r}$ which consists of those distributions in $\mathcal{S}'_h$ such that

$$\|u\|_{\dot{B}^s_{p,r}} := \big(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\dot{\Delta}_j u\|^r_{L^p}\big)^{\frac{1}{r}} < \infty.$$

From the definition we immediately know the norm of $\dot{H}^s$ coincides with $\dot{B}^s_{2,2}$.

Lemma 3.10. *For $0 < \gamma < 1$,*

$$\|u \otimes u\|_{\dot{B}^{2\gamma-\frac{1}{2}}_{2,1}} \lesssim \|u\|^2_{\dot{H}^{\frac{1}{2}+\gamma}}$$

Proof. See Corollary 2.55 of [1].

□

Remark 3.11.

1. *By definition $\dot{\Delta}_j$ commutates with $\Gamma(t)$ and $\Gamma^{-1}(t)$. That is, for $u \in L^p$, we have*
$$\Gamma\dot{\Delta}_j u = \dot{\Delta}_j\Gamma u \text{ and } \Gamma^{-1}\dot{\Delta}_j u = \dot{\Delta}_j\Gamma^{-1}u. \tag{3.10}$$

2. *From (3.9), we know for any p, L^p is dense in $\mathcal{S}'_h$, thus we can extend $\Gamma(t)$ and $\Gamma^{-1}(t)$ continuously and uniquely to an operator from $\mathcal{S}'_h$ to $\mathcal{S}'_h$.*

3. *We claim for $u \in \dot{B}^s_{p,r}$,*
$$\Gamma u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j\Gamma u = \sum_{j\in\mathbb{Z}} \Gamma\dot{\Delta}_j u, \tag{3.11}$$
and
$$\|\Gamma(t)\|_{L(\dot{B}^s_{p,r},\dot{B}^s_{p,r})} \le \|\Gamma(t)\|_{L(L^p,L^p)}, \tag{3.12}$$
where $L(X, X)$ denotes the operator norm from X to X.
Indeed, first we note that (3.10) still holds for $u \in \dot{B}^s_{p,r}$, since by the way of expansion, we have
$$\Gamma u := \sum_{k\in\mathbb{Z}} \Gamma\dot{\Delta}_k u,$$
and
$$\begin{aligned}\Gamma\dot{\Delta}_j u &= \sum_{k\in\mathbb{Z}} \Gamma\dot{\Delta}_k\dot{\Delta}_j u \\ &= \sum_{k\in\mathbb{Z}} \Gamma\dot{\Delta}_j\dot{\Delta}_k u \\ &= \sum_{k\in\mathbb{Z}} \dot{\Delta}_j\Gamma\dot{\Delta}_k u \\ &= \dot{\Delta}_j\Gamma u,\end{aligned}$$

where the third inequality is due to (3.10) in L^p. Thus we finish the proof of (3.11). For the proof of (3.12), immediately follows by the definition of Besov norms we have

$$\begin{aligned}\|\Gamma u\|_{\dot{B}^s_{p,r}} &= \Big(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\dot{\Delta}_j\Gamma u\|^r_{L^p}\Big)^{\frac{1}{r}} \\ &= \Big(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\Gamma\dot{\Delta}_j u\|^r_{L^p}\Big)^{\frac{1}{r}} \\ &\le \Big(\sum_{j\in\mathbb{Z}} 2^{rjs}\|\Gamma\|^r_{L(L^p,L^p)}\|\dot{\Delta}_j u\|^r_{L^p}\Big)^{\frac{1}{r}} \\ &\le \|\Gamma\|_{L(L^p,L^p)}\|u\|_{\dot{B}^s_{p,r}}.\end{aligned}$$

Roughly speaking, the above preparation is to show how 'good' the operators Γ and Γ^{-1} are. They could be expanded to any Besov (hence Sobolev) space and could commutate with derivatives and Littlewood-Paley operators. After the preparation, we now show the following lemma, which is the crucial step of the proof of the main theorem.

3.3.2 Proof of Main Theorem

Define

$$F(y)(t) = -\int_0^t e^{(t-s)\Delta}\Gamma^{-1}Q(\Gamma y, \Gamma y)ds.$$

Lemma 3.12. *There exists some constant C, which depends on the path ω, such that*

$$\|F(y)\|_{\mathcal{Z}^\gamma_T} \le CT^{\frac{\gamma}{2}}\|y\|^2_{\mathcal{Z}^\gamma_T}.$$

Proof. The key point of the proof is based on the contraction property of the semigroup $e^{t\Delta}$. By definition

$$\begin{aligned}\|F(y)(t)\|_{\dot{H}^{\frac{1}{2}+\gamma}} &\le \int_0^t \|e^{(t-s)\Delta}\Gamma^{-1}Q(\Gamma y(s), \Gamma y(s))\|_{\dot{H}^{\frac{1}{2}+\gamma}}ds \\ &\lesssim \int_0^t \|e^{(t-s)\Delta}\Gamma^{-1}(\Gamma y(s)\otimes\Gamma y(s))\|_{\dot{H}^{\frac{3}{2}+\gamma}}ds,\end{aligned}$$

where the second inequality is due to the reason that $Q(\Gamma y, \Gamma y)$ is the sum of the first order derivatives of $\Gamma y\otimes\Gamma y$. Moreover, by Lemma 2.4 of [1], there exists some constant

c,

$$\begin{aligned}
&\|\dot{\Delta}_j[e^{(t-s)\Delta}\Gamma^{-1}(\Gamma y\otimes\Gamma y)]\|_{L^2}\\
&\lesssim e^{-c(t-s)2^{2j}}\|\dot{\Delta}_j[\Gamma^{-1}(\Gamma y\otimes\Gamma y)]\|_{L^2}\\
&\lesssim e^{-c(t-s)2^{2j}}\|\Gamma^{-1}(s)\|_{L(L^2,L^2)}2^{-j(2\gamma-\frac{1}{2})}d_j\|\Gamma y\otimes\Gamma y\|_{\dot{B}^{2\gamma-\frac{1}{2}}_{2,1}}\\
&\lesssim e^{-c(t-s)2^{2j}}\|\Gamma^{-1}(s)\|_{L(L^2,L^2)}2^{-j(2\gamma-\frac{1}{2})}d_j\|\Gamma y\|^2_{\dot{H}^{\frac{1}{2}+\gamma}}\\
&\lesssim \eta_s e^{-c(t-s)2^{2j}}2^{-j(2\gamma-\frac{1}{2})}\|y(s)\|^2_{\dot{H}^{\frac{1}{2}+\gamma}}d_j,
\end{aligned} \tag{3.13}$$

where d_j is a sequence in ℓ^1, the third inequality is due to Lemma 3.10 and the last inequality is due to the reason that from Remark 3.11, we know

$$\|\Gamma(t)\|_{L(\dot{H}^{\frac{1}{2}+\gamma},\dot{H}^{\frac{1}{2}+\gamma})}\leq\|\Gamma(t)\|_{L(L^2,L^2)}.$$

Thus we have

$$\|e^{(t-s)\Delta}\Gamma^{-1}(\Gamma y\otimes\Gamma y)\|_{\dot{H}^{\frac{3}{2}+\gamma}}\lesssim\eta_s\|y(s)\|^2_{\dot{H}^{\frac{1}{2}+\gamma}}\sup_j[e^{-c(t-s)2^{2j}}2^{j(2-\gamma)}].$$

Since there exists some constant $C(\gamma)$, such that

$$e^{-c(t-s)2^{2j}}\leq C(\gamma)[(t-s)2^{2j}]^{-1+\frac{\gamma}{2}},$$

we obtain

$$\|e^{(t-s)\Delta}\Gamma^{-1}(\Gamma y\otimes\Gamma y)\|_{\dot{H}^{\frac{3}{2}+\gamma}}\lesssim\eta_s\|y(s)\|^2_{\dot{H}^{\frac{1}{2}+\gamma}}(t-s)^{-1+\frac{\gamma}{2}}. \tag{3.14}$$

Therefore,

$$\|F(y)(t)\|_{L^\infty([0,T];\dot{H}^{\frac{1}{2}+\gamma})}\lesssim T^{\frac{\gamma}{2}}\|y(s)\|^2_{L^\infty([0,T];\dot{H}^{\frac{1}{2}+\gamma})}\sup_{t\geq0}\eta_t. \tag{3.15}$$

On the other hand,

$$\|\nabla F(y)(t)\|_{\dot{H}^{\frac{1}{2}+\gamma}}\leq\int_0^t\|e^{(t-s)\Delta}\Gamma^{-1}(\Gamma y(s)\otimes\Gamma y(s))\|_{\dot{H}^{\frac{5}{2}+\gamma}}ds. \tag{3.16}$$

Following the same way that we obtain (3.14), by replacing γ by $\gamma+1$, we obtain

$$\|e^{(t-s)\Delta}\Gamma^{-1}(\Gamma y(s)\otimes\Gamma y(s))\|_{\dot{H}^{\frac{5}{2}+\gamma}}\lesssim\eta_s\|y(s)\|^2_{\dot{H}^{\frac{3}{2}+\gamma}}(t-s)^{\frac{\gamma-1}{2}}. \tag{3.17}$$

Therefore, by combining (3.16) and (3.17), we deduce

$$\begin{aligned}\int_0^T \|\nabla F(y)(t)\|^2_{\dot H^{\frac12+\gamma}} dt &\le \int_0^T \Big(\int_0^t \eta_s \|y\|^2_{\dot H^{\frac32+\gamma}} (t-s)^{\frac{\gamma-1}{2}} ds\Big)^2 dt \\ &\lesssim \sup_{t\ge 0} \eta_t^2 \int_0^T \big(\int_0^t \|y(s)\|^2_{\dot H^{\frac32+\gamma}} ds\ t^{\frac{\gamma-1}{2}}\big)^2 dt \\ &\lesssim \sup_{t\ge 0} \eta_t^2 \int_0^T t^{\gamma-1} dt \int_0^T \|y(t)\|^2_{\dot H^{\frac32+\gamma}} dt \\ &\lesssim \sup_{t\ge 0} \eta_t^2\ T^\gamma \int_0^T \|y(t)\|^2_{\dot H^{\frac32+\gamma}} dt.\end{aligned}$$

That is,

$$\|\nabla F(y)(t)\|_{L^2([0,T];\dot H^{\frac12+\gamma})} \lesssim T^{\frac{\gamma}{2}} \|\nabla y(s)\|^2_{L^2([0,T];\dot H^{\frac12+\gamma})} \sup_{t\ge 0} \eta_t. \tag{3.18}$$

The conclusion follows directly from (3.15) and (3.18).

□

Lemma 3.13. *For any $T > 0$ and $0 < \gamma < 1$,*

$$\|e^{t\Delta} u_0\|_{\mathcal{Z}_T^\gamma} \le \|u_0\|_{\dot H^{\frac12+\gamma}}.$$

Proof. The proof is trivial. We write a simple proof here for completeness.
Set $v = e^{t\Delta} u_0$, then v is the solution of the following heat equations

$$\begin{cases} \partial_t u = \Delta u \\ u\ |_{t=0} = u_0. \end{cases} \tag{3.19}$$

Taking $\dot H^{\frac12+\gamma}$ inner product of both sides with u immediately yields the result.

□

The following fixed point theorem comes from Lemma 5.5 of [1]:

Lemma 3.14. *Let E be a Banach space, $\mathcal{B}$ a continuous bilinear map from $E \times E$ to E, and α a positive real number such that*

$$\alpha < \frac{1}{4\|\mathcal{B}\|} \text{ with } \|\mathcal{B}\| := \sup_{\|u\|\le 1, \|v\|\le 1} \|\mathcal{B}(u,v)\|.$$

For any a in the ball $B(0,\alpha)$ (i.e., with center 0 and radius α) in E, a unique x then exists in $B(0,2\alpha)$ such that

$$x = a + \mathcal{B}(x,x).$$

Proof of Theorem 3.6
The result comes immediately by Lemma 3.12, Lemma 3.13 and Lemma 3.14.

□

Appendix A

Lemma 3.1 in [9]

Lemma A.1. *Let $\mathcal{O}$ be a domain of $\mathbb{R}^2$ with smooth boundary. Let H be the space of vector field $L^2(\mathcal{O}, R^2)$ with divergence free. Let V be the space of vector field $H^1(\mathcal{O}, R^2)$ with divergence free. Let U' be a space such that H can be embedded compactly in U'.*

Define

$$\tilde{\mathcal{Z}} := C([0,T];U') \cap L^2_w([0,T];V) \cap L^2([0,T];H_{loc})$$

and let $\tilde{\mathcal{T}}$ be the supremum of the corresponding topologies. Then, a set $\mathcal{K} \subset \tilde{\mathcal{Z}}$ is $\tilde{\mathcal{T}}$ -relatively compact if the following two conditions hold:

1. $\sup\limits_{u\in\mathcal{K}} \int_0^T \|u(t)\|_V^2 dt < \infty$
2. $\lim\limits_{\delta\to 0} \sup\limits_{u\in\mathcal{K}} \sup\limits_{t,s\in[0,T],|t-s|\le\delta} \|u(t)-u(s)\|_{U'} = 0.$

Appendix B

Standard Borel Spaces

First we introduce the following notions of countably generated Borel Space and standard Borel space.

Definition B.1 (Countably generated Borel space, see [77] Chapter V Definition 2.1)**.** *A Borel space $(X, \mathscr{B})$ is said to be countably generated if there exists a denumerable class $\mathcal{D} \subset \mathscr{B}$ such that $\mathcal{D}$ generates $\mathscr{B}$.*

Definition B.2 (Standard Borel space , see [77] Chapter V Definition 2.2)**.** *A countably generated Borel space $(X, \mathscr{B})$ is called standard if there exists a complete separable metric space Y such that the σ-algebras $\mathscr{B}$ and $\mathscr{B}(Y)$ are σ-isomorphic.*

Moreover, we will introduce the following theorem, which is in Theorem 2.4 of Chapter V of [77].

Theorem B.3. *Let $(X, \mathscr{B})$ be standard, $(Y, \mathscr{C})$ countably generated and φ a one-one map of X into Y which is measurable. Then $Y' = \varphi(X) \in \mathscr{C}$ and φ is a Borel isomorphism between the Borel spaces $(X, \mathscr{B})$ and $(Y', \mathscr{C}_{Y'})$.*

By the Theorem B.3 we will show the following theorem.

Theorem B.4. *Let $(X, \mathscr{B})$ be any standard Borel space. Assume that $\{f_n\}_{n\geq 1}$ is a sequence of $\mathscr{B}-$measurable functions from X to $\mathbb{R}$ which separate the points of X. Denote by $\sigma_0(X)$ the σ-algebra generated by $\{f_n\}_{n\geq 1}$. Then $\sigma_0(X) = \mathscr{B}$.*

Proof. Consider the identity map id:

$$(X, \mathscr{B}) \longrightarrow (X, \sigma_0(X)).$$

Since each f_i is measurable, it is obvious that id is measurable. Hence by Theorem B.3 we know that id is a Borel isomorphism, which finishes our proof.

□

Recall that in Chapter 1 and Chapter 2, we have defined the spaces

$$\mathcal{X}_T = C([0,T];H^{-1}) \cap L^2([0,T];H) \cap L^2_w([0,T];H^{1,1}) \cap L^\infty_{w*}([0,T];H^{0,1})$$

and

$$\mathcal{Z}_T = C([0,T];U') \cap L^2_w([0,T];\tilde{H}^1) \cap L^2([0,T];H_{loc}) \cap C([0,T];H^1_w),$$

where U is chosen to be another Hilbert space which is dense in H^1 and can be compactly embedded in H^1, thus H^{-1} can be compactly and densely embedded in U'.

In the next two sections we will show that both $\mathcal{X}_T$ and $\mathcal{Z}_T$ are standard.

B.1 $\mathcal{Z}_T$ is Standard

We first prove that for $\mathcal{Z}_T$ since it is more complicated. Recall

$$\mathcal{Z}_T = C([0,T];U') \cap L^2_w([0,T];\tilde{H}^1) \cap L^2([0,T];H_{loc}) \cap C([0,T];H^1_w).$$

Let $Y := C([0,T];U') \cap L^2_w([0,T];\tilde{H}^1) \cap L^2([0,T];H_{loc})$. Let τ_z be the topology generated by intersecting of the four topologies, i.e. the open sets of τ_z are defined as the intersection of four open sets which belong to each intersected space respectively. Let $\mathscr{B}(\tau_z)$ be the corresponding Borel σ-algebra.

Theorem B.5. *$(\mathcal{Z}_T, \mathscr{B}(\tau_z))$ is standard.*

Proof. Consider $Y := L^2([0,T];U')$ with Borel σ-algebra $\mathscr{B}(Y)$. Then by Theorem 2.3 of Chapter V of [77], since Y is standard, it suffices to show

(i) $\mathcal{Z}_T \subset Y$ continuously.

(ii) $\mathcal{Z}_T \in \mathscr{B}(Y)$.

(iii) $\mathscr{B}(\tau_z) = \mathscr{B}(Y) \cap \mathcal{Z}_T$.

(i) is trivial since $C([0,T];U') \subset L^2([0,T];U')$ continuously.
(ii)+(iii): From (i) we immediately have

$$\mathscr{B}(Y) \cap \mathcal{Z}_T \subset \mathscr{B}(\tau_z). \tag{2.1}$$

For $N \in \mathbb{N}$ let

$$(B_N^{H^1})_w := \{x \in H^1; \|x\|_{H^1} \leq N\},$$

equipped with the H^1-weak topology. Then $(B_N^{H^1})_w$ is weakly compact and completely metrizable, hence separable. Define

$$C_N := C([0,T];(B_N^{H^1})_w)$$

to be all H^1-weakly continuous paths in $(B_N^{H^1})_w$. Since $(B_N^{H^1})_w$ is a compact (hence complete) separable metric space (let q be the metric) and a convex convex subset of H^1. C_N is also a complete separable metric space with metric d_1 defined by

$$d_1(w_1, w_2) = \sup_{t\in[0,T]} q(w_1(t), w_2(t)).$$

Define

$$(\tilde{C}_N)_w := \{w \in L^2([0,T]; \tilde{H}^1); \|w\|_{L^2([0,T];\tilde{H}^1)} \le N\},$$

equipped with the weak topology on $L^2([0,T]; \tilde{H}^1)$. Then $(\tilde{C}_N)_w$ is compact and metrizable, hence separable and complete with some metric d_2. Define

$$Z^{(N)} := C_N \cap (\tilde{C}_N)_w \cap C([0,T]; U') \cap L^2([0,T]; H_{loc}).$$

Denote these four intersected spaces by Y_1, Y_2, Y_3 and Y_4 respectively. Note that Y_1, Y_2, Y_3 and Y_4 are complete separable metric spaces with metric d_1,d_2 d_3 and d_4 respectively. Let $Z^{(N)}$ be equipped with the metric $d = max(d_1, d_2, d_3, d_4)$. Then $Z^{(N)}$ is separable (see the Lemma B.7 below). To show that $Z^{(N)}$ is complete it is enough to show that if $w_k \in Z^{(N)}$, $k \in \mathbb{N}$ and w_k converges to $w^{(i)}$ in d_i for $i = 1, 2, 3, 4$, then $w^{(1)} = w^{(2)} = w^{(3)} = w^{(4)}$. First we note that clearly that $w^{(2)} = w^{(4)}$, since obviously w_k converges to $w^{(2)}$ and $w^{(4)}$ in $L^2([0,T]; L^2(K))$ weakly for every compact set K. Moreover, $w^{(2)} = w^{(3)}$ since w_k converges weakly in $L^2([0,T]; U')$. Thus we have $w^{(2)} = w^{(3)} = w^{(4)}$. For $w^{(1)}$, since w_k converges to $w^{(1)}$ in metric d_1, we know for any $t \in [0,T]$,

$$w_k(t) \to w^{(1)}(t) \text{ in } H^1_w. \tag{2.2}$$

Since

$$\sup_{t\in[0,T]} |w_k(t)|_{H^1} \le N, \ \forall t \in [0,T],$$

we have that $\{w_k\}$ is bounded in $L^2([0,T]; H^1)$, hence there exists subsequences $(k_n)_{k\in\mathbb{N}}$, $(N_l)_{l\in\mathbb{N}}$ of $(n)_{n\in\mathbb{N}}$ and $(N)_{N\in\mathbb{N}}$ and $w \in L^2([0,T]; H^1)$, such that

$$\frac{1}{N_l}\sum_{n=1}^{N_l} w_{k_n} \to w \text{ in } L^2([0,T]; H^1), \text{and } dt-a.e. \text{ in } H^1\text{strongly, hence weakly.} \tag{2.3}$$

By (2.2) it follows that $w^{(1)} = w$. Meanwhile, we have

$$\frac{1}{N_l}\sum_{n=1}^{N_l} w_{k_n} \to w^{(4)} \text{ in } L^2([0,T]; H_{loc}).$$

Thus we have $w^{(1)} = w^{(2)} = w^{(3)} = w^{(4)} = w$. Hence that $Z^{(N)}$ is a complete separable metric space. Furthermore, the following embeddings are continuous:

$$(Z^{(N)}, d) \subset (\mathcal{Z}_T, \tau_z) \subset Y,$$

hence measurable. Thus by Theorem 2.4 in Chapter V of [77], we have $Z^{(N)} \in \mathscr{B}(Y)$ and

$$\mathscr{B}((Z^{(N)}, d)) = \mathscr{B}(Y) \cap Z^{(N)}, \tag{2.4}$$

hence $\mathcal{Z}_T = \bigcup_{N=1}^{\infty} Z^{(N)} \in \mathscr{B}(Y)$ and (ii) is proved.

It remains to prove (iii). The following lemma will be applied.

Lemma B.6. *$Z^{(N)}$ is the closed subset of $\mathcal{Z}_T$.*

Proof. Let $w_i \in Z^{(N)}$, $i \in I$, where I is a net, such that $w_i \xrightarrow{I} w \in \mathcal{Z}_T$ w.r.t. τ_z. Then it is obvious that w is in Y_2, Y_3 and Y_4 . Now we only need to show that $w \in C_N$. We recall that the topology on $C([0,T]; H^1_w)$ is defined to be the weakest topology such that for any $y \in H^1$, the maps

$$\begin{aligned} C([0,T]; H^1_w) &\longrightarrow C([0,T]; \mathbb{R}) \\ w &\mapsto \langle w(\cdot), y\rangle_{H^1} \end{aligned} \tag{2.5}$$

are continuous. This implies that for all $y \in H$, $\|y\|_{H^1} = 1$,

$$\sup_{t\in[0,T]} \langle w_i(t), y\rangle_{H^1} \xrightarrow{I} \langle w(t), y\rangle_{H^1}.$$

But then

$$\begin{aligned} \sup_{t\in[0,T]} \|w(t)\|_{H^1} &= \sup_{t\in[0,T]} \sup_{\|y\|_{H^1}=1} \langle w(t), y\rangle_{H^1} \\ &= \sup_{\|y\|_{H^1}=1} \sup_{t\in[0,T]} \langle w(t), y\rangle_{H^1} \\ &= \sup_{\|y\|_{H^1}=1} \lim_{i\in I} \sup_{t\in[0,T]} \langle w_i(t), y\rangle_{H^1} \\ &\leq N. \end{aligned} \tag{2.6}$$

Thus we have $w \in C_N$, which completes the proof.

□

Proof of (iii)

Lemma B.6 implies that $Z^{(N)} \in \mathscr{B}(\tau_z)$, which in turn implies that

$$\mathscr{B}(\tau_z) \cap Z^{(N)} = \{B \in \mathscr{B}(\tau_z) | B \subset Z^{(N)}\},$$

and hence

$$\mathscr{B}(\tau_z) = \bigcup_{N=1}^{\infty} \{B \in \mathscr{B}(\tau_z) | B \subset Z^{(N)}\}.$$

We note that obviously $(Z^{(N)}, d) \subset (Z^{(N)}, \tau_z \cap Z^{(N)})$ continuously. Hence

$$\mathscr{B}(\tau_z \cap Z^{(N)}) \subset \mathscr{B}((Z^{(N)}, d)). \tag{2.7}$$

Furthermore, let $A \subset \mathscr{Z}_T$, A τ_z-closed. Then $A \cap Z^{(N)})$ is τ_z-closed, hence

$$\begin{aligned} A \cap Z^{(N)} &\in \mathscr{B}(\tau_z \cap Z^{(N)}) \\ &\subset \mathscr{B}((Z^{(N)}, d)) \\ &= \mathscr{B}(Y) \cap Z^{(N)}, \end{aligned}$$

where the '$\subset$' is due to (2.7) and '$=$' is due to (2.4).
Since $Z^{(N)} \in \mathscr{B}(Y)$,

$$\mathscr{B}(Y) \cap Z^{(N)} = \{B \in \mathscr{B}(\tau_z) | B \subset Z^{(N)}\} \subset \{B \in \mathscr{B}(\tau_z) | B \subset Z\} \subset \mathscr{B}(Y) \cap \mathscr{Z}_T.$$

Hence

$$A = \bigcup_{N=1}^{\infty} A \cap Z^{(N)} \in \mathscr{B}(Y) \cap \mathscr{Z}_T.$$

Hence $\mathscr{B}(\tau_z) \subset \mathscr{B}(Y) \cap \mathscr{Z}_T$.
By (2.1), we have $\mathscr{B}(\tau_z) = \mathscr{B}(Y) \cap \mathscr{Z}_T$,which is (iii). □

Lemma B.7. *The intersection of finite separable metric spaces (with the maximal metric) is a separable metric space.*

Proof. It suffices to prove the result for the intersection of two separable spaces. Let X and Y be two separable metric spaces with $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ to be the countable dense subset respectively.

Claim that

$$\{B_X(x_n, r_k) \cap B_Y(y_n, q_l)\}_{r_k, q_l \in \mathbb{Q}}, \tag{2.8}$$

where $B_X(x_n, r_k)$ denotes the ball in the space X with centre x_n and radius r_k and $B_Y(y_n, q_l)$ denotes the ball in the space Y with centre y_n and radius q_l, is the countable topological basis of $X \cap Y$.
In fact, let U be any open set in $X \cap Y$, it suffices to prove that for any $a \in U$, there exists an element $V \in$ (2.8), such that $a \in V \subset U$. Since U is an open set, there exists some r_0 (obviously we can let it be a rational number), such that

$$B_{X\cap Y}(a, r_0) \subset U.$$

Since X and Y are both separable, we can find n_1 and n_2, such that $d_X(a, x_{n_1}) < \frac{r_0}{2}$ and $d_Y(a, y_{n_2}) < \frac{r_0}{2}$. Thus we have

$$a \in B_X(x_{n_1}, \frac{r_0}{2}) \cap B_Y(y_{n_2}, \frac{r_0}{2}) \subset B_{X\cap Y}(a, r_0) \subset U.$$

Thus we have proved that $X \cap Y$ is second countable, thus separable. □

B.2 $\mathcal{X}_T$ is Standard

Now we prove a similar result for $\mathcal{X}_T$. Recall

$$\mathcal{X}_T = C([0,T];H^{-1}) \cap L^2([0,T];H) \cap L^2_w([0,T];H^{1,1}) \cap L^\infty_{w*}([0,T];H^{0,1}).$$

Before we prove the standard property of $\mathcal{X}_T$ we note that $L^\infty_{w*}([0,T];H^{0,1})$ is a complete separable space since the closed unit ball of the space $L^\infty_{w*}([0,T];H^{0,1})$ is compact separable metric space, where the compactness follows by Banach-Alaoglu theorem and the separable property is because $L^\infty([0,T];H^{0,1})$ is the dual space of $L^1([0,T];H^{0,-1})$, which is separable.

Theorem B.8. *Let τ_X be the open sets of $\mathcal{X}_T$ and $\mathscr{B}(\tau_X)$ be the corresponding Borel $\sigma-$algebra with this topology. Then $(\mathcal{X}_T, \mathscr{B}(\tau_X))$ is standard.*

Proof. The proof is mainly the repeat of the proof of the Theorem B.5. We write it again for completeness. Consider $Y := L^2([0,T];H^{-1})$ with Borel σ-algebra $\mathscr{B}(Y)$. Then by Theorem 2.3 of Chapter V of [77], since Y is standard, it suffices to show

(i) $\mathcal{X}_T \subset Y$ continuously.

(ii) $\mathcal{X}_T \in \mathscr{B}(Y)$.

(iii) $\mathscr{B}(\tau_X) = \mathscr{B}(Y) \cap \mathcal{X}_T$.

(i) is trivial since $C([0,T];H^{-1}) \subset L^2([0,T];H^{-1})$ continuously.
(ii)+(iii): From (i) we immediately have

$$\mathscr{B}(Y) \cap \mathcal{X}_T \subset \mathscr{B}(\tau_X). \tag{2.9}$$

Similarly, we can write it as the following

$$\mathcal{X}_T = \bigcup_{N=1}^{\infty} X^{(N)},$$

where

$$X^{(N)} = C([0,T];H^{-1}) \cap L^2([0,T];H) \cap (D_N)_w \cap (E_N)_{w*}$$
$$D_N = \{x \in L^2([0,T];H^{1,1}); \|x\|_{L^2([0,T];H^{1,1})} \le N\}$$

and

$$E_N = \{x \in L^\infty([0,T];H^{0,1}); \|x\|_{L^\infty([0,T];H^{0,1})} \le N\}.$$

In other word, $(D_N)_w$ and $(E_N)_{w*}$ are two balls of radius N in the space $L^2([0,T];H^{1,1})$ and $L^\infty([0,T];H^{0,1})$ with weak and weak star topology, respectively. Therefore, they are both compact and metrizable, hence separable. Let d_5, d_6 be two metrics of $(D_N)_w$ and

$(E_N)_{w*}$ which induce their topology, respectively.
To prove: $X^{(N)}$ is complete. Let $\{x_n\}_{n\geq 1}$ be any Cauchy sequence in $X^{(N)}$. Since each of the intersected spaces of $X^{(N)}$ is complete, it suffices to prove the limits in each space are the same. Let a_1, a_2, a_3 and a_4 be the limits of $\{x_n\}_{n\geq 1}$ in the four spaces respectively. Obviously we have $a_1 = a_2 = a_3$ since x_n converges to a_1, a_2 and a_3 weakly in Y. For any $\phi \in L^2$,

$$\langle x_n, \phi\rangle \to \langle a_2, \phi\rangle \text{ in } L^1([0,T],\mathbb{R}).$$

$$\langle x_n, \phi\rangle \to \langle a_4, \phi\rangle \text{ in } L^1([0,T],\mathbb{R}).$$

Hence $a_2 = a_4$. Therefore, we have $a_1 = a_2 = a_3 = a_4$, which means $X^{(N)}$ is complete. By Theorem B.7 $X^{(N)}$ is a complete separable metric space. Denote by d its metric. By Theorem 2.4 in Chapter V of [77], we have $X^{(N)} \in \mathscr{B}(Y)$ and

$$\mathscr{B}((X^{(N)}, d)) = \mathscr{B}(Y) \cap X^{(N)}, \tag{2.10}$$

hence $\mathcal{X}_T = \bigcup_{N=1}^{\infty} X^{(N)} \in \mathscr{B}(Y)$ and (ii) is proved.
Note that $X^{(N)}$ is a closed subset of $\mathcal{X}_T$ with the induced topology. For any $A \in \mathscr{B}(\tau_X)$,

$$A = \bigcup_{N=1}^{\infty} (A \cap X^{(N)}).$$

Since $A \cap X^{(N)} \in \mathscr{B}((X^{(N)}, d)) = \mathscr{B}(Y) \cap X^{(N)}$, we obtain $A \in \mathscr{B}(Y) \cap \mathcal{X}_T$. □

B.3 Consequences

Corollary B.9. *The σ-algebra generated by a sequence of $\mathscr{B}(\mathcal{X}_T)$-measurable functions which separate the points in $\mathcal{X}_T$ is exactly $\mathscr{B}(\mathcal{X}_T)$. The σ-algebra generated by a sequence of $\mathscr{B}(\mathcal{Z}_T)$-measurable functions which separate the points in $\mathcal{Z}_T$ is exactly $\mathscr{B}(\mathcal{Z}_T)$.*

Proof. The results come from Theorem B.4, Theorem B.5 and Theorem B.8.

□

Appendix C

the Completeness of the Normed Space $\mathcal{Z}_T^\gamma$

We prove the completeness of the space $\mathcal{Z}_T^\gamma$ when $0 < \gamma < 1$.
Firstly, recall that the norm of $\mathcal{Z}_T^\gamma$ is:

$$\|u\|^2_{\mathcal{Z}_T^\gamma} := \|u\|^2_{L^\infty([0,T];\dot{H}^{\frac{1}{2}+\gamma})} + \int_0^T \|\nabla u(t)\|^2_{\dot{H}^{\frac{1}{2}+\gamma}} dt.$$

Let $\{u_n\}_{n\geq 1}$ be a Cauchy sequence in $\mathcal{Z}_T^\gamma$, which means $\{u_n\}_{n\geq 1}$ is a Cauchy sequence in $L^\infty([0,T];\dot{H}^{\frac{1}{2}+\gamma})$ and $\{\nabla u_n\}_{n\geq 1}$ is a Cauchy sequence in $L^2([0,T];\dot{H}^{\frac{1}{2}+\gamma})$.

Our aim is to find some $u \in \mathcal{Z}_T^\gamma$, such that u_n converges to u in the norm of $\mathcal{Z}_T^\gamma$. By Prop 1.34 of [1], we know that when $0 < \gamma < 1$, the homogeneous Sobolev space $\dot{H}^{\frac{1}{2}+\gamma}$ is a Hilbert space. Therefore, both $L^\infty([0,T];\dot{H}^{\frac{1}{2}+\gamma})$ and $L^2([0,T];\dot{H}^{\frac{1}{2}+\gamma})$ are complete. Therefore, there exists v_1 and v_2, such that

$$u_n \longrightarrow v_1 \text{ in } L^\infty([0,T];\dot{H}^{\frac{1}{2}+\gamma}),$$

and

$$\nabla u_n \longrightarrow v_2 \text{ in } L^2([0,T];\dot{H}^{\frac{1}{2}+\gamma}).$$

Now note that if we can prove that $\nabla v_1(t) = v_2(t)$ for $a.e.t \in [0,T]$, (of course the derivatives mean weak derivatives, which can be at least defined for those $v_1(t)$ which satisfy $v_1(t) \in \dot{H}^{\frac{1}{2}+\gamma}$ and the derivatives are at least in the space $\mathcal{S}'$),
then we immediately have

$$\nabla u_n \longrightarrow \nabla v_1 \text{ in } L^2([0,T];\dot{H}^{\frac{1}{2}+\gamma}),$$

hence $v_1 \in \mathcal{Z}_T^\gamma$ and u_n converges to v_1 in the norm of $\mathcal{Z}_T^\gamma$.
To prove: $\nabla v_1(t) = v_2(t)$ **for** $a.e.t \in [0,T]$**:**

Since L^p convergence implies convergence in probability, hence we can find a subsequence n_k, such that

$$u_{n_k}(t) \longrightarrow v_1(t) \text{ in } \dot{H}^{\frac{1}{2}+\gamma} a.e.,$$

and

$$\nabla u_{n_k}(t) \longrightarrow v_2(t) \text{ in } \dot{H}^{\frac{1}{2}+\gamma} a.e..$$

Therefore, there exists a subset A of the interval $[0, T]$ which has the full Lebesgue measure T, such that for any $t \in A$,

$$u_{n_k}(t) \longrightarrow v_1(t) \text{ in } \dot{H}^{\frac{1}{2}+\gamma},$$

and

$$\nabla u_{n_k}(t) \longrightarrow v_2(t) \text{ in } \dot{H}^{\frac{1}{2}+\gamma}.$$

Then for any $\phi \in (C_c^\infty(\mathbb{R}^3))^3$, and $t \in A$, we have

$$\begin{aligned}
\langle v_2(t), \phi \rangle &= \lim_{k\to\infty} \langle \nabla u_{n_k}(t), \phi \rangle \\
&= -\lim_{k\to\infty} \langle u_{n_k}(t), \operatorname{div} \phi \rangle \\
&= -\langle v_1(t), \operatorname{div} \phi \rangle \\
&= \langle \nabla v_1(t), \phi \rangle,
\end{aligned}$$

where same as before, derivatives mean weak derivatives, $\langle \cdot, \cdot \rangle$ is the duality bracket and the second and the fourth equalities are due to the definition of the weak derivatives. Therefore, we have shown $\nabla v_1(t) = v_2(t)$ for $a.e.t \in [0, T]$, which finishes our proof.

Bibliography

[1] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Grundlehren der mathematischen Wissenschaften, **343** Springer-Verlag Berlin Heidelberg, 2011.

[2] V Barbu and M Röckner. Global solutions to random 3D vorticity equations for small initial data. *Journal of Differential Equations*, **263(9)** 5395 – 5411, 2017.

[3] H. Bessaih and B. Ferrario. Inviscid limit of stochastic damped 2D Navier–Stokes equations. *Nonlinearity*, **27(1)** 1–15, Dec 2013.

[4] H. Bessaih and B. Ferrario, Invariant measures for stochastic damped 2D Euler equations. *arXiv:1909.00424v1*

[5] Z. Brzeźniak. On stochastic convolution in Banach spaces and applications. *Stochastics An International Journal of Probability and Stochastic Processes*, **61** 245–295, 11 1997.

[6] H. Bessaih and A. Millet, On stochastic modified 3D Navier–Stokes equations with anisotropic viscosity, *Journal of Mathematical Analysis and Applications*, **462** , 915-956, 2018.

[7] Z. Brzeźniak and Y. Li, Asymptotic compactness and absorbing sets for 2D stochastic Navier–Stokes equations on some unbounded domains. *Transactions of the American Mathematical Society*, **358(12)** 5587–5629, 2006.

[8] Z. Brzeźniak and E. Motyl, The existence of martingale solutions to the stochastic Boussinesq equations. *Global and stochastic analysis*, **1** 175–216, 07 2014.

[9] Z. Brzeźniak and E. Motyl. Existence of a martingale solution of the stochastic Navier–Stokes equations in unbounded 2D and 3D domains. *Journal of Differential Equations*, **254(4)** 1627 – 1685, 2013.

[10] Z. Brzeźniak, E. Motyl and M. Ondreját, Invariant measure for the stochastic Navier–Stokes equations in unbounded 2D domains. *Ann. Probab.*, 45(5):3145–3201, 09 2017.

[11] Z. Brzeźniak and M. Ondreját, Stochastic geometric wave equations with values in compact Riemannian homogeneous spaces. *Ann. Probab.*, 41(3B):1938–1977, 05 2013.

[12] Z. Brzeźniak and S. Peszat, Stochastic two dimensional Euler Equations. *Ann. Probab.*, **29(4)** 1796–1832, 10 2001.

[13] T. Buckmaster, S. Shkoller, and V. Vicol, Nonuniqueness of weak solutions to the SQG equation *Communications on Pure and Applied Mathematics*, 72:1809-1874, 2019.

[14] M. Cannone. A generalization of a theorem by kato on Navier–Stokes equations. *Revista Matematica Iberoamericana*, 13:515–541, 1997.

[15] L. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, *Annals of Mathematics*, **171** , 1903–1930, 2010.

[16] D. Cao, G. Qin, W. Zhan, C. Zou , On the global classical solutions for the generalized SQG equation , arXiv preprint arXiv:2104.11048, 2021.

[17] D. Chae, P. Constantin, D. Córdoba, F. Gancedo, and Jiahong Wu, Generalized surface quasi-geostrophic equations with singular velocities, Comm. Pure Appl. Math. **65** no. 8, 1037-1066, 2012.

[18] D. Chae, P. Constantin, and J. Wu, Inviscid models generalizing the two-dimensional Euler and the surface quasi-geostrophic equations, Arch. Ration. Mech. Anal. **202** no. 1, 35-62, 2011.

[19] D. Chae, P. Constantin, and J. Wu, Dissipative models generalizing the 2D Navier–Stokes and surface quasi-geostrophic equations, Indiana Univ. Math. J. **61** , 5, 1997-2018, 2012.

[20] J. -Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Fluids with anisotropic viscosity, *ESAIM: Mathematical Modelling and Numerical Analysis*, **34** 315–335, 2000.

[21] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, *Mathematical geophysics: An Introduction to Rotating Fluids and the Navier–Stokes Equations*, **32** Oxford University Press on Demand, 2006.

[22] J. -Y. Chemin and I. Gallagher, A Nonlinear Estimate of the Life Span of solutions of the three dimensional Navier–Stokes equations, *Tunisian J. Math.*, 1(2):273–293, 2019.

[23] J. -Y. Chemin and P. Zhang, On the global wellposedness to the 3-D incompressible anisotropic Navier–Stokes equations, *Communications in Mathematical Physics*, **272** 529–566, 2007.

[24] P. Constantin, D. Cordoba, and J. Wu, On the critical dissipative quasi-geostrophic equation, Indiana Univ. Math. J. **50** (2001), no. Special Issue, 97–107, Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000).

[25] P. Constantin, Andrew J. Majda, and E. Tabak, *Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar*, Nonlinearity **7** no. 6, 1495-1533, 1994.

[26] P. Constantin and F. Ramos, Inviscid Limit for Damped and Driven Incompressible Navier–Stokes Equations in $\mathbb{R}^2$, *Communications in Mathematical Physics*, **275** 529–551, 2007.

[27] P. Constantin, A. Tarfulea, and V. Vicol, *Long time dynamics of forced critical SQG*, Communications in Mathematical Physics, **335** 93-141, 2015.

[28] P. Constantin and V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, Geometric and Functional Analysis, **22** 1289–1321,2012.

[29] P. Constantin and J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, SIAM Journal on Mathematical Analysis, **30** 937-948, 1999.

[30] A. Córdoba and D. Córdoba, A maximum principle applied to quasi-geostrophic equations, *Communications in Mathematical Physics,* **249** 511-528, 2004.

[31] D. Córdoba, C. Fefferman, and J L. Rodrigo, Almost sharp fronts for the surface quasi-geostrophic equation, Proc. Natl. Acad. Sci. USA **101** no. 9, 2687-2691, 2004.

[32] D. Córdoba, J. Gómez-Serrano, and A D. Ionescu, Global solutions for the generalized SQG patch equation, *Arch Rational Mech Anal* , **233** ,1211-1251 ,2019.

[33] G. Da Prato, Kolmogorov equations for stochastic PDEs. Birkhäuser, 2004.

[34] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, 1992.

[35] H. C. Davies and H. Wernli. Quasi-geostrophic theory. In James R. Holton, editor, *Encyclopedia of Atmospheric Sciences*, pages 1787–1794. Academic Press, Oxford, 2003.

[36] V. W. Ekman and others, *On the influence of the earth's rotation on ocean-currents*, Almqvist & Wiksells boktryckeri, A.-B., 1905.

[37] F. Flandoli. Weak vorticity formulation of 2D Euler equations with white noise initial condition. *Communications in Partial Differential Equations*, 43(7):1102–1149, 2018.

[38] F. Flandoli and D. Gatarek, Martingale and stationary solutions for stochastic Navier–Stokes equations. *Probability Theory and Related Fields*, 102(3):367–391, Sep 1995.

[39] F. Flandoli, D. Luo, Convergence of transport noise to Ornstein-Uhlenbeck for 2D Euler equations under the enstrophy measure. *Ann. Probab.* 48(1):264 – 295,2020.

[40] F. Flandoli and B. Maslowski, Ergodicity of the 2D Navier–Stokes equation under random perturbations. *Communications in Mathematical Physics*, 172(1):119–141, Aug 1995.

[41] F. Flandoli, M. Saal, mSQG equations in distributional spaces and point vortex approximation. *J. Evol. Equ.* **19** no. 4, 1071–1090,2019.

[42] H. Fujita and T. Kato, On the Navier–Stokes initial value problem. i. *Archive for Rational Mechanics and Analysis*, 16:269–315, 1964.

[43] C. Garcia. Vortex patches choreography for active scalar equations. *Journal of Nonlinear Science*, **31:75** 2021.

[44] C. Geldhauser and M. Romito. Point vortices for inviscid generalized surface quasi-geostrophic models. *Am. Ins. Math. Sci.*, **25(7)** 2583–2606.

[45] N. Glatt-Holtz, V. Šverák, and V.Vicol, On inviscid limits for the stochastic Navier–Stokes equations and related models. *Archive for Rational Mechanics and Analysis*, **217(2)**619–649, Aug 2015.

[46] N. Glatt-Holtz and V. Vicol, Local and global existence of smooth solutions for the stochastic Euler equations with multiplicative noise, *The Annals of Probability* , **42** 80-145, 2014.

[47] B. Goldys, M. Röckner and X. Zhang, Martingale solutions and Markov selections for stochastic partial differential equations, *Stochastic Processes and their Applications*, **119** 1725–1764, 2009.

[48] E. Grenier and N. Masmoudi, Ekman layers of rotating fluids, the case of well prepared initial data, *Communications in Partial Differential Equations*, **22** 213–218, 1997.

[49] M. Gubinelli, M. Turra, Hyperviscous stochastic Navier–Stokes equations with white noise invariant measure in two dimensions, arXiv:1912.06881v1.

[50] M. Hairer and J. C. Mattingly, Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing. *Annals of Mathematics*, 993–1032, 2006.

[51] I.M. Held, R.T. Pierrehumbert, S.T. Garner and K.L. Swanson, Surface quasi-geostrophic dynamics. *J. Fluid Mech.* **282** 1-20,1995.

[52] T. Hmidi and J. Mateu. Existence of corotating and counter-rotating vortex pairs for active scalar equations. *Comm. Math. Phys.*, 350(2):699–747, 2017.

[53] J. Hunter, J. Shu and Q. Zhang. Two-Front Solutions of the SQG Equation and its Generalizations *Communications in Mathematical Sciences* , 18(6), 1685-1741, 2020.

[54] J. Hunter, J. Shu and Q. Zhang. Global solutions for a family of GSQG front equations. *arXiv:2005.09154*, 2020.

[55] A. Jakubowski, Short Communication:The Almost Sure Skorokhod Representation for Subsequences in Nonmetric Spaces, *Theory of Probability & Its Applications*, **42** 209-216, 1998.

[56] G. Kallianpur and J. Xiong, Stochastic differential equations in infinite-dimensional spaces: An extended version of the lectures delivered by Gopinath Kallianpur as part of the 1993 Barrett Lectures at the University of Tennessee, Knoxville, TN. *Institute of Mathematical Statistics*, 1995.

[57] T. Kato, Strong L^p-solutions of the Navier–Stokes equation in $\mathbb{R}^m$, with applications to weak solutions. *Mathematische Zeitschrift*, 187:471-480, 1984.

[58] A. Kechris, Classical Descriptive Set Theory. *Springer-Verlag New York* , 1995.

[59] A. Kiselev, F. Nazarov, and A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, *Inventiones mathematicae*, **167** 445-453,2007.

[60] H. Koch and D.Tataru, Well-posedness for the Navier–Stokes equations. *Advances in Mathematics*, **157** 22-35, 2001.

[61] S B. Kuksin, The Eulerian limit for 2D statistical hydrodynamics. *Journal of Statistical Physics*, **115(1)** 469-492, Apr 2004.

[62] O.A. Ladyzhenskaya, Solution“in the large” of the nonstationary boundary value problem for the Navier–Stokes system in two space variables *Comm. Pure Appl. Math.* **12** 427-433,1959.

[63] L.D. Landau and E.M. Lifshitz. *Fluid Mechanics (Second Edition)*. Pergamon, Second edition, 1987.

[64] G. Lapeyre. Surface quasi-geostrophy. *Fluids*, 2(1):7, 2017.

[65] P. G. Lemarié-Rieusset, Recent developments in the Navier–Stokes problem, *CRC Press*, 2002.

[66] J. Leray, Essai sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Matematica*, **63** 193-248, 1933.

[67] S. Liang, P. Zhang, and R. Zhu. Remark on the Lifespan of Solutions to 3-D Anisotropic Navier Stokes Equations. *Communications in Mathematical Research*, **36** 31-50, 2020.

[68] S. Liang, P. Zhang, and R. Zhu. Deterministic and Stochastic 2D Navier–Stokes equations with anisotropic viscosity. *Journal of Differential Equations*, **275**:473–508, 2021.

[69] W. Liu and M. Röckner, *Stochastic partial differential equations: an introduction*, Springer ,2015.

[70] D. Luo and M. Saal. Regularization by noise for the point vortex model of mSQG equations. *Acta Mathematica Sinica, English Series*, **37**: 408-422, 2021.

[71] D. Luo and R. Zhu, Stochastic mSQG equations with multiplicative transport noises: white noise solutions and scaling limit, *Stochastic Processes and their Applications*, **140**, 236-286, 2021.

[72] F. Marchand, *Existence and regularity of weak solutions to the quasi-geostrophic equations in the spaces L^p or $\dot{H}^{1/2}$*, Communications in Mathematical Physics, **277** , 45-67,2008.

[73] J. Mourrat and H. Weber, Global well-posedness of the dynamic Φ^4 model in the plane. *The Annals of Probability*, **45(4)** 2398-2476, 2017.

[74] I. Munteanu and M. Röckner, Global solutions for random vorticity equations perturbed by gradient dependent noise, in two and three dimensions . *Journal of Evolution Equations*, **20** 1173-1194,2020.

[75] C. Odasso, Exponential mixing for stochastic pdes: The non-additive case. *Probability Theory and Related Fields*, **140** 06 2005.

[76] M. Paicu, Équation anisotrope de Navier–Stokes dans des espaces critiques, *Revista Matematica Iberoamericana*, **21** 179–235,2005.

[77] K.R. Parthasarathy, *Probability Measures on Metric Spaces.* Academic Press, 1967.

[78] J. Pedlovsky, *Geophysical Fluid Dynamics,* Springer 1979.

[79] F. Planchon. Asymptotic behavior of global solutions to the Navier–Stokes equations in $\mathbb{R}^3$. *Revista Matematica Iberoamericana*, 14:71–93, 1998.

[80] E. Poulon About the behaviour of regular Navier–Stokes solutions near the blow up, *Bull. Soc. Math. France,* **146**(2) 355-390, 2018.

[81] S. Resnick, Dyanmical Problems in Non-Linear Advective Partial Differential Equations, PhD thesis, University of Chicago, 1995.

[82] J. C. Robinson J. L. Rodrigo and W. Sadowski, The three-dimensional Navier–Stokes equations: classical theory, *Cambridge University Press*,2016.

[83] M. Röckner and X. Zhang, Stochastic tamed 3D Navier–Stokes equations: existence, uniqueness and ergodicity *Probability Theory and Related Fields* , **145** 211-267,2009.

[84] M. Röckner, R. C. Zhu and X. C. Zhu, Sub- and supercritical stochastic quasi-geostrophic equation, *Annals of Probability*, **43** 1202-1273, 2015.

[85] M. Rosenzweig, Justification of the point vortex approximation for modified surface quasigeostrophic equations, *SIAM J. Math. Anal.* , **52** no. 2, 1690-1728,2020.

[86] M. Röckner, R. C. Zhu and X. C. Zhu, A remark on global solutions to random 3D vorticity equations for small initial data, *Discrete & Continuous Dynamical Systems - B*, **24(8)**, 4021-4030,2019.

[87] R.Temam, *Navier–Stokes Equations and Nonlinear Functional Analysis,* Society for Industrial and Applied Mathematics,1995.

[88] H. Triebel,*Theory of Function Spaces,* Birkhäuser, Basel, 1983.

[89] H. Triebel. *Theory of Function Spaces III.* 01 2006.

[90] G K Vallis. *Atmospheric and oceanic fluid dynamics.* Cambridge University Press, 2017.