Low regularity local well-posedness for the Zakharov system and KP-I type equations

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Chapter 1

Introduction

"A journey of a thousand miles begins with a single step."
-Lao Tzu

The subject matter of this thesis is *low regularity local well-posedness* for *dispersive* partial differential equations. The notion of well-posedness devised by Hadamard includes

(i) existence of solutions,

(ii) uniqueness of the solution,

(iii) continuous dependence of the solution on the initial data.

By local well-posedness, we mean that the time $T > 0$ of existence of the solution depends on the initial data. The equations dealt with in this thesis are the *Zakharov system* and the *dispersion generalised KP-I equation* – two models occurring in plasma physics and fluid mechanics, respectively. We postpone a brief physical motivation for these equations to later chapters and focus on the mathematical aspects of these. The solutions to the linear counterparts of these equations disperse over space as time progresses. We consider Cauchy problems with initial data in L^2 based[1] Sobolev spaces $H^s(\mathbb{R}^d)$. Without taking into account the dispersive character of these equations[2], well-posedness results can be obtained for smoother classes of initial data, see, for instance, Section 2.3. However, it is not straight-forward to obtain well-posedness results for initial data lying in low regularity spaces (larger class of initial data). The low regularity spaces are of physical interest since various conserved quantities like mass and energy for a system are defined at these regularities. Consequently, (if proved) local well-posedness results translate to global results in many cases. Our aim is to lower the regularity threshold as much as possible to include broader class of initial data. To achieve this, we need to exploit the particular structure of the equations.

Both the problems considered in the thesis exhibit similar properties in the sense that they have non-trivial *resonant* sets. This requires a careful analysis of the equation to obtain multilinear estimates. Moreover, the *transversality* of the waves in the resonant interaction plays an important role in both the equations. However, we shall also see that the function spaces required to handle the equations are different – each adapted to handle the particular form of the equation. We shall refer to an equation as *semilinear* if it can be handled by a fixed point argument, and *quasilinear*, otherwise. In the case of a

[1] isotropic or anisotropic

[2] e.g. using the Banach algebra property of Sobolev spaces and energy methods

semilinear problem, the nonlinearity is considered as a perturbation of the solution to the linear problem in a suitably defined norm. After writing the problem in an equivalent integral form, the problem reduces to finding a function space, say $X_T \hookrightarrow C([0,T];H^s(\mathbb{R}^d))$, so that the integral is a contraction in a suitable ball in X_T. As a by-product of this approach, we obtain that the data-to-solution map is as smooth as the nonlinearity. For the Zakharov system, we prove new local well-posedness results by defining new function spaces to implement the aforementioned approach. For the dispersion generalised KP-I equation, in the low dispersion regime, we prove a priori estimates for the solutions by localising the solution to smaller time intervals. The size of these intervals depends on the size of the spatial frequency of the solution. Function spaces based on this localisation enable to prove the estimates required to conclude an a priori estimate on the solution. To prove the continuity of the data-to-solution map, we use the equation satisfied by the difference of two solutions. In the high dispersion case, we invoke the fixed point argument. We give the details of the techniques in the following.

The Zakharov system

Consider the Cauchy problem for the Zakharov system in spatial dimension $d \leqslant 3$:

$$\begin{cases} i\partial_t u + \Delta u &= nu, \\ \partial_t^2 n - \Delta n &= \Delta|u|^2, \\ (u(0,x), n(0,x), \partial_t n(0,x)) &= (u_0(x), n_0(x), n_1(x)). \end{cases} \tag{1.0.1}$$

Here $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ and $n : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$. The initial data $(u_0(x), n_0(x), n_1(x)) \in H^s(\mathbb{R}^d) \times H^l(\mathbb{R}^d) \times H^{l-1}(\mathbb{R}^d)$. Counterexamples arising from the failure of continuity (or C^2-differentiability) of the data-to-solution map [4, 33, 23] suggest the problem to be locally well-posed for s and l satisfying the following (see Figure 1.1):

$$l \geqslant -\frac{1}{2}, \quad \max\left(l-1, \frac{l}{2}+\frac{1}{4}\right) \leqslant s \leqslant l+2. \tag{1.0.2}$$

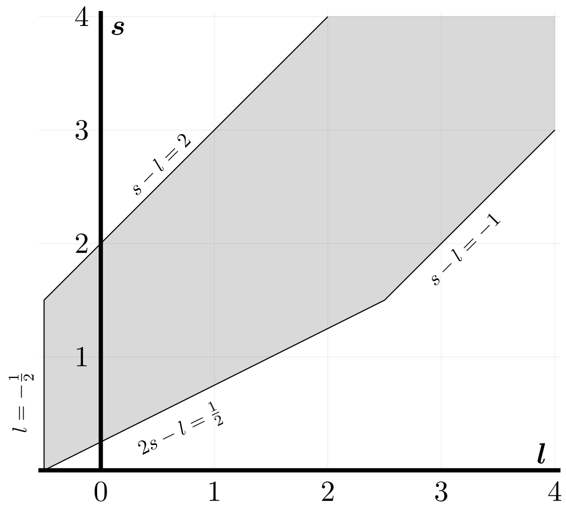

Figure 1.1: Conjectured region of local well-posedness for $d \leqslant 3$

With the goal of proving a local well-posedness result in the above region using Banach's fixed point theorem, we construct new function spaces that are motivated by [14]. In [14], the authors consider the Cauchy problem for the Zakharov system in dimension $d \geqslant 4$ and prove an optimal local well-posedness result. Using the function spaces constructed in [14], one cannot prove a well-posedness result for dimensions $d \leqslant 3$. This inability can be attributed to the fact that waves have lesser directions to disperse in lower dimensions. If $e^{it\Delta}u_0$ denotes the solution to the linear Schrödinger equation $i\partial_t u + \Delta u = 0$ with initial data u_0, then

$$\|e^{it\Delta}u_0\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}}\|u_0\|_{L^1(\mathbb{R}^d)},$$

which quantifies the aforementioned statement on the dispersion of the solution to the linear equation.

To validate the choice of our function spaces, we consider two types of interactions:

(i) Resonant interactions,

(ii) Non-resonant interactions.

Roughly speaking, resonant interactions are the ones in which one cannot gain any derivatives[3] through integration by parts, while for the non-resonant interactions, a gain of derivatives is possible. For instance, if we substitute in the nonlinearity the solutions to the linear Schrödinger and (half) wave equations, the spatial Fourier transform of the integral formulation of the first equation in the system (1.0.1) reads:

$$\hat{u}(t,\xi) = e^{-it|\xi|^2}\hat{u}_0(\xi) - ie^{-it|\xi|^2}\int_0^t \int_{\mathbb{R}^d} e^{is\Omega_{ZS}}\hat{u}_0(\xi-\eta)\hat{v}_0(\eta)d\eta ds,$$

where $\Omega_{ZS}(\xi,\eta) = |\xi|^2 + |\eta| - |\xi-\eta|^2$. If $|\eta| \lesssim |\xi| \sim |\xi-\eta|$, the size of the function Ω_{ZS} can become very small. In this case, there is little oscillation in time, hence less cancellation. The resonant interactions dictate the lowest possible regularity at which a local-well posedness result can be proved, while the non-resonant interactions decide the other constraints on the regularities s and l of the initial data. Specifically, we expect that values s and l cannot be very far apart for the system (1.0.1) to be well-posed.

For $d = 2, 3$, the solution to treat the resonant interactions has been provided in [4, 3] by observing that the waves leading to the resonant interaction propagate in directions that are *transverse*. Informally speaking, these waves do not interact much. This allows to achieve multilinear estimates with smoothing. In this context, *bilinear Strichartz* estimates play an important role. Consider the interaction of a high frequency (N_1) Schrödinger solution with that of a low frequency ($N_2 \ll N_1$) Schrödinger solution, we find that the difference in the group velocities:

$$|\nabla\omega(\xi_1) - \nabla\omega(\xi_2)| \sim \max(|\xi_1|, |\xi_2|) = N_1, \quad \omega(\xi) = -|\xi|^2 \text{ for Schrödinger equation}$$

provides good bilinear estimates, see Section 3.4. The same holds for a Schrödinger-wave interaction. The latter is sufficient to handle the resonant case for $d = 1$. A more refined estimate than bilinear Strichartz estimates can be proved when we consider the interaction of three waves. Ignoring other technical assumptions, this can be understood as follows: let f_{iN_i} denote the Fourier transform of projection onto frequency around N_i of the function

[3] a gain of derivatives is required to control the nonlinearity

f_i, $i = 1, 2, 3$ having Fourier support 'close'[4] to the characteristic hypersurface. Then, for $N_1 \sim N_2 \gg N_3$, we have

$$\Big| \int (f_{1N_1} * f_{2N_2}) \cdot f_{3N_3} \Big| \lesssim \frac{1}{N_1^{\frac{1}{2}}} \prod_{i=1}^{3} \|f_{iN_i}\|_{L^2}. \tag{1.0.3}$$

The characteristic hypersurface[5] is the distributional support of the solution to a linear equation. Although sharp, the above observations are not sufficient to prove an optimal result for the system in $d = 1, 2, 3$. We need to construct new function spaces that accommodate the resonant and non-resonant interactions simultaneously. We achieve this by defining different norms for the low and high modulation parts of the solutions, i.e. parts concentrated near to or far from the characteristic hypersurfaces. We use Fourier restriction ($X^{s,\theta}$) norms which capture the variation of a solution from the linear solution via the weight $\langle \tau - \omega(\xi) \rangle$:

$$\|u\|_{X^{s,\theta}(\mathbb{R}\times\mathbb{R}^d)} = \|\langle \xi \rangle^s \langle \tau - \omega(\xi) \rangle^\theta \hat{u}(\tau, \xi)\|_{L^2_{\tau,\xi}(\mathbb{R}\times\mathbb{R}^d)}. \tag{1.0.4}$$

For the resonant interactions, we still want to use estimates from [4, 3]. Thus, for the low modulation part of the solution, we stick to the $X^{s,\theta}$ norm defined above with the parameter s being the regularity of the initial data and θ – a number slightly greater than $\frac{1}{2}$. For the high modulation norms we have flexibility in choosing the exponents s and θ, and we deviate from the usual choice. We vary these exponents by introducing additional parameters that come into action in different parts of the region in Figure 1.1. Moreover, there is a weight in the temporal frequency added to the norms. The particular choice of the parameters is a result of handling each possible interaction depending on the size of the modulation and the convolution constraints, see Section 3.6. As stated, we expect well-posedness for values s, l which are not too far. The additional parameters precisely accomplish this task of not allowing the Schrödinger and wave solutions to be too far in their spatial regularities.

Using the newly defined function spaces, we prove that for $d \leqslant 3$, (1.0.1) is locally well-posed in $H^s(\mathbb{R}^d) \times H^l(\mathbb{R}^d) \times H^{l-1}(\mathbb{R}^d)$ provided

$$l > -\frac{1}{2}, \quad \max\Big(l - 1, \frac{l}{2} + \frac{1}{4}\Big) < s < l + 2. \tag{1.0.5}$$

As substantiated by the counterexamples, the result is optimal up to the boundaries. The boundaries cannot be covered owing to the choice of the parameters and the restrictions imposed on them to be able to prove the required estimates.

The dispersion generalised KP-I equation

We consider the Cauchy problem for the dispersion generalised KP-I equation (fKP-I) in the high dispersion regime[6]:

$$\begin{cases} \partial_t u - D_x^\alpha \partial_x u - \partial_x^{-1} \partial_y^2 u &= u \partial_x u, \quad (t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \\ u(0, x, y) &= u_0(x, y) \in H^{s,0}(\mathbb{R}^2), \end{cases} \tag{1.0.6}$$

[4] this is referred to as low modulation

[5] e.g. the characteristic hypersurface for the Schrödinger equation is the paraboloid $\{(\tau, \xi) : \tau = -|\xi|^2\}$

[6] compared to the KP-I equation which corresponds to $\alpha = 2$

where $2 < \alpha < 4$, and D_x^α is defined as a Fourier multiplier by

$$(D_x^\alpha f)^\wedge(\xi,\eta) = |\xi|^\alpha \hat{f}(\xi,\eta).$$

For $2 < \alpha \leqslant \frac{5}{2}$ we only consider real-valued solutions; for $\alpha > \frac{5}{2}$ we also treat complex-valued solutions. The space for initial data are the anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$ defined by[7]

$$H^{s_1,s_2}(\mathbb{R}^2) := \{\phi \in L^2(\mathbb{R}^2) : \|\phi\|_{H^{s_1,s_2}(\mathbb{R}^2)} = \|\hat{\phi}(\xi,\eta)(1+|\xi|^2)^{\frac{s_1}{2}}(1+|\eta|^2)^{\frac{s_2}{2}}\|_{L^2_{\xi,\eta}} < \infty\}.$$

To begin, we observe that for $\alpha < \frac{7}{3}$, the data-to-solution map for (1.0.6) fails to be C^2-differentiable at zero from $H^{s_1,s_2}(\mathbb{R}^2)$ to $H^{s_1,s_2}(\mathbb{R}^2)$. This failure is attributed to the resonant interaction – when a high (x) frequency interacts with a low (x) frequency, the size of the resonance function

$$\Omega_{fKP}(\xi_1,\eta_1,\xi_2,\eta_2) = |\xi_1+\xi_2|^\alpha(\xi_1+\xi_2) - |\xi_1|^\alpha\xi_1 - |\xi_2|^\alpha\xi_2 - \frac{(\eta_1\xi_2-\eta_2\xi_1)^2}{\xi_1\xi_2(\xi_1+\xi_2)}$$

becomes small. However, as in the case of the Zakharov system, this interaction is transverse. Via the nonlinear Loomis-Whitney inequality from [39], we find that the following analogue of (1.0.3) holds:

$$\Big|\int (f_{1N_1} * f_{2N_2}) \cdot f_{3N_3}\Big| \lesssim N_1^{-\frac{3\alpha}{4}+\frac{1}{2}} N_2^{-\frac{1}{2}} \prod_{i=1}^{3} \|f_{iN_i}\|_{L^2}, \tag{1.0.7}$$

where f_{iN_i} is the Fourier transform of f_i having x frequency support around N_i. In case $N_2 \gtrsim N_1^{-\kappa}$, for some $\kappa > 0$, the above estimate allows us to remedy the derivative loss occurring in the nonlinearity $u\partial_x u$. For $N_2 \lesssim N_1^{-\kappa}$, we invoke another form of transversality via the bilinear Strichartz estimate:

$$\|P_{N_1}uP_{N_2}v\|_{L^2_{t,x,y}} \lesssim N_1^{-\frac{\alpha}{4}} N_2^{\frac{1}{2}} \|P_{N_1}u\|_{L^2}\|P_{N_2}v\|_{L^2}, \tag{1.0.8}$$

for functions localised in x frequency and 'close' to the characteristic surface. Note that both the estimates above improve as α increases. We observe that for $\alpha > \frac{5}{2}$, using a combination of the above estimates, we can remedy the derivative loss in the nonlinearity completely. For the non-resonant case, the size of the modulation and the linear Strichartz estimates suffice. Employing $X^{s,\theta}$ spaces given by (1.0.4) $\big($with $\omega(\xi,\eta) = |\xi|^\alpha\xi - \frac{\eta^2}{\xi}\big)$ to run a contraction mapping argument, we prove that for $\frac{5}{2} < \alpha < 4$, (1.0.6) is locally well-posed in $H^{s,0}(\mathbb{R}^2)$ for $s > \frac{5}{4} - \frac{\alpha}{2}$. As a consequence of the conservation of mass

$$M(u)(t) = \int_{\mathbb{R}^2} (u(x,y))^2 dxdy = M(u)(0),$$

we also obtain the global well-posedness of the solutions in $L^2(\mathbb{R}^2)$ for real-valued initial data in this case.

To handle the low dispersion case, namely $2 < \alpha \leqslant \frac{5}{2}$, we follow the strategy of [34] – we use function spaces which are based on *frequency dependent time localisation*. For this, let us review the case of the KP-I equation. For $N_2 \lesssim N_1$, and frequency localised

[7] (ξ,η) denote the Fourier variables corresponding to (x,y)

free solutions $P_{N_1}S(t)u_0$ and $P_{N_2}S(t)v_0$ to the KP-I equation, we have, using Hölder's inequality and the bilinear Strichartz estimate (1.0.8)

$$\begin{aligned}&\|\partial_x(P_{N_1}S(t)u_0P_{N_2}S(t)v_0)\|_{L^1([0,T];L^2_{x,y}(\mathbb{R}^2))}\\ &\lesssim N_1T^{\frac{1}{2}}\|P_{N_1}S(t)u_0P_{N_2}S(t)v_0\|_{L^2([0,T];L^2_{x,y}(\mathbb{R}^2))}\\ &\lesssim N_1T^{\frac{1}{2}}\left(\frac{N_2}{N_1}\right)^{\frac{1}{2}}\|P_{N_1}u_0\|_{L^2}\|P_{N_2}v_0\|_{L^2},\end{aligned}$$

which suggests that the derivative loss can be remedied if the functions are localised to time intervals of size $T(N_1) = N_1^{-1}$. This is the rough idea behind the functions spaces introduced in [34]. For the dispersion generalised KP-I equation, we choose the time scale by interpolating between the cases $\alpha = 2$ and $\alpha = \frac{5}{2}+$, i.e. $T(N) = N^{(2\alpha-5)-}$. Then we work in two steps:

1. Prove an a priori estimate for the solution,
2. Prove the continuity of the data-to-solution map.

We prove the following set of estimates for function spaces $F^{s,0}(T)$, $\mathcal{N}^{s,0}(T)$ and the energy space $E^{s,0}(T)$:

$$\begin{cases}\|u\|_{F^{s,0}(T)} & \lesssim \|u\|_{E^{s,0}(T)} + \|\partial_x(u^2)\|_{\mathcal{N}^{s,0}(T)},\\ \|\partial_x(u^2)\|_{\mathcal{N}^{s,0}(T)} & \lesssim \|u\|_{F^{s,0}(T)}\|u\|_{F^{s,0}(T)},\\ \|u\|^2_{E^{s,0}(T)} & \lesssim \|u_0\|^2_{H^{s,0}} + \|u\|^3_{F^{s,0}(T)}.\end{cases}$$

The first estimate is analogous to the linear estimate that one requires in a semilinear problem, but with a different norm than the norm of the initial data on the right-hand side. The second estimate is the nonlinear estimate in short-time Fourier restriction spaces. This is again proved by employing (1.0.7) and (1.0.8). The (additional) last estimate known as the energy estimate controls the energy norm. Using a bootstrap argument, we conclude an a priori estimate for the solution provided that the initial data is small enough.

To prove the continuous dependence of the solution on the initial data, we consider the equation satisfied by the difference of two solutions (say u_1 and u_2) to (1.0.6) viz. $v = u_1 - u_2$:

$$\begin{cases}\partial_t v - D_x^\alpha\partial_x v - \partial_x^{-1}\partial_y v & = \partial_x(v(u_1+u_2)/2),\\ v(0) & = u_1(0) - u_2(0),\end{cases} \tag{1.0.9}$$

and prove a priori estimates for the same. We construct the data-to-solution map as an extension of the data-to-solution map for smooth initial data. This is accomplished by showing that the sequence of solutions corresponding to smooth initial data is a Cauchy sequence in the Banach space $C([0,T];H^{s,0}(\mathbb{R}^d))$. The continuity of the data-to-solution map for smooth initial data comes in handy. Then, we consider the difference of solutions corresponding to smooth initial data and low frequency smooth initial data to arrive at the conclusion that the solution depends only *continuously* on the initial data as it depends on the profile of the initial data.

Using the above strategy for $2 < \alpha \leqslant \frac{5}{2}$ and real-valued initial data, we prove that (1.0.6) is locally well-posed in $H^{s,0}(\mathbb{R}^2)$ for $s > 5 - 2\alpha$.

The above is based on a joint work with Robert Schippa.

Outline of the thesis

Chapter 2 is preliminary and introduces the reader to the basics. We collect known results and provide some tools from harmonic analysis. We familiarise the reader with the notion of well-posedness for smooth initial data. Fourier restriction spaces or $X^{s,\theta}$ spaces are introduced and their properties are listed.

In Chapter 3, we start with known well-posedness and ill-posedness results on the Zakharov system and then describe heuristic features of the Zakharov system. After defining the new function spaces, we state the known multilinear estimates from [4, 3]. The linear estimates are then proved by handling the norms in different regions depending on parameters defining the norms. The proofs of the multilinear estimates follow via a case-by-case analysis. Finally, we conclude the proof of local well-posedness.

Chapter 4 begins with the introduction to the KP-I family of equations. After describing known results on the same, we prove the semilinear ill-posedness result. Linear Strichartz estimates are proved, followed by the analysis of the resonance function to obtain mutilinear estimates. Next, the well-posedness proof in the low dispersion regime follows. To do so, the short-time function spaces are defined along with the required properties. The rest of the chapter deals with the proof of the well-posedness result in the high dispersion case.

We conclude the thesis by giving further directions which can be pursued based on the present work.

Chapter 2

Notation and preliminaries

"You can't go back and change the beginning, but you can start where you are and change the ending."
-CS Lewis

In this chapter, we collect basic tools from harmonic analysis and set the notation. We also introduce Fourier restriction ($X^{s,\theta}$) spaces that form the understructure for the forthcoming analysis. Standard references for the material presented in this chapter are [22, 21, 64].

2.1 Fourier transform and Sobolev spaces

Definition 2.1.1. A complex-valued smooth function f defined on $\mathbb{R}^d$ is called *Schwartz* if

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty,$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^d$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The space of all Schwartz functions on $\mathbb{R}^d$ will be denoted by $\mathcal{S}(\mathbb{R}^d)$.

Definition 2.1.2. The dual space of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is called the space of *tempered distributions*, denoted by $\mathcal{S}'(\mathbb{R}^d)$. We write

$$u(\phi) = \langle u, \phi \rangle$$

for $u \in \mathcal{S}'(\mathbb{R}^d)$, $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Definition 2.1.3. The *Fourier transform* of $f \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx,$$

where $x \cdot \xi = \sum_{i=1}^d x_i \xi_i$ for $x = (x_1, x_2, ..., x_d), \xi = (\xi_1, \xi_2, ..., \xi_d) \in \mathbb{R}^d$.

The Fourier transform is a homeomorphism on the Schwartz space with its inverse given by

$$\check{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\cdot\xi} d\xi.$$

We define the Fourier transform and the inverse Fourier transform of $u \in \mathcal{S}'(\mathbb{R}^d)$ by

$$\langle \hat{u}, \phi \rangle = \langle u, \hat{\phi} \rangle, \quad \langle \check{u}, \phi \rangle = \langle u, \check{\phi} \rangle,$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$.
We shall use $\mathcal{F}f$ and $\hat{f}$ to denote the Fourier transform of f. Occasionally, we shall append subscripts t and/or x to avoid confusion.

Definition 2.1.4. The convolution of $f, g \in L^1(\mathbb{R}^d)$ is defined as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y)g(x-y)dy.$$

Important properties of the Fourier transform are

(i) Parseval's relation

$$\int_{\mathbb{R}^d} f(x)\bar{h}(x)dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{h}(\xi)}d\xi$$

(ii) Plancherel's identity

$$\|f\|_{L^2} = \frac{1}{(2\pi)^{\frac{d}{2}}}\|\hat{f}\|_{L^2}$$

(iii) Convolution

$$(f * h)^\wedge(\xi) = \hat{f}(\xi)\hat{h}(\xi)$$

Definition 2.1.5. Let $s \in \mathbb{R}$ and $\langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}}$. The L^2 based *inhomogeneous Sobolev spaces* $H^s(\mathbb{R}^d)$ are defined as

$$H^s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \langle \xi \rangle^s \hat{f} \in L^2(\mathbb{R}^d)\},$$

and we have

$$\|f\|^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \langle \xi \rangle^{2s}|\hat{f}(\xi)|^2 d\xi.$$

The *homogeneous Sobolev spaces* $\dot{H}^s(\mathbb{R}^d)$ are defined via

$$\|f\|^2_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2s}|\hat{f}(\xi)|^2 d\xi.$$

Definition 2.1.6. Let $s \in \mathbb{R}$ be a real number and $1 < p < \infty$. The *inhomogeneous Sobolev spaces* $W^{s,p}(\mathbb{R}^d)$ are defined as

$$W^{s,p}(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : ((1+|\cdot|^2)^{s/2}\hat{f})^\vee \in L^p(\mathbb{R}^d)\}.$$

Theorem 2.1.7 ([22, Theorem 6.2.4], Sobolev embedding theorem)**.** *Let $1 < p < \infty$.*

(i) Let $0 < s < \frac{d}{p}$. Then the Sobolev space $W^{s,p}(\mathbb{R}^d)$ embeds continuously in $L^q(\mathbb{R}^d)$ when

$$\frac{1}{p} - \frac{1}{q} = \frac{s}{d}.$$

(ii) Let $0 < s < \frac{d}{p}$. Then the Sobolev space $W^{s,p}(\mathbb{R}^d)$ embeds continuously in $L^q(\mathbb{R}^d)$ for any $\frac{d}{s} < q < \infty$.

(iii) Let $\frac{d}{p} < s < \infty$. Then every element of $W^{s,p}(\mathbb{R}^d)$ can be modified on a set of measure zero so that the resulting function is bounded and uniformly continuous.

In a similar spirit, we can define anisotropic L^2 based Sobolev spaces:

Definition 2.1.8. Let $\bar{s} = (s_1, s_2, ..., s_d) \in \mathbb{R}^d$. The *anisotropic Sobolev spaces* $H^{\bar{s}}(\mathbb{R}^d)$ are defined as

$$H^{\bar{s}}(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d) : ((1+|\xi_1|^2)^{\frac{s_1}{2}}(1+|\xi_2|^2)^{\frac{s_2}{2}}...(1+|\xi_d|^2)^{\frac{s_d}{2}}\hat{f}) \in L^2(\mathbb{R}^d)\}, \quad (2.1.1)$$

where ξ_i denotes the ith component of the Fourier variable $\xi \in \mathbb{R}^d$. The homogeneous counterparts of anisotropic Sobolev spaces can be defined analogously. For brevity, whenever it is clear from the context, we shall drop the domain $\mathbb{R}^d$ from the notation.

Definition 2.1.9. For a measurable function $f : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ and $1 \leqslant q, r < \infty$, we define the mixed Lebesgue norms as follows:

$$\|f\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} = \Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^d}|f(t,x)|^r dx\Big)^{\frac{q}{r}}dt\Big)^{\frac{1}{q}},$$

with obvious modifications for $q = \infty$ and $r = \infty$. When restricting the time interval to $[0, T]$, for some $T > 0$, we write $\|f\|_{L_t^q([0,T];L_x^r(\mathbb{R}^d))}$ instead.

For $1 \leqslant r \leqslant q \leqslant \infty$, Minkowski's inequality reads

$$\|f\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \leqslant \|f\|_{L_x^r L_t^q(\mathbb{R}^d\times\mathbb{R})}.$$

We use $\mathbf{1}_A$ to denote the indicator function of the set A. We write $A \lesssim B$, if there exists a constant $C > 0$ such that $A \leqslant CB$. If $0 < C < 1$, we write $A \ll B$. If $A \lesssim B$ and $B \lesssim A$, then we use $A \sim B$. We use $a+$ to denote $a + \varepsilon$, $\varepsilon > 0$ sufficiently small. $a-$ is defined similarly.

2.2 Notion of well-posedness

The problems we consider are posed on $\mathbb{R}\times\mathbb{R}^d$. We will refer to $t \in \mathbb{R}$ as the time variable and $x \in \mathbb{R}^d$ as the spatial variable. Occasionally, the spatial variables will also be denoted by $(x, y) \in \mathbb{R}^2$. By $C([0, T]; H^s(\mathbb{R}^d))$, we denote the space of all functions $f = f(t, x)$ such that the map

$$t \mapsto f(t) \in H^s(\mathbb{R}^d)$$

is continuous for all $t \in [0, T]$. For $k \in \mathbb{R}$, $C^k([0, T]; H^s(\mathbb{R}^d))$ is defined analogously.

Remark 2.2.1. (i) Let I be a time interval and L a linear operator. In the following chapters, we will deal with evolution equations of the following form:

$$\begin{cases} \partial_t u - Lu &= F(u), \quad t \in I \\ u(0,x) &= u_0 \in H^s(\mathbb{R}^d). \end{cases} \quad (2.2.1)$$

We will, instead, consider the following integral formulation of (2.2.1) for $u \in C([0, T]; H^s(\mathbb{R}^d))$ and $F(u) \in L^1(I; H^s(\mathbb{R}^d))$:

$$u(t) = e^{tL}u_0 + \int_0^t e^{(t-s)L}(F(u))(s)ds, \quad t \in I. \quad (2.2.2)$$

(ii) For concrete applications of the above integral formulation later, we will use a frequency localised function f for F above, the former is sufficiently regular to make the evolution well-defined.

Definition 2.2.2. $u \in C([0,T]; H^s(\mathbb{R}^d))$ satisfying (2.2.2) is called a *mild solution* of the Cauchy problem (2.2.1).

Definition 2.2.3. The Cauchy problem (2.2.1) is said to be *(locally) well-posed* in $H^s(\mathbb{R}^d)$ if the following hold:

- for any $u_0 \in H^s(\mathbb{R}^d)$, there exists a time $T > 0$, an open ball $B \subseteq H^s(\mathbb{R}^d)$ and a space $X_T \hookrightarrow C([0,T]; H^s(\mathbb{R}^d))$ such that for each $u_0 \in B$, there exists a unique solution $u \in X_T$ to (2.2.1),
- the map $u_0 \mapsto u$ is continuous.

Remark 2.2.4. (i) In the above definition, if the data-to-solution map $u_0 \mapsto u$ is uniformly continuous, we call the problem to be *semilinear*. If it is only continuous, the problem is called a *quasilinear* problem.

(ii) In the above definition, if we can choose T arbitrarily large, we say that the problem is *globally well-posed*.

(iii) If $X_T = C([0,T]; H^s(\mathbb{R}^d))$ in the above definition, the problem is said to be *unconditionally well-posed*.

2.3 Local well-posedness for smooth initial data

As a primer, we illustrate the application of Banach's fixed point theorem to show the local well-posedness of the Cauchy problem for the nonlinear Schrödinger (NLS) equation. In subsequent chapters, we will conclude local well-posedness results for the Zakharov system and the dispersion generalised KP-I equation via the same means.

For $s > \frac{d}{2}$ and p odd, consider

$$\begin{cases} i\partial_t u + \Delta u &= \pm |u|^{p-1} u \\ u(0,x) &= u_0(x) \in H^s(\mathbb{R}^d). \end{cases} \tag{2.3.1}$$

Using Duhamel's formula, the above is equivalent to

$$u(t) = e^{it\Delta} u_0 \mp i \int_0^t e^{i(t-s)\Delta} (|u|^p u)(s) ds, \tag{2.3.2}$$

where

$$\mathcal{F}_x(e^{it\Delta} u_0)(\xi) = e^{-it|\xi|^2} \hat{u}_0(\xi).$$

In order to use the Banach fixed point theorem, we write (2.3.2) as

$$u(t) =: \Gamma_{u_0}(u)(t). \tag{2.3.3}$$

For a given $u_0 \in H^s(\mathbb{R}^d)$, $s > \frac{d}{2}$, we shall show that there exists $u \in C([-T,T]; H^s(\mathbb{R}^d))$, $T > 0$ which is the unique fixed point of Γ_{u_0}. The main ingredient is the algebra property of Sobolev spaces $H^s(\mathbb{R}^d)$ for $s > \frac{d}{2}$, i.e.

$$\|fg\|_{H^s(\mathbb{R}^d)} \lesssim \|f\|_{H^s(\mathbb{R}^d)} \|g\|_{H^s(\mathbb{R}^d)}.$$

Thus, we need to show that the map Γ_{u_0} is well-defined and a contraction in a suitable ball in the space $C([-T,T];H^s(\mathbb{R}^d))$. For the sake of brevity, we shall denote the space $C([-T,T];H^s(\mathbb{R}^d))$ by X_T in this section.

- Γ_{u_0} *is well-defined:* Using the unitarity of $e^{it\Delta}$, (see Section 2.5) and Minkowski's integral inequality, we have

$$\begin{aligned}\|\Gamma_{u_0}(u)\|_{X_T} &\leqslant \|e^{it\Delta}u_0\|_{X_T} + \Big\| \int_0^t e^{i(t-s)\Delta}(|u|^{p-1}u)(s)ds\Big\|_{X_T} \\ &\leqslant \|u_0\|_{H^s(\mathbb{R}^d)} + \int_0^T \||u|^{p-1}u\|_{X_T}dt \\ &\leqslant \|u_0\|_{H^s(\mathbb{R}^d)} + TC\|u\|_{X_T}^p.\end{aligned}$$

If we choose the closed ball $\bar{B}_R \subseteq X_T$ with radius R such that $R = 2\|u_0\|_{H^s(\mathbb{R}^d)}$ and T such that $TCR^p = \frac{R}{2}$, then

$$\|\Gamma_{u_0}(u)\|_{X_T} \leqslant \frac{R}{2} + TCR^p = R.$$

This implies that Γ_{u_0} is well-defined on $\bar{B}_R$.

- Γ_{u_0} *is a contraction:* We first note that for $p = p_1 + p_2$,

$$|u|^{p-1}u-|v|^{p-1}v = u^{p_1}\bar{u}^{p_2}-v^{p_1}\bar{v}^{p_2} = (u-v)u^{p_1-1}\bar{u}^{p_2}+v(u-v)u^{p_1-2}\bar{u}^{p_2}+...+v^{p_1}\bar{v}^{p_2-1}(\bar{u}-\bar{v}).$$

Now, using the above

$$\begin{aligned}\|\Gamma_{u_0}(u) - \Gamma_{u_0}(v)\|_{X_T} &\lesssim \int_0^t \||u|^{p-1}u - |v|^{p-1}v\|_{X_T}dt \\ &\lesssim T\Big(\sum_{j=0}^{p-1} \|u\|_{X_T}^{p-1-j}\|v\|_{X_T}^{j}\Big)\|u-v\|_{X_T} \\ &\leqslant C'T(\|u\|_{X_T}^{p-1} + \|v\|_{X_T}^{p-1})\|u-v\|_{X_T},\end{aligned}$$

where we used the inequality $xy \leqslant \frac{x^p}{p} + \frac{y^q}{q}$ for $\frac{1}{p} + \frac{1}{q} = 1$ in the last step. Hence Γ_{u_0} becomes a contraction if we choose T such that $C'TR^{p-1} \leqslant \frac{1}{2}$. Using Banach's fixed point theorem, we conclude the existence of a unique solution u on the time interval $[-T,T]$ where $T = \min\Big(\frac{R^{1-p}}{2C}, \frac{R^{1-p}}{2C'}\Big)$. Imitating the same arguments, we also conclude local Lipschitz continuity of the data-to-solution map $u_0 \mapsto u$.

2.4 Scaling symmetry and criticality

We explain the notion of criticality via the NLS (2.3.1). If u solves (2.3.1) on $[-T,T]$, then

$$u_\lambda(t,x) = \frac{1}{\lambda^{\frac{2}{p-1}}}u\Big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\Big)$$

also solves NLS on the time interval $[-\lambda^2 T, \lambda^2 T]$ with scaled initial data

$$u_{0\lambda}(x) = \frac{1}{\lambda^{\frac{2}{p-1}}} u_0\Big(\frac{x}{\lambda}\Big), \quad \lambda > 0.$$

Furthermore,

$$\|u_{0\lambda}\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{-s+\frac{d}{2}-\frac{2}{p-1}} \|u_0\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{-s+s_c} \|u_0\|_{\dot{H}^s(\mathbb{R}^d)},$$

where

$$s_c = \frac{d}{2} - \frac{2}{p-1}.$$

The Sobolev regularity exponent s can be classified as

(i) $s > s_c$ (Subcritical): As $\lambda \to \infty$, $\|u_{0\lambda}\|_{\dot{H}^s(\mathbb{R}^d)} \to 0$ and the time of existence of the solution, i.e. $\lambda^2 T \to \infty$. This is consistent with the proof of local well-posedness where the time of existence is inversely proportional to the norm of the initial datum. Hence, this is the most favourable situation to prove a well-posedness result.

(ii) $s = s_c$ (Critical): The norm of the scaled initial data remains invariant but $\lambda^2 T \to \infty$. This is a tender situation where increase in the existence time could lead to global well-posedness or blowup.

(iii) $s < s_c$ (Supercritical): As $\lambda \to \infty$, the norm of the scaled initial data as well as the time of existence tend to infinity. We do not expect well-posedness in this case.

2.5 Dispersion and Strichartz estimates

Roughly speaking, dispersion means that waves with different frequencies travel with different velocities. Consider the Airy's equation,

$$i\partial_t u + \partial_x^3 u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{2.5.1}$$

which on taking the space-time Fourier transform becomes

$$(i\tau - i\xi^3)\hat{u}(\tau, \xi) = 0. \tag{2.5.2}$$

This shows that the distributional support of u is $\{(\tau, \xi) : \ \tau = \xi^3\}$. Let $\omega(\xi) := \xi^3$. The phase velocity is given by

$$c_p = \frac{\omega(\xi)}{\xi} = \xi^2,$$

while the group velocity is given by

$$c_g = \frac{dw}{d\xi} = 3\xi^2.$$

This difference in the phase and the group velocities allows the waves to disperse. Below we give a mathematical description of the same for the linear Schrödinger equation

$$\left\{ \begin{array}{ll} \partial_t u & = i\Delta u \\ u(0,x) & = u_0(x), \end{array} \right. \tag{2.5.3}$$

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$. The solution to the above is given by

$$e^{it\Delta} u_0 := (e^{-it|\xi|^2} \hat{u}_0)^\vee = \frac{1}{(2\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i|x-y|^2/2t} u_0(y)dy. \tag{2.5.4}$$

The family of operators $e^{it\Delta}$ exhibits the following properties:

(i) For all $t \in \mathbb{R}, e^{it\Delta} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is an isometry, i.e.

$$\|e^{it\Delta} g\|_{L^2} = \|g\|_{L^2}. \tag{2.5.5}$$

(ii) $e^{it\Delta} e^{is\Delta} = e^{i(t+s)\Delta}$, $(e^{it\Delta})^{-1} = e^{-it\Delta} = (e^{it\Delta})^*$, and $e^{i0\Delta} = 1$.

(iii) For a fixed $g \in L^2(\mathbb{R}^d)$, $\Delta_g : \mathbb{R} \mapsto L^2(\mathbb{R}^d)$ defined by $\Delta_g(t) = e^{it\Delta} g$ is a continuous function.

We have the following result for the linear propagator $e^{it\Delta}$ which we will use to show that the decay of initial data is translated to smoothing property for the solution.

Lemma 2.5.1. *Let* $t \neq 0, \frac{1}{p} + \frac{1}{p'} = 1, 2 \leqslant p \leqslant \infty$, *then*

$$\|e^{it\Delta} g\|_{L^p(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}\left(\frac{1}{p'} - \frac{1}{p}\right)} \|g\|_{L^{p'}(\mathbb{R}^d)}. \tag{2.5.6}$$

Proof. From property (i), we have

$$\|e^{it\Delta} g\|_{L^2(\mathbb{R}^d)} = \|g\|_{L^2(\mathbb{R}^d)},$$

and from (2.5.4), we have

$$\|e^{it\Delta} g\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}} \|g\|_{L^1(\mathbb{R}^d)}.$$

Interpolating the above inequalities using the Riesz-Thorin theorem, we get the desired result. □

Combining the above dispersive estimate with the following (Hardy-Littlewood-Sobolev) inequality, we can obtain a set of very useful space-time estimates called as Strichartz estimates.

Lemma 2.5.2 (cf. [21, Theorem 6.1.3])**.** *Let* $1 < p, q, r < \infty$ *be such that*

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}. \tag{2.5.7}$$

Then,

$$\left\| |x|^{-\frac{d}{p}} * f \right\|_{L^r(\mathbb{R}^d)} \lesssim \|f\|_{L^q(\mathbb{R}^d)}. \tag{2.5.8}$$

Lemma 2.5.3 (Strichartz estimates for Schrödinger equation)**.** *Let* $d \geqslant 1$. *We call a pair* (q, r) *of exponents Schrödinger admissible if*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leqslant q, r \leqslant \infty, \quad (q, r, d) \neq (2, \infty, 2). \tag{2.5.9}$$

For any admissible exponents $(q,r),(\tilde{q},\tilde{r})$, we have

$$\begin{aligned}\|e^{it\Delta}u_0\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)} &\lesssim \|u_0\|_{L_x^2(\mathbb{R}^d)},\\ \Big\|\int_0^t e^{i(t-s)\Delta}F(s)ds\Big\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)} &\lesssim \|F\|_{L_t^{\tilde{q}}L_x^{\tilde{r}}(\mathbb{R}\times\mathbb{R}^d)}.\end{aligned} \tag{2.5.10}$$

Proof. We give an outline of the proof for the non-endpoint case while the endpoint case has been proved in [37]. Let H be a Hilbert space and B be a Banach space. For an operator $T : H \to B$, we have

$$\|T\| < \infty \iff \|T^*\| < \infty \iff \|TT^*\| < \infty, \tag{2.5.11}$$

where T^* is the adjoint of T. Now define the operator T as

$$Tf(t,x) := e^{it\Delta}f(x), \tag{2.5.12}$$

then,

$$T^*G(x) = \int_{\mathbb{R}} e^{-it\Delta}G(t,x)dt \text{ and } TT^*F(t,x) := \int_{\mathbb{R}} e^{i(t-s)\Delta}F(s)ds, \tag{2.5.13}$$

and the following are equivalent:

$$\begin{aligned}\|Tf\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)} &\lesssim \|f\|_{L_x^2(\mathbb{R}^d)},\\ \|T^*G\|_{L_x^2(\mathbb{R}^d)} &\lesssim \|G\|_{L_t^{\tilde{q}}L_x^{\tilde{r}}(\mathbb{R}\times\mathbb{R}^d)},\\ \|TT^*F\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)} &\lesssim \|F\|_{L_t^{\tilde{q}}L_x^{\tilde{r}}(\mathbb{R}\times\mathbb{R}^d)}.\end{aligned}$$

Using Minkowski's inequality, Lemma 2.5.1 and Lemma 2.5.2, respectively, we have

$$\begin{aligned}\|TT^*F\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)} &\leqslant \Big\|\int_{\mathbb{R}} \|S(t-s)F(s)\|_{L_x^r(\mathbb{R}^d)}ds\Big\|_{L_t^q(\mathbb{R})}\\ &\lesssim \Big\|\int_{\mathbb{R}} \frac{1}{|t-s|^{d(\frac{1}{2}-\frac{1}{r})}}\|F(s)\|_{L_x^{r'}(\mathbb{R}^d)}ds\Big\|_{L_t^q(\mathbb{R})}\\ &\lesssim \|F\|_{L_t^{q'}L_x^{r'}(\mathbb{R}\times\mathbb{R}^d)}.\end{aligned} \tag{2.5.14}$$

To prove the retarded estimate in (2.5.10), we use Christ-Kiselev lemma [64, Lemma 2.4]. □

2.6 Littlewood-Paley projectors

To analyse the interactions occurring between functions, we need to localise the functions according to their spatial and temporal frequencies, distance from the characteristic hypersurfaces. We use Littlewood-Paley theory to accomplish this.

Let $\chi \in C_c^\infty(\mathbb{R})$ be a symmetric, non-negative function such that $\chi(\xi) = 1$ for $|\xi| \leqslant 1$ and $\chi(\xi) = 0$ for $|\xi| \geqslant 2$. Set

$$\phi(\xi) = \chi(\xi) - \chi(2\xi), \quad \phi_\lambda(\xi) = \phi\Big(\frac{\xi}{\lambda}\Big).$$

Then

$$\chi(\xi) + \sum_{\lambda \in 2^{\mathbb{N}}} \phi_\lambda(\xi) = 1.$$

For $\lambda \in 2^{\mathbb{N}}, \lambda > 1$, we define the spatial frequency localisation Fourier multipliers with the following symbols:

$$P_\lambda = \phi_\lambda(|\nabla|) \text{ and } P_1 = \chi(|\nabla|).$$

In other words,

$$(P_\lambda f)^\wedge(\xi) = \phi_\lambda(|\xi|)\hat{f}(\xi), \quad \lambda > 1.$$

Thus, P_λ is a Fourier multiplier which localises the spatial frequencies to the set $\{\frac{\lambda}{2} \leqslant |\xi| \leqslant 2\lambda\}$. We also define

$$P_{\leqslant \lambda} f = \sum_{\mu \leqslant \lambda} P_\mu f,$$

$P_{\geqslant \lambda}$ is defined analogously. In the following chapter, we shall introduce more Fourier multipliers to localise the temporal frequency supports and the space-time Fourier supports of functions.

2.6.1 Bernstein's inequalities

Localisation on the frequency side allows us to trade integrability and regularity via inequalities referred to as Bernstein's inequalities.

Lemma 2.6.1. *Let $f \in \mathcal{S}(\mathbb{R}^d)$ with supp$\hat{f} \subseteq B(0,\lambda)$. For any $1 \leqslant p \leqslant q \leqslant \infty$, we have*

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \lambda^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)}.$$

Proof. Fix $\phi \in \mathcal{S}(\mathbb{R}^d)$ such that $\phi(\xi) = 1$ for $|\xi| \leqslant 1$. Define $\phi_\lambda(\xi) := \phi(\frac{\xi}{\lambda})$. Then

$$\hat{f}(\xi) = \hat{f}(\xi) \cdot \phi_\lambda(\xi), \quad f(x) = (f * \phi_\lambda)^\vee(x).$$

Using Young's inequality for $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and $(\phi_\lambda)^\vee(x) = \lambda^d \phi^\vee(\lambda x)$, we obtain

$$\|f\|_{L^q(\mathbb{R}^d)} \lesssim \|(\phi_\lambda)^\vee\|_{L^r(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)} \lesssim \lambda^{d\left(1-\frac{1}{r}\right)} \|f\|_{L^p(\mathbb{R}^d)} = \lambda^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^d)}.$$

□

From the above, we also obtain the following.

Lemma 2.6.2 (Bernstein's inequalities, [64, pp 333])**.** *Let $s \geqslant 0$ and $1 \leqslant p \leqslant q \leqslant \infty$. Then*

(i) $\|P_{\geqslant \lambda} f\|_{L^p(\mathbb{R}^d)} \lesssim \lambda^{-s} \||\nabla|^s P_{\geqslant \lambda} f\|_{L^p(\mathbb{R}^d)}$

(ii) $\|P_{\leqslant \lambda} |\nabla|^s f\|_{L^p(\mathbb{R}^d)} \lesssim \lambda^s \|P_{\leqslant \lambda} f\|_{L^p(\mathbb{R}^d)}$

(iii) $\|P_\lambda |\nabla|^{\pm s} f\|_{L^p(\mathbb{R}^d)} \lesssim \lambda^{\pm s} \|P_\lambda f\|_{L^p(\mathbb{R}^d)}$

(iv) $\|P_{\leqslant \lambda} f\|_{L^q(\mathbb{R}^d)} \leqslant \lambda^{d(\frac{1}{p}-\frac{1}{q})} \|P_{\leqslant \lambda} f\|_{L^p(\mathbb{R}^d)}$

(v) $\|P_\lambda f\|_{L^q(\mathbb{R}^d)} \leqslant \lambda^{d(\frac{1}{p}-\frac{1}{q})} \|P_\lambda f\|_{L^p(\mathbb{R}^d)}$.

2.7 $X^{s,\theta}$ spaces

$X^{s,\theta}$ spaces or Fourier restriction spaces introduced by Bourgain [9] are efficient tools to handle dispersive equations in low regularity settings.

Definition 2.7.1. Let $s, \theta \in \mathbb{R}$ and $\omega : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. The space $X^{s,\theta}$ is defined as the closure of Schwartz functions with respect to the following norm

$$\|u\|_{X^{s,\theta}(\mathbb{R}\times\mathbb{R}^d)} := \|\langle\xi\rangle^s \langle\tau - \omega(\xi)\rangle^\theta \hat{u}(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R}\times\mathbb{R}^d)} = \|S(-t)u\|_{H^\theta_t H^s_x(\mathbb{R}\times\mathbb{R}^d)}, \tag{2.7.1}$$

where $S(t)$ is the propagator corresponding to the linear equation.

$X^{s,\theta}$ spaces are Banach spaces invariant under space and time translations. For $s_1 \leqslant s_2, \theta_1 \leqslant \theta_2$, we have $X^{s_2,\theta_2} \subseteq X^{s_1,\theta_1}$. The dual of the space $X^{s,\theta}$ is given by $(X^{s,\theta}_{\tau=\omega(\xi)})' = X^{-s,-\theta}_{\tau=-\omega(-\xi)}$ while the behaviour under complex conjugation is given by $\|\overline{u}\|_{X^{s,\theta}_{\tau=-\omega(-\xi)}} = \|u\|_{X^{s,\theta}_{\tau=\omega(\xi)}}$.

We record the properties of $X^{s,\theta}$ spaces in the following. The first result says that these spaces are well-suited to the solution of a linear equation when one localises in time.

Lemma 2.7.2 ([64, Lemma 2.8])**.** *For $s \in \mathbb{R}, g \in H^s(\mathbb{R}^d)$ and $\eta \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\|\eta(t)S(t)g\|_{X^{s,\theta}} \lesssim \|g\|_{H^s(\mathbb{R}^d)}. \tag{2.7.2}$$

With the following result, we can transfer estimates in H^s to that in $X^{s,\theta}$ spaces.

Lemma 2.7.3. *Let $\theta > \frac{1}{2}$ and Y be a Banach space of space-time functions such that*

$$\|e^{it\tau}S(t)g\|_Y \lesssim \|g\|_{H^s_x(\mathbb{R}^d)}, \tag{2.7.3}$$

for all $g \in H^s(\mathbb{R}^d), \tau \in \mathbb{R}$. Then we have

$$\|u\|_Y \lesssim \|u\|_{X^{s,\theta}(\mathbb{R}\times\mathbb{R}^d)}. \tag{2.7.4}$$

In particular, the above lemma holds for Lebesgue spaces $L^q_t L^r_x$ and we have the following.

Corollary 2.7.4. *Let q, r satisfy*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leqslant r < \infty,$$

then,

$$\|u\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u\|_{X^{0,\theta}_\Delta(\mathbb{R}\times\mathbb{R}^d)}, \tag{2.7.5}$$

where the Δ in the subscript denotes that the space in consideration is for the Schrödinger equation. For the half-wave equation,

$$i\partial_t u + |\nabla|u = 0,$$

we have,

$$\|u\|_{L^\infty_t L^2_x} \lesssim \|u\|_{X^{0,\theta}_{|\nabla|}}. \tag{2.7.6}$$

For $\theta > \frac{1}{2}$, $X^{s,\theta} \hookrightarrow C(\mathbb{R}; H^s(\mathbb{R}^d))$:

Lemma 2.7.5. *Let $\theta > \frac{1}{2}, s \in \mathbb{R}$ and $\omega : \mathbb{R}^d \to \mathbb{R}$, we have*

$$\|u\|_{C_t H_x^s(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u\|_{X^{s,\theta}(\mathbb{R}\times\mathbb{R}^d)}. \tag{2.7.7}$$

We note some Sobolev type embeddings for the $X^{s,\theta}$ spaces.

Lemma 2.7.6. *(i) For $2 \leqslant q < \infty$, $\theta \geqslant \frac{1}{2} - \frac{1}{q}$, we have*

$$\|u\|_{L_t^q H_x^s(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u\|_{X^{s,\theta}(\mathbb{R}\times\mathbb{R}^d)}. \tag{2.7.8}$$

(ii) For $2 \leqslant q, r < \infty$, $\theta \geqslant \frac{1}{2} - \frac{1}{q}, s \geqslant \frac{1}{2} - \frac{1}{r}$, we have

$$\|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u\|_{X^{s,\theta}(\mathbb{R}\times\mathbb{R}^d)}. \tag{2.7.9}$$

(iii) For $1 < q \leqslant 2$, $\theta \leqslant \frac{1}{2} - \frac{1}{q}$, we have

$$\|u\|_{X^{s,\theta}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u\|_{L_t^q H_x^s(\mathbb{R}\times\mathbb{R}^d)}. \tag{2.7.10}$$

The following lemma relates the solution to the nonlinearity and helps in achieving a large data result by localising to a suitably small time interval. We redefine η -let $\eta \in C_c^\infty(\mathbb{R})$ be a non-negative symmetric smooth time cut-off supported in $(-2, 2)$ and $\eta = 1$ on $[-1, 1]$. Also, define $\eta_T(t) = \eta\big(\frac{t}{T}\big)$.

Lemma 2.7.7 ($X^{s,\theta}$ energy estimate, cf. [64, Lemma 2.8])**.** *Let v be a solution to $\partial_t v = Lv + F$, with $L = ih(\nabla/i)$ and $v(0) = 0$. For $-\frac{1}{2} < \theta' \leqslant 0 \leqslant \theta \leqslant \theta' + 1$, we have*

$$\|\eta_T(t)v\|_{X^{s,\theta}_{\tau=h(\xi)}} \lesssim T^{1+\theta'-\theta}\|F\|_{X^{s,\theta'}_{\tau=h(\xi)}}. \tag{2.7.11}$$

Remark 2.7.8. In literature, the $X^{s,\theta}$ spaces are usually referred to as $X^{s,b}$ spaces. However, we deviate from this notation to avoid confusion because of the multitude of parameters in Chapter 3.

Chapter 3

Local well-posedness for the Zakharov system in dimension $d \leqslant 3$

"Musicians are like mathematicians. Every part has to be right for it to work."
-Jeff Beck

This chapter is an extended version of [56].

3.1 Introduction and previous results

Plasma oscillation is the organised motion of electrons or ions in a plasma. At equilibrium, there is an equal number of positive ions and electrons in any volume of the plasma. The charge density is zero, and there is no large scale electric field in the plasma. A displacement of the electrons by a small amount while keeping the ions fixed causes the electrons to accelerate and gain kinetic energy. Plasma oscillations, also known as Langmuir waves (named after Irving Langmuir), are rapid oscillations of the electron density in conducting media such as plasmas or metals. The Zakharov system, derived by V.E. Zakharov [70] in 1972 is a simplified model for the description of the nonlinear interactions between u, the envelope of the electric field, and n, the mean of the ionic fluctuations of density in the plasma. The sources [66, 25] can be consulted for more details on the physical aspects of the Zakharov system.

We consider the Cauchy problem for the Zakharov system

$$\begin{cases} i\partial_t u + \Delta u &= nu \\ \partial_t^2 n - \Delta n &= \Delta |u|^2 \end{cases} \tag{3.1.1}$$

with initial data

$$u(0,x) = u_0(x), \quad n(0,x) = n_0(x), \quad \partial_t n(0,x) = n_1(x), \tag{3.1.2}$$

where $(u_0, n_0, n_1) \in H^s(\mathbb{R}^d) \times H^l(\mathbb{R}^d) \times H^{l-1}(\mathbb{R}^d)$. Here $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is complex-valued while $n : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is real-valued. Sufficiently regular solutions to the Zakharov system satisfy the conservation of mass

$$M(u)(t) = \int_{\mathbb{R}^d} |u(t)|^2 dx = \int_{\mathbb{R}^d} |u_0|^2 dx = M(u_0),$$

and the conservation of energy

$$\begin{aligned} E(u, n, \partial_t n)(t) &= \int_{\mathbb{R}^d} \left(|\nabla u(t)|^2 + \frac{1}{4}||\nabla|^{-1}\partial_t n(t)|^2 + \frac{1}{4}|n(t)|^2 + \frac{1}{2}n(t)|u(t)|^2\right) dx \\ &= E(u_0, n_0, n_1). \end{aligned}$$

In terms of its mathematical analysis, the Zakharov system has attracted considerable attention. The first results concerning the local well-posedness for the same came in 90s by Ozawa-Tsutsumi in [51] where they proved that the system is locally well-posed in $H^2(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ for $d = 1, 2, 3$. Smoothing for the Schrödinger evolution was also shown in the same. Bourgain-Colliander [12] considered the system in spatial dimensions two and three and showed that the system is well-posed in the energy space, namely $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. Thanks to the conservation of the energy, a global result with a smallness in $H^1(\mathbb{R}^d)$ norm of the initial data follows. In [20], the authors considered the system (3.1.1) in a general space dimension $d \geqslant 1$ and proved local well-posedness results for indices (s, l) depending on d. They use a contraction mapping argument in Fourier restriction spaces and prove local well-posedness for (3.1.1)-(3.1.2) in the following cases:

- $d = 1 : -\frac{1}{2} < s - l \leqslant 1, 2s - l \geqslant \frac{1}{2}$,
- $d = 2, 3 : l \geqslant 0, 2s - l \geqslant 1, 0 \leqslant s - l \leqslant 1$,
- $d \geqslant 4 : l \geqslant \frac{d}{2} - 2, 2s - l \geqslant \frac{d}{2} - 1, 0 \leqslant s - l \leqslant 1$.

For $d = 2$, in [4] local well-posedness is proved at $(s, l) = (0, -\frac{1}{2})$. In [3], the authors deal with $d = 3$ and prove that the system is locally well-posed for

$$l > -\frac{1}{2}, l \leqslant s \leqslant l + 1, 2s - l > \frac{1}{2},$$

see Figure 3.1. These results exploit the transversality of the characteristic hypersurfaces [6, 46] in the resonant interactions to obtain sharp multilinear estimates. We shall rely on the estimates from [4, 3] to prove local well-posedness in the low regularity region for $d = 2, 3$.

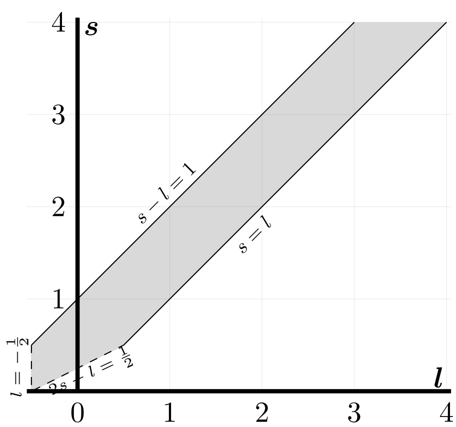

Figure 3.1: Region of local well-posedness for $d = 3$ in [3]

For the Zakharov system in one dimension, global well-posedness without any smallness assumption on the initial data is proved in the space $H^s(\mathbb{R}) \times L^2(\mathbb{R})$ for $\frac{9}{10} < s < 1$ in [52]. In [53], the author considers the one dimensional Zakharov system and proves local well-posedness in the spaces $\widehat{H^{s,p}}(\mathbb{R})$ for $1 < p < 2$ where

$$\|g\|_{\widehat{H^{s,p}}(\mathbb{R})} = \|\langle\xi\rangle^s \hat{g}\|_{L_\xi^{p'}(\mathbb{R})}, \ \frac{1}{p} + \frac{1}{p'} = 1.$$

Here, the regularity of the initial data can be lowered so that $s < 0, l < -\frac{1}{2}$ compared to the lowest possible regularity in L^2 based Sobolev spaces. The author uses a modified Fourier restriction norm to prove the same. Global well-posedness using mass conservation for $d = 1$ is proved in [17] at the point $(s, l) = (0, -\frac{1}{2})$ which is the largest space where the system is locally well-posed. In [2], global well-posedness and scattering results are proved for dimension $d = 4$ using normal from reduction, see Remark 3.2.2 and Appendix A. The interested reader is referred to [28, 17, 19] and references therein for more results on global well-posedness and scattering.

The present work derives motivation from [14] where the Zakharov system is considered in spatial dimensions $d \geqslant 4$. The result in [14] completely answers the question of local well-posedness for $d \geqslant 4$. The authors prove the following.

Theorem 3.1.1 ([14, Theorem 1.1]). *Let $d \geqslant 4$. The Zakharov system (3.1.1)-(3.1.2) is locally well-posed with a real analytic flow map, if and only if $(s, l) \in \mathbb{R}^2$ satisfies*

$$l \geqslant \frac{d}{2} - 2, \ \max\Big(l - 1, \frac{l}{2} + \frac{d-2}{4}\Big) \leqslant s \leqslant l + 2, \ (s, l) \neq \Big(\frac{d}{2}, \frac{d}{2} - 1\Big), \Big(\frac{d}{2}, \frac{d}{2} + 1\Big).$$

The question to prove an optimal local well-posedness result, however, remains open for $d \leqslant 3$.

To conjecture the expected region of local well-posedness for the Zakharov system in dimension $d \leqslant 3$, we throw some light on ill-posedness results. In [33], for the 1D system, it is shown that the constraint $2s - l \geqslant \frac{1}{2}$ is optimal in the sense that norm inflation occurs otherwise. For $l < -\frac{3}{2}$ and $s = 0$, it is shown that the data-to-solution map fails to be uniformly continuous from $H^s(\mathbb{R}) \times H^l(\mathbb{R}) \times H^{l-1}(\mathbb{R})$ to $C([0,T]; H^s(\mathbb{R})) \times C([0,T]; H^l(\mathbb{R})) \times C([0,T]; H^{l-1}(\mathbb{R}))$. This nearly matches the result of [7] where the authors prove ill-posedness for $s < 0$ and $l \leqslant -\frac{3}{2}$ for the 1D system. In [14, Section 9], it is also proved that for a general dimension $d \geqslant 1$, the flow map is not C^2-differentiable for s, l and d satisfying

$$l < \frac{d}{2} - 2, \ s - l > 2, \ 2s - l < \frac{d-2}{2}, \ s - l < -1.$$

For $d = 2$, stronger counterexamples in [4, Section 6] prove that the flow map is not C^2 for $2s - l < \frac{1}{2}$ or $l < -\frac{1}{2}$. In [20, Proposition 3.2], using the close relation of the system (3.1.1)-(3.1.2) to the cubic nonlinear Schrödinger equation, the authors prove that the data-to-solution map for the system fails to be Lipschitz continuous at the point $(s, l) = (\frac{1}{2}, -\frac{1}{2})$. Using power series expansion for the solutions, as done in [41] for the Schrödinger equation, norm inflation results have been proved for the Zakharov system in dimension $d \geqslant 1$ in [23] and in the updated article [24].

For the Cauchy problem (2.2.1) we say *norm-inflation* occurs in $H^s(\mathbb{K}^d)$ if for any $\delta > 0$, there exists $u_0 \in H^\infty(\mathbb{K}^d)$ and $T > 0$ such that

$$\|u_0\|_{H^s(\mathbb{K}^d)} < \delta, \quad 0 < T < \delta,$$

and the corresponding smooth solution u to (2.2.1) exists on $[0, T]$ and

$$\|u(T)\|_{H^s(\mathbb{K}^d)} > \delta^{-1}.$$

In particular, in [24], for $d \leqslant 3$, it is proved that all the boundaries but $l = -\frac{1}{2}$ are sharp. The failure of C^2-differentiability of the data-to-solution map implies that the problem is not amenable to an iterative approach, see [67] for details. Other methods can still be applied to prove a well-posedness result. In the case of the Zakharov system, one would also expect some quasilinear behaviour, but we can rule out this possibility because the data-to-solution map fails to be even continuous beyond the lines $s - l = 2$, $s - l = -1$ and $2s - l = \frac{1}{2}$. Nevertheless, it is still possible that the system exhibits quasilinear behaviour for $l < -\frac{1}{2}$ and $d = 1, 2$.

The interested reader can refer to [18] for a summarised account of semilinear ill-posedness results for the Zakharov system in dimension $d \geqslant 1$. For $d \leqslant 3$, the ill-posedness results are summarised in the following figure:[1]

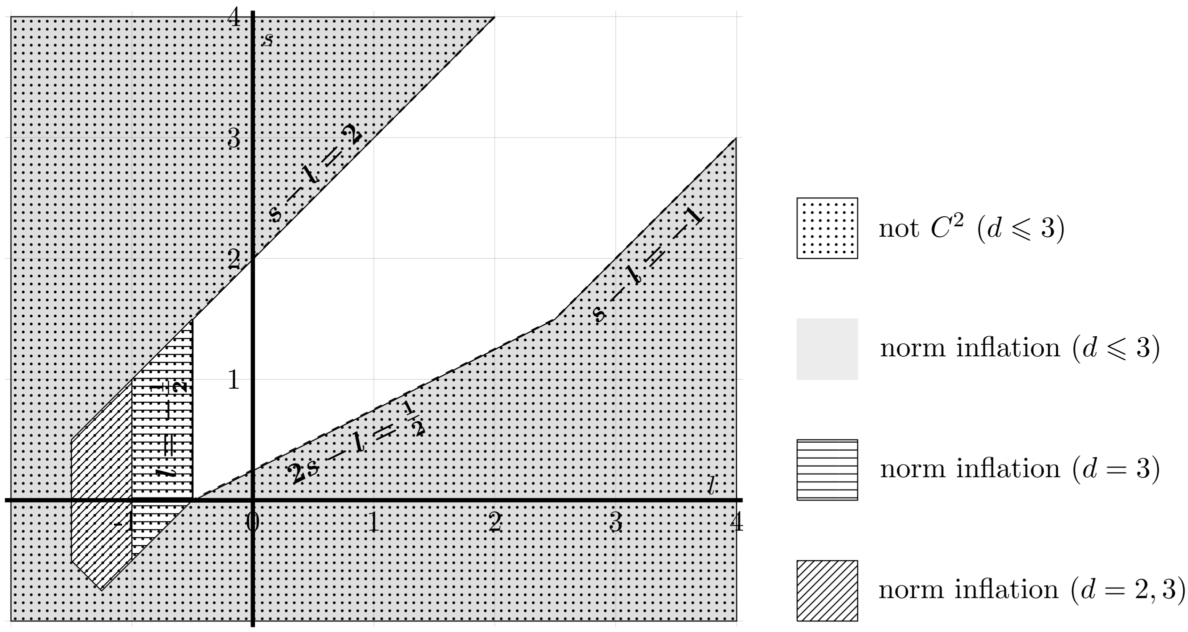

Figure 3.2: Ill-posedness for $d \leqslant 3$

Conjecture: The above leads us to conjecture that for $d \leqslant 3$, (3.1.1) − (3.1.2) is locally well-posed in $H^s(\mathbb{R}^d) \times H^l(\mathbb{R}^d) \times H^{l-1}(\mathbb{R}^d)$ for (s, l) satisfying

$$l \geqslant -\frac{1}{2}, \quad \max\left(l - 1, \frac{l}{2} + \frac{1}{4}\right) \leqslant s \leqslant l + 2.$$

Now we present some heuristics for the Zakharov system.

[1] the behaviour at the boundaries except $s - l = -1$ for $d = 2, 3$ is not know for $d \leqslant 3$, see Appendix A

3.1.1 Criticality for the Zakharov system

Since the wave and Schrödinger equations have different scaling invariance, the Zakharov system cannot have a straightforward scaling invariance. However, in [20], the authors define a notion of scaling criticality for the system. We give the details: consider (3.1.1) and split n into its positive and negative frequency parts as follows:

$$n_\pm = n \pm i|\nabla|^{-1}\partial_t n, \tag{3.1.3}$$

where $|\nabla| = (-\Delta)^{\frac{1}{2}}$. The Zakharov system then reads

$$\begin{cases} i\partial_t u + \Delta u &= +\frac{1}{2}(n_+ + n_-)u \\ (i\partial_t \mp |\nabla|)n_\pm &= \pm|\nabla||u|^2. \end{cases} \tag{3.1.4}$$

If one ignores the term $\mp|\nabla|$ in the second equation, the system becomes invariant under the following scaling:

$$u_\lambda(t,x) = \frac{1}{\lambda^{\frac{3}{2}}}u\Big(\frac{t}{\lambda^2},\frac{x}{\lambda}\Big), \quad n_\lambda(t,x) = \frac{1}{\lambda^2},n\Big(\frac{t}{\lambda^2},\frac{x}{\lambda}\Big). \tag{3.1.5}$$

On calculating the $\dot{H}^s$ (and $\dot{H}^l$) norm of the corresponding scaled initial data, i.e.

$$\Big(\frac{1}{\lambda^{\frac{3}{2}}}u_0\Big(\frac{x}{\lambda}\Big),\ \frac{1}{\lambda^2}n_0\Big(\frac{x}{\lambda}\Big)\Big),$$

the critical exponents for the system are

$$(s^c, l^c) = \Big(\frac{d-3}{2},\frac{d-4}{2}\Big).$$

This is the criticality for the Zakharov system and can be verified from the fact that the $\mp|\nabla|$ term in the second equation in (3.1.4) is intermediate between the terms $|\nabla|^0 = 1$ and $|\nabla|^2$. The former can be eliminated using a gauge transform $n_\pm \to e^{\pm it}n_\pm$, at the cost of some additional factors in the nonlinearity which are harmless, while for the latter, the system is actually invariant under the transformation (3.1.5). Implicitly, the following analysis does not make use of the $\mp|\nabla|$ as Strichartz estimates for the wave equation are not used, except the $L_t^\infty L_x^2$ estimate.

Optimality of the line $s - l = \frac{1}{2}$: The point $(s,l) = (0,-\frac{1}{2})$ lies on the line $s - l = \frac{1}{2}$. For $d = 3$, this corresponds to the scaling critical value, while for $d = 1,2$, it is far from the corresponding values (s^c, l^c). On examining the system (3.1.4), we find that there is a loss of derivative occurring in the first equation because of the derivative nonlinearity in the second equation. We expect that the system behaves the best when this loss is equally shared by both the equations. This occurs when $s - l = \frac{1}{2}$, which also justifies the scaling critical values (s^c, l^c).

3.1.2 Comparison with its NLS limit

The Zakharov system relates to the (cubic) NLS equation in the following way [61]. Consider the Zakharov system with wave speed c:

$$\begin{cases} i\partial_t u + \Delta u &= nu \\ \frac{1}{c^2}\partial_t^2 n - \Delta n &= \Delta|u|^2. \end{cases} \tag{3.1.6}$$

Then, as $c \to \infty$, the system formally converges to

$$i\partial_t u + \Delta u + |u|^2 u = 0.$$

The critical regularity for the latter is $s = \frac{d}{2} - 1$, (see Section 2.4) which compared to the Zakharov system is half a derivative high. This can be seen as a 'smoothing' effect provided by the wave equation. In particular, for $d = 3$, one can almost reach the point $(s, l) = (0, -\frac{1}{2})$ which is supercritical ($s = 0$) for the cubic NLS. More details on the convergence to the NLS are found in [1].

3.2 Local well-posedness result

We now state the main result of this chapter.

Theorem 3.2.1. *Let $d \leqslant 3$. The system (3.1.1) with initial conditions (3.1.2) is locally well-posed in $H^s(\mathbb{R}^d) \times H^l(\mathbb{R}^d) \times H^{l-1}(\mathbb{R}^d)$ provided*

$$l > -\frac{1}{2}, \quad \max\left(l - 1, \frac{l}{2} + \frac{1}{4}\right) < s < l + 2. \tag{3.2.1}$$

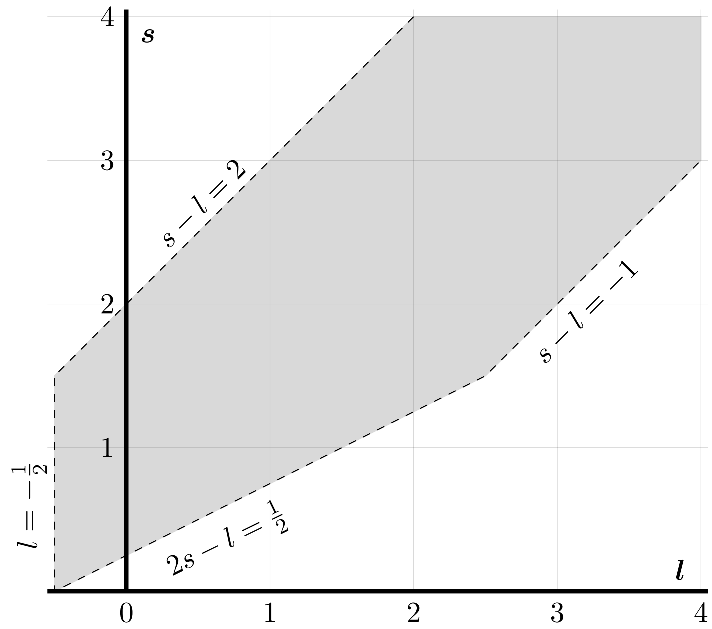

Figure 3.3: New region of local well-posedness for $d \leqslant 3$

Following are some remarks concerning the result:

Remark 3.2.2. (i) From the ill-posedness results concerning the failure of C^2-differentiability, one concludes that Theorem 3.2.1 is sharp up to the boundaries.

(ii) From the existing results, it follows that $(s, l) = (0, -\frac{1}{2})$ is the optimal (lowest regularity) point at which local well-posedness can be proved in dimension $d \leqslant 3$ by a contraction mapping argument. This also corresponds to the scaling critical regularity for the three dimensional system. Our result does not cover this because of the choice of our function spaces and losses which arise from the application of the estimates in Lemma 3.4.6. Hence the endpoint remains an open problem for $d = 3$.

(iii) The regularity $(s,l)=(1,0)$ which corresponds to the energy space for the system, lies on the boundary of the region in the results proved in [20, 3]. The inclusion of the same in the interior of the region of local well-posedness is noteworthy.

(iv) Using the normal form reduction as done in [29], one can 'improve' the local well-posedness result for the Zakharov system. This was achieved in [55] for $d \geqslant 2$. However, the result does not cover the negative regularity region corresponding to the wave regularity l. The recent article [16] is in the same direction. Here, the lower right boundary $s-l=-1$ has been covered, unlike in our result. In Appendix A, we give details of the reduction from [29], see also Chapter 5.

3.2.1 Reduced system and integral formulation

The Zakharov system admits a first order reduction which is simpler to work with. For $|\nabla| = (-\Delta)^{\frac{1}{2}}$, we use the transformation

$$v = n - i|\nabla|^{-1}\partial_t n$$

to reduce the system (3.1.1) to

$$\begin{cases} i\partial_t u + \Delta u &= uRe(v) \\ i\partial_t v + |\nabla| v &= -|\nabla||u|^2. \end{cases} \tag{3.2.2}$$

If (u,v) solves the reduced system, we observe that $(u, Re(v))$ solves the original system. Using Duhamel's formula, we write the above system as the following system of integral equations:

$$\begin{aligned} u(t) &= e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}(uRe(v))(s)ds, \\ v(t) &= e^{it|\nabla|}v_0 + i\int_0^t e^{i(t-s)|\nabla|}(|\nabla||u|^2)(s)ds. \end{aligned} \tag{3.2.3}$$

In the sequel, we shall work with the system (3.2.2) and (3.2.3).

3.3 Frequency localisation and function spaces

In addition to the spatial frequency, we need to localise the temporal frequencies and the space-time Fourier supports of the functions. We accomplish this as follows: with χ and ϕ_λ as defined in Chapter 2, we define

$$P_\lambda^{(t)} = \phi_\lambda(|\partial_t|),\, C_\lambda = \phi_\lambda(|i\partial_t + \Delta|),\, Q_\lambda = \phi_\lambda(|i\partial_t \pm |\nabla||),$$

and for $\lambda = 1$, we define

$$P_1^{(t)} = \chi(|\partial_t|),\, C_1 = \chi(|i\partial_t + \Delta|),\, Q_1 = \chi(|i\partial_t \pm |\nabla||).$$

$P_\lambda^{(t)}$ localises the temporal frequencies to the set $\{\frac{\lambda}{2} \leqslant |\tau| \leqslant 2\lambda\}$. In a similar spirit, C_λ and Q_λ localise the space-time Fourier support to distances $\sim \lambda$ from the paraboloid and the cone, respectively. More precisely, for $L \in 2^{\mathbb{N}}$ the space-time Fourier supports of C_L, C_1, Q_L and Q_1 can be respectively, written as

$$\mathcal{S}_L = \left\{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^d : \frac{L}{2} \leqslant |\tau + |\xi|^2| \leqslant 2L\right\}$$

$$\mathcal{S}_1 = \Big\{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^d : |\tau + |\xi|^2| \leqslant 2\Big\}$$
$$\mathcal{W}_L^{\pm} = \Big\{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^d : \frac{L}{2} \leqslant |\tau \pm |\xi|| \leqslant 2L\Big\}$$
$$\mathcal{W}_1^{\pm} = \Big\{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^d : |\tau \pm |\xi|| \leqslant 2\Big\}.$$

To restrict the Fourier support to larger sets, we use the notation

$$P_{\leqslant\lambda} = \sum_{\mu\in 2^{\mathbb{N}_0},\mu\leqslant\lambda} \phi_\mu(|\nabla|), \quad P^{(t)}_{\leqslant\lambda} = \sum_{\mu\in 2^{\mathbb{N}_0},\mu\leqslant\lambda} \phi_\mu(|\partial_t|),$$
$$C_{\leqslant\lambda} = \sum_{\mu\in 2^{\mathbb{N}_0},\mu\leqslant\lambda} \phi_\mu(|i\partial_t + \Delta|), \quad Q_{\leqslant\lambda} = \sum_{\mu\in 2^{\mathbb{N}_0},\mu\leqslant\lambda} \phi_\mu(|i\partial_t \pm |\nabla||).$$

For $\lambda \in 2^{\mathbb{N}_0}$, we use the shorthand $f_\lambda = P_\lambda f$.

3.3.1 New function spaces

We introduce the function spaces that we will use as auxiliary spaces to run the contraction mapping argument on. We set $\theta := \frac{1}{2}+$. Given $s, l \in \mathbb{R}$, we define the parameters $0 \leqslant a, b < 2-2\theta, \theta', s'$ and β as follows:

$$\begin{aligned}
a &:= \begin{cases} s-l-3+4\theta, & s-l\geqslant 1 \\ 0, & s-l<1 \end{cases}, \\
b &:= \begin{cases} 0, & s-l\geqslant 0 \\ (l-s)+2\theta-1, & s-l<0 \end{cases}, \\
\theta' &:= \begin{cases} \theta\ , s-l\geqslant 0 \\ 1\ , s-l<0 \end{cases}, \\
s' &:= \begin{cases} s, & s-l\geqslant 0 \\ s+2\theta-2, & s-l<0 \end{cases} \quad \text{and} \\
\beta &:= \begin{cases} l+a+2\theta-1, & s-l\geqslant 1 \\ l, & -\frac{1}{2}\leqslant s-l<1 \\ s-\theta+1, & s-l<-\frac{1}{2} \end{cases}.
\end{aligned} \tag{3.3.1}$$

For $\lambda \in 2^{\mathbb{N}_0}$, $s, a, b, \theta \in \mathbb{R}$, we control the frequency localised Schrödinger component of the Zakharov evolution by the following:

$$\begin{aligned}
\|u_\lambda\|_{S_\lambda^{s,l,\theta}} = \lambda^s \|u_\lambda\|_{L_t^\infty L_x^2} &+ \lambda^s \|\langle\tau+|\xi|^2\rangle^\theta \mathcal{F}(C_{\ll\lambda^2}u_\lambda)\|_{L^2_{\tau,\xi}} \\
&+ \lambda^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda+|\partial_t|)^a C_{\gtrsim\lambda^2}u_\lambda)\|_{L^2_{\tau,\xi}}.
\end{aligned}$$

To control the frequency localised Schrödinger nonlinearity, we define

$$\begin{aligned}
\|F_\lambda\|_{N_\lambda^{s,l,\theta-1}} = \lambda^{s+2\theta-3}\|P^{(t)}_{\ll\lambda^2}F_\lambda\|_{L_t^\infty L_x^2} &+ \lambda^s\|\langle\tau+|\xi|^2\rangle^{\theta-1}\mathcal{F}(C_{\ll\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}} \\
&+ \lambda^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda+|\partial_t|)^a C_{\gtrsim\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}}.
\end{aligned}$$

For $l, \beta, a, \theta \in \mathbb{R}$, the evolution of the frequency localised wave component is controlled by the following norm

$$\begin{aligned}\|v_\lambda\|_{W_\lambda^{l,s,\theta}} = \lambda^l \|v_\lambda\|_{L_t^\infty L_x^2} + \lambda^{l-a} \|\langle \tau - |\xi|\rangle^\theta \mathcal{F}((\lambda + |\partial_t|)^a Q_{\ll \lambda^2} v_\lambda)\|_{L^2_{\tau,\xi}} \\ + \lambda^{\beta-1} \|\langle \tau - |\xi|\rangle \mathcal{F}(Q_{\gtrsim \lambda^2} v_\lambda)\|_{L^2_{\tau,\xi}},\end{aligned}$$

and the right-hand side (RHS) of the wave equation is controlled by

$$\begin{aligned}\|G_\lambda\|_{R_\lambda^{l,s,\theta-1}} = \lambda^{l+2\theta-3} \|G_\lambda\|_{L_t^\infty L_x^2} + \lambda^{l-a} \|\langle \tau - |\xi|\rangle^{\theta-1} \mathcal{F}((\lambda + |\partial_t|)^a Q_{\ll \lambda^2} G_\lambda)\|_{L^2_{\tau,\xi}} \\ + \lambda^{\beta-1} \|\mathcal{F}(Q_{\gtrsim \lambda^2} G_\lambda)\|_{L^2_{\tau,\xi}}.\end{aligned}$$

When there is no confusion, we will drop the superscripts $l, s, \theta, \theta - 1$ from the function spaces. We observe that the norms S_λ and W_λ have three components:

- The $L_t^\infty L_x^2$ norms justify the use of S_λ and W_λ norms as auxiliary spaces. We cannot use Lemma 2.7.5 to achieve this because in the low temporal frequency case and $a > 0$, the Schrödinger solution does not have full s derivatives. A similar argument suggests the requirement of the $L_t^\infty L_x^2$ term for the wave norm, owing to the choice of the parameter β.

- The second terms are for the low modulation components of the solutions. In particular, these are the spaces for the respective linear solutions. One clearly sees that this is exactly the $X^{s,\theta}$ norm for $\theta > \frac{1}{2}$. Note that in the norm for the wave solution, we have the additional weight $(\lambda + |\partial_t|)^a$. For a free solution, this weight is ~ 1. A norm like this for the free solutions enables to use the estimates from [4, 3] for the resonant interaction. However, the choice $\theta = \frac{1}{2}+$ does not allow us to reach the endpoints, namely the lines $l = -\frac{1}{2}$ and $2s - l = \frac{1}{2}$ in $d = 2$. We also note that for $d = 3$, there is an additional logarithmic loss in the trilinear estimate derived from the nonlinear Loomis-Whitney inequality. This is another hurdle to reaching the endpoints.

- The third terms of the norms are for the high modulation components. This is where the parameters and the weights come into play. We also observe that the exponent of the modulation is different from $\frac{1}{2}+$, i.e. θ' for the Schrödinger solution and 1 for the wave solution. The Schrödinger "regularity" s also changes to s' and varies in parallel to θ'.

The norms for the nonlinearities are the defined in accordance with the $X^{s,\theta}$ energy estimate, i.e. Lemma 2.7.7.

Choice of the parameters: The parameters a, b, s', β and θ' are required to prove the bilinear estimates in the full region described by (3.2.1) and can be understood as follows:

- a measures the loss of regularity for the Schrödinger component in the low ($\ll \lambda^2$) temporal frequency region as can be seen from the weight

$$m_S(\tau) := \Big(\frac{\lambda + |\tau|}{\lambda^2}\Big)^a, 0 \leqslant a < 1.$$

Note that

$$|m_S(\tau)| \sim \begin{cases} \lambda^{-a}, & |\tau| \lesssim \lambda, \\ 1, & |\tau| \sim \lambda^2, \end{cases} \quad \text{and} \quad |m_S(\tau)| \gg 1, \quad |\tau| \gg \lambda^2,$$

while

$$|m_W(\tau)| := \Big(\frac{\lambda + |\tau|}{\lambda}\Big)^a \gtrsim 1.$$

In the low temporal frequency ($|\tau| \ll \lambda^2$) regime and $a > 0$, we allow the solution u_λ to be 'a' derivatives less regular. However, in this case, we have $\langle \tau + |\xi|^2 \rangle \sim |\xi|^2$, i.e. we are in the non-resonant case. It is the size of this weight that allows to prove bilinear estimates in this case.

- The parameter $b \geqslant 0$ gives a gain in regularity to the Schrödinger evolution in the high modulation regime and helps to achieve bilinear estimates for the wave equation in the region (3.2.1).
- a and b are not non-zero simultaneously and $0 \leqslant a, b < 1$. In addition to the choice of θ, it is this restriction on the upper bounds for the parameters a and b that does not allow us to achieve the boundaries ($s - l = 2, -1$) of the region described by (3.2.1).
- The parameters s' and β which can be understood as the regularity for the high modulation Schrödinger part and high modulation wave part, respectively, are chosen depending on s and l as explained below.

The idea of choosing the parameters is to make sure that the Schrödinger and wave regularities are not too far from each other in the region described by (3.2.1). In the region $l + 1 \leqslant s < l + 2$, where the Schrödinger component is more regular, we choose $a > 0$. In the 'balanced' region $l < s < l + 1$, we choose $a = 0 = b$. In the final regime i.e. $l - 1 < s \leqslant l$, where the wave component is more regular, we choose $a = 0, b > 0$. Similarly, for β, we have $\beta \approx s - 1$ when $1 \leqslant s - l < 2$, which is greater than or equal to the "ideal" regularity l. In the balanced regime, it is chosen to be l, while in the thin strip $-1 < s - l \leqslant -\frac{1}{2}$, where the wave is more regular, we choose β to be $\approx s + \frac{1}{2}$ which is less than l. For the regularity s' of the high modulation Schrödinger norm, we see that the change in the exponent of the modulation weight viz θ', compensates for the loss (s' is smaller than s while $\theta' = 1 > \frac{1}{2}+ = \theta$).

Remark 3.3.1. (i) The choice of the temporal weights in the function spaces defined above is motivated from [14] and has been modified accordingly to cater to spatial dimensions $d \leqslant 3$.

(ii) Equation (3.3.1) provides us with one possible choice for the parameters. We reckon other choices might be plausible too.

The evolution of the full Schrödinger solution and nonlinearity, respectively, is controlled by the following norms

$$\|u\|_{S^{s,l,\theta}} = \Big(\sum_{\lambda \in 2^{\mathbb{N}_0}} \|u_\lambda\|^2_{S^{s,l,\theta}_\lambda} \Big)^{\frac{1}{2}}, \quad \|F\|_{N^{s,l,\theta-1}} = \Big(\sum_{\lambda \in 2^{\mathbb{N}_0}} \|F_\lambda\|^2_{N^{s,l,\theta-1}_\lambda} \Big)^{\frac{1}{2}},$$

and that of the wave solution and wave nonlinearity is controlled by

$$\|v\|_{W^{l,s,\theta}} = \Big(\sum_{\lambda \in 2^{\mathbb{N}_0}} \|v_\lambda\|^2_{W^{l,s,\theta}_\lambda} \Big)^{\frac{1}{2}}, \quad \|G\|_{R^{l,s,\theta-1}} = \Big(\sum_{\lambda \in 2^{\mathbb{N}_0}} \|G_\lambda\|^2_{R^{l,s,\theta-1}_\lambda} \Big)^{\frac{1}{2}},$$

respectively. For $0 < T \leqslant 1$, we localise the norms in time by defining

$$\|u\|_{S^{s,l,\theta}(T)} = \inf\{\|\tilde{u}\|_{S^{s,l,\theta}} : \tilde{u} \in S^{s,l,\theta}, \tilde{u}|_{(0,T)\times\mathbb{R}^d} = u\}.$$

The norms $N^{s,l,\theta-1}(T), W^{l,s,\theta}(T)$ and $R^{l,s,\theta-1}(T)$ are defined similarly. We end the section by stating a product rule for fractional derivatives in time which we will use to prove linear and nonlinear estimates in the next sections.

3.3.2 A product estimate for fractional derivatives

In the new function spaces defined, the temporal weight $(\lambda+|\partial_t|)^a$ plays a crucial role. The following result from [14] enables us to distribute this derivative on to the two functions with a gain of λ^{-a}.

Lemma 3.3.2 ([14, Lemma 2.7])**.** *Let* $a \in \mathbb{R}, 1 \leqslant \tilde{p}, \tilde{q}, \tilde{r}, p, q, r \leqslant \infty$ *with*

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r} \quad \textit{and} \quad \frac{1}{\tilde{p}} = \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}}.$$

Then, for all $\mu > 0$,

$$\|(\mu+|\partial_t|)^a(vu)\|_{L^{\tilde{p}}_t L^p_x} \lesssim \mu^{-|a|}\|(\mu+|\partial_t|)^{|a|}v\|_{L^{\tilde{r}}_t L^r_x}\|(\mu+|\partial_t|)^{|a|}u\|_{L^{\tilde{q}}_t L^q_x}.$$

3.4 Resonance and transversality

The crucial building blocks to the well-posedness result are multilinear estimates. In this section, we recall some estimates that are already known. Consider the following interaction for the Zakharov system: when two high frequency Schrödinger solutions and a low frequency wave solution interact. The dispersion relations for the Schrödinger equation and the half wave equations are given by

$$\omega_S(\xi) = -|\xi|^2,$$
$$\omega_W(\xi) = |\xi|.$$

The resonance function

$$|\Omega_{ZS}(\xi_1, \xi_2)| = ||\xi_1|^2 - |\xi_2|^2 - |\xi_1 + \xi_2||, \tag{3.4.1}$$

can become very small for $|\xi_1 + \xi_2| \lesssim |\xi_1| \sim |\xi_2|$. In this case, we exploit the transversality of the interacting waves to obtain good multilinear estimates. In the present case, we have two paraboloids and a cone as the characteristic hypersurfaces which are the distributional supports of the waves leading to a resonant interaction. The determinant of the normals at any points on these hypersurfaces is large. This allows to prove a trilinear estimate with smoothing. This was proved in [4] for $d = 2$ and in [3] for $d = 3$. To prove the result, one requires a nonlinear version of the *Loomis-Whitney inequality.* The classical Loomis-Whitney inequality is an isoperimetric inequality which bounds the d-dimensional measure of a subset O of $\mathbb{R}^d$ in terms of the $(d-1)$-dimensional measures of its projections onto the coordinate hyperplanes, i.e. $|O| \lesssim |\partial O|^{\frac{d}{d-1}}$. It can also be formulated as follows:

Theorem 3.4.1 (Classical Loomis-Whitney inequality, [46]). *For $x \in \mathbb{R}^d$, let $\pi_j : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be given by $\pi_j(x) = (x_1, x_2, ..., \tilde{x_j}, ..., x_d)$*[2]*, i.e. the orthogonal projection onto the coordinate hyperplane $e_j^\perp$. Then for all $f_j \in L^{d-1}(\mathbb{R}^{d-1})$, we have*

$$\int_{\mathbb{R}^d} f_1(\pi_1(x))...f_d(\pi_d(x))dx \leqslant \|f_1\|_{L^{d-1}(\mathbb{R}^{d-1})}...\|f_d\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

When the coordinate hyperplanes $e_j^\perp$ are replaced by (bounded) sufficiently smooth hypersurfaces in $\mathbb{R}^d$, the above is referred to as nonlinear Loomis-Whitney inequality [6]. These nonlinear generalisations become useful in the context of dispersive partial differential equations when one tries to control a bilinear interaction.

Definition 3.4.2. Let $\{S_i\}_{i=1}^d$ be a family of hypersurfaces in $\mathbb{R}^d$ and $\nu_i : S_i \to \mathbb{S}^{d-1}$ be the associated Gauss map. We say that the family of hypersurfaces is *κ-transversal* for some $\kappa > 0$ if

$$|det(\nu_1(x_1), ..., \nu_d(x_d))| \geqslant \kappa, \text{ whenever } x_i \in S_i, 1 \leqslant i \leqslant d.$$

Now, we state the setting and result from [5].

Assumption 3.4: For $i = 1, 2, 3$, there exist $0 < \beta \leqslant 1, b > 0$, $\theta \geqslant 0$ and $\Sigma_i^* \subset \mathbb{R}^3$ such that

(i) the oriented surface Σ_i^* is given as

$$\Sigma_i^* = \{\sigma_i \in U_i \big| \Phi_i(\sigma_i) = 0, \nabla\Phi_i \neq 0, \Phi_i \in C^{1,\beta}(U_i)\},$$

for a convex $U_i \subset \mathbb{R}^3$ and Σ_i is an open and bounded subset of Σ_i^* such that $dist(\Sigma_i, U_i^c) \geqslant diam(\Sigma_i)$.

(ii) the unit normal vector field $\mathfrak{n}_i$ on Σ_i^* satisfies the Hölder condition

$$\sup_{\sigma,\tilde{\sigma}\in\Sigma_i^*} \frac{|\mathfrak{n}_i(\sigma) - \mathfrak{n}_i(\tilde{\sigma})|}{|\sigma - \tilde{\sigma}|^\beta} + \frac{|\mathfrak{n}_i(\sigma)(\sigma - \tilde{\sigma})|}{|\sigma - \tilde{\sigma}|^{1+\beta}} \leqslant b. \tag{3.4.2}$$

(iii) the matrix $N(\sigma_1, \sigma_2, \sigma_3) = (\mathfrak{n}_1(\sigma_1), \mathfrak{n}_2(\sigma_2), \mathfrak{n}_3(\sigma_3))$ of the unit normals to the surfaces satisfies the transversality condition

$$\theta \leqslant detN(\sigma_1, \sigma_2, \sigma_3) \leqslant 1, \tag{3.4.3}$$

for all $(\sigma_1, \sigma_2, \sigma_3) \in \Sigma_1^* \times \Sigma_2^* \times \Sigma_3^*$.

Theorem 3.4.3 ([5, Theorem 1.2]). *Let $\Sigma_1, \Sigma_2, \Sigma_3$ be surfaces in $\mathbb{R}^3$ which satisfy Assumption 3.4 with parameters $0 < \beta \leqslant 1, b = 1$ and $\theta = \frac{1}{2}$ with diam $\Sigma_i \leqslant 1$. Then for each $f \in L^2(\Sigma_1)$ and $g \in L^2(\Sigma_2)$, the restriction of the convolution $f * g$ to Σ_3 is a well-defined $L^2(\Sigma_3)$ function which satisfies*

$$\|f * g\|_{L^2(\Sigma_3)} \leqslant C\|f\|_{L^2(\Sigma_1)}\|g\|_{L^2(\Sigma_2)}, \tag{3.4.4}$$

[2] $\tilde{x_j}$ denotes omitting x_j

the constant C depending only on β.

As a consequence, we have

Theorem 3.4.4 ([4, Proposition 4.5])**.** *Let C_1, C_2, C_3 be cubes in $\mathbb{R}^3$ of diameter $2R > 0$. Consider two paraboloids in $\mathbb{R}^3$ which are graphs of $\phi_1, \phi_2 \in C^{1,1}$ and a cone in $\mathbb{R}^3$ which is a graph of $\phi_3 \in C^{1,1}$ within C_3, such that the homogeneous semi-norms satisfy $[\phi_j]_{C^{1,1}} \lesssim 1$. Moreover, assume that they are transverse in the sense that the determinant of every triple of unit normals to the points on the surfaces within these cubes is of size $\theta > 0$ and suppose that $R \lesssim \theta$. Then, for given subsets $\Sigma_1, \Sigma_2, \Sigma_3$ of the above surfaces which are contained in the $\frac{1}{2}$-shrinked cubes with the same center and for each $f \in L^2(\Sigma_1)$ and $g \in L^2(\Sigma_2)$, the restriction of the convolution $f * g$ to Σ_3 is a well-defined $L^2(\Sigma_3)$ function which satisfies*

$$\|f * g\|_{L^2(\Sigma_3)} \leqslant \frac{C}{\theta^{1/2}} \|f\|_{L^2(\Sigma_1)} \|g\|_{L^2(\Sigma_2)}.$$

The above result leads to the following:

Lemma 3.4.5 ([4, Proposition 4.4])**.** *Let $d = 2$ and $f, g_1, g_2 \in L^2(\mathbb{R}^3)$ be such that $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$. For $k = 1, 2$ let*

$$\begin{aligned} supp(f) &\subset \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 : \frac{\lambda}{2} \leqslant |\xi| \leqslant 2\lambda \right\} \cap \mathcal{W}_L^{\pm} \\ supp(g_k) &\subset \left\{ (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 : \frac{\lambda_k}{2} \leqslant |\xi| \leqslant 2\lambda_k \right\} \cap \mathcal{S}_{L_k}, \end{aligned}$$

where the frequencies $\lambda, \lambda_1, \lambda_2$ and the modulations L, L_1, L_2 satisfy

$$1 \ll \lambda \lesssim \lambda_1 \sim \lambda_2, \quad L, L_1, L_2 \lesssim \lambda_1^2.$$

Then, for

$$I(f, g_1, g_2) = \int f(\zeta_1 - \zeta_2) g_1(\zeta_1) g_2(\zeta_2) d\zeta_1 d\zeta_2, \quad \zeta_i = (\tau_i, \xi_i), i = 1, 2, \tag{3.4.5}$$

the following estimate holds

$$|I(f, g_1, g_2)| \lesssim \frac{(L L_1 L_2)^{\frac{1}{2}}}{\lambda_1^{\frac{1}{2}}}. \tag{3.4.6}$$

An alternate proof of the above is provided in Appendix E using the nonlinear Loomis-Whitney inequality without locality assumptions on Σ_i from [39]. To handle the $d = 3$ case, we state the following result from [3] which is another application of the nonlinear Loomis-Whitney inequality.

Lemma 3.4.6 ([3, Corollary 3.6])**.** *Let $d = 3$ and the assumptions of Lemma 3.4.5 hold. Then, the following estimate holds:*

$$|I(f, g_1, g_2)| \lesssim \frac{(L L_1 L_2)^{\frac{1}{2}}}{\lambda_1^{\frac{1}{2}}} \log \lambda_1. \tag{3.4.7}$$

Remark 3.4.7. Comparing (3.4.6) and (3.4.7), we note the additional log term in the latter, i.e. for $d = 3$. In Section 3.6, when treating the resonant cases, we do not differentiate $d = 2$ and $d = 3$ and use the version with the logarithmic term to treat $d = 2$

and $d = 3$ simultaneously. This does not affect the analysis because we do not cover the endpoints.

Now, we prove another set of important estimates, namely the bilinear Strichartz estimates. The $L^2_{t,x}$ estimate for the Schrödinger equation was proved by Bourgain, [11, Lemma 111]. In general, we have the following:

Lemma 3.4.8 ([13, Lemma 2.6])**.** *Let $0 < r < 1$ and $f, g \in L^2_x(\mathbb{R}^d)$. Assume that the spatial Fourier supports of f and g are contained in balls of radius r and $\forall \xi \in supp(\hat{f}), \eta \in supp(\hat{g})$,*

$$|\nabla h_1(\xi) - \nabla h_2(\eta)| \geqslant C_0. \tag{3.4.8}$$

Then,

$$\|e^{ith_1(\nabla/i)} f e^{ith_2(\nabla/i)} g\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim \left(\frac{r^{d-1}}{C_0}\right)^{\frac{1}{2}} \|f\|_{L^2_x(\mathbb{R}^d)} \|g\|_{L^2_x(\mathbb{R}^d)}.$$

In spirit of the above, we have the following bilinear estimates.

Lemma 3.4.9 (Bilinear Strichartz estimates, [4, Proposition 4.3], [3, Proposition 3.3])**.** *For $i = 1, 2$, let $u_i, v \in L^2(\mathbb{R} \times \mathbb{R}^d)$. Let $\hat{u}_i, \hat{v}$ denote the space-time Fourier transforms of u_i, v, respectively. Then*

(a) (Schrödinger-Schrödinger) Let $d \in \{2, 3\}$ and u_i be dyadically Fourier-localised such that

$$supp(\hat{u}_i) \subset \left\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \frac{\lambda_i}{2} \leqslant |\xi| \leqslant 2\lambda_i\right\} \cap \mathcal{S}_{L_i}$$

for $L_i, \lambda_i \geqslant 1$. Then, the following estimate holds:

$$\|u_1 u_2\|_{L^2_{t,x}} \lesssim \frac{\min(\lambda_1, \lambda_2)^{\frac{d-1}{2}}}{\max(\lambda_1, \lambda_2)^{\frac{1}{2}}} (L_1 L_2)^{\frac{1}{2}} \|u_1\|_{L^2_{t,x}} \|u_2\|_{L^2_{t,x}}. \tag{3.4.9}$$

(b) (Schrödinger-Wave) Let $d \leqslant 3$ and u, v be dyadically Fourier-localised such that

$$supp(\hat{u}) \subset \left\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \frac{\lambda_1}{2} \leqslant |\xi| \leqslant 2\lambda_1\right\} \cap \mathcal{S}_{L_1},$$
$$supp(\hat{v}) \subset \left\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d : \frac{\lambda_2}{2} \leqslant |\xi| \leqslant 2\lambda_2\right\} \cap \mathcal{W}^{\pm}_{L_2}$$

for $L_i, \lambda_i \geqslant 1$. Then, the following estimate holds:

$$\|uv\|_{L^2_{t,x}} \lesssim \frac{\min(\lambda_1, \lambda_2)^{\frac{d-1}{2}}}{\lambda_1^{\frac{1}{2}}} (L_1 L_2)^{\frac{1}{2}} \|u\|_{L^2_{t,x}} \|v\|_{L^2_{t,x}}. \tag{3.4.10}$$

Before proving the above estimates, we prove an elementary result.

Lemma 3.4.10. *Let I, J be intervals in $\mathbb{R}$ and $f : J \to \mathbb{R}$ be a smooth function. Then*

$$|\{x : f(x) \in I\}| \leqslant \frac{|I|}{\inf_y |f'(y)|}. \tag{3.4.11}$$

Proof. The proof follows from the mean value theorem. Let $x_1, x_2 \in J$ be such that

$f(x_1), f(x_2) \in I$. Then

$$|x_1 - x_2| = \frac{|f(x_1) - f(x_2)|}{|f'(\xi)|} \leqslant \frac{|I|}{\inf_y |f'(y)|}.$$

□

Proof of Lemma 3.4.9. *(a)* We prove the above for $d = 2$ in the case $\lambda_2 \ll \lambda_1$; the proof for $d = 3$ being analogous. Using Plancherel's identity and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|u_1 u_2\|_{L^2_{t,x}(\mathbb{R}^3)} &= \Big\| \int \hat{u}_1(\tau_1, \xi_1)\hat{u}_2(\tau - \tau_1, \xi - \xi_1) d\tau_1 d\xi_1 \Big\|_{L^2_{\tau,\xi}(\mathbb{R}^3)} \\ &\lesssim \sup_{\tau,\xi} |E(\tau, \xi)|^{\frac{1}{2}} \|u_1\|_{L^2} \|u_2\|_{L^2},\end{aligned}$$

where $E(\tau, \xi) \subset \mathbb{R}^3$ is the set

$$E(\tau, \xi) = \{(\tau_1, \xi_1) \in \textit{supp } \hat{u}_1 : (\tau - \tau_1, \xi - \xi_1) \in \textit{supp } \hat{u}_2\}.$$

We use Fubini's theorem to estimate the volume of the set E. Let

$$\underline{L} := \min\{L_1, L_2\} \quad \text{and} \quad \bar{L} := \max\{L_1, L_2\}.$$

Then

$$|E(\tau, \xi)| \lesssim \underline{L}\ |\{\xi_1 :\ |\tau \pm |\xi_1|^2 + |\xi - \xi_1|^2| \lesssim \bar{L},\ |\xi_1| \sim \lambda_1, |\xi - \xi_1| \sim \lambda_2\}| =: \underline{L}|E_1(\xi)|.$$

$E_1 \subset \mathbb{R}^2$ is contained in a cube of length $m = \min(\lambda_1, \lambda_2)$. Using this observation and orthogonality, we prove the estimate in the case where either λ_1 or λ_2 is of size ~ 1. If this is not the case, suppose that the first component ξ_{11} of $\xi_1 = (\xi_{11}, \xi_{12})$ is fixed. This gives that the second component is confined to an interval of length m. We have

$$|\partial_{\xi_{11}}(\tau \pm |\xi_1|^2 + |\xi - \xi_1|^2)| \sim |\xi_{11} - (\xi - \xi_1)_1| \gtrsim \lambda_1, \tag{3.4.12}$$

in the subset where $|\xi_{11}| \gtrsim \lambda_1$. Using Lemma 3.4.10, this gives

$$|E_1(\xi)| = \Big| \int \Big(\int \mathbf{1}_{E_1(\xi)}(\xi_{11}, \xi_{12}) d\xi_{11} \Big) d\xi_{12} \Big| \lesssim \frac{m\bar{L}}{\lambda_1}.$$

If $|\xi_{12}| \gtrsim \lambda_1$, we fix the second component ξ_{12} and compute the derivative in (3.4.12) with respect to ξ_{12} and obtain the same bound on the measure of the set $E_1(\xi)$. Hence,

$$|E(\tau, \xi)| = \frac{m\underline{L}\bar{L}}{\lambda_1}$$

which implies the result.
(b) The proof follows on the same lines as for part *(a)*. In this case, we need to estimate the volume of the set

$$E(\tau, \xi) = \{(\tau_1, \xi_1) \in \textit{supp } \hat{u} |\ (\tau - \tau_1, \xi - \xi_1) \in \textit{supp } \hat{v}\}.$$

As in the previous case, we have

$$|E(\tau,\xi)| \lesssim \underline{L}\ |\{\xi_1 \big| |\tau \pm |\xi_1| + |\xi-\xi_1|^2| \lesssim \bar{l}, |\xi_1| \sim \lambda, |\xi-\xi_1| \sim \lambda_1\}|,$$
$$\text{and } |\{\xi_1 \big| |\tau \pm |\xi_1| + |\xi-\xi_1|^2| \lesssim \bar{l}, |\xi_1| \sim \lambda, |\xi-\xi_1| \sim \lambda_1\}| \lesssim \frac{m\bar{L}}{\lambda_1},$$

since $|\partial_{\xi_{11}}(\tau \pm |\xi_1| + |\xi-\xi_1|^2)| \gtrsim \lambda_1$ in the case where $|(\xi-\xi_1)_1| \gtrsim \lambda_1$, if the first component ξ_{11} is kept fixed. This gives the desired result. □

Remark 3.4.11. (i) If the frequencies λ_1 and λ_2 are such that $\lambda_1 \ll \lambda_2$ (or $\lambda_2 \ll \lambda_1$), then the estimate (3.4.9) holds for $d = 1$ as well.

(ii) The estimates (3.4.9) and (3.4.10) remain valid if we replace the functions on the left-hand side (LHS) by their complex conjugates.

(iii) An alternative proof of Lemma 3.4.9 *(a)* can be found in [65].

(iv) For $d = 2, 3$, the estimate (3.4.9) also follows as a consequence of Bernstein's inequality, Strichartz estimates (2.5.10) and Lemma 2.7.3.

3.5 Linear estimates

Let $\mathcal{I}_S$ and $\mathcal{I}_W$ be the solution operators for the inhomogeneous Schrödinger equation and the half wave equation, respectively, i.e.

$$\mathcal{I}_S[F](t) = -i\int_0^t e^{i(t-s)\Delta}F(s)ds, \quad \mathcal{I}_W[G](t) = i\int_0^t e^{i(t-s)|\nabla|}G(s)ds.$$

Recall the definition of η from Chapter 2: $\eta \in C_c^\infty(\mathbb{R})$ is a non-negative smooth function supported on $(-2,2)$ such that $\eta(t) = 1$ for $t \in [-1,1]$. We prove the following linear estimates.

Lemma 3.5.1 (Energy inequality for Schrödinger equation)**.** *Let $a, b, s', \theta' \in \mathbb{R}$ be as defined in (3.3.1). For any $\lambda \in 2^{\mathbb{N}_0}$, η as above and $e^{it\Delta}u_0$ supported in frequencies of size $\sim \lambda$, the following estimates hold:*

$$\|\eta(t)e^{it\Delta}u_0\|_{S_\lambda^{s,l,\theta}} \lesssim \lambda^s\|u_0\|_{L_x^2},$$
$$\|\eta(t)\mathcal{I}_S[F_\lambda]\|_{S_\lambda^{s,l,\theta}} \lesssim \|F_\lambda\|_{N_\lambda^{s,l,\theta-1}}.$$

Proof. We first note the following property of the linear group $e^{it\Delta}$ with respect to temporal frequency and modulation localisation, which we shall use repeatedly, (see Appendix B for a proof)

$$C_\star f = e^{tL}P_\star^{(t)}(e^{-tL}f). \tag{3.5.1}$$

The $S_\lambda^{s,l,\theta}$ norm for the time localised free solution is given by

$$\|\eta(t)e^{it\Delta}u_0\|_{S_\lambda^{s,l,\theta}} = \lambda^s\|\eta(t)e^{it\Delta}u_0\|_{L_t^\infty L_x^2} + \lambda^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\lambda^2}\eta(t)e^{it\Delta}u_0)\|_{L^2_{\tau,\xi}}$$
$$+ \lambda^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda+|\partial_t|)^a C_{\gtrsim\lambda^2}\eta(t)e^{it\Delta}u_0)\|_{L^2_{\tau,\xi}}.$$

Using (2.7.5) and (2.7.2), we have

$$\lambda^s\|\eta(t)e^{it\Delta}u_0\|_{L_t^\infty L_x^2} \lesssim \lambda^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(\eta(t)e^{it\Delta}u_0)\|_{L^2_{\tau,\xi}} \lesssim \lambda^s\|u_0\|_{L_x^2},$$

and for the second term using (3.5.1) and (2.7.2), we have

$$\begin{aligned}\lambda^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\lambda^2}\eta(t)e^{it\Delta}u_0)\|_{L^2_{\tau,\xi}} &= \lambda^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(e^{it\Delta}P^{(t)}_{\ll\lambda^2}(\eta(t)u_0))\|_{L^2_{\tau,\xi}} \\ &\lesssim \lambda^s\|u_0\|_{L_x^2}.\end{aligned}$$

A similar computation gives

$$\begin{aligned}&\lambda^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda+|\partial_t|)^aC_{\gtrsim\lambda^2}\eta(t)e^{it\Delta}u_0)\|_{L^2_{\tau,\xi}} \\ &= \lambda^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'+a}\mathcal{F}(e^{it\Delta}P^{(t)}_{\gtrsim\lambda^2}\eta(t)u_0)\|_{L^2_{\tau,\xi}} \\ &\lesssim \lambda^s\|u_0\|_{L_x^2},\end{aligned}$$

where we used $s'+b\leqslant s$ to obtain the last inequality. Now, we consider the S_λ norm of the Duhamel integral. The $L_t^\infty L_x^2$ term is decomposed as follows:

$$\lambda^s\Big\|\eta(t)\int_0^t e^{i(t-s)\Delta}C_{\ll\lambda^2}F_\lambda(s)ds\Big\|_{L_t^\infty L_x^2} + \lambda^s\Big\|\eta(t)\int_0^t e^{i(t-s)\Delta}C_{\gtrsim\lambda^2}F_\lambda(s)ds\Big\|_{L_t^\infty L_x^2}. \quad (3.5.2)$$

Using equation (2.7.5) and Lemma 2.7.7 respectively, the first term can be bounded by

$$\lambda^s\|\langle\tau+|\xi|^2\rangle^{\theta-1}\mathcal{F}(C_{\ll\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}}.$$

Before proceeding further, we note that the following holds, [14, Lemma 2.4]:

$$\|\mathcal{I}_S[C_{\gtrsim\mu}G]\|_{L_t^\infty L_x^2} \lesssim \mu^{-1}\|C_{\gtrsim\mu}G\|_{L_t^\infty L_x^2}. \quad (3.5.3)$$

Now, the second term in (3.5.2) can be further written as

$$\lambda^s\Big\|\eta(t)\int_0^t e^{i(t-s)\Delta}C_{\sim\lambda^2}P^{(t)}_{\ll\lambda^2}F_\lambda(s)ds\Big\|_{L_t^\infty L_x^2} + \lambda^s\Big\|\eta(t)\int_0^t e^{i(t-s)\Delta}C_{\gtrsim\lambda^2}P^{(t)}_{\gtrsim\lambda^2}F_\lambda(s)ds\Big\|_{L_t^\infty L_x^2}.$$

Using (3.5.3), the first term above can be bounded by

$$\lambda^{s-2}\|P^{(t)}_{\ll\lambda^2}F_\lambda\|_{L_t^\infty L_x^2},$$

which is sufficient since $\theta>\frac{1}{2}$. For the second term, (2.7.5) and Lemma 2.7.7 can be used to obtain the bound

$$\lambda^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda+|\partial_t|)^aC_{\gtrsim\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}}$$

by noting that the temporal weight $|m_S(\tau)|\gtrsim 1$ and $b\geqslant 0$, while $s'=s+2\theta-2$ when $\theta'=1$.

The low modulation norm of the Duhamel integral is decomposed as follows:

$$\lambda^s\Big\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\lambda^2}\eta(t)\int_0^t e^{i(t-s)\Delta}C_{\ll\lambda^2}F_\lambda(s)ds)\Big\|_{L^2_{\tau,\xi}}$$

$$+ \lambda^s \Big\| \langle \tau + |\xi|^2 \rangle^\theta \mathcal{F}(C_{\ll \lambda^2} \eta(t) \int_0^t e^{i(t-s)\Delta} C_{\gtrsim \lambda^2} F_\lambda(s) ds) \Big\|_{L^2_{\tau,\xi}} =: \text{(I)}+\text{(II)}.$$

An application of Lemma 2.7.7 gives that we can bound (I) by

$$\lambda^s \| \langle \tau + |\xi|^2 \rangle^{\theta-1} \mathcal{F}(C_{\ll \lambda^2} F_\lambda) \|_{L^2_{\tau,\xi}}.$$

To handle (II), we consider the following cases pertaining to the high modulation norm for the Schrödinger nonlinearity:
(i) $\boldsymbol{a = 0 = b}$ **:** A straightforward application of Lemma 2.7.7 gives

$$\text{(II)} \lesssim \lambda^s \| \langle \tau + |\xi|^2 \rangle^{\theta-1} \mathcal{F}(C_{\gtrsim \lambda^2} F_\lambda) \|_{L^2_{\tau,\xi}}.$$

(ii) $\boldsymbol{a > 0, b = 0}$ **:** We note that we can bound (II) by

$$\lambda^{s'-2a} \| \langle \tau + |\xi|^2 \rangle^{\theta'-1} \mathcal{F}((\lambda + |\partial_t|)^a C_{\gtrsim \lambda^2} F_\lambda) \|_{L^2_{\tau,\xi}},$$

using Lemma 2.7.7, provided the temporal frequencies of F_λ are of size $\gtrsim \lambda^2$ as this allows us to insert the weight $m_S(\tau)$ to the norm. We consider the case when the temporal frequencies of F_λ are of size $\ll \lambda^2$. Note that this implies that F_λ has a modulation of size $\sim \lambda^2$. We use (3.5.1) and Sobolev embedding to obtain

$$\begin{aligned}
\text{(II)} &= \lambda^s \Big\| P^{(t)}_{\ll \lambda^2} (\eta(t) \int_0^t P^{(s)}_{\sim \lambda^2} e^{-is\Delta} F_\lambda(s) ds) \Big\|_{H^\theta_t L^2_x} \\
&\lesssim \lambda^s \Big\| P^{(t)}_{\ll \lambda^2} (\eta(t) \int_0^t P^{(s)}_{\sim \lambda^2} e^{-is\Delta} F_\lambda(s) ds) \Big\|_{W^{1,p}_t L^2_x}, \quad p = \frac{2}{3-2\theta} \\
&\lesssim \lambda^s \Big(\Big\| P^{(t)}_{\ll \lambda^2} (\eta(t) \int_0^t P^{(s)}_{\sim \lambda^2} e^{-is\Delta} F_\lambda(s) ds) \Big\|_{L^p_t L^2_x} \\
&\quad + \Big\| P^{(t)}_{\ll \lambda^2} (\eta(t) \int_0^t P^{(s)}_{\sim \lambda^2} e^{-is\Delta} F_\lambda(s) ds)' \Big\|_{L^p_t L^2_x} \Big) \\
&=: \text{(II.1)}+\text{(II.2)},
\end{aligned}$$

where the prime denotes derivative with respect to time. Using Hölder's inequality, unitarity of $e^{it\Delta}$ and (3.5.3), we have

$$\text{(II.1)} \lesssim \lambda^s \|\eta(t)\|_{L^p_t L^\infty_x} \Big\| \int_0^t P^{(s)}_{\sim \lambda^2} e^{-is\Delta} F_\lambda(s) ds \Big\|_{L^\infty_t L^2_x} \lesssim \lambda^{s-2} \| P^{(t)}_{\ll \lambda^2} F_\lambda \|_{L^\infty_t L^2_x}.$$

Applying the product rule, (II.2) is further written as

$$\text{(II.2)} \lesssim \lambda^s \Big(\Big\| P^{(t)}_{\ll \lambda^2} (\eta'(t) \int_0^t P^{(s)}_{\sim \lambda^2} e^{-is\Delta} F_\lambda(s) ds) \Big\|_{L^p_t L^2_x} + \| P^{(t)}_{\ll \lambda^2} (\eta(t) P^{(t)}_{\sim \lambda^2} e^{-it\Delta} F_\lambda) \|_{L^p_t L^2_x} \Big).$$

The first term above can be handled like (II.1). The second term is equivalent to

$$\lambda^s \| P^{(t)}_{\ll \lambda^2} (P^{(t)}_{\sim \lambda^2} \eta(t) P^{(t)}_{\sim \lambda^2} e^{-it\Delta} F_\lambda) \|_{L^p_t L^2_x}.$$

We use Bernstein's and Hölder's inequalities to obtain

$$\lambda^s \| P^{(t)}_{\ll \lambda^2} (P^{(t)}_{\sim \lambda^2} \eta(t) P^{(t)}_{\sim \lambda^2} e^{-it\Delta} F_\lambda) \|_{L^p_t L^2_x} \lesssim \lambda^{s+2\theta-1} \| P^{(t)}_{\ll \lambda^2} (P^{(t)}_{\sim \lambda^2} \eta(t) P^{(t)}_{\sim \lambda^2} e^{-it\Delta} F_\lambda) \|_{L^1_t L^2_x}$$

$$\begin{aligned}&\lesssim \lambda^{s+2\theta-1}\|P^{(t)}_{\sim\lambda^2}\eta(t)\|_{L^1_tL^\infty_x}\|P^{(t)}_{\sim\lambda^2}e^{-it\Delta}F_\lambda\|_{L^\infty_tL^2_x}\\&\lesssim \lambda^{s+2\theta-3}\|C_{\sim\lambda^2}F_\lambda\|_{L^\infty_tL^2_x}\\&\lesssim \|F_\lambda\|_{N^{s,l,\theta-1}_\lambda},\end{aligned}$$

where in the last inequality we use that the L^1 norm of a time cut-off at high temporal frequencies ($\gtrsim \lambda^2$) is $\lesssim \lambda^{-2}$ (see Appendix D).
(iii) $\boldsymbol{b>0, a=0}$ **:** As for case (i), we have

$$\text{(II)} \lesssim \lambda^s\|\langle\tau+|\xi|^2\rangle^{\theta-1}\mathcal{F}(C_{\gtrsim\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}} \lesssim \lambda^{s+2\theta-2}\|\mathcal{F}(C_{\gtrsim\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}} \lesssim \|F_\lambda\|_{N^{s,l,\theta-1}_\lambda},$$

where the last inequality follows by noting that $b>0$ and $s'=s+2\theta-2$ for the given case.
For the high modulation norm of the Duhamel integral, we again use the decomposition

$$\begin{aligned}&\lambda^{s'-2a+b}\Big\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda+|\partial_t|)^aC_{\gtrsim\lambda^2}\eta(t)\int_0^t e^{i(t-s)\Delta}C_{\gtrsim\lambda^2}F_\lambda(s)ds)\Big\|_{L^2_{\tau,\xi}}\\&\quad+\lambda^{s'-2a+b}\Big\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda+|\partial_t|)^aC_{\gtrsim\lambda^2}\eta(t)\int_0^t e^{i(t-s)\Delta}C_{\ll\lambda^2}F_\lambda(s)ds)\Big\|_{L^2_{\tau,\xi}}\\&=:\text{(III)+(IV)}.\end{aligned}$$

For (III), Lemma 2.7.7 suffices. For (IV), we consider three cases:
(i) $\boldsymbol{a=0=b}$ **:** Using the definition of the norm and Lemma 2.7.7,

$$\text{(IV)} \lesssim \lambda^s\|\langle\tau+|\xi|^2\rangle^{\theta-1}\mathcal{F}(C_{\ll\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}} \lesssim \|F_\lambda\|_{N^{s,l,\theta-1}_\lambda}.$$

(ii) $\boldsymbol{a>0, b=0}$ **:** We prove the estimate for $a=1$. Interpolation with the case $a=0$ then leads us to the desired result. Using the definition of the S_λ norm for $a=1$, property (3.5.1) and Sobolev embedding, we obtain

$$\begin{aligned}\text{(IV)} &= \lambda^{s-2}\Big\|\langle\tau+|\xi|^2\rangle^{\theta}\mathcal{F}((\lambda+|\partial_t|)C_{\gtrsim\lambda^2}\eta(t)\int_0^t e^{i(t-s)\Delta}C_{\ll\lambda^2}F_\lambda(s)ds)\Big\|_{L^2_{\tau,\xi}}\\&\lesssim \lambda^{s-2}\Big\|(\lambda+|\partial_t|)(P^{(t)}_{\gtrsim\lambda^2}\eta(t)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is\Delta}F_\lambda(s)ds)\Big\|_{H^\theta_tL^2_x}\\&\lesssim \lambda^{s-2}\Big\|(\lambda+|\partial_t|)(P^{(t)}_{\gtrsim\lambda^2}\eta(t)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is\Delta}F_\lambda(s)ds)\Big\|_{W^{1,p}_tL^2_x},\quad p=\frac{2}{3-2\theta}\\&\lesssim \lambda^{s-2}\Big\|(\lambda+|\partial_t|)(P^{(t)}_{\gtrsim\lambda^2}\eta(t)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is\Delta}F_\lambda(s)ds)\Big\|_{L^p_tL^2_x}\\&\quad+\lambda^{s-2}\Big\|(\lambda+|\partial_t|)(P^{(t)}_{\gtrsim\lambda^2}\eta(t)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is\Delta}F_\lambda(s)ds)'\Big\|_{L^p_tL^2_x}\\&=:\text{(IV.1)+(IV.2)},\end{aligned}$$

where the prime denotes derivative with respect to t.
Using the product estimate (Lemma 3.3.2) for $a=1$, $\frac{1}{p}=\frac{1}{2}+\frac{1}{q}$, we have

$$\begin{aligned}\text{(IV.1)} &\lesssim \lambda^{s-3}\|(\lambda+|\partial_t|)\eta(t)\|_{L^q_tL^\infty_x}\Big\|(\lambda+|\partial_t|)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is\Delta}F_\lambda(s)ds\Big\|_{L^2_{t,x}}\\&\lesssim \lambda^s\|\langle\tau+|\xi|^2\rangle^{\theta-1}\mathcal{F}(C_{\ll\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}}.\end{aligned}$$

On applying the product rule to (IV.2), we find that the term for which the derivative hits the smooth time cut-off η is similar to (IV.1). Using the product estimate for $\frac{1}{p} = \frac{1}{q} + \frac{1}{2}$, the second term can be bounded by

$$\begin{aligned}
&\lambda^{s-2}\|(\lambda + |\partial_t|)P^{(t)}_{\gtrsim\lambda^2}(\eta(t)P^{(t)}_{\ll\lambda^2}e^{-it\Delta}F_\lambda)\|_{L^p_tL^2_x}\\
&\lesssim \lambda^{s-3}\|(\lambda + |\partial_t|)\eta(t)\|_{L^q_tL^\infty_x}\|(\lambda + |\partial_t|)P^{(t)}_{\ll\lambda^2}e^{-it\Delta}F_\lambda\|_{L^2_{t,x}}\\
&\lesssim \lambda^{s-1}\|C_{\ll\lambda^2}F_\lambda\|_{L^2_{t,x}}\\
&\lesssim \lambda^{s+1-2\theta}\|\langle\tau + |\xi|^2\rangle^{\theta-1}\mathcal{F}(C_{\ll\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}}\\
&\lesssim \|F_\lambda\|_{N^{s,l,\theta-1}_\lambda}.
\end{aligned}$$

(iii) $\boldsymbol{b > 0, a = 0}$ **:** In this case, the term (IV) reads

$$\lambda^{s+2\theta-2+b}\Big\|\langle\tau + |\xi|^2\rangle\mathcal{F}(C_{\gtrsim\lambda^2}\eta(t)\int_0^t e^{i(t-s)\Delta}C_{\ll\lambda^2}F_\lambda(s)ds)\Big\|_{L^2_{\tau,\xi}}.$$

Using (3.5.1), with the prime denoting derivative with respect to time, we have

$$\begin{aligned}
\text{(IV)} &\lesssim \lambda^{s+2\theta-2+b}\Big\|P^{(t)}_{\gtrsim\lambda^2}(\eta(t)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is\Delta}F_\lambda(s)ds)\Big\|_{L^2_{t,x}}\\
&+ \lambda^{s+2\theta-2+b}\Big\|(P^{(t)}_{\gtrsim\lambda^2}(\eta(t)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is\Delta}F_\lambda(s)ds))'\Big\|_{L^2_{t,x}}\\
&=: \text{(IV.3)}+\text{(IV.4)}.
\end{aligned}$$

Unitarity of $e^{it\Delta}$ and an application of Lemma 2.7.7 give

$$\begin{aligned}
\text{(IV.3)} &\lesssim \lambda^{s+2\theta-2+b}\Big\|e^{-it\Delta}\eta(t)\int_0^t e^{i(t-s)\Delta}C_{\ll\lambda^2}F_\lambda(s)ds\Big\|_{L^2_{t,x}}\\
&\lesssim \lambda^{s+2\theta-2+b}\|\langle\tau + |\xi|^2\rangle^{-1}\mathcal{F}(C_{\ll\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda^{s+2\theta-2+b}\|\langle\tau + |\xi|^2\rangle^{\theta-1}\mathcal{F}(C_{\ll\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda^{s}\|\langle\tau + |\xi|^2\rangle^{\theta-1}\mathcal{F}(C_{\ll\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}}.
\end{aligned}$$

Using the product rule, we obtain

$$\begin{aligned}
\text{(IV.4)} &\lesssim \lambda^{s+b+2\theta-2}\Big\|P^{(t)}_{\gtrsim\lambda^2}(\eta'(t)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is\Delta}F_\lambda(s)ds)\Big\|_{L^2_{t,x}}\\
&+ \lambda^{s+b+2\theta-2}\|P^{(t)}_{\gtrsim\lambda^2}(\eta(t)P^{(t)}_{\ll\lambda^2}e^{-it\Delta}F_\lambda)\|_{L^2_{t,x}} =: \text{(IV.41)}+\text{(IV.42)}.
\end{aligned}$$

(IV.41) can be handled exactly in the same way as (IV.3) while for (IV.42) is equivalent to

$$\lambda^{s+b+2\theta-2}\|P^{(t)}_{\gtrsim\lambda^2}(P^{(t)}_{\gtrsim\lambda^2}\eta(t)P^{(t)}_{\ll\lambda^2}e^{-it\Delta}F_\lambda)\|_{L^2_{t,x}}.$$

We control the above as follows:

$$\begin{aligned}
\text{(IV.42)} &\lesssim \lambda^{s+b+2\theta-2}(\|(P^{(t)}_{\gtrsim\lambda^2}\eta(t)-\eta(t)P^{(t)}_{\gtrsim\lambda^2})P^{(t)}_{\ll\lambda^2}e^{-it\Delta}F_\lambda\|_{L^2_{t,x}}+\|\eta(t)P^{(t)}_{\gtrsim\lambda^2}P^{(t)}_{\ll\lambda^2}e^{-it\Delta}F_\lambda\|_{L^2_{t,x}})\\
&\lesssim \lambda^{s+b+2\theta-4}\|P^{(t)}_{\ll\lambda^2}e^{-it\Delta}F_\lambda\|_{L^2_{t,x}}+\lambda^{s+b+2\theta-2}\|\eta(t)P^{(t)}_{\sim\lambda^2}e^{-it\Delta}F_\lambda\|_{L^2_{t,x}}\\
&\lesssim \lambda^{s+2\theta-3}\|e^{-it\Delta}C_{\ll\lambda^2}F_\lambda\|_{L^2_{t,x}}+\lambda^{s'+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}(C_{\sim\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda^{s}\|\langle\tau+|\xi|^2\rangle^{\theta-1}\mathcal{F}(C_{\ll\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}}+\lambda^{s'+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}(C_{\sim\lambda^2}F_\lambda)\|_{L^2_{\tau,\xi}}.
\end{aligned}$$

where we used the commutator estimate to obtain the first inequality for the first term (see Appendix C). □

We prove a similar inequality for the wave equation.

Lemma 3.5.2 (Energy inequality for the wave equation)**.** *Let $l,\beta,\theta,a\in\mathbb{R}$ be as defined in (3.3.1). For any $\lambda\in 2^{\mathbb{N}}$, η as before and $e^{it|\nabla|}v_0$ supported in frequencies of size $\sim\lambda$, the following estimates hold:*

$$\|\eta(t)e^{it|\nabla|}v_0\|_{W^{l,s,\theta}_\lambda}\lesssim\lambda^l\|v_0\|_{L^2_x},$$
$$\|\eta(t)\mathcal{I}_W[G_\lambda]\|_{W^{l,s,\theta}_\lambda}\lesssim\|G_\lambda\|_{R^{l,s,\theta-1}_\lambda}.$$

Proof. We will be brief here as most of the steps will be same as in Lemma 3.5.1.

$$\begin{aligned}
\|\eta(t)e^{it|\nabla|}v_0\|_{W^{l,s,\theta}_\lambda}=\lambda^l&\|\eta(t)e^{it|\nabla|}v_0\|_{L^\infty_tL^2_x}\\
&+\lambda^{l-a}\|\langle\tau-|\xi|\rangle^\theta\mathcal{F}((\lambda+|\partial_t|)^aQ_{\ll\lambda^2}\eta(t)e^{it|\nabla|}v_0)\|_{L^2_{\tau,\xi}}\\
&+\lambda^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\lambda^2}\eta(t)e^{it|\nabla|}v_0)\|_{L^2_{\tau,\xi}}.
\end{aligned}$$

The first term above can be controlled using (2.7.6) and (2.7.2), respectively, as follows

$$\lambda^l\|\eta(t)e^{it|\nabla|}v_0\|_{L^\infty_tL^2_x}\lesssim\lambda^l\|\langle\tau-|\xi|\rangle^\theta\mathcal{F}(\eta(t)e^{it|\nabla|}v_0)\|_{L^2_{\tau,\xi}}\lesssim\lambda^l\|v_0\|_{L^2_x}.$$

For the second term, using (3.5.1) we obtain

$$\begin{aligned}
&\lambda^{l-a}\|\langle\tau-|\xi|\rangle^\theta\mathcal{F}((\lambda+|\partial_t|)^aQ_{\ll\lambda^2}\eta(t)e^{it|\nabla|}v_0)\|_{L^2_{\tau,\xi}}\\
&=\lambda^{l-a}\|\langle\tau-|\xi|\rangle^\theta(\lambda+|\tau|)^a\mathcal{F}(e^{it|\nabla|}P^{(t)}_{\ll\lambda^2}(\eta(t)v_0))\|_{L^2_{\tau,\xi}}\\
&\lesssim\max\big(\lambda^{l-a}\|\langle\tau-|\xi|\rangle^{\theta+2a}\mathcal{F}(e^{it|\nabla|}P^{(t)}_{\ll\lambda^2}(\eta(t)v_0))\|_{L^2_{\tau,\xi}},\\
&\qquad\lambda^l\|\langle\tau-|\xi|\rangle^\theta\mathcal{F}(e^{it|\nabla|}P^{(t)}_{\ll\lambda^2}(\eta(t)v_0))\|_{L^2_{\tau,\xi}}\big)\\
&\lesssim\lambda^l\|P^{(t)}_{\ll\lambda^2}(\eta(t)v_0)\|_{H^{\theta+2a}_tL^2_x}\\
&\lesssim\lambda^l\|v_0\|_{L^2_x}.
\end{aligned}$$

Similarly, for the last term, using the choice of the parameter β, we obtain

$$\begin{aligned}
\lambda^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\lambda^2}\eta(t)e^{it|\nabla|}v_0)\|_{L^2_{\tau,\xi}}&=\lambda^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(e^{it|\nabla|}P^{(t)}_{\gtrsim\lambda^2}(\eta(t)v_0))\|_{L^2_{\tau,\xi}}\\
&=\lambda^{\beta-1}\|P^{(t)}_{\gtrsim\lambda^2}(\eta(t)v_0)\|_{H^1_tL^2_x}\\
&\lesssim\lambda^l\|v_0\|_{L^2_x}.
\end{aligned}$$

We consider the W_λ norm for the Duhamel integral now. For the $L_t^\infty L_x^2$ norm, we use the same technique as in the previous lemma. We decompose the nonlinearity:

$$\begin{aligned}
&\lambda^l\Big\|\eta(t)\int_0^t e^{i(t-s)|\nabla|}G_\lambda(s)ds\Big\|_{L_t^\infty L_x^2}\\
&\lesssim \lambda^l\Big\|\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\ll\lambda^2}G_\lambda(s)ds\Big\|_{L_t^\infty L_x^2}+\lambda^l\Big\|\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\gtrsim\lambda^2}G_\lambda(s)ds\Big\|_{L_t^\infty L_x^2}.
\end{aligned}\tag{3.5.4}$$

For the first term in (3.5.4), we have

$$\begin{aligned}
&\lambda^l\Big\|\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\ll\lambda^2}G_\lambda(s)ds\Big\|_{L_t^\infty L_x^2}\\
&\lesssim \lambda^l\Big\|\langle\tau-|\xi|\rangle^\theta\mathcal{F}\Big(\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\ll\lambda^2}G_\lambda(s)ds\Big)\Big\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda^{l-a}\|\langle\tau-|\xi|\rangle^{\theta-1}\mathcal{F}(\lambda+|\partial_t|)^aQ_{\ll\lambda^2}G_\lambda\|_{L^2_{\tau,\xi}},
\end{aligned}$$

where in the last equality, we use the fact that $|m_W(\tau)|\gtrsim 1$. For the second term in (3.5.4), we use (3.5.3) to obtain

$$\lambda^l\Big\|\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\gtrsim\lambda^2}G_\lambda(s)ds\Big\|_{L_t^\infty L_x^2}\lesssim\lambda^{l-2}\|Q_{\gtrsim\lambda^2}G_\lambda\|_{L_t^\infty L_x^2}\lesssim\lambda^{l+2\theta-3}\|G_\lambda\|_{L_t^\infty L_x^2}.$$

The low modulation norm of the Duhamel integral is written as

$$\begin{aligned}
&\lesssim \lambda^{l-a}\Big\|\langle\tau-|\xi|\rangle^\theta\mathcal{F}((\lambda+|\partial_t|)^aQ_{\ll\lambda^2}(\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\ll\lambda^2}G_\lambda(s)ds))\Big\|_{L^2_{\tau,\xi}}\\
&\quad+\lambda^{l-a}\Big\|\langle\tau-|\xi|\rangle^\theta\mathcal{F}((\lambda+|\partial_t|)^aQ_{\ll\lambda^2}(\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\gtrsim\lambda^2}G_\lambda(s)ds))\Big\|_{L^2_{\tau,\xi}}\\
&=: \text{(I.1)+(I.2)}.
\end{aligned}$$

(I.1) is controlled using Lemma 3.5.1. For (I.2), we consider two cases:
(i) $\boldsymbol{a>0}$: Using Lemma 3.5.1 and the choice of the parameters a and β, we have

$$\begin{aligned}
\text{(I.2)}&\lesssim\lambda^{l-a}\Big\|\langle\tau-|\xi|\rangle^\theta(\lambda+|\tau|)^a\mathcal{F}(Q_{\ll\lambda^2}\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\gtrsim\lambda^2}G_\lambda(s)ds)\Big\|_{L^2_{\tau,\xi}}\\
&\lesssim\lambda^{l+a+2\theta}\|\langle\tau-|\xi|\rangle^{-1}\mathcal{F}(Q_{\gtrsim\lambda^2}G_\lambda)\|_{L^2_{\tau,\xi}}\\
&\lesssim\lambda^{l+a+2\theta-2}\|\mathcal{F}(Q_{\gtrsim\lambda^2}G_\lambda)\|_{L^2_{\tau,\xi}}\\
&\lesssim\lambda^{\beta-1}\|\mathcal{F}(Q_{\gtrsim\lambda^2}G_\lambda)\|_{L^2_{\tau,\xi}}.
\end{aligned}$$

(ii) $\boldsymbol{a=0}$: We need to control the following term

$$\text{(I.2)}=\lambda^l\Big\|\langle\tau-|\xi|\rangle^\theta\mathcal{F}(Q_{\ll\lambda^2}(\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\gtrsim\lambda^2}G_\lambda(s)ds))\Big\|_{L^2_{\tau,\xi}},$$

which, using (3.5.1) and Sobolev embedding can be bounded by

$$\lambda^l\Big\|P^{(t)}_{\ll\lambda^2}(\eta(t)\int_0^t P^{(s)}_{\gtrsim\lambda^2}e^{-is|\nabla|}G_\lambda(s)ds)\Big\|_{W_t^{1,p}L_x^2},\quad p=\frac{2}{3-2\theta}.$$

The above is written in an equivalent norm as

$$\lambda^l\Big\|P^{(t)}_{\ll\lambda^2}(\eta(t)\int_0^t P^{(s)}_{\gtrsim\lambda^2}e^{-is|\nabla|}G_\lambda(s)ds)\Big\|_{L^p_tL^2_x}+\lambda^l\Big\|(P^{(t)}_{\ll\lambda^2}(\eta(t)\int_0^t P^{(s)}_{\gtrsim\lambda^2}e^{-is|\nabla|}G_\lambda(s)ds))'\Big\|_{L^p_tL^2_x}$$
$$=: \text{(I.21)+(I.22)},$$

where the prime denotes derivative in t. Using Hölder's inequality and (3.5.3), we have

$$\text{(I.21)}\lesssim\lambda^l\|\eta(t)\|_{L^p_tL^\infty_x}\Big\|\int_0^t P^{(s)}_{\gtrsim\lambda^2}e^{-is|\nabla|}G_\lambda(s)ds\Big\|_{L^\infty_tL^2_x}\lesssim\lambda^{l-2}\|Q_{\gtrsim\lambda^2}G_\lambda\Big\|_{L^\infty_tL^2_x}.$$

Product rule gives

$$\text{(I.22)}\lesssim\lambda^l\Big\|(P^{(t)}_{\ll\lambda^2}(\eta'(t)\int_0^t P^{(s)}_{\gtrsim\lambda^2}e^{-is|\nabla|}G_\lambda(s)ds)\Big\|_{L^p_tL^2_x}+\lambda^l\|P^{(t)}_{\ll\lambda^2}(\eta(t)P^{(t)}_{\gtrsim\lambda^2}e^{-it|\nabla|}G_\lambda)\|_{L^p_tL^2_x}$$
$$=: \text{(I.221)+(I.222)}.$$

(I.221) can be treated like (I.21) while for (I.222), we decompose the time cut-off to obtain

$$\text{(I.222)}\lesssim\lambda^l\|P^{(t)}_{\ll\lambda^2}(P^{(t)}_{\ll\lambda^2}\eta(t)+P^{(t)}_{\gtrsim\lambda^2}\eta(t))P^{(t)}_{\gtrsim\lambda^2}e^{-it|\nabla|}G_\lambda\|_{L^p_tL^2_x}.$$

The first term in the above sum vanishes and for the second, we use Bernstein's and Hölder's inequalities to obtain the bound

$$\begin{aligned}&\lambda^l\|P^{(t)}_{\ll\lambda^2}(P^{(t)}_{\gtrsim\lambda^2}\eta(t)P^{(t)}_{\gtrsim\lambda^2}e^{-it|\nabla|}G_\lambda)\|_{L^p_tL^2_x}\\ \lesssim\ &\lambda^{l+2\theta-1}\|P^{(t)}_{\ll\lambda^2}(P^{(t)}_{\gtrsim\lambda^2}\eta(t)P^{(t)}_{\gtrsim\lambda^2}e^{-it|\nabla|}G_\lambda)\|_{L^1_tL^2_x}\\ \lesssim\ &\lambda^{l+2\theta-3}\|Q_{\gtrsim\lambda^2}G_\lambda\|_{L^\infty_tL^2_x}.\end{aligned}$$

In the last inequality, we again used that the L^1 norm of a time cut-off at high temporal frequencies is $\sim\lambda^{-2}$.

To treat the high modulation norm of Duhamel integral, we again decompose it as follows:

$$\begin{aligned}&\lambda^{\beta-1}\Big\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\lambda^2}(\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\gtrsim\lambda^2}G_\lambda(s)ds))\Big\|_{L^2_{\tau,\xi}}\\ &\quad+\lambda^{\beta-1}\Big\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\lambda^2}(\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\ll\lambda^2}G_\lambda(s)ds))\Big\|_{L^2_{\tau,\xi}}.\end{aligned}\tag{3.5.5}$$

The first term in (3.5.5) is controlled using Lemma 2.7.7, i.e.

$$\lambda^{\beta-1}\Big\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\lambda^2}(\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\gtrsim\lambda^2}G_\lambda(s)ds))\Big\|_{L^2_{\tau,\xi}}\lesssim\lambda^{\beta-1}\|\mathcal{F}(Q_{\gtrsim\lambda^2}G_\lambda)\|_{L^2_{\tau,\xi}}.$$

For the second term in (3.5.5), using similar computation as before, we obtain the bound

$$\begin{aligned}&\lambda^{\beta-1}\Big\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\lambda^2}(\eta(t)\int_0^t e^{i(t-s)|\nabla|}Q_{\ll\lambda^2}G_\lambda(s)ds))\Big\|_{L^2_{\tau,\xi}}\\ =\ &\lambda^{\beta-1}\Big\|P^{(t)}_{\gtrsim\lambda^2}(\eta(t)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is|\nabla|}G_\lambda(s)ds)\Big\|_{H^1_tL^2_x}\end{aligned}$$

$$\begin{aligned}&\lesssim \lambda^{\beta-1}\Big\|P^{(t)}_{\gtrsim\lambda^2}(\eta(t)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is|\nabla|}G_\lambda(s)ds)\Big\|_{L^2_{t,x}}\\&\quad+\lambda^{\beta-1}\Big\|P^{(t)}_{\gtrsim\lambda^2}(\eta(t)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is|\nabla|}G_\lambda(s)ds)'\Big\|_{L^2_{t,x}}\\&=: \text{(II)}+\text{(III)}.\end{aligned}$$

Using Lemma 3.5.1, the choice of β and $|m_W|\gtrsim 1$,

$$\begin{aligned}\text{(II)}&\lesssim \lambda^{\beta-1}\Big\|\eta(t)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is|\nabla|}G_\lambda(s)ds\Big\|_{L^2_{t,x}}\\&\lesssim \lambda^{\beta-1}\|\langle\tau-|\xi|\rangle^{-1}\mathcal{F}(Q_{\ll\lambda^2}G_\lambda)\|_{L^2_{\tau,\xi}}\\&\lesssim \lambda^{l-a}\|\langle\tau-|\xi|\rangle^{\theta-1}\mathcal{F}((\lambda+|\partial_t|)^aQ_{\ll\lambda^2}G_\lambda)\|_{L^2_{\tau,\xi}}.\end{aligned}$$

For (III), we have

$$\text{(III)}\lesssim \lambda^{\beta-1}\Big\|P^{(t)}_{\gtrsim\lambda^2}(\eta'(t)\int_0^t P^{(s)}_{\ll\lambda^2}e^{-is|\nabla|}G_\lambda(s)ds)\Big\|_{L^2_{t,x}}+\lambda^{\beta-1}\|P^{(t)}_{\gtrsim\lambda^2}(\eta(t)P^{(t)}_{\ll\lambda^2}e^{-it|\nabla|}G_\lambda)\|_{L^2_{t,x}}.$$

The first term of (III) is treated like (II) and the second reduces to

$$\begin{aligned}&\lambda^{\beta-1}\|(P^{(t)}_{\gtrsim\lambda^2}(P^{(t)}_{\gtrsim\lambda^2}\eta(t)P^{(t)}_{\ll\lambda^2}e^{-it|\nabla|}G_\lambda)\|_{L^2_{t,x}}\\&\lesssim \lambda^{\beta-1}\|(P^{(t)}_{\gtrsim\lambda^2}\eta(t)-\eta(t)P^{(t)}_{\gtrsim\lambda^2})P^{(t)}_{\ll\lambda^2}e^{-it|\nabla|}G_\lambda)\|_{L^2_{t,x}}+\lambda^{\beta-1}\|\eta(t)P^{(t)}_{\gtrsim\lambda^2}P^{(t)}_{\ll\lambda^2}e^{-it|\nabla|}G_\lambda)\|_{L^2_{t,x}}.\end{aligned}$$

For the first above, we use the commutator estimate to bound it by

$$\lambda^{\beta-3}\|P^{(t)}_{\ll\lambda^2}e^{-it|\nabla|}G_\lambda)\|_{L^2_{t,x}}\lesssim \lambda^{l-a}\|\langle\tau-|\xi|\rangle^{\theta-1}\mathcal{F}((\lambda+|\partial_t|)^aQ_{\ll\lambda^2}G_\lambda)\|_{L^2_{\tau,\xi}},$$

where the choice of the parameter β and $|m_W|\gtrsim 1$ ensure that the last inequality holds. For the remaining term, we have

$$\lambda^{\beta-1}\|\eta(t)P^{(t)}_{\sim\lambda^2}e^{-it|\nabla|}G_\lambda\|_{L^2_{t,x}}\lesssim \|G_\lambda\|_{R^{l,s,\theta-1}_\lambda}.$$

□

3.6 Nonlinear estimates

We control the Schrödinger nonlinearity and the wave nonlinearity in their respective norms in this section.

3.6.1 Schrödinger nonlinearity

Theorem 3.6.1. *Let $d\leqslant 3$ and s,l satisfy (3.2.1). Then, there exist $a,b,s,\ l,\beta,\theta\in\mathbb{R}$ such that the estimate*

$$\|uRe(v)\|_{N^{s,l,\theta-1}}\lesssim \|u\|_{S^{s,l,\theta}}\|v\|_{W^{l,s,\theta}} \tag{3.6.1}$$

holds.

Proof. We choose the parameters a,b,s',l,β,θ' as in (3.3.1) and begin by noting the following characterisation of the $N^{s,l,\theta-1}$ norm for the nonlinearity in the case $0 \leqslant a < \frac{1}{2}$:

$$\begin{aligned}\|F_\lambda\|_{N_\lambda^{s,l,\theta-1}} \approx \lambda^s \|\langle \tau + |\xi|^2\rangle^{\theta-1} \mathcal{F}(C_{\ll\lambda^2} F_\lambda)\|_{L^2_{\tau,\xi}} \\ + \lambda^{s'-2a+b}\|\langle \tau + |\xi|^2\rangle^{\theta'-1} \mathcal{F}((\lambda + |\partial_t|)^a C_{\gtrsim\lambda^2} F_\lambda)\|_{L^2_{\tau,\xi}}.\end{aligned} \tag{3.6.2}$$

For $\frac{1}{r} = \frac{1}{2} - a$, using Bernstein's inequality (in x) and Sobolev embedding (in t), we have

$$\begin{aligned}\lambda^{s+2\theta-3}\|P^{(t)}_{\ll\lambda^2} F_\lambda\|_{L^\infty_t L^2_x} &\lesssim \lambda^{s+2\theta-3+\frac{2}{r}}\|P^{(t)}_{\ll\lambda^2} F_\lambda\|_{L^r_t L^2_x} \\ &\lesssim \lambda^{s+2\theta-3+\frac{2}{r}}\|P^{(t)}_{\ll\lambda^2} F_\lambda\|_{L^2_x L^r_t} \\ &\lesssim \lambda^{s+2\theta-2-a}\|P^{(t)}_{\ll\lambda^2} F_\lambda\|_{L^2_{t,x}} \\ &\lesssim \lambda^{s+2\theta-2-2a}\|(\lambda + |\partial_t|)^a P^{(t)}_{\ll\lambda^2} F_\lambda\|_{L^2_{t,x}} \\ &\lesssim \lambda^{s+2\theta-2-2a+b}\|\mathcal{F}((\lambda + |\partial_t|)^a P^{(t)}_{\ll\lambda^2} C_{\approx\lambda^2} F_\lambda)\|_{L^2_{\tau,\xi}} \\ &\lesssim \lambda^{s'-2a+b}\|\langle \tau + |\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda + |\partial_t|)^a C_{\gtrsim\lambda^2} F_\lambda)\|_{L^2_{\tau,\xi}},\end{aligned}$$

where we obtain the penultimate inequality by noting that $b \geqslant 0$.
So, for $0 \leqslant a < \frac{1}{2}$, it suffices to prove the following:

$$\Big(\sum_{\lambda_0 \in 2^{\mathbb{N}_0}} \lambda_0^{2s}\|\langle \tau + |\xi|^2\rangle^{\theta-1}\mathcal{F}(C_{\ll\lambda_0^2} P_{\lambda_0}(uv))\|^2_{L^2_{\tau,\xi}}\Big)^{\frac{1}{2}} \lesssim \|u\|_{S^{s,l,\theta}}\|v\|_{W^{l,s,\theta}}, \tag{3.6.3}$$

$$\Big(\sum_{\lambda_0 \in 2^{\mathbb{N}_0}} \lambda_0^{2(s'-2a+b)}\|\langle \tau + |\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda_0 + |\partial_t|)^a C_{\gtrsim\lambda_0^2} P_{\lambda_0}(uv))\|^2_{L^2_{\tau,\xi}}\Big)^{\frac{1}{2}} \lesssim \|u\|_{S^{s,l,\theta}}\|v\|_{W^{l,s,\theta}}. \tag{3.6.4}$$

In the case $\frac{1}{2} \leqslant a < 1$, we shall additionally prove the following:

$$\Big(\sum_{\lambda_0 \in 2^{\mathbb{N}_0}} \lambda_0^{2(s+2\theta-3)}\|P^{(t)}_{\ll\lambda_0^2} P_{\lambda_0}(uv)\|^2_{L^\infty_t L^2_x}\Big)^{\frac{1}{2}} \lesssim \|u\|_{L^\infty_t H^s_x}\|v\|_{L^\infty_t H^l_x}. \tag{3.6.5}$$

We now proceed to prove (3.6.3) and (3.6.4). Since $Re(v) = \frac{v+\bar{v}}{2}$, we treat the nonlinearity uv, the case $u\bar{v}$ being analogous. We decompose uv as

$$uv = \sum_{\lambda_0 \in 2^{\mathbb{N}_0}} P_{\lambda_0}(uv).$$

Further, we distinguish the high-low, low-high and the balanced interactions as follows:

$$P_{\lambda_0}(uv) = \sum_{\lambda_1,\lambda_0 \in 2^{\mathbb{N}_0}} P_{\lambda_0}(u_{\lambda_1} v) = \sum_{\lambda_1 \ll \lambda_0} P_{\lambda_0}(u_{\lambda_1} v) + \sum_{\lambda_0 \ll \lambda_1} P_{\lambda_0}(u_{\lambda_1} v) + \sum_{\lambda_0 \sim \lambda_1} P_{\lambda_0}(u_{\lambda_1} v).$$

We set some notation that we will use in this section. High modulation for the spatially localised Schrödinger solution $u_{|\xi|}(|\xi| = \lambda_0, \lambda_1, \lambda_2)$ means $\langle \tau + |\xi|^2\rangle \gtrsim |\xi|^2$ while low modulation means $\langle \tau + |\xi|^2\rangle \ll |\xi|^2$. For the spatially localised wave $v_{|\xi|}$, high modulation refers to $\langle \tau - |\xi|\rangle \gtrsim |\xi|^2$, while low modulation refers to $\langle \tau - |\xi|\rangle \ll |\xi|^2$. For both the cases, we abbreviate high modulation by H and low modulation by L. Using the convolution constraint, the output temporal frequencies are given by $\tau_0 = \tau_1 + \tau_2$. For the

Schrödinger solution, low modulation occurs when the temporal frequencies are of size $\sim |\xi|^2(\tau = -|\xi|^2)$. For the wave solution, temporal frequencies of size $\sim |\xi|(\tau = |\xi|)$ lead to a free wave solution. Other than that, the sizes of the modulation and the temporal frequencies go hand in hand for a frequency localised wave solution.
All the possible interactions are treated individually. In cases with low output modulation, the required bilinear estimates are reduced to trilinear estimates by observing

$$\int_{\mathbb{R}^d}(f*g)\cdot h = \int_{\mathbb{R}^d}(\tilde{f}*h)\cdot g = \int_{\mathbb{R}^d}(\tilde{g}*h)\cdot f,$$

where $\tilde{f}(\cdot) = f(-\cdot)$ and duality of $X^{s,\theta}$ spaces, see Section 2.7. To distinguish the dual term, we use $-s$ and $-l$ instead of s and l, respectively, in the superscripts of the norms for the Schrödinger and wave components. Each subcase can be summed up depending on the size of the interacting frequencies. The constraints required to obtain the estimate for the full norms are listed at the end of each subcase. We do not explicitly mention the exclusion of the points with logarithmic losses, since the endpoints are not covered, eventually.

Remark 3.6.2. In the following, we treat the case of high spatial (and temporal) frequencies ($|\xi_i|, |\tau_i| \gg 1, i = 0, 1, 2$). In most of the cases, the estimates for the low frequencies (all the frequencies are of size ~ 1 or the size of the lowest frequency in the interaction is ~ 1) follow using the same arguments without having to decompose the space-time Fourier supports of the interacting solutions. Hence, we do not mention it explicitly for each case. However, in some cases the arguments need to be modified for the low frequency cases. We shall do it wherever required.

Case I. Low to high interaction ($\lambda_1 \ll \lambda_0$)
We decompose u_{λ_1} and v_{λ_0} as follows:

$$u_{\lambda_1} = C_{\ll\lambda_1^2}u_{\lambda_1} + C_{\gtrsim\lambda_1^2}u_{\lambda_1}, \quad v_{\lambda_0} = Q_{\ll\lambda_0^2}v_{\lambda_0} + Q_{\gtrsim\lambda_0^2}v_{\lambda_0}.$$

The following interactions can be distinguished on the basis of the size of the modulation:

1. $\boldsymbol{H \times H \to H}$
We require to prove (3.6.4). Using the size of the modulation, the product estimate and Bernstein's inequality, respectively, we have

$$\begin{aligned}
&\lambda_0^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda_0+|\partial_t|)^a C_{\gtrsim\lambda_0^2}(C_{\gtrsim\lambda_1^2}u_{\lambda_1}Q_{\gtrsim\lambda_0^2}v_{\lambda_0}))\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda_0^{s'-3a+b+2\theta'-2}\lambda_1^{\frac{d}{2}}\|(\lambda_0+|\partial_t|)^a C_{\gtrsim\lambda_1^2}u_{\lambda_1}\|_{L^\infty_t L^2_x}\|(\lambda_0+|\partial_t|)^a Q_{\gtrsim\lambda_0^2}v_{\lambda_0}\|_{L^2_{t,x}}\\
&\lesssim \lambda_0^{s'-3a+b+2\theta'-\beta-1}\lambda_1^{\frac{d}{2}-s+2a}\sup_{|\tau_1|}\Big(\frac{\lambda_0+|\tau_1|}{\lambda_1+|\tau_1|}\Big)^a\sup_{|\tau_2|\gtrsim\lambda_0^2}\frac{(\lambda_0+|\tau_2|)^a}{\langle\tau_2-\lambda_0\rangle}\\
&\quad\times\lambda_1^{s'-2a}\|(\lambda_1+|\partial_t|)^a C_{\gtrsim\lambda_1^2}u_{\lambda_1}\|_{L^\infty_t L^2_x}\ \lambda_0^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\lambda_0^2}v_{\lambda_0})\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda_0^{s'+b+2\theta'-\beta-3}\lambda_1^{\frac{d}{2}-s+a}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}.
\end{aligned}$$

The last inequality above follows from Lemma 2.7.5. Since $s'+2\theta' = s+2\theta$, we can sum up the subcase for spatial frequencies $\lambda_1 \ll \lambda_0$ to obtain (3.6.4) provided $s-\beta \leqslant 3-2\theta-b$ and $\beta \geqslant -3+\frac{d}{2}+2\theta+a+b$.

2. $\boldsymbol{H}\times\boldsymbol{H}\to\boldsymbol{L}$

We consider two cases for the temporal frequencies:

a. $|\tau_1|\lesssim\lambda_0^2$, $|\tau_2|\sim\lambda_0^2$

We prove (3.6.3) using duality. Consider the expression I defined in (3.4.5) and use Cauchy-Schwarz inequality as follows:

$$\begin{aligned}&|I(\mathcal{F}(Q_{\sim\lambda_0^2}v_{\lambda_0}),\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0}),\mathcal{F}(C_{\lambda_1^2\lesssim\cdot\lesssim\lambda_0^2}u_{\lambda_1}))|\\ &\lesssim\|Q_{\sim\lambda_0^2}v_{\lambda_0}\|_{L^2_{t,x}}\|C_{\lambda_1^2\lesssim\cdot\lesssim\lambda_0^2}u_{\lambda_1}\overline{C_{\ll\lambda_0^2}w_{\lambda_0}}\|_{L^2_{t,x}}.\end{aligned}\tag{3.6.6}$$

On decomposing the space-time Fourier supports of $C_{\lambda_1^2\lesssim\cdot\lesssim\lambda_0^2}u_{\lambda_1}$, $\overline{C_{\ll\lambda_0^2}w_{\lambda_0}}$ into pieces of size L_1 and L_0, respectively and applying the bilinear estimate (3.4.9), we get

$$\begin{aligned}\text{RHS of (3.6.6)}&\lesssim\|Q_{\sim\lambda_0^2}v_{\lambda_0}\|_{L^2_{t,x}}\sum_{\substack{L_0\ll\lambda_0^2\\ L_1\lesssim\lambda_0^2}}(L_0L_1)^{\frac12}\frac{\lambda_1^{\frac{d-1}{2}}}{\lambda_0^{\frac12}}\|C_{L_1}u_{\lambda_1}\|_{L^2_{t,x}}\|C_{L_0}w_{\lambda_0}\|_{L^2_{t,x}}\\ &\lesssim\lambda_0^{-\beta-\frac52+s+2\theta}\lambda_1^{\frac{d-1}{2}-s+a}\lambda_1^{s'-2a}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda_1+|\partial_t|)^aC_{\lambda_1^2\lesssim\cdot\lesssim\lambda_0^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}\\ &\quad\times\lambda_0^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\sim\lambda_0^2}v_{\lambda_0})\|_{L^2_{\tau,\xi}}\lambda_0^{-s}\|\langle\tau+|\xi|^2\rangle^{1-\theta}\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0})\|_{L^2_{\tau,\xi}}\\ &\lesssim\lambda_0^{-\beta-\frac52+s+2\theta}\lambda_1^{\frac{d-1}{2}-s+a}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.\end{aligned}$$

We require $s-\beta\leqslant\frac52-2\theta$ and $\beta\geqslant-3+2\theta+\frac d2+a$ to sum the above up and obtain the estimate for the full norms.

In case $\lambda_1\sim\lambda_0\sim1$, we have

$$\begin{aligned}&|I(\mathcal{F}(Q_{\sim\lambda_0^2}v_{\lambda_0}),\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0}),\mathcal{F}(C_{\lambda_1^2\lesssim\cdot\lesssim\lambda_0^2}u_{\lambda_1}))|\\ &\lesssim\|Q_{\sim\lambda_0^2}v_{\lambda_0}\|_{L^2_{t,x}}\|C_{\lambda_1^2\lesssim\cdot\lesssim\lambda_0^2}u_{\lambda_1}\|_{L^\infty_tL^2_x}\|C_{\ll\lambda_0^2}w_{\lambda_0}\|_{L^2_{t,x}}\\ &\lesssim\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.\end{aligned}$$

b. $|\tau_1|,|\tau_2|\gg\lambda_0^2$

Using Hölder's and Bernstein's inequality, we have

$$\begin{aligned}&|I(\mathcal{F}(Q_{\gg\lambda_0^2}v_{\lambda_0}),\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0}),\mathcal{F}(C_{\gg\lambda_0^2}u_{\lambda_1}))|\\ &\lesssim\lambda_1^{\frac d2}\|C_{\gg\lambda_0^2}u_{\lambda_1}\|_{L^2_{t,x}}\|Q_{\gg\lambda_0^2}v_{\lambda_0}\|_{L^2_{t,x}}\|C_{\ll\lambda_0^2}w_{\lambda_0}\|_{L^\infty_tL^2_x}\\ &\lesssim\lambda_1^{\frac d2-s'+2a}\lambda_0^{-2a-\beta-2+2(\theta-\theta')+s}\lambda_1^{s'-2a}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda_1+|\partial_t|)^aC_{\gg\lambda_0^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}\\ &\quad\times\lambda_0^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gg\lambda_0^2}v_{\lambda_0})\|_{L^2_{\tau,\xi}}\lambda_0^{-s}\|C_{\ll\lambda_0^2}w_{\lambda_0}\|_{L^p_tL^2_x}\\ &\lesssim\lambda_1^{\frac d2-s'+2a}\lambda_0^{-2a-\beta-2+2(\theta-\theta')+s}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.\end{aligned}$$

Note that the penultimate inequality follows from Bernstein's inequality for $\frac1p=\theta-\frac12$ while the ultimate comes from the embedding (2.7.8). For summability, we require $s-\beta\leqslant2+2(\theta'-\theta)+2a$ and $\beta\geqslant-2+\frac d2$.

3. $\boldsymbol{H}\times\boldsymbol{L}\to\boldsymbol{H}$

Using the size of the modulation, the product estimate and Bernstein's inequality, we obtain

$$
\begin{aligned}
&\lambda_0^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda_0+|\partial_t|)^a C_{\gtrsim\lambda_0^2}(C_{\gtrsim\lambda_1^2}u_{\lambda_1}Q_{\ll\lambda_0^2}v_{\lambda_0}))\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda_0^{s'-3a+b+2\theta'-2}\lambda_1^{\frac{d}{2}}\|(\lambda_0+|\partial_t|)^a C_{\gtrsim\lambda_1^2}u_{\lambda_1}\|_{L^2_{t,x}}\|(\lambda_0+|\partial_t|)^a Q_{\ll\lambda_0^2}v_{\lambda_0}\|_{L^\infty_t L^2_x}\\
&\lesssim \lambda_0^{s'-2a+b+2\theta'-2-l}\lambda_1^{\frac{d}{2}-s+2a-2\theta}\sup_{|\tau_1|}\Big(\frac{\lambda_0+|\tau_1|}{\lambda_1+|\tau_1|}\Big)^a\lambda_0^{l-a}\|(\lambda_0+|\partial_t|)^a Q_{\ll\lambda_0^2}v_{\lambda_0}\|_{L^\infty_t L^2_x}\\
&\quad\times\lambda_1^{s'-2a}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda_1+|\partial_t|)^a C_{\gtrsim\lambda_1^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda_0^{s'-a+b+2\theta'-2-l}\lambda_1^{\frac{d}{2}-s+a-2\theta}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}.
\end{aligned}
$$

Provided $s-l\leqslant 2-2\theta+a-b$ and $l\geqslant -2+\frac{d}{2}+b$, we can sum the above to obtain (3.6.4).

4. $\boldsymbol{H}\times\boldsymbol{L}\to\boldsymbol{L}$

From the relation

$$|\tau_0+|\xi_0|^2|=|\tau_1-\tau_2-|\xi_2|+|\xi_2|+|\xi_0|^2|\ll\lambda_0^2,$$

and $|\tau_2-|\xi_2||\ll\lambda_0^2$, we conclude that $|\tau_1|\sim\lambda_0^2$. Using Hölder's inequality, we obtain

$$|I(\mathcal{F}(Q_{\ll\lambda_0^2}v_{\lambda_0}),\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0}),\mathcal{F}(C_{\sim\lambda_0^2}u_{\lambda_1}))|\lesssim\|C_{\sim\lambda_0^2}u_{\lambda_1}\|_{L^2_{t,x}}\|P_{\lambda_1}(\overline{Q_{\ll\lambda_0^2}v_{\lambda_0}}C_{\ll\lambda_0^2}w_{\lambda_0})\|_{L^2_{t,x}}. \tag{3.6.7}$$

The spatial frequency support of $C_{\sim\lambda_0^2}u_{\lambda_1}$ is of size $\sim\lambda_1$. Using orthogonality, we can reduce the estimate to the case when the spatial supports of $Q_{\ll\lambda_0^2}v_{\lambda_0}$ and $C_{\ll\lambda_0^2}w_{\lambda_0}$ are also of size $\sim\lambda_1$. Noting this, decomposing the space-time Fourier supports of $\overline{Q_{\ll\lambda_0^2}v_{\lambda_0}}$ and $C_{\ll\lambda_0^2}w_{\lambda_0}$ into pieces L_2 and L_0 respectively and applying the bilinear estimate (3.4.10), we can bound the right-hand side of (3.6.7) by

$$
\begin{aligned}
&\|C_{\sim\lambda_0^2}u_{\lambda_1}\|_{L^2_{t,x}}\sum_{L_0,L_2\ll\lambda_0^2}(L_0L_2)^{\frac12}\frac{\lambda_1^{\frac{d-1}{2}}}{\lambda_0^{\frac12}}\|Q_{L_2}v_{\lambda_0}\|_{L^2_{t,x}}\|C_{L_0}w_{\lambda_0}\|_{L^2_{t,x}}\\
&\lesssim\lambda_0^{s-l-\frac32+2(\theta-\theta')-2a}\lambda_1^{\frac{d-1}{2}-s'+2a}\lambda_1^{s'-2a}\|\langle\tau+|\xi|^2\rangle^{\theta}\mathcal{F}((\lambda_1+|\partial_t|)^aC_{\sim\lambda_0^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}\\
&\quad\times\lambda_0^{l-a}\|\langle\tau-|\xi|\rangle^{\theta}\mathcal{F}((\lambda_0+|\partial_t|)^aQ_{\ll\lambda_0^2}v_{\lambda_0})\|_{L^2_{\tau,\xi}}\lambda_0^{-s}\|\langle\tau+|\xi|^2\rangle^{1-\theta}\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0})\|_{L^2_{\tau,\xi}}\\
&\lesssim\lambda_0^{s-l-\frac32+2(\theta-\theta')-2a}\lambda_1^{\frac{d-1}{2}-s'+2a}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.
\end{aligned}
$$

We require $s-l\leqslant\frac32-2(\theta-\theta')+2a$ and $l\geqslant-2+\frac{d}{2}$ for the summability of the above estimate.

5. $\boldsymbol{L}\times\boldsymbol{H}\to\boldsymbol{H}$

Using the size of the modulation, the product estimate and Bernstein's inequality, we have

$$
\begin{aligned}
&\lambda_0^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda_0+|\partial_t|)^aC_{\ll\lambda_1^2}u_{\lambda_1}Q_{\gtrsim\lambda_0^2}v_{\lambda_0})\|_{L^2_{\tau,\xi}}\\
&\lesssim\lambda_0^{s'-3a+b+2\theta'-2}\lambda_1^{\frac{d}{2}}\|(\lambda_0+|\partial_t|)^aC_{\ll\lambda_1^2}u_{\lambda_1}\|_{L^\infty_tL^2_x}\|(\lambda_0+|\partial_t|)^aQ_{\gtrsim\lambda_0^2}v_{\lambda_0}\|_{L^2_{t,x}}\\
&\lesssim\lambda_0^{s'-3a+b+2\theta'-1-\beta}\lambda_1^{\frac{d}{2}-s}(\lambda_0+\lambda_1^2)^a\sup_{|\tau_2|\gtrsim\lambda_0^2}\frac{(\lambda_0+|\tau_2|)^a}{\langle\tau_2-\lambda_0\rangle}\lambda_1^s\|C_{\ll\lambda_1^2}u_{\lambda_1}\|_{L^\infty_tL^2_x}
\end{aligned}
$$

$$
\begin{aligned}
&\times \lambda_0^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\lambda_0^2}v_{\lambda_0})\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda_0^{s'-a+b+2\theta'-3-\beta}\lambda_1^{\frac{d}{2}-s}(\lambda_0+\lambda_1^2)^a\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}\\
&\lesssim \begin{cases} \lambda_0^{s'+b+2\theta'-3-\beta}\lambda_1^{\frac{d}{2}-s}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}, & \lambda_0+\lambda_1^2\sim\lambda_0\\ \lambda_0^{s'-a+b+2\theta'-3-\beta}\lambda_1^{\frac{d}{2}-s+2a}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}, & \lambda_0+\lambda_1^2\sim\lambda_1^2. \end{cases}
\end{aligned}
$$

Both the cases above can be summed up provided $s-\beta\leqslant 3-2\theta-b$ and $\beta\geqslant -3+\frac{d}{2}+2\theta+a+b$.

6. $\boldsymbol{L\times H\to L}$

If $|\tau_2|\gg\lambda_0^2$, the output will have a high modulation. Hence it suffices to consider $|\tau_2|\sim\lambda_0^2$. Then, we have

$$
|I(\mathcal{F}(Q_{\sim\lambda_0^2}v_{\lambda_0}),\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0}),\mathcal{F}(C_{\ll\lambda_1^2}u_{\lambda_1}))|\lesssim\|Q_{\sim\lambda_0^2}v_{\lambda_0}\|_{L^2_{t,x}}\|C_{\ll\lambda_1^2}u_{\lambda_1}\overline{C_{\ll\lambda_0^2}w_{\lambda_0}}\|_{L^2_{t,x}}. \tag{3.6.8}
$$

We decompose the space-time Fourier supports of $C_{\ll\lambda_1^2}u_{\lambda_1}$ and $\overline{C_{\ll\lambda_0^2}w_{\lambda_0}}$ into pieces L_1 and L_0 respectively, and apply the bilinear estimate (3.4.9) to obtain

$$
\begin{aligned}
\text{RHS of (3.6.8)} &\lesssim \|Q_{\sim\lambda_0^2}v_{\lambda_0}\|_{L^2_{t,x}}\sum_{\substack{L_0\ll\lambda_0^2,\\ L_1\ll\lambda_1^2}}(L_0L_1)^{\frac12}\frac{\lambda_1^{\frac{d-1}{2}}}{\lambda_0^{\frac12}}\|C_{L_1}u_{\lambda_1}\|_{L^2_{t,x}}\|C_{L_0}w_{\lambda_0}\|_{L^2_{t,x}}\\
&\lesssim \lambda_0^{-\beta-\frac52+s+2\theta}\lambda_1^{\frac{d-1}{2}-s}\ \lambda_1^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\lambda_1^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}\\
&\quad\times\lambda_0^{-s}\|\langle\tau+|\xi|^2\rangle^{1-\theta}\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0})\|_{L^2_{\tau,\xi}}\lambda_0^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\sim\lambda_0^2}v_{\lambda_0})\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda_0^{-\beta-\frac52+s+2\theta}\lambda_1^{\frac{d-1}{2}-s}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.
\end{aligned}
$$

We require $s-\beta\leqslant\frac52-2\theta$ and $\beta\geqslant -3+\frac{d}{2}+2\theta$ to sum the above estimate and obtain (3.6.3). For $\lambda_1\sim\lambda_0\sim1, d=1$, we have

$$
\begin{aligned}
|I(\mathcal{F}(Q_{\sim\lambda_0^2}v_{\lambda_0}),\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0}),\mathcal{F}(C_{\ll\lambda_1^2}u_{\lambda_1}))| &\lesssim \|Q_{\sim\lambda_0^2}v_{\lambda_0}\|_{L^2_{t,x}}\|C_{\ll\lambda_1^2}u_{\lambda_1}\|_{L^\infty_tL^2_x}\|C_{\ll\lambda_0^2}w_{\lambda_0}\|_{L^2_{t,x}}\\
&\lesssim \|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.
\end{aligned}
$$

7. $\boldsymbol{L\times L\to H}$

$\boldsymbol{d=3:}$ Using the size of the modulation, the product estimate and the endpoint Strichartz space $L^2_tL^6_x$, we have

$$
\begin{aligned}
&\lambda_0^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda_0+|\partial_t|)^aC_{\gtrsim\lambda_0^2}(C_{\ll\lambda_1^2}u_{\lambda_1}Q_{\ll\lambda_0^2}v_{\lambda_0}))\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda_0^{s'-3a+b+2\theta'-2}\lambda_1^{\frac12}\|(\lambda_0+|\partial_t|)^aC_{\ll\lambda_1^2}u_{\lambda_1}\|_{L^2_tL^6_x}\|(\lambda_0+|\partial_t|)^aQ_{\ll\lambda_0^2}v_{\lambda_0}\|_{L^\infty_tL^2_x}\\
&\lesssim \begin{cases}\lambda_0^{s'-2a+b+2\theta'-2-l}\lambda_1^{\frac12-s+2a}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}, & \lambda_0+\lambda_1^2\sim\lambda_1^2\\ \lambda_0^{s'-a+b+2\theta'-2-l}\lambda_1^{\frac12-s}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}, & \lambda_0+\lambda_1^2\sim\lambda_0.\end{cases}
\end{aligned}
$$

The constraints $s-l\leqslant 2-2\theta+a-b$ and $l\geqslant -\frac32+2\theta+b$ are required to sum the above cases.

$\boldsymbol{d=2:}$ We consider two subcases for the size of the wave modulation:
a. $\boldsymbol{\langle \tau - |\xi| \rangle \ll \lambda_0}$: For $\lambda_0 \ll |\tau_2| \ll \lambda_0^2$, the wave has a modulation of size $\gg \lambda_0$, so it suffices to consider $|\tau_2| \sim \lambda_0$. Using the size of the output modulation and the bilinear estimate (3.4.10), we have

$$\begin{aligned}
&\lambda_0^{s'-2a+b} \|\langle \tau + |\xi|^2 \rangle^{\theta'-1} \mathcal{F}((\lambda_0 + |\partial_t|)^a C_{\gtrsim \lambda_0^2}(C_{\ll \lambda_1^2} u_{\lambda_1} P^{(t)}_{\sim \lambda_0} Q_{\ll \lambda_0^2} v_{\lambda_0}))\|_{L^2_{\tau,\xi}} \\
&\lesssim \lambda_0^{s'-2a+b+2\theta'-2} (\lambda_0 + \lambda_1^2)^a \sum_{\substack{L_1 \ll \lambda_1^2, \\ L_2 \ll \lambda_0}} (L_1 L_2)^{\frac12} \frac{\lambda_1^{\frac12}}{\lambda_1^{\frac12}} \|C_{L_1} u_{\lambda_1}\|_{L^2_{t,x}} \|Q_{L_2} P^{(t)}_{\sim \lambda_0} v_{\lambda_0}\|_{L^2_{t,x}} \\
&\lesssim \begin{cases} \lambda_0^{s'-a+b+2\theta'-2-l} \lambda_1^{-s} \|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}} \|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}, & \lambda_0 + \lambda_1^2 \sim \lambda_0 \\ \lambda_0^{s'-2a+b+2\theta'-2-l} \lambda_1^{-s+2a} \|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}} \|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}, & \lambda_0 + \lambda_1^2 \sim \lambda_1^2. \end{cases}
\end{aligned}$$

In the case $1 \sim \lambda_1 \ll \lambda_0$, we have

$$\begin{aligned}
&\lambda_0^{s'-2a+b} \|\langle \tau + |\xi|^2 \rangle^{\theta'-1} \mathcal{F}((\lambda_0 + |\partial_t|)^a C_{\gtrsim \lambda_0^2}(C_{\ll \lambda_1^2} u_{\lambda_1} P^{(t)}_{\sim \lambda_0} Q_{\ll \lambda_0^2} v_{\lambda_0}))\|_{L^2_{\tau,\xi}} \\
&\lesssim \lambda_0^{s'-a+b+2\theta'-2-l} \|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}} \|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}.
\end{aligned}$$

b. $\boldsymbol{\lambda_0 \lesssim \langle \tau - |\xi| \rangle \ll \lambda_0^2}$: We use the size of the modulation, the product estimate and Bernstein's inequality to obtain

$$\begin{aligned}
&\lambda_0^{s'-2a+b} \|\langle \tau + |\xi|^2 \rangle^{\theta'-1} \mathcal{F}((\lambda_0 + |\partial_t|)^a C_{\gtrsim \lambda_0^2}(C_{\ll \lambda_1^2} u_{\lambda_1} Q_{\lambda_0 \lesssim \cdot \ll \lambda_0^2} v_{\lambda_0}))\|_{L^2_{\tau,\xi}} \\
&\lesssim \lambda_0^{s'-3a+b+2\theta'-2} \lambda_1 \|(\lambda_0 + |\partial_t|)^a C_{\ll \lambda_1^2} u_{\lambda_1}\|_{L^\infty_t L^2_x} \|(\lambda_0 + |\partial_t|)^a Q_{\lambda_0 \lesssim \cdot \ll \lambda_0^2} v_{\lambda_0}\|_{L^2_{t,x}} \\
&\lesssim \lambda_0^{s'-2a+b+2\theta'-2-l-\theta} \lambda_1^{1-s} (\lambda_0 + \lambda_1^2)^a \lambda_1^s \|C_{\ll \lambda_1^2} u_{\lambda_1}\|_{L^\infty_t L^2_x} \\
&\quad \times \lambda_0^{l-a} \|\langle \tau - |\xi| \rangle^\theta \mathcal{F}((\lambda_0 + |\partial_t|)^a Q_{\lambda_0 \lesssim \cdot \ll \lambda_0^2} v_{\lambda_0})\|_{L^2_{\tau,\xi}} \\
&\lesssim \begin{cases} \lambda_0^{s'-a+b+2\theta'-2-l-\theta} \lambda_1^{1-s} \|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}} \|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}, & \lambda_0 + \lambda_1^2 \sim \lambda_0 \\ \lambda_0^{s'-2a+b+2\theta'-2-l-\theta} \lambda_1^{1-s+2a} \|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}} \|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}, & \lambda_0 + \lambda_1^2 \sim \lambda_1^2. \end{cases}
\end{aligned}$$

Cases a and b can be summed up provided $s - l \leqslant 2 - \theta + a - b$ and $l \geqslant -1 + \theta + b$.

$\boldsymbol{d=1:}$ As for $d = 3$, we have

$$\begin{aligned}
&\lambda_0^{s'-2a+b} \|\langle \tau + |\xi|^2 \rangle^{\theta'-1} \mathcal{F}((\lambda_0 + |\partial_t|)^a C_{\gtrsim \lambda_0^2}(C_{\ll \lambda_1^2} u_{\lambda_1} Q_{\ll \lambda_0^2} v_{\lambda_0}))\|_{L^2_{\tau,\xi}} \\
&\lesssim \lambda_0^{s'-3a+b+2\theta'-2} \lambda_1^{\frac12} \|(\lambda_0 + |\partial_t|)^a C_{\ll \lambda_1^2} u_{\lambda_1}\|_{L^\infty_t L^2_x} \|(\lambda_0 + |\partial_t|)^a Q_{\ll \lambda_0^2} v_{\lambda_0}\|_{L^2_{t,x}} \\
&\lesssim \lambda_0^{s'-2a+b+2\theta'-2-l} \lambda_1^{\frac12-s} (\lambda_0 + \lambda_1^2)^a \lambda_1^s \|C_{\ll \lambda_1^2} u_{\lambda_1}\|_{L^\infty_t L^2_x} \\
&\quad \times \lambda_0^{l-a} \|\langle \tau - |\xi| \rangle^\theta \mathcal{F}((\lambda_0 + |\partial_t|)^a Q_{\ll \lambda_0^2} v_{\lambda_0})\|_{L^2_{\tau,\xi}} \\
&\lesssim \begin{cases} \lambda_0^{s'-2a+b+2\theta'-2-l} \lambda_1^{\frac12-s+2a} \|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}} \|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}, & \lambda_0 + \lambda_1^2 \sim \lambda_1^2 \\ \lambda_0^{s'-a+b+2\theta'-2-l} \lambda_1^{\frac12-s} \|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}} \|v_{\lambda_0}\|_{W^{l,s,\theta}_{\lambda_0}}, & \lambda_0 + \lambda_1^2 \sim \lambda_0. \end{cases}
\end{aligned}$$

Provided $s - l \leqslant 2 - 2\theta + a - b$ and $l \geqslant -\frac32 + 2\theta + b$, we can sum the above estimates to obtain (3.6.4).

We prove (3.6.5) as follows:

$$\lambda_0^{s+2\theta-3}\|P_{\ll\lambda_0^2}(u_{\lambda_1}v_{\lambda_0})\|_{L_t^\infty L_x^2} \lesssim \lambda_0^{s+2\theta-3}\lambda_1^{\frac{d}{2}}\|u_{\lambda_1}\|_{L_t^\infty L_x^2}\|v_{\lambda_0}\|_{L_t^\infty L_x^2}$$
$$\lesssim \lambda_0^{s+2\theta-3-l}\lambda_1^{\frac{d}{2}-l}\|u_{\lambda_1}\|_{L_t^\infty H_x^s}\|v_{\lambda_0}\|_{L_t^\infty H_x^l},$$

which can be summed up to obtain (3.6.5) if $s-l<3-2\theta$ and $l>-3+2\theta+\frac{d}{2}$.
From the conclusions made at the end of each subcase, we see that following conditions on the parameters are required to ensure the validity of the estimates (3.6.3) and (3.6.4) for $d\leqslant 3$:

$$\bullet\ l\geqslant -\frac{3}{2}+2\theta+b \quad \bullet\ \beta\geqslant -\frac{3}{2}+2\theta+a+b$$
$$\bullet\ s-l\leqslant \min(2-2\theta+a-b,3-2\theta) \quad \bullet\ s-\beta\leqslant \min\Big(3-2\theta-b,\frac{5}{2}-2\theta\Big). \tag{3.6.9}$$

Case II. High to high interaction ($\lambda_1\sim\lambda_0$)
For $\mu\lesssim\lambda_1$, we decompose u_{λ_1} and v_μ as follows:

$$u_{\lambda_1}=C_{\ll\lambda_1^2}u_{\lambda_1}+C_{\gtrsim\lambda_1^2}u_{\lambda_1},\quad v_\mu=Q_{\ll\mu^2}v_\mu+Q_{\gtrsim\mu^2}v_\mu.$$

The following interactions can be distinguished on the basis of the size of the modulation:

1. $\boldsymbol{H\times H\to H}$

We consider two cases for the temporal frequencies:
a. $|\tau_1|\lesssim\lambda_0^2$ or $|\tau_2|\lesssim\lambda_0^2$
Since at least one of the temporal frequencies τ_S or τ_W has size $\lesssim\lambda_0^2$, we can apply Bernstein's inequality in the time variable. We use the size of the modulation, the product estimate and Bernstein's inequality in space and time variables to obtain

$$\begin{aligned}
&\lambda_0^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda_0+|\partial_t|)^aC_{\gtrsim\lambda_0^2}(C_{\gtrsim\lambda_1^2}u_{\lambda_1}Q_{\gtrsim\mu^2}v_\mu))\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda_0^{s'-3a+b+2\theta'-1}\mu^{\frac{d}{2}}\|(\lambda_1+|\partial_t|)^aC_{\gtrsim\lambda_1^2}u_{\lambda_1}\|_{L^2_{t,x}}\|(\lambda_0+|\partial_t|)^aQ_{\gtrsim\mu^2}v_\mu\|_{L^2_{t,x}}\\
&\lesssim \lambda_0^{-a+b-1}\mu^{\frac{d}{2}+1-\beta}\sup_{|\tau_2|\gtrsim\mu^2}\frac{(\lambda_0+|\tau_2|)^a}{\langle\tau_2-\mu\rangle}\lambda_1^{s'-2a}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda_1+|\partial_t|)^aC_{\gtrsim\lambda_1^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}\\
&\quad\times\mu^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\mu^2}v_\mu)\|_{L^2_{\tau,\xi}}\\
&\lesssim\begin{cases}\lambda_0^{b-1}\mu^{\frac{d}{2}-1-\beta}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_\mu\|_{W^{l,s,\theta}_\mu}, & |\tau_2|\lesssim\lambda_0\\ \lambda_0^{-a+b-1}\mu^{\frac{d}{2}-\beta+2a-1}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_\mu\|_{W^{l,s,\theta}_\mu}, & \lambda_0\lesssim|\tau_2|.\end{cases}
\end{aligned}$$

Since $b<1$ and $-a+b<1$, we require $\beta\geqslant -2+\frac{d}{2}+a+b$ to sum the above estimates.

b. $|\tau_1|,|\tau_2|\gg\lambda_0^2$
We employ the same steps as in the previous case but cannot use Bernstein's inequality wrt the temporal frequencies. Instead, we make use of the high modulation of the wave.

$$\begin{aligned}
&\lambda_0^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda_0+|\partial_t|)^aC_{\gtrsim\lambda_0^2}(C_{\gg\lambda_0^2}u_{\lambda_1}Q_{\gg\lambda_0^2}v_\mu))\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda_0^{s'-3a+b+2\theta'-2}\mu^{\frac{d}{2}}\|(\lambda_1+|\partial_t|)^aC_{\gg\lambda_1^2}u_{\lambda_1}\|_{L_t^\infty L_x^2}\|(\lambda_0+|\partial_t|)^aQ_{\gg\lambda_0^2}v_\mu\|_{L^2_{t,x}}
\end{aligned}$$

$$\lesssim \lambda_0^{-a+b+2\theta-2}\mu^{\frac{d}{2}-\beta+1}\sup_{|\tau_2|\gg\lambda_0^2}\frac{(\lambda_0+|\tau_2|)^a}{\langle\tau_2-\lambda_0\rangle}\lambda_1^{s'-2a}\|(\lambda_1+|\partial_t|)^a C_{\gg\lambda_1^2}u_{\lambda_1}\|_{L_t^\infty L_x^2}$$
$$\times\,\mu^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gg\lambda_0^2}v_\mu)\|_{L^2_{\tau,\xi}}$$
$$\lesssim \lambda_0^{a+b+2\theta-4}\mu^{\frac{d}{2}-\beta+1}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_\mu\|_{W^{l,s,\theta}_\mu}.$$

We can sum the above up provided $\beta \geqslant -3+\frac{d}{2}+2\theta+a+b$.

2. $\boldsymbol{H\times H\to L}$
We consider the expression I and apply Hölder's and Bernstein's inequality to obtain

$$|I(\mathcal{F}(Q_{\gtrsim\mu^2}v_\mu),\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0}),\mathcal{F}(C_{\gtrsim\lambda_1^2}u_{\lambda_1}))|$$
$$\lesssim \mu^{\frac{d}{2}}\|C_{\gtrsim\lambda_1^2}u_{\lambda_1}\|_{L^2_{t,x}}\|Q_{\gtrsim\mu^2}v_\mu\|_{L^2_{t,x}}\|C_{\ll\lambda_0^2}w_{\lambda_0}\|_{L_t^\infty L_x^2}$$
$$\lesssim \mu^{\frac{d}{2}-1-\beta}\lambda_1^{a-1}\lambda_1^{s'-2a}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda_1+|\partial_t|)^a C_{\gtrsim\lambda_1^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}$$
$$\times\,\mu^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\mu^2}v_\mu)\|_{L^2_{\tau,\xi}}\lambda_0^{-s}\|C_{\ll\lambda_0^2}w_{\lambda_0}\|_{L_t^pL_x^2}$$
$$\lesssim \lambda_1^{a-1}\mu^{\frac{d}{2}-1-\beta}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_\mu\|_{W^{l,s,\theta}_\mu}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.$$

To obtain the last second inequality, we use Bernstein's inequality for $\frac{1}{p}=\theta-\frac{1}{2}$ and the last inequality follows from (2.7.8). We require $\beta\geqslant -2+\frac{d}{2}+a$ to sum this estimate to obtain (3.6.3).

3. $\boldsymbol{H\times L\to H}$
Using the size of the modulation weight, the product estimate and Bernstein's inequality, we get

$$\lambda_0^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda_0+|\partial_t|)^a C_{\gtrsim\lambda_0^2}(C_{\gtrsim\lambda_1^2}u_{\lambda_1}Q_{\ll\mu^2}v_\mu))\|_{L^2_{\tau,\xi}}$$
$$\lesssim \lambda_0^{s'-3a+b+2\theta'-2}\mu^{\frac{d}{2}}\|(\lambda_1+|\partial_t|)^a C_{\gtrsim\lambda_1^2}u_{\lambda_1}\|_{L^2_{t,x}}\|(\lambda_0+|\partial_t|)^a Q_{\ll\mu^2}v_\mu\|_{L_t^\infty L_x^2}$$
$$\lesssim \lambda_0^{-a+b-2}\mu^{\frac{d}{2}-l+a}\sup_{|\tau_2|\ll\mu^2}\left(\frac{\lambda_0+|\tau_2|}{\mu+|\tau_2|}\right)^a\lambda_1^{s'-2a}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda_1+|\partial_t|)^a C_{\gtrsim\lambda_1^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}$$
$$\times\,\mu^{l-a}\|(\mu+|\partial_t|)^a Q_{\ll\mu^2}v_\mu\|_{L_t^\infty L_x^2}$$
$$\lesssim \begin{cases}\lambda_0^{b-2}\mu^{\frac{d}{2}-l}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_\mu\|_{W^{l,s,\theta}_\mu}, & |\tau_2|\lesssim\lambda_0,\\ \lambda_0^{-a+b-2}\mu^{\frac{d}{2}-l+a}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_\mu\|_{W^{l,s,\theta}_\mu}, & \lambda_0\lesssim|\tau_2|\end{cases}$$
$$\lesssim \lambda_0^{b-2}\mu^{\frac{d}{2}-l}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_\mu\|_{W^{l,s,\theta}_\mu}.$$

The constraint $l\geqslant -2+\frac{d}{2}+b$ is required to sum the above up.

4. $\boldsymbol{L\times H\to H}$
Using the size of the modulation, the product estimate and Bernstein's inequality, we have

$$\lambda_0^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda_0+|\partial_t|)^a C_{\gtrsim\lambda_0^2}(C_{\ll\lambda_1^2}u_{\lambda_1}Q_{\gtrsim\mu^2}v_\mu))\|_{L^2_{\tau,\xi}}$$
$$\lesssim \lambda_0^{s'-3a+b+2\theta'-2}\mu^{\frac{d}{2}}\|(\lambda_1+|\partial_t|)^a C_{\ll\lambda_1^2}u_{\lambda_1}\|_{L_t^\infty L_x^2}\|(\lambda_1+|\partial_t|)^a Q_{\gtrsim\mu^2}v_\mu\|_{L^2_{t,x}}$$

$$
\begin{aligned}
&\lesssim \lambda_0^{-a+b+2\theta-2}\mu^{\frac{d}{2}-\beta+1}\sup_{|\tau_2|\gtrsim\mu^2}\frac{(\lambda_0+|\tau_2|)^a}{\langle\mu-\tau_2\rangle}\ \lambda_1^s\|C_{\ll\lambda_1^2}u_{\lambda_1}\|_{L_t^\infty L_x^2}\\
&\quad\times\mu^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\mu^2}v_\mu)\|_{L^2_{\tau,\xi}}\\
&\lesssim\begin{cases}\lambda_0^{b+2\theta-2}\mu^{\frac{d}{2}-\beta-1}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_\mu\|_{W^{l,s,\theta}_\mu}, & |\tau_2|\lesssim\lambda_0\\ \lambda_0^{-a+b+2\theta-2}\mu^{\frac{d}{2}-\beta-1+2a}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_\mu\|_{W^{l,s,\theta}_\mu}, & \lambda_0\lesssim|\tau_2|.\end{cases}
\end{aligned}
$$

With $b\leqslant 2-2\theta$ and $\beta\geqslant -3+\frac{d}{2}+2\theta+a+b$, we can sum this estimate to obtain (3.6.4).

5. $\boldsymbol{L\times H\to L}$

Since u_{λ_1} has temporal frequencies of size $\sim\lambda_1^2$, we observe that the temporal frequencies τ_2 are such that $\mu^2\lesssim|\tau_2|\lesssim\lambda_0^2$.

$\boldsymbol{d=2,3:}$ We have the standard decomposition:

$$
\begin{aligned}
&|I(\mathcal{F}(Q_{\mu^2\lesssim\cdot\lesssim\lambda_0^2}v_\mu),\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0}),\mathcal{F}(C_{\ll\lambda_1^2}u_{\lambda_1}))|\\
&\lesssim\Big|I\Big(\sum_{\mu^2\lesssim L_2\lesssim\lambda_0^2}\mathcal{F}(Q_{L_2}v_\mu),\sum_{L_0\ll\lambda_0^2}\mathcal{F}(C_{L_0}w_{\lambda_0}),\sum_{L_1\ll\lambda_1^2}\mathcal{F}(C_{L_1}u_{\lambda_1})\Big)\Big|\\
&\lesssim\sum_{\substack{L_1,L_0\ll\lambda_0^2,\\ \mu^2\lesssim L_2\lesssim\lambda_0^2}}\Big|I(\mathcal{F}(Q_{L_2}v_\mu),\mathcal{F}(C_{L_0}w_{\lambda_0}),\mathcal{F}(C_{L_1}u_{\lambda_1}))\Big|.
\end{aligned}\tag{3.6.10}
$$

We apply Lemma 3.4.6 to the above to obtain

$$
\begin{aligned}
\text{RHS of (3.6.10)}&\lesssim\sum_{\substack{L_1,L_0\ll\lambda_0^2,\\ \mu^2\lesssim L_2\lesssim\lambda_0^2}}\lambda_1^{-\frac{1}{2}}\log\lambda_1(L_0L_1L_2)^{\frac{1}{2}}\|C_{L_1}u_{\lambda_1}\|_{L^2_{t,x}}\|Q_{L_2}v_\mu\|_{L^2_{t,x}}\|C_{L_0}w_{\lambda_0}\|_{L^2_{t,x}}\\
&\lesssim\mu^{-\beta}\lambda_0^{2\theta-\frac{3}{2}}\log\lambda_1\log\mu\ \lambda_1^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\lambda_1^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}\\
&\quad\times\mu^\beta\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\mu^2\lesssim\cdot\lesssim\lambda_0^2})v_\mu\|_{L^2_{\tau,\xi}}\lambda_0^{-s}\|\langle\tau+|\xi|^2\rangle^{1-\theta}\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0})\|_{L^2_{\tau,\xi}}\\
&\lesssim\mu^{-\beta}\lambda_0^{2\theta-\frac{3}{2}}\log\lambda_1\log\mu\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_\mu\|_{W^{l,s,\theta}_\mu}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.
\end{aligned}
$$

$\boldsymbol{d=1:}$ We apply Hölder's inequality and bilinear Strichartz estimate for wave-Schrödinger interaction (3.4.10) as follows:

$$
\begin{aligned}
&|I(\mathcal{F}(Q_{\mu^2\lesssim\cdot\lesssim\lambda_0^2}v_\mu),\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0}),\mathcal{F}(C_{\ll\lambda_1^2}u_{\lambda_1}))|\\
&\lesssim\|C_{\ll\lambda_1^2}u_{\lambda_1}\|_{L^2_{t,x}}\|\overline{Q_{\mu^2\lesssim\cdot\lesssim\lambda_0^2}v_\mu}C_{\ll\lambda_0^2}w_{\lambda_0}\|_{L^2_{t,x}}\\
&\lesssim\|C_{\ll\lambda_1^2}u_{\lambda_1}\|_{L^2_{t,x}}\sum_{\substack{L_0\ll\lambda_0^2,\\ \mu^2\lesssim L_2\lesssim\lambda_0^2}}(L_0L_2)^{\frac{1}{2}}\frac{1}{\lambda_0^{\frac{1}{2}}}\|Q_{L_2}v_\mu\|_{L^2_{t,x}}\|C_{L_0}w_{\lambda_0}\|_{L^2_{t,x}}\\
&\lesssim\lambda_0^{-\frac{3}{2}+2\theta}\log\mu\ \mu^{-\beta}\lambda_1^s\|\langle\tau+|\xi|^2\rangle^\theta C_{\ll\lambda_1^2}u_{\lambda_1}\|_{L^2_{t,x}}\mu^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\mu^2\lesssim\cdot\lesssim\lambda_0^2}v_\mu)\|_{L^2_{\tau,\xi}}\\
&\quad\times\lambda_0^{-s}\|\langle\tau+|\xi|^2\rangle^{1-\theta}\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0})\|_{L^2_{\tau,\xi}}\\
&\lesssim\lambda_0^{-\frac{3}{2}+2\theta}\log\mu\ \mu^{-\beta}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_\mu\|_{W^{l,s,\theta}_\mu}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.
\end{aligned}
$$

The constraint $\beta>-\frac{3}{2}+2\theta$ enables us to sum the above estimate for $d\leqslant 3$.

For $\mu \sim \lambda_1 \sim 1$ in $d \leqslant 3$, we have

$$\begin{aligned}
&|I(\mathcal{F}(Q_{\mu^2 \lesssim \cdot \lesssim \lambda_0^2} v_\mu), \mathcal{F}(C_{\ll \lambda_0^2} w_{\lambda_0}), \mathcal{F}(C_{\ll \lambda_1^2} u_{\lambda_1}))| \\
&\lesssim \|C_{\ll \lambda_1^2} u_{\lambda_1}\|_{L_t^\infty L_x^2} \|Q_{\mu^2 \lesssim \cdot \lesssim \lambda_0^2} v_\mu\|_{L^2_{t,x}} \|C_{\ll \lambda_0^2} w_{\lambda_0}\|_{L^2_{t,x}} \\
&\lesssim \|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}} \|v_\mu\|_{W^{l,s,\theta}_\mu} \|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.
\end{aligned}$$

Remark 3.6.3. In case the wave temporal frequencies are of size $\sim \lambda_0^2$, a simpler argument applies.

6. $\boldsymbol{L \times L \to L}$

$\boldsymbol{d = 2, 3}$ **:** We consider the expression I, decompose the space-time Fourier supports of $C_{\ll \lambda_1^2} u_{\lambda_1}, Q_{\ll \lambda_0^2} v_\mu$ and $C_{\ll \lambda_0^2} w_{\lambda_0}$ into pieces of size L_1, L_2 and L_0, respectively to obtain

$$\begin{aligned}
&|I(\mathcal{F}(Q_{\ll \lambda_0^2} v_\mu), \mathcal{F}(C_{\ll \lambda_0^2} w_{\lambda_0}), \mathcal{F}(C_{\ll \lambda_1^2} u_{\lambda_1}))| \\
&\lesssim \sum_{L_1, L_2, L_0 \ll \lambda_0^2} \Big| I(\mathcal{F}(Q_{L_2} v_\mu), \mathcal{F}(C_{L_0} w_{\lambda_0}), \mathcal{F}(C_{L_1} u_{\lambda_1})) \Big|.
\end{aligned} \tag{3.6.11}$$

Using Lemma 3.4.6, we can bound the right-hand side of (3.6.11) by

$$\begin{aligned}
&\sum_{L_1, L_2, L_0 \ll \lambda_0^2} \lambda_1^{-\frac{1}{2}} \log \lambda_1 (L_0 L_1 L_2)^{\frac{1}{2}} \|C_{L_1} u_{\lambda_1}\|_{L^2_{t,x}} \|Q_{L_2} v_\mu\|_{L^2_{t,x}} \|C_{L_0} w_{\lambda_0}\|_{L^2_{t,x}} \\
&\lesssim \mu^{-l} \lambda_0^{2\theta - \frac{3}{2}} \log \lambda_1 \ \lambda_1^s \|\langle \tau + |\xi|^2 \rangle^\theta \mathcal{F}(C_{\ll \lambda_1^2} u_{\lambda_1})\|_{L^2_{\tau,\xi}} \\
&\quad \times \mu^{l-a} \|\langle \tau - |\xi| \rangle^\theta (\mu + |\partial_t|)^a \mathcal{F}(Q_{\ll \lambda_0^2} v_\mu)\|_{L^2_{\tau,\xi}} \lambda_0^{-s} \|\langle \tau + |\xi|^2 \rangle^{1-\theta} \mathcal{F}(C_{\ll \lambda_0^2} w_{\lambda_0})\|_{L^2_{\tau,\xi}} \\
&\lesssim \mu^{-l} \lambda_0^{2\theta - \frac{3}{2}} \log \lambda_1 \|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}} \|v_\mu\|_{W^{l,s,\theta}_\mu} \|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.
\end{aligned}$$

$\boldsymbol{d = 1}$: Using Cauchy-Schwarz inequality, we have

$$|I(\mathcal{F}(Q_{\ll \lambda_0^2} v_\mu), \mathcal{F}(C_{\ll \lambda_0^2} w_{\lambda_0}), \mathcal{F}(C_{\ll \lambda_1^2} u_{\lambda_1}))| \lesssim \|C_{\ll \lambda_1^2} u_{\lambda_1}\|_{L^2_{t,x}} \|\overline{Q_{\ll \lambda_0^2} v_\mu} C_{\ll \lambda_0^2} w_{\lambda_0}\|_{L^2_{t,x}}. \tag{3.6.12}$$

We decompose the space-time Fourier support of $\overline{Q_{\ll \lambda_0^2} v_\mu}$ and $C_{\ll \lambda_0^2} w_{\lambda_0}$ into pieces of size L_2 and L_0 respectively and apply the bilinear estimate (3.4.10) for Schrödinger-wave interaction to obtain that the right-hand side of (3.6.12) can be bounded by

$$\begin{aligned}
&\|C_{\ll \lambda_1^2} u_{\lambda_1}\|_{L^2_{t,x}} \sum_{L_0, L_2 \ll \lambda_0^2} (L_0 L_2)^{\frac{1}{2}} \frac{1}{\lambda_0^{\frac{1}{2}}} \|Q_{L_2} v_\mu\|_{L^2_{t,x}} \|C_{L_0} w_{\lambda_0}\|_{L^2_{t,x}} \\
&\lesssim \lambda_0^{-\frac{3}{2} + 2\theta} \mu^{-l} \ \lambda_1^s \|\langle \tau + |\xi|^2 \rangle^\theta \mathcal{F}(C_{\ll \lambda_1^2} u_{\lambda_1})\|_{L^2_{\tau,\xi}} \lambda_0^{-s} \|\langle \tau + |\xi|^2 \rangle^{1-\theta} \mathcal{F}(C_{\ll \lambda_0^2} w_{\lambda_0})\|_{L^2_{\tau,\xi}} \\
&\quad \times \mu^{l-a} \|\langle \tau - |\xi| \rangle^\theta \mathcal{F}((\lambda_0 + |\partial_t|)^a Q_{\ll \lambda_0^2} v_\mu)\|_{L^2_{\tau,\xi}} \\
&\lesssim \lambda_0^{-\frac{3}{2} + 2\theta} \mu^{-l} \|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}} \|v_\mu\|_{W^{l,s,\theta}_\mu} \|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.
\end{aligned}$$

For $l > -\frac{3}{2} + 2\theta$, we can sum the estimate for $d \leqslant 3$ to obtain (3.6.3).

We prove (3.6.5) as follows:

$$\lambda_0^{s+2\theta-3}\|P_{\ll\lambda_0^2}(u_{\lambda_1}v_\mu)\|_{L_t^\infty L_x^2} \lesssim \lambda_0^{s+2\theta-3}\mu^{\frac{d}{2}}\|u_{\lambda_1}\|_{L_t^\infty L_x^2}\|v_\mu\|_{L_t^\infty L_x^2}$$
$$\lesssim \lambda_0^{2\theta-3}\mu^{\frac{d}{2}-l}\|u_{\lambda_1}\|_{L_t^\infty H_x^s}\|v_\mu\|_{L_t^\infty H_x^l},$$

which requires $l > -3+2\theta+\frac{d}{2}$ for summability.

For $\mu \sim \lambda_0 \sim 1$, the estimate (3.6.3) holds with trivial modification.
We conclude that we require

$$\bullet\ l > \max\Big(-\frac{1}{2}+b, -\frac{3}{2}+2\theta\Big) \quad \bullet\ \beta > -\frac{3}{2}+2\theta+a+b \tag{3.6.13}$$

to obtain the estimates (3.6.3) and (3.6.4) when $\lambda_0 \sim \lambda_1$ for $d \leqslant 3$.

Case III. Low to high interaction ($\lambda_0 \ll \lambda_1$)
We decompose u_{λ_1} and v_{λ_1} as follows:

$$u_{\lambda_1} = C_{\ll\lambda_1^2}u_{\lambda_1} + C_{\gtrsim\lambda_1^2}u_{\lambda_1}, \quad v_{\lambda_1} = Q_{\ll\lambda_1^2}v_{\lambda_1} + Q_{\gtrsim\lambda_1^2}v_{\lambda_1}$$

1. ***H* × *H* → *H***
Using the size of the modulation, Bernstein's inequality and the product estimate, we get

$$\lambda_0^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda_0+|\partial_t|)^a C_{\gtrsim\lambda_0^2}(C_{\gtrsim\lambda_1^2}u_{\lambda_1}Q_{\gtrsim\lambda_1^2}v_{\lambda_1}))\|_{L^2_{\tau,\xi}}$$
$$\lesssim \lambda_0^{s'-2a+b+2\theta'-2+\frac{d}{2}}\lambda_1^{-a}\|(\lambda_1+|\partial_t|)^a C_{\gtrsim\lambda_1^2}u_{\lambda_1}\|_{L_t^\infty L_x^2}\|(\lambda_1+|\partial_t|)^a Q_{\gtrsim\lambda_1^2}v_{\lambda_1}\|_{L^2_{t,x}}$$
$$\lesssim \lambda_0^{s-2a+b+2\theta-2+\frac{d}{2}}\lambda_1^{-s+a-\beta+1}\sup_{|\tau_2|\gtrsim\lambda_1^2}\frac{(\lambda_1+|\tau_2|)^a}{\langle\tau_2-\lambda_1\rangle}\lambda_1^{s'-2a}\|(\lambda_1+|\partial_t|)^a C_{\gtrsim\lambda_1^2}u_{\lambda_1}\|_{L_t^\infty L_x^2}$$
$$\times\ \lambda_1^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\lambda_1^2}v_{\lambda_1})\|_{L^2_{\tau,\xi}}$$
$$\lesssim \lambda_0^{s-2a+b+2\theta-2+\frac{d}{2}}\lambda_1^{-s-\beta+3a-1}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_1}\|_{W^{l,s,\theta}_{\lambda_1}}.$$

If $s+\beta \geqslant -1+3a$ and $\beta \geqslant -3+\frac{d}{2}+2\theta+a+b$, we can obtain (3.6.4) by summing up the above estimate.

2. ***H* × *H* → *L***
Since $|\tau_2| \gtrsim \lambda_1^2$, from the relation $|\tau_0+|\xi_0|^2| = |\tau_1+\tau_2+|\xi_0|^2| \ll \lambda_0^2$, it follows that $|\tau_1| \gtrsim \lambda_1^2$. We consider the expression I and use Hölder's and Bernstein's inequality as follows:

$$|I(\mathcal{F}(Q_{\gtrsim\lambda_1^2}v_{\lambda_1}),\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0}),\mathcal{F}(C_{\gtrsim\lambda_1^2}u_{\lambda_1}))|$$
$$\lesssim \lambda_0^{\frac{d}{2}}\|C_{\gtrsim\lambda_1^2}u_{\lambda_1}\|_{L^2_{t,x}}\|Q_{\gtrsim\lambda_1^2}v_{\lambda_1}\|_{L^2_{t,x}}\|C_{\ll\lambda_0^2}w_{\lambda_0}\|_{L_t^\infty L_x^2}$$
$$\lesssim \lambda_0^{\frac{d}{2}+s+2\theta-1}\lambda_1^{-s-2\theta-\beta-1}\lambda_1^{s'-2a}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda_1+|\partial_t|)^a C_{\gtrsim\lambda_1^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}$$
$$\times\ \lambda_1^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\lambda_1^2}v_{\lambda_1})\|_{L^2_{\tau,\xi}}\lambda_0^{-s}\|C_{\ll\lambda_0^2}w_{\lambda_0}\|_{L_t^p L_x^2}$$
$$\lesssim \lambda_0^{\frac{d}{2}+s+2\theta-1}\lambda_1^{-s-2\theta-\beta-1}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_1}\|_{W^{l,s,\theta}_{\lambda_1}}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}},$$

where the last inequality follows from the embedding (2.7.8) applied for $\frac{1}{p} = \theta - \frac{1}{2}$. Since $s + \frac{d}{2} + 2\theta - 1 > 0$, we require $\beta \geqslant -2 + \frac{d}{2}$ in order to obtain (3.6.3).

3. $\boldsymbol{H \times L \to H}$

Using the size of the modulation, the product estimate and Bernstein's inequality, we have

$$\begin{aligned}
&\lambda_0^{s'-2a+b} \|\langle \tau + |\xi|^2\rangle^{\theta'-1} \mathcal{F}((\lambda_0 + |\partial_t|)^a C_{\gtrsim\lambda_0^2}(C_{\gtrsim\lambda_1^2} u_{\lambda_1} Q_{\ll\lambda_1^2} v_{\lambda_1}))\|_{L^2_{\tau,\xi}} \\
&\lesssim \lambda_0^{s'-2a+b+2\theta'-2+\frac{d}{2}} \lambda_1^{-a} \|(\lambda_1 + |\partial_t|)^a C_{\gtrsim\lambda_1^2} u_{\lambda_1}\|_{L^2_{t,x}} \|(\lambda_1 + |\partial_t|)^a Q_{\ll\lambda_1^2} v_{\lambda_1}\|_{L^\infty_t L^2_x} \\
&\lesssim \lambda_0^{s-2a+b+2\theta-2+\frac{d}{2}} \lambda_1^{-s-2\theta-l+2a} \lambda_1^{s'-2a} \|\langle \tau + |\xi|^2\rangle^{\theta'} \mathcal{F}((\lambda_1 + |\partial_t|)^a C_{\gtrsim\lambda_1^2} u_{\lambda_1})\|_{L^2_{\tau,\xi}} \\
&\quad\times \lambda_1^{l-a} \|(\lambda_1 + |\partial_t|)^a Q_{\ll\lambda_1^2} v_{\lambda_1}\|_{L^\infty_t L^2_x} \\
&\lesssim \lambda_0^{s-2a+b+2\theta-2+\frac{d}{2}} \lambda_1^{-s-2\theta-l+2a} \|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}} \|v_{\lambda_1}\|_{W^{l,s,\theta}_{\lambda_1}}.
\end{aligned}$$

The estimate can be summed up provided $s + l \geqslant -2\theta + 2a$ and $l \geqslant -2 + \frac{d}{2} + b$.

4. $\boldsymbol{H \times L \to L}$

Since low modulation for the wave v_{λ_1} implies $|\tau_2| \ll \lambda_1^2$, we observe from the relation

$$|\tau_0 + |\xi_0|^2| = |\tau_1 + \tau_2 - |\xi_2| + |\xi_2| + |\xi_0|^2| \ll \lambda_0^2,$$

and

$$|\tau_2 - |\xi_2|| \ll \lambda_1^2, \quad |\xi_2| + |\xi_0|^2 \sim \max(\lambda_1, \lambda_0^2),$$

that the temporal frequencies $|\tau_1| \ll \lambda_1^2$. Using this information, we consider the following subcases for the size of the temporal frequencies τ_1 and τ_2 and make conclusions in the last two columns:

$\lvert\tau_1\rvert$	$\lvert\tau_2\rvert$	$\lvert\tau_0\rvert$	**Conclusion**
$\lesssim \lambda_1$	$\lesssim \lambda_1$	$\lesssim \lambda_1$	$\lambda_0^2 \lesssim \lambda_1$
$\lesssim \lambda_1$	$\lambda_1 \ll \cdot \ll \lambda_1^2$	$\lambda_1 \ll \cdot \ll \lambda_1^2$	$\lvert\tau_W\rvert \sim \lvert\tau_0\rvert,\ \lambda_1 \ll \lambda_0^2$
$\lambda_1 \ll \cdot \ll \lambda_1^2$	$\lesssim \lambda_1$	$\lambda_1 \ll \cdot \ll \lambda_1^2$	$\lvert\tau_S\rvert \sim \lvert\tau_0\rvert,\ \lambda_1 \ll \lambda_0^2$
$\lambda_1 \ll \cdot \ll \lambda_1^2$	$\lambda_1 \ll \cdot \ll \lambda_1^2$	$\lambda_1 \ll \cdot \ll \lambda_1^2$	$\lvert\tau_S\rvert, \lvert\tau_W\rvert \sim \lambda_0^2,\ \lambda_1 \ll \lambda_0^2$
$\lambda_1 \ll \cdot \ll \lambda_1^2$	$\lambda_1 \ll \cdot \ll \lambda_1^2$	$\ll \lambda_1$	$\lambda_0^2 \ll \lambda_1$

Table 3.1: $H \times L \to L$

In all the above cases, we can bound the weight $(\lambda_1 + |\tau_1|)^{-a}(\lambda_1 + |\tau_2|)^{-a}$ by $\lambda_1^{-a}\lambda_0^{-2a}$. We proceed to prove (3.6.3) by duality and consider the expression I. Using Cauchy-Schwarz inequality, we get

$$|I(\mathcal{F}(Q_{\ll\lambda_1^2} v_{\lambda_1}), \mathcal{F}(C_{\ll\lambda_0^2} w_{\lambda_0}), \mathcal{F}(C_{\sim\lambda_1^2} u_{\lambda_1}))| \lesssim \|C_{\sim\lambda_1^2} u_{\lambda_1}\|_{L^2_{t,x}} \|\overline{Q_{\ll\lambda_1^2} v_{\lambda_1}} C_{\ll\lambda_0^2} w_{\lambda_0}\|_{L^2_{t,x}}. \tag{3.6.14}$$

In order to apply the bilinear estimate (3.4.10), we decompose the space-time Fourier supports of the two terms in the right-hand side of the above display into pieces of size

L_2 and L_0, respectively and obtain the following for the right-hand side of (3.6.14):

$$\begin{aligned}
&\lesssim \|C_{\sim\lambda_1^2}u_{\lambda_1}\|_{L^2_{t,x}} \sum_{\substack{L_0\ll\lambda_0^2\\ L_2\ll\lambda_1^2}} \frac{\lambda_0^{\frac{d-1}{2}}}{\lambda_0^{\frac{1}{2}}}(L_0L_2)^{1/2}\|Q_{L_2}v_{\lambda_1}\|_{L^2_{t,x}}\|C_{L_0}w_{\lambda_0}\|_{L^2_{t,x}}\\
&\lesssim \lambda_0^{\frac{d}{2}-2+2\theta+s-2a}\lambda_1^{-s-l+2a-2\theta}\lambda_1^{s'-2a}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda_1+|\partial_t|)^a C_{\sim\lambda_1^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}\\
&\quad\times \lambda_1^{l-a}\|\langle\tau-|\xi|\rangle^{\theta}\mathcal{F}((\lambda_1+|\partial_t|)^a Q_{\ll\lambda_1^2}v_{\lambda_1})\|_{L^2_{\tau,\xi}}\lambda_0^{-s}\|\langle\tau+|\xi|^2\rangle^{1-\theta}\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0})\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda_0^{\frac{d}{2}-2+2\theta+s-2a}\lambda_1^{-s-l+2a-2\theta}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_1}\|_{W^{l,s,\theta}_{\lambda_1}}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.
\end{aligned}$$

$s+l\geqslant -2\theta+2a$ and $l\geqslant -2+\frac{d}{2}$ are the requirements for the summability of the estimate.

5. $\boldsymbol{L\times H\to L}$

We observe that the temporal frequencies $|\tau_2|\sim\lambda_1^2$. Using Hölder's inequality, we have

$$|I(\mathcal{F}(Q_{\sim\lambda_1^2}v_{\lambda_1}),\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0}),\mathcal{F}(C_{\ll\lambda_1^2}u_{\lambda_1}))|\lesssim \|Q_{\sim\lambda_1^2}v_{\lambda_1}\|_{L^2_{t,x}}\|C_{\ll\lambda_1^2}u_{\lambda_1}\overline{C_{\ll\lambda_0^2}w_{\lambda_0}}\|_{L^2_{t,x}}. \tag{3.6.15}$$

We apply the bilinear estimate (3.4.9) to the above by decomposing the space-time Fourier supports of $C_{\ll\lambda_1^2}u_{\lambda_1}$ and $\overline{C_{\ll\lambda_0^2}w_{\lambda_0}}$ into pieces of size L_1 and L_0 respectively and bound the right-hand side of (3.6.15) by

$$\begin{aligned}
&\|Q_{\sim\lambda_1^2}v_{\lambda_1}\|_{L^2_{t,x}}\sum_{\substack{L_0\ll\lambda_0^2\\ L_1\ll\lambda_1^2}}(L_0L_1)^{\frac{1}{2}}\frac{\lambda_0^{\frac{d-1}{2}}}{\lambda_1^{\frac{1}{2}}}\|C_{L_1}u_{\lambda_1}\|_{L^2_{t,x}}\|C_{L_0}w_{\lambda_0}\|_{L^2_{t,x}}\\
&\lesssim \lambda_0^{\frac{d-3}{2}+s+2\theta}\lambda_1^{-\beta-s-\frac{3}{2}}\lambda_1^{s}\|\langle\tau+|\xi|^2\rangle^{\theta}\mathcal{F}(C_{\ll\lambda_1^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}\\
&\quad\times\lambda_1^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\sim\lambda_1^2}v_{\lambda_1})\|_{L^2_{\tau,\xi}}\lambda_0^{-s}\|\langle\tau+|\xi|^2\rangle^{1-\theta}\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0})\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda_0^{\frac{d-3}{2}+s+2\theta}\lambda_1^{-\beta-s-\frac{3}{2}}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_1}\|_{W^{l,s,\theta}_{\lambda_1}}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.
\end{aligned}$$

One can sum the above estimate to obtain (3.6.3) provided $s+\beta\geqslant-\frac{3}{2}$ and $\beta\geqslant-3+\frac{d}{2}+2\theta$. For $d=1$, in the case $\lambda_0\sim\lambda_1\sim1$, we have

$$|I(\mathcal{F}(Q_{\sim\lambda_1^2}v_{\lambda_1}),\mathcal{F}(C_{\ll\lambda_0^2}w_{\lambda_0}),\mathcal{F}(C_{\ll\lambda_1^2}u_{\lambda_1}))|\lesssim\|v_{\lambda_1}\|_{W^{l,s,\theta}_{\lambda_1}}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|w_{\lambda_0}\|_{S^{-s,l,1-\theta}_{\lambda_0}}.$$

6. $\boldsymbol{L\times L\to H}$

From the constraint $\tau_0=\tau_1+\tau_2$, we note that the output temporal frequencies are of size $\sim\lambda_1^2$. This implies that the output has a modulation of size $\sim\lambda_1^2$. We use the size of the modulation and the temporal frequencies and employ the bilinear estimate for wave-Schrödinger interaction. By orthogonality arguments, we reduce the estimates to the case when the spatial Fourier supports of $C_{\ll\lambda_1^2}u_{\lambda_1}$ and $Q_{\ll\lambda_1^2}v_{\lambda_1}$ are of size $\sim\lambda_0$.

$$\begin{aligned}
&\lambda_0^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda_0+|\partial_t|)^a C_{\gtrsim\lambda_0^2}(C_{\ll\lambda_1^2}u_{\lambda_1}Q_{\ll\lambda_1^2}v_{\lambda_1}))\|_{L^2_{\tau,\xi}}\\
&\lesssim\lambda_0^{s'-2a+b}\lambda_1^{2\theta'-2+2a}\|C_{\ll\lambda_1^2}u_{\lambda_1}Q_{\ll\lambda_1^2}v_{\lambda_1}\|_{L^2_{t,x}}
\end{aligned}$$

$$\lesssim \lambda_0^{s'-2a+b}\lambda_1^{2\theta'-2+2a} \sum_{L_1,L_2\ll\lambda_1^2} \frac{\lambda_0^{\frac{d-1}{2}}}{\lambda_1^{\frac{1}{2}}}(L_1L_2)^{\frac{1}{2}}\|C_{L_1}u_{\lambda_1}\|_{L^2_{t,x}}\|Q_{L_2}v_{\lambda_1}\|_{L^2_{t,x}}$$
$$\lesssim \lambda_0^{s'-2a+b+\frac{d-1}{2}}\lambda_1^{2\theta'-\frac{5}{2}+2a-s-l}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_1}\|_{W^{l,s,\theta}_{\lambda_1}}.$$

To sum this estimate, we require $s+l \geqslant -\frac{5}{2}+2\theta'+2a$ and $l \geqslant -3+\frac{d}{2}+2\theta+b$.

7. $\boldsymbol{L \times H \to H}$

With Bernstein's inequality and the product estimate, we get

$$\lambda_0^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'-1}\mathcal{F}((\lambda_0+|\partial_t|)^a C_{\gtrsim\lambda_0^2}(C_{\ll\lambda^2}u_{\lambda_1}Q_{\gtrsim\lambda_1^2}v_{\lambda_1}))\|_{L^2_{\tau,\xi}}$$
$$\lesssim \lambda_0^{s'-2a+b+2\theta'-2+\frac{d}{2}}\lambda_1^{-a}\|(\lambda_1+|\partial_t|)^a C_{\ll\lambda^2}u_{\lambda_1}\|_{L^\infty_t L^2_x}\|(\lambda_1+|\partial_t|)^a Q_{\gtrsim\lambda_1^2}v_{\lambda_1}\|_{L^2_{t,x}}$$
$$\lesssim \lambda_0^{s'-2a+b+2\theta'-2+\frac{d}{2}}\lambda_1^{-s+a-\beta+1}\sup_{|\tau_2|\gtrsim\lambda_1^2}\frac{(\lambda_1+|\tau_2|)^a}{\langle\tau_2-\lambda_1\rangle}\,\lambda_1^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\lambda^2}u_{\lambda_1})\|_{L^2_{\tau,\xi}}$$
$$\times\,\lambda_1^{\beta-1}\|\langle\tau-|\xi|\rangle\mathcal{F}(Q_{\gtrsim\lambda_1^2}v_{\lambda_1})\|_{L^2_{\tau,\xi}}$$
$$\lesssim \lambda_0^{s'-2a+b+2\theta'-2+\frac{d}{2}}\lambda_1^{-s-\beta+3a-1}\|u_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|v_{\lambda_1}\|_{W^{l,s,\theta}_{\lambda_1}}.$$

If $s+\beta \geqslant -1+3a$ and $\beta \geqslant -3+\frac{d}{2}+2\theta+a+b$, we can obtain (3.6.4) by summing up the above estimate.

We prove (3.6.5) as follows:

$$\lambda_0^{s+2\theta-3}\|P^{(t)}_{\ll\lambda_0^2}(u_{\lambda_1}v_{\lambda_0})\|_{L^\infty_t L^2_x} \lesssim \lambda_0^{s+2\theta-3+\frac{d}{2}}\|u_{\lambda_1}\|_{L^\infty_t L^2_x}\|v_{\lambda_0}\|_{L^\infty_t L^2_x}$$
$$\lesssim \lambda_0^{s+2\theta-3+\frac{d}{2}}\lambda_1^{-s-l}\|u_{\lambda_1}\|_{L^\infty_t H^s_x}\|v_{\lambda_0}\|_{L^\infty_t H^l_x},$$

Hence, for $\lambda_0 \ll \lambda_1$, we conclude that the following constraints are required for summability in dimension $d \leqslant 3$:

$$\begin{aligned} &\bullet\; l \geqslant -\frac{3}{2}+2\theta+b \quad \bullet\; \beta \geqslant -\frac{3}{2}+2\theta+a+b \\ &\bullet\; s+l \geqslant \max\Big(-2\theta+2a, -\frac{5}{2}+2\theta'+2a\Big) \quad \bullet\; s+\beta \geqslant -1+3a. \end{aligned} \tag{3.6.16}$$

For $\frac{1}{2} \leqslant a < 1$, the above can be summed up provided $l > -\frac{3}{2}+2\theta, s+l > 0$ and $s-l < 3-2\theta$. It can be proved by imitating the proof of the product estimate $\|fg\|_{H^{s-3+2\theta}} \lesssim \|f\|_{H^s}\|g\|_{H^l}$ for each case above.
From (3.6.9), (3.6.13) an (3.6.16), we conclude that

$$\begin{aligned} &\bullet\; l > -\frac{3}{2}+2\theta+b \quad \bullet\; \beta > -\frac{3}{2}+2\theta+a+b \\ &\bullet\; s-l \leqslant \min\{2-2\theta+a-b, 3-2\theta\} \quad \bullet\; s-\beta \leqslant \min\Big(3-2\theta-b, \frac{5}{2}-2\theta\Big) \\ &\bullet\; s+l \geqslant \max\Big(-\frac{5}{2}+2\theta'+2a, -2\theta+2a\Big) \quad \bullet\, s+\beta \geqslant -1+3a \end{aligned}$$

are the requirements for the estimate (3.6.1) to hold. □

3.6.2 Wave nonlinearity

Theorem 3.6.4. *Let $d \leqslant 3$ and s, l satisfy (3.2.1). Then, there exist $a, b, s, l, \beta, \theta \in \mathbb{R}$ such that the estimate*

$$\||\nabla|(\overline{\varphi}\psi)\|_{R^{l,s,\theta-1}} \lesssim \|\varphi\|_{S^{s,l,\theta}}\|\psi\|_{S^{s,l,\theta}} \tag{3.6.17}$$

holds.

Proof. We choose the parameters as in (3.3.1). Using the definition of the norms, it suffices to prove the following:

$$\Big(\sum_{\mu\in 2^{\mathbb{N}}}\mu^{2(l-a+1)}\|\langle\tau-|\xi|\rangle^{\theta'-1}\mathcal{F}((\mu+|\partial_t|)^a Q_{\ll\mu^2}P_\mu(\bar{\varphi}\psi))\|^2_{L^2_{\tau,\xi}}\Big)^{\frac{1}{2}} \lesssim \|\varphi\|_{S^{s,l,\theta}}\|\psi\|_{S^{s,l,\theta}} \tag{3.6.18}$$

$$\Big(\sum_{\mu\in 2^{\mathbb{N}}}\mu^{2\beta}\|\mathcal{F}(Q_{\gtrsim\mu^2}P_\mu(\bar{\varphi}\psi))\|^2_{L^2_{\tau,\xi}}\Big)^{\frac{1}{2}} \lesssim \|\varphi\|_{S^{s,l,\theta}}\|\psi\|_{S^{s,l,\theta}} \tag{3.6.19}$$

$$\Big(\sum_{\mu\in 2^{\mathbb{N}}}\mu^{2(l+2\theta-2)}\|P_\mu(\bar{\varphi}\psi)\|^2_{L^\infty_t L^2_x}\Big)^{\frac{1}{2}} \lesssim \|\varphi\|_{S^{s,l,\theta}}\|\psi\|_{S^{s,l,\theta}} \tag{3.6.20}$$

The definition of high and low modulation remains the same as in Section 3.6.1, so do the abbreviations H and L. We now append 1 or 2 as subscripts with τ to denote the temporal frequencies of φ and ψ. The same is done for the spatial frequencies.
Employing the frequency trichotomy, we decompose the nonlinearity $P_\mu(\bar{\varphi}\psi)$ into high-low, balanced (high-high) and low-high interactions as follows:

$$\begin{aligned} P_\mu(\overline{\varphi}\psi) &= P_\mu(\overline{\varphi}\psi_{\ll\mu}) + \sum_{\mu\lesssim\lambda_1\sim\lambda_2} P_\mu(\overline{\varphi_{\lambda_1}}\psi_{\lambda_2}) + P_\mu(\overline{\varphi_{\ll\mu}}\psi) \\ &= \sum_{\lambda\ll\mu} P_\mu(\overline{\varphi_\mu}\psi_\lambda) + \sum_{\mu\lesssim\lambda_1\sim\lambda_2} P_\mu(\overline{\varphi_{\lambda_1}}\psi_{\lambda_2}) + \sum_{\lambda\ll\mu} P_\mu(\overline{\varphi_\lambda}\psi_\mu) \end{aligned}$$

It suffices to consider the first two interactions above.

Case I. High-low interaction ($\mu \gg \lambda$)

We decompose the spatially localised waves $\overline{\varphi_\mu}$ and ψ_λ as follows:

$$\overline{\varphi_\mu} = \overline{C_{\ll\mu^2}\varphi_\mu} + \overline{C_{\gtrsim\mu^2}\varphi_\mu}, \quad \psi_\lambda = C_{\ll\lambda^2}\psi_\lambda + C_{\gtrsim\lambda^2}\psi_\lambda.$$

The following cases can be identified from the size of the modulation.

1. $H \times H \to H$

Since $\mu^2 \lesssim |\tau_0 - |\xi_0|| = |\tau_1 - \tau_2 - |\xi_0||$, we conclude that at least one of the temporal frequencies τ_1 or τ_2 has size $\gtrsim \mu^2$. Since assuming $|\tau_2| \gtrsim \mu^2$ gives the same estimate, we let $|\tau_1| \gtrsim \mu^2$. Using Hölder's and Bernstein's inequalities, we obtain

$$\begin{aligned} &\mu^\beta\|\mathcal{F}(Q_{\gtrsim\mu^2}(\overline{C_{\gtrsim\mu^2}\varphi_\mu}C_{\gtrsim\lambda^2}\psi_\lambda))\|_{L^2_{\tau,\xi}} \\ &\lesssim \mu^\beta\lambda^{\frac{d}{2}}\|\overline{C_{\gtrsim\mu^2}\varphi_\mu}\|_{L^2_{t,x}}\|C_{\gtrsim\lambda^2}\psi_\lambda\|_{L^\infty_t L^2_x} \\ &\lesssim \mu^{\beta-s-2\theta-b}\lambda^{\frac{d}{2}-s+a}\mu^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\mu+|\partial_t|)^a C_{\gtrsim\mu^2}\varphi_\mu)\|_{L^2_{\tau,\xi}} \\ &\quad\times\lambda^{s'-2a}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda+|\partial_t|)^a C_{\gtrsim\lambda^2}\psi_\lambda)\|_{L^2_{\tau,\xi}} \end{aligned}$$

$$\lesssim \mu^{\beta-s-2\theta-b}\lambda^{\frac{d}{2}-s+a}\|\varphi_\mu\|_{S^{s,l,\theta}_\mu}\|\psi_\lambda\|_{S^{s,l,\theta}_\lambda}.$$

We require $s-\beta \geqslant -2\theta-b$ and $2s-\beta \geqslant \frac{d}{2}-2\theta+a-b$.

2. $H \times H \to L$

We consider the expression I and employ Hölder's and Bernstein's inequalities to obtain

$$\begin{aligned}
&|I(\mathcal{F}(Q_{\ll\mu^2}\eta_\mu),\mathcal{F}(C_{\gtrsim\lambda^2}\psi_\lambda),\mathcal{F}(C_{\gtrsim\mu^2}\varphi_\mu))| \\
&\lesssim \lambda^{\frac{d}{2}}\|C_{\gtrsim\mu^2}\varphi_\mu\|_{L^2_{t,x}}\|C_{\gtrsim\lambda^2}\psi_\lambda\|_{L^2_{t,x}}\|Q_{\ll\mu^2}\eta_\mu\|_{L^\infty_t L^2_x} \\
&\lesssim \lambda^{\frac{d}{2}-s+a-2\theta}\mu^{l-s+a-b}\mu^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\mu+|\partial_t|)^a C_{\gtrsim\mu^2}\varphi_\mu)\|_{L^2_{\tau,\xi}} \\
&\quad\times \lambda^{s'-2a}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda+|\partial_t|)^a C_{\gtrsim\lambda^2}\psi_\lambda)\|_{L^2_{\tau,\xi}}\ \mu^{-l-1}\|Q_{\ll\mu^2}\eta_\mu\|_{L^p_t L^2_x} \\
&\lesssim \lambda^{\frac{d}{2}-s+a-2\theta}\mu^{l-s+a-b}\|\varphi_\mu\|_{S^{s,l,\theta}_\mu}\|\psi_\lambda\|_{S^{s,l,\theta}_\lambda}\|\eta_\mu\|_{W^{-l,s,1-\theta}_\mu}.
\end{aligned}$$

The last inequality follows from (2.7.8) for $\frac{1}{p}=\theta-\frac{1}{2}$. We require $s-l \geqslant a-b$, $2s-l \geqslant \frac{d}{2}-1+2a-b$ to obtain (3.6.18).

3. $H \times L \to H$

Using the relation $\mu^2 \lesssim |\tau_0-|\xi_0||=|\tau_1-\tau_2-\mu|$ and $|\tau_2|\sim\lambda^2$, we find that $|\tau_1|\gtrsim\mu^2$. Employing Hölder's and Bernstein's inequality, we obtain

$$\begin{aligned}
&\mu^\beta\|\mathcal{F}(Q_{\gtrsim\mu^2}(\overline{P^{(t)}_{\gtrsim\mu^2}C_{\gtrsim\mu^2}\varphi_\mu}C_{\ll\lambda^2}\psi_\lambda))\|_{L^2_{\tau,\xi}} \\
&\lesssim \mu^\beta\lambda^{\frac{d}{2}}\|\overline{P^{(t)}_{\gtrsim\mu^2}C_{\gtrsim\mu^2}\varphi_\mu}\|_{L^2_{t,x}}\|C_{\ll\lambda^2}\psi_\lambda\|_{L^\infty_t L^2_x} \\
&\lesssim \mu^{\beta-s-b-2\theta}\lambda^{\frac{d}{2}-s}\mu^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\mu+|\partial_t|)^a C_{\gtrsim\mu^2}\varphi_\mu)\|_{L^2_{\tau,\xi}}\lambda^s\|C_{\ll\lambda^2}\psi_\lambda\|_{L^\infty_t L^2_x} \\
&\lesssim \mu^{\beta-s-b-2\theta}\lambda^{\frac{d}{2}-s}\|\varphi_\mu\|_{S^{s,l,\theta}_\mu}\|\psi_\lambda\|_{S^{s,l,\theta}_\lambda}.
\end{aligned}$$

For $s-\beta \geqslant -b-2\theta$ and $2s-\beta \geqslant \frac{d}{2}-2\theta-b$, we can sum this estimate and obtain (3.6.19).

4. $H \times L \to L$

From the relation $|\tau_0|=|\tau_1-\tau_2| \ll \mu^2$ and $|\tau_2|\sim\lambda^2$, we conclude that the temporal frequencies τ_1 are such that $|\tau_1|\ll\mu^2$. We use Cauchy-Schwarz inequality for the expression I as follows

$$|I(\mathcal{F}(Q_{\ll\mu^2}\eta_\mu),\mathcal{F}(C_{\ll\lambda^2}\psi_\lambda),\mathcal{F}(C_{\sim\mu^2}\varphi_\mu))| \lesssim \|C_{\sim\mu^2}\varphi_\mu\|_{L^2_{t,x}}\|C_{\ll\lambda^2}\psi_\lambda\overline{Q_{\ll\mu^2}\eta_\mu}\|_{L^2_{t,x}}. \tag{3.6.21}$$

To employ the bilinear estimate (3.4.10) for the last term in the above display, we decompose the space-time Fourier supports of $C_{\ll\lambda^2}\psi_\lambda$ and $\overline{Q_{\ll\mu^2}\eta_\mu}$ into pieces of size L_2 and L_0, respectively and bound the right-hand side of (3.6.21) by

$$\begin{aligned}
&\|C_{\sim\mu^2}\varphi_\mu\|_{L^2_{t,x}}\sum_{\substack{L_2\ll\lambda^2\\ L_0\ll\mu^2}}\frac{\lambda^{\frac{d-1}{2}}}{\lambda^{\frac{1}{2}}}(L_1L_0)^{\frac{1}{2}}\|C_{L_1}\psi_\lambda\|_{L^2_{t,x}}\|Q_{L_0}\eta_\mu\|_{L^2_{t,x}} \\
&\lesssim \mu^{-s+a-b+l}\lambda^{\frac{d}{2}-1-s}\ \mu^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\mu+|\partial_t|)^a C_{\sim\mu^2}\varphi_\mu)\|_{L^2_{\tau,\xi}}
\end{aligned}$$

$$\times \lambda^s \|\langle \tau + |\xi|^2\rangle^\theta \mathcal{F}(C_{\ll \lambda^2}\psi_\lambda)\|_{L^2_{\tau,\xi}} \mu^{-l-1} \|\langle \tau - |\xi|\rangle^{1-\theta} \mathcal{F}(Q_{\ll \mu^2}\eta_\mu)\|_{L^2_{\tau,\xi}}$$
$$\lesssim \mu^{-s+a-b+l}\lambda^{\frac{d}{2}-1-s} \, \|\varphi_\mu\|_{S^{s,l,\theta}_\mu} \|\psi_\lambda\|_{S^{s,l,\theta}_\lambda} \|\eta_\mu\|_{W^{-l,s,1-\theta}_\mu}.$$

Provided $s - l \geqslant a - b$ and $2s - l \geqslant \frac{d}{2} - 1 + a - b$, we can sum this estimate.

5. $L \times H \to H$

We consider two cases for the temporal frequencies τ_2:

a. $|\tau_2| \lesssim \mu^2$: We apply the bilinear estimate (3.4.9) directly by decomposing the space time Fourier supports of $\overline{C_{\ll \mu^2}\varphi_\mu}$ and $C_{\gtrsim \lambda^2}\psi_\lambda$ into pieces of size L_1 and L_2 respectively.

$$\mu^\beta \|\mathcal{F}(Q_{\sim \mu^2}(\overline{C_{\ll \mu^2}\varphi_\mu} C_{\lambda^2 \lesssim \cdot \lesssim \mu^2}\psi_\lambda))\|_{L^2_{\tau,\xi}}$$
$$\lesssim \mu^\beta \sum_{\substack{L_1 \ll \mu^2 \\ L_2 \lesssim \mu^2}} \frac{\lambda^{\frac{d-1}{2}}}{\mu^{\frac{1}{2}}} (L_1 L_2)^{\frac{1}{2}} \|C_{L_1}\varphi_\mu\|_{L^2_{t,x}} \|C_{L_2}\psi_\lambda\|_{L^2_{t,x}}$$
$$\lesssim \mu^{\beta - \frac{1}{2} - s}\lambda^{\frac{d-1}{2} - s + a - b} \; \mu^s \|\langle \tau + |\xi|^2\rangle^\theta \mathcal{F}(C_{\ll \mu^2}\varphi_\mu)\|_{L^2_{\tau,\xi}}$$
$$\times \lambda^{s'-2a+b} \|\langle \tau + |\xi|^2\rangle^{\theta'} \mathcal{F}((\lambda + |\partial_t|)^a C_{\lambda^2 \lesssim \cdot \lesssim \mu^2}\psi_\lambda)\|_{L^2_{\tau,\xi}}$$
$$\lesssim \mu^{\beta - \frac{1}{2} - s}\lambda^{\frac{d-1}{2} - s + a - b} \|\varphi_\mu\|_{S^{s,l,\theta}_\mu} \|\psi_\lambda\|_{S^{s,l,\theta}_\lambda}.$$

Provided $s - \beta \geqslant -\frac{1}{2}$ and $2s - \beta \geqslant \frac{d}{2} - 1 + a - b$, we can sum this estimate to obtain (3.6.19). For very small spatial frequencies, i.e. $\lambda \sim \mu \sim 1$, using Hölder's inequality, we have

$$\mu^\beta \|\mathcal{F}(Q_{\sim \mu^2}(\overline{C_{\ll \mu^2}\varphi_\mu} C_{\lambda^2 \lesssim \cdot \lesssim \mu^2}\psi_\lambda))\|_{L^2_{\tau,\xi}} \lesssim \|\varphi_\mu\|_{S^{s,l,\theta}_\mu} \|\psi_\lambda\|_{S^{s,l,\theta}_\lambda}.$$

b. $|\tau_2| \gg \mu^2$: We make use of the high modulation of ψ_λ by using Hölder's and Bernstein's inequalities:

$$\mu^\beta \|\mathcal{F}(Q_{\gg \mu^2}(\overline{C_{\ll \mu^2}\varphi_\mu} C_{\gg \mu^2}\psi_\lambda))\|_{L^2_{\tau,\xi}}$$
$$\lesssim \mu^\beta \lambda^{\frac{d}{2}} \|C_{\ll \mu^2}\varphi_\mu\|_{L^\infty_t L^2_x} \|C_{\gg \mu^2}\psi_\lambda\|_{L^2_{t,x}}$$
$$\lesssim \mu^{\beta - 2\theta - 2a - s}\lambda^{\frac{d}{2} - s + 2a - b} \mu^s \|C_{\ll \mu^2}\varphi_\mu\|_{L^\infty_t L^2_x} \lambda^{s'-2a+b} \|\langle \tau + |\xi|^2\rangle^{\theta'} \mathcal{F}((\lambda + |\partial_t|)^a C_{\gg \mu^2}\psi_\lambda)\|_{L^2_{\tau,\xi}}$$
$$\lesssim \mu^{\beta - 2\theta - 2a - s}\lambda^{\frac{d}{2} - s + 2a - b} \|\varphi_\mu\|_{S^{s,l,\theta}_\mu} \|\psi_\lambda\|_{S^{s,l,\theta}_\lambda}.$$

We require $s - \beta \geqslant -2\theta - 2a$ and $2s - \beta \geqslant \frac{d}{2} - 2\theta - b$ for summability.

6. $L \times H \to L$

From the relation $|\tau_0| = |\tau_1 - \tau_2| \ll \mu^2$ nd $|\tau_1| \sim \mu^2$, we conclude that the temporal frequencies $|\tau_2| \sim \mu^2$. Cauchy-Schwarz inequality then gives

$$|I(\mathcal{F}(Q_{\ll \mu^2}\eta_\mu), \mathcal{F}(C_{\sim \mu^2}\psi_\lambda), \mathcal{F}(C_{\ll \mu^2}\varphi_\mu))| \lesssim \|C_{\sim \mu^2}\psi_\lambda\|_{L^2_{t,x}} \|P_\lambda(C_{\ll \mu^2}\varphi_\mu Q_{\ll \mu^2}\eta_\mu)\|_{L^2_{t,x}}. \tag{3.6.22}$$

In order to apply the bilinear estimate (3.4.10), we decompose the Fourier supports of the last two terms in the above display into pieces of size L_1 and L_0 respectively. Since the spatial support of ψ is localised to frequencies of size $\sim \lambda$, the estimate reduces to the case where the spatial supports of φ and η are also of size $\sim \lambda$. Using this reduction, we have

$$\begin{aligned}
\text{RHS of (3.6.22)} &\lesssim \|C_{\sim\mu^2}\psi_\lambda\|_{L^2_{t,x}} \sum_{L_0,L_1\ll\mu^2} \frac{\lambda^{\frac{d-1}{2}}}{\mu^{\frac{1}{2}}}(L_1L_0)^{\frac{1}{2}}\|C_{L_1}\varphi_\mu\|_{L^2_{t,x}}\|Q_{L_0}\eta_\mu\|_{L^2_{t,x}} \\
&\lesssim \mu^{-\frac{1}{2}+2(\theta-\theta')-2a+l-s}\lambda^{\frac{d-1}{2}-s'-b+2a}\mu^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\mu^2}\varphi_\mu)\|_{L^2_{\tau,\xi}} \\
&\quad\times\lambda^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle\mathcal{F}(C_{\sim\mu^2}\psi_\lambda)\|_{L^2_{\tau,\xi}}\mu^{-l-1}\|\langle\tau-|\xi|\rangle^{1-\theta}\mathcal{F}(Q_{\ll\mu^2}\eta_\mu)\|_{L^2_{\tau,\xi}} \\
&\lesssim \mu^{-\frac{1}{2}+2(\theta-\theta')-2a+l-s}\lambda^{\frac{d-1}{2}-s'-b+2a}\|\varphi_\mu\|_{S^{s,l,\theta}_\mu}\|\psi_\lambda\|_{S^{s,l,\theta}_\lambda}\|\eta_\mu\|_{W^{-l,s,1-\theta}_\mu}.
\end{aligned}$$

Provided $s-l\geqslant -\frac{1}{2}+2(\theta-\theta')-2a$ and $2s-l\geqslant\frac{d}{2}-1-b$, we can obtain (3.6.18) by summing up the above estimate. For very small spatial frequencies $\lambda\sim\mu\sim 1$, we use Hölder's inequality to conclude

$$|I(\mathcal{F}(Q_{\ll\mu^2}\eta_\mu),\mathcal{F}(C_{\sim\mu^2}\psi_\lambda),\mathcal{F}(C_{\ll\mu^2}\varphi_\mu))| \lesssim \|\varphi_\mu\|_{S^{s,l,\theta}_\mu}\|\psi_\lambda\|_{S^{s,l,\theta}_\lambda}\|\eta_\mu\|_{W^{-l,s,1-\theta}_\mu}. \tag{3.6.23}$$

7. $L\times L\to H$

We employ the bilinear Strichartz estimate (3.4.9) by decomposing the space-time Fourier support of $\overline{C_{\ll\mu^2}\psi_\mu}$ and $C_{\sim\lambda^2}\psi_\lambda$ into pieces of size L_1 and L_2, respectively:

$$\begin{aligned}
&\mu^\beta\|\mathcal{F}(Q_{\sim\mu^2}(\overline{C_{\ll\mu^2}\psi_\mu}C_{\ll\lambda^2}\psi_\lambda))\|_{L^2_{\tau,\xi}} \\
&\lesssim \mu^\beta\sum_{\substack{L_1\ll\mu^2\\L_2\ll\lambda^2}}\frac{\lambda^{\frac{d-1}{2}}}{\mu^{\frac{1}{2}}}(L_1L_2)^{\frac{1}{2}}\|C_{L_1}\varphi_\mu\|_{L^2_{t,x}}\|C_{L_2}\psi_\lambda\|_{L^2_{t,x}} \\
&\lesssim \mu^{\beta-\frac{1}{2}-s}\lambda^{\frac{d-1}{2}-s}\ \mu^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\mu^2}\varphi_\mu)\|_{L^2_{\tau,\xi}}\lambda^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\lambda^2}\psi_\lambda)\|_{L^2_{\tau,\xi}} \\
&\lesssim \mu^{\beta-\frac{1}{2}-s}\lambda^{\frac{d-1}{2}-s}\ \|\varphi_\mu\|_{S^{s,l,\theta}_\mu}\|\psi_\lambda\|_{S^{s,l,\theta}_\lambda}.
\end{aligned}$$

The constraints $2s-\beta\geqslant\frac{d}{2}-1$ and $s-\beta\geqslant-\frac{1}{2}$ are required for summability. The argument in (3.6.23) can be mimicked for the vey small spatial frequency case.
To prove (3.6.20), we have

$$\begin{aligned}
\mu^{l+2\theta-3}\|\overline{\varphi_\mu}\psi_\lambda\|_{L^\infty_tL^2_x} &\lesssim \mu^{l+2\theta-3-s}\lambda^{\frac{d}{2}}\|\varphi_\mu\|_{L^\infty_tL^2_x}\|\psi_\lambda\|_{L^\infty_tL^2_x} \\
&\lesssim \mu^{l+2\theta-3-s}\lambda^{\frac{d}{2}-s}\|\varphi_\mu\|_{L^\infty_tH^s_x}\|\psi_\lambda\|_{L^\infty_tH^s_x}.
\end{aligned}$$

$l>-3+\frac{d}{2}+2\theta$ is the requirement for the summability of this estimate.

We conclude that the following are the requirements for the validity of the estimates in the case $\lambda\lesssim\mu$ for $d\leqslant 3$:

$$\begin{aligned}
&\bullet\ s-\beta\geqslant-\frac{1}{2}\quad\bullet\ s-l\geqslant\max\Big(a-b,-\frac{1}{2}+2(\theta-\theta')-2a\Big)\\
&\bullet\ 2s-l\geqslant\frac{1}{2}+2a-b\quad\bullet\ 2s-\beta\geqslant\frac{1}{2}+a-b.
\end{aligned}\tag{3.6.24}$$

Case II. High-high interaction ($\mu\lesssim\lambda_1\sim\lambda_2$)

We decompose $\overline{\varphi_{\lambda_1}}$ and ψ_{λ_2} as follows:

$$\overline{\varphi_{\lambda_1}}=\overline{C_{\ll\lambda_1^2}\varphi_{\lambda_1}}+\overline{C_{\gtrsim\lambda_1^2}\varphi_{\lambda_1}},\quad \psi_{\lambda_2}=C_{\ll\lambda_2^2}\psi_{\lambda_2}+C_{\gtrsim\lambda_2^2}\psi_{\lambda_2}$$

The following cases can be distinguished on the basis of the size of the modulation:

1. $H \times H \to H$
We use Hölder's and Bernstein's inequality as follows:

$$\begin{aligned}
&\mu^{\beta}\|\mathcal{F}(Q_{\gtrsim\mu^2}(\overline{C_{\gtrsim\lambda_1^2}\varphi_{\lambda_1}}C_{\gtrsim\lambda_2^2}\psi_{\lambda_2}))\|_{L^2_{\tau,\xi}} \\
&\lesssim \mu^{\beta+\frac{d}{2}}\lambda_1^{-2s-2\theta+a-b}\lambda_1^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda_1+|\partial_t|)^a C_{\gtrsim\lambda_1^2}\varphi_{\lambda_1})\|_{L^2_{\tau,\xi}}\lambda_2^s\|C_{\gtrsim\lambda_2^2}\psi_{\lambda_2}\|_{L^\infty_t L^2_x} \\
&\lesssim \mu^{\beta+\frac{d}{2}}\lambda_1^{-2s-2\theta+a-b}\|\varphi_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|\psi_{\lambda_2}\|_{S^{s,l,\theta}_{\lambda_2}}.
\end{aligned}$$

We can sum the estimate in spatial frequencies $\mu \ll \lambda_1 \sim \lambda_2$ provided $2s-\beta \geqslant \frac{d}{2}-2\theta+a-b$, by noting that $\beta+\frac{d}{2}>0$ for $d \geqslant 1$ and β as in (3.3.1).

2. $H \times H \to L$
We use Hölder's and Bernstein's inequality to obtain

$$\begin{aligned}
&|I(\mathcal{F}(Q_{\ll\mu^2}\eta_\mu),\mathcal{F}(C_{\gtrsim\lambda_2^2}\psi_{\lambda_2}),\mathcal{F}(C_{\gtrsim\lambda_1^2}\varphi_{\lambda_1}))| \\
&\lesssim \mu^{\frac{d}{2}}\|C_{\gtrsim\lambda_1^2}\varphi_{\lambda_1}\|_{L^2_{t,x}}\|C_{\gtrsim\lambda_2^2}\psi_{\lambda_2}\|_{L^2_{t,x}}\|Q_{\ll\mu^2}\eta_\mu\|_{L^\infty_t L^2_x} \\
&\lesssim \mu^{\frac{d}{2}+l+2\theta}\lambda_1^{-2s-4\theta+2a-b}\lambda_1^{s'-2a+b}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda_1+|\partial_t|)^a C_{\gtrsim\lambda_1^2}\varphi_{\lambda_1})\|_{L^2_{\tau,\xi}} \\
&\quad\times\lambda_2^{s'-2a}\|\langle\tau+|\xi|^2\rangle^{\theta'}\mathcal{F}((\lambda_2+|\partial_t|)^a C_{\gtrsim\lambda_2^2}\psi_{\lambda_2})\|_{L^2_{\tau,\xi}}\ \mu^{-l-1}\|Q_{\ll\mu^2}\eta_\mu\|_{L^p_t L^2_x} \\
&\lesssim \mu^{\frac{d}{2}+l+2\theta}\lambda_1^{-2s-4\theta+2a-b}\|\varphi_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|\psi_{\lambda_2}\|_{S^{s,l,\theta}_{\lambda_2}}\|\eta_\mu\|_{W^{-l,s,1-\theta}_\mu},
\end{aligned}$$

where the last inequality follows from the embedding (2.7.8).
By noting that $\frac{d}{2}+l+2\theta>0$ we conclude that we require $2s-l \geqslant \frac{d}{2}-2\theta+2a-b$ for summability.

3. $H \times L \to H$
From Hölder's and Bernstein's inequality, we get

$$\begin{aligned}
&\mu^{\beta}\|\mathcal{F}(\overline{C_{\gtrsim\lambda_1^2}\varphi_{\lambda_1}}C_{\ll\lambda_2^2}\psi_{\lambda_2})\|_{L^2_{\tau,\xi}} \\
&\lesssim \mu^{\beta+\frac{d}{2}}\lambda_1^{-2s+a-b-2\theta}\lambda_1^{s'-2a+b}\|\langle\tau+|\xi|\rangle^{\theta'}\mathcal{F}((\lambda_1+|\partial_t|)^a C_{\gtrsim\lambda_1^2}\varphi_{\lambda_1})\|_{L^2_{\tau,\xi}}\lambda_2^s\|C_{\ll\lambda_2^2}\psi_{\lambda_2}\|_{L^\infty_t L^2_x} \\
&\lesssim \mu^{\beta+\frac{d}{2}}\lambda_1^{-2s+a-b-2\theta}\|\varphi_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|\psi_{\lambda_2}\|_{S^{s,l,\theta}_{\lambda_2}}.
\end{aligned}$$

Provided $2s-\beta \geqslant \frac{d}{2}-2\theta+a-b$, we can obtain (3.6.19).

4. $L \times H \to H$
This interaction can be handled like case 3 with the roles of φ and ψ reversed.

5. $L \times L \to H$
We treat $d=1,2,3$ separately.
$d=3$: We use Hölder's and Bernstein's inequalities and the endpoint Strichartz space $L^2_t L^6_x$.

$$\mu^{\beta}\|\mathcal{F}(C_{\gtrsim\mu^2}(\overline{C_{\ll\lambda_1^2}\varphi_{\lambda_1}}C_{\ll\lambda_1^2}\psi_{\lambda_2}))\|_{L^2_{\tau,\xi}}$$

$$
\begin{aligned}
&\lesssim \mu^{\beta+\frac{1}{2}} \|C_{\ll\lambda_1^2}\varphi_{\lambda_1}\|_{L_t^2L_x^6}\|C_{\ll\lambda_1^2}\psi_{\lambda_2}\|_{L_t^\infty L_x^2}\\
&\lesssim \mu^{\beta+\frac{1}{2}}\lambda_1^{-2s}\|\varphi_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|\psi_{\lambda_2}\|_{S^{s,l,\theta}_{\lambda_2}}.
\end{aligned}
$$

$\boldsymbol{d=2:}$ We employ the bilinear Strichartz estimate (3.4.9) as follows by decomposing the space-time Fourier supports of $\overline{C_{\ll\lambda_1^2}\varphi_{\lambda_1}}$ and $C_{\ll\lambda_1^2}\psi_{\lambda_2}$ into pieces of size L_1 and L_2, respectively:

$$
\begin{aligned}
&\mu^\beta\|\mathcal{F}(C_{\gtrsim\mu^2}(\overline{C_{\ll\lambda_1^2}\varphi_{\lambda_1}}C_{\ll\lambda_1^2}\psi_{\lambda_2}))\|_{L^2_{\tau,\xi}}\\
&\lesssim \mu^\beta\sum_{L_1,L_2\ll\lambda_1^2}(L_1L_2)^{\frac{1}{2}}\|C_{L_1}\varphi_{\lambda_1}\|_{L^2_{t,x}}\|C_{L_2}\psi_{\lambda_2}\|_{L^2_{t,x}}\\
&\lesssim \mu^\beta\lambda_1^{-2s}\|\varphi_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|\psi_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_2}}.
\end{aligned}
$$

$\boldsymbol{d=1:}$ An application of Hölder's and Bernstein's inequality gives

$$
\begin{aligned}
\mu^\beta\|\mathcal{F}(C_{\gtrsim\mu^2}(\overline{C_{\ll\lambda_1^2}\varphi_{\lambda_1}}C_{\ll\lambda_1^2}\psi_{\lambda_2}))\|_{L^2_{\tau,\xi}} &\lesssim \mu^{\beta+\frac{1}{2}}\|C_{\ll\lambda_1^2}\varphi_{\lambda_1}\|_{L_t^\infty L_x^2}\|C_{\ll\lambda_1^2}\psi_{\lambda_2}\|_{L^2_{t,x}}\\
&\lesssim \mu^{\beta+\frac{1}{2}}\lambda_1^{-2s}\|\varphi_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|\psi_{\lambda_2}\|_{S^{s,l,\theta}_{\lambda_2}}.
\end{aligned}
$$

For $d\leqslant 3$, we require $s\geqslant 0,\ 2s-\beta\geqslant\frac{1}{2}$ to sum the above subcase.

6. $\boldsymbol{L\times L\to L}$

$\boldsymbol{d=2,3:}$ We decompose the Fourier supports of $C_{\ll\lambda_1^2}\varphi_{\lambda_1}, C_{\ll\lambda_2^2}\psi_{\lambda_2}$ and $Q_{\ll\mu^2}\eta_\mu$ into pieces of size L_1, L_2 and L_0, respectively and then apply Lemma 3.4.6:

$$
\begin{aligned}
&|I(\mathcal{F}(Q_{\ll\mu^2}\eta_\mu),\mathcal{F}(C_{\ll\lambda_2^2}\psi_{\lambda_2}),\mathcal{F}(C_{\ll\lambda_1^2}\varphi_{\lambda_1}))|\\
&\lesssim \sum_{\substack{L_1,L_2\ll\lambda_1^2\\ L_0\ll\mu^2}}|I(\mathcal{F}(Q_{L_0}\eta_\mu),\mathcal{F}(C_{L_2}\psi_{\lambda_2}),\mathcal{F}(C_{L_1}\varphi_{\lambda_1}))|\\
&\lesssim \sum_{\substack{L_1,L_2\ll\lambda_1^2\\ L_0\ll\mu^2}}\frac{(L_0L_1L_2)^{\frac{1}{2}}}{(\lambda_1)^{\frac{1}{2}}}\log\lambda_1\|C_{L_1}\varphi_{\lambda_1}\|_{L^2_{t,x}}\|C_{L_2}\psi_{\lambda_2}\|_{L^2_{t,x}}\|Q_{L_0}\eta_\mu\|_{L^2_{t,x}}\\
&\lesssim \lambda_1^{-\frac{1}{2}-2s}\mu^{2\theta+l}\log\lambda_1\ \lambda_1^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\lambda_1^2}\varphi_{\lambda_1})\|_{L^2_{\tau,\xi}}\\
&\quad\times\lambda_2^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\lambda_2^2}\psi_{\lambda_2})\|_{L^2_{\tau,\xi}}\mu^{-l-1}\|\langle\tau-|\xi|\rangle^{1-\theta}\mathcal{F}(Q_{\ll\mu^2}\eta_\mu)\|_{L^2_{\tau,\xi}}\\
&\lesssim \lambda_1^{-\frac{1}{2}-2s}\mu^{2\theta+l+a}\log\lambda_1\ \|\varphi_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|\psi_{\lambda_2}\|_{S^{s,l,\theta}_{\lambda_2}}\|\eta_\mu\|_{W_\mu^{-l,s,1-\theta}}.
\end{aligned}
$$

$\boldsymbol{d=1:}$ We apply Hölder's inequality and bilinear Strichartz estimate (3.4.10) by decomposing the space-time Fourier supports of $\overline{C_{\ll\lambda_2^2}\psi_{\lambda_2}}$ and $Q_{\ll\mu^2}\eta_\mu$ into pieces of size L_2 and L_0, respectively.

$$
\begin{aligned}
&|I(\mathcal{F}(Q_{\ll\mu^2}\eta_\mu),\mathcal{F}(C_{\ll\lambda_2^2}\psi_{\lambda_2}),\mathcal{F}(C_{\ll\lambda_1^2}\varphi_{\lambda_1}))|\\
&\lesssim \|C_{\ll\lambda_1^2}\varphi_{\lambda_1}\|_{L^2_{t,x}}\|\overline{C_{\ll\lambda_2^2}\psi_{\lambda_2}}Q_{\ll\mu^2}\eta_\mu\|_{L^2_{t,x}}\\
&\lesssim \|C_{\ll\lambda_1^2}\varphi_{\lambda_1}\|_{L^2_{t,x}}\sum_{\substack{L_0\ll\mu^2\\ L_2\ll\lambda_2^2}}\frac{1}{\lambda_2^{1/2}}(L_0L_2)^{1/2}\|C_{L_2}\psi_{\lambda_2}\|_{L^2_{t,x}}\|Q_{L_0}\eta_\mu\|_{L^2_{t,x}}
\end{aligned}
$$

$$\lesssim \mu^{l+2\theta}\lambda_1^{-1/2-2s}\lambda_1^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\lambda_1^2}\varphi_{\lambda_1})\|_{L^2_{\tau,\xi}}\lambda_2^s\|\langle\tau+|\xi|^2\rangle^\theta\mathcal{F}(C_{\ll\lambda_2^2}\psi_{\lambda_2})\|_{L^2_{\tau,\xi}}$$
$$\times\,\mu^{-l-1}\|\langle\tau-|\xi|\rangle^{1-\theta}\mathcal{F}(Q_{\ll\mu^2}\eta_\mu)\|_{L^2_{\tau,\xi}}$$
$$\lesssim \mu^{l+2\theta+a}\lambda_1^{-1/2-2s}\|\varphi_{\lambda_1}\|_{S^{s,l,\theta}_{\lambda_1}}\|\psi_{\lambda_2}\|_{S^{s,l,\theta}_{\lambda_2}}\|\eta_\mu\|_{W^{-l,s,1-\theta}_\mu}.$$

One requires $2s-l>-\frac{1}{2}+2\theta+a$ to sum the subcase for $d\leqslant 3$.

The estimate (3.6.20) is proved as follows:

$$\mu^{l+2\theta-3}\|\overline{\varphi_{\lambda_1}}\psi_{\lambda_2}\|_{L^\infty_tL^2_x}\lesssim \mu^{l+2\theta-3+\frac{d}{2}}\|\varphi_{\lambda_1}\|_{L^\infty_tL^2_x}\|\psi_{\lambda_2}\|_{L^\infty_tL^2_x}$$
$$\lesssim \mu^{l+2\theta-3+\frac{d}{2}}\lambda_1^{-2s}\|\varphi_{\lambda_1}\|_{L^\infty_tH^s_x}\|\psi_{\lambda_2}\|_{L^\infty_tH^s_x}.$$

The above is summable for $s>0$ and $2s-l>-3+2\theta+\frac{d}{2}$.

To conclude,

$$\bullet\ 2s-l>\max\Big(-\frac{1}{2}+2\theta+a,\frac{3}{2}-2\theta+2a-b\Big)\quad\bullet\ 2s-\beta>\max\Big(\frac{3}{2}-2\theta+a-b,\frac{1}{2}\Big)\qquad(3.6.25)$$

are the requirements for summability in this case.
The estimate (3.6.20) holds provided $s-l\geqslant-2+2\theta, s\geqslant 0, 2s-l\geqslant-\frac{1}{2}+2\theta$, and follows by imitating the proof of the product estimate $\|fg\|_{H^{l-2+2\theta}}\lesssim\|f\|_{H^s}\|g\|_{H^s}$ for both the cases above.
Finally, using (3.6.24) and (3.6.25), we list the required constraints on the parameters for the estimate (3.6.17) to hold true for $d\leqslant 3$:

- $s-l\geqslant\max(-2+2\theta,a-b,-\frac{1}{2}+2(\theta-\theta')-2a)$ • $s-\beta\geqslant-\frac{1}{2}$
- $2s-l>\max\Big(-\frac{1}{2}+2\theta+2a-b,-\frac{1}{2}+2\theta+a\Big)$ • $2s-\beta\geqslant\max\Big(\frac{3}{2}-2\theta+a-b,\frac{1}{2}\Big)$.

□

3.7 Proof of Theorem 3.2.1

Given the nonlinear estimates proved in Sections 3.6.1 and 3.6.2, we can achieve a small data local well-posedness result. A simplified argument for the same is provided in [14, Section 5]. To achieve a large data result, we need to extract a small power of T on the right-hand side of the nonlinear estimates. We can then shrink our time interval to suit large data.

We start with proving a slightly weaker form of the energy inequalities but with a small power of T on the right-hand side. Note that for the $X^{s,\theta}$ type part of the norms, the required factor comes from Lemma 2.7.7. It remains to extract this factor for the $L^\infty_tL^2_x$ part of the norm. To that end, we note for $0<T<1$ and a smooth time cut-off η_T

$$\lambda^s\|\eta_T\mathcal{I}_S[C_{\sim\lambda^2}P^{(t)}_{\ll\lambda^2}F_\lambda]\|_{L^\infty_tL^2_x}\lesssim\lambda^s\|\eta_{4T}C_{\sim\lambda^2}P^{(t)}_{\ll\lambda^2}F_\lambda\|_{L^1_tL^2_x}\lesssim\lambda^sT\|C_{\sim\lambda^2}P^{(t)}_{\ll\lambda^2}F_\lambda\|_{L^\infty_tL^2_x}.$$

Interpolating the above estimate with

$$\lambda^s\|\eta_T\mathcal{I}_S[C_{\sim\lambda^2}P^{(t)}_{\ll\lambda^2}F_\lambda]\|_{L^\infty_tL^2_x}\lesssim\lambda^{s-2}\|P^{(t)}_{\ll\lambda^2}F_\lambda\|_{L^\infty_tL^2_x},$$

we obtain for some $0 < \delta_1 \leqslant -\frac{1}{2} + \theta \ll 1$,

$$\lambda^s \|\eta_T \mathcal{I}_S[C_{\sim\lambda^2} P^{(t)}_{\ll\lambda^2} F_\lambda]\|_{L^\infty_t L^2_x} \lesssim \lambda^{s-2+2\delta_1} T^{\delta_1} \|C_{\sim\lambda^2} P^{(t)}_{\ll\lambda^2} F_\lambda\|_{L^\infty_t L^2_x} \lesssim T^{\delta_1} \|F_\lambda\|_{N^{s,l,\theta-1}_\lambda} \tag{3.7.1}$$

Note that we considered only that part of the nonlinearity with low ($\ll \lambda^2$) temporal frequency. This is because for the for the other case ($|\tau| \gtrsim \lambda^2$), we can again use the energy inequality, viz. Lemma 2.7.7 since $|m_S(\tau)| \gtrsim 1$.
As stated, for $\delta_2 > 0$, $-\frac{1}{2} < \theta - 1 < \theta - 1 + \delta_2 < \frac{1}{2}$, we have *only* for the $X^{s,\theta}$ part of the N norm

$$\|F\|_{N^{s,l,\theta-1}(T)} \lesssim T^{\delta_2} \|F\|_{N^{s,l,\theta-1+\delta_2}(T)}. \tag{3.7.2}$$

From (3.7.1) and (3.7.2) we conclude

$$\|\eta_T \mathcal{I}_S[F]\|_{S^{s,l,\theta}(T)} \lesssim T^{\delta} \|F\|_{N^{s,l,\tilde{\theta}}(T)} \tag{3.7.3}$$

where $\delta = \min(\delta_1, \delta_2) > 0$ and $\tilde{\theta} = \theta - 1 + \delta$. The bilinear estimates are not affected as they hold only up to the boundaries. Using the same arguments for the wave solution and wave nonlinearity, we have

$$\|\eta_T \mathcal{I}_W[G]\|_{W^{l,s,\theta}(T)} \lesssim T^{\delta} \|G\|_{R^{l,s,\tilde{\theta}}(T)}. \tag{3.7.4}$$

We now head to prove Theorem 3.2.1.

Proof. We call $(u, v) \in S^{s,l,\theta}(T) \times W^{l,s,\theta}(T)$ a solution to the system (3.2.2) with initial data $(u_0, v_0) \in H^s \times H^l$ if for all $0 < T < 1$, we have

$$\begin{aligned} u(t) &= \eta(e^{it\Delta} u_0 + \eta_T \mathcal{I}_S[uRe(v)])(t) \\ v(t) &= \eta(e^{it|\nabla|} v_0 + \eta_T \mathcal{I}_W[|\nabla||u|^2])(t). \end{aligned} \tag{3.7.5}$$

To apply the contraction mapping argument, we write (3.7.5) as

$$\Gamma(u, v)(t) = (u, v)(t).$$

We prove that Γ is a contraction in a closed ball B_R of the space $S^{s,l,\theta}(T) \times W^{l,s,\theta}(T)$ for a suitably chosen T. In the following, we shall drop the superscripts on the norms for notational convenience. First, we show the map Γ is well-defined on $B_R \subseteq S(T) \times W(T)$. using Lemma 3.5.1, Lemma 3.5.2, Theorem 3.6.1, Theorem 3.6.4, (3.7.3) and (3.7.4) we obtain

$$\begin{aligned} \|\Gamma(u, v)\|_{S(T)\times W(T)} &= \|\eta(t) e^{it\Delta} u_0\|_{S(T)} + \|\eta_T(t) \mathcal{I}_S[uRe(v)]\|_{S(T)} \\ &\quad + \|\eta(t) e^{it|\nabla|} v_0\|_{W(T)} + \|\eta_T(t) \mathcal{I}_W[|\nabla||u|^2]\|_{W(T)} \\ &\lesssim \|u_0\|_{H^s_x} + T^{\delta} \|u\|_{S(T)} \|v\|_{W(T)} + \|v_0\|_{H^l_x} + T^{\delta} \|u\|^2_{S(T)} \\ &\leqslant C(\|u_0\|_{H^s_x} + \|v_0\|_{H^l_x} + T^{\delta}(\|(u, v)\|_{S(T)\times W(T)} + \|(u, v)\|^2_{S(T)\times W(T)})). \end{aligned}$$

Choose R such that

$$T^{\delta}(R + R^2) = \|u_0\|_{H^s_x} + \|v_0\|_{H^l_x}$$

and T such that

$$CT^\delta(1+R) < \frac{1}{2}.$$

Then Γ becomes a well-defined map on the closed ball B_R of radius R. In order to show that Γ is a contraction, we consider

$$\begin{aligned}
&\|\Gamma(u_1,v_1) - \Gamma(u_2,v_2)\|_{S(T)\times W(T)} \\
&= \|\eta_T(t)\mathcal{I}_S[u_1 Re(v_1) - u_2 Re(v_2)]\|_{S(T)} + \|\eta_T(t)\mathcal{I}_W[|\nabla|(|u_1|^2 - |u_2|^2)]\|_{W(T)} \\
&\lesssim T^\delta(\|u_1 Re(v_1) - u_2 Re(v_1) + u_2 Re(v_1) - u_2 Re(v_2)\|_{N^{s,l,\tilde{\theta}}(T)} + \||u_1|^2 - |u_2|^2\|_{R^{l,s,\tilde{\theta}}(T)}) \\
&\lesssim T^\delta(\|v_1\|_{W(T)}\|u_1 - u_2\|_{S(T)} + \|u_2\|_{S(T)}\|v_1 - v_2\|_{W(T)} + \|u_1 - u_2\|_{S(T)}\|u_1 + u_2\|_{S(T)}) \\
&\lesssim T^\delta(\|u_1\|_{S(T)} + \|v_1\|_{W(T)} + \|u_2\|_{S(T)} + \|v_2\|_{W(T)})(\|u_1 - u_2\|_{S(T)} + \|v_1 - v_2\|_{W(T)}) \\
&= C'T^\delta(\|(u_1,v_1)\|_{S\times W(T)} + \|(u_2,v_2)\|_{S\times W(T)})(\|(u_1 - u_2, v_1 - v_2)\|_{S\times W(T)}).
\end{aligned}$$

If we choose T such that

$$C'T^\delta R < \frac{1}{2},$$

then Γ becomes a contraction on the closed ball B_R of radius R in $S(T)\times W(T)$. Using Banach's fixed point theorem, we conclude the existence of a unique solution to the system (3.1.1)-(3.1.2) on the interval $[-T,T]$ where

$$T = \min\Big(\Big(\frac{1}{2C(1+R)}\Big)^{1/\delta}, \Big(\frac{1}{2C'R}\Big)^{1/\delta}\Big).$$

Similar arguments as above show that the data-to-solution map is Lipschitz continuous. This concludes the proof of the theorem.

□

Chapter 4

Well-posedness for the dispersion generalised KP-I equation

"To believe in something, and not to live it, is dishonest."
-Mahatma Gandhi

4.1 Introduction and results

The Kadomtsev-Petviashvili (KP) equations [36] were derived in 1970 by physicists Boris Kadomtsev and Vladimir Petviashvili to study the transverse stability of the solitary wave solution of the Korteweg-de Vries (KdV) equation. The KdV equation reads

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0.$$

The idea is to consider a weakly transverse perturbation of the KdV equation by a Taylor series approximation of the dispersion relation of the two-dimensional wave equation in the regime $\left|\frac{\eta}{\xi}\right| \ll 1$. The dispersion relation for the wave equation is given by

$$\omega(\xi, \eta) = \pm(\xi^2 + \eta^2)^{\frac{1}{2}}.$$

The mentioned approximation is given by

$$(\xi^2 + \eta^2)^{\frac{1}{2}} \sim \pm\xi\Big(1 + \frac{\eta^2}{2\xi^2}\Big).$$

On the physical side, this amounts to adding the non-local term $\partial_x^{-1}\partial_y^2 u$ to the KdV equation and results in the KP equations

$$\partial_t u + u\partial_x u + \partial_x^3 u + \kappa\partial_x^{-1}\partial_y^2 = 0,$$

where $\kappa = 1$ corresponds to the KP-II equation and $\kappa = -1$ to the KP-I equation. These equations model long weakly nonlinear waves propagating essentially along the x direction with a small dependence on the y variable. Following the same strategy for the fractional KdV equation

$$\partial_t u + u\partial_x u \pm D_x^\alpha \partial_x u = 0$$

leads to the fractional KP equations.

In this chapter, we consider the Cauchy problem for the dispersion generalised Kadomtsev-Petviashvili I (fKP-I) equation

$$\begin{cases} \partial_t u - D_x^\alpha \partial_x u - \partial_x^{-1}\partial_y^2 u &= u\partial_x u, \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}, \\ u(0,x,y) &= u_0(x,y) \in H^{s_1,s_2}(\mathbb{R}^2). \end{cases} \tag{4.1.1}$$

We consider α in the range $2 < \alpha < 4$ and D_x^α is given by

$$\mathcal{F}_{x,y}(D_x^\alpha f)(\xi,\eta) = |\xi|^\alpha \hat{f}(\xi,\eta).$$

The non-local term can be interpreted as a Fourier multiplier

$$(\partial_x^{-1} f)^\wedge(\xi) = \frac{1}{i\xi}\hat{f}(\xi).$$

For $2 < \alpha \leqslant \frac{5}{2}$ we only consider real-valued solutions; for $\alpha > \frac{5}{2}$ we also treat complex-valued solutions. Note that the solution stays real-valued provided that the initial data is real-valued. We study the Cauchy problem with initial data from anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$, which are defined by (2.1.1). The following quantities are conserved for real-valued solutions:

$$M(u)(t) = \int_{\mathbb{R}^2} u(x,y)^2 dxdy, \tag{4.1.2}$$

$$E_\alpha(u)(t) = \int_{\mathbb{R}^2} \Big(\frac{1}{2}|D_x^{\frac{\alpha}{2}} u|^2 + \frac{1}{2}|\partial_x^{-1}\partial_y u|^2 + \frac{1}{6}u^3\Big)dxdy. \tag{4.1.3}$$

Hence, the natural energy space is given by

$$\mathbf{E}^\alpha(\mathbb{R}^2) = \{\phi \in L^2(\mathbb{R}^2) : \|\phi\|_{E^\alpha(\mathbb{R}^2)} := \|p(\xi,\eta)\hat{\phi}(\xi,\eta)\|_{L^2_{\xi,\eta}} < \infty\}, \tag{4.1.4}$$

where

$$p(\xi,\eta) := 1 + |\xi|^{\frac{\alpha}{2}} + \frac{|\eta|}{|\xi|}.$$

The present analysis yields global well-posedness in the energy space $\mathbf{E}^\alpha$ as well; we focus on anisotropic Sobolev spaces, which are larger.

This chapter is an extended version of [57] which is joint work with Robert Schippa. I proved the bilinear Strichartz estimate and the trilinear estimate using the nonlinear Loomis-Whitney inequality. Robert Schippa found the optimal way to use the bilinear Strichartz estimate and the trilinear estimate in the resonant case. This led to the lowering of the regularity threshold and the gap between the C^2 ill-posedness and the semilinear well-posedness becoming small.

Scaling and criticality

If u solves (4.1.1) with initial data ϕ, then u_λ given by

$$u_\lambda(t,x,y) = \frac{1}{\lambda^\alpha} u\Big(\frac{t}{\lambda^{\alpha+1}}, \frac{x}{\lambda}, \frac{y}{\lambda^{\frac{\alpha+2}{2}}}\Big) \tag{4.1.5}$$

also solves the same with scaled initial data

$$\phi_\lambda = \frac{1}{\lambda^\alpha}\phi\Big(\frac{x}{\lambda}, \frac{y}{\lambda^{\frac{\alpha+2}{2}}}\Big). \tag{4.1.6}$$

Moreover,

$$\|\phi_\lambda\|_{\dot{H}^{s_1,s_2}(\mathbb{R}^2)} = \lambda^{\frac{3\alpha}{4}-1+s_1+(\frac{\alpha}{2}+1)s_2}\|\phi\|_{\dot{H}^{s_1,s_2}(\mathbb{R}^2)}. \tag{4.1.7}$$

This shows that for $\alpha = \frac{4}{3}$, (4.1.1) is L^2-critical and for $\alpha > 2$, (4.1.1) is L^2-subcritical. In the following, we shall deal with subcritical anisotropic Sobolev regularity $(s, 0)$.

Known results

The range of dispersion considered presently $(2 < \alpha < 4)$ almost starts with the classical KP-I equation

$$\begin{cases} \partial_t u - D_x^2\partial_x u - \partial_x^{-1}\partial_y^2 u &= u\partial_x u, \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}, \\ u(0) &= u_0 \in H^{s_1,s_2}(\mathbb{R}^2), \end{cases} \tag{4.1.8}$$

which has been extensively studied and goes up to the fifth order KP-I equation. Using standard energy methods, well-posedness is proved for the KP equations in Sobolev spaces of high regularity in [68, 35]. In [48], global well-posedness is obtained for the KP-I equation in the space Z, where

$$\|\phi\|_Z = \|\phi\|_{L^2} + \|\partial_x^3\phi\|_{L^2} + \|\partial_y\phi\|_{L^2} + \|\partial_x\partial_y\phi\|_{L^2} + \|\partial_x^{-1}\partial_y\phi\|_{L^2} + \|\partial_x^{-2}\partial_y^2\phi\|_{L^2}.$$

The space Z is related to the second energy for the KP-I equation. Kenig [38] obtained local well-posedness in the space Y^s for $s > \frac{3}{2}$, where Y^s is defined by

$$\|\phi\|_{Y^s} = \|\phi\|_{L^2} + \|D_x^s\phi\|_{L^2} + \|\partial_y\partial_x^{-1}\phi\|_{L^2}.$$

A global well-posedness result related to the KP-I conservation laws is also obtained.

We remark that it is well-known that the fifth order KP-I equation

$$\begin{cases} \partial_t u - D_x^4\partial_x u - \partial_x^{-1}\partial_y^2 u &= u\partial_x u, \quad (t,x,y) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}, \\ u(0) &= u_0 \in H^{s_1,s_2}(\mathbb{R}^2) \end{cases} \tag{4.1.9}$$

can be solved via Picard iteration as pointed out by Saut-Tzvetkov [58, 59]. In [47], Molinet-Saut-Tzvetkov proved that the KP-I equation cannot be solved by Picard iteration in Sobolev spaces H^{s_1,s_2} (or H^s). More precisely, they show that the data-to-solution map fails to be C^2-differentiable for the KP-I equation. This failure is due to the derivative in the nonlinearity and the resonance function being very small in the case of a high-low interaction. Hence, the approach to find an auxiliary function space for the solutions, say X_T, which embeds into $C([0,T]; H^{s_1,s_2})$ and a space N_T for the nonlinearity which satisfy the following:

$$\text{Linear estimate: } \|u\|_{X_T^s} \lesssim \|u_0\|_{H^s} + \|\partial_x(u^2)\|_{N_T^s},$$
$$\text{Nonlinear estimate: } \|\partial_x(uv)\|_{N_T^s} \lesssim \|u\|_{X_T^s}\|v\|_{X_T^s}$$

does *not* work. In the seminal article [34], Ionescu-Kenig-Tataru proved global well-posedness in the natural energy space (corresponding to $\alpha = 2$ in (4.1.4)) by employing short-time Fourier restriction spaces. The following estimates on short-time function spaces F and $\mathcal{N}$ and energy space E give the existence and uniqueness of solutions:

$$\begin{aligned}\text{Linear estimate: } & \|u\|_{F^s(T)} \lesssim \|u\|_{E^s(T)} + \|\partial_x(u^2)\|_{\mathcal{N}^s(T)},\\ \text{Nonlinear estimate: } & \|\partial_x(u^2)\|_{\mathcal{N}^s(T)} \lesssim \|u\|_{F^s(T)}\|u\|_{F^s(T)},\\ \text{Energy estimate: } & \|u\|^2_{E^s(T)} \lesssim \|u_0\|^2_{H^s} + \|u\|^3_{F^s(T)}.\end{aligned}$$

The continuity of the solutions with respect to the initial data follows from a variant of the Bona-Smith approximation [8], see Section 4.6.3. [54, 27] are two applications of this framework, among many. Other results of interest for the KP-I equation in the dispersion generalised case on $\mathbb{R}^2$ are summarised[1] in Table 4.1. In addition, in [15], the authors prove a local well-posedness result for the fifth order KP-I equation in the space $E_s, 0 < s \leqslant 1$ of functions for which

$$\|f\|_{E_s} = \left\| \left(1 + |\xi|^2 + \frac{|\eta|}{|\xi|}\right)^s \hat{f}(\xi,\eta) \right\|_{L^2_{\xi,\eta}} < \infty.$$

We also mention the article [42] by Klein-Saut which focuses on the numerical aspects of KP type equations but provides a good overview of results on the family of KP equations.

$\alpha \geqslant 4$	LWP in H^{s_1,s_2} for $s_1 > \frac{-(\alpha-1)}{4}$, $s_2 \geqslant 0$	Yan *et al.* [69]
$4 \leqslant \alpha \leqslant 5$	GWP in $H^{s_1,0}$ with $s_1 > -\frac{(\alpha-1)(3\alpha-4)}{4(5\alpha+3)}$	Yan *et al.* [69]
$\alpha > 5$	GWP in $H^{s_1,0}$ for $s_1 > \frac{-\alpha(3\alpha-4)}{4(5\alpha+4)}$	Yan *et al.* [69]
$\alpha = 4$	LWP in H^{s_1,s_2} for $s_1 \geqslant 1$, $s_2 \geqslant 0$	Saut-Tzvetkov [59]
$\alpha = 4$	GWP in H^{s_1,s_2} for $s_1, s_2 \geqslant 0$	Li-Xiao [44]
$\alpha = 2$	GWP in the energy space	Ionescu *et al.* [34]
$\alpha = 2$	LWP in $H^{1,0}$	Z. Guo *et al.* [30]
$\alpha = 4$	LWP in $H^{s,0}$ for $s \geqslant -\frac{3}{4}$	B. Guo *et al.* [26]

Table 4.1: Well-posedness results for the dispersion generalised KP-I equation

Main results

We shall follow the strategy of [34] to obtain local well-posedness results in the low dispersion regime. Since this strategy involves energy estimates, the results in [34, 30] require real-valued solutions. Likewise, the results we prove for small dispersion require real-valued initial data:

Theorem 4.1.1. *Let $2 < \alpha \leqslant \frac{5}{2}$. Then, (4.1.1) is locally well-posed in $H^{s,0}(\mathbb{R}^2)$ for $s > 5 - 2\alpha$ and real-valued initial data.*

[1] GWP - Global well-posedness, LWP - Local well-posedness

We give a technically more detailed version of the above theorem in Section 4.6.

By modifying the counterexample provided in [47], Linares-Pilod-Saut prove the failure of C^2-differentiability of the data-to-solution map for the dispersion generalised KP-I equation in [45] for $\alpha < 2$, see also [43]. It turns out that the argument extends to $\alpha < \frac{7}{3}$:

Theorem 4.1.2. *Let $\alpha < \frac{7}{3}$, $\bar{s} \in \mathbb{R}^2$. Then, there exists no $T > 0$ such that there is a function space $X_T \hookrightarrow C([0,T]; H^{\bar{s}}(\mathbb{R}^2))$, in which (4.1.1) admits a unique local solution such that the flow-map for (4.1.1)*

$$\Gamma_t : \phi \mapsto u(t), \quad t \in [-T, T],$$

is C^2-differentiable at zero from $H^{\bar{s}}(\mathbb{R}^2)$ to $H^{\bar{s}}(\mathbb{R}^2)$.

As stated, the problematic nonlinear interaction is a resonant $High \times Low$-interaction, (see Section 4.2) in which a solution with high x frequencies interacts with a solution at low x frequencies leading to the size of the resonance function being small. However, we shall see that in the resonant case, we can argue that the interaction between the two nonlinear waves and the dual factor with low modulation is strongly transverse, which we quantify via a nonlinear Loomis-Whitney inequality. This transversality was already observed in [34], while in the proof in [34] this is not related to nonlinear Loomis-Whitney. We believe that pointing out the connection with nonlinear Loomis-Whitney inequalities makes the proof more systematic.

We exploit transversality to its fullest by using the nonlinear Loomis-Whitney inequality and the bilinear Strichartz estimate. Let N_1 and N_2 be the size of the high and the low frequency, respectively. While the trilinear estimate provided by the nonlinear Loomis-Whitney inequality is better in case $N_2 \gtrsim N_1^{-\kappa}$, for some $\kappa > 0$, the bilinear Strichartz estimate is better in the other case. A combination of the aforementioned estimates leads us to conclude that (4.1.1) is semilinearly well-posed for $\alpha > \frac{5}{2}$:

Theorem 4.1.3. *Let $\frac{5}{2} < \alpha < 4$. Then, (4.1.1) is analytically locally well-posed in $H^{s,0}(\mathbb{R}^2)$ for $s > \frac{5}{4} - \frac{\alpha}{2}$.*

The analyticity of the data-to-solution mapping is a consequence of Banach's fixed point theorem and the analyticity of the nonlinearity. By conservation of mass, i.e. (4.1.2), the following corollary is immediate:

Corollary 4.1.4. *Let $\frac{5}{2} < \alpha < 4$. Then, (4.1.1) is globally well-posed in $L^2(\mathbb{R}^2)$ for real-valued initial data.*

In the limiting cases of α considered, we recover the currently best local well-posedness results in anisotropic Sobolev spaces: for $\alpha \downarrow 2$, we recover the result from [30] and for $\alpha \uparrow 4$, we arrive at the result from [69]. Still, we note that there is a mismatch between the range of dispersion, for which we can show failure of Picard iteration and for which we actually use frequency dependent time localisation. It is unclear to us whether one has to improve the counterexample or the argument to show semilinear local well-posedness.

4.2 C^2 ill-posedness

In this section, we prove that the data-to-solution map for (4.1.1) fails to be C^2-differentiable for a given choice of initial data. First, we define the linear propagator $U_\alpha(t)$ as a Fourier multiplier acting on functions $\phi \in \mathcal{S}(\mathbb{R}^2)$ whose Fourier transform is supported away from the origin

$$(U_\alpha(t)\phi)^\wedge(\xi,\eta) = e^{it(|\xi|^\alpha\xi+\frac{\eta^2}{\xi})}\hat{\phi}(\xi,\eta). \tag{4.2.1}$$

Since $U_\alpha(t)$ is a linear isometric mapping on H^{s_1,s_2}, the above definition extends by density. The dispersion relation for (4.1.1) is given by

$$\omega_{fKP}(\xi,\eta) := |\xi|^\alpha\xi + \frac{\eta^2}{\xi}, \quad \xi \in \mathbb{R}\backslash\{0\}, \eta \in \mathbb{R}, \tag{4.2.2}$$

and we define the resonance function by

$$\begin{aligned}\Omega_{fKP}(\xi_1,\eta_1,\xi_2,\eta_2) &:= \omega_{fKP}(\xi_1+\xi_2,\eta_1+\eta_2) - \omega_{fKP}(\xi_1,\eta_1) - \omega_{fKP}(\xi_2,\eta_2)\\ &= |\xi_1+\xi_2|^\alpha(\xi_1+\xi_2) - |\xi_1|^\alpha\xi_1 - |\xi_2|^\alpha\xi_2 - \frac{(\eta_1\xi_2-\eta_2\xi_1)^2}{(\xi_1+\xi_2)\xi_1\xi_2}\\ &=: \Omega^1_{fKP} - \Omega^2_{fKP},\end{aligned}$$

where

$$\begin{aligned}\Omega^1_{fKP}(\xi_1,\eta_1,\xi_2,\eta_2) &= |\xi_1+\xi_2|^\alpha(\xi_1+\xi_2) - |\xi_1|^\alpha\xi_1 - |\xi_2|^\alpha\xi_2, \quad \text{and}\\ \Omega^2_{fKP}(\xi_1,\eta_1,\xi_2,\eta_2) &= \frac{(\eta_1\xi_2-\eta_2\xi_1)^2}{(\xi_1+\xi_2)\xi_1\xi_2}.\end{aligned}$$

In case $|\xi_1+\xi_2| \sim |\xi_1| \gg |\xi_2|$, due to similar size of Ω^1_{fKP} and Ω^2_{fKP}, the size of the resonance function $|\Omega_{fKP}|$ can become much smaller than $|\Omega^1_{fKP}|$, i.e.

$$|\Omega_{fKP}| \ll ||\xi_1+\xi_2|^\alpha(\xi_1+\xi_2) - |\xi_1|^\alpha\xi_1 - |\xi_2|^\alpha\xi_2|, \tag{4.2.3}$$

which we refer to as *resonant* case. Owing to the resonant interaction, one cannot employ Picard iteration to prove well-posedness results for (4.1.1).

To prove Theorem 4.1.2, we shall require some preliminary results which we prove in the following. These results have been proved for the KP-I equation in [47].

Lemma 4.2.1. *The following identity holds*

$$\int_0^t U_\alpha(t-s)F(s,x,y)ds = c\int_{\mathbb{R}^3} e^{ix\xi+iy\eta+it(\xi|\xi|^\alpha+\frac{\eta^2}{\xi})}\frac{e^{it(\tau-\xi|\xi|^\alpha-\frac{\eta^2}{\xi})}-1}{\tau-\xi|\xi|^\alpha-\frac{\eta^2}{\xi}}\hat{F}(\tau,\xi,\eta)d\tau d\xi d\eta, \tag{4.2.4}$$

whenever both terms are well-defined.

Proof. We set

$$v(t,x,y) = \int_0^t U_\alpha(t-s)F(s,x,y)ds = U_\alpha(t)\int_0^t U_\alpha(-s)F(s,x,y)ds,$$

and

$$H(s,x,y) = U_\alpha(-s)F(s,x,y).$$

Then, using Parseval's identity, we have

$$\int_0^t U_\alpha(-s)F(s,x,y)ds = \int_0^t H(s,x,y)ds = \int_{\mathbb{R}} \mathbf{1}_{(0,t)} H(s,x,y)ds$$
$$= c\int_{\mathbb{R}} \frac{e^{it\tau}-1}{i\tau}\mathcal{F}_s(H)(\tau,x,y)d\tau.$$

After setting $G(t,s,x,y) = U_\alpha(t-s)F(s,x,y)$, we can write v as

$$v(t,x,y) = cU_\alpha(t)\int_{\mathbb{R}} \frac{e^{it\tau}-1}{i\tau}\mathcal{F}_s(H)(\tau,x,y)d\tau = c\int_{\mathbb{R}} \frac{e^{it\tau}-1}{i\tau}\mathcal{F}_s(G)(t,\tau,x,y)d\tau.$$

Taking inverse Fourier transform with respect to the spatial variables (x,y), we get

$$v(t,x,y) = c\int_{\mathbb{R}^3} \frac{e^{it\tau}-1}{\tau}\mathcal{F}_{s,x,y}(G)(t,\tau,\xi,\eta)e^{ix\xi+iy\eta}d\tau d\xi d\eta.$$

Using the definition of G, we have

$$\mathcal{F}_{x,y}G(t,s,\xi,\eta) = e^{i(t-s)(\xi|\xi|^\alpha+\eta^2/\xi)}\mathcal{F}_{x,y}(F)(s,\xi,\eta),$$

which gives

$$\mathcal{F}_{s,x,y}(G)(t,\tau,\xi,\eta) = \int_{\mathbb{R}} e^{i(t-s)(\xi|\xi|^\alpha+\eta^2/\xi)}e^{-is\tau}\mathcal{F}_{x,y}(F)(s,\xi,\eta)ds$$
$$= e^{it(\xi|\xi|^\alpha+\eta^2/\xi)}\int_{\mathbb{R}} e^{-is(\xi|\xi|^\alpha+\eta^2/\xi)}e^{-is\tau}\mathcal{F}_{x,y}(F)(s,\xi,\eta)ds$$
$$= e^{it(\xi|\xi|^\alpha+\eta^2/\xi)}\hat{F}(\tau+\xi|\xi|^\alpha+\frac{\eta^2}{\xi},\xi,\eta).$$

Substituting this into the expression for $v(t,x,y)$, we get

$$v(t,x,y) = c\int_{\mathbb{R}^3} e^{ix\xi+iy\eta+it(\xi|\xi|^\alpha+\eta^2/\xi)}\frac{e^{it\tau}-1}{\tau}\hat{F}(\tau+\xi|\xi|^\alpha+\frac{\eta^2}{\xi},\xi,\eta)d\tau d\xi d\eta.$$

Using a change of variables $\tau+\xi|\xi|^\alpha+\frac{\eta^2}{\xi}\to\tau'$, we obtain

$$v(t,x,y) = c\int_{\mathbb{R}^3} e^{ix\xi+iy\eta+it(\xi|\xi|^\alpha+\eta^2/\xi)}\frac{e^{it(\tau-\xi|\xi|^\alpha-\eta^2/\xi)}-1}{\tau-\xi|\xi|^\alpha-\eta^2/\xi}\hat{F}(\tau,\xi,\eta)d\tau d\xi d\eta,$$

which proves the lemma. □

Proof of Theorem 4.1.2. We consider the following Cauchy problem for $\gamma\in\mathbb{R}$:

$$\begin{cases} \partial_t u - D_x^\alpha\partial_x u - \partial_x^{-1}\partial_y^2 u &= u\partial_x u, \quad (t,x,y)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}, \\ u(0,x,y) &= \gamma\phi(x,y)\in H^{\bar{s}}(\mathbb{R}^2). \end{cases} \tag{4.2.5}$$

Suppose that $u(\gamma,t,x,y)$ solves (4.2.5). Fix $t\neq 0$ such that the data-to-solution map Γ_t is C^2-differentiable. Then,

$$u(\gamma,t,x,y) = \gamma U_\alpha(t)\phi(x,y) - \int_0^t U_\alpha(t-s)u(\gamma,s,x,y)\partial_x u(\gamma,s,x,y)ds. \tag{4.2.6}$$

We have

$$\begin{aligned} \frac{\partial u}{\partial \gamma}(0,t,x,y) &= U_\alpha(t)\phi(x,y) =: u_1(t,x,y) \\ \frac{\partial^2 u}{\partial \gamma^2}(0,t,x,y) &= -2\int_0^t U_\alpha(t-s)u_1(s,x,y)\partial_x u_1(s,x,y)ds =: u_2(t,x,y). \end{aligned} \tag{4.2.7}$$

The C^2 assumption on Γ_t and $u(0,t,x,y)=0$ enable us to write

$$u(\gamma,t,x,y) = \gamma u_1(t,x,y) + \frac{\gamma^2}{2!}u_2(t,x,y) + O(\gamma^3),$$

and

$$\|u_2(t,\cdot,\cdot)\|_{H^{\bar{s}}(\mathbb{R}^2)} \lesssim \|\phi\|^2_{H^{\bar{s}}(\mathbb{R}^2)}. \tag{4.2.8}$$

We show that there exists initial data ϕ such that (4.2.8) fails. We define the initial data as

$$\phi = \phi_1 + \phi_2,$$

where ϕ_i, $i=1,2$ are defined via their Fourier transform as follows:

$$\begin{aligned} \hat{\phi}_1(\xi_1,\eta_1) &= \sigma^{-\frac{3}{2}}\mathbf{1}_{D_1}(\xi_1,\eta_1), \\ \hat{\phi}_2(\xi_2,\eta_2) &= \sigma^{-\frac{3}{2}}N^{-s_1-(1+\frac{\alpha}{2})s_2}\mathbf{1}_{D_2}(\xi_2,\eta_2), \end{aligned} \tag{4.2.9}$$

$D_i = \tilde{D}_i \cup (-\tilde{D}_i)$ and $\tilde{D}_i$ are defined as follows:

$$\begin{aligned} \tilde{D}_1 &:= \left[\frac{\sigma}{2},\sigma\right] \times [-\sqrt{1+\alpha}\ \sigma^2, \sqrt{1+\alpha}\ \sigma^2], \\ \tilde{D}_2 &:= [N,N+\sigma] \times [\sqrt{1+\alpha}\ N^{\frac{\alpha+2}{2}}, \sqrt{1+\alpha}\ N^{\frac{\alpha+2}{2}} + \sigma^2]. \end{aligned} \tag{4.2.10}$$

Here $N,\sigma>0$ are real numbers such that $N \gg 1, \sigma \ll 1$, which will be chosen later. We also note that $|D_i| \sim \sigma^3$. A simple computation gives us the $H^{\bar{s}}(\mathbb{R}^2)$ norm of the initial data is

$$\|\phi\|_{H^{\bar{s}}(\mathbb{R}^2)} \sim 1.$$

Using Lemma 4.2.1 we can write the second Picard iterate u_2 as

$$u_2(t,x,y) = -2\int_{\mathbb{R}^3} \xi e^{ix\xi+iy\eta+it(|\xi|^\alpha\xi+\frac{\eta^2}{\xi})}\ \frac{e^{it(\tau-|\xi|^\alpha\xi-\eta^2/\xi)}-1}{\tau-|\xi|^\alpha\xi-\frac{\eta^2}{\xi}}(\hat{u}_1 * \hat{u}_1)(\tau,\xi,\eta)d\tau d\xi d\eta.$$

Moreover, we have

$$\begin{aligned} (\hat{u}_1 * \hat{u}_1)(\tau,\xi,\eta) = \int_{\mathbb{R}^2} &\delta\big(\tau_1 - |\xi_1|^\alpha\xi_1 - \frac{\eta_1^2}{\xi_1}\big)\delta\big(\tau-\tau_1-|\xi-\xi_1|^\alpha(\xi-\xi_1) - \frac{(\eta-\eta_1)^2}{\xi-\xi_1}\big) \\ &\times \hat{\phi}(\xi_1,\eta_1)\hat{\phi}(\xi-\xi_1,\eta-\eta_1)d\xi_1 d\eta_1. \end{aligned}$$

Hence, we can write u_2 as

$$\begin{aligned} u_2(t,x,y) &= -2\int_{\mathbb{R}^4} \xi e^{ix\xi+iy\eta+it(|\xi|^\alpha\xi+\frac{\eta^2}{\xi})}\frac{e^{-it\Omega_{fKP}(\xi_1,\eta_1,\xi-\xi_1,\eta-\eta_1)}-1}{\Omega_{fKP}(\xi_1,\eta_1,\xi-\xi_1,\eta-\eta_1)}d\xi_1 d\eta_1 d\xi d\eta \\ &=: -2\int_{\mathbb{R}^4} \Phi(t,x,y,\xi,\eta,\xi_1,\eta_1)d\xi_1 d\eta_1 d\xi d\eta, \end{aligned}$$

where

$$\Phi(t,x,y,\xi,\eta,\xi_1,\eta_1) = \xi e^{ix\xi+iy\eta+it(|\xi|^\alpha\xi+\frac{\eta^2}{\xi})}\frac{e^{-it\Omega_{fKP}(\xi_1,\eta_1,\xi-\xi_1,\eta-\eta_1)}-1}{\Omega_{fKP}(\xi_1,\eta_1,\xi-\xi_1,\eta-\eta_1)}.$$

We decompose u_2 into three parts depending on the initial data:

$$u_2(t,x,y) = c(f_1(t,x,y)+f_2(t,x,y)+f_3(t,x,y)),$$

where

$$\begin{aligned}
f_1(t,x,y) &= \frac{c}{\sigma^3}\int_{\substack{(\xi_1,\eta_1)\in D_1,\\(\xi-\xi_1,\eta-\eta_1)\in D_1}}\Phi(t,x,y,\xi,\eta,\xi_1,\eta_1)d\xi_1 d\eta_1 d\xi d\eta,\\
f_2(t,x,y) &= \frac{c}{\sigma^3 N^{2s_1+2(1+\frac{\alpha}{2})s_2}}\int_{\substack{(\xi_1,\eta_1)\in D_2,\\(\xi-\xi_1,\eta-\eta_1)\in D_2}}\Phi(t,x,y,\xi,\eta,\xi_1,\eta_1)d\xi_1 d\eta_1 d\xi d\eta,\\
f_3(t,x,y) &= \frac{c}{\sigma^3 N^{s_1+(1+\frac{\alpha}{2})s_2}}\int_{\substack{(\xi_1,\eta_1)\in D_1,\\(\xi-\xi_1,\eta-\eta_1)\in D_2}}\Phi(t,x,y,\xi,\eta,\xi_1,\eta_1)d\xi_1 d\eta_1 d\xi d\eta\\
&\quad+\frac{c}{\sigma^3 N^{s_1+(1+\frac{\alpha}{2})s_2}}\int_{\substack{(\xi_1,\eta_1)\in D_2,\\(\xi-\xi_1,\eta-\eta_1)\in D_1}}\Phi(t,x,y,\xi,\eta,\xi_1,\eta_1)d\xi_1 d\eta_1 d\xi d\eta.
\end{aligned}$$

We shall focus on f_3, i.e. the high-low term leading to resonant interaction. Since the Fourier supports of $f_i, i=1,2,3$ are disjoint for fixed t, we have

$$\|u_2(t,\cdot,\cdot)\|_{H^{\bar{s}}(\mathbb{R}^2)} \geqslant \|f_3(t,\cdot,\cdot)\|_{H^{\bar{s}}(\mathbb{R}^2)},$$

and we compute the right-hand side of the above display. The spatial Fourier transform of f_3 is given by

$$\begin{aligned}
\hat{f}_3(t,\xi,\eta) = \frac{c\xi e^{it(|\xi|^\alpha\xi+\frac{\eta^2}{\xi})}}{\sigma^3 N^{s_1+(1+\frac{\alpha}{2})s_2}}\Big(&\int_{\substack{(\xi_1,\eta_1)\in D_1,\\(\xi-\xi_1,\eta-\eta_1)\in D_2}}\frac{e^{-it\Omega_{fKP}(\xi_1,\eta_1,\xi-\xi_1,\eta-\eta_1)}-1}{\Omega_{fKP}(\xi_1,\eta_1,\xi-\xi_1,\eta-\eta_1)}d\xi_1 d\eta_1\\
+&\int_{\substack{(\xi_1,\eta_1)\in D_2,\\(\xi-\xi_1,\eta-\eta_1)\in D_1}}\frac{e^{-it\Omega_{fKP}(\xi_1,\eta_1,\xi-\xi_1,\eta-\eta_1)}-1}{\Omega_{fKP}(\xi_1,\eta_1,\xi-\xi_1,\eta-\eta_1)}d\xi_1 d\eta_1\Big).
\end{aligned}$$

To obtain a lower bound on the $H^{\bar{s}}$ norm of f_3, we estimate the size of the resonance function in the following.

Lemma 4.2.2 (Size of the resonance function)**.** *Let $(\xi_1,\eta_1)\in D_1$ and $(\xi_2,\eta_2)\in D_2$ (or $(\xi_1,\eta_1)\in D_2$ and $(\xi_2,\eta_2)\in D_1$), then*

$$|\Omega^1_{fKP}(\xi_1,\eta_1,\xi_2,\eta_2)| \sim N^\alpha\sigma.$$

Proof. We carry out a case-by-case analysis.

(i) $\xi_1>0,\xi_2>0$: Using the mean value theorem,

$$(\xi_1+\xi_2)^{\alpha+1}-\xi_2^{\alpha+1} = (\alpha+1)\xi_1\xi_*^\alpha, \quad \xi_*\in(\xi_2,\xi_2+\xi_1).$$

This gives

$$|\Omega^1_{fKP}| \sim |\xi_1((\alpha+1)(\xi_*^\alpha-\xi_1^\alpha))| \sim N^\alpha\sigma.$$

(ii) $|\xi_1| < |\xi_2|, \xi_1 > 0, \xi_2 < 0$: We define

$$\xi_2^{'} = -\xi_2, \text{ i.e. } \xi_2^{'} > 0.$$

Hence

$$\begin{aligned}\Omega^1_{fKP}(\xi_1, \eta_1, \xi_1, \eta_2) &= |\xi_1 + \xi_2|^\alpha(\xi_1 + \xi_2) - |\xi_1|^\alpha \xi_1 - |\xi_2|^\alpha \xi_2 \\ &= -(\xi_2^{'} - \xi_1)^{\alpha+1} - \xi_1^{\alpha+1} + (\xi_2^{'})^{\alpha+1} \\ &= (\xi_2^{'})^{\alpha+1} - (\xi_2^{'} - \xi_1)^{\alpha+1} - \xi_1^{\alpha+1}.\end{aligned}$$

This is the same form as obtained in case (i). Using the mean value theorem, we conclude the same for this case. Finally, using

$$\Omega^1_{fKP}(\xi_1, \eta_1, \xi_2, \eta_2) = \Omega^1_{fKP}(\xi_2, \eta_2, \xi_1, \eta_1),$$

the claim of the lemma follows. □

Remark 4.2.3. The above computation can also be adapted to determine the size of the resonance function in other general cases.

Proof of Theorem 4.1.2 (ctd). Using Taylor's theorem, we have

$$\begin{aligned}|\Omega^1_{fKP}(\xi_1, \eta_1, \xi_2, \eta_2)| &= N^\alpha \sigma + O(N^{\alpha-1}\sigma^2), \\ \text{and } |\Omega^2_{fKP}(\xi_1, \eta_1, \xi_2, \eta_2)| &= N^\alpha \sigma + O(N^{\frac{\alpha}{2}}\sigma^2).\end{aligned}$$

So, for $\alpha > 2$ we obtain

$$|\Omega_{fKP}(\xi_1, \eta_1, \xi_2, \eta_2)| = |(\Omega^1_{fKP} - \Omega^2_{fKP})(\xi_1, \eta_1, \xi_2, \eta_2)| \sim N^{\alpha-1}\sigma^2.$$

We choose $\sigma = N^{-\frac{\alpha-1}{2}-\theta}, \theta > 0$, which makes the resonance small. Since the size of the resonance function is small, we have

$$\Big|\frac{e^{-it\Omega_{fKP}} - 1}{\Omega_{fKP}}\Big| = |t| + O(N^{-\varepsilon}),$$

where $N^{-\varepsilon}$ denotes the small size of the resonance function for $\varepsilon > 0$. This gives

$$|\hat{f}_3(t, \xi, \eta)| \sim \frac{N}{N^{s_1+(1+\frac{\alpha}{2})s_2}},$$

for the given choice of initial data. Consequently, the $H^{\bar{s}}(\mathbb{R}^2)$ norm of $f_3(t, \cdot, \cdot)$ is given by

$$\|f_3(t, \cdot, \cdot)\|_{H^{\bar{s}}(\mathbb{R}^2)} \sim N\sigma^{\frac{3}{2}} = N^{\frac{7}{4}-\frac{3\alpha}{4}-\frac{3\theta}{2}}.$$

For (4.2.8) to hold, we require

$$1 \sim \|\phi\|^2_{H^{\bar{s}}(\mathbb{R}^2)} \gtrsim N^{\frac{7}{4}-\frac{3\alpha}{4}-\frac{3\theta}{2}}.$$

The above is true only for $\alpha \geqslant \frac{7}{3}$. This completes the proof. □

Remark 4.2.4. (i) As is the case with KP-I and KP-II equations (Bourgain [10] proved global well-posedness in $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{T}^2)$), one can contrast the fKP-I equation with the fKP-II equation (with the $-$ sign on $D_x^\alpha \partial_x$ replaced by $+$ in (4.1.1)). The resonance function for the fKP-II equation reads

$$\Omega_{fKPII}(\xi_1,\eta_1,\xi_2,\eta_2) = |\xi_1+\xi_2|^\alpha(\xi_1+\xi_2) - |\xi_1|^\alpha \xi_1 - |\xi_2|^\alpha \xi_2 + \frac{(\eta_1\xi_2-\eta_2\xi_1)^2}{(\xi_1+\xi_2)\xi_1\xi_2},$$

which can be bounded from below. This leads to the fKP-II equation being semilinear and exhibiting better behaviour than the fKP-I equation, see, for instance [31, Theorem 1.2].

(ii) The result of [10] has been improved in [63, 62, 32], subsequently.

4.3 Frequency dependent time localisation and function spaces

Noting that (4.1.1) is not amenable to Picard iteration[2] when $\alpha \leqslant \frac{5}{2}$, for $N_2 \lesssim N_1$, we consider the following term which needs to be controlled to handle the Duhamel integral:

$$\begin{aligned}&\|\partial_x(P_{N_1}U_\alpha(t)u_0 \cdot P_{N_2}U_\alpha(t)v_0)\|_{L^1([0,T];L_x^2(\mathbb{R}^2))}\\ \lesssim\ & T^{\frac12}N_1\|P_{N_1}U_\alpha(t)u_0 \cdot P_{N_2}U_\alpha(t)v_0\|_{L^2([0,T];L_x^2(\mathbb{R}^2))}\\ \lesssim\ & T^{\frac12}N_1^{1-\frac{\alpha}{4}}N_2^{\frac12}\|P_{N_1}u_0\|_{L^2}\|P_{N_2}v_0\|_{L^2},\end{aligned}$$

where the last inequality follows from (4.5.8). If we choose $T = T(N_1) = N_1^{\frac{\alpha}{2}-2}$, we see that the derivative loss is completely remedied. Note that this also corresponds to the choice of time scale in [34] for $\alpha = 2$. However, we additionally observe that for $\alpha > \frac{5}{2}$, using the estimates (4.5.8) and (4.5.4) for $N_2 \lesssim N_1^{\frac{1-\alpha}{2}}$ and $N_2 \gtrsim N_1^{\frac{1-\alpha}{2}}$, respectively, we have for $X^{s,\theta}$ spaces

$$\|\partial_x(P_{N_1}u \cdot P_{N_2}v)\|_{X^{0,\theta-1}} \lesssim N_1^{\frac54-\frac{\alpha}{2}+}\|P_{N_1}u\|_{X^{0,\theta}}\|P_{N_2}v\|_{X^{0,\theta}},$$

for $\theta > \frac12$. Hence, we conclude that for $\alpha > \frac52$, (4.1.1) is semilinearly well-posed. This suggests that the correct choice of time scale is the one obtained by interpolating $(\alpha, T(N)) = (2, N^{-1})$ and $(\alpha, T(N)) = \left(\frac52+, 1\right)$, i.e. $T(N) = N^{(2\alpha-5)-}$. Next, we define the short-time function spaces corresponding to this time localisation.

4.3.1 Function spaces

The proofs of the forthcoming results can be found in [34] while [60, Section 2.5] can be referred for an overview of the properties of short-time function spaces.

We adapt the Littlewood-Paley projectors to the present case. We recall the definition of χ from Chapter 2. Let $\chi \in C_c^\infty(\mathbb{R})$ be a symmetric, non-negative function such that $\chi(\xi) = 1$ for $|\xi| \leqslant 1$ and $\chi(\xi) = 0$ for $|\xi| \geqslant 2$. For $N \in 2^{\mathbb{N}}$, we let

$$\phi_N(\xi) = \chi\Big(\frac{\xi}{N}\Big) - \chi\Big(\frac{2\xi}{N}\Big).$$

[2]we recall that Theorem 4.1.2 suggests semilinear ill-posedness for $\alpha < \frac73$, where $\frac73 < \frac52$

We have

$$\chi(\xi) + \sum_{N \in 2^{\mathbb{N}}} \phi_N(\xi) \equiv 1.$$

For $f \in \mathcal{S}'(\mathbb{R}^d)$ and $N \in 2^{\mathbb{N}_0}$, we define

$$(P_N f)^\wedge(\xi, \eta) = \phi_N(\xi) \hat{f}(\xi, \eta).$$

For $N \in 2^{\mathbb{N}}$, let

$$A_N = \Big\{ (\xi, \eta) \in \mathbb{R}^2 : \frac{N}{8} \leqslant |\xi| \leqslant 8N \Big\},$$

with the obvious modification for A_1. Moreover, for $N \in 2^{\mathbb{Z}}$, we let

$$\tilde{A}_N = \Big\{ (\xi, \eta) \in \mathbb{R}^2 : \frac{N}{8} \leqslant |\xi| \leqslant 8N \Big\}.$$

Additionally, for $N \in 2^{\mathbb{N}_0}$, $L \in 2^{\mathbb{N}}$, we define

$$D_{N,L} = \{ (\tau, \xi, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : (\xi, \eta) \in A_N, \, |\tau - \omega(\xi, \eta)| \sim L \}, \quad D_{N, \leqslant L} = \bigcup_{L'=1}^{L} D_{N,L'},$$
$$D_{N,1} = \{ (\tau, \xi, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : (\xi, \eta) \in A_N, \, |\tau - \omega(\xi, \eta)| \leqslant 2 \}.$$

For $N \in 2^{\mathbb{Z}}$, $L \in 2^{\mathbb{N}_0}$, we define

$$\tilde{D}_{N,L} = \{ (\tau, \xi, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : (\xi, \eta) \in \tilde{A}_N, \, |\tau - \omega(\xi, \eta)| \sim L \}.$$

In the following we write for notational convenience, in order to distinguish modulation and spatial frequencies, $\eta_L(\tau) = \phi_L(\tau)$ for $L \in \mathbb{N}_0$, and let

$$\eta_{\leqslant L}(\tau) = \sum_{L'=1}^{L} \eta_{L'}(\tau).$$

We define the dyadic $X^{s,b}$ spaces by

$$X_N = \{ f \in L^2(\mathbb{R} \times \mathbb{R}^2) : f \text{ is supported in } \mathbb{R} \times A_N, \, \|f\|_{X_N} < \infty \},$$

and

$$\|f\|_{X_N} = \sum_{L=1}^{\infty} L^{\frac{1}{2}} \| \eta_L(\tau - \omega(\xi, \eta)) f \|_{L^2_{\tau, \xi, \eta}}.$$

Decomposing f as

$$f = \sum_{L \geqslant 1} \eta_L(\tau - \omega(\xi, \eta)) f,$$

we obtain

$$\Big\| \int_{\mathbb{R}} |f(\tau, \xi, \eta)| d\tau \Big\|_{L^2_{\xi, \eta}} \lesssim \|f\|_{X_N}.$$

Moreover, we have the estimate

$$\begin{aligned}&\sum_{L'\geqslant L} L'^{\frac{1}{2}}\Big\|\eta_{L'}(\tau-\omega(\xi,\eta))\int_{\mathbb{R}}|f(\tau',\xi,\eta)|L'^{-1}(1+L'^{-1}|\tau-\tau'|)^{-4}d\tau'\Big\|_{L^2_{\tau,\xi,\eta}}\\&\quad+L^{\frac{1}{2}}\Big\|\eta_{\leqslant L}(\tau-\omega(\xi,\eta))\int_{\mathbb{R}}|f(\tau',\xi,\eta)|L^{-1}(1+L^{-1}|\tau-\tau'|)^{-4}d\tau'\Big\|_{L^2_{\tau,\xi,\eta}}\\&\lesssim\|f\|_{X_N}.\end{aligned}\tag{4.3.1}$$

We find for Schwartz functions $\gamma\in\mathcal{S}(\mathbb{R})$, $M,N\in 2^{\mathbb{N}_0}$, $t_0\in\mathbb{R}$, $f\in X_N$, the estimate

$$\|\mathcal{F}_{t,x,y}[\gamma(M(t-t_0))\mathcal{F}^{-1}_{t,x,y}(f)]\|_{X_N}\lesssim_\gamma\|f\|_{X_N}.$$

We define

$$E_N=\{\phi:\mathbb{R}^2\to\mathbb{R}\,:\,\hat{\phi}\text{ is supported in }A_N,\ \|\phi\|_{E_N}=\|\phi\|_{L^2}<\infty\}.$$

Let $\eta_0:\mathbb{R}\to[0,1]$ be an even smooth function supported in $[-\frac{8}{5},\frac{8}{5}]$ which is equal to 1 in $[-\frac{5}{4},\frac{5}{4}]$. For $\alpha\in\Big(2,\frac{5}{2}\Big]$ and dyadic frequency $N\in 2^{\mathbb{N}}$, corresponding to the time localisation $N^{-(5-2\alpha)-}$, we define

$$F_N=\{u_N\in C(\mathbb{R};E_N)\,:\,\|u_N\|_{F_N}=\sup_{t_N\in\mathbb{R}}\|\mathcal{F}_{t,x,y}[u_N\cdot\eta_0(N^{(5-2\alpha)+}(t-t_N))]\|_{X_N}<\infty\}.$$

We place the solution into these short-time function spaces after dyadic frequency localisation. To ease the notation, the dependence on α is suppressed. For the nonlinearity, we consider correspondingly

$$\begin{aligned}\mathcal{N}_N=\{u_N\in C(\mathbb{R};E_N)\,:\,\|u_N\|_{\mathcal{N}_N}=\sup_{t_N\in\mathbb{R}}&\|(\tau-\omega(\xi,\eta)+iN^{(5-2\alpha)+})^{-1}\\&\times\mathcal{F}_{t,x,y}[u_N\cdot\eta_0(N^{(5-2\alpha)+}(t-t_N))]\|_{X_N}<\infty\}.\end{aligned}$$

We localize the spaces in time by the usual means: for $T\in(0,1]$, let

$$F_N(T)=\{u_N\in C([-T,T];E_N)\,:\,\|u_N\|_{F_N(T)}=\inf_{\substack{\tilde{u}_N=u_N\text{ in}\\ [-T,T]\times\mathbb{R}^2}}\|\tilde{u}_N\|_{F_N}<\infty\},$$
$$\mathcal{N}_N(T)=\{u_N\in C([-T,T];E_N)\,:\,\|u_N\|_{\mathcal{N}_N(T)}=\inf_{\substack{\tilde{u}_N=u_N\text{ in}\\ [-T,T]\times\mathbb{R}^2}}\|\tilde{u}_N\|_{\mathcal{N}_N}<\infty\}.$$

Let $H^{\infty,0}(\mathbb{R}^2)=\bigcap_{s\geqslant 0}H^{s,0}(\mathbb{R}^2)$. We assemble the spaces $F^{s,0}(T)$, $\mathcal{N}^{s,0}(T)$, and $E^{s,0}(T)$ via Littlewood-Paley decomposition

$$F^{s,0}(T)=\{u\in C([-T,T];H^{\infty,0}(\mathbb{R}^2))\,:\,\|u\|^2_{F^{s,0}(T)}=\sum_{N\in 2^{\mathbb{N}_0}}N^{2s}\|P_Nu\|^2_{F_N(T)}<\infty\},$$
$$\mathcal{N}^{s,0}(T)=\{u\in C([-T,T];H^{\infty,0}(\mathbb{R}^2))\,:\,\|u\|^2_{\mathcal{N}^{s,0}(T)}=\sum_{N\in 2^{\mathbb{N}_0}}N^{2s}\|P_Nu\|^2_{\mathcal{N}_N(T)}<\infty\},$$
$$\begin{aligned}E^{s,0}(T)=\{u\in C([-T,T];H^{\infty,0}(\mathbb{R}^2))\,:&\\&\|u\|^2_{E^{s,0}(T)}=\sum_{N\in 2^{\mathbb{N}_0}}N^{2s}\sup_{t\in[-T,T]}\|P_Nu(t)\|^2_{E_N(T)}<\infty\}.\end{aligned}$$

We state the multiplier properties of admissible time-multiplication. For $N \in 2^{\mathbb{N}_0}$, we define the set S_N of N-acceptable time multiplication factors for $d = (5-2\alpha)+$

$$S_N = \{m_N : \mathbb{R} \to \mathbb{R} : \|m_N\|_{S_N} = \sum_{j=0}^{10} N^{-dj} \|\partial_j m_N\|_{L^\infty} < \infty\}.$$

We have, for any $s \geqslant 0$ and $T \in (0,1]$:

$$\begin{cases} \|\sum_{N\in 2^{\mathbb{N}_0}} m_N(t) P_N(u)\|_{F^{s,0}(T)} & \lesssim \big(\sup_{N\in 2^{\mathbb{N}_0}} \|m_N\|_{S_N}\big) \|u\|_{F^{s,0}(T)}, \\ \|\sum_{N\in 2^{\mathbb{N}_0}} m_N(t) P_N(u)\|_{\mathcal{N}^{s,0}(T)} & \lesssim \big(\sup_{N\in 2^{\mathbb{N}_0}} \|m_N\|_{S_N}\big) \|u\|_{\mathcal{N}^{s,0}(T)}, \\ \|\sum_{N\in 2^{\mathbb{N}_0}} m_N(t) P_N(u)\|_{E^{s,0}(T)} & \lesssim \big(\sup_{N\in 2^{\mathbb{N}_0}} \|m_N\|_{S_N}\big) \|u\|_{E^{s,0}(T)}. \end{cases} \tag{4.3.2}$$

The following statements were proved for the KP-I equation in [34]. The first results says that the $F^{s,0}(T)$ norm controls the $L^\infty_T H^{s,0}$ norm, which justifies the use of $F^{s,0}(T)$ spaces as auxiliary spaces.

Lemma 4.3.1 (cf. [34, Lemma 3.1])**.** *Let $T \in (0,1]$. If $u \in F^{s,0}(T)$, then for all $s \in \mathbb{R}$, the following estimate holds:*

$$\sup_{t\in[-T,T]} \|u(t)\|_{H^{s,0}} \lesssim \|u\|_{F^{s,0}(T)}. \tag{4.3.3}$$

The next statement associates the solution to the nonlinearity as follows:

Lemma 4.3.2 (cf. [34, Proposition 3.2])**.** *Let $T \in (0,1]$, $u, v \in C([-T,T]; H^{\infty,0})$ and*

$$\partial_t u - D_x^\alpha \partial_x u - \partial_x^{-1}\partial_y^2 u = v, \quad (x,y) \in \mathbb{R}^2, \quad t \in (-T,T).$$

Then, for all $s \geqslant 0$, the following estimate holds:

$$\|u\|_{F^{s,0}(T)} \lesssim \|u\|_{E^{s,0}(T)} + \|v\|_{\mathcal{N}^{s,0}(T)}. \tag{4.3.4}$$

4.4 Linear Strichartz estimates and consequences

The following Strichartz estimates are due to Hadac [31] for dispersion generalised KP-II equations, but it is easy to see that the argument transpires to the KP-I equations. We observe the smoothing effect pertaining to the higher dispersion for $\alpha > 2$. These estimates enable us to handle the non-resonant interactions.

Theorem 4.4.1 (Linear Strichartz estimate, cf. [31, Theorem 3.1])**.** *Let $\alpha \geqslant 2$, $2 < q \leqslant \infty$, and*

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{2}, \quad \gamma := \Big(1 - \frac{2}{r}\Big)\Big(\frac{1}{2} - \frac{\alpha}{4}\Big).$$

Then, we have

$$\|D_x^{-\gamma} U_\alpha(t) u_0\|_{L_t^q L_{xy}^r(\mathbb{R}\times\mathbb{R}^2)} \lesssim \|u_0\|_{L^2_{xy}(\mathbb{R}^2)}. \tag{4.4.1}$$

We record a second linear Strichartz estimate for low x frequencies.

Lemma 4.4.2 (Strichartz estimate for low frequencies)**.** *Let $N, K \in 2^{\mathbb{Z}}$, $I \subseteq \mathbb{R}$ be an interval of length $|I| \sim K$, and $|\xi| \sim N$ for any $\xi \in I$. Suppose that $\hat{u}_0(\xi,\eta) = 0$, if $\xi \notin I$.*

Then, the following estimate holds:

$$\|U_\alpha(t)u_0\|_{L_t^4([0,1];L_{xy}^4(\mathbb{R}^2))} \lesssim K^{\frac{1}{4}} N^{\frac{1}{8}} \|u_0\|_{L^2(\mathbb{R}^2)}. \tag{4.4.2}$$

Proof. We use Bernstein's inequality in x, Plancherel's theorem, and Minkowski's inequality to find

$$\begin{aligned}
&\Big\| \int e^{i(x\xi+y\eta+t(\xi|\xi|^\alpha+\frac{\eta^2}{\xi})}\hat{u}_0(\xi,\eta)d\xi d\eta\Big\|_{L_t^4([0,1];L_{xy}^4(\mathbb{R}^2))} \\
&\lesssim K^{\frac{1}{4}}\Big\| \int e^{i(y.\eta+t\frac{\eta^2}{\xi})}\hat{u}_0(\xi,\eta)d\eta\Big\|_{L_t^4([0,1];L_y^4L_\xi^2)} \\
&\lesssim K^{\frac{1}{4}}\Big(\int_I d\xi\Big\| \int e^{i(y.\eta+t\frac{\eta^2}{\xi})}\hat{u}_0(\xi,\eta)d\eta\Big\|^2_{L_t^4([0,1];L_y^4(\mathbb{R}))}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Thus, it remains to prove

$$\Big\| \int e^{i(y.\eta+t\frac{\eta^2}{\xi})}\hat{u}_0(\xi,\eta)d\eta\Big\|_{L_t^4([0,1];L_y^4(\mathbb{R}))} \lesssim N^{\frac{1}{8}}\|\hat{u_0}(\xi,\cdot)\|_{L_\eta^2(\mathbb{R})}. \tag{4.4.3}$$

By a change of variables $t \to t'\xi$ and using Hölder's inequality in time, we find

$$\begin{aligned}
\Big\| \int e^{i(y.\eta+t\frac{\eta^2}{\xi})}\hat{u}_0(\xi,\eta)d\eta\Big\|_{L_t^4([0,1];L_y^4(\mathbb{R}))} &\lesssim N^{\frac{1}{4}}\Big\| \int e^{i(y.\eta+t\eta^2)}\hat{u}_0(\xi,\eta)d\eta\Big\|_{L_t^4([0,\xi^{-1}];L_y^4(\mathbb{R}))} \\
&\lesssim N^{\frac{1}{8}}\Big\| \int e^{i(y.\eta+t\eta^2)}\hat{u}_0(\xi,\eta)d\eta\Big\|_{L_t^8([0,\xi^{-1}];L_y^4(\mathbb{R}))} \\
&\lesssim N^{\frac{1}{8}}\|\hat{u_0}(\xi,\cdot)\|_{L_\eta^2(\mathbb{R})}.
\end{aligned}$$

The ultimate estimate is an application of the $L_t^8L_y^4$ Strichartz estimate for the one-dimensional Schrödinger equation, see Lemma 2.5.3. □

As a consequence of the transfer principle, we have the following:

Corollary 4.4.3. *Let $u \in X_N$ be supported in $\tilde{D}_{N,L}$. Then*

$$\|\mathcal{F}^{-1}(u)\|_{L_{t,x,y}^4} \lesssim \big(\max(1,N)\big)^{\frac{1}{4}-\frac{\alpha}{8}} L^{\frac{1}{2}}\|u\|_{L^2}. \tag{4.4.4}$$

Proof. We first note that for $q = r = 4$, (4.4.1) gives

$$\|D_x^{\frac{\alpha}{8}-\frac{1}{4}}U_\alpha(t)u_0\|_{L_{t,x,y}^4} \lesssim \|u_0\|_{L_{x,y}^2}. \tag{4.4.5}$$

We have

$$\begin{aligned}
\mathcal{F}^{-1}_{t,x,y}(u)(t,x,y) &= \int_{\mathbb{R}^3}\hat{u}(\tau,\xi,\eta)e^{ix\xi+iy\eta+it\tau}d\tau d\xi d\eta \\
&= \int_{\mathbb{R}^3}\hat{u}(\tau+\omega(\xi,\eta),\xi,\eta)e^{ix\xi+iy\eta+it\tau}e^{it\omega(\xi,\eta)}d\tau d\xi d\eta \\
&= \int_{\mathbb{R}} e^{it\tau}\Big(\int_{\mathbb{R}^2} e^{it\omega(\xi,\eta)}e^{ix\xi+iy\eta}\hat{u}(\tau+\omega(\xi,\eta),\xi,\eta)d\xi d\eta\Big)d\tau.
\end{aligned}$$

Using (4.4.5),(4.4.2) and Cauchy-Schwarz inequality,

$$\|\mathcal{F}^{-1}(u)\|_{L^4_{t,x,y}} \lesssim \max(1, N^{\frac{1}{4}-\frac{\alpha}{8}})\Big\|\|u(\tau+\omega(\xi,\eta),\cdot,\cdot)\|_{L^2}\Big\|_{L^1_\tau} \leqslant \max(1, N^{\frac{1}{4}-\frac{\alpha}{8}})L^{\frac{1}{2}}\|u\|_{L^2}. \tag{4.4.6}$$

This concludes the proof. □

4.5 Resonance and transversality

We analyse the resonance function and use it to obtain trilinear estimates via the nonlinear Loomis-Whitney inequality. Transversality in the resonant case also allows us to obtain good bilinear estimates. We recall the resonant condition, i.e.

$$|\Omega_{fKP}| \ll ||\xi_1+\xi_2|^\alpha(\xi_1+\xi_2) - |\xi_1|^\alpha\xi_1 - |\xi_2|^\alpha\xi_2|.$$

Suppose that we have $N_{\max} \sim |\xi_1+\xi_2| \sim |\xi_1| \gtrsim |\xi_2| \sim N_{\min}$, then from the computation done in Lemma 4.2.2, we get that the right-hand side of the above equation has size $N_{max}^\alpha N_{min}$, i.e. in the resonant case,

$$N_{\max}^\alpha N_{\min} \sim \Big|\frac{(\eta_1\xi_2-\eta_2\xi_1)^2}{\xi_1\xi_2(\xi_1+\xi_2)}\Big|.$$

This can be further simplified to

$$|\eta_1\xi_2-\eta_2\xi_1| \sim N_{\max}^{\frac{\alpha}{2}+1}N_{\min}. \tag{4.5.1}$$

Now consider the gradient of the dispersion relation

$$\nabla\omega_{fKP}(\xi,\eta) = \Big(|\xi|^\alpha - \frac{\eta^2}{\xi^2}, \frac{2\eta}{\xi}\Big).$$

Using (4.5.1), we have

$$|\nabla\omega_{fKP}(\xi_1,\eta_1) - \nabla\omega_{fKP}(\xi_2,\eta_2)| \gtrsim \Big|\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}\Big| \sim N_{\max}^{\frac{\alpha}{2}}. \tag{4.5.2}$$

The above relation we shall employ to obtain good multilinear estimates via the nonlinear Loomis-Whitney inequality and bilinear Strichartz estimates.

4.5.1 Nonlinear Loomis-Whitney inequality

In this section, we state the setting and prove the trilinear estimate in the resonant case via the nonlinear Loomis-Whitney inequality from [39]. This leads to a sharp estimate in the case $N_{\min} \gtrsim N_{\max}^{-\kappa}$, for some $\kappa > 0$. We recall the assumptions on the parametrisations.
Assumption 4.5.1: For $i = 1, 2, 3$, there exist $0 < \beta \leqslant 1$, $b > 0$, $A \geqslant 1$, $F_i \in C^{1,\beta}(\mathcal{U}_i)$, where the $\mathcal{U}_i$ denote open and convex sets in $\mathbb{R}^2$ and $G_i \in O(3)$ such that

(i) the oriented surfaces S_i are given by

$$S_i = G_i gr(F_i), \quad gr(F_i) = \{(x,y,z) \in \mathbb{R}^3 : \ z = F_i(x,y), (x,y) \in \mathcal{U}_i\}.$$

(ii) the unit normal vector field $\mathbf{n}_i$ on S_i satisfies the Hölder condition

$$\sup_{\sigma,\tilde{\sigma}\in S_i} \frac{|\mathbf{n}_i(\sigma)-\mathbf{n}_i(\tilde{\sigma})|}{|\sigma-\tilde{\sigma}|^{\beta}} + \frac{|\mathbf{n}_i(\sigma)(\sigma-\tilde{\sigma})|}{|\sigma-\tilde{\sigma}|^{1+\beta}} \leqslant b;$$

(iii) the matrix $N(\sigma_1,\sigma_2,\sigma_3) = (\mathbf{n}_1(\sigma_1),\mathbf{n}_2(\sigma_2),\mathbf{n}_3(\sigma_3))$ satisfies the transversality condition

$$A^{-1} \leqslant \det\ N(\sigma_1,\sigma_2,\sigma_3) \leqslant 1,$$

for all $(\sigma_1,\sigma_2,\sigma_3) \in S_1 \times S_2 \times S_3$.

For $\varepsilon > 0$, by $S_i(\varepsilon)$ we denote

$$S_i(\varepsilon) := G_i\{(x,y,z) \in \mathcal{U}_i \times \mathbb{R} : |z - F_i(x,y)| < \varepsilon\}.$$

Remark 4.5.1. Note that Assumption 4.5.1 is same as Assumption 3.4 up to the boundedness of the surfaces. We state it again for convenience.

Theorem 4.5.2 ([39, Theorem 4.3])**.** *Let A be dyadic and $f_i \in L^2(S_i(\varepsilon))$, $i = 1,2$. Suppose that $(S_i)_{i=1}^3$ satisfies Assumption 4.5.1. Then, for $\varepsilon > 0$, we find the following estimate to hold:*

$$\|f_1 * f_2\|_{L^2(S_3(\varepsilon))} \lesssim \varepsilon^{\frac{3}{2}} A^{\frac{1}{2}} \|f_1\|_{L^2(S_1(\varepsilon))} \|f_2\|_{L^2(S_2(\varepsilon))}, \tag{4.5.3}$$

where the implicit constant is independent of β and b.

In the following, we apply Theorem 4.5.2 in the resonant case to obtain a trilinear estimate:

Lemma 4.5.3. *Let $\alpha > 0$, and $N_1, N_2, N_3 \in 2^{\mathbb{Z}}$ be such that $N_2 \lesssim N_1 \sim N_3$, $N_1 \gtrsim 1$ and $L_1, L_2, L_3 \in 2^{\mathbb{N}_0}$ be such that $L_1, L_2, L_3 \leqslant N_1^{\alpha} N_2$. Let $f, g, h : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}_+$ be L^2 functions supported in $\tilde{D}_{N_1,L_1}, \tilde{D}_{N_2,L_2}$ and $\tilde{D}_{N_3,L_3}$, respectively. Then*

$$\Big| \int (f * g) \cdot h \Big| \lesssim N_1^{-\frac{3\alpha}{4}+\frac{1}{2}} N_2^{-\frac{1}{2}} (L_1 L_2 L_3)^{\frac{1}{2}} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}. \tag{4.5.4}$$

Proof. To account for the localisation of the functions to D_{N_i,L_i}, for $N_2 \lesssim N_1 \sim N_3$, we denote the left-hand side of (4.5.4) by

$$\Big| \int (f_{N_1,L_1} * g_{N_2,L_2}) \cdot h_{N_3,L_3} \Big|.$$

We estimate the above in the resonant case, i.e. where $L_1, L_2, L_3 \leqslant N_1^{\alpha} N_2$. Furthermore, we can decompose f_{N_1,L_1}, g_{N_2,L_2}, and h_{N_3,L_3} into L_1, L_2, L_3 number of pieces, respectively, such that the decomposed pieces are supported in a (translated) neighbourhood of size 1 of the characteristic surface. For example, we write $f_{N_1,L_1} = \sum_i f_i$ with f_i supported on

$$a_i \leqslant \tau - \omega(\xi,\eta) \leqslant a_i + 1,$$

for some $a_i \in \mathbb{R}$. To lighten the notation, we denote the decomposed pieces by f_{N_1,L_1}, g_{N_2,L_2} and h_{N_3,L_3}. After a harmless translation, we can suppose that f_{N_1,L_1}, g_{N_2,L_2} and h_{N_3,L_3} are supported in the unit neighbourhood of the characteristic surface. Then, it suffices to prove the following:

$$\Big| \int (f_{N_1,L_1} * g_{N_2,L_2}) \cdot h_{N_3,L_3} \Big| \lesssim N_1^{-\frac{3\alpha}{4}+\frac{1}{2}} N_2^{-\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N_3,L_3}\|_{L^2}. \tag{4.5.5}$$

A lengthy, but straight-forward computation, for which details are provided in Appendix F, yields

$$B = |\det(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)| \sim N_1^{\frac{3\alpha}{2}-1} N_2.$$

We make an additional inhomogeneous decomposition in the η support with $|\eta_i| \lesssim N_i$ or $|\eta_i| \sim K_i \in 2^{\mathbb{Z}}$ for $K_i \gtrsim N_i$:

$$\int (f_{N_1,L_1} * g_{N_2,L_2}) h_{N_3,L_3} \leqslant \sum_{K_i \geqslant N_i} \int (f_{N_1,K_1,L_1} * g_{N_2,K_2,L_2}) h_{N_3,K_3,L_3}$$

Let $K_1^* \geqslant K_2^* \geqslant K_3^*$ denote a decreasing rearrangement. By convolution constraint, we have

$$\sum_{K_i \geqslant N_i} \int (f_{N_1,K_1,L_1} * g_{N_2,K_2,L_2}) h_{N_3,K_3,L_3} = \sum_{K_1^* \sim K_2^* \gtrsim K_3^*} \int (f_{N_1,K_1,L_1} * g_{N_2,K_2,L_2}) h_{N_3,K_3,L_3}.$$

Let e.g. $K_1^* = K_1$, $K_2^* = K_2$. We subsume

$$\begin{aligned} &\sum_{\substack{K_3^* \leqslant K_1^* \sim K_2^*, \\ (K_1^*, K_2^*) = (K_1, K_2)}} \int (f_{N_1,K_1,L_1} * g_{N_2,K_2,L_2}) h_{N_3,K_3,L_3} \\ &= \sum_{K_1 \sim K_2} \int (f_{N_1,K_1,L_1} * g_{N_2,K_2,L_2}) h_{N_3,\lesssim K_1,L_3}. \end{aligned}$$

Now it suffices to prove

$$\begin{aligned} &\sum_{K_1 \sim K_2} \int \left| (f_{N_1,K_1,L_1} * g_{N_2,K_2,L_2}) h_{N_3,\lesssim K_1,L_3} \right| \\ &\lesssim \sum_{K_1 \sim K_2 \gtrsim N_1} C(N_1,N_2,N_3) \| f_{N_1,K_1,L_1} \|_{L^2} \| g_{N_2,K_2,L_2} \|_{L^2} \| h_{N_3,L_3} \|_{L^2} \end{aligned} \tag{4.5.6}$$

because the sum over $K_1 \sim K_2$ is estimated by the Cauchy-Schwarz inequality. To ease the notation, we denote the functions in (4.5.6) by f, g and h. Henceforth, we suppose that $|(\xi_i, \eta_i)| \sim K_i$. We normalise the normals by the size of the maximum total frequency $K^* = \max(K_1, K_2, K_3)$. Define

$$\xi_i' = \frac{\xi_i}{K^*}, \quad \eta_i' = \frac{\eta_i}{(K^*)^{\frac{\alpha}{2}+1}}, \quad \tau_i' = \frac{\tau_i}{(K^*)^{\alpha+1}}.$$

Let $B' = \det(\mathbf{n}_1', \mathbf{n}_2', \mathbf{n}_3')$, where $\mathbf{n}_i', i = 1,2,3$ is computed for the normalised variables. It is easy to see that

$$B' = \frac{B}{(K^*)^{\frac{3\alpha}{2}}}.$$

Define

$$\begin{aligned} f'(\tau,\xi,\eta) &= f((K^*)^{\alpha+1}\tau, K^*\xi, (K^*)^{\frac{\alpha}{2}+1}\eta), \\ g'(\tau,\xi,\eta) &= g((K^*)^{\alpha+1}\tau, K^*\xi, (K^*)^{\frac{\alpha}{2}+1}\eta), \\ h'(\tau,\xi,\eta) &= h((K^*)^{\alpha+1}\tau, K^*\xi, (K^*)^{\frac{\alpha}{2}+1}\eta). \end{aligned}$$

Then f', g' and h' are supported in a $(K^*)^{-(\alpha+1)}$ neighbourhood of the characteristic surface. Using Theorem 4.5.2, we have

$$\begin{aligned}\Big|\int(f*g)\cdot h\Big| &= (K^*)^{2(\frac{3\alpha}{2}+3)}\Big|\int(f'*g')\cdot h'\Big| \\ &\lesssim (K^*)^{2(\frac{3\alpha}{2}+3)}B'^{-\frac{1}{2}}(K^*)^{-\frac{3}{2}(\alpha+1)}\|f'\|_{L^2}\|g'\|_{L^2}\|h'\|_{L^2} \\ &\lesssim (K^*)^{2(\frac{3\alpha}{2}+3)}B'^{-\frac{1}{2}}(K^*)^{-\frac{3}{2}(\alpha+1)}(K^*)^{-\frac{3}{2}(\frac{3\alpha}{2}+3)}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2} \\ &\lesssim N_1^{-\frac{3\alpha}{4}+\frac{1}{2}}N_2^{-\frac{1}{2}}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}.\end{aligned} \tag{4.5.7}$$

This suffices to establish (4.5.4) by the almost orthogonality argued above. The proof is complete. □

Remark 4.5.4. $(B')^{-1} \gtrsim 1$ serves as A in Theorem 4.5.2.

4.5.2 Bilinear Strichartz estimates

As a consequence of transversality in the resonant case, we also obtain sharp bilinear estimates. This we pursue in the following.

Proposition 4.5.5. *Let $\alpha > 0$, suppose that $u, v \in L^2(\mathbb{R}\times\mathbb{R}^2)$ have their Fourier supports in $\tilde{D}_{N_1,L_1}$ and $\tilde{D}_{N_2,L_2}$, respectively, and that for $(\tau_1, \xi_1, \eta_1) \in supp(\hat{u})$ and $(\tau_2, \xi_2, \eta_2) \in supp(\hat{v})$, the resonance condition (4.2.3) holds. Then,*

$$\|uv\|_{L^2} \lesssim (L_1L_2)^{\frac{1}{2}}\frac{\min(N_1,N_2)^{\frac{1}{2}}}{\max(N_1,N_2)^{\frac{\alpha}{4}}}\|u\|_{L^2}\|v\|_{L^2}. \tag{4.5.8}$$

Proof. Let $\bar{L} = \max(L_1, L_2)$, $\underline{L} = \min(L_1, L_2)$. Using Plancherel's identity and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|uv\|_{L^2_{t,x,y}} &= \Big\|\int_{\mathbb{R}\times\mathbb{R}^2}\hat{u}(\tau_1,\xi_1,\eta_1)\hat{v}(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1)d\tau_1 d\xi_1 d\eta_1\Big\|_{L^2_{\tau,\xi,\eta}} \\ &\lesssim \underline{L}^{\frac{1}{2}}|E(\xi,\eta)|^{\frac{1}{2}}\|u\|_{L^2}\|v\|_{L^2},\end{aligned} \tag{4.5.9}$$

where the set E is given by

$$\begin{aligned}E(\xi,\eta) := \{(\xi_1,\eta_1)\in\tilde{A}_{N_1} : |\tau-\omega_{fKP}(\xi_1,\eta_1)-\omega_{fKP}(\xi-\xi_1,\eta-\eta_1)| \lesssim \bar{L}, \\ (\xi-\xi_1,\eta-\eta_1)\in\tilde{A}_{N_2}\}.\end{aligned}$$

The measure of this set can be estimated by Fubini's theorem. Using (4.5.2), Lemma 3.4.10 and orthogonality, we have

$$|E(\xi,\eta)| = \Big|\int\Big(\int \mathbf{1}_{E(\xi,\eta)}(\xi_1,\eta_1)d\eta_1\Big)d\xi_1\Big| \lesssim \min(N_1,N_2)\frac{\bar{L}}{\max(N_1,N_2)^{\frac{\alpha}{2}}}.$$

Substituting this in (4.5.9), we obtain

$$\|uv\|_{L^2} \lesssim (L_1L_2)^{\frac{1}{2}}\frac{\min(N_1,N_2)^{\frac{1}{2}}}{\max(N_1,N_2)^{\frac{\alpha}{4}}}\|u\|_{L^2}\|v\|_{L^2}.$$

□

Remark 4.5.6. The estimate (4.5.8) still holds if we replace the functions on the left-hand side of (4.5.8) by their complex conjugates.

The next lemma allows us to handle the non-resonant case when the smallest frequency has size $\lesssim 1$.

Lemma 4.5.7. *Let $\alpha > 0$, $N_1, N_2, N_3 \in 2^{\mathbb{Z}}$ be such that $N_1 \ll N_2 \sim N_3$, and $L, L_1, L_2 \in 2^{\mathbb{N}_0}$. For $i = 1, 2, 3$, let $f_i : \mathbb{R}^3 \to \mathbb{R}_+$ be L^2 functions supported in D_{N_i,L_i}. If $\max(L_1, L_2, L_3) \gtrsim N_1 N_2^{\alpha}$, the following estimate holds:*

$$\int_{\mathbb{R}^3} (f_1 * f_2) \cdot f_3 \lesssim \frac{(L_1 L_2 L_3)^{1/2}}{\max(L_1, L_2, L_3)^{1/4}} N_2^{-\frac{\alpha}{2}} N_1^{\frac{1}{4}} \|f\|_{L^2} \|f_1\|_{L^2} \|f_2\|_{L^2}. \tag{4.5.10}$$

The proof is a generalisation of the proof of [30, Lemma 3.1] to the case $\alpha > 2$. We provide the details for the sake of completeness.

Proof. Using a change of variables, we can write the left-hand side of (4.5.10) as

$$\begin{aligned}
&\int_{\mathbb{R}^6} f_1(\tau_1 + \omega(\xi_1, \eta_1), \xi_1, \eta_1) f_2(\tau_2 + \omega(\xi_2, \eta_2), \xi_2, \eta_2) \\
&\qquad \times f_3(\tau_1 + \tau_2 + \omega(\xi_1, \eta_1) + \omega(\xi_2, \eta_2), \xi_1 + \xi_2, \eta_1 + \eta_2) \prod_{i=1}^{2} d\tau_i d\xi_i d\eta_i, \\
&= \int_{\mathbb{R}^6} f_1^{\#}(\tau_1, \xi_1, \eta_1) f_2^{\#}(\tau_2, \eta_2, \xi_2) \\
&\qquad \times f_3^{\#}(\tau_1 + \tau_2 + \Omega((\xi_1, \eta_1), (\xi_2, \eta_2)), \xi_1 + \xi_2, \eta_1 + \eta_2) \prod_{i=1}^{2} d\tau_i d\xi_i d\eta_i, ,
\end{aligned}$$

where

$$f_i^{\#}(\tau, \xi, \eta) := f_i(\tau + \omega(\xi, \eta), \xi, \eta), \ \ i = 1, 2, 3,$$

and

$$\begin{aligned}
\Omega((\xi_1, \eta_1), (\xi_2, \eta_2)) &= -\omega(\xi_1 + \xi_2, \eta_1 + \eta_2) + \omega(\xi_1, \eta_1) + \omega(\xi_1, \eta_2) \\
&= -|\xi_1 + \xi_2|^{\alpha}(\xi_1 + \xi_2) + |\xi_1|^{\alpha}\xi_1 + |\xi_2|^{\alpha}\xi_2 + \frac{(\eta_2\xi_1 - \eta_1\xi_2)^2}{(\xi_1 + \xi_2)\xi_1\xi_2}.
\end{aligned}$$

Note that for $i = 1, 2, 3$, $\|f_i^{\#}\|_{L^2} = \|f_i\|_{L^2}$ and $f_i^{\#}$ are functions supported in

$$\{(\tau_i, \xi_i, \eta_i) : |\tau_i| \sim L_i, (\xi, \eta) \in A_{N_i}\}.$$

Using Cauchy-Schwarz inequality, it is sufficient to prove

$$\begin{aligned}
&\int_{\mathbb{R}^4} g_1(\xi_1, \eta_1) g_2(\xi_2, \eta_2) g(\Omega((\xi_1, \eta_1), (\xi_2, \eta_2)), \xi_1 + \xi_2, \eta_1 + \eta_2) d\xi_1 d\xi_2 d\eta_1 d\eta_2 \\
&\qquad \lesssim L_{\max}^{\frac{1}{4}} N_2^{-\frac{\alpha}{2}} N_1^{\frac{1}{4}} \|g_1\|_{L^2} \|g_2\|_{L^2} \|g\|_{L^2},
\end{aligned} \tag{4.5.11}$$

where $g_i : \mathbb{R}^2 \to \mathbb{R}_+$ are L^2 functions supported in $\tilde{A}_{N_i}$, $i = 1, 2$, and $g : \mathbb{R}^3 \to \mathbb{R}$ is an L^2 function supported in $[-L_{\max}, L_{\max}] \times \tilde{A}_{N_3}$. Using a change of variables

$$\xi_2 \to \xi_2 - \xi_1, \quad \eta_2 \to \eta_2 + \eta_1,$$

the left-hand side of (4.5.11) becomes

$$\int_{\mathbb{R}^4} g_1(\xi_1,\eta_1)g_2(\xi_2-\xi_1,\eta_2-\eta_1)g(\Omega((\xi_1,\eta_1),(\xi_2-\xi_1,\eta_2-\eta_1)),\xi_2,\eta_2)\prod_{i=1}^{2} d\xi_i d\eta_i. \quad (4.5.12)$$

By Cauchy-Schwarz inequality, we find that we can bound the above quantity by

$$\begin{aligned}&\int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}|g_1(\xi_1,\eta_1)g(\xi_2-\xi_1,\eta_2-\eta_1)|^2 d\xi_1 d\eta_1\Big)^{\frac12}\\ &\quad\times\Big(\int_{\mathbb{R}^2}|g(\Omega((\xi_1,\eta_1),(\xi_2-\xi_1,\eta_2-\eta_1)),\xi_2,\eta_2)|^2 d\xi_1 d\eta_1\Big)^{\frac12} d\xi_2 d\eta_2.\end{aligned} \quad (4.5.13)$$

Define

$$\begin{aligned}\beta_1(\eta_1) &= \Omega((\xi_1,\eta_1),(\xi_2-\xi_1,\eta_2-\eta_1))\\ &= |\xi_2-\xi_1|^\alpha(\xi_2-\xi_1)-|\xi_2|^\alpha\xi_2+|\xi_1|^\alpha\xi_1+\frac{(\eta_1\xi_2-\eta_2\xi_1)^2}{\xi_1\xi_2(\xi_2-\xi_1)},\\ \beta_2(\xi_1) &= -(|\xi_2-\xi_1|^\alpha(\xi_2-\xi_1)-|\xi_2|^\alpha\xi_2+|\xi_1|^\alpha\xi_1).\end{aligned}$$

We have $|\beta_1|\lesssim L_{\max}$, $|\beta_2|\lesssim L_{\max}$, $|\beta_2|\lesssim N_2^\alpha N_1$ and using

$$\beta_1+\beta_2=\frac{(\eta_1\xi_2-\eta_2\xi_1)^2}{\xi_1\xi_2(\xi_2-\xi_1)},$$

we obtain

$$d\xi_1 d\eta_1=\frac{\xi_1^{\frac12}(\xi_2-\xi_1)^{\frac12}}{2(\alpha+1)(\beta_1+\beta_2)^{\frac12}\xi_2^{\frac12}[(\xi_2-\xi_1)^\alpha-\xi_1^\alpha]}d\beta_1 d\beta_2.$$

Using $|\xi_2|\sim|\xi_2-\xi_1|$ and Fubini's theorem, we get

$$\Big(\int_{\mathbb{R}^2}|g(\Omega((\xi_1,\eta_1),(\xi_2-\xi_1,\eta_2-\eta_1)),\xi_2,\eta_2)|^2 d\xi_1 d\eta_1\Big)^{\frac12}\lesssim\frac{N_1^{\frac14}L_{\max}^{\frac14}}{N_2^{\frac{\alpha}{2}}}\|g(\cdot,\xi_2,\eta_2)\|_{L^2}.$$

Substituting the above in (4.5.13) and an application of the Cauchy-Schwarz inequality completes the proof. □

We are now set to prove Theorem 4.1.1.

4.6 Quasilinear well-posedness

This section is devoted to the proof of the theorem below, which yields Theorem 4.1.1.

Theorem 4.6.1. *Let $u_0\in H^{\infty,0}(\mathbb{R}^2)$ and $s>5-2\alpha$. Then, there exists $T=T(\|u_0\|_{H^{s,0}})>0$ such that there is a unique solution*

$$u=S_T^\infty(u_0)\in C([-T,T];H^{\infty,0}(\mathbb{R}^2))$$

of (4.1.1). In addition, for $s'\geqslant s$

$$\sup_{|t|\leqslant T}\|S_T^\infty(u_0)(t)\|_{H^{s',0}(\mathbb{R}^2)}\lesssim C(T,s',\|u_0\|_{H^{s',0}}).$$

Moreover, the mapping

$$S_T^\infty : H^{\infty,0}(\mathbb{R}^2) \to C([-T,T]; H^{\infty,0}(\mathbb{R}^2))$$

extends uniquely to a continuous mapping

$$S_T^{s'} : H^{s',0}(\mathbb{R}^2) \to C([-T,T]; H^{s',0}(\mathbb{R}^2)).$$

The existence of local-in-time solutions for initial data in $H^{2,0}(\mathbb{R}^2)$ to the KP-I equation was proved by Molinet-Saut-Tzvetkov [49]. The proof is a non-trivial variant of the energy method, which relies on commutator estimates. Also, persistence of regularity is discussed in [49]. These arguments transpire to the case of dispersion generalised KP-I equation and show the existence of a mapping S_T^∞.

4.6.1 Short-time bilinear estimates

In this subsection we prove short-time bilinear estimates which we need to control the nonlinearity in (4.1.1).

Proposition 4.6.2. *Let $2 < \alpha \leqslant \frac{5}{2}$, $T \in (0,1]$, $s' \geqslant 0$ and $u, v \in F^{s',0}(T)$. Then, the following estimate holds:*

$$\|\partial_x(uv)\|_{\mathcal{N}^{s',0}(T)} \lesssim \|u\|_{F^{0,0}(T)}\|v\|_{F^{s',0}(T)} + \|v\|_{F^{0,0}(T)}\|u\|_{F^{s',0}(T)}. \tag{4.6.1}$$

This proposition will be proved by means of dyadic estimates which we prove in the following. We first consider the *High* $\times$ *Low* $\to$ *High* interaction.

Lemma 4.6.3 (*High* $\times$ *Low* $\to$ *High*)**.** *Let $N_1, N_2, N \in 2^{\mathbb{N}_0}$ be such that $N_2 \ll N_1 \sim N$ and $u_{N_1} \in F_{N_1}$, $v_{N_2} \in F_{N_2}$. Then,*

$$\|P_N(\partial_x(u_{N_1}v_{N_2}))\|_{\mathcal{N}_N} \lesssim N_1^{0-}\|u_{N_1}\|_{F_{N_1}}\|v_{N_2}\|_{F_{N_2}}. \tag{4.6.2}$$

Proof. Using the definition of the $\mathcal{N}_N$ norm, we can bound the left-hand side of (4.6.2) by

$$\sup_{t_N \in \mathbb{R}} \|(\tau - \omega(\xi,\eta) + iN^{(5-2\alpha)+})^{-1} N \mathbf{1}_{A_N}(\xi)\mathcal{F}[u_{N_1} \cdot \eta_0(N^{(5-2\alpha)+}(t-t_N))] \\ * \mathcal{F}[v_{N_2} \cdot \eta_0(N^{(5-2\alpha)+}(t-t_N))]\|_{X_N}.$$

Let

$$f_{N_1} := \mathcal{F}[u_{N_1} \cdot \eta_0(N^{(5-2\alpha)+}(t-t_N))],$$
$$g_{N_2} := \mathcal{F}[v_{N_2} \cdot \eta_0(N^{(5-2\alpha)+}(t-t_N))].$$

Using the properties (4.3.1) and (4.3.2), it suffices to prove that if $L_1, L_2 \geqslant N^{(5-2\alpha)+}$ and

$$f_{N_1,L_1}, g_{N_2,L_2} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}_+$$

are functions supported in D_{N_1,L_1} and D_{N_2,L_2}, respectively, then

$$N \sum_{L \geqslant N^{(5-2\alpha)+}} L^{-\frac{1}{2}} \|\mathbf{1}_{D_{N,L}}(f_{N_1,L_1} * g_{N_2,L_2})\|_{L^2} \lesssim N_1^{0-} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2}\, L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2}. \tag{4.6.3}$$

We also note that by duality, it suffices to prove:

$$\Big|\int (f_{N_1,L_1} * g_{N_2,L_2}) \cdot h_{N,L}\Big| \lesssim N_1^{-1-} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2}\; L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2}\; L^{\frac{1}{2}-} \|h_{N,L}\|_{L^2}, \qquad (4.6.4)$$

where $h_{N,L}$ is supported in $D_{N,L}$.
Let $L_{\max} = \max(L_1, L_2, L)$. We consider two cases:

- $L_{\max} \leqslant N_2 N_1^{\alpha}$: In case $N_2 = 1$, we make an additional dyadic decomposition in the low frequencies. Now, by abusing the notation, we let $N_2 \in 2^{\mathbb{Z}}$ denote the dyadic frequency. Depending on the size of N_1 and N_2, we consider further subcases:
  $\star$ $N_2 \gtrsim N_1^{(3-\frac{3\alpha}{2})+}$: Using the nonlinear Loomis-Whitney inequality (4.5.4), the left-hand side of (4.6.4) can be bounded by

$$\begin{aligned} &N_1^{-\frac{3}{4}\alpha+\frac{1}{2}} N_2^{-\frac{1}{2}} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2}\; L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2} L^{\frac{1}{2}} \|h_{N,L}\|_{L^2} \\ \lesssim\; &N_1^{-1-} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2}\; L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2} L^{\frac{1}{2}-} \|h_{N,L}\|_{L^2}. \end{aligned}$$

  $\star$ $N_2 \lesssim N_1^{(3-\frac{3\alpha}{2})+}$: Using the bilinear Strichartz estimate (4.5.8), we have

$$\begin{aligned} \text{LHS of (4.6.3)} &\lesssim N N^{(\alpha-\frac{5}{2})-} \frac{N_2^{\frac{1}{2}}}{N_1^{\frac{\alpha}{4}}} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2} \\ &\lesssim N_1^{0-} N_2^{0+} \|f_{N_1}\|_{F_{N_1}} \|g_{N_2}\|_{F_{N_2}}. \end{aligned}$$

  The additional factor N_2^{0+} is used to carry out summation in N_2.

- $L_{\max} \geqslant N_1^{\alpha} N_2$: In case the size of the low frequency satisfies $N_2 \gtrsim 1$, for $L_{\max} = L_2$, using Parseval's identity and Hölder's inequality, we have

$$\begin{aligned} \text{LHS of (4.6.4)} &\lesssim \|g_{N_2,L_2}\|_{L^2} \|\mathcal{F}^{-1}(f_{N_1,L_1})\|_{L^4} \|\mathcal{F}^{-1}(h_{N,L})\|_{L^4} \\ &\lesssim N_1^{-\frac{\alpha}{2}+} N_2^{-\frac{1}{2}} N_1^{\frac{1}{4}-\frac{\alpha}{8}} N^{\frac{1}{4}-\frac{\alpha}{8}} L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} L^{\frac{1}{2}-} \|h_{N,L}\|_{L^2} \\ &\lesssim N_1^{\frac{1}{2}-\frac{3\alpha}{4}+} N_2^{-\frac{1}{2}} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2} L^{\frac{1}{2}-} \|h_{N,L}\|_{L^2}, \end{aligned}$$

  which proves (4.6.4) since $\alpha > 2$. Now, we let $L_{\max} = L$, i.e. $L \geqslant N_1^{\alpha} N_2$. We use (4.4.4) as follows:

$$\begin{aligned} \text{LHS of (4.6.3)} &\lesssim N N_1^{-\frac{\alpha}{2}} N_2^{-\frac{1}{2}} \|\mathcal{F}^{-1}(f_{N_1,L_1})\|_{L^4} \|\mathcal{F}^{-1}(g_{N_2,L_2})\|_{L^4} \\ &\lesssim N N_1^{\frac{2-\alpha}{8}} N_2^{\frac{2-\alpha}{8}} N_1^{-\frac{\alpha}{2}} N_2^{-\frac{1}{2}} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2} \\ &= N_1^{\frac{5}{4}-\frac{5\alpha}{8}} N_2^{-\frac{1}{4}-\frac{\alpha}{8}} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2}, \end{aligned}$$

  which is sufficient since $\alpha > 2$ and $N_2 \gtrsim 1$. The case $L_{\max} = L_1$ leads to the same estimate. For $N_2 \lesssim 1$, without loss of generality, we assume $L_{\max} = L$. Moreover, we make a decomposition of N_2 such that $N_2 \in 2^{\mathbb{Z}}$ and accordingly, g_{N_2,L_2} is supported in $\tilde{D}_{N_2,L_2}$. We use the estimate (4.5.10):

$$\begin{aligned}
&\Big|\int (f_{N_1,L_1} * g_{N_2,L_2}) \cdot h_{N,L}\Big| \\
&\lesssim N_2^{\frac{1}{4}} N_1^{-\frac{\alpha}{2}} (L_1 L_2)^{\frac{1}{2}} L^{\frac{1}{4}} \|f_{N_1,L_2}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2} \\
&\lesssim N_2^{\frac{1}{4}} N_1^{-\frac{\alpha}{2}} N_2^{-\frac{1}{4}+} N_1^{-\frac{\alpha}{4}+} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2}\ L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2}\ L^{\frac{1}{2}-} \|h_{N,L}\|_{L^2} \\
&\lesssim N_1^{-\frac{3\alpha}{4}+} N_2^{0+} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2}\ L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2}\ L^{\frac{1}{2}-} \|h_{N,L}\|_{L^2},
\end{aligned}$$

which is sufficient to obtain (4.6.2) after summing up.

□

Next, we consider the *High* × *High* → *Low* interaction.

Lemma 4.6.4 (*High* × *High* → *Low*)**.** *Let $N_1, N_2, N \in 2^{\mathbb{N}_0}$ be such that $N_1 \geqslant 2^{10}$, $N \ll N_1 \sim N_2$, and $u_{N_1} \in F_{N_1}, v_{N_2} \in F_{N_2}$. Then, we have*

$$\|P_N(\partial_x(u_{N_1} v_{N_2}))\|_{\mathcal{N}_N} \lesssim N_1^{0-} \|u_{N_1}\|_{F_{N_1}} \|v_{N_2}\|_{F_{N_2}}. \tag{4.6.5}$$

Proof. Let $\gamma : \mathbb{R} \to [0,1]$ be a smooth function supported in $[-1,1]$ such that

$$\sum_{n\in\mathbb{Z}} \gamma^2(t-n) \equiv 1, \quad t \in \mathbb{R}.$$

We need to further localise the nonlinearity to intervals of size $N_1^{(2\alpha-5)-}$. Moreover, we carry out an additional decomposition in case $N = 1$ into very low frequencies. By abuse of notation, now let $N \in 2^{\mathbb{Z}}$ and $N_+ = \max(N, 1)$. Using the definition of the $\mathcal{N}_N$ norm, the left-hand side of (4.6.5) is dominated by

$$\begin{aligned}
&\sup_{t_N\in\mathbb{R}} \big\|(\tau - \omega(\xi,\eta) + iN_+^{(5-2\alpha)+})^{-1} N \mathbf{1}_{\tilde{A}_N}(\xi,\eta) \\
&\sum_{|m|\lesssim (\frac{N_1}{N_+})^{(5-2\alpha)+}} \mathcal{F}[u_{N_1} \cdot \eta_0(N_+^{(5-2\alpha)+}(t-t_N))\gamma(N_1^{(5-2\alpha)+}(t-t_N)-m)] \\
&* \mathcal{F}[v_{N_2} \cdot \eta_0(N_+^{(5-2\alpha)+}(t-t_N))\gamma(N_1^{(5-2\alpha)+}(t-t_N)-m)]\big\|_{X_N}.
\end{aligned} \tag{4.6.6}$$

Hence, it suffices to prove that if $f_{N_1,L_1}, g_{N_2,L_2} : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}_+$ are functions supported in D_{N_1,L_1} and D_{N_2,L_2}, respectively, then

$$N\Big(\frac{N_1}{N_+}\Big)^{(5-2\alpha)+} \sum_{L\geqslant 1} L^{-\frac{1}{2}} \|\mathbf{1}_{D_{N,L}}(f_{N_1,L_1} * g_{N_2,L_2})\|_{L^2} \lesssim N_1^{0-} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2}\ L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2}. \tag{4.6.7}$$

By duality, it suffices to show

$$\begin{aligned}
\Big|\int (f_{N_1,L_1} * g_{N_2,L_2}) \cdot h_{N,L}\Big| &\lesssim N^{-1} N_1^{0-} \Big(\frac{N_+}{N_1}\Big)^{(5-2\alpha)+} N_1^{0-}\ (L_1 L_2 L^{1-})^{\frac{1}{2}} \\
&\qquad \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2},
\end{aligned} \tag{4.6.8}$$

where $h_{N,L}$ is supported in $\tilde{D}_{N,L}$. For $L_{\max} = \max(L, L_1, L_2)$, we consider two cases:

- $\underline{L_{\max} \leqslant N_1^{\alpha} N}$: We consider two subcases:
$\star$ $N \lesssim N_1^{3-\frac{3\alpha}{2}}$: Note that since $\alpha > 2$, we have $N \lesssim 1$. We assume $L_{\max} = L_2$. We use the bilinear Strichartz estimate (4.5.8) as follows:

$$\begin{aligned}\Big| \int (f_{N_1,L_1} * g_{N_2,L_2}) \cdot h_{N,L} \Big| &\lesssim \|f_{N_1,L_1} * h_{N,L}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \\ &\lesssim \frac{N^{\frac{1}{2}}}{N_1^{\frac{\alpha}{4}}} (LL_1)^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|h_{N,L}\|_{L^2} \|g_{N_2,L_2}\|_{L_2} \\ &\lesssim N^{\frac{1}{2}} N_1^{\frac{3\alpha}{4} - \frac{5}{2}+} (L_1 L_2 L^{1-})^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2},\end{aligned}$$

which is sufficient since $\alpha > 2$. The other cases, namely $L_{\max} = L_1$ or $L_{\max} = L$ can be treated with the application of the estimate (4.5.8) for a high-low interaction, leading to the same estimate.
$\star$ $N \gtrsim N_1^{3-\frac{3\alpha}{2}}$: For $N \gtrsim 1$, the coefficient on the right-hand side of (4.6.8) becomes $N^{(4-2\alpha)+} N_1^{(-5+2\alpha)-}$. We use (4.5.4) to bound the left-hand side of (4.6.8) by

$$\begin{aligned}&N_1^{(-\frac{3\alpha}{4}+\frac{1}{2})} N^{-\frac{1}{2}} (L_1 L_2 L)^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L_2} \\ \lesssim\ & N^{(4-2\alpha)+} N_1^{(-5+2\alpha)-} \Big(\frac{N_1}{N}\Big)^{(4-2\alpha)+} (L_1 L_2 L^{1-})^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L_2}.\end{aligned}$$

The above is sufficient for $\alpha > 2$.

Remark 4.6.5. Note that in case $N \gtrsim N_1^{3-\frac{3\alpha}{2}}$, we can still have $N \lesssim 1$. However, we find that the above estimate is sufficient.

- $\underline{L_{\max} \geqslant N_1^{\alpha} N}$: First, we consider the case $N \gtrsim 1$.
$\underline{\text{If } L \geqslant N_1^{\alpha} N}$, we can apply two linear Strichartz estimates to estimate the left-hand side of (4.6.7) as follows:

$$\begin{aligned}&\lesssim N_1^{(5-2\alpha)+} N^{(-4+2\alpha)-} (N_1^{\alpha} N)^{-\frac{1}{2}} \|\mathcal{F}^{-1}(f_{N_1,L_1})\|_{L^4} \|\mathcal{F}^{-1}(g_{N_2,L_2})\|_{L^4} \\ &\lesssim N_1^{(5-2\alpha)+} N^{(-4+2\alpha)-} N^{-\frac{1}{2}} N_1^{-\frac{\alpha}{2}} N_1^{\frac{2-\alpha}{4}} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2}\ L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2} \\ &\lesssim N^{(-\frac{9}{2}+2\alpha)-} N_1^{(\frac{11}{2}-\frac{11\alpha}{4})+} (L_1 L_2)^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2}.\end{aligned}$$

For $L \leqslant N_1^{\alpha} N$, we suppose that $L_{\max} = L_2$ and bound the left-hand side of (4.6.8) by

$$\begin{aligned}&\|\mathcal{F}^{-1}(f_{N_1,L_1})\|_{L^4} \|\mathcal{F}^{-1}(h_{N,L})\|_{L^4} \|g_{N_2,L_2}\|_{L^2} \\ \lesssim\ & N^{-\frac{1}{2}} N_1^{-\frac{\alpha}{2}} N^{\frac{2-\alpha}{8}} N_1^{\frac{2-\alpha}{8}} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2}\ L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2}\ L^{\frac{1}{2}} \|h_{N,L}\|_{L^2} \\ \lesssim\ & N^{-\frac{17}{4}+\frac{15\alpha}{8}+} N_1^{\frac{21}{4}-\frac{21\alpha}{8}+} N_1^{-5+2\alpha+} N^{4-2\alpha+} (L_1 L_2 L^{-1})^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2} \\ \lesssim\ & N^{1-\frac{3\alpha}{4}} N_1^{(-5+2\alpha)+} N^{(4-2\alpha)+} (L_1 L_2 L^{-1})^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2}.\end{aligned}$$

The above implies the required estimate. The case $L_{\max} = L_1$ is similar.
In case $N \lesssim 1$, we use the estimate (4.5.10) to obtain

$$\begin{aligned}\Big| \int (f_{N_1,L_1} * g_{N_2,L_2} \cdot h_{N,L}) \Big| &\lesssim N_1^{-\frac{\alpha}{2}} N^{\frac{1}{4}} L_{\max}^{-\frac{1}{4}} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2}\ L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2}\ L^{\frac{1}{2}} \|h_{N,L}\|_{L^2} \\ &\lesssim N_1^{-\frac{3\alpha}{4}+} N^{0+} L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2}\ L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2}\ L^{\frac{1}{2}-} \|h_{N,L}\|_{L^2},\end{aligned}$$

which is sufficient to prove the required estimate.

□

The case when the three frequencies are comparable is treated in the following lemma.

Lemma 4.6.6 (*High × High → High*)**.** *Let $N_1, N_2, N \in 2^{\mathbb{N}}$ be such that $N_1 \sim N_2 \sim N \gg 1$. Let $u_{N_1} \in F_{N_1}$, $v_{N_2} \in F_{N_2}$. Then, the following estimate holds:*

$$\|P_N(\partial_x(u_{N_1}v_{N_2}))\|_{\mathcal{N}_N} \lesssim N_1^{(1-\frac{3\alpha}{4})+}\|u_{N_1}\|_{F_{N_1}}\|v_{N_2}\|_{F_{N_2}}. \tag{4.6.9}$$

Proof. Using the same reductions as in the previous lemmata, it is sufficient to prove

$$N\sum_{L\geqslant N^{(5-2\alpha)+}} L^{-\frac{1}{2}}\|\mathbf{1}_{D_{N,L}}(f_{N_1,L_1}*g_{N_2,L_2})\|_{L^2} \lesssim N_1^{(1-\frac{3\alpha}{4})+}L_1^{\frac{1}{2}}\|f_{N_1,L_1}\|_{L^2}\ L_2^{\frac{1}{2}}\|g_{N_2,L_2}\|_{L^2}, \tag{4.6.10}$$

for functions f_{N_1,L_1}, g_{N_2,L_2} supported in D_{N_1,L_1} and D_{N_2,L_2}, respectively. Moreover, by duality, it is sufficient to prove

$$\Big|\int(f_{N_1,L_1}*g_{N_2,L_2})\cdot h_{N,L}\Big| \lesssim N_1^{-\frac{3\alpha}{4}+}(L_1L_2L^{1-})^{\frac{1}{2}}\|f_{N_1,L_1}\|_{L^2}\|g_{N_2,L_2}\|_{L^2}\|h_{N,L}\|_{L^2},$$

for $h_{N,L}$ supported in $D_{N,L}$.
For $L_{\max}$ as before, we consider two cases:

- $\underline{L_{\max} \leqslant N_1^{\alpha+1}}$: Using (4.5.4) directly, we obtain

$$\Big|\int(f_{N_1,L_1}*g_{N_2,L_2})\cdot h_{N,L}\Big| \lesssim N_1^{-\frac{3\alpha}{4}+}(L_1L_2L^{1-})^{\frac{1}{2}}\|f_{N_1,L_1}\|_{L^2}\|g_{N_2,L_2}\|_{L^2}\|h_{N,L}\|_{L^2}.$$

- $\underline{L_{\max} \geqslant N_1^{\alpha+1}}$: For $L \geqslant N_1^{\alpha+1}$, using the L^4 Strichartz estimate and the size of L, we find

$$\begin{aligned}\text{LHS of (4.6.10)} &\lesssim NN^{-\frac{\alpha+1}{2}}N_1^{\frac{2-\alpha}{8}}N_2^{\frac{2-\alpha}{8}}L_1^{\frac{1}{2}}\|f_{N_1,L_1}\|_{L^2}\ L_2^{\frac{1}{2}}\|g_{N_2,L_2}\|_{L^2}\\ &\lesssim N_1^{1-\frac{3\alpha}{4}}L_1^{\frac{1}{2}}\|f_{N_1,L_1}\|_{L^2}\ L_2^{\frac{1}{2}}\|g_{N_2,L_2}\|_{L^2}.\end{aligned}$$

  For $N_1^{(5-2\alpha)+} \leqslant L \leqslant N_1^{\alpha+1}$, we find the above estimate up to N_1^{0+} by two L^4 Strichartz estimates involving the dual function. This concludes the proof of the lemma.

□

Finally, we consider the very low frequency case.

Lemma 4.6.7 (Very low frequencies)**.** *Let $N_1, N_2, N \in 2^{\mathbb{N}_0}$ be such that $N_1, N_2, N \lesssim 1$. Let $u_{N_1} \in F_{N_1}$ and $v_{N_2} \in F_{N_2}$. Then, the following estimate holds:*

$$\|P_N(\partial_x(u_{N_1}v_{N_2}))\|_{\mathcal{N}_N} \lesssim \|u_{N_1}\|_{F_{N_1}}\|v_{N_2}\|_{F_{N_2}}. \tag{4.6.11}$$

Proof. Using the definitions of the function spaces, it is sufficient to prove the following:

$$N\sum_{L\geqslant 1} L^{-\frac{1}{2}}\|\mathbf{1}_{\tilde{D}_{N,L}}(f_{N_1,L_1}*g_{N_2,L_2})\|_{L^2} \lesssim L_1^{\frac{1}{2}}\|u_{N_1}\|_{L^2}\ L_2^{\frac{1}{2}}\|v_{N_2}\|_{L^2}, \tag{4.6.12}$$

for $L_1, L_2 \geqslant 1$ and $f_{N_1,L_1}, g_{N_2,L_2} : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}_+$, supported in $\tilde{D}_{N_1,L_1}, \tilde{D}_{N_2,L_2}$, respectively. Using (4.4.4), we have

$$\text{LHS of (4.6.12)} \lesssim N L_1^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2}\ L_2^{\frac{1}{2}} \|g_{N_2,L_2}\|_{L^2},$$

which is sufficient to conclude the proof. □

Proof of Proposition 4.6.2. We decompose the nonlinearity $\partial_x(uv)$ as follows:

$$\partial_x(uv) = \Big(\sum_{N_1 \ll N_2 \sim N} + \sum_{N_2 \ll N_1 \sim N} + \sum_{N \ll N_1 \sim N_2} + \sum_{N_1 \sim N_2 \sim N \gg 1} + \sum_{N, N_1, N_2 \lesssim 1} \Big) \times P_N \partial_x (P_{N_1} u \cdot P_{N_2} v).$$

Of the first two summands above, it is sufficient to consider only the first one if we assume that the derivative hits the high frequency. Each of the terms can then be separately handled by Lemma 4.6.3, Lemma 4.6.4, Lemma 4.6.6, and Lemma 4.6.7, respectively. We multiply each of the estimates in the lemmata by $N^{2s'}$ and sum up dyadically over the spatial (x) frequencies using Cauchy-Schwarz inequality on the right-hand side. This concludes the proof. □

Remark 4.6.8. As a particular case of Proposition 4.6.2, we obtain

$$\|\partial_x(uv)\|_{\mathcal{N}^{0,0}(T)} \lesssim \|u\|_{F^{0,0}(T)} \|v\|_{F^{0,0}(T)}. \tag{4.6.13}$$

4.6.2 Energy estimates

We prove the energy estimates for the solution and the difference of the solutions in this section. The former is crucial to conclude an a priori estimate for the solution while the latter is required to prove the continuity of the data-to-solution map.

Energy estimate for the solution

Consider the following equation:

$$\begin{cases} \partial_t u - D_x^{\alpha} \partial_x u - \partial_x^{-1} \partial_y^2 u & = u \partial_x u, \quad (t,x,y) \in (-T,T) \times \mathbb{R} \times \mathbb{R}, \\ u(0) & = u_0 \in H^{s_1,s_2}(\mathbb{R}^2), \end{cases} \tag{4.6.14}$$

for Littlewood-Paley pieces $P_N u$. Multiplying this equation with $P_N u$ and integrating, we obtain

$$\int_{[0,t_N]\times\mathbb{R}^2} \frac{\partial_t (P_N u)^2}{2}\, dtdxdy = \int_{[0,t_N]\times\mathbb{R}^2} P_N u \cdot P_N (u \partial_x u)\, dtdxdy.$$

Using the fundamental theorem of calculus, the above yields

$$\sup_{t_N \in [-T,T]} \|P_N u(t_N)\|_{L^2}^2 \leqslant \|P_N u_0\|_{L^2}^2 + \sup_{t_N \in [-T,T]} \Big| \int_{[0,t_N]\times\mathbb{R}^2} P_N u \cdot P_N (u \partial_x u)\, dtdxdy \Big|. \tag{4.6.15}$$

We need to control the last term in the above display. We consider

$$P_N u P_N(\partial_x u^2) = 2P_N u P_N(u\partial_x u) = 2P_N u P_N(P_{\gtrsim N} u \cdot \partial_x u) + 2P_N u P_N(P_{\ll N} u \cdot \partial_x u)$$
$$=: 2(I + II).$$

We can further decompose I as

$$I = P_N u P_N(P_{\gg N} u \cdot P_{\gg N}\partial_x u) + P_N u P_N(P_{\gtrsim N} u \cdot P_{\lesssim N}\partial_x u) =: a + b,$$

while II can be written as

$$II = P_N u P_{\ll N} u P_N \partial_x u + P_N u[P_N(P_{\ll N} u \cdot \partial_x u) - P_{\ll N} u \cdot P_N \partial_x u] =: c + d.$$

We have

$$2a + 2c = P_N u \cdot \partial_x (P_{\gg N} u)^2 + \partial_x (P_N u)^2 \cdot P_{\ll N} u.$$

For b, we observe that the derivative already hits the low frequency term, while for $a + c$, using integration by parts, we have

$$\int_{[0,t_N]\times\mathbb{R}^2} 2(a+c)\; dxdydt = \sum_{N \ll N_1} \int_{[0,t_N]\times\mathbb{R}^2} \partial_x P_N u \cdot P_{N_1} u \cdot \tilde{P}_{N_1} u\; dtdxdy$$
$$+ \sum_{N_1 \ll N} \int_{[0,t_N]\times\mathbb{R}^2} \partial_x P_{N_1} u \cdot P_N u \cdot \tilde{P}_N u\; dtdxdy$$

where $\tilde{P}_{N_1} = \sum_{N' \sim N_1} P_{N'}$ (and the multiplier is denoted by $\tilde{\phi}_{N_1}$).
Next, we treat d by the same argument as in [34, eq. 6.10] to transfer the derivative to the low frequency factor. We have

$$d = P_N u \sum_{N_1 \ll N} [P_N(P_{N_1} u \cdot \partial_x u) - P_{N_1} u \cdot P_N \partial_x u].$$

We fix an extension of u which we still denote by u. We have

$$\begin{aligned}
&\mathcal{F}[P_N(P_{N_1} u \partial_x u) - P_{N_1} u P_N \partial_x u](\tau, \xi, \eta) \\
&= \int \phi_N(\xi)(\xi - \xi_1)\hat{u}(\tau - \tau_1, \xi - \xi_1, \eta - \eta_1)\phi_{N_1}(\xi_1)\hat{u}(\tau_1, \xi_1, \eta_1) \\
&\quad - \phi_{N_1}(\xi_1)\hat{u}(\tau_1, \xi_1, \eta_1)(\xi - \xi_1)\phi_N(\xi - \xi_1)\hat{u}(\tau - \tau_1, \xi - \xi_1, \eta - \eta_1)\; d\tau_1 d\xi_1 d\eta_1 \\
&= \int \frac{(\xi - \xi_1)}{\xi_1}(\tilde{\phi}_N(\xi) - \tilde{\phi}_N(\xi - \xi_1))\widehat{P_N u}(\tau - \tau_1, \xi - \xi_1, \eta - \eta_1) \\
&\quad \times \widehat{P_{N_1}\partial_x u}(\tau_1, \xi_1, \eta_1)\; d\tau_1 d\xi_1 d\eta_1 \\
&=: M(P_N u, P_{N_1}\partial_x u),
\end{aligned}$$

where the symbol of the (bilinear) multiplier M is

$$m(\xi, \xi_1) = \frac{(\xi - \xi_1)}{\xi_1}(\tilde{\phi}_N(\xi) - \tilde{\phi}_N(\xi - \xi_1))\tilde{\phi}_{N_1}(\xi_1).$$

Using the mean value theorem,

$$(\tilde{\phi}_N(\xi) - \tilde{\phi}_N(\xi - \xi_1)) = -\int_0^1 \tilde{\phi}_N'(\xi - h\xi_1)\xi_1 \, dh,$$

we obtain the uniform boundedness of the multiplier

$$|m(\xi, \xi_1)| \lesssim \int_0^1 \Big| \frac{\xi - \xi_1}{N} \tilde{\phi}' \Big(\frac{\xi - h\xi_1}{N} \Big) \Big| \, dh \lesssim 1.$$

To conclude, we have shown that by taking the advantage of the form of the nonlinearity, we can transfer the derivative to the low frequency in all the cases. More precisely, we can assume that our integrand is of the form

$$P_N u \cdot P_{N_2} u \cdot (P_{N_1} \partial_x u) \text{ or } P_N u \cdot M(P_N u, P_{N_1} \partial_x u) \quad \text{with } N_1 \lesssim N \sim N_2, \tag{4.6.16}$$

where M is a bilinear Fourier multiplier M with bounded symbol m.

Now we localise the functions in time to intervals of size $N_{\max}^{(2\alpha-5)-}$. To begin, we assume that $T \in (0, 1]$, $N_1, N_2, N_3 \in 2^{\mathbb{Z}}$ with $\max(N_1, N_2, N_3) \geqslant 1$, $u_i, \in F_{N_i}(T)$, $i = 1, 2, 3$. Without any loss of generality, we assume that $N_1 \leqslant N_2 \leqslant N_3$. Let $\gamma : \mathbb{R} \to [0, 1]$ denote a smooth function supported in $[-1, 1]$ with the property that

$$\sum_{n \in \mathbb{Z}} \gamma^3(t - n) \equiv 1, \quad t \in \mathbb{R}.$$

We fix extensions $\tilde{u}_i$ of u_i, $i = 1, 2, 3$ such that $\|\tilde{u}_i\|_{F_{N_i}} \leqslant 2\|u_i\|_{F_{N_i}(T)}$. Then, we use the function γ to divide the time interval to sub-intervals of size $\sim N_3^{(2\alpha-5)-}$

$$\begin{aligned}
&\Big| \int_{[0,T] \times \mathbb{R}^2} u_1 u_2 u_3 \, dt dx dy \Big| \\
\lesssim &\sum_{|n| \leqslant C N_3^{(5-2\alpha)+}} \Big| \int_{\mathbb{R} \times \mathbb{R}^2} (\gamma(N_3^{(5-2\alpha)+} t - n) \mathbf{1}_{[0,T]}(t) \tilde{u}_1)(\gamma(N_3^{(5-2\alpha)+} t - n) \mathbf{1}_{[0,T]}(t) \tilde{u}_2) \\
&\qquad \times (\gamma(N_3^{(5-2\alpha)+} t - n) \mathbf{1}_{[0,T]}(t) \tilde{u}_3) \, dt dx dy \Big| \\
= &\sum_{|n| \leqslant C N_3^{(5-2\alpha)+}} \Big| \int_{\mathbb{R} \times \mathbb{R}^2} \mathcal{F}((\gamma(N_3^{(5-2\alpha)+} t - n) \mathbf{1}_{[0,T]}(t) \tilde{u_1})) \\
&\quad * \mathcal{F}((\gamma(N_3^{(5-2\alpha)+} t - n) \mathbf{1}_{[0,T]}(t) \tilde{u_2})) \mathcal{F}((\gamma(N_3^{(5-2\alpha)+} t - n) \mathbf{1}_{[0,T]}(t) \tilde{u_3})) \, d\tau d\xi d\eta \Big| \\
=: &\sum_{|n| \leqslant C N_3^{(5-2\alpha)+}} \Big| \int_{\mathbb{R} \times \mathbb{R}^2} (f_1 * f_2) \cdot f_3 \, d\tau d\xi d\eta \Big|,
\end{aligned} \tag{4.6.17}$$

i.e.

$$f_i := \mathcal{F}((\gamma(N_3^{(5-2\alpha)+} t - n) \mathbf{1}_{[0,T]}(t) \tilde{u_i})), \quad i = 1, 2, 3. \tag{4.6.18}$$

In the above summation over $n \in \mathbb{Z}$, we consider two sets as follows:

$$\begin{aligned} A &= \{n \in \mathbb{Z}, |n| \leqslant C N_3^{(5-2\alpha)+} : \gamma(N_3^{(5-2\alpha)+}t - n)\mathbf{1}_{[0,T]}(t) = \gamma(N_3^{(5-2\alpha)+}t - n)\}, \\ A^c &= \{n \in \mathbb{Z}, |n| \leqslant C N_3^{(5-2\alpha)+} : 0 \in \operatorname{supp}(\gamma(N_3^{(5-2\alpha)+} \cdot - n)) \\ &\qquad \vee T \in \operatorname{supp}(\gamma(N_3^{(5-2\alpha)+} \cdot - n))\}. \end{aligned}$$

Since $T \in (0,1]$ and γ is supported in $[-1,1]$, we have that $|A| \lesssim N_3^{(5-2\alpha)+}$ while $|A^c| \leqslant 4$. On the physical side, the temporal support of f_i, $i = 1,2,3$ is of size $\sim N_3^{(2\alpha-5)-}$. We can further decompose

$$f_i = \sum_{L_i \geqslant N_3^{(5-2\alpha)+}} f_{i,L_i}. \tag{4.6.19}$$

In the following computations, we shall assume that we have already made the above reduction. For $n \in A^c$, we use the following estimate to substitute for (4.3.1) (cf. [34, p. 291]),

$$\sup_{L \geqslant 1} L^{\frac{1}{2}} \|\eta_L(\tau - \omega(\xi,\eta)) \cdot f_N^I\|_{L^2} \lesssim \|f_N\|_{X_N},$$

where $f_N^I = \mathcal{F}(\mathbf{1}_I(t) f_N \cdot \mathcal{F}^{-1}(f_N))$ for an interval $I \subset \mathbb{R}$ (in our case I is an interval of length $\min(1, N_{\max}^{(2\alpha-5)-})$.

To handle the summation in A^c, we consider two cases depending on the size of L:

(i) $L \leqslant N_{\max}^\alpha N_{\min}$: In this case, we have

$$\sum_{1 \leqslant L \leqslant N_{\max}^\alpha N_{\min}} L^{\frac{1}{2}} \lesssim \log(N_{\max}^\alpha N_{\min}) \sup L^{\frac{1}{2}},$$

(ii) $L \geqslant N_{\max}^\alpha N_{\min}$: In this case, we can always spare a small negative exponent of L and have

$$\sum_{L \geqslant N_{\max}^\alpha N_{\min}} L^{-a} < \infty \text{ for any } a > 0.$$

We shall focus on $n \in A$ in the following.

Proposition 4.6.9. *Let $2 < \alpha \leqslant \frac{5}{2}$ and $u \in C([-T,T]; H^{\infty,0})$ be a solution to (4.1.1) on $(-T,T) \times \mathbb{R}^2$. Then, for all $T \in (0,1]$ and $s' \geqslant s > 5 - 2\alpha$, the following estimate holds:*

$$\|u\|_{E^{s',0}(T)}^2 \lesssim \|u_0\|_{H^{s',0}}^2 + \|u\|_{F^{s,0}(T)} \|u\|_{F^{s',0}(T)}^2.$$

Proof. We recall

$$\sup_{t_N \in [-T,T]} \|P_N u(t_N)\|_{L^2}^2 \leqslant \|P_N u_0\|_{L^2}^2 + \sup_{t_N \in [-T,T]} \Big| \int_{[0,t_N] \times \mathbb{R}^2} P_N u P_N(u \partial_x u) \, dt dx dy \Big|. \tag{4.6.20}$$

From the reductions performed in the beginning of subsection 4.6.2, we have

$$\begin{aligned} \Big| \int_0^{t_N} P_N u \cdot P_N(u\partial_x u) \, dtdxdy \Big| &\sim \Big| \int_0^{t_N} (P_N u \cdot P_{N_2} u \cdot P_{N_1} \partial_x u) \, dtdxdy \Big| \\ &\lesssim \sum_{|n| \leqslant cN^{(5-2\alpha)+}} \Big| \int_{\mathbb{R}\times\mathbb{R}^2} (\tilde{f}_1 * f_2) \cdot f_3 \, d\tau d\xi d\eta \Big|, \end{aligned} \tag{4.6.21}$$

where $N_1 \lesssim N \sim N_2$ and

$$\begin{aligned} \tilde{f}_1 &= \mathcal{F}(\gamma(N^{(5-2\alpha)+}t-n)\mathbf{1}_{[0,T]}(t)\partial_x P_{N_1}u), \\ f_2 &= \mathcal{F}(\gamma(N^{(5-2\alpha)+}t-n)\mathbf{1}_{[0,T]}(t)P_{N_2}u), \\ f_3 &= \mathcal{F}(\gamma(N^{(5-2\alpha)+}t-n)\mathbf{1}_{[0,T]}(t)P_{N}u). \end{aligned}$$

We remark that at first, we have $N_1 \in 2^{\mathbb{N}_0}$ to take into account the definition of the function spaces. For $N_1 = 1$, we carry out an additional dyadic decomposition into very low frequencies $N_1 \in 2^{\mathbb{Z}}$ to take advantage of the derivative, which is smoothing for $N_1 \ll 1$. Note that in the estimates proved below, we always have summability for $N_1 \lesssim 1$. Taking into account the derivative in $\tilde{f}_1$ and summing up in n, we obtain

$$\sum_{|n|\leqslant cN^{(5-2\alpha)+}} \Big| \int_{\mathbb{R}\times\mathbb{R}^2} (\tilde{f}_1 * f_2)\cdot f_3 \, d\tau d\xi d\eta \Big| \lesssim N_1 N^{(5-2\alpha)+} \Big| \int_{\mathbb{R}\times\mathbb{R}^2} (f_1 * f_2)\cdot f_3 \, d\tau d\xi d\eta \Big|, \tag{4.6.22}$$

where

$$f_1 := \mathcal{F}(\gamma(N^{(5-2\alpha)+}t-n)\mathbf{1}_{[0,T]}(t)P_{N_1}u).$$

With $L_{\max} = \max(L_1, L_2, L_3)$, we consider the following:

- $\underline{L_{\max} \leqslant N^\alpha N_1}$: For f_{i,L_i} as defined in (4.6.19) and using (4.5.4), we have

$$\begin{aligned} &N_1 N^{(5-2\alpha)+} \Big| \int_{\mathbb{R}\times\mathbb{R}^2} (f_1 * f_2)\cdot f_3 \, d\tau d\xi d\eta \Big| \\ &\lesssim N_1 N^{(5-2\alpha)+} \sum_{N^{(5-2\alpha)+} \leqslant L_i \leqslant N^\alpha N_1} \Big| \int_{\mathbb{R}\times\mathbb{R}^2} (f_{1,L_1} * f_{2,L_2})\cdot f_{3,L_3} \, d\tau d\xi d\eta \Big| \\ &\lesssim N_1 N^{(5-2\alpha)+} N^{-\frac{3\alpha}{4}+\frac{1}{2}} N_1^{-\frac{1}{2}} \prod_{i=1}^{3} \sum_{L_i \geqslant N^{(5-2\alpha)+}} L_i^{\frac{1}{2}} \|f_{i,L_i}\|_{L^2} \\ &\lesssim N_1^{\frac{1}{2}} N^{\frac{11}{2}-\frac{11\alpha}{4}+} \|u_{N_1}\|_{F_{N_1}(T)} \|u_{N_2}\|_{F_{N_2}(T)} \|u_N\|_{F_N(T)}. \end{aligned} \tag{4.6.23}$$

Remark 4.6.10. We note the bilinear Strichartz estimate (4.5.8) gives the same estimate if $N_1 \lesssim N^{3-\frac{3\alpha}{2}}$.

- $\underline{L_{\max} \geqslant N^\alpha N_1}$: If $L_{\max} = L_1$, using (4.4.1), we obtain

$$\begin{aligned} &N_1 N^{(5-2\alpha)+} \Big| \int_{\mathbb{R}\times\mathbb{R}^2} (f_1 * f_2)\cdot f_3 \, d\tau d\xi d\eta \Big| \\ &\lesssim N_1 N^{(5-2\alpha)+} \sum_{\substack{L_2, L_3 \geqslant N^{(5-2\alpha)+} \\ L_1 \geqslant N^\alpha N_1}} \int_{\mathbb{R}\times\mathbb{R}^2} \big|(f_{1,L_1} * f_{2,L_2})\cdot f_{3,L_3}\big| \, d\tau d\xi d\eta \\ &\lesssim N_1 N^{(5-2\alpha)+} N^{\frac{1}{4}-\frac{\alpha}{8}} N_2^{\frac{1}{4}-\frac{\alpha}{8}} N^{-\frac{\alpha}{2}} N_1^{-\frac{1}{2}} \prod_{i=1}^{3} \sum_{L_i \geqslant N^{(5-2\alpha)+}} L_i^{\frac{1}{2}} \|f_{i,L_i}\|_{L^2} \\ &\lesssim N_1^{\frac{1}{2}} N^{\frac{11}{2}-\frac{11\alpha}{4}+} \|u_N\|_{F_N(T)} \|u_{N_1}\|_{F_{N_1}(T)} \|u_{N_2}\|_{F_{N_2}(T)}. \end{aligned}$$

For $L_{\max} = L_2$ we proceed similarly but use the linear Strichartz estimate (4.4.1) for one high and one low frequency. We have

$$\begin{aligned}
&N_1 N^{(5-2\alpha)+}\Big|\int_{\mathbb{R}\times\mathbb{R}^2}(f_1 * f_2)\cdot f_3 \; d\tau d\xi d\eta\Big| \\
&\lesssim N_1 N^{(5-2\alpha)+}\sum_{\substack{L_1,L_3\geqslant N^{(5-2\alpha)+}\\ L_2\geqslant N^{\alpha}N_1}}\int_{\mathbb{R}\times\mathbb{R}^2}\big|(f_{1,L_1} * f_{2,L_2})\cdot f_{3,L_3}\big| \; d\tau d\xi d\eta \\
&\lesssim N_1 N^{(5-2\alpha)+}N^{\frac{2-\alpha}{8}}N_1^{\frac{2-\alpha}{8}}N^{-\frac{\alpha}{2}}N_1^{-\frac{1}{2}}\prod_{i=1}^{3}\sum_{L_i\geqslant N^{(5-2\alpha)+}}L_i^{\frac{1}{2}}\|f_{i,L_i}\|_{L^2} \\
&\lesssim N_1^{\frac{3}{4}-\frac{\alpha}{8}}N^{\frac{21}{4}-\frac{21\alpha}{8}+}\|u_N\|_{F_N(T)}\|u_{N_1}\|_{F_{N_1}(T)}\|u_{N_2}\|_{F_{N_2}(T)}.
\end{aligned}$$

Substituting the above estimate(s) in (4.6.20), multiplying by $N^{2s'}$ and summing up with respect to the spatial frequencies, we obtain the desired estimate. This completes the proof.

□

Energy estimate for the difference equation

Let u_1, u_2 solve (4.6.14) with initial data ϕ_1 and ϕ_2, respectively. The difference of the solutions, i.e. $v = u_1 - u_2$ satisfies the following:

$$\begin{cases} \partial_t v - D_x^{\alpha}\partial_x v - \partial_x^{-1}\partial_y^2 v &= \partial_x(v(u_1+u_2))/2, \quad (t,x,y)\in(-T,T)\times\mathbb{R}\times\mathbb{R}, \\ v(0) &= \phi_1-\phi_2 =: \phi. \end{cases} \tag{4.6.24}$$

Proposition 4.6.11. *Let u_1, u_2 and v be as above. Then, for all $T\in(0,1]$ and $s>5-2\alpha$, the following estimates hold:*

$$\|v\|_{E^{0,0}(T)}^2 \lesssim \|\phi\|_{L^2}^2 + \|v\|_{F^{0,0}(T)}^2(\|u_1\|_{F^{s,0}(T)}+\|u_2\|_{F^{s,0}(T)}), \tag{4.6.25}$$

$$\begin{aligned}\|v\|_{E^{s,0}(T)}^2 \lesssim{}& \|\phi\|_{H^{s,0}}^2 + \|v\|_{F^{s,0}(T)}^3 \\ &+ (\|v\|_{F^{s,0}(T)}^2\|u_2\|_{F^{s,0}(T)} + \|v\|_{F^{0,0}(T)}\|v\|_{F^{s,0}(T)}\|u_2\|_{F^{2s,0}(T)}).\end{aligned} \tag{4.6.26}$$

Proof. As in the previous section, we consider (4.6.24) for Littlewood-Paley pieces $P_N v$. Multiplying the same with $P_N v$ and integrating, we obtain

$$\sup_{t_N\in[-T,T]}\|P_N v(t_N)\|_{L^2}^2 \lesssim \|P_N\phi\|_{L^2}^2 + \sup_{t_N\in[-T,T]}\Big|\int_{[0,t_N]\times\mathbb{R}^2}P_N v P_N(\partial_x(v(u_1+u_2)))dtdxdy\Big|. \tag{4.6.27}$$

We require to handle the last term in the above display. We treat the term $P_N v P_N(\partial_x(vu_1))$ since the other term, namely $P_N v P_N(\partial_x(vu_2))$ and be treated similarly. We have

$$\begin{aligned}P_N(vu_1) &= P_N(P_{\ll N}v\cdot u_1) + P_N(P_{\gtrsim N}v\cdot u_1) \\ &\sim P_N(P_{\ll N}v\cdot P_N u_1) + P_N(P_{\gtrsim N}v\cdot P_{\gtrsim N}u_1).\end{aligned} \tag{4.6.28}$$

Corresponding to the integrand in the last term of (4.6.27), we need to consider

$$P_N v\cdot\partial_x P_N(P_{\ll N}v\cdot P_N u_1) = P_N v\cdot\partial_x P_{\ll N}v\cdot P_N u_1 + P_N v\cdot P_{\ll N}v\cdot\partial_x P_N u_1 \tag{4.6.29}$$

and

$$P_N v \cdot \partial_x P_N (P_{\gtrsim N} v \cdot P_{\gtrsim N} u_1) = \sum_{N_2 \sim N_1 \gtrsim N} P_N v \cdot \partial_x P_N (P_{N_1} v \cdot P_{N_2} u_1). \tag{4.6.30}$$

The first term on the right-hand side of (4.6.29) can be estimated easily as the derivative hits the low frequency term. However, for the second term, the derivative hits the high frequency term and the resulting term is not amenable to an integration by parts argument to transfer the derivative to the low frequency term. (4.6.30) is easy to handle and we treat it as follows: we fix extensions of $P_N v$, $P_{N_1} v$ and $P_{N_2} u_1$ and still denote them by $P_N v$, $P_{N_1} v$ and $P_{N_2} u_1$ to lighten the notation. Then, using Parseval's identity and using the reductions explained before, we have

$$\begin{aligned} &\Big| \int_{[0,t_N]\times\mathbb{R}^2} P_N v \cdot \partial_x P_N (P_{N_1} v \cdot P_{N_2} u_1) \; dt dx dy \Big| \\ \lesssim\; & N \Big| \int_{\mathbb{R}\times\mathbb{R}^2} \widehat{P_N v} \cdot (\widehat{P_{N_1} v} * \widehat{P_{N_1} u_1}) \; d\tau d\xi d\eta \Big| \\ \lesssim\; & N \sum_{|n| \leqslant C N_1^{(5-2\alpha)+}} \Big| \int_{\mathbb{R}\times\mathbb{R}^2} (f_1 * f_2) \cdot f_3 \; d\tau d\xi d\eta \Big|, \end{aligned}$$

where

$$\begin{aligned} f_1 &= \mathcal{F}(\gamma(N_1^{(5-2\alpha)+} t - n) \mathbf{1}_{[0,T]}(t) P_{N_1} v), \\ f_2 &= \mathcal{F}(\gamma(N_1^{(5-2\alpha)+} t - n) \mathbf{1}_{[0,T]}(t) P_{N_2} u_1), \\ f_3 &= \mathcal{F}(\gamma(N_1^{(5-2\alpha)+} t - n) \mathbf{1}_{[0,T]}(t) P_N v). \end{aligned}$$

After summing up in n, we need to control the following term:

$$N N_1^{(5-2\alpha)+} \Big| \int_{\mathbb{R}\times\mathbb{R}^2} (f_1 * f_2) \cdot f_3 \; d\tau d\xi d\eta \Big|, \qquad \text{where } N \lesssim N_1 \sim N_2. \tag{4.6.31}$$

We use the same notation as before for the decomposition in modulation for functions f_i, $i = 1, 2, 3$ and consider the following cases:

- $\underline{L_{\max} \leqslant N N_1^{\alpha}}$: We further consider two subcases depending on the size of the high and low x frequencies. This is done to optimise the gain from transversality via nonlinear Loomis-Whitney inequality and bilinear Strichartz estimate.

  ⋆ $N^{\frac{1}{2}} \leqslant N_1^{\frac{3}{2} - \frac{3\alpha}{4}}$: After decomposing the functions in modulation, an application of the bilinear Strichartz estimate (4.5.8) to a high-low interaction gives

$$\begin{aligned} (4.6.31) &\lesssim N N_1^{(5-2\alpha)+} \sum_{L_i \leqslant N N_1^{\alpha}} \Big| \int_{\mathbb{R}\times\mathbb{R}^2} (f_{1,L_1} * f_{2,L_2}) \cdot f_{3,L_3} d\tau d\xi d\eta \Big| \\ &\lesssim N N_1^{(5-2\alpha)+} \frac{N^{\frac{1}{2}}}{N_1^{\frac{\alpha}{4}}} N_1^{\left(\frac{2\alpha-5}{2}\right)-} \prod_{i=1}^{3} \sum_{L_i \geqslant N_1^{(5-2\alpha)+}} L_i^{\frac{1}{2}} \|f_{i,L_i}\|_{L^2} \\ &\lesssim N_1^{(7-\frac{7\alpha}{2})+} \|P_{N_1} v\|_{F_{N_1}(T)} \|P_{N_2} u_1\|_{F_{N_2}(T)} \|P_N v\|_{F_{N_1}(T)}. \end{aligned} \tag{4.6.32}$$

$\star$ $N_1^{\frac{3}{2}-\frac{3\alpha}{4}} \leqslant N^{\frac{1}{2}}$: In this case, an application of (4.5.4) gives

$$(4.6.31) \lesssim NN_1^{(5-2\alpha)+}N_1^{-\frac{3\alpha}{4}+\frac{1}{2}}N^{-\frac{1}{2}}\prod_{i=1}^{3}\sum_{L_i\geqslant N_1^{(5-2\alpha)+}}L_i^{\frac{1}{2}}\|f_{i,L_i}\|_{L^2}$$
$$\lesssim N_1^{(6-\frac{11\alpha}{4})+}\|P_Nv\|_{F_N(T)}\|P_{N_2}u_1\|_{F_{N_2}(T)}\|P_{N_1}v\|_{F_{N_1}(T)}.$$

- $\underline{L_{\max} \geqslant NN_1^{\alpha}}$: For $L_{\max} = L_3$, using the L^4 Strichartz estimate, we obtain

$$(4.6.31) \lesssim NN_1^{(5-2\alpha)+}\sum_{\substack{L_1\geqslant NN_1^{\alpha}\\ L_2,L_3\geqslant N_1^{(5-2\alpha)+}}}\Big|\int_{\mathbb{R}\times\mathbb{R}^2}(f_{1,L_1}*f_{2,L_2})\cdot f_{3,L_3}d\tau d\xi d\eta\Big|$$
$$\lesssim NN_1^{(5-2\alpha)+}(NN_1^{\alpha})^{-\frac{1}{2}}N_1^{\frac{1}{4}-\frac{\alpha}{8}}N_2^{\frac{1}{4}-\frac{\alpha}{8}}\prod_{i=1}^{3}\sum_{L_i\geqslant N_1^{(5-2\alpha)+}}L_i^{\frac{1}{2}}\|f_{i,L_i}\|_{L^2}$$
$$\lesssim N^{\frac{1}{2}}N_1^{\frac{11}{2}-\frac{11\alpha}{4}+}\|P_Nv\|_{F_N(T)}\|P_{N_2}u_1\|_{F_{N_2}(T)}\|P_{N_1}v\|_{F_{N_1}(T)}.$$

If $L_{\max} = L_2$, we apply the L^4 Strichartz estimate to f_{1,L_1} and f_{3,L_3} and utilise the modulation gain from f_{2,L_2} as follows:

$$(4.6.31) \lesssim NN_1^{(5-2\alpha)+}N^{\frac{1}{4}-\frac{\alpha}{8}}N_1^{\frac{1}{4}-\frac{\alpha}{8}}N_1^{-\frac{\alpha}{2}}N^{-\frac{1}{2}}\prod_{i=1}^{3}\sum_{L_i\geqslant N_1^{(5-2\alpha)+}}L_i^{\frac{1}{2}}\|f_{i,L_i}\|_{L^2}$$
$$\lesssim N^{\frac{3}{4}-\frac{\alpha}{8}}N_1^{\frac{21}{4}-\frac{21\alpha}{8}+}\|P_Nv\|_{F_N(T)}\|P_{N_2}u_1\|_{F_{N_2}(T)}\|P_{N_1}v\|_{F_{N_1}(T)}.$$

The case $L_{\max} = L_1$ can be treated similarly.

The proof in this case is concluded by summing up in the x frequencies. The decisive term is the following:

$$P_Nv\cdot P_{N_1}v\cdot\partial_xP_{N_2}u_1, \quad N_1 \ll N_2 \sim N,$$

which corresponds to the second term in (4.6.29). Using the notation and reductions explained in the beginning of this section, we define

$$f_1 = \mathcal{F}(\gamma(N^{(5-2\alpha)+}t-n)\mathbf{1}_{[0,T]}(t)P_{N_1}v),$$
$$\tilde{f}_2 = \mathcal{F}(\gamma(N^{(5-2\alpha)+}t-n)\mathbf{1}_{[0,T]}(t)\partial_xP_{N_2}u_1),$$
$$f_3 = \mathcal{F}(\gamma(N^{(5-2\alpha)+}t-n)\mathbf{1}_{[0,T]}(t)P_Nv),$$

After considering the derivative in $\tilde{f}_2$, carrying out summation in n, we require to handle the following term:

$$NN^{(5-2\alpha)+}\Big|\int_{\mathbb{R}\times\mathbb{R}^2}(f_1*f_2)\cdot f_3d\tau d\xi d\eta\Big|,$$

where

$$f_2 = \mathcal{F}(\gamma(N^{(5-2\alpha)+}t-n)\mathbf{1}_{[0,T]}(t)P_{N_2}u_1).$$

Furthermore, decomposing f_i in modulation L_i, $i = 1,2,3$, we reduce to estimating a term of the form

$$NN^{(5-2\alpha)+}\sum_{N_1^{(5-2\alpha)+}\leqslant L_i\leqslant N_1^{\alpha}N}\Big|\int_{\mathbb{R}\times\mathbb{R}^2}\big|(f_{1,L_1}*f_{2,L_2})\cdot f_{3,L_3}\;d\tau d\xi d\eta\Big|. \tag{4.6.33}$$

Depending on the size of $L_{\max}$, we consider the following cases:

- $\underline{L_{\max}\leqslant N^{\alpha}N_1}$: We further consider two subcases depending on the size of the high and low x frequencies.

  $\star$ $N_1^{\frac{1}{2}}\leqslant N_2^{\frac{3}{2}-\frac{3\alpha}{4}}$: An application of the bilinear Strichartz estimate (4.5.8) gives

$$\begin{aligned}(4.6.33)&\lesssim N^{(5-2\alpha)+}N\frac{N_1^{\frac{1}{2}}}{N_2^{\frac{\alpha}{4}}}N_2^{\left(\frac{2\alpha-5}{2}\right)-}\prod_{i=1}^{3}\sum_{L_i\geqslant N_2^{(5-2\alpha)+}}L_i^{\frac{1}{2}}\|f_{i,L_i}\|_{L^2}\\&\lesssim N^{(5-2\alpha)+}\|P_{N_1}v\|_{F_{N_1}(T)}\|P_{N_2}u_1\|_{F_{N_2}(T)}\|P_Nv\|_{F_{N_1}(T)}.\end{aligned} \tag{4.6.34}$$

  $\star$ $N_2^{\frac{3}{2}-\frac{3\alpha}{4}}\leqslant N_1^{\frac{1}{2}}$: We use (4.5.4) to obtain

$$\begin{aligned}(4.6.33)&\lesssim N^{(5-2\alpha)+}NN^{-\frac{3\alpha}{4}+\frac{1}{2}}N_1^{-\frac{1}{2}}\prod_{i=1}^{3}\sum_{L_i\geqslant N_2^{(5-2\alpha)+}}L_i^{\frac{1}{2}}\|f_{i,L_i}\|_{L^2}\\&\lesssim N^{(5-2\alpha)+}\|P_Nv\|_{F_N(T)}\|P_{N_2}u_1\|_{F_{N_2}(T)}\|P_{N_1}v\|_{F_{N_1}(T)}.\end{aligned}$$

  Now we handle the non-resonant case.

- $\underline{L_{\max}\geqslant N^{\alpha}N_1}$: We apply the estimate (4.5.10) by assuming that $L_{\max}=L_3$. Note that in this case the small frequency N_1 can have size $\lesssim 1$.

$$\begin{aligned}(4.6.33)&\lesssim NN^{(5-2\alpha)+}N_1^{\frac{1}{4}}N^{-\frac{\alpha}{2}}L_{\max}^{-\frac{1}{4}}\prod_{i=1}^{3}\sum_{\substack{L_i\geqslant N^{(5-2\alpha)+},\\ L_{\max}\geqslant N_1N^{\alpha}}}L_i^{\frac{1}{2}}\|f_{i,L_i}\|_{L^2}\\&\lesssim N^{(6-\frac{11\alpha}{4})+}\|P_Nv\|_{F_N(T)}\|P_{N_2}u_1\|_{F_{N_2}(T)}\|P_{N_1}v\|_{F_{N_1}(T)}.\end{aligned}$$

  The other assumptions, namely $L_{\max}=L_1$ or $L_{\max}=L_2$ lead to the same conclusion.

The proof of (4.6.25) follows by substituting the obtained estimates in (4.6.27) and carrying out a summation in the x frequencies. For (4.6.26), we multiply the same by N^{2s} and sum up. Noting that $u_1=v+u_2$ leads to (4.6.26). □

Remark 4.6.12. One notices a difference in the regularity in the energy estimates for the solution and the difference of the solutions, i.e. in Propositions 4.6.9 and 4.6.11. It is clear that the difference arises because it is inevitable to spend a derivative to control the trilinear term in (4.6.34). This difference is $\big(5-2\alpha-(6-\frac{11\alpha}{4})\big)+=\frac{3\alpha}{4}-1+$ which non-negative for for $\alpha>\frac{4}{3}$, and can be seen when estimating the trilinear term in (4.6.34) in the case $N_{\max}^{3-\frac{3\alpha}{2}}\lesssim N_{\min}$. Comparing this to the factor $N_{\min}^{\frac{1}{2}}N_{\max}^{\frac{11}{4}-\frac{11\alpha}{4}+}$ obtained in (4.6.23) (energy estimate for the solution) we see that the ratio is $N_{\max}^{1-\frac{3\alpha}{4}}$.

4.6.3 Proof of Theorem 4.6.1

We conclude the proof of Theorem 4.6.1 in this section. We recall that the existence of solutions corresponding to initial data in $H^{2,0}(\mathbb{R}^2)$ is obtained in [49, Proposition 7] for the KP-I equation. In the first step, we show a priori estimates. Then, Lipschitz continuous dependence on the initial data is shown, provided the initial data at high regularity is small. This enables to invoke the Bona-Smith argument to prove continuity of the data-to-solution map after an a priori bound on the difference of the solutions is established.

A priori estimates: Let $\alpha \in \left(2, \frac{5}{2}\right]$, $s > 6 - \frac{11\alpha}{4}$, and $u_0 \in H^{\infty,0}$. Since $(s,0)$ is subcritical, we can rescale the initial data so that

$$\|u_0\|_{H^{s,0}} \leqslant \varepsilon_0 \ll 1,$$

where ε_0 will be determined later. Let the time of existence of the solutions in $H^{2,0}$ be $T_{\max} = T_{\max}(\|u_0\|_{H^{2,0}})$.

By Lemma 4.3.2, Proposition 4.6.2, and Proposition 4.6.9, we have the following set of estimates for $T \leqslant \min(T_{\max}, 1)$:

$$\begin{cases} \|u\|_{F^{s,0}(T)} & \lesssim \|u\|_{E^{s,0}(T)} + \|\partial_x(u^2)\|_{\mathcal{N}^{s,0}(T)}, \\ \|\partial_x(u^2)\|_{\mathcal{N}^{s,0}(T)} & \lesssim \|u\|^2_{F^{s,0}(T)}, \\ \|u\|^2_{E^{s,0}(T)} & \lesssim \|u_0\|^2_{H^{s,0}} + \|u\|^3_{F^{s,0}(T)}. \end{cases}$$

Let $X(T) = \|u\|_{E^{s,0}(T)} + \|\partial_x(u^2)\|_{\mathcal{N}^{s,0}(T)}$. Then, from the above set of estimates, we obtain

$$X(T)^2 \lesssim \|u\|^2_{E^{s,0}(T)} + \|\partial_x(u^2)\|^2_{\mathcal{N}^{s,0}(T)} \lesssim \|u_0\|^2_{H^{s,0}} + X(T)^3 + X(T)^4.$$

Moreover, we have (cf. [34, Lemma 4.2, p. 279])

$$\lim_{T\downarrow 0} \|u\|_{E^{s,0}(T)} \lesssim \|u_0\|_{H^{s,0}}, \quad \lim_{T\downarrow 0} \|\partial_x(u^2)\|_{\mathcal{N}^{s,0}(T)} = 0.$$

Hence, by choosing ε_0 small enough and the continuity of $X(T)$, we obtain

$$\lim_{T\downarrow 0} X(T) \lesssim \|u_0\|_{H^{s,0}}.$$

This implies

$$\|u\|_{F^{s,0}(T)} \lesssim \|u_0\|_{H^{s,0}}. \tag{4.6.35}$$

for $T = \min(T_{\max}, 1)$. Another application of Lemma 4.3.2 and Propositions 4.6.2 and 4.6.9 yields

$$\begin{cases} \|u\|_{F^{2,0}(T)} & \lesssim \|u\|_{E^{2,0}(T)} + \|\partial_x(u^2)\|_{\mathcal{N}^{2,0}(T)}, \\ \|\partial_x(u^2)\|_{\mathcal{N}^{2,0}(T)} & \lesssim \|u\|_{F^{2,0}(T)}\|u\|_{F^{s,0}(T)}, \\ \|u\|^2_{E^{2,0}(T)} & \lesssim \|u_0\|^2_{H^{2,0}} + \|u\|^2_{F^{2,0}(T)}\|u\|_{F^{s,0}(T)}. \end{cases}$$

This set of estimates yields

$$\|u\|^2_{F^{2,0}(T)} \lesssim \|u_0\|^2_{H^{2,0}} + \|u\|_{F^{s,0}(T)}\|u\|^2_{F^{2,0}(T)} + \|u\|^2_{F^{2,0}(T)}\|u\|^2_{F^{s,0}(T)},$$

and therefore, for $\|u\|_{F^{s,0}(T)} \lesssim \varepsilon_0$ we have $\|u\|_{F^{2,0}(T)} \lesssim \|u_0\|_{H^{2,0}}$. Consequently, we have existence up to $T = 1$ choosing ε_0 sufficiently small.

Using the same arguments as in [49] for the KP-I equation, we can conclude the local well-posedness of solutions in $H^{2,0}(\mathbb{R}^2)$. Then, the above a priori bound allows us to conclude the existence of solutions in $H^{(6-\frac{11\alpha}{4})+}(\mathbb{R}^2)$.

Remark 4.6.13. We note that for $\alpha > \frac{24}{11}$, we have $s < 0$ above. By the above a priori estimates and the conservation of mass, i.e. (4.1.2), we conclude the *global existence* of solutions in $L^2(\mathbb{R}^2)$.

Lipschitz continuous dependence in L^2: Let $s > 5 - 2\alpha$ and u, w denote two local-in-time solutions with initial data $\|u(0)\|_{H^{s,0}} \leqslant \varepsilon_0$ and $\|w(0)\|_{H^{s,0}} \leqslant \varepsilon_0$, respectively. By the above argument, we have for $s' \geqslant s$

$$\|u\|_{F^{s',0}(1)} \lesssim \|u_0\|_{H^{s',0}}. \tag{4.6.36}$$

Let $v = u - w$ denote the solution to the difference equation, namely

$$\partial_t v - D_x^\alpha \partial_x v - \partial_x^{-1}\partial_y^2 v = \partial_x(v(u+w))/2.$$

From Lemma 4.3.2, Proposition 4.6.2, Proposition 4.6.11, we have

$$\begin{cases} \|v\|_{F^{0,0}(1)} & \lesssim \|v\|_{E^{0,0}(1)} + \|\partial_x(v(u+w))\|_{\mathcal{N}^{0,0}(1)} \\ \|\partial_x(v(u+w))\|_{\mathcal{N}^{0,0}(1)} & \lesssim \|v\|_{F^{0,0}(1)}(\|u\|_{F^{0,0}(1)} + \|w\|_{F^{0,0}(1)}) \\ \|v\|^2_{E^{0,0}(1)} & \lesssim \|v(0)\|^2_{L^2} + \|v\|^2_{F^{0,0}(1)}(\|u\|_{F^{s,0}(1)} + \|w\|_{F^{s,0}(1)}) \end{cases} \tag{4.6.37}$$

This enables us to conclude

$$\|v\|_{F^{0,0}(1)} \lesssim \|v(0)\|_{L^2}, \tag{4.6.38}$$

since $\|u\|_{F^{s,0}(1)}, \|w\|_{F^{s,0}(1)} \lesssim \varepsilon_0$ are chosen sufficiently small.

Continuity of the data-to-solution mapping: Also, from Lemma 4.3.2, Proposition 4.6.2, and Proposition 4.6.11, we have

$$\begin{cases} \|v\|_{F^{s,0}(T)} & \lesssim \|v\|_{E^{s,0}(T)} + \|\partial_x(v(u+w))\|_{\mathcal{N}^{s,0}(T)} \\ \|\partial_x(v(u+w))\|_{\mathcal{N}^{s,0}(T)} & \lesssim \|v\|_{F^{s,0}(T)}(\|u\|_{F^{s,0}(T)} + \|w\|_{F^{s,0}(T)}) \\ \|v\|^2_{E^{s,0}(T)} & \lesssim \|v(0)\|^2_{H^{s,0}} + \|v\|^3_{F^{s,0}(T)} \\ & \quad + \|v\|_{F^{0,0}(T)}\|v\|_{F^{s,0}(T)}\|w\|_{F^{2s,0}(T)}. \end{cases} \tag{4.6.39}$$

From the above set of estimates, we can conclude a priori estimates for $\|v\|_{F^{s,0}(T)}$:

$$\|v\|^2_{F^{s,0}(T)} \lesssim \|v(0)\|^2_{H^{s,0}} + \|v\|^3_{F^{s,0}(T)} + \|v\|_{F^{0,0}(T)}\|v\|_{F^{s,0}(T)}\|w\|_{F^{2s,0}(T)}. \tag{4.6.40}$$

Note that again the smallness of $\|u(0)\|_{H^{s,0}}, \|w(0)\|_{H^{s,0}}$ becomes useful to absorb the term from the nonlinear estimate.
For $s > 5 - 2\alpha$, let $\phi \in H^{s,0}$ be fixed and $\{\phi_n\}_{n=1}^\infty \in H^{\infty,0}$ be such that

$$\lim_{n\to\infty} \phi_n = \phi. \tag{4.6.41}$$

By subcriticality (see equation (4.1.7)) and rescaling, we can again assume that

$$\|\phi\|_{H^{s,0}} \leqslant \varepsilon_0 \ll 1 \text{ and } \|\phi_n\|_{H^{s,0}} \leqslant \varepsilon_0 \ll 1 \text{ for all } n \in \mathbb{N}.$$

Let u be the solution corresponding to initial data ϕ_n, and w be the solution corresponding to initial data $P_{\leqslant K}\phi_n$. We construct the data-to-solution mapping as an extension of the data-to-solution mapping for smooth initial data.

For $u = S_T^\infty(\phi_n) \in C([-1,1]; H^{\infty,0})$, we can take the existence time as 1 by the a priori estimates and persistence of regularity property argued above. To prove the continuity of the data-to-solution map, we shall show that the sequence $S_T^\infty(\phi_n) \in C([-1,1]; H^{\infty,0})$ is a Cauchy sequence in the space $C([-1,1]; H^{s,0})$, $s > 5 - 2\alpha$. Hence, it suffices to show that for any $\delta > 0$, there exists $M_\delta \in \mathbb{N}$ such that

$$\|S_T^\infty(\phi_n) - S_T^\infty(\phi_m)\|_{C([-1,1];H^{s,0})} \leqslant \delta, \quad \text{for all } m, n \geqslant M_\delta. \tag{4.6.42}$$

For $K \in 2^{\mathbb{N}_0}$, let $\phi_n^K := P_{\leqslant K}\phi_n$. We have

$$\begin{aligned}\|S_T^\infty(\phi_n) - S_T^\infty(\phi_m)\|_{C([-1,1];H^{s,0})} \leqslant\ & \|S_T^\infty(\phi_n) - S_T^\infty(\phi_n^K)\|_{C([-1,1];H^{s,0})} \\ & + \|S_T^\infty(\phi_m) - S_T^\infty(\phi_m^K)\|_{C([-1,1];H^{s,0})} \\ & + \|S_T^\infty(\phi_n^K) - S_T^\infty(\phi_m^K)\|_{C([-1,1];H^{s,0})}\end{aligned} \tag{4.6.43}$$

The third term can be handled by using the continuity of the data-to-solution map for smooth data in $H^{2,0}$, i.e.

$$\|S_T^\infty(\phi_n^K) - S_T^\infty(\phi_m^K)\|_{C([-1,1];H^{s,0})} \leqslant \|S_T^\infty(\phi_n^K) - S_T^\infty(\phi_m^K)\|_{H^{2,0}} \to 0, \tag{4.6.44}$$

for $m, n \to \infty$ because

$$\|\phi_m^K - \phi_n^K\|_{H^{2,0}} \to 0, \qquad \text{as } n, m \to \infty.$$

Let $v = S_T^\infty(\phi_n) - S_T^\infty(\phi_n^K)$. We observe that v is the solution corresponding to initial data $P_{>K}\phi_n$. From (4.6.38), we have

$$\|v\|_{F^{0,0}(T)} \lesssim \|\phi_n - \phi_n^K\|_{L^2} = \|P_{>K}\phi_n\|_{L^2} \lesssim K^{-s}\|P_{>K}\phi_n\|_{H^{s,0}}.$$

From (4.6.36) for w, we have,

$$\|w\|_{F^{2s,0}(T)} \lesssim \|\phi_n^K\|_{H^{2s,0}} \lesssim K^s\|\phi_n\|_{H^{s,0}}. \tag{4.6.45}$$

Combining the above with (4.6.40), we conclude an a priori estimate for v which now depends on the profile of the initial data, namely on $P_{>K}\phi_n$. We have

$$\|S_T^\infty(\phi_n) - S_T^\infty(\phi_m)\|_{C([-1,1];H^{s,0})} \lesssim \|P_{>K}\phi_n\|_{H^{s,0}} + \|P_{>K}\phi_m\|_{H^{s,0}} + C(m,n,K).$$

By the convergence of ϕ_n and choosing K large enough so that

$$\|P_{>K}\phi_n\|_{H^{s,0}} + \|P_{>K}\phi_m\|_{H^{s,0}} < \varepsilon,$$

we conclude that $\{S_T^\infty(\phi_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([-1,1]; H^{s,0})$. This shows that S_T^∞ extends to a continuous map $S_T : H^{s,0} \to C([-1,1]; H^{s,0})$. □

Remark 4.6.14. With the above analysis, we also obtain global well-posedness in the energy space (4.1.4) of (4.1.1).

4.7 Semilinear well-posedness

For $\alpha > \frac{5}{2}$, we observed in Section 4.3 that via estimates (4.5.4) and (4.5.8), we can remedy the derivative loss completely without having to invoke short-time function spaces, using a fixed point argument. This we carry out within the standard $X^{s,\theta}$ spaces as our auxiliary spaces, see Chapter 2, Section 2.7. Let $s, \theta \in \mathbb{R}$. The space $X^{s,\theta}$ corresponding to the equation (4.1.1) is defined as the closure of Schwartz functions with respect to the norm

$$\|u\|_{X^{s,\theta}(\mathbb{R}\times\mathbb{R}^2)} := \|\langle\xi\rangle^s \langle\tau - \omega_{fKP}(\xi,\eta)\rangle^\theta \hat{u}(\tau,\xi)\|_{L^2_{\tau,\xi,\eta}(\mathbb{R}\times\mathbb{R}^2)},$$

where ω_{fKP} is defined in (4.2.2). We localize in time as usual by setting

$$X^{s,\theta}_T = \{f : [0,T]\times\mathbb{R}^2 \to \mathbb{C} \mid \exists\, \tilde{f} : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C},\ \tilde{f}\big|_{[0,T]} = f,\ \tilde{f} \in X^{s,\theta}\} \tag{4.7.1}$$

endowed with norm

$$\|f\|_{X^{s,\theta}_T} = \inf_{\tilde{f}|_{[0,T]}=f} \|\tilde{f}\|_{X^{s,\theta}},$$

with $\tilde{f}$ as defined in (4.7.1).

With the function spaces introduced, we give a precise version of Theorem 4.1.3.

Theorem 4.7.1. *Let $\alpha > \frac{5}{2}$ and $s > \frac{5}{4} - \frac{\alpha}{2}$. Then, there exists $\theta > 1/2$ such that for $T = T(\|u_0\|_{H^{s,0}})$, (4.1.1) is analytically locally well-posed in $H^{s,0}(\mathbb{R}^2)$ with the solution lying in $X^{s,\theta}_T \hookrightarrow C([0,T]; H^{s,0})$.*

This section is devoted to the proof of Theorem 4.7.1. Recall from Section 2.7 that using the energy estimate for $X^{s,\theta}$ spaces, the proof reduces to proving the following bilinear estimate:

$$\|\partial_x(uv)\|_{X^{s,\theta-1}} \lesssim \|u\|_{X^{s,\theta}}\|v\|_{X^{s,\theta}}$$

for some $\theta > 1/2$. For frequency and modulation localisation operators we use the same notation as in Section 4.3.1.

4.7.1 Bilinear estimate

We prove the estimate in a fixed time interval $[0,1]$ so that we do not have to keep track of additional decompositions in modulation or gain of small powers in T.

Proposition 4.7.2. *Let $\frac{5}{2} < \alpha < 4$. Then, for $s > \frac{5}{4} - \frac{\alpha}{2}$, there is some $\theta > \frac{1}{2}$ such that the following estimate holds:*

$$\|\partial_x(uv)\|_{X^{s,\theta-1}_1} \lesssim \|u\|_{X^{s,\theta}_1}\|v\|_{X^{s,\theta}_1}.$$

Proof. By duality and Plancherel's theorem, we can reduce the above to proving

$$\Big|\int_{\mathbb{R}^3} \xi\widehat{(uv)} \cdot \bar{\hat{w}}\, d\tau d\xi d\eta\Big| \lesssim \|u\|_{X^{s,\theta}_1}\|v\|_{X^{s,\theta}_1}\|w\|_{X^{-s,1-\theta}_1}. \tag{4.7.2}$$

Let $N, N_i \in 2^{\mathbb{Z}}$, $L, L_i \in 2^{\mathbb{N}_0}, i = 1,2$. For functions f_{N_1,L_1}, g_{N_2,L_2} and $h_{N,L}$ supported in $\tilde{D}_{N_1,L_1}, \tilde{D}_{N_2,L_2}$ and $\tilde{D}_{N,L}$, respectively, we focus on dyadic estimates, namely

$$\Big|\int (f_{N_1,L_1} * g_{N_2,L_2}) \cdot h_{N,L}\Big| \lesssim (L_1L_2L^{1-})^{\frac{1}{2}} C(N_1,N_2,N)\|f_{N_1,L_1}\|_{L^2}\|g_{N_2,L_2}\|_{L^2}\|h_{N,L}\|_{L^2} \tag{4.7.3}$$

for a suitable $C(N_1,N_2,N)$. Summing up the above in the spatial frequency and modulation leads to the desired estimate. We turn to a case-by-case analysis depending on the size of the spatial x frequencies.

1. High $\times$ Low $\to$ High ($N_2 \ll N_1 \sim N$): We shall prove (4.7.3) with $C(N_1, N_2) = N_1^{\left(\frac{1}{4}-\frac{\alpha}{2}\right)+} N_2^{0+}$. Taking into account the additional derivative, this proves (4.7.2) for $s > \frac{5}{4} - \frac{\alpha}{2}$. For $L_{\max} = \max(L_1, L_2, L)$, two cases arise:

- $\underline{L_{\max} \leqslant N_1^{\alpha} N_2}$: We consider two subcases:
$\star$ $\underline{N_2 \lesssim N_1^{\frac{1-\alpha}{2}}}$: We use Cauchy-Schwarz inequality and the bilinear Strichartz estimate (4.5.8). Note that since $\alpha > \frac{5}{2}$, the size of the small frequencies $N_2 \lesssim 1$, necessarily.

$$\begin{aligned}\text{LHS of (4.7.3)} &\lesssim \|f_{N_1,L_1} * g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2} \\ &\lesssim (L_1 L_2)^{\frac{1}{2}} \frac{N_2^{\frac{1}{2}}}{N_1^{\frac{\alpha}{4}}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2} \\ &\lesssim (L_1 L_2)^{\frac{1}{2}} L^{\frac{1}{2}-} N_1^{(\frac{1}{4}-\frac{\alpha}{2})+} N_2^{0+} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2}.\end{aligned}$$

$\star$ $N_2 \gtrsim N_1^{\frac{1-\alpha}{2}}$: Using the trilinear estimate (4.5.4), we have

$$\begin{aligned}\text{LHS of (4.7.3)} &\lesssim (L L_1 L_2)^{\frac{1}{2}} N_1^{-\frac{3\alpha}{4}+\frac{1}{2}} N_2^{-\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2} \\ &\lesssim (L_1 L_2)^{\frac{1}{2}} L^{\frac{1}{2}-} N_1^{(\frac{1}{4}-\frac{\alpha}{2})+} N_2^{0+} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2}.\end{aligned}$$

- $\underline{L_{\max} \geqslant N_1^{\alpha} N_2}$: Two subcases are as follows:
$\star$ $N_2 \gtrsim 1$: First, we assume $L_{\max} = L_2$. We employ Parseval's identity, Cauchy-Schwarz inequality and the L^4 Strichartz estimate to obtain

$$\begin{aligned}\text{LHS of (4.7.3)} &\lesssim \|g_{N_2,L_2}\|_{L^2} \|\mathcal{F}^{-1}(f_{N_1,L_1})\|_{L^4} \|\mathcal{F}^{-1}(h_{N,L})\|_{L^4} \\ &\lesssim N_1^{-\frac{\alpha}{2}+} N_2^{-\frac{1}{2}+} N_1^{\frac{1}{4}-\frac{\alpha}{8}} N^{\frac{1}{4}-\frac{\alpha}{8}} (L_1 L_2 L^{1-})^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2} \\ &\lesssim N_1^{\frac{1}{2}-\frac{3\alpha}{4}+} N_2^{-\frac{1}{2}+} (L_1 L_2 L^{1-})^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2}.\end{aligned}$$

For $L_{\max} = L$, using the same steps as before, we obtain

$$\begin{aligned}\text{LHS of (4.7.3)} &\lesssim \|\mathcal{F}^{-1}(f_{N_1,L_1})\|_{L^4} \|\mathcal{F}^{-1}(g_{N_2,L_2})\|_{L^4} \|h_{N,L}\|_{L^2} \\ &\lesssim N_1^{\frac{1}{4}-\frac{\alpha}{8}} N_2^{\frac{1}{4}-\frac{\alpha}{8}} N_1^{-\frac{\alpha}{2}+} N_2^{-\frac{1}{2}+} (L_1 L_2 L^{1-})^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2} \\ &= N_1^{(\frac{1}{4}-\frac{5\alpha}{8})+} N_2^{(-\frac{1}{4}-\frac{\alpha}{8})+} (L_1 L_2 L^{1-})^{\frac{1}{2}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2}.\end{aligned}$$

The assumption $L_{\max} = L_1$ leads to the same estimate as above, since $N_1 \sim N$.
$\star$ $N_2 \lesssim 1$: We use the estimate (4.5.10) as follows:

$$\begin{aligned}\text{LHS of (4.7.3)} &\lesssim (L_1 L_2)^{\frac{1}{2}} L^{\frac{1}{4}} N_1^{-\frac{\alpha}{2}} N_2^{\frac{1}{4}} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2} \\ &\lesssim (L_1 L_2)^{\frac{1}{2}} L^{\frac{1}{2}-} N_1^{-\frac{3\alpha}{4}+} N_2^{0+} \|f_{N_1,L_1}\|_{L^2} \|g_{N_2,L_2}\|_{L^2} \|h_{N,L}\|_{L^2},\end{aligned}$$

which is sufficient since $1 - \frac{3\alpha}{4} + < 0$ for $\alpha > \frac{5}{2}$.

2. High $\times$ High $\to$ Low ($N \ll N_1 \sim N_2$): This case is dual to Case 1 above and can be handled similarly with slight modification.

3. Three comparable frequencies ($N_1 \sim N_2 \sim N \gtrsim 1$): We shall prove the estimate (4.7.3) with $C(N_1, N_2, N) = N^{1-\frac{3\alpha}{4}+}$ by considering the following two subcases:

- $\underline{L_{\max} \leqslant N_1^{\alpha+1}}$: We use the estimate (4.5.4):

$$\text{LHS of (4.7.3)} \lesssim N_1^{-\frac{3\alpha}{4}+}(L_1L_2)^{\frac{1}{2}}L^{\frac{1}{2}-}\|f_{N_1,L_1}\|_{L^2}\|g_{N_2,L_2}\|_{L^2}\|h_{N,L}\|_{L^2}.$$

- $\underline{L_{\max} \geqslant N_1^{\alpha+1}}$: We assume $L_{\max} = L$ and employ the linear Strichartz estimate via (4.4.4):

$$\begin{aligned}\text{LHS of (4.7.3)} &\lesssim \|\mathcal{F}^{-1}(f_{N_1,L_1})\|_{L^4}\|\mathcal{F}^{-1}(g_{N_2,L_2})\|_{L^4}\|h_{N,L}\|_{L^2}\\ &\lesssim N_1^{\frac{2-\alpha}{8}}N_2^{\frac{2-\alpha}{8}}(L_1L_2)^{\frac{1}{2}}\|f_{N_1,L_1}\|_{L^2}\|g_{N_2,L_2}\|_{L^2}\|h_{N,L}\|_{L^2}\\ &\lesssim N_1^{-\frac{3\alpha}{4}+}(L_1L_2)^{\frac{1}{2}}L^{\frac{1}{2}-}\|f_{N_1,L_1}\|_{L^2}\|g_{N_2,L_2}\|_{L^2}\|h_{N,L}\|_{L^2}.\end{aligned}$$

  Other assumptions, namely $L_{\max} = L_1$ or $L_{\max} = L_2$ lead to the same estimate.

4. Very low frequencies ($N_1 \sim N_2 \sim N \lesssim 1$): After using Plancherel's identity and Cauchy-Schwarz inequality, we use the L^4 Strichartz estimate via (4.4.4)

$$\begin{aligned}\left|\int (f_{N_1,L_1} * g_{N_2,L_2}) \cdot h_{N,L}\right| &= \left|\int \mathcal{F}^{-1}(f_{N_1,L_1})\mathcal{F}^{-1}(g_{N_2,L_2})\mathcal{F}^{-1}(h_{N,L})\right|\\ &\lesssim \|\mathcal{F}^{-1}(f_{N_1,L_1})\|_{L^4}\|\mathcal{F}^{-1}(g_{N_2,L_2})\|_{L^4}\|h_{N,L}\|_{L^2}\\ &\lesssim (L_1L_2)^{\frac{1}{2}}L^{\frac{1}{2}-}\|f_{N_1,L_1}\|_{L^2}\|g_{N_2,L_2}\|_{L^2}\|h_{N,L}\|_{L^2},\end{aligned}$$

which is sufficient for (4.7.3).
In all the cases considered above, we can sum up the dyadic estimates in frequency and modulation for $\alpha > \frac{5}{2}$ owing to $C(N_1, N_2, N)$. This proves the estimate (4.7.2). □

4.7.2 Proof of Theorem 4.7.1

We prove Theorem 4.7.1 using Lemma 2.7.7 and Proposition 4.7.2. First, we prove the result on the time interval $[0, 1]$ for small initial data. Thereafter, we argue by scaling and subcriticality that the solution also exists for large initial data on a time interval $[0, T]$ where $T = T(\|u_0\|_{H^{s,0}})$.

Proof of Theorem 4.7.1. Let $\eta \in C_c^\infty(\mathbb{R})$ be a non-negative symmetric smooth time cut-off supported in $(-2, 2)$ and $\eta = 1$ on $[-1, 1]$. We define Γ as follows:

$$\Gamma(u)(t) = \eta(t)U_\alpha(t)u_0 + \eta(t)\int_0^t U_\alpha(t-s)(u\partial_x u)(s)ds,$$

where U_α is defined in (4.2.1). We shall prove that u is a fixed point of the map Γ in a closed ball $\bar{B}_R \subseteq X_1^{s,\theta}$ of radius R for initial data with sufficiently small norm. We first show that Γ is well-defined. For $u \in \bar{B}_R$, we have, using Lemma 2.7.2, Lemma 2.7.7 and

Proposition 4.7.2

$$\begin{aligned}\|\Gamma(u)\|_{X_1^{s,\theta}} &\lesssim \|\eta(t)U_\alpha(t)u_0\|_{X_1^{s,\theta}} + \left\|\eta(t)\int_0^t U_\alpha(t-s)(u\partial_x u)(s)ds\right\|_{X_1^{s,\theta}} \\ &\lesssim \|u_0\|_{H^{s,0}} + \|u\partial_x u\|_{X_1^{s,\theta-1}} \\ &\leqslant C(\|u_0\|_{H^{s,0}} + \|u\|^2_{X_1^{s,\theta}}).\end{aligned}$$

If we choose the radius R of the ball such that $C\|u_0\|_{H^{s,0}} = \frac{R}{2}$, then

$$\|\Gamma(u)\|_{X_1^{s,\theta}} \leqslant \frac{R}{2} + CR^2 \leqslant R, \text{ if } C(2C\|u_0\|_{H^{s,0}}) < \frac{1}{2}, \tag{4.7.4}$$

which shows that Γ is well-defined. To show that Γ is a contraction, for $u_1, u_2 \in \bar{B}_R$, using Lemma 2.7.7 and Proposition 4.7.2, we have

$$\begin{aligned}\|\Gamma(u_1) - \Gamma(u_2)\|_{X_1^{s,\theta}} &\lesssim \left\|\eta(t)\int_0^t U_\alpha(t-s)(u_1\partial_x u_1 - u_2\partial_x u_2)(s)ds\right\|_{X_1^{s,\theta}} \\ &\lesssim \|\partial_x(u_1+u_2)(u_1-u_2)\|_{X_1^{s,\theta-1}} \\ &\lesssim \|u_1+u_2\|_{X_1^{s,\theta}}\|u_1-u_2\|_{X_1^{s,\theta}} \\ &\leqslant 2C_1R\|u_1-u_2\|_{X_1^{s,\theta}}.\end{aligned}$$

Γ becomes a contraction on $\bar{B}_R \subseteq X_1^{s,\theta}$ if $\|u_0\|_{H^{s,0}}$ is such that

$$2C_1(2C\|u_0\|_{H^{s,0}}) < 1. \tag{4.7.5}$$

Hence, by Banach's fixed point theorem, we conclude the existence of a unique solution to (4.1.1) in $X_1^{s,\theta}$ where the norm of the initial data is chosen as the minimum of that given by (4.7.4) and (4.7.5).

Now suppose that $\|u_0\|_{H^{s,0}} \leqslant \varepsilon$ for $\varepsilon \ll 1$ and we have obtained a solution on corresponding to this small initial data on the time interval $[0,1]$. For $\alpha > \frac{5}{2}$ and $s > \frac{5}{4} - \frac{\alpha}{2}$, from (4.1.7), we observe that the anisotropic Sobolev regularity $(s,0)$ is subcritical. Thus any large initial data, say $\|u_0\|_{H^{s,0}} \geqslant \varepsilon$, can be scaled to small data via (4.1.6). We then invoke the above argument to obtain a unique solution to (4.1.1) on a time interval $[0,T]$ where T depends only on the norm of the large initial data, s, and α. Using similar arguments, we also conclude that the solution is Lipschitz continuous with respect to the initial data. The proof is complete. □

Chapter 5

Summary and outlook

"Isn't it nice to think that tomorrow is a new day with no mistakes in it yet?"
-L.M. Montgomery

We summarise the results obtained in this thesis and delve into further directions that can be pursued.

Zakharov system

We obtain a local well-posedness result for initial data in $H^s(\mathbb{R}^d)\times H^l(\mathbb{R}^d)\times H^{l-1}(\mathbb{R}^d),\ d \geqslant 3$ for the following s and l:

$$l > -\frac{1}{2}, \quad \max\left(l-1, \frac{l}{2}+\frac{1}{4}\right) < s < l+2.$$

Inclusion of the boundaries

The inequalities in the above display correspond to the boundaries which also include the endpoint $(s,l) = \left(0, -\frac{1}{2}\right)$ and have not been covered. In particular, for $d = 3$, $\left(0, -\frac{1}{2}\right)$ corresponds to the scaling critical exponent. We remark that the result in [16] covers the line $s - l = -1$. Hence it is worthwhile to check if the same function spaces can be tweaked to cover the same line. Whether the system is ill-posed or well-posed at the other boundaries is still unknown.

Local well-posedness in the periodic setting

It is also interesting to check if the same function spaces can be used in the periodic setting, i.e. for initial data in $H^s(\mathbb{T}^d) \times H^l(\mathbb{T}^d) \times H^{l-1}(\mathbb{T}^d)$ and improve on the result of Kishimoto [40] for $d = 2, 3$. For $d = 3$, it is certain that additional effort will be required to bring the regularity threshold to the energy space which corresponds to $(s,l) = (1,0)$.

Dispersion generalised KP-I equation

Scattering in L^2

For $\alpha > \frac{5}{2}$, we obtain a global well-posedness result for the dispersion generalised KP-I equation in L^2. We could investigate the long-time behaviour, in particular, to see if the solutions to (4.1.1) scatter in L^2.

Well-posedness in the periodic setting

We proved well-posedness results for the fractional KP-I equation in anisotropic Sobolev spaces $H^{s,0}(\mathbb{R}^2)$. In [71], it is proved that the KP-I equation on $\mathbb{T}^2$ is well-posed in a Besov version ($B^1_{2,1}$) of the natural energy space of the equation. We expect higher dispersion to yield better results in the periodic case as well.

To conclude, in this thesis we try to complete the picture for the local well-posedness of the Zakharov system in L^2 based Sobolev spaces on $\mathbb{R}^d$ ($d \leqslant 3$) by constructing new functions spaces to employ the available estimates optimally. For the fractional KP-I equations, we prove sharp multilinear estimates to prove new well-posedness results by utilising the function spaces that are already known.

Appendices

A Normal form approach to the Zakharov system

Below we revisit the normal form reduction to contrast the results obtained in [16] and [56]. A detailed exposition can be found in [29, 2, 16]. With $\alpha > 0$ being the wave speed, we consider the system

$$\begin{cases} i\partial_t u + \Delta u &= nu \\ \frac{1}{\alpha^2}\partial_t^2 n - \Delta n &= \Delta|u|^2, \end{cases} \tag{A.1}$$

with initial data

$$u(0,x) = u_0(x), \quad n(0,x) = n_0(x), \quad \partial_t n(0,x) = n_1(x). \tag{A.2}$$

Set $D := \sqrt{-\Delta}$ and $N = n - \frac{iD^{-1}\partial_t u}{\alpha}$; the system can then be written as

$$\begin{cases} i\partial_t u + \Delta u &= Nu/2 + \bar{N}u/2 \\ i\partial_t N + \alpha DN &= -\alpha D|u|^2. \end{cases} \tag{A.3}$$

Since the term $\bar{N}u$ can be treated in the same way as the term Nu, we can consider the right-hand side of the first equation to be Nu. The system after the normal form reduction reads

$$\begin{cases} (i\partial_t + \Delta)(u + \Omega(N,u)) &= (Nu)_{HH+LH+\alpha L} - \Omega(D|u|^2, u) + \Omega(N, Nu) \\ (i\partial_t + \alpha D)(N + D\tilde{\Omega}(u,u)) &= D(u\bar{u})_{HH+\alpha L+L\alpha} + D\tilde{\Omega}(u, Nu) + D\tilde{\Omega}(Nu, u), \end{cases} \tag{A.4}$$

where

$$(uv)_{LH} := \sum_{N\in 2^{\mathbb{N}}} P_{\leqslant \frac{N}{32}} u P_N v, \qquad (uv)_{HL} := (vu)_{LH}, \qquad (uv)_{HH} := \sum_{N_1 \sim N_2} P_{N_1} P_{N_2},$$

$$(uv)_{\alpha L} := \sum_{|\log N - \log_2 \alpha| \leqslant 1} P_N u P_{\leqslant \frac{N}{32}} v, \quad \text{and} \quad (uv)_{XL} := \sum_{|\log N - \log_2 \alpha| > 1} P_N u P_{\leqslant \frac{N}{32}} v.$$

Then, we have

$$uv = (uv)_{HH} + (uv)_{LH} + (uv)_{HL} = (uv)_{HH} + (uv)_{LH} + (uv)_{\alpha L} + (uv)_{XL}.$$

Furthermore,

$$\Omega(f,g) = \mathcal{F}^{-1} \int \mathcal{P}_{XL} \frac{\hat{f}(\xi - \eta)\hat{g}(\eta)}{|\eta|^2 - |\xi|^2 + \alpha|\xi - \eta|} d\eta,$$

$$\tilde{\Omega}(f,g) = \mathcal{F}^{-1} \int \mathcal{P}_{XL+LX} \frac{\alpha \hat{f}(\xi - \eta)\hat{g}(\eta)}{|\xi - \eta|^2 - |\eta|^2 - \alpha|\xi|} d\eta.$$

Now one can prove multilinear estimates for (A.4) to obtain local well-posedness for the Zakharov system.

Theorem A.1 ([16, Theorems 1.1, 1.2])**.** *The Cauchy problem for the Zakharov system is locally well-posed in* $H^s(\mathbb{R}^d)\times H^l(\mathbb{R}^d)\times H^{l-1}(\mathbb{R}^d)$ *provided*

$$\begin{aligned} l\geqslant 0, \max\Big(\frac{l+1}{2}, l-1\Big)\leqslant s\leqslant l+\frac{3}{2}, \quad d=2,\\ l\geqslant 0, \max\Big(\frac{l+1}{2}, l-1\Big)\leqslant s\leqslant l+\frac{5}{4}, \quad d=3. \end{aligned} \tag{A.5}$$

The following figures show the local well-posedness results obtained in [16]. Note the line $s-l=-1$ which cannot be reached in [56].

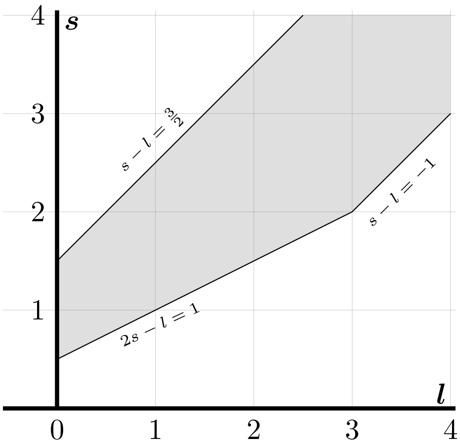

Figure 5.1: Local well-posedness for $d=2$

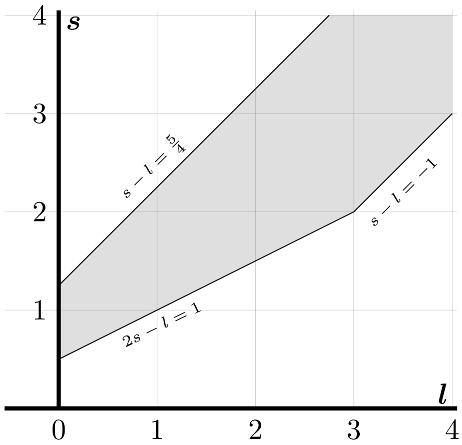

Figure 5.2: Local well-posedness for $d=3$

The inability of the normal form approach to achieve the negative regularity regime can be attributed to the lack of smoothing in the resonant case where estimates via transversality have been applied to prove results in [4, 3, 56].

B Property of the linear propagator

Lemma B.1. *Let* L *be a linear operator such that* $L=ih(\nabla/i)$*. For operators* $P_\star^{(t)}$ *and* $C_\star$ *defined in Chapter 3, we have*

$$C_\star f = e^{tL}P_\star^{(t)}e^{-tL}f.$$

Proof. The proof is a trivial implication of the properties of the Fourier transform. Let $P_\star^{(t)}e^{-tL}f=:g$. The space-time Fourier transform of $e^{tL}g$ is given by

$$(e^{tL}g)^\wedge(\tau,\xi)=\hat{g}(\tau-h(\xi),\xi).$$

Let $e^{-tL}f=:h$, then

$$\hat{h}(\tau,\xi)=\hat{f}(\tau+h(\xi),\xi),$$

and this gives using the definition of the Fourier multiplier $P_\star^{(t)}$,

$$(P_*^{(t)}e^{-tL}f)^\wedge(\tau,\xi)=\chi\big(\frac{\tau}{\star}\big)\hat{f}(\tau+h(\xi),\xi),$$

Hence, we get

$$(e^{tL}P_\star^{(t)}e^{-tL}f)^\wedge(\tau,\xi) = \chi\Big(\frac{\tau - h(\xi)}{\star}\Big)\hat{f}(\tau,\xi),$$

which completes the proof by using the definition of $C_\star$. □

C A commutator estimate

We prove the commutator estimate that has been employed in Chapter 3 to prove the linear and multilinear estimates.

Lemma C.1. *For any $f \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$ and a smooth time cut-off η, the following estimate holds*

$$\|(C_{\gtrsim\lambda^2}\eta(t) - \eta(t)C_{\gtrsim\lambda^2})f\|_{L^2_{t,x}} \lesssim \lambda^{-2}\|f\|_{L^2_{t,x}}. \tag{C.1}$$

Proof. Using the definition of the Fourier multiplier C, we have

$$\mathcal{F}(C_{\ll\lambda^2}f)(\tau,\xi) = \chi\Big(\frac{\tau - h(\xi)}{\lambda^2}\Big)\hat{f}(\tau,\xi).$$

We consider the commutator $C_{\gtrsim\lambda^2}\eta(t) - \eta(t)C_{\gtrsim\lambda^2}$ and write it in integral form with the kernel K, i.e.

$$\begin{aligned}
&(C_{\gtrsim\lambda^2}\eta(t) - \eta(t)C_{\gtrsim\lambda^2})f(t,x) = \eta(t)C_{\ll\lambda^2}f(t,x) - C_{\ll\lambda^2}(\eta f)(t,x)\\
&= Kf(t,x)\\
&= \int_{t',x'} \eta(t)C(t-t',x-x')f(t',x') - C(t-t',x-x')\eta(t')f(t',x')dt'dx'.
\end{aligned}$$

Using the mean value theorem, we have

$$\eta(t) - \eta(t') = (t-t')\int_0^1 \eta'(t + a(t'-t))da.$$

Also, the definition of C gives

$$C(t-t',x-x') = \int e^{i(t-t')\tau + i(x-x')\cdot\xi}\chi\Big(\frac{\tau - h(\xi)}{\lambda^2}\Big)d\xi d\tau.$$

Integrating by parts and using the support properties of χ, we have

$$\begin{aligned}
(t'-t)C(t-t',x-x') &= \frac{i}{\lambda^2}\int \chi'\Big(\frac{\tau - h(\xi)}{\lambda^2}\Big)e^{i(t-t')\tau + (x-x')\cdot\xi}d\tau d\xi\\
&=: \frac{1}{\lambda^2}\tilde{K}(x-x',t-t').
\end{aligned}$$

This gives

$$\|(C_{\gtrsim\lambda^2}\eta(t) - \eta(t)C_{\gtrsim\lambda^2})f\|_{L^2_{t,x}} = \frac{1}{\lambda^2}\|\tilde{K}*f\|_{L^2_{t,x}} \lesssim \frac{1}{\lambda^2}\|\tilde{K}\|_{L^1_{t,x}}\|f\|_{L^2_{t,x}} \lesssim \lambda^{-2}\|f\|_{L^2_{t,x}},$$

where the last inequality follows by noting that the L^1 norm of $\tilde{K}$ is finite. □

D Property of time cut-offs

In the following we give the proof of a bound on the L^1 norm of the indicator function of an interval. The following proof from [50] is included for the sake of completeness.
Let $T > 0$ and denote by $\mathbf{1}_T$ the indicator function of the interval $[0, T]$. We decompose

$$\mathbf{1}_T = \mathbf{1}_{T,R}^{low} + \mathbf{1}_{T,R}^{high}.$$

where

$$\widehat{\mathbf{1}_{T,R}^{low}}(\tau) = \chi\Big(\frac{\tau}{R}\Big)\widehat{\mathbf{1}_T}(\tau), \text{ for some } R > 0.$$

Lemma D.1. *For any $R > 0$ and $T > 0$,*

$$\|\mathbf{1}_{T,R}^{high}\|_{L^1} \lesssim \min(T, R^{-1}), \quad \|\mathbf{1}_{T,R}^{low}\|_{L^\infty} \lesssim 1.$$

Proof. From the definition of $\mathbf{1}_{T,R}^{low}$, we have

$$\mathbf{1}_{T,R}^{low}(t) = \Big(\chi\Big(\frac{\cdot}{R}\Big)^\vee * \mathbf{1}_T\Big)(t) = \int \check{\chi}(s)\mathbf{1}_T\Big(t - \frac{s}{R}\Big)ds.$$

This gives

$$\begin{aligned}\mathbf{1}_{T,R}^{high}(t) = \mathbf{1}_T(t) - \int \check{\chi}(s)\mathbf{1}_T\Big(t - \frac{s}{R}\Big)ds &\lesssim \int \Big|\int \Big(\mathbf{1}_T(t) - \mathbf{1}_T\Big(t - \frac{s}{R}\Big)\Big)dt\Big|\check{\chi}(s)ds \\ &\lesssim \int \check{\chi}(s) \min\Big(T, \frac{s}{R}\Big)ds \\ &\lesssim \min(T, R^{-1}).\end{aligned}$$

The L^∞ bound follows from the definition and Young's inequality:

$$\|\mathbf{1}_{T,R}^{low}\|_{L^\infty} = R\|\check{\chi}(\cdot R) * \mathbf{1}_T\|_{L^\infty} \lesssim R\|\check{\chi}(\cdot R)\|_{L^1}\|\mathbf{1}_T\|_{L^\infty} \lesssim 1.$$

□

E Determinant for the Zakharov system

Let S_i, $i = 1, 2, 3$ be the hypersurface given by

$$\begin{aligned}S_1 &= \{(\tau_1, \xi_1) = (\tau_1, \xi_{11}, \xi_{12}) \in \mathbb{R}^3 : \tau_1 = |\xi_1|^2 = \xi_{11}^2 + \xi_{12}^2\}, \\ S_2 &= \{(\tau_2, \xi_2) = (\tau_2, \xi_{21}, \xi_{22}) \in \mathbb{R}^3 : \tau_2 = -|\xi_2|^2 = -\xi_{21}^2 - \xi_{22}^2\}, \\ S_3 &= \Big\{(\tau_3, \xi_3) = (\tau_3, \xi_{31}, \xi_{32}) \in \mathbb{R}^3 : \tau_3 = \frac{|\xi_3|}{\lambda_1} = \frac{(\xi_{31}^2 + \xi_{32}^2)^{\frac{1}{2}}}{\lambda_1}\Big\}.\end{aligned}$$

The unit normals $\mathbf{n}_i$, $i = 1, 2, 3$ to these hypersurfaces are given by

$$\mathbf{n}_1 = \Big(\frac{2\xi_{11}}{\langle 2\xi_1\rangle}, \frac{2\xi_{12}}{\langle 2\xi_1\rangle}, \frac{1}{\langle 2\xi_1\rangle}\Big), \mathbf{n}_2 = \Big(\frac{2\xi_{21}}{\langle 2\xi_2\rangle}, \frac{2\xi_{22}}{\langle 2\xi_2\rangle}, -\frac{1}{\langle 2\xi_2\rangle}\Big), \mathbf{n}_3 = \Big(\frac{\xi_{31}}{|\xi_3|\langle\lambda_1\rangle}, \frac{\xi_{32}}{|\xi_3|\langle\lambda_1\rangle}, \frac{\lambda_1}{\langle\lambda_1\rangle}\Big).$$

The determinant of the normals is given by $d = d_1 + d_2 + d_3$, where

$$d_1 = \frac{\lambda_1}{\langle\lambda_1\rangle}\begin{vmatrix}\frac{2\xi_{11}}{\langle 2\xi_1\rangle} & \frac{2\xi_{21}}{\langle 2\xi_2\rangle} \\ \frac{2\xi_{12}}{\langle 2\xi_1\rangle} & \frac{2\xi_{22}}{\langle 2\xi_2\rangle}\end{vmatrix}, d_2 = -\frac{\xi_{32}}{|\xi_3|\langle\lambda_1\rangle}\begin{vmatrix}\frac{2\xi_{11}}{\langle 2\xi_1\rangle} & \frac{2\xi_{21}}{\langle \xi_1\rangle} \\ \frac{1}{\langle 2\xi_1\rangle} & -\frac{1}{\langle 2\xi_2\rangle}\end{vmatrix}, d_3 = \frac{\xi_{31}}{|\xi_3|\langle\lambda_1\rangle}\begin{vmatrix}\frac{2\xi_{12}}{\langle 2\xi_1\rangle} & \frac{2\xi_{22}}{\langle \xi_1\rangle} \\ \frac{1}{\langle 2\xi_1\rangle} & -\frac{1}{\langle 2\xi_2\rangle}\end{vmatrix}.$$

Using $|\xi_1| \sim |\xi_2| \sim \lambda_1 \ll |\xi_3| \sim \lambda$, we conclude that $d_2 + d_3 \lesssim \lambda_1^{-2}$ while for d_1, we find

$$d_1 = \frac{\lambda_1}{\langle \lambda_1 \rangle} \frac{2|\xi_1|}{\langle 2\xi_1 \rangle} \frac{2|\xi_2|}{\langle 2\xi_2 \rangle} \sin \angle\Big(\frac{\xi_1}{|\xi_1|}, \frac{\xi_2}{|\xi_2|}\Big) =: \frac{\lambda_1}{\langle \lambda_1 \rangle} \frac{2|\xi_1|}{\langle 2\xi_1 \rangle} \frac{2|\xi_2|}{\langle 2\xi_2 \rangle} \sin \tilde{\theta},$$

where $\tilde{\theta}$ denotes the angle between the Schrödinger frequencies. In case of a transverse interaction, we have $\sin \tilde{\theta} \gtrsim \lambda_1^{-1}$. Hence, $d \sim \lambda_1^{-1}$. We provide a short proof of Lemma 3.4.5 via the nonlinear Loomis-Whitney inequality from [39].

Proof of Lemma 3.4.5. After changing ζ_2 to $-\zeta_2$, the left-hand side of (3.4.5) becomes

$$\int f(\zeta_1 + \zeta_2) g_1(\zeta_1) g_2(-\zeta_2) d\zeta_1 d\zeta_2,$$

where $g_2(\cdot) = g_2(-\cdot)$ is now supported on

$$\Big\{(\tau_2, \xi_2) \in \mathbb{R} \times \mathbb{R}^2 : \frac{\lambda_2}{2} \leqslant |\xi_2| \leqslant 2\lambda_2\Big\} \cap \Big\{(\tau_2, \xi_2) \in \mathbb{R} \times \mathbb{R}^2 : \frac{L_2}{2} \leqslant |\tau_2 - |\xi_2|^2| \leqslant 2L_2\Big\}.$$

By decomposing f, g_1, g_2 into L, L_1, L_2 pieces respectively and using Cauchy-Schwarz inequality, it suffices to prove

$$\Big| \int g_1(\tau_1, \xi_1) g_2(\tau_2, \xi_2) f(\tau_1 + \tau_2, \xi_1 + \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \Big| \lesssim \lambda_1^{-\frac{1}{2}} \|g_1\|_{L^2} \|g_2\|_{L^2} \|f\|_{L^2}, \quad \text{(E.1)}$$

where f is supported in $c \leqslant \tau - |\xi| \leqslant c + 1$, g_1 on $c_1 \leqslant \tau_1 + |\xi_1|^2 \leqslant c_1 + 1$ and g_2 in $c_2 \leqslant |\tau_2 - |\xi_2|^2| \leqslant c_2 + 1$. We use parabolic rescaling to define

$$\tilde{f}(\tau, \xi) = f(\lambda_1^2 \tau, \lambda_1 \xi), \tilde{g}_1(\tau_1, \xi_1) = g(\lambda_1^2 \tau_1, \lambda_1 \xi_1), \text{ and } \tilde{g}_2(\tau_2, \xi_2) = g_2(\lambda_1^2 \tau_2, \lambda_1 \xi_2).$$

Now $\tilde{g}_1$ and $\tilde{g}_2$ are supported in a λ_1^{-2} neighbourhood of the paraboloids $\tau = \pm|\xi|^2$, and f is supported in a λ_1^{-2} neighbourhood of the surface with the parametric representation $(\tau, \xi) = \Big(\frac{|\xi|}{\lambda_1}, \xi\Big)$. With this, (E.1) reduces to showing

$$\Big| \int \tilde{g}_1(\tau_1, \xi_1) \tilde{g}_2(\tau_2, \xi_2) \tilde{f}(\tau_1 + \tau_2, \xi_1 + \xi_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \Big| \lesssim \lambda_1^{-\frac{5}{2}} \|\tilde{g}_1\|_{L^2} \|\tilde{g}_2\|_{L^2} \|\tilde{f}\|_{L^2}. \quad \text{(E.2)}$$

Using the nonlinear Loomis-Whitney inequality from [39], we find for $\epsilon = \lambda_1^{-2}$ and $A = \lambda_1$ that the left-hand side of (E.2) is dominated by

$$\lambda_1^{\frac{1}{2}} (\lambda_1^{-2})^{\frac{3}{2}} \|\tilde{g}_1\|_{L^2} \|\tilde{g}_2\|_{L^2} \|\tilde{f}\|_{L^2}$$

which concludes the proof. □

F Determinant for the dispersion generalised KP-I equation

Let S_i, $i = 1, 2, 3$ be the surface given by

$$S_i = \Big\{(\tau_i, \xi_i, \eta_i) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} : \tau_i = \xi_i |\xi_i|^\alpha + \frac{\eta_i^2}{\xi_i}\Big\}.$$

The normal to $S_i, i = 1, 2$ is given by

$$\mathbf{n}_i = \Big((\alpha+1)|\xi_i|^\alpha - \frac{\eta_i^2}{\xi_i^2}, \frac{2\eta_i}{\xi_i}, 1\Big).$$

Due to the convolution constraints, S_3 will have a normal vector given by

$$\mathbf{n}_3 = \Big((\alpha+1)|\xi_1+\xi_2|^\alpha - \frac{(\eta_1+\eta_2)^2}{(\xi_1+\xi_2)^2}, \frac{2(\eta_1+\eta_2)}{\xi_1+\xi_2}, 1\Big).$$

We compute the determinant of these normals. Let

$$B := \begin{vmatrix} (\alpha+1)|\xi_1|^\alpha - \frac{\eta_1^2}{\xi_1^2} & (\alpha+1)|\xi_2|^\alpha - \frac{\eta_2^2}{\xi_2^2} & (\alpha+1)(|\xi_1+\xi_2|)^\alpha - \frac{(\eta_1+\eta_2)^2}{(\xi_1+\xi_2)^2} \\ \frac{2\eta_1}{\xi_1} & \frac{2\eta_2}{\xi_2} & \frac{2(\eta_1+\eta_2)}{\xi_1+\xi_2} \\ 1 & 1 & 1 \end{vmatrix}$$

We compute by multilinearity for $\tilde{B} = \frac{\xi_1\xi_2(\xi_1+\xi_2)B}{2}$:

$$\begin{aligned}
\tilde{B} &= \begin{vmatrix} (\alpha+1)|\xi_1|^\alpha - \frac{\eta_1^2}{\xi_1^2} & (\alpha+1)|\xi_2|^\alpha - \frac{\eta_2^2}{\xi_2^2} & (\alpha+1)(|\xi_1+\xi_2|)^\alpha - \frac{(\eta_1+\eta_2)^2}{(\xi_1+\xi_2)^2} \\ \eta_1\xi_2(\xi_1+\xi_2) & \eta_2\xi_1(\xi_1+\xi_2) & (\eta_1+\eta_2)\xi_1\xi_2 \\ 1 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} (\alpha+1)|\xi_1|^\alpha - \frac{\eta_1^2}{\xi_1^2} & (\alpha+1)|\xi_2|^\alpha - \frac{\eta_2^2}{\xi_2^2} & (\alpha+1)(|\xi_1+\xi_2|^\alpha - |\xi_1|^\alpha) - \frac{(\eta_1+\eta_2)^2}{(\xi_1+\xi_2)^2} + \frac{\eta_1^2}{\xi_1^2} \\ \eta_1\xi_2(\xi_1+\xi_2) & \eta_2\xi_1(\xi_1+\xi_2) & \xi_2(\eta_2\xi_1 - \eta_1\xi_2) \\ 1 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} (\alpha+1)(|\xi_1|^\alpha - |\xi_2|^\alpha) - \frac{\eta_1^2}{\xi_1^2} + \frac{\eta_2^2}{\xi_2^2} & (\alpha+1)|\xi_2|^\alpha - \frac{\eta_2^2}{\xi_2^2} & (\alpha+1)(|\xi_1+\xi_2|^\alpha - |\xi_1|^\alpha) - \frac{(\eta_1+\eta_2)^2}{(\xi_1+\xi_2)^2} + \frac{\eta_1^2}{\xi_1^2} \\ (\eta_1\xi_2 - \eta_2\xi_1)(\xi_1+\xi_2) & \eta_2\xi_1(\xi_1+\xi_2) & \xi_2(\eta_2\xi_1 - \eta_1\xi_2) \\ 0 & 1 & 0 \end{vmatrix} \\
&= -(\eta_1\xi_2 - \eta_2\xi_1)\begin{vmatrix} (\alpha+1)(|\xi_1|^\alpha - |\xi_2|^\alpha) + \frac{\eta_2^2}{\xi_2^2} - \frac{\eta_1^2}{\xi_1^2} & (\alpha+1)(|\xi_1+\xi_2|^\alpha - |\xi_1|^\alpha) + \frac{(\eta_1+\eta_2)^2}{(\xi_1+\xi_2)^2} \\ \xi_1+\xi_2 & -\xi_2 \end{vmatrix} \\
&= -(\eta_1\xi_2 - \eta_2\xi_1)\Big((\alpha+1)(|\xi_1|^\alpha\xi_1 + |\xi_2|^\alpha\xi_2 - |\xi_1+\xi_2|^\alpha(\xi_1+\xi_2)) - \frac{(\eta_1\xi_2 - \eta_2\xi_1)^2}{\xi_1\xi_2(\xi_1+\xi_2)}\Big).
\end{aligned}$$

This gives

$$B = -\frac{2(\eta_1\xi_2 - \eta_2\xi_1)}{\xi_1\xi_2(\xi_1+\xi_2)}\Big((\alpha+1)(|\xi_1|^\alpha\xi_1 + |\xi_2|^\alpha\xi_2 - |\xi_1+\xi_2|^\alpha(\xi_1+\xi_2)) - \frac{(\eta_1\xi_2 - \eta_2\xi_1)^2}{\xi_1\xi_2(\xi_1+\xi_2)}\Big).$$

From (4.5.1), we have that the first factor is

$$\Big|\frac{\eta_1\xi_2 - \eta_2\xi_1}{\xi_1\xi_2(\xi_1+\xi_2)}\Big| \sim \frac{N_{\max}^{\frac{\alpha}{2}+1}N_{\min}}{N_{\max}^2 N_{\min}} \sim N_{\max}^{\frac{\alpha}{2}-1}.$$

By the resonance condition, we find for the second factor

$$\begin{aligned}
&\Big|\alpha(|\xi_1|^\alpha\xi_1 + |\xi_2|^\alpha\xi_2 - |\xi_1+\xi_2|^\alpha(\xi_1+\xi_2)) \\
&\quad + \big(|\xi_1|^\alpha\xi_1 + |\xi_2|^\alpha\xi_2 - |\xi_1+\xi_2|^\alpha(\xi_1+\xi_2) - \frac{(\eta_1\xi_2 - \eta_2\xi_1)^2}{\xi_1\xi_2(\xi_1+\xi_2)}\big)\Big| \\
&\sim \big||\xi_1|^\alpha\xi_1 + |\xi_2|^\alpha\xi_2 - |\xi_1+\xi_2|^\alpha(\xi_1+\xi_2)\big| \sim N_{\max}^\alpha N_{\min}.
\end{aligned}$$

This implies

$$B \sim N_{\max}^{\frac{3\alpha}{2}-1}N_{\min}.$$

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