

# Contents

Foreword — VII

Foreword to the second edition — IX

Introduction — XI

## Part I: Basics

- 1 Model examples — 3**
  - 1.1 Noncommutative tori — 3
    - 1.1.1 Geometric definition — 3
    - 1.1.2 Analytic definition — 4
    - 1.1.3 Algebraic definition — 5
    - 1.1.4 Abstract noncommutative torus — 5
    - 1.1.5 First properties of the  $\mathcal{A}_\theta$  — 6
  - 1.2 Elliptic curves — 7
    - 1.2.1 Weierstrass and Jacobi normal forms — 7
    - 1.2.2 Complex tori — 8
    - 1.2.3 Weierstrass uniformization — 9
    - 1.2.4 Complex multiplication — 10
  - 1.3 Functor  $\mathcal{E}_\tau \rightarrow \mathcal{A}_\theta$  — 10
  - 1.4 Ranks of elliptic curves — 13
    - 1.4.1 Symmetry of complex and real multiplication — 14
    - 1.4.2 Arithmetic complexity of the  $\mathcal{A}_{\text{RM}}$  — 14
  - 1.5 Classification of surface automorphisms — 15
    - 1.5.1 Anosov automorphisms — 16
    - 1.5.2 Functor  $F$  — 16
    - 1.5.3 Handelman's invariant — 17
    - 1.5.4 Module determinant — 18
    - 1.5.5 Numerical examples — 18
- 2 Categories and functors — 23**
  - 2.1 Categories — 23
  - 2.2 Functors — 26
  - 2.3 Natural transformations — 28

<b>3</b>	<b><math>C^*</math>-algebras — 31</b>
3.1	Basic definitions — 31
3.2	Crossed products — 33
3.3	K-theory of the $C^*$ -algebras — 35
3.4	Noncommutative tori — 39
3.4.1	$n$ -dimensional noncommutative tori — 39
3.4.2	2-dimensional noncommutative tori — 41
3.5	AF-algebras — 43
3.5.1	Generic AF-algebras — 43
3.5.2	Stationary AF-algebras — 46
3.6	UHF-algebras — 47
3.7	Cuntz–Krieger algebras — 49

## **Part II: Noncommutative invariants**

<b>4</b>	<b>Topology — 55</b>
4.1	Classification of the surface automorphisms — 55
4.1.1	Pseudo-Anosov automorphisms of a surface — 55
4.1.2	Functors and invariants — 57
4.1.3	Jacobian of measured foliations — 59
4.1.4	Equivalent foliations — 61
4.1.5	Proofs — 62
4.1.6	Anosov maps of the torus — 67
4.2	Torsion in the torus bundles — 70
4.2.1	Cuntz–Krieger functor — 70
4.2.2	Proof of Theorem 4.2.1 — 71
4.2.3	Noncommutative invariants of torus bundles — 73
4.3	Obstruction theory for Anosov’s bundles — 75
4.3.1	Fundamental AF-algebra — 75
4.3.2	Proofs — 78
4.3.3	Obstruction theory — 83
4.4	Cluster $C^*$ -algebras and knot polynomials — 86
4.4.1	Invariant Laurent polynomials — 86
4.4.2	Birman–Hilden Theorem — 88
4.4.3	Cluster $C^*$ -algebras — 89
4.4.4	Jones and HOMFLY polynomials — 91
4.4.5	Proof of Theorem 4.4.1 — 92
4.4.6	Examples — 97
<b>5</b>	<b>Algebraic geometry — 103</b>
5.1	Elliptic curves — 103

5.1.1	Noncommutative tori via Sklyanin algebras —	104
5.1.2	Noncommutative tori via measured foliations —	109
5.2	Algebraic curves of genus $g \geq 1$ —	114
5.2.1	Toric AF-algebras —	115
5.2.2	Proof of Theorem 5.2.1 —	116
5.3	Projective varieties of dimension $n \geq 1$ —	120
5.3.1	Serre $C^*$ -algebras —	121
5.3.2	Proof of Theorem 5.3.4 —	124
5.3.3	Example —	127
5.4	Tate curves and UHF-algebras —	129
5.4.1	Elliptic curve over $p$ -adic numbers —	129
5.4.2	Proof of Theorem 5.4.1 —	130
5.4.3	Example —	134
5.5	Mapping class group —	135
5.5.1	Harvey's conjecture —	135
5.5.2	Proof of Theorem 5.5.1 —	136
<b>6</b>	<b>Number theory —</b>	<b>143</b>
6.1	Isogenies of elliptic curves —	143
6.1.1	Symmetry of complex and real multiplication —	144
6.1.2	Proof of Theorem 6.1.1 —	145
6.1.3	Proof of Theorem 6.1.2 —	148
6.1.4	Proof of Theorem 6.1.3 —	150
6.2	Ranks of elliptic curves —	155
6.2.1	Arithmetic complexity of $\mathcal{A}_{\text{RM}}$ —	155
6.2.2	Mordell AF-algebra —	156
6.2.3	Proof of Theorem 6.2.1 —	157
6.2.4	Numerical examples —	161
6.3	Transcendental number theory —	162
6.3.1	Algebraic values of $\mathcal{J}(\theta, \varepsilon) = e^{2\pi i\theta + \log \log \varepsilon}$ —	162
6.3.2	Proof of Theorem 6.3.1 —	163
6.3.3	Comments on a note by M. Waldschmidt —	165
6.4	Class field theory —	167
6.4.1	Hilbert class field of a real quadratic field —	167
6.4.2	AF-algebra of the Hecke eigenform —	169
6.4.3	Proof of Theorem 6.4.1 —	170
6.4.4	Examples —	173
6.5	Noncommutative reciprocity —	175
6.5.1	$L$ -function of noncommutative tori —	175
6.5.2	Proof of Theorem 6.5.1 —	176
6.5.3	Supplement: Größencharacters, units, and $\pi(n)$ —	182
6.6	Langlands Conjecture for $\mathcal{A}_{\text{RM}}^{2n}$ —	185

6.6.1	$L(\mathcal{A}_{RM}^{2n}, s)$ —	<b>185</b>
6.6.2	Proof of Theorem 6.6.1 —	<b>189</b>
6.6.3	Supplement: Artin $L$ -function —	<b>191</b>
6.7	Projective varieties over finite fields —	<b>193</b>
6.7.1	Traces of Frobenius endomorphisms —	<b>193</b>
6.7.2	Proof of Theorem 6.7.1 —	<b>195</b>
6.7.3	Examples —	<b>200</b>
<b>7</b>	<b>Arithmetic topology —</b>	<b>207</b>
7.1	Arithmetic topology of 3-manifolds —	<b>207</b>
7.1.1	Braids, links, and Galois covering —	<b>209</b>
7.1.2	Proof of Theorem 7.1.1 —	<b>210</b>
7.1.3	Punctured torus —	<b>213</b>
7.2	Arithmetic topology of 4-manifolds —	<b>215</b>
7.2.1	4-dimensional manifolds —	<b>216</b>
7.2.2	Galois theory for noncommutative fields —	<b>217</b>
7.2.3	Uchida map —	<b>218</b>
7.2.4	Proofs —	<b>219</b>
7.2.5	Rokhlin and Donaldson's Theorems revisited —	<b>222</b>
7.3	Untying knots in 4D and Wedderburn's Theorem —	<b>225</b>
7.3.1	Wedderburn's Theorem —	<b>225</b>
7.3.2	Proof of Theorem 7.3.1 —	<b>226</b>
7.4	Dynamical ideals of noncommutative rings —	<b>227</b>
7.4.1	Topological dynamics —	<b>230</b>
7.4.2	Cyclic division algebras —	<b>230</b>
7.4.3	Piergallini covering —	<b>231</b>
7.4.4	Proof of Theorem 7.4.1 —	<b>232</b>
7.4.5	Knotted surfaces in 4-manifolds —	<b>234</b>
7.5	Etesi $C^*$ -algebras —	<b>238</b>
7.5.1	Minkowski group —	<b>239</b>
7.5.2	Gompf's Theorem —	<b>240</b>
7.5.3	Proofs —	<b>241</b>
<b>8</b>	<b>Quantum arithmetic —</b>	<b>249</b>
8.1	Langlands reciprocity for $C^*$ -algebras —	<b>249</b>
8.1.1	Trace cohomology —	<b>250</b>
8.1.2	Langlands reciprocity —	<b>251</b>
8.1.3	Proofs —	<b>252</b>
8.1.4	Pimsner–Voiculescu embedding —	<b>257</b>
8.2	$K$ -theory of rational quadratic forms —	<b>259</b>
8.2.1	Algebraic groups over adèles —	<b>260</b>
8.2.2	Proofs —	<b>261</b>

- 8.2.3 Binary quadratic forms — 264
- 8.3 Quantum dynamics of elliptic curves — 265
- 8.3.1  $C^*$ -dynamical systems — 267
- 8.3.2 Abelian extensions of quadratic fields — 267
- 8.3.3 Shafarevich–Tate group of elliptic curves — 268
- 8.3.4 Proofs — 268
- 8.4 Shafarevich–Tate groups of abelian varieties — 275
- 8.4.1 Abelian varieties — 276
- 8.4.2 Weil–Châtelet group — 277
- 8.4.3 Localization formulas — 278
- 8.4.4 Proof of Theorem 8.4.1 — 279
- 8.4.5 Abelian varieties with complex multiplication — 283
- 8.5 Noncommutative geometry of elliptic surfaces — 285
- 8.5.1 Brock–Elkies–Jordan variety — 286
- 8.5.2 Elliptic surfaces — 288
- 8.5.3 Proofs — 288
- 8.5.4 Picard numbers — 294
- 8.6 Class field towers and minimal models — 295
- 8.6.1 Algebraic surfaces — 297
- 8.6.2 Proofs — 297

### Part III: Brief survey of NCG

- 9 Finite geometries — 305
- 9.1 Axioms of projective geometry — 305
- 9.2 Projective spaces over skew fields — 306
- 9.3 Desargues and Pappus axioms — 307
- 10 Continuous geometries — 309
- 10.1  $W^*$ -algebras — 309
- 10.2 Von Neumann geometry — 311
- 11 Connes geometries — 313
- 11.1 Classification of type III factors — 313
- 11.1.1 Tomita–Takesaki theory — 313
- 11.1.2 Connes invariants — 314
- 11.2 Noncommutative differential geometry — 315
- 11.2.1 Hochschild homology — 315
- 11.2.2 Cyclic homology — 316
- 11.2.3 Novikov Conjecture for hyperbolic groups — 317
- 11.3 Connes’ Index Theorem — 318

- 11.3.1 Atiyah–Singer Theorem for families of elliptic operators — **318**
- 11.3.2 Foliated spaces — **319**
- 11.3.3 Index Theorem for foliated spaces — **320**
- 11.4 Bost–Connes dynamical system — **321**
- 11.4.1 Hecke  $C^*$ -algebra — **321**
- 11.4.2 Bost–Connes Theorem — **322**
  
- 12 Index Theory — 325**
- 12.1 Atiyah–Singer Theorem — **325**
- 12.1.1 Fredholm operators — **325**
- 12.1.2 Elliptic operators on manifolds — **326**
- 12.1.3 Index Theorem — **327**
- 12.2 K-homology — **328**
- 12.2.1 Topological K-theory — **328**
- 12.2.2 Atiyah’s realization of K-homology — **330**
- 12.2.3 Brown–Douglas–Fillmore Theory — **331**
- 12.3 Kasparov’s KK-theory — **332**
- 12.3.1 Hilbert modules — **333**
- 12.3.2 KK-groups — **334**
- 12.4 Applications of Index Theory — **335**
- 12.4.1 Novikov Conjecture — **335**
- 12.4.2 Baum–Connes Conjecture — **337**
- 12.4.3 Positive scalar curvature — **337**
- 12.5 Coarse geometry — **338**
  
- 13 Jones polynomials — 341**
- 13.1 Subfactors — **341**
- 13.2 Braids — **342**
- 13.3 Trace invariant — **344**
  
- 14 Quantum groups — 347**
- 14.1 Manin’s quantum plane — **348**
- 14.2 Hopf algebras — **349**
- 14.3 Operator algebras and quantum groups — **350**
  
- 15 Noncommutative algebraic geometry — 353**
- 15.1 Serre isomorphism — **353**
- 15.2 Twisted homogeneous coordinate rings — **355**
- 15.3 Sklyanin algebras — **356**
  
- 16 Trends in noncommutative geometry — 359**
- 16.1 Derived categories — **359**

- 16.2 Noncommutative thickening — 360
- 16.3 Deformation quantization of Poisson manifolds — 361
- 16.4 Algebraic geometry of noncommutative rings — 362

**Bibliography — 365**

**Index — 375**