Approaches to Ample Stability for Quiver Moduli

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Introduction

Motivation

One of the main problems of representation theory is to classify the representations of a given finite-dimensional algebra, up to isomorphism. Finite-dimensional algebras can be described in terms of quivers. Namely, to any quiver (i.e. oriented graph), we associate an algebra generated by the paths in the quiver, called the path algebra. The modules of the path algebras correspond to the representations of the quiver [ASS06] and, therefore, the study of representations of finite-dimensional algebras reduces to the study of quiver representations.

The Krull–Schmidt theorem states that every quiver representation can be uniquely expressed as a direct sum of indecomposable representations, up to isomorphism. Therefore, in order to classify all the representations of a given quiver, it is enough to classify the indecomposable representations.

The first complete classification result in the representation theory of quivers is due to Gabriel [Gab72] in the case of Dynkin quivers, i.e. the underlying unoriented graph is a disjoint union of Dynkin graphs of type $A_n$, $D_n$, $E_6$, $E_7$, or $E_8$. These are precisely the quivers that have finitely many isomorphism classes of indecomposable representations.

The problem of classifying quivers admitting an infinite number of indecomposable representations is only known for extended Dynkin quivers due to Nazarova [Naz73] and to Donovan and Freislich [DF73] (see also [DR76] and [ASS06]).

Although there are infinitely many indecomposable representations for quivers of extended Dynkin type, they are classified by a discrete parameter (its dimension vector), together with at most one family of continuous parameters. All other quivers with infinitely many indecomposable representations are considered wild, in the sense that the classification problem depends on arbitrarily many continuous parameters, as shown by Kac in [Kac80] and in [Kac83]. Refer also to [KR80] and to [KJ16]. The problem of classifying the indecomposable representations of wild quivers is still open, since it cannot be tackled with most of the classical tools of the representation theory of algebras.

However, through a geometric approach developed by King in [Kin94], we can find spaces whose points naturally correspond to isomorphism classes of quiver representations, which
encode all the continuous parameters in the classification problem. These spaces lead to a qualitative understanding of the classification problem as exposed in [Rei08].

To begin with, a finite quiver $Q$ consists of a finite set of vertices $Q_0$ and a finite set of arrows $Q_1$ denoted by $\alpha : i \rightarrow j$ for $i, j \in Q_0$. Fix a dimension vector $d = (d_i)_{i \in Q_0} \in \mathbb{N}Q_0$ for $Q$, and assign a $\mathbb{C}$-vector space $V_i$ of dimension $d_i$ to every vertex $i$. Consider the space of complex representations of $Q$ of dimension vector $d$

$$R_d(Q) = \bigoplus_{\alpha : i \rightarrow j} \text{Hom}_{\mathbb{C}}(V_i, V_j),$$

on which the reductive algebraic group

$$G_d = \prod_{i \in Q_0} \text{GL}(V_i)$$

naturally acts via the base change action $g \cdot v = (g_i)_{i \in Q_0} \cdot (V_\alpha)_{\alpha \in Q_1} = (g_j V_\alpha g_i^{-1})_{\alpha : i \rightarrow j \in Q_1}$.

By definition, the isomorphism classes of representations of $Q$ of dimension vector $d$ are in bijection with the orbits of $G_d$ in $R_d(Q)$. Thus, the problem of classifying quiver representations translates into parametrizing the $G_d$-orbits on $R_d(Q)$. This geometric problem can be reformulated as follows:

**Moduli Problem:** Find a Zariski open subset $U \subset R_d(Q)$, as large as possible, together with an algebraic variety $X$ and a morphism $\pi : U \rightarrow X$ such that the fibres of $\pi$ are precisely the orbits of $G_d$ in $U$.

In order to find such a subset, we need to fix a stability vector $\Theta \in (\mathbb{Z}Q_0)^*$ such that $\Theta(d) = \sum_{i \in Q_0} \Theta(d_i) = 0$. A representation $V$ of $Q$ of dimension vector $d$ is said to be $\Theta$-semistable (resp. $\Theta$-stable) if, for all dimension vectors $e$ of non-zero proper subrepresentations of $V$, we have that $\Theta(e) \leq 0$ (resp. $\Theta(e) < 0$).

We let $R_d^{\Theta-(s)st}(Q)$ denote the $\Theta$-(semi)stable locus in $R_d(Q)$. The relative GIT quotient

$$M_d^{\Theta-(s)st}(Q) = R_d^{\Theta-(s)st}(Q) // G_d$$

is a variety that parametrizes isomorphism classes of $\Theta$-polystable representations of $Q$ of dimension vector $d$. These representations are defined as direct sums of $\Theta$-stable representations.

The relative GIT quotient restricts to a geometric quotient

$$M_d^{\Theta-st}(Q) = R_d^{\Theta-st}(Q) / G_d$$

called the moduli space of $\Theta$-stable representations. It parametrizes isomorphism classes of $\Theta$-stable representations of $Q$ of dimension vector $d$. 
A dimension vector \( \mathbf{d} \) is said to be stable if there exists a \( \Theta \)-stable representation of \( Q \) of dimension vector \( \mathbf{d} \); that is, if the moduli space \( M_{\mathbf{d}}^{\Theta-\text{st}}(Q) \) is non-empty. In this case, it is an irreducible smooth complex quasiprojective variety of dimension \( 1 - \langle \mathbf{d}, \mathbf{d} \rangle \).

The quiver moduli space \( M_{\mathbf{d}}^{\Theta-\text{st}}(Q) \) provides a solution to the Moduli Problem, see [Rei08].

We devote Chapter 1 to recalling all the necessary facts about quiver representations and their module spaces.

When studying a moduli problem, the best possible solution we can aim for is a fine moduli space; that is, a moduli space that comes with a universal (or at least tautological) family of objects [Bal10], [New78].

Following [Kin94], [Rei08] and [FRS20], in the context of quiver moduli, a tautological family of quiver representations on \( M_{\mathbf{d}}^{\Theta-\text{st}}(Q) \) is constructed as follows: For each \( i \in Q_0 \), we consider the trivial vector bundles \( R_{\mathbf{d}}^{\Theta-\text{st}}(Q) \times V_i \to R_{\mathbf{d}}^{\Theta-\text{st}}(Q) \), and for any \( \alpha : i \to j \) we consider the natural vector bundle maps

\[
V_{\alpha} : R_{\mathbf{d}}^{\Theta-\text{st}}(Q) \times V_i \to R_{\mathbf{d}}^{\Theta-\text{st}}(Q) \times V_j
\]

\[
(V, v_i) \mapsto (V, V_{\alpha}(v_i)).
\]

The subgroup of scalar matrices \( \Delta = \{(\lambda I_d)_{i \in Q_0} | \lambda \in \mathbb{C}^\times \} \subset G_{\mathbf{d}} \) is the stabilizer of each point in the open \( \Theta \)-stable locus \( R_{\mathbf{d}}^{\Theta-\text{st}}(Q) \). Hence, in order to descend these bundles to bundles on the geometric quotient \( M_{\mathbf{d}}^{\Theta-\text{st}}(Q) \), we need to twist the \( G_{\mathbf{d}} \) action on \( R_{\mathbf{d}}^{\Theta-\text{st}}(Q) \times V_i \) by a character \( \chi_\Theta \), which takes the value \( \lambda^{-1} \) when restricted to \( \Delta \), so that the group \( \Delta \) acts trivially along the fibres of the bundle, i.e. the factors \( V_i \). For such a character to exist, we need to assume that the dimension vector \( \mathbf{d} \) is indivisible, namely \( \gcd(d_i | i \in Q_0) = 1 \).

Thus, we can choose integers \( s_i \) for \( i \in Q_0 \) such that \( \sum_{i \in Q_0} s_i d_i = 1 \), and then we define the character to be

\[
\chi_\Theta((g_i)_i) = \prod_{i \in Q_0} \det(g_i)^{-s_i}.
\]

Therefore, via the quotient map \( R_{\mathbf{d}}^{\Theta-\text{st}}(Q) \to M_{\mathbf{d}}^{\Theta-\text{st}}(Q) \), we get vector bundles \( M_{\mathbf{d}}^{\Theta-\text{st}}(Q) \times V_i \to M_{\mathbf{d}}^{\Theta-\text{st}}(Q) \) of rank \( d_i \) for all \( i \in Q_0 \) and map of vector bundles

\[
V_{\alpha} : M_{\mathbf{d}}^{\Theta-\text{st}}(Q) \times V_i \to M_{\mathbf{d}}^{\Theta-\text{st}}(Q) \times V_j
\]

for all arrows \( \alpha : i \to j \) in \( Q_1 \) so that the following holds:

Given a point \( |V| \in M_{\mathbf{d}}^{\Theta-\text{st}}(Q) \), the induced quiver representation in the fibre over \( |V| \) is isomorphic to \( V \); that is

\[
((|V_i|_i), (|V_i|_\alpha)_i) \cong V.
\]

In particular, any fibrewise \( \Theta \)-stable representation of \( Q \) in vector bundles of rank vector \( \mathbf{d} \) can be obtained as a pullback of this tautological family of bundles. In fact, this makes \( M_{\mathbf{d}}^{\Theta-\text{st}}(Q) \) a fine moduli space when \( \mathbf{d} \) is an indivisible dimension vector, see [Kin94, Proposition 5.3].
The non-existence of universal bundles for moduli of vector bundles on curves was proved by Balaji, Biswas, Gabber and Nagaraj [VBN07] using obstruction in Brauer groups. Subsequently, in [ER09], Engel and Reineke constructed projective bundles $P_n \rightarrow M_{d}^{\Theta,\text{st}}(Q)$, and Reineke and Schröer [RS17] adapted the arguments of [VBN07] to quiver moduli. In particular, they formulated the following statement:

**Conjecture**: ([RS17]) If $d$ is stable, then the Brauer group of $M_{d}^{\Theta,\text{st}}(Q)$ is cyclic of order $\gcd(d_i \mid i \in Q_0)$ and the class of every $P_n$ for $n \neq 0$ is a generator.

This leads to the non-existence of tautological representations on $M_{d}^{\Theta,\text{st}}(Q)$ when $\gcd(d_i \mid i \in Q_0) \geq 2$; in other words, if the Conjecture holds, the moduli space $M_{d}^{\Theta,\text{st}}(Q)$ is not fine for divisible dimension vectors $d$; see [RS17, Theorem 4.4].

The main result of [RS17], Theorem 4.2, proves the Conjecture under the assumption of $\Theta$-ample stability on $d$: A dimension vector $d$ is $\Theta$-amplly stable if there are no divisors in the unstable locus of $R_{d}(Q)$, which means that

$$\text{codim}_{R_{d}(Q)} \left( R_{d}(Q) \setminus R_{d}^{\Theta,\text{st}}(Q) \right) \geq 2.$$

**Numerical criterion for $\Theta$-ample stability** [RS17, Proposition 5.1]: A $\Theta$-stable dimension vector $d$ is $\Theta$-amplly stable if, for all proper decompositions $d = e + f$ such that $\Theta(e) \geq \Theta(f)$, we have $(e, f) \leq -2$. This is known as $\Theta$-numerical ample stability (abbreviated by $\Theta$-nas).

In [RS17], it was verified that for $m$-loop quivers and $m$-Kronecker quivers, all $\Theta$-stable dimension vectors are $\Theta$-nas, with the exception of $L_2$ and $K_3$:

$$L_2: \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array} \quad K_3: \begin{array}{c}
\circ \\
\circ \\
\circ
\end{array}$$

Reineke and Schröer showed that the Brauer groups of the two special cases $M_{(2)}^{\Theta,\text{st}}(L_2)$ and $M_{(2,2)}^{\Theta,\text{st}}(K_3)$ are isomorphic to the cyclic group $\mathbb{Z}/2\mathbb{Z}$. Along with this, they proved that their Conjecture also holds for multiple loop and generalized Kronecker quivers.

We introduce the key notion of ample stability and the sufficient, purely numerical, criterion for determining whether a dimension vector is amply stable in Chapter 2.

The condition of being $\Theta$-nas also appears as an important condition in [ERS20]. There the setup is a bit more special: They are only interested in dimension vectors which are $\Theta$-coprime, meaning that $\Theta(d) \neq 0$ for all non-zero proper dimension vectors $e \leq d$, and they are only interested in the canonical stability

$$\Theta^{\text{can}} = \{d, \_\} = (d, \_) - (\_, \_).$$

More precisely, Franzen, Reineke, and Sabatini proved that, under the assumption of $\Theta^{\text{can}}$-coprimality on $d$, $\Theta^{\text{can}}$-ample stability implies that the moduli space $M_{d}^{\Theta^{\text{can}}\text{-st}}(Q)$
is a projective Fano variety of known dimension, Picard rank and index. We review the relevant notions of this result in Section 3.4. Furthermore, the results obtained in Section 3.3 provide examples of arbitrarily high-dimensional Fano quiver moduli spaces.

One of the main goals of this thesis is to explore the numerical ample stability condition, since it allows the calculation of algebraic-geometric invariants for moduli spaces of $\Theta$-stable representations.

For the proof of [RS17, Proposition 6.2], which verifies that the $\Theta$-nas condition is fulfilled in the case of generalized Kronecker quivers, one key step is the reduction to dimension vectors in the fundamental domain $F_Q$. This is the set of dimension vectors $d$ such that 

$$\langle d, i \rangle + \langle i, d \rangle \leq 0$$

for all $i \in Q_0$, and such that the full subquiver of $Q$ with vertices 

$$\text{supp}(d) = \{ i \in Q_0 \mid d_i \neq 0 \}$$

is connected.

This reduction is possible since, for these quivers, the so-called reflection functors induce isomorphisms of moduli spaces, see [Wei13]. This technique is only available for generalized Kronecker quivers, hence another main goal of this thesis is to see if this can be done for more general quivers.

**Overview of the main results**

Let $Q$ be a quiver and let us consider the canonical stability $\Theta^{\text{can}}$. Then every dimension vector in the fundamental domain $F_Q$ is $\Theta^{\text{can}}$-stable. This result is due to Kac [Kac82] and Schofield [Sch92].

In Chapter 3, we focus on studying the nas condition, with respect to $\Theta^{\text{can}}$, only for dimension vectors in $F_Q$. In this chapter, we prove that the $\Theta^{\text{can}}$ condition simplifies if we assume that $d \in F_Q$. Namely, if $d$ is a fundamental dimension vector for $Q$, then $d$ is $\Theta^{\text{can}}$-nas if and only if there is no decomposition $d = e + f$ such that $\langle e, f \rangle = -1$ and $(f, e) \in \{-1, 0\}$.

This allowed us to verify the $\Theta^{\text{can}}$-nas condition for other classes of quivers. For example, we consider the two-vertex quivers $K_{m,n}$:
In this case, we find the following:

**Theorem 1:** (Theorem 3.3.1). Let \( \mathbf{d} \) be a fundamental dimension vector for \( K_{m,n} \). Then \( \mathbf{d} \) is \( \Theta^{\text{can}} \)-nas except for

- \( m + n = 3 \) and \( \mathbf{d} = (2,2) \),
- \( m = n \) and \( d_2 = md_1 \).

In case \( m = n \) and \( d_2 = md_1 \), we show that there is an isomorphism of moduli spaces (Lemma 3.3.7):

\[
M_{(d_1,md_1)}(K_{m,m}) \cong M_{d_1}(L_{m^2}).
\]

It follows that the Brauer group of \( M_{(d_1,md_1)}(Q) \) is cyclic of order two, explicit calculations are given in [RS17].

As another example, we consider the three-vertex quiver \( Q_{2,2} \):

\[
\begin{array}{c}
\bullet & \overset{i_1}{\underset{i_2}{\longrightarrow}} & \overset{i_3}{\bullet} \\
\end{array}
\]

for which we find:

**Theorem 2:** (Theorem 3.3.2). Let \( \mathbf{d} \) be a fundamental dimension vector for \( Q_{2,2} \). Then \( \mathbf{d} \) is \( \Theta^{\text{can}} \)-nas except if it is not of the form \((1, a + 1, a), (a, a + 1, 1)\) for some \( a \geq 1 \) or \((2,2,2)\).

In case \( \mathbf{d} = (2,2,2) \), the moduli space \( M_{(2,2,2)}(Q_{2,2}) \) is isomorphic to the locally closed subset of \( \mathbb{P}^9 \) with coordinates

\[
(P_{0,0} : P_{0,1} : P_{0,2} : P_{1,0} : P_{1,1} : P_{1,2} : P_{2,0} : P_{2,1} : P_{2,2} : Q)
\]

such that

\[
P_{i,j}P_{k,l} = P_{i,l}P_{k,j} \quad \forall i, j, k, l \in \{0, 1, 2\},
\]

\[
Q^2 - (2P_{0,2} + P_{1,1} + 2P_{2,0})Q + (P_{0,2} + P_{2,0})^2 + (P_{0,1} + P_{1,0})(P_{1,2} + P_{2,1}) \neq 0.
\]

We prove this statement in Lemma 3.3.8

A natural step would be to compute the Brauer group of this moduli space, which is expected to be cyclic of order two. This will not be part of this work, but it suggests a potential further direction.

It is hard to check the \( \Theta^{\text{can}} \)-nas condition for more general three-vertex quivers, however, in Chapter 3 we formulate two statements, Conjecture 1 and Conjecture 2 for the three-vertex quivers \( Q_{m,n} \) and the symmetric quiver \( Q \) given, respectively, by:

\[
\begin{array}{c}
\bullet & \overset{i_1}{\underset{i_2}{\longrightarrow}} & \overset{i_3}{\bullet} \\
vi
\end{array}
\]
In Section 3.3.3 we verify the $\Theta$-nas condition for the $m$-subspace quiver (with vertices $i_1, \ldots, i_m, j$ and arrows $i_k \to j$ for $k \in \{1, \ldots, m\}$) and the dimension vector $d = i_1 + \ldots + i_k + 2i_{k+1} + \ldots + 2i_m + dj$, in the case that $d$ is $\Theta$-can-coprime.

Let $Q$ be an acyclic quiver, let $d$ be a fixed dimension vector for $Q$ and let $i$ be a sink in $Q$. We fix a stability $\Theta$ such that $\Theta(d) = 0$ and such that $\Theta_i < 0$. Denote by $P(i)$ the indecomposable projective representations for all $i \in Q_0$.

In Chapter 4, writing $\Theta = \langle \alpha, \rangle$, where $\alpha = \sum_{i \in Q_0} \Theta_i \dim P(i)$, and using reflection functors and duality, we show that there is a bijection between isomorphism classes of $\Theta$-semistable representations of $Q$ of dimension vector $d$ and isomorphism classes of $S_i \Theta$-semistable representations of $s_iQ$ of dimension vector $s_id$ (see Theorem 4.3.3), and that there is an isomorphism of moduli spaces of semistable representations

$$M^{\Theta-\sst}_d(Q) \cong M^{S_i \Theta-\sst}_{s_id}(s_iQ),$$

where $s_iQ$ is the reflected quiver, $s_id$ is the reflection of the dimension vector $d$, and $S_i \Theta = \langle s_i \alpha, \rangle_{s_iQ}$.

Furthermore, in Lemma 4.3.4, we prove that the $\Theta$-nas condition is also compatible with reflection functors, i.e. $d$ $\Theta$-stable is $\Theta$-nas if and only if $s_id$ is $S_i \Theta$-nas.

In particular, we show that all this hold for dimension vectors in $F_Q$ and for $\Theta$-can, see Corollary 4.3.6.
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Chapter 1

Preliminaries

In this chapter, we fix standard notation and introduce the main notions that will be used in this thesis. We start by presenting some basics of the theory of quiver representations, then we will expose an algebro-geometric approach to the problem of classifying isomorphism classes of representations of quivers and, making use of Geometric Invariant Theory, we will finally construct moduli spaces in the quiver setting.

1.1 Quivers and their representations

In this section, we introduce one of the essential concepts of this thesis: quivers and quiver representations, as well as the main facts and results that will be needed throughout. Refer to the books [ASS06], [Sch14] or to [KJ16], which contain a detailed exposition of this theory.

Definition 1.1.1: A quiver $Q = (Q_0, Q_1, s, t)$ consists of

- two sets: a set of vertices $Q_0$, a set of arrows $Q_1$,
- and two maps $s, t: Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ the vertices $s(\alpha)$ and $t(\alpha)$, called the source and the target of $\alpha$, respectively.

We will write $\alpha: i \to j$ or $i \xrightarrow{\alpha} j$ to indicate that the arrow $\alpha$ has source $i$ and target $j$.

Throughout, we assume that the set of vertices and arrows are finite, and that $Q$ is connected. The following are some standard examples of finite and connected quivers:

Note that a quiver is nothing but a directed graph where loops, oriented cycles, and multiple arrows between two vertices are allowed.
We call \( Q' = (Q'_0, Q'_1, s', t') \) a subquiver of \( Q = (Q_0, Q_1, s, t) \) if \( Q'_0 \subseteq Q_0, Q'_1 \subseteq Q_1 \) and \( s', t' \) are respectively the restrictions of \( s, t \) to \( Q'_1 \). Such a subquiver is said to be full if

\[
Q'_1 = \{ \alpha \in Q_1 \mid s(\alpha), t(\alpha) \in Q'_0 \}.
\]

**Example 1.1.2:** The quiver \( A_n \)

\[
\begin{array}{ccccccc}
& & * & & & & * \\
& 1 & 2 & 3 & \cdots & n-1 & n \\
\end{array}
\]

is a subquiver of the quiver \( \tilde{A}_n \)

\[
\begin{array}{ccccccc}
& & * & & & & * \\
& 1 & 2 & 3 & \cdots & n-1 & n \\
\end{array}
\]

and a full subquiver of the quiver \( \tilde{A}_{n+1} \)

\[
\begin{array}{ccccccc}
& & * & & & & * \\
& 1 & 2 & 3 & \cdots & n-1 & n & n+1 \\
\end{array}
\]

We denote by \( \mathbb{N}Q_0 \) the abelian group generated by \( Q_0 \), whose elements \( d \) associate to each vertex \( i \in Q_0 \) a natural number \( d_i \), i.e.

\[
d = \sum_{i \in Q_0} d_i i.
\]

\( d \in \mathbb{N}Q_0 \) is called a *dimension vector* for \( Q \) and \( i \in \mathbb{N}Q_0 \) is the coordinate vector at \( i \in Q_0 \).

The *Euler form* of \( Q \) is a non-symmetric bilinear form, defined by

\[
(d, e) = \sum_{i \in Q_0} d_i e_i - \sum_{(\alpha: i \to j) \in Q_1} d_i e_j,
\]

for \( d, e \in \mathbb{N}Q_0 \). It is represented by the matrix \( E = (E_{ij}) \) given by

\[
E_{ij} = \delta_{ij} - |\{ (\alpha: i \to j) \in Q_1 \}|,
\]

where \( \delta_{ij} \) is the Kronecker delta symbol.

Indeed, identifying the dimension vectors with column vectors, we have

\[
(d, e) = d'E e.
\]

We obtain on \( \mathbb{N}Q_0 \) the *symmetrized Euler form* by defining

\[
(d, e) = (d, e) + (e, d),
\]

and its corresponding quadratic form, called the *Tits form*

\[
q(d) = \frac{1}{2} (d, d) = (d, d).
\]
1.1. Quivers and their representations

The theory of quiver representations can be done over an arbitrary field. For simplicity, we let the ground field \( k \) be algebraically closed unless specified otherwise.

**Definition 1.1.3:** A representation \( V \) of a quiver \( Q \) is an assignment of

- a \( k \)-vector space \( V_i \) to each vertex \( i \in Q_0 \),
- a \( k \)-linear map \( V_\alpha : V_i \rightarrow V_j \) to each arrow \((\alpha : i \rightarrow j) \in Q_1\).

Such a representation is denoted by \( V = ((V_i)_{i \in Q_0}, (V_\alpha)_{\alpha \in Q_1}) \), or simply \( V = ((V_i, (V_\alpha)_i)) \).

A representation \( V \) is called **finite-dimensional** if each \( k \)-vector space \( V_i \) is finite-dimensional. In this case, the dimension vector \( \dim V \in \mathbb{N}Q_0 \) of \( V \) is defined as

\[
\dim V = \sum_{i \in Q_0} \dim V_i.
\]

Let \( Q \) be a quiver and let \( V = ((V_i)_i, (V_\alpha)_\alpha), W = ((W_i)_i, (W_\alpha)_\alpha) \) be two representations of \( Q \). A **morphism** of representations \( f : V \rightarrow W \) is a collection of \( k \)-linear maps \((f_\alpha : V_\alpha \rightarrow W_\alpha)_{\alpha \in Q_1}\) such that for each arrow \( \alpha : i \rightarrow j \in Q_1 \) the diagram

\[
\begin{array}{ccc}
V_i & \xrightarrow{V_\alpha} & V_j \\
\downarrow{f_i} & & \downarrow{f_j} \\
W_i & \xrightarrow{W_\alpha} & W_j
\end{array}
\]

commutes, that is, \( W_\alpha \circ f_i(v) = f_j \circ V_\alpha(v) \) for all \( v \in V_i \). A morphism is said to be a **monomorphism**, **epimorphism**, **isomorphism**, etc. if every \( f_i \) has the corresponding property as a linear map.

The finite-dimensional representations of \( Q \) together with the morphisms of representations form a category, which we denote by \( \text{rep}_k Q \).

**Notation 1.1.4:** Throughout the course of this work, we consider only representations in \( \text{rep}_k Q \). Thus, the finite-dimensional representations of \( Q \) will be simply referred to as representations of \( Q \).

For \( V, W \) representations in the category \( \text{rep}_k Q \), we define the following:

1. The **direct sum** of \( V \) and \( W \) in \( \text{rep}_k Q \) is given by the representation

\[
V \oplus W = \left((V_i \oplus W_i)_{i \in Q_0}, \begin{bmatrix} V_\alpha & 0 \\ 0 & W_\alpha \end{bmatrix}_{(\alpha : i \rightarrow j) \in Q_1}\right).
\]
2. A subrepresentation $U \subseteq V$ is a collection of vector subspaces $(U_i \subseteq V_i)_{i \in Q_0}$ such that for all $(\alpha: i \rightarrow j) \in Q_1$ we have $V_{\alpha}(U_i) \subseteq U_j$, the restriction of $V_\alpha$ to $U_i$ is denoted by $U_\alpha$. Moreover, the quotient of $V$ by a subrepresentation $U$ is the collection of vector spaces $(V_i/U_i)_{i \in Q_0}$ with $k$-linear maps given by

$$(V/U)_\alpha: V_i/U_i \rightarrow V_j/U_j,$$

for any $(\alpha: i \rightarrow j) \in Q_1$.

3. Let $f: V \rightarrow W$ be a morphism in $\text{rep}_kQ$. Then the subrepresentation $\text{Ker}f$ of $V$ defined by

$$(\text{Ker}f)_i = \text{Ker}(f_i: V_i \rightarrow W_i)$$

is the kernel of $f$ in $\text{rep}_kQ$, and the subrepresentation $\text{Im}f$ of $W$ defined by

$$(\text{Im}f)_i = \text{Im}(f_i: V_i \rightarrow W_i)$$

is the image of $f$ in $\text{rep}_kQ$.

With these definitions, we can check that $\text{rep}_kQ$ is a $k$-linear abelian category. A detailed proof of this fact can be found in [ASS06, Lemma 1.3. Chapter III].

The following is one of the main problems in the theory of finite-dimensional quiver representations.

**Classification Problem:** Classify all representations in $\text{rep}_kQ$, up to isomorphism.

We will now briefly explain why all notions and results from the theory of modules over associative algebras can be translated into the language of quiver representations. In particular, we will rephrase the notion of indecomposability and reduce the Classification Problem to classifying the isomorphism classes of indecomposable representations.

Let $Q$ be a quiver. A path of length $l$ in $Q$ is a sequence of arrows $(\alpha_l, \ldots, \alpha_1)$ such that $s(\alpha_{i+1}) = t(\alpha_i)$ for all $i \in \{1, \ldots, l-1\}$. We also associate a trivial path $e_i$ of length 0 to each vertex $i \in Q_0$, for which $s(e_i) = t(e_i) = i$.

**Definition 1.1.5:** The path algebra $kQ$ of a quiver $Q$ is the $k$-algebra with basis consisting of all paths of length $l \geq 0$ in $Q$ and with multiplication defined via concatenation

$$(\alpha_l, \ldots, \alpha_1)(\beta_r, \ldots, \beta_1) = \begin{cases} (\alpha_l, \ldots, \alpha_1, \beta_r, \ldots, \beta_1), & \text{if } s(\alpha_1) = t(\beta_r), \\ 0, & \text{otherwise}. \end{cases}$$

This multiplication extends by linearity to make $kQ$ an associative $k$-algebra with unit $1 = \sum_{i \in Q_0} e_i$. 

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Theorem 1.1.6: Let $Q$ be a quiver. Then there is an equivalence of categories

$$\text{rep}_k Q \simeq kQ\text{-mod},$$

whose functors are given by

$$(V_i, V_\alpha) \mapsto \bigoplus_{i \in Q_0} V_i$$

and

$$M \mapsto (e_i M)_{i \in Q_0}.$$

For further details and a complete proof of this result, we refer the reader to [ASS06, Theorem 1.6. Chapter III].

In this way, a given representation $V \in \text{rep}_k Q$ is called

- **simple**, or **irreducible**, if it contains exactly two subrepresentations, namely $0$ and $V$ itself.
- **semisimple** if it is isomorphic to a direct sum of simple representations.
- **indecomposable** if $V \neq 0$ and, whenever $V \cong U_1 \oplus U_2$ with $U_1, U_2 \subseteq V$ subrepresentations, either $U_1 = 0$ or $U_2 = 0$.

In order to classify representations, we assume $Q$ to be acyclic, that is, it has no oriented cycles.

For every $i \in Q_0$, define the **simple representation** $S(i)$ by

$$S(i)_j = \begin{cases} k, & \text{if } j = i, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and}$$

$$S(i)_\alpha = 0 \quad \text{for all } \alpha \in Q_1.$$

The proposition below states that all simple representations of $Q$ are isomorphic to $S(i)$, for some $i \in Q_0$. The proof of which can be found in [Sch14, Proposition 2.9].

Proposition 1.1.7: Let $Q$ be acyclic as above. Then $\{S(i) \mid i \in Q_0\}$ form a complete set of simple representations in $\text{rep}_k Q$.

Knowing the representations of a given quiver $Q$ is not enough to tackle the **Classification Problem**. However, the following theorem, due to Krull and Schmidt [Sch14, Theorem 1.2], shows that it is sufficient to classify all indecomposable representations.

Theorem 1.1.8: Any representation in $\text{rep}_k Q$ can be written as a direct sum of indecomposable representations, and such a decomposition is unique up to ordering.
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For every $i \in Q_0$, define the projective representation $P(i)$ to be the representation corresponding to the $kQ$-module given by

$$P(i) = kQ e_i,$$

where $(P_i)_j$ is the $k$-vector space with basis the set of all paths from $i$ to $j$ in $Q$ and, if $(\alpha: j \rightarrow l) \in Q_1$, then $P(i)_\alpha: P(i)_j \rightarrow P(i)_l$ is the $k$-linear map defined on the basis by composing the paths from $i$ to $j$ with $\alpha$.

**Remark 1.1.9:** For any $V \in \text{rep}_kQ$ and any $i \in Q_0$, we have $\text{Hom}(P(i), V) = V_i$.

As a consequence of this remark, for $Q$ acyclic, we can classify the projective indecomposables in $\text{rep}_kQ$ up to isomorphism as stated in the theorem below. For more details, see [KJ16, Theorem 1.18].

**Theorem 1.1.10:** Assume that $Q$ has no oriented cycles. Then $\{P(i) \mid i \in Q_0\}$ form the complete set of non-zero indecomposable projective representations in $\text{rep}_kQ$.

Given a representation $V = ((V_i), (V_\alpha)_\alpha)$ of $Q$, with $V_i$ vector spaces of dimension $d_i$ for all $i \in Q_0$, define the projective $kQ$-modules

$$P_0 = \bigoplus_{i \in Q_0} P(i) \otimes V_i,$$

$$P_1 = \bigoplus_{\alpha: i \rightarrow j} P(j) \otimes V_i.$$

Then there exists a short exact sequence [Sch14, Theorem 2.15]

$$0 \longrightarrow P_1 \overset{f}{\longrightarrow} P_0 \overset{g}{\longrightarrow} V \longrightarrow 0,$$

where the maps $f$ and $g$ are defined by

$$g(p \otimes v) = pv \quad \text{for } p \in P(i), v \in V_i,$$

$$f(p \otimes v) = p\alpha \otimes v - p \otimes \alpha v \quad \text{for } p \in P(t(\alpha)), v \in V_{s(\alpha)}.$$

This exact sequence is called the standard projective resolution of $V$.

**Remark 1.1.11:** Since any representation $V$ of $Q$ admits a standard projective resolution of length one, we have that the category $\text{rep}_kQ$ is hereditary, that is, $\text{Ext}_Q^1(V, W) = 0$ for all $V, W \in \text{rep}_kQ$ and all $i > 1$.

Consider now representations $V, W$ of $Q$ and apply the functor $\text{Hom}_Q(-, W)$ to the standard projective resolution of $V$ to obtain the exact sequence

$$0 \longrightarrow \text{Hom}_Q(V, W) \longrightarrow \text{Hom}_Q(P_0, W) \longrightarrow \text{Hom}_Q(P_1, W) \longrightarrow \text{Ext}_Q^1(V, W) \longrightarrow 0,$$

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1.2 Orbits in the representation space

which, by Remark 1.1.9, yields the following exact sequence (see [Bri08, Section 1.4]):

\[ 0 \rightarrow \text{Hom}_\mathbb{Q}(V, W) \rightarrow \bigoplus_{i \in Q_0} \text{Hom}_k(V_i, W_i) \rightarrow \bigoplus_{\alpha: i \rightarrow j} \text{Hom}_k(V_i, W_j) \rightarrow \text{Ext}^1_{\mathbb{Q}}(V, W) \rightarrow 0. \]

This leads to a characterization of the Euler form for dimension vectors of representations of \( Q \) as follows:

\[
\dim \text{Hom}_\mathbb{Q}(V, W) - \dim \text{Ext}^1_{\mathbb{Q}}(V, W) = \dim \bigoplus_{i \in Q_0} \text{Hom}_k(V_i, W_i) - \sum_{\alpha: i \rightarrow j} \dim V_i \dim W_j = \langle \dim V, \dim W \rangle.
\]

(1.4)

Example 1.1.12: Let \( i \in Q_0 \) and let \( V \) be a representation of \( Q \). Then, Remark 1.1.9 implies that \( \langle \dim P(i), \dim V \rangle = \dim V_i \) and thus, \( \langle \dim P(i), \dim S(j) \rangle = \delta_{ij} \) for any \( i, j \in Q_0 \).

1.2 Orbits in the representation space

In this section the set of isomorphism classes of representations in \( \text{rep} \, Q \) will be endowed with a geometric structure, identifying them with the orbits of a certain group action.

Notation 1.2.1: From now on, we consider representations of \( Q \) over the field \( k = \mathbb{C} \) of the complex numbers, so the field is omitted from the notation.

Let \( Q \) be a quiver, fix a dimension vector \( d \) for \( Q \) and fix complex vector spaces \((V_i)_{i \in Q_0}\) of dimension \((d_i)_{i \in Q_0}\). The space of representations of \( Q \) of dimension vector \( d \) is defined as the affine variety

\[ R_d(Q) = \bigoplus_{\alpha: i \rightarrow j} \text{Hom}_\mathbb{C}(V_i, V_j). \]

Fixing a base for each vector space \( V_i \), we have the identifications \( V_i \cong \mathbb{C}^{d_i} \), for all \( i \in Q_0 \), and we have that \( V_\alpha \) determines a linear map \( \mathbb{C}^{d_i} \rightarrow \mathbb{C}^{d_j} \), for each arrow \( (\alpha: i \rightarrow j) \in Q_1 \). Thus, any point \( v \in R_d(Q) \) defines a representation \( V = ((k_{ij}), (V_\alpha)_\alpha) \) of \( Q \) of dimension vector \( d = (d_i)_{i \in Q_0} \). Conversely, any representation \( V = ((V_i), (V_\alpha)_\alpha) \) of \( Q \) of dimension vector \( d \) defines a point \( v = (V_\alpha)_{\alpha \in Q_1} \) in \( R_d(Q) \), hence we can write \( V \in R_d(Q) \).

Consider now the reductive algebraic group

\[ G_d = \prod_{i \in Q_0} \text{GL}(V_i) \]

where \( \text{GL}(V_i) \) denotes the general linear group of invertible \( d_i \times d_i \) matrices with entries in \( \mathbb{C} \). Thus, by definition, we have

\[
\dim G_d - \dim R_d(Q) = \sum_{i \in Q_0} d_i^2 - \sum_{\alpha: i \rightarrow j} d_id_j = \langle d, d \rangle.
\]

(1.5)
There is a natural action of $G_d$ on the space $R_d(Q)$ by base change
\[ g \cdot v = (g_i)_{i \in Q_0} \cdot (V_\alpha)_{\alpha \in Q_1} = (g_j V_\alpha g_i^{-1})_{(\alpha : i \rightarrow j) \in Q_1}, \]
so that the $G_d$-orbits in $R_d(Q)$ are in bijection with the isomorphism classes of representations of $Q$ of dimension vector $\mathbf{d}$.

**Lemma 1.2.2:** Let $Q$ be a quiver with dimension vector $\mathbf{d}$ and let $V$ be a representation in $R_d(Q)$. Then the orbit $G_d \cdot V$ corresponds to the isomorphism class of $V$, i.e.
\[ G_d \cdot V = \{ W \in R_d(Q) \mid V \cong W \}. \]

**Proof.** Assume that $V = ((V_i)_i, (V_\alpha)_\alpha)$ and $W = ((W_i)_i, (W_\alpha)_\alpha)$ are representations in the same $G_d$-orbit. Then there exists $g = (g_i)_{i \in Q_0}$ such that $W = g \cdot V$, meaning that for each arrow $(\alpha : i \rightarrow j) \in Q_1$ the diagram
\[
\begin{array}{ccc}
V_i & \xrightarrow{V_\alpha} & V_j \\
\downarrow{g_i} & & \downarrow{g_j} \\
W_i & \xrightarrow{W_\alpha} & W_j
\end{array}
\]
commutes. Since $g_i$ is an isomorphism for each $i \in Q_0$, it follows that $g : V \rightarrow W$ is an isomorphism.

On the other hand, if $V \cong W$ is an isomorphism of representations of $Q$, then by definition there are isomorphisms $(g_i)_{i \in Q_0}$ such that $W_\alpha \circ g_i = g_j \circ V_\alpha$ for all $(\alpha : i \rightarrow j) \in Q_1$. Thus, we have that $g_i$ belongs to $GL(V_i)$ for each $i \in Q_0$ and therefore $W = g \cdot V$. \qed

Hence the problem of classifying representations of $Q$ of dimension vector $\mathbf{d}$ (see the **Classification Problem**) is translated to a geometric problem which will allow us to use algebro-geometric methods to approach it.

**Geometric formulation of the Classification Problem:** Classify all $G_d$-orbits in $R_d(Q)$.

Next, we collect some properties of $G_d$-orbits without proving them. Proofs and further details can be found, for instance, in [KJ16] or in [OV90].

Let $v = (V_\alpha)_{\alpha \in Q_1}$ be a point in $R_d(Q)$ and let $V$ be the corresponding representation. Then, since $G_d$ is a reductive algebraic group, there exists a natural isomorphism
\[ G_d \cdot V \cong G_d/(G_d)_v, \]
where $(G_d)_v = \{ g \in G_d \mid g \cdot v = v \}$ is a closed algebraic subgroup in $G_d$ called the stabilizer of $v$. 
1.3. GIT and moduli spaces for quiver representations

From now on, the representation space \( R_d(Q) \) is considered to be an algebraic variety endowed with the Zariski topology, so the notions below refer to the aforementioned topology.

- The orbit \( G_d \cdot V \) is open if and only if \( \text{Ext}^1_Q(V, V) = 0 \).
- The orbit \( G_d \cdot V \) is closed if and only if \( V \) is the semisimple representation of \( Q \) of dimension vector \( d \).

1.3 GIT and moduli spaces for quiver representations

We are interested in finding a space encoding all the parameters in the Geometric formulation of the Classification Problem. This means that we want to obtain an orbit space for the action of the algebraic group \( G_d \) on the affine algebraic variety \( R_d(Q) \). To start with, let us give a precise description of these spaces following [New78].

**Definition 1.3.1:** Let \( G \) be an algebraic group acting on a variety \( X \). An orbit space is a variety \( Y \) together with a morphism \( \pi : X \to Y \) such that

1. \( \pi \) is constant on the orbits of the action, i.e. \( \pi \) is \( G \)-invariant.
2. For any variety \( Z \) and any \( G \)-invariant morphism \( \phi : X \to Z \), there exists a unique morphism \( \psi : Y \to Z \) such that \( \phi = \psi \circ \pi \); that is, the following diagram is commutative:

\[ 
\begin{array}{ccc}
X & \xrightarrow{\pi} & Y \\
\downarrow \phi & & \downarrow \psi \\
Z & \xrightarrow{k} & Y
\end{array}
\]

3. The fibre \( \pi^{-1}(y) \) is a single orbit, up to isomorphism, for all \( y \in Y \).

A categorical quotient of \( G \) by \( X \) is a variety \( Y \) together with a morphism \( \pi \) satisfying only conditions 1 and 2.

As \( \pi \) is constant on the orbits of the action, it is also constant on the closure of the orbits. Hence, the existence of an orbit space implies that the \( G \)-action on \( X \) is closed, i.e. the orbits \( G \cdot x \) are closed for all \( x \in X \).

**Definition 1.3.2:** Let \( G \) and \( X \) be as before. A variety \( Y \) together with a morphism \( \pi : X \to Y \) is a good quotient for the action of \( G \) on \( X \) if

1. \( \pi \) is \( G \)-invariant.
2. \( \pi \) is surjective.
3. For every open subset \( U \subseteq Y \), the morphism \( \mathbb{C}[U] \to \mathbb{C}[\pi^{-1}(U)] \) is an isomorphism of \( \mathbb{C}[U] \) onto \( \mathbb{C}[\pi^{-1}(U)]^G \).
4. For every \( G \)-invariant closed subset \( V \subseteq X \), we have that \( \pi(V) \) is closed in \( Y \).
5. If $V_1$ and $V_2$ are disjoint $G$-invariant closed subsets, then $\pi(V_1) \cap \pi(V_2) = \emptyset$.

A geometric quotient is a good quotient which is also an orbit space. Any good quotient is a categorical quotient [New78, Proposition 3.11].

**Notation 1.3.3:** It is conventional to write $Y = X//G$ when $Y$ is a categorical quotient and $Y = X/G$ when $Y$ is a geometric quotient to emphasise that it is an orbit space.

The following is a criterion for determining when a variety (together with a morphism) is a quotient. It is due to Kraft and can be found in [Kra84].

**Theorem 1.3.4 (Quotient criterion):** Let $G$ be an algebraic group acting on a variety $X$ and $\pi: X \rightarrow Y$ a $G$-invariant map. If $Y$ is normal and $\pi$ surjective, and there is a dense subset $U \subset Y$ such that for all $y \in U$ the fiber $\pi^{-1}(y)$ consists of exactly one $G$-orbit, then $Y$ is a categorical quotient $X//G$.

We can relate geometric and categorical quotients for actions of algebraic groups on algebraic varieties as follows:

**Lemma 1.3.5:** Let $G, H$ be two algebraic groups and let $X$ be a $G \times H$-variety such that there exists a geometric quotient $\pi_H: X \rightarrow X/H$ of $X$ by $H$, and a categorical quotient $\pi_{G \times H}: X \rightarrow X/((G \times H)$ of $X$ by $G \times H$. Then $X/H$ is naturally a $G$-variety and the categorical quotient of $X/H$ by $G$ is isomorphic to $X/((G \times H)$.

**Proof.** A $G \times H$-action on $X$ induces actions of $G$ and $H$ on $X$ such that $g \cdot (h \cdot x) = h \cdot (g \cdot x)$ for all $g \in G$, $h \in H$ and $x \in X$. Hence, the multiplication map $m: G \times X \rightarrow X$ is $H$-equivariant and, since $\pi_H$ is $H$-invariant, the morphism $\pi_H \circ m: G \times X \rightarrow X/H$ is $H$-invariant. Thus, we obtain the commutative diagram

\[
\begin{array}{ccc}
G \times X & \xrightarrow{m} & X \\
\downarrow & & \downarrow \pi_H \\
G \times X/H = (G \times X)/H & \rightarrow & X/H
\end{array}
\]

giving the desired action of $G$ on $X/H$.

On the other hand, the morphism $\pi_{G \times H}$ is clearly $H$-invariant. Thus, by the universal property of categorical quotients, there exists a unique morphism $\pi_G: X/H \rightarrow X/((G \times H)$ such that $\pi_G \circ \pi_H = \pi_{G \times H}$. Because $\pi_{G \times H}$ is $G \times H$-invariant, $\pi_G$ is $G$-invariant. To verify that $\pi_G$ is indeed a categorical quotient, let $f: X/H \rightarrow Y$ be a $G$-invariant morphism. Then $f \circ \pi_H$ is $G \times H$-invariant and thus there exists a unique morphism $f': X/((G \times H) \rightarrow Y$ such that

$$f \circ \pi_H = f' \circ \pi_{G \times H} = f' \circ \pi_G \circ \pi_H.$$  

Since $\pi_H$ is a geometric quotient, it is surjective, and thus $f = f' \circ \pi_G$. Therefore, the categorical quotient of $X/H$ by $G$ is the variety $X/((G \times X)$ together with $\pi_G$. 

\[\square\]
1.3. GIT and moduli spaces for quiver representations

Going back to the $G_d$-action on $R_d(Q)$ we have that the set-theoretic quotient space

$$R_d(Q) / \text{set } G_d = \{ G_d \cdot V \mid V \in R_d(Q) \}$$

does not exist in general as an orbit space because $G_d$-actions on $R_d(Q)$ usually have some non-closed orbits. This can be seen in an explicit example in what follows.

**Example 1.3.6:** Let $Q$ be the quiver

\[
\begin{array}{c}
\bullet \\
1 \\
\end{array} \quad \begin{array}{c}
\bullet \\
2 \\
\end{array}
\]

with dimension vector $d = (1, 1)$. Consider the action of $G_d = \mathbb{C}^\times \times \mathbb{C}^\times$ on $R_d(Q) \cong \mathbb{C}$ by $(g_1, g_2)(v) = g_2 v g_1^{-1}$. In this case, there are exactly two orbits, namely $G_d \cdot 0 = \{0\}$ and $G_d \cdot 1 = \mathbb{C}^\times$ corresponding to the representations

\[
\begin{array}{cccc}
\circ & 0 & 1 \\
\mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C}
\end{array}
\]

respectively. These orbits are such that $G_d \cdot 0 = G_d \cdot 0$ and $G_d \cdot 1 = G_d \cdot 0 \cup G_d \cdot 1$. So the quotient $R_d(Q) / \text{set } G_d$ consists of two points, but only one of these points is closed, thus it is not an orbit space.

Hence the problem of finding spaces that parameterize orbits can be rephrased as follows.

**Moduli Problem:** Find a Zariski open subset $U \subset R_d(Q)$, as large as possible, together with an algebraic variety $X$ and a morphism $\pi : U \rightarrow X$ such that the fibres of $\pi$ are precisely the $G_d$-orbits in $U$.

In order to construct a set $U$ as described in the **Moduli Problem** we need to use techniques of Geometric Invariant Theory (GIT), which gives a way of constructing quotients by group actions in algebraic geometry. The standard reference for GIT is Mumford’s book [MFK94], however the books [Muk03] and [New78] also contain a detailed and clear explanation of this theory.

In the remainder part of this section, we will recall the main definitions and results of GIT in the setting of representation theory and apply them to the study of moduli spaces of quiver representations. These results are due to King [Kin94] and can be found in an explicit form in [KJ16] and in a well-summarized form in [Rei08].

### 1.3.1 The GIT quotient and moduli spaces of (semi)simple representations

As in Section 1.2 let $Q$ be a quiver with fixed dimension vector $d$ and let $G_d$ be a reductive group acting on $R_d(Q)$.
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Let $\mathbb{C}[R_d(Q)]$ be the ring of polynomial functions on $R_d(Q)$ with values in $\mathbb{C}$. Since $G_d$ is a linearly reductive group ($G_d$ is reductive and $\text{char}(\mathbb{C}) = 0$), by a theorem due to Hilbert [Muk03, Theorem 4.51], the ring

$$\mathbb{C}[R_d(Q)]^{G_d} := \{ f \in \mathbb{C}[R_d(Q)] \mid f(g \cdot v) = f(v) \text{ for all } g \in G_d \text{ and } v \in R_d(Q) \}$$

of $G_d$-invariant polynomial functions on $R_d(Q)$ is finitely generated and, thanks to Le Bruyn and Procesi, it is possible to give an explicit description of its generators. This result is formulated here and proved in [LBP90, Theorem 1].

Theorem 1.3.7 (Le Bruyn–Procesi): Let $Q$ be a quiver with dimension vector $d$. Then the ring of invariant functions for the action $G_d$ on $R_d(Q)$ is generated by the elements $\text{tr}_\sigma$ for $\sigma$ an oriented cycle in $Q$.

Given an oriented cycle $\sigma = (\alpha_n, \cdots, \alpha_1)$ in $Q$, the function $\text{tr}_\sigma : R_d(Q) \rightarrow \mathbb{C}$ assigns to a point $V = (V_\sigma)$ the trace $\text{tr}(V_\alpha \cdot \cdots \cdot V_1)$, which is in fact a $G_d$-invariant.

A particular consequence of Hilbert’s theorem already mentioned is that the set of maximal ideals $\text{Spec} \left( \mathbb{C}[R_d(Q)]^{G_d} \right)$ has the structure of an affine variety, and the embedding $\mathbb{C}[R_d(Q)]^{G_d} \hookrightarrow \mathbb{C}[R_d(Q)]$ induces the $G_d$-invariant quotient map

$$R_d(Q) \rightarrow \text{Spec} \left( \mathbb{C}[R_d(Q)]^{G_d} \right).$$

Our first attempt to approach the Moduli Problem is to define the GIT quotient of $R_d(Q)$ by the action $G_d$ as the good quotient

$$R_d(Q)/G_d := \text{Spec} \left( \mathbb{C}[R_d(Q)]^{G_d} \right).$$

A reference that provide a detailed proof about the spectrum of the ring of invariant functions being a good quotient is, for example, [Muk03].

Theorem 1.3.8: Consider the quotient $R_d(Q)/_{\text{top}} G_d$ in the category of topological spaces (usually non-Hausdorff), then there exists a natural surjective map [Muk03, Theorem 5.9]

$$R_d(Q)/_{\text{top}} G_d \twoheadrightarrow R_d(Q)/G_d$$

which identifies points in the GIT quotient with closure equivalence classes of $G_d$-orbits in $R_d(Q)$.

For a complete proof of this theorem, we refer to [Muk03 Theorem 5.9].

Meaning that, seen as a topological space, the GIT quotient $R_d(Q)/G_d = R_d(Q)/\sim$ is equipped with the equivalence relation given by

$$[v] \sim [w] \text{ if and only if } G_d \cdot V \cap G_d \cdot W \neq \emptyset. \quad (1.7)$$

where $V, W$ are representations defined by the points $v, w \in R_d(Q)$. 

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The GIT quotient, as a topological space, admits an explicit description. In fact, since $G_d$ is a reductive algebraic group, the closure of any $G_d$-orbit contains a unique closed orbit. Then, we obtain a bijection between $R_d(Q)/\sim$ and the set of all closed $G_d$-orbits in $R_d(Q)$ by associating an element $[v]$ in the GIT quotient to the unique closed orbit contained in $G_d \cdot V$. Furthermore, by the properties of orbits of Section 1.2, closed $G_d$-orbits in the representation space correspond to semisimple representations of $Q$ of dimension vector $d$, yielding the following result.

**Theorem 1.3.9:** There exists a bijection

$$R_d(Q)/G_d \longleftrightarrow \{ [V] \in \text{rep } Q \mid V \text{ is semisimple and dim } V = d \}.$$ 

Therefore, the GIT quotient parametrizes isomorphism classes of semisimple representations of $Q$ of dimension vector $d$. Thus, it will be called the moduli space of semisimple representations and denoted by $M_{d}^\text{simp}(Q)$.

We will now define a subset of $R_d(Q)$ for which the GIT quotient restricts to an orbit space.

**Definition 1.3.10:** A point $v \in R_d(Q)$ is called stable if its orbit $G_d \cdot V$ is closed and the stabilizer $(G_d)_v$ is zero-dimensional.

**Remark 1.3.11:** Note that the subgroup of scalar matrices $\Delta = \{(\lambda I_d)_{i \in Q_0} \mid \lambda \in \mathbb{C}^\times \} \subset G_d$ acts trivially on $R_d(Q)$ and therefore it fixes any point $v$, i.e. $\mathbb{C}^\times \subset (G_d)_v$ for all $v \in R_d(Q)$. Hence, in order to admit orbits with zero-dimensional stabilizers we have to consider the induced action of the factor group $\varphi G_d = G_d/\Delta$ on $R_d(Q)$. Furthermore, we note that the $G_d$-orbits and the $\varphi G_d$-orbits in $R_d(Q)$ coincide.

We denote the stable locus by $R_d(Q)^{\text{st}}$. It is an open (but possibly empty) subset of $R_d(Q)$ such that $R_d(Q)^{\text{st}}/G_d \cong R_d(Q)^{\text{st}}/G_d$ and the restriction $\pi: R_d(Q)^{\text{st}} \rightarrow R_d(Q)^{\text{st}}/G_d$, which is a geometric quotient, fulfills the condition of the Moduli Problem.

Points in $R_d(Q)^{\text{st}}$ correspond to simple representations of dimension vector $d$, hence the space $M_{d}^\text{simp}(Q) := R_d(Q)^{\text{st}}/G_d$ is a moduli space of simple representations. However, note that if $Q$ has no oriented cycles the only simple representations are $S(i)$, for $i \in Q_0$, as stated in Proposition 1.1.7. Therefore, the moduli space $R_d(Q)^{\text{st}}/G_d$ is either empty or a point. When it is non-empty, $G_d \cdot 0$ is the unique closed orbit in $R_d(Q)^{\text{st}}/G_d$, see [KJ16].

Although the GIT quotient of the stable locus is the first approximation to solve our problem, it is not an optimal solution since it loses so much information about the action by only parametrizing closed orbits. So we need to find a larger set to deal with this problem.
1.3.2 The relative GIT quotient

In order to recover some information about non-closed orbits, we generalize the construction of the GIT quotient. For this reason, we need the variety to be projective rather than affine.

Let \( \Theta = (\Theta_i)_{i \in \mathbb{Q}_0} \in (\mathbb{Z}Q_0)^* \) be a functional called the \textit{stability vector} and associate to it a character of \( G_d \)

\[ \chi_\Theta : G_d \rightarrow \mathbb{C}^\times \]

\[ (g_i)_i \mapsto \prod_{i \in \mathbb{Q}_0} \det(g_i)^{-\Theta_i} . \]

We denote by \( X(G_d) \) the abelian \textit{group of characters of} \( G_d \).

\textbf{Remark 1.3.12:} The minus sign differs from King’s convention [Kin94] to guarantee the inequality in our definition of \( \Theta \)-(semi)stable representations.

Since we are considering the induced action of the factor group as described in Remark 1.3.11 in order for the character to be well-defined we must have

\[ \chi_\Theta(\Delta) = \prod_{i \in \mathbb{Q}_0} \det(\lambda I_d)^{-\Theta_i} = 1 , \]

for any \( \lambda \in \mathbb{C}^\times \). Thus we require that \( \Theta(d) = \sum_{i \in \mathbb{Q}_0} \Theta_i d_i = 0 \) for all dimension vectors \( d \) of \( Q \).

For a character \( \chi_\Theta \) of \( G_d \), we define the space of \( \chi_\Theta \)-semiinvariant polynomial functions of degree \( n \) on \( R_d(Q) \) by

\[ \mathbb{C}[R_d(Q)]^{G_d, \chi_\Theta^n} = \{ f \in \mathbb{C}[R_d(Q)] | f(g \cdot v) = \chi_\Theta(g)^n f(v) \text{ for all } g \in G_d \text{ and } v \in R_d(Q) \} , \]

and we denote the naturally induced graded algebra by

\[ \mathbb{C}[R_d(Q)]^{G_d, \chi_\Theta}_n = \bigoplus_{n \geq 0} \mathbb{C}[R_d(Q)]^{G_d, \chi_\Theta^n} . \tag{1.8} \]

Given a character \( \chi_\Theta \), we lift the action of \( G_d \) on \( R_d(Q) \) to an action on \( R_d(Q) \times \mathbb{C} \) by

\[ g \cdot (v, \lambda) = (g \cdot v, \chi(g)^{-1} \lambda) . \]

The \( G_d \)-invariant polynomial functions \( \hat{f} \in \mathbb{C}[R_d(Q) \times \mathbb{C}^\times]^{G_d} \) under the lifted action are of the form

\[ \hat{f}(v, \lambda) = \sum_{n \geq 0} f_n(v) \lambda^n , \]

where \( f_n \in \mathbb{C}[R_d(Q)]^{G_d, \chi_\Theta^n} \). Then we have that the algebra (1.8) is precisely the algebra \( \mathbb{C}[R_d(Q) \times \mathbb{C}^\times]^{G_d} \).
Hence, it follows from Hilbert’s Theorem that it is finitely generated. For a more detailed exposition, refer to [Muk03, Section 6.1] or to [MFK94, Section 1.3].

Thus, we can define the relative GIT quotient as the corresponding quasi-projective variety

\[ R_d(Q)/(\mathcal{G}_d) \cong \text{Proj} \left( \mathbb{C}[R_d(Q)]^{\mathcal{G}_d}_{\mathcal{G}_d} \right), \]

which is also a good quotient [MFK94].

Since \( \mathbb{C}[R_d(Q)]^{\mathcal{G}_d}_{\mathcal{G}_d} = \mathbb{C}[R_d(Q)]^{\mathcal{G}_d_d} \), the embedding \( \mathbb{C}[R_d(Q)]^{\mathcal{G}_d} \rightarrow \mathbb{C}[R_d(Q)]^{\mathcal{G}_d_d} \) induces a projective morphism

\[ R_d(Q)/(\mathcal{G}_d) \rightarrow R_d(Q)/(\mathcal{G}_d) \tag{1.9} \]

The relative GIT quotient also has the structure of a topological space, and therefore it can be described in terms of equivalence classes of orbits, as described for the GIT quotient in Theorem 1.3.8. For this, we need to introduce the notion of \( \mathcal{G}_d \)-semistability in \( R_d(Q) \).

**Definition 1.3.13:** Let \( v \in R_d(Q) \) and let \( V \) be its corresponding representation.

- The point \( v \) is called \( \mathcal{G}_d \)-semistable if, for some \( n \geq 1 \), there exists a polynomial function \( f \in \mathbb{C}[R_d(Q)]^{\mathcal{G}_d} \) such that \( f(v) \neq 0 \). Denote by \( R_d(Q)^{\mathcal{G}_d}_{\mathcal{G}_d,\mathcal{G}_d} \) the Zariski open \( \mathcal{G}_d \)-semistable locus in \( R_d(Q) \).

- The point \( v \) is called \( \mathcal{G}_d \)-stable if it is \( \mathcal{G}_d \)-semistable, its orbit \( \mathcal{G}_d \cdot V \) is closed in \( R_d(Q)^{\mathcal{G}_d}_{\mathcal{G}_d,\mathcal{G}_d} \) and its stabilizer \( (\mathcal{G}_d)_v \) is zero-dimensional. Denote by \( R_d(Q)^{\mathcal{G}_d}_{\mathcal{G}_d,\mathcal{G}_d} \subseteq R_d(Q)^{\mathcal{G}_d}_{\mathcal{G}_d,\mathcal{G}_d} \) the Zariski open \( \mathcal{G}_d \)-stable locus in \( R_d(Q) \).

With this definition, we can upgrade Theorem 1.3.8 to the setting of the relative GIT quotient as follows.

**Theorem 1.3.14 ([Muk03]):** There exists a natural surjective map

\[ R_d(Q)^{\mathcal{G}_d}_{\mathcal{G}_d,\mathcal{G}_d} \rightarrow R_d(Q)/(\mathcal{G}_d) \tag{1.10} \]

which identifies points in the relative GIT quotient with closure-equivalence classes of \( \mathcal{G}_d \)-orbits in \( R_d(Q)^{\mathcal{G}_d}_{\mathcal{G}_d,\mathcal{G}_d} \).

Therefore, we obtain an analogue of the equivalence class \([1.7]\) for the topological space \( R_d(Q)/(\mathcal{G}_d) \cong R_d(Q)^{\mathcal{G}_d}_{\mathcal{G}_d,\mathcal{G}_d} / \sim \) in the following way:

\[ [v] \sim [w] \quad \text{if and only if} \quad \mathcal{G}_d \cdot V \cap \mathcal{G}_d \cdot W \cap R_d(Q)^{\mathcal{G}_d}_{\mathcal{G}_d,\mathcal{G}_d} \neq \emptyset, \]

for any \( \mathcal{G}_d \)-semistable points \( v, w \) of \( R_d(Q) \).

We can give an explicit description of the projective map \([1.9]\) in terms of orbits, by mapping an element \([v] \in R_d(Q)^{\mathcal{G}_d}_{\mathcal{G}_d,\mathcal{G}_d} / \sim \) to the unique closed orbit contained in \( \mathcal{G}_d \cdot V \), which lies in \( R_d(Q)/(\mathcal{G}_d) \). This yields an isomorphism

\[ R_d(Q)^{\mathcal{G}_d}_{\mathcal{G}_d,\mathcal{G}_d} / \sim \cong \{ \text{closed } \mathcal{G}_d \text{-orbits in } R_d(Q)^{\mathcal{G}_d}_{\mathcal{G}_d,\mathcal{G}_d} \}. \]
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As a consequence, the relative GIT quotient \( R_d(Q) / \chi_\Theta G_d \) parametrizes the closed \( G_d \)-orbits in \( R_d(Q)^{\chi_\Theta \text{-sst}} \) and the restriction \( \pi: R_d(Q)^{\chi_\Theta \text{-sst}} \to R_d(Q)^{\chi_\Theta \text{-sst}} / G_d \) of the map (1.10) has as fibres precisely the \( G_d \)-orbits in \( R_d(Q)^{\chi_\Theta \text{-sst}} \). This means that the relative GIT quotient restricts to a geometric quotient.

Hence, given a character \( \chi_\Theta \) of \( G_d \), the open locus of \( \chi_\Theta \)-stable points \( R_d(Q)^{\chi_\Theta \text{-st}} \) is a candidate for the Moduli Problem.

Remark 1.3.15: Notice that the case \( R_d(Q)^{\text{st}} \) of Section 1.3.1 corresponds to the trivial character \( \chi_\Theta = 1 \), for which \( R_d(Q)^{\chi_\Theta \text{-sst}} = R_d(Q) \). In this case, the morphism (1.9) is an isomorphism and therefore the relative GIT quotient coincides with the GIT quotient.

1.3.3 Moduli spaces of (semi)stable representations

King interpreted the notions of \( \chi_\Theta \)-(semi)stability in the language of representation theory, allowing us to define moduli spaces of stable and semistable quiver representations \([Kin94]\).

Definition 1.3.16: Let \( Q \) be a quiver, fix a dimension vector \( d \) and fix a stability \( \Theta \in (\mathbb{Z}Q_0)^* \) such that \( \Theta(d) = 0 \) as in the previous section. Let \( V \) be a representation of \( Q \) with \( \dim V = d \).

- \( V \) is called \( \Theta \)-semistable if \( \Theta(e) \leq 0 \) for all dimension vectors \( e \) of proper subrepresentations of \( V \).
- \( V \) is called \( \Theta \)-stable if \( \Theta(e) < 0 \) for all dimension vectors \( e \) of non-zero proper subrepresentations of \( V \).
- \( V \) is called \( \Theta \)-polystable if \( V \) is the direct sum of \( \Theta \)-stable representations.

These new concepts of \( \Theta \)-(semi)stability are indeed equivalent to the \( \chi_\Theta \)-(semi)stability conditions arising in the setting of relative GIT quotients, see \([Kin94\), Proposition 3.1].

Proposition 1.3.17: Let \( Q \) be a quiver with dimension vector \( d \) and fix the stability \( \Theta \in (\mathbb{Z}Q_0)^* \) such that \( \Theta(d) = 0 \). A point \( v \in R_d(Q) \) is \( \chi_\Theta \)-(semi)stable if and only if the corresponding representation \( V \) is \( \Theta \)-(semi)stable.

Thus, in light of Section 1.3.2 we will denote the corresponding \( \Theta \)-(semi)stable locus by

\[ R_d^{\Theta \text{-sst}}(Q) = R_d(Q)^{\chi_\Theta \text{-sst}} \quad \text{and} \quad R_d^{\Theta \text{-st}}(Q) = R_d(Q)^{\chi_\Theta \text{-st}}. \]

Example 1.3.18: For \( \Theta = 0 \) we have that any representation \( V \) corresponding to a point \( v \in R_d(Q) \) is \( \Theta \)-semistable as seen in Remark 1.3.15. In this case, the \( \Theta \)-stable representations are exactly the simple representations and the \( \Theta \)-polystable representations are the semisimple ones.
In [Kin94, Proposition 3.2], King proved that closed $G_d$-orbits in $R_d^{\Theta\text{-sst}}(Q)$ correspond to $\Theta$-polystable representations. Hence, from Section 1.3.2, it follows that the relative GIT quotient parametrizes isomorphism classes of $\Theta$-polystable representations of $Q$ of dimension vector $d$. It will be denoted by $M_d^{\Theta\text{-sst}}(Q)$ and called the moduli space of $\Theta$-semistable representations.

Moreover, the quotient variety $M_d^{\Theta\text{-st}}(Q) := R_d^{\Theta\text{-st}}(Q)/G_d$ parametrizes isomorphism classes of $\Theta$-stable representations of $Q$ of dimension vector $d$; that is, it is a moduli space of $\Theta$-stable representations.

**Remark 1.3.19:** From Example 1.3.18 we have that for $\Theta = 0$ the moduli space $M_d^{\Theta\text{-st}}(Q)$ corresponds to $\text{Spec} \left( \mathbb{C}[R_d(Q)]^{G_d} \right)$.

The following diagram synthesizes the geometric properties of (relative) GIT quotients exposed in this section, Section 1.3.1 and Section 1.3.2:

\[
\begin{array}{cccccc}
R_d^{\Theta\text{-st}}(Q) & \subset & R_d^{\Theta\text{-sst}}(Q) & \subset & R_d(Q) & \subset \n \nR_d^{\Theta\text{-st}}(Q)/G_d & \subset & R_d^{\Theta\text{-sst}}(Q)/\chi \otimes G_d & \longrightarrow & R_d(Q)/\chi \otimes G_d & \subset \n \nM_d^{\Theta\text{-st}}(Q) & \longrightarrow & \text{Proj} \left( \mathbb{C}[R_d(Q)]^{G_d} \right) & \longrightarrow & \text{Spec} \left( \mathbb{C}[R_d(Q)]^{G_d} \right) & \subset \n \nR_d^{\Theta\text{-sst}}(Q)/\sim & \longrightarrow & R_d(Q)/\sim & \subset \n \nM_d^{\Theta\text{-sst}}(Q) & \subset & M_d^{\text{simp}}(Q)
\end{array}
\]

- $R_d^{\Theta\text{-st}}(Q) \subset R_d^{\Theta\text{-sst}}(Q) \subset R_d(Q)$ is a chain of open inclusions.
- If non-empty, the variety $M_d^{\Theta\text{-st}}(Q)$ is smooth of dimension
  \[
  \dim M_d^{\Theta\text{-st}}(Q) = \dim R_d(Q) - \dim G_d + 1 \quad \text{(see (1.5))}
  \]
  \[
  = 1 - \langle d, d \rangle. \quad \text{(1.11)}
  \]
  as shown in [KJ16, Theorem 10.8].
Chapter 2

Ample stability for a dimension vector

This chapter will be devoted to introducing a slightly modified version of Θ-(semi)stability, which considers representations with arbitrary dimension vectors, and we will see some useful properties of this stability notion. Next, we will define what it means to be an amply stable dimension vector, and we will present a key ingredient for the development of this work: a sufficient, purely numerical, criterion for ample stability. Finally, we will exhibit a particular example for which the numerical ample stability condition is almost always fulfilled.

2.1 Algebraic properties of (semi)stability

Fixing a stability Θ, the Θ-(semi)stability condition defined in Section 1.3.3 only considers representations of fixed dimension vector d. Here, we will define a notion of (semi)stability for non-zero representations of Q of different dimension vectors. This new concept of stability characterizes the (semi)stable representations algebraically, and it coincides with the notion of Θ-(semi)stability when considering representations in Rd(Q). More details on this direction can be found in [Rei08, Section 4] and in [HdlP02].

As in the previous chapter, let Q be a quiver and fix a stability Θ : ZQ0 → Z. Consider the linear function dim : ZQ0 → Z given by dim d = ∑i di, which is called the total dimension of d, and define a slope function µ on (NQ0)* by

\[ \mu(d) := \frac{\Theta(d)}{\dim d} \in \mathbb{Q}. \]

Remark 2.1.1: More generally, the slope function can be defined by replacing the functional dim for any functional κ on (ZQ0)* such that κ(d) > 0 for all non-zero dimension vectors d, and modifying the stability Θ by a stability parameter aΘ + bκ, for positive integers a, b. That is, we can write µ in the form µ = aΘ + bκ.

Note that the slope of the zero representation is not defined.
Chapter 2. Ample stability for a dimension vector

**Notation 2.1.2:** We write \( \mu(V) := \mu(\dim V) \) for any non-zero representations \( V \) of \( Q \).

**Definition 2.1.3:** A representations \( V \) of \( Q \) (of arbitrary dimension vector) is called \( \mu \)-semistable if for all non-zero subrepresentations \( U \) of \( V \) we have

\[
\mu(U) \leq \mu(V).
\]

We call \( V \) a \( \mu \)-stable representation if \( \mu(U) < \mu(V) \) for all non-zero proper subrepresentations \( U \) of \( V \).

In the light of the previous definition, a representation is said to be (semi)stable if there exists a slope \( \mu \) such that it is a \( \mu \)-(semi)stable representation.

If we fix a dimension vector \( d \) of \( Q \) and consider a representation \( V \in R_d(Q) \) we have that \( \mu(V) = 0 \). Therefore, in this case, a representation in \( R_d(Q) \) is \( \Theta \)-(semi)stable if and only if it is \( \mu \)-(semi)stable.

For any non-zero representation \( V \in R_d(Q) \) and any subrepresentation \( U \in R_e(Q) \) there is a short exact sequence in \( \text{rep}Q \) given by

\[
0 \to U \to V \to W \to 0,
\]

where \( W \cong V/U \in R_{d-e}(Q) \) is called the factor representation of \( V \).

The next result gives us an alternative characterization of the slope (semi)stability in terms of factor representations.

**Lemma 2.1.4:** Given a short exact sequence \( 0 \to U \to V \to W \to 0 \) of non-zero representations of \( Q \) as above and a slope function \( \mu \), we have

\[
\mu(U) \leq \mu(V) \quad \text{if and only if} \quad \mu(U) \leq \mu(W) \quad \text{if and only if} \quad \mu(V) \leq \mu(W).
\]

**Proof.** Since \( \dim W = d - e \), its slope equals

\[
\mu(W) = \frac{\Theta(d) - \Theta(e)}{\dim d - \dim e}.
\]

An easy calculation shows that in fact

\[
\frac{\Theta(e)}{\dim e} \leq \frac{\Theta(d)}{\dim d} \iff \frac{\Theta(e)}{\dim e} \leq \frac{\Theta(d) - \Theta(e)}{\dim d - \dim e} \iff \frac{\Theta(d)}{\dim d} \leq \frac{\Theta(d) - \Theta(e)}{\dim d - \dim e}.
\]

The same holds for strict inequality, and assuming without loss of generality that \( \mu(U) = \min(\mu(U), \mu(W)) \) and that \( \mu(W) = \max(\mu(U), \mu(W)) \) we immediately obtain

\[
\min(\mu(U), \mu(W)) \leq \mu(V) \leq \max(\mu(U), \mu(W)). \tag{2.1}
\]

As a consequence of (2.1) and the lemma above, we get the next result.
Corollary 2.1.5: If $\mu(U) = \mu(V) = \mu(W)$, then $V$ is $\mu$-semistable if and only if $U$ and $W$ are $\mu$-semistable.

We denote by $\text{rep}^{\mu}Q$ the full subcategory of $\text{rep}Q$ whose objects are the semistable representations of $Q$ of slope $\mu \in \mathbb{Q}$. Corollary 2.1.5 implies that this category is closed under extensions.

The category $\text{rep}^{\mu}Q$ is abelian for all slopes $\mu$. Indeed, a morphism $f: U \rightarrow V$ in $\text{rep}^{\mu}Q$ induces the short exact sequences

$$
0 \rightarrow \text{Ker} f \rightarrow U \rightarrow \text{Im} f \rightarrow 0, \\
0 \rightarrow \text{Im} f \rightarrow V \rightarrow \text{Coker} f \rightarrow 0,
$$

from which we get that $\mu(\text{Im} f) = \mu(U) = \mu(V) = \mu(\text{Ker} f) = \mu(\text{Coker} f)$ and therefore the image, the kernel and the cokernel of $f$ are all semistable representations of slope $\mu$.

The definition of $\mu$-stability and an application of Schur’s Lemma leads to the following result, see [Rei08, Lemma 4.2].

Lemma 2.1.6: The simple objects in $\text{rep}^{\mu}Q$ correspond to the $\mu$-stable representations, which satisfy the following properties:

1. They are indecomposable.
2. Their endomorphism ring is trivial.
3. If $U$ and $V$ are $\mu$-stable representations with $\mu(U) \geq \mu(V)$, then a morphism $f: U \rightarrow V$ is either zero or an isomorphism.

From this lemma and Definition 1.3.16, it follows that the semisimple objects in the abelian category $\text{rep}^{\mu}Q$ are exactly the polystable representations of fixed slope $\mu$.

Lemma 2.1.7: Let $U, V$ be representations in $\text{rep}^{\mu}Q$. If $f: U \rightarrow V$ is a non-trivial homomorphism, then $\mu(U) \leq \mu(V)$. In addition, we have that $\mu > \nu$ implies $\text{Hom}(\text{rep}^{\mu}Q, \text{rep}^{\nu}Q) = 0$.

Proof. Let $f: U \rightarrow V$ be a non-zero morphism of $\mu$-semistable representations, and consider the short exact sequences as in (2.2). Because $V$ is a $\mu$-semistable representation, we have that $\mu(\text{Im} f) \leq \mu(V)$, and assuming that $\mu(U) > \mu(V)$, we obtain $\mu(\text{Im} f) < \mu(U)$. Hence, by (2.1), we get $\mu(U) < \mu(\text{Ker} f)$, contradicting the semistability of $U$.

On the other hand, consider a $\mu$-semistable representation $U$, a $\nu$-semistable representation $V$ and a non-zero morphism $f: U \rightarrow V$. As before, by stability, from (2.2) we get $\mu = \mu(U) \leq \mu(\text{Im} f) \leq \mu(V) = \nu$. \qed
Chapter 2. Ample stability for a dimension vector

Lemma 2.1.8 ([Rei08]): Let $Q$ be a quiver and let $\mu$ be a slope. Then for any representation $V$ of $Q$ there exists a unique subrepresentation $U$ such that

1. $\mu(W) \leq \mu(U)$ for all subrepresentations $W$ of $V$.

2. If $W$ is a proper subrepresentation of $V$ with $\mu(U) = \mu(W)$, then either $W = U$ or $\dim W < \dim U$.

Moreover, $U$ is $\mu$-semistable.

Remark 2.1.9: Clearly, such a subrepresentation is precisely $V$ when $V \in \text{rep}^\mu Q$.

Thus, each representation of $Q$ admits a unique subrepresentation of maximal slope, and it is of maximal dimension among all subrepresentations satisfying this property. This subrepresentation yields a uniquely defined filtration for each representation of $Q$ (see, for example, [Rei08, Lemma 4.7]).

Lemma: Given a slope function $\mu$, every representation $V$ of $Q$ has a unique filtration

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_{s-1} \subset V_s = V$$

such that the quotient representations $V_k/V_{k-1}$ satisfy the following conditions:

1. $V_k/V_{k-1}$ are $\mu$-semistable for $k \in \{1, \ldots, s\}$.

2. $\mu(V_1/V_0) > \mu(V_2/V_1) > \cdots > \mu(V_s/V_{s-1})$.

This filtration is called the Harder–Narasimhan (HN) filtration of $V$, and it will be denoted by $V_s$.

Remark 2.1.10: In the HN filtration of $V$, the factor representation $V_k/V_{k-1}$ is the maximal $\Theta^k$-semistable subrepresentation of $V/V_{k-1}$, with

$$\Theta^k = \Theta - \mu(d^k)$$

the stability associated to $d^k = \dim V_k/V_{k-1}$, for all $1 \leq k \leq s$.

The Harder–Narasimhan stratification

Using the uniqueness of the HN filtration, in [Rei03, Proposition 3.4], Reineke showed that there is a stratification by HN types, which is called the HN stratification. Here, the term stratification is used in a weak sense; that is, to refer to a finite decomposition of the variety of representations into locally closed and irreducible subvarieties.

A HN type is a tuple $d^* = (d^1, \ldots, d^s)$ such that $R_{d^k}^\text{sst}(Q) \neq \emptyset$ for all $k \in \{1, \ldots, s\}$ and such that $\mu(d^1) > \cdots > \mu(d^s)$. The weight of $d^*$ is the sum $\sum_{k=1}^s d^k$. 
2.2. Numerical ample stability

A HN filtration $V_*$ is of type $d^*$ if each $d^k$ is the dimension vector of the semistable representations $V_k/V_{k-1}$ as in Remark 2.1.10. The trivial HN type of $V$ is $d^* = (d)$ and the representations of trivial HN type are precisely the representations in $\mathrm{rep}^\mu Q$.

There is a decomposition of the variety of representations

$$R_d(Q) = \bigcup_{d^*} R_{d^*}^{HN}(Q)$$

(2.3)

where $R_{d^*}^{HN}(Q)$ are irreducible, locally closed subvarieties consisting of all representations with HN type $d^*$ and weight $d$.

$R_{d^*}^{HN}(Q)$ is called the HN stratum for the HN type $d^*$. In particular, the stratum corresponding to the trivial HN type $d^* = (d)$ is exactly $R_{d^*}^{sst}(Q)$.

We refer to the decomposition (2.3) as the HN stratification.

Proposition 2.1.11 (Rei03): The codimension of the HN stratum $R_{d^*}^{HN}(Q)$ in $R_d(Q)$ is given by

$$\text{codim}_{R_d(Q)} \left( R_{d^*}^{HN}(Q) \right) = - \sum_{1 \leq k < l \leq s} \langle d^k, d^l \rangle.$$ 

The Harder–Narasimhan stratification together with the codimension above yield a recursive criterion for the existence of semistable representations as follows:

Proposition 2.1.12: $R_{d^*}^{sst}(Q)$ is non-empty if and only if there are no non-trivial decomposition $d = d^1 + \ldots + d^s$ such that the following conditions hold:

- $R_{d^k}^{sst}(Q) \neq \emptyset$ for all $k \in \{1, \ldots, s\}$.
- $\mu(d^1) > \ldots > \mu(d^s)$.
- $\langle d^k, d^l \rangle = 0$ for all $1 \leq k < l \leq s$.

For more details on the proposition and for its proof, refer to Rei03 and to Rei08.

2.2 Numerical ample stability

In this section we introduce the notion of ample stability. In particular, we present the main notion of this thesis: a sufficient, but not necessary, numerical condition for ample stability. The material covered in this section can be found in RS17, where Reineke and Schröer defined this concept.

Let $Q$ be a quiver, let $d$ be a dimension vector for $Q$, and fix a stability $\Theta$ with associated slope function $\mu$.

The dimension vector $d$ is called $\Theta$-stable if there exists a $\Theta$-stable representation of $Q$ of dimension vector $d$; that is, if $R_{d^*}^{\Theta \text{-st}}(Q) \neq \emptyset$. 

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**Definition 2.2.1:** We say that $d$ is **θ-amply stable** if the unstable locus has codimension at least 2. This is,

$$\text{codim}_{R_d(Q)} \left( R_d(Q) \setminus R_d^{\Theta-\text{st}}(Q) \right) \geq 2.$$  

**Definition 2.2.2:** A θ-stable dimension vector $d$ is called **θ-numerically amply stable (θ-nas)** if we have $(e, f) \leq -2$, for all proper decompositions $d = e + f$ such that $\mu(e) \geq \mu(f)$.

We will proceed to show that, in fact, this is a sufficient numerical criterion for ample stability. One of the key notions used in the proof is the codimension of the Harder–Narasimhan strata of Proposition 2.1.11.

**Proposition 2.2.3:** If $d$ is θ-nas, then it is θ-amply stable.

**Proof.** We will consider

$$R_d(Q) \setminus R_d^{\Theta-\text{st}}(Q) = (R_d(Q) \setminus R_d^{\Theta-\text{st}}(Q)) \cup (R_d^{\Theta-\text{st}}(Q) \setminus R_d^{\Theta-\text{st}}(Q)),$$

and we will prove that $d$ is θ-amply stable by checking that the codimension of both subsets satisfy the desired inequality.

We first consider the case $R_d(Q) \setminus R_d^{\Theta-\text{st}}(Q)$. Assume that there is a proper decomposition $d = d_1 + \ldots + d_s$ such that it is a HN type, and such that the codimension of the HN stratum $R_d^{HN}(Q)$ equals 1, i.e. $\sum_{k<l} (d_k, d_l) = -1$. For any $1 \leq k < l \leq s$, we have that $R_d^{HN}(Q), R_d^{HN}(Q) \neq \emptyset$, we can thus choose semistable representations $U$ and $V$ of dimension vector $d_k$ and $d_l$, respectively. By assumption $\mu(U) > \mu(V)$, then $\text{Hom}(U, V) = 0$, because of Lemma 2.1.7. Therefore, $(d_k, d_l) \leq 0$ for all $k < l$. Then, there is exactly one pair $k_0 < l_0$ such that $(d_{k_0}, d_{l_0}) = -1$, and $(d_k, d_l) = 0$ for any other pair $k < l$. Writing $e = d_1 + \ldots + d_{k_0}$ and $f = d_{k_0+1} + \ldots + d_s$, we get a proper decomposition $d = e + f$ with $\mu(e) > \mu(f)$ such that $(e, f) = -1$, a contradiction to our hypothesis. We conclude that $\text{codim}_{R_d(Q)} \left( R_d(Q) \setminus R_d^{\Theta-\text{st}}(Q) \right) \geq 2$.

On the other hand, for $R_d^{\Theta-\text{st}}(Q) \setminus R_d^{\Theta-\text{st}}(Q)$, fix a proper decomposition $d = e + f$ where $\Theta(d) = \Theta(e) = \Theta(f)$. Moreover, consider the product of Grassmannians of complex vector spaces $W_i$ of dimension $d_i$ for all $i \in Q_0$ written as

$$\text{Gr}_e(d) = \prod_{i \in Q_0} \text{Gr}_{e_i}(W_i), \quad (2.4)$$

whose dimension equals $\sum_{i \in Q_0} e_i(d_i - e_i)$, and consider the closed subscheme $X_{e,f}$ of $R_d^{\Theta-\text{st}}(Q) \times \text{Gr}_e(d)$ given by

$$X_{e,f} := \left\{ ((V_{a_1}, (U_i)) \in R_d^{\Theta-\text{st}}(Q) \times \text{Gr}_e(d) \mid V_a(U_i) \subset U_j \forall (\alpha : i \to j) \in Q_1 \right\}.$$

It is equipped with the projections

$$\xymatrix{ X_{e,f} \ar[rr]^{p_1} & & R_d^{\Theta-\text{st}}(Q) \ar[rr]^{p_2} & & \text{Gr}_e(d).}$$
2.2. Numerical ample stability

The projection $p_2$ realizes $X_{e,f}$ as the total space of a homogeneous vector bundle of rank

$$r = \sum_{\alpha : \beta \to \gamma} (e_{\beta}e_{\gamma} + f_{\beta}f_{\gamma} + f_{\beta}e_{\gamma})$$

over $Gr_e(d)$, see [CIFR12, Section 2].

The image of the projection $p_1$ is given by all $\Theta$-semistable representations containing a subrepresentation of dimension vector $e$. Then, by definition, we have

$$R_d^{\Theta\text{-sst}}(Q)\setminus R_d^{\Theta\text{-st}}(Q) = \bigcup_{e,f} p_1(X_{e,f})$$

where

$$\text{codim}_{R_d^{\Theta\text{-sst}}(Q)}(p_1(X_{e,f})) \geq \dim R_d(Q) - \dim Gr_e(d) - r$$

$$= \sum_{\alpha : \beta \to \gamma} d_{\alpha}d_{\gamma} - \sum_{i \in Q_0} e_i(d_i - e_i) - \sum_{\alpha : \beta \to \gamma} (e_{\beta}e_{\gamma} + f_{\beta}f_{\gamma} + f_{\beta}e_{\gamma})$$

$$= -\langle e, f \rangle.$$

By assumption $\langle e, f \rangle \leq -2$ and we get $\text{codim}_{R_d^{\Theta\text{-sst}}(Q)}(R_d^{\Theta\text{-sst}}(Q)\setminus R_d^{\Theta\text{-st}}(Q)) \geq 2$, as desired.

The $\Theta$-nas condition will play an important role in the remainder of this work since, in most cases, dimension vectors that are $\Theta$-stable are already $\Theta$-nas. An example is the multiple loop quiver, for which the $\Theta$-nas condition always holds except in one particular case.

**Example 2.2.4:** Let $L_m$ be the $m$-loop quiver with one vertex and $m \geq 0$ loops

$$\bullet \bigcirc (m)$$

For a fixed integer $d \geq 1$ we consider the dimension vector $d = (d)$, and fix the trivial stability $\Theta = 0$.

In this case, all representations are $\Theta$-semistable, the $\Theta$-stable representations correspond precisely to the simple ones and the $\Theta$-polystable representations correspond to the semisimple ones, as seen in Example 1.3.18. Thus, $M_d^{\Theta\text{-st}}(L_m)$ equals the moduli space of $d$-dimensional simples, and $M_d^{\Theta\text{-sst}}(L_m)$ equals the moduli space of $d$-dimensional semisimples.

If $m = 0$ or $m = 1$, a representation $V$ of $L_m$ is simple if and only if $\dim V = (1)$ (calculations can be found in [BarLe]). Therefore, for $m = 0$ and $m = 1$, the only non-empty moduli spaces are $M_d^{\Theta\text{-st}}(L_0) \cong \{ \ast \}$ and $M_d^{\Theta\text{-st}}(L_1) \cong \mathbb{A}^1$. Refer to [RST17].

We can thus assume that $m \geq 2$ and that $d \geq 2$. Moreover, assume that $d = e + f = (e) + (f)$ is a decomposition with $e, f \geq 1$ such that it contradicts $\Theta$-nas, i.e. such that $\langle e, f \rangle = (1 - m)ef \geq -1$, which holds if and only if $m = 2$ and $e, f = 1$. Hence, $d$ is $\Theta$-nas if it is different to the case $m = 2$ and $d = 2$.
Now we will compute the moduli space of semistable representations of $L_2$ for the special case $d = (2)$.

Let $V$ be a representation of $L_2$ of dimension vector $(2)$. Choosing a basis for $V$ we obtain an isomorphic representation $(\mathbb{C}^2, (A, B))$ where $(A, B) \in M_2(\mathbb{C}) \times M_2(\mathbb{C})$. The group $G_{(2)}$ is just a copy of $\text{PGL}_2(\mathbb{C})$, which acts on the space of 2-tuples of $2 \times 2$-matrices via simultaneous conjugation.

Kraft and Procesi ([KP96, Section 2.4]) proved that the invariants of pairs of $2 \times 2$-matrices under simultaneous conjugation are generated by the algebraically independent functions

$$
\text{tr}(A), \text{tr}(A^2), \text{tr}(B), \text{tr}(B^2), \text{tr}(AB).
$$

(2.5)

We then have $M^{\Theta-\text{sst}}_{(2)}(L_2) \cong \mathbb{A}^5$ with coordinates corresponding to these invariants.

The representation defined by $A$ and $B$ is non-semisimple if and only if $A$ and $B$ are upper-triangular $2 \times 2$-matrices with repeated eigenvalues. So we can suppose that $A$ and $B$ have the form

$$
A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} c & d \\ 0 & c \end{pmatrix}.
$$

Thus, the five invariants read

$$
\text{tr}(A) = 2a, \ \text{tr}(A^2) = 2a^2, \ \text{tr}(B) = 2b, \ \text{tr}(B^2) = 2b^2, \ \text{tr}(AB) = 2ab,
$$

and they fulfill the relation

$$
\text{tr}(AB)^2 - \text{tr}(AB)\text{tr}(A)\text{tr}(B) = \text{tr}(A^2)\text{tr}(B^2) - \frac{1}{2} (\text{tr}(A^2)\text{tr}(B)^2 + \text{tr}(A)^2\text{tr}(B^2)).
$$

Therefore, $M^{\Theta-\text{sst}}_{(2)}(L_2)$ is isomorphic to the open subset of $\mathbb{A}^5$ with coordinates (2.5) such that

$$
\text{tr}(AB)^2 - \text{tr}(AB)\text{tr}(A)\text{tr}(B) \neq \text{tr}(A^2)\text{tr}(B^2) - \frac{1}{2} (\text{tr}(A^2)\text{tr}(B)^2 + \text{tr}(A)^2\text{tr}(B^2)).
$$
Chapter 3

Approaches to ample stability

The numerical ample stability condition is a powerful criterion that enables us to verify the ample stability condition combinatorially. In fact, in [RS17], their strategy was to check the nas condition for dimension vectors in general (for fixed $Q$ and $\Theta$). This strategy worked out for multiple loop and generalized Kronecker quivers. However, in general, for any dimension and stability vector it seems to be a very difficult problem.

So, the focus of this chapter is on a particular class of dimension vectors called fundamentals, and on the so-called canonical stability vector. Combining a result of Schofield with a result of Kac, we have that this class of dimension vectors is stable for this specific stability vector. Another justification for this approach is that the numerical ample stability condition for $\Theta$ canonical is considerably simplified by assuming that the dimension vector is fundamental. This makes the verification of $\Theta$-nas much easier, allowing us to provide a list of examples where we check if this condition is met.

3.1 Schur roots and the fundamental domain

In this section, we discuss results of Kac [Kac80, Kac82, Kac83] and Schofield [Sch92] about Schur roots and general representations. Here, we will also introduce the so-called fundamental domain and the canonical stability vector. Moreover, we will show that Schur roots can be characterized by general subdimension vectors, and that every fundamental dimension vector is a Schur root. Finally, we will see that these two facts yield the main result of this section: Fundamental dimension vectors are stable for the canonical stability vector.

Let $Q$ be a quiver and let $d$ be a dimension vector for $Q$. We denote by $\text{supp}(d)$ the support of $d$; it is the full subquiver of $Q$ with set of vertices $\text{supp}(d)_0 = \{i \in Q_0 \mid d_i \neq 0\}$.

For each loop-free vertex $i \in Q_0$ there is a reflection $s_i : \mathbb{Z}Q_0 \rightarrow \mathbb{Z}Q_0$ defined by

$$s_i(d) := d - (d,i)i,$$

where $(\cdot, \cdot)$ is the symmetrized Euler form introduced in (1.3). The reflection $s_i$ is known as the simple reflection at $i$. 

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Moreover, define the Weyl group of $Q$, denoted by $W(Q)$, as the subgroup of $GL(ZQ_0)$ generated by the reflections $s_i$ for $i \in Q_0$.

The coordinate vector $i \in NQ_0$ is called a simple root if there are no loops at vertex $i$. By $\Pi_Q$ we denote the set of all simple roots for $Q$.

**Definition 3.1.1:** The fundamental domain of $Q$ is the set

$$F_Q := \{ d \in NQ_0 \setminus \{0\} | \text{supp}(d) \text{ is connected and } (d,i) \leq 0 \text{ for all } i \in Q_0 \}. \quad (3.2)$$

The dimension vectors $d \in F_Q$ are called fundamental dimension vectors.

Note that, if there is a loop at $i \in Q_0$, then $(d,i) \leq 0$ is satisfied. It thus suffices to show that $d$ has connected support and that $(d,i) \leq 0$ for all $i \in \Pi_Q$, in order to check that $d$ belongs to $F_Q$.

The set of positive real roots for $Q$

$$\Delta^+_\text{re}(Q) := W(Q)\Pi_Q \cap NQ_0$$

is defined as the set of dimension vectors that arise from reflecting simple roots, and the set of positive imaginary roots for $Q$

$$\Delta^+_\text{im}(Q) := W(Q)F_Q$$

is the set of dimension vectors that arise from reflecting elements of the fundamental domain.

The set of positive roots is defined as the union $\Delta_+(Q) := \Delta^+_\text{re}(Q) \sqcup \Delta^+_\text{im}(Q)$, and it is also known as the positive root system associated to $Q$.

Given $e \in \Delta^+_\text{re}(Q)$, we write $s_e \in W(Q)$ for the reflection defined by

$$s_e(d) = d - (d,e)e.$$

Note that $s_e(d)$ is equal to $d - \sum_{i \in Q_0} (d,i)e_i$.

We have that $F_Q \subset \Delta_+(Q)$. Indeed, every dimension vector in $F_Q$ is a positive imaginary root, see [Kac80].

Additional details, and results concerning root systems, may be found in [Kac80] and [Kac83].

**Definition 3.1.2:** A representation $V$ of $Q$ is said to be a Schur representation if its endomorphism ring is trivial. A dimension vector $d$ is called a Schur root if there exists a Schur representation of dimension vector $d$. Every Schur representation must be indecomposable.

Next, we will present the notion of general representations, which was introduced by Schofield in [Sch92].
3.1. Schur roots and the fundamental domain

Let $d$ be a dimension vector for $Q$, and let $P$ be a property of representations. Following Schofield, we say that a \textit{general representation} of dimension vector $d$ has property $P$ if and only if there exists a Zariski-dense subset $U \subset R_d(Q)$ such that property $P$ holds for every representation $V = ((V_i)_i, (V_\alpha)_\alpha)$, with $(V_\alpha)_\alpha \in U$.

This means that if a property is independent of the point chosen in some Zariski-dense subset $U \subset R_d(Q)$, then this property is true for a general representation of dimension vector $d$.

For dimension vectors $d$ and $e$ of a quiver $Q$, we denote by $\text{ext}_Q(d, e)$ the minimal value of $\dim \text{Ext}^1_Q(V, W)$ where $V$ and $W$ are representations of dimension vectors $d$ and $e$, respectively. Similarly, we define $\text{hom}_Q(d, e)$ to be the minimal value of $\dim \text{Hom}_Q(V, W)$ for representations $V$ and $W$ of dimension vectors $d$ and $e$, respectively. These minimal values are attained in a Zariski-open subset of $R_d(Q) \times R_e(Q)$.

By Schofield, we have that if $((V_\alpha)_\alpha, (W_\alpha)_\alpha) \in R_d(Q) \times R_e(Q)$ is a general pair of representations, then $\dim \text{Ext}^1_Q(V, W) = \text{ext}_Q(d, e)$ and $\dim \text{Hom}_Q(V, W) = \text{hom}_Q(d, e)$. Thus, from (1.4), and from the definition of general representations, there is a Zariski-open subset $U_1 \times U_2 \subset R_d(Q) \times R_e(Q)$ such that

$$\langle d, e \rangle = \text{hom}_Q(d, e) - \text{ext}_Q(d, e),$$

for all $((V_\alpha)_\alpha, (W_\alpha)_\alpha) \in U_1 \times U_2$.

The following definition and result involving general representations are due to Kac. The proof of this result can be found in [Kac82].

\textbf{Definition 3.1.3:} Let $d$ be a dimension vector for $Q$. If $d = d^1 + \ldots + d^s$, and a general representation of dimension vector $d$ has a decomposition $V^1 \oplus \cdots \oplus V^s$ where each $V^k$ is an indecomposable representation with $\text{dim} V^k = d^k$, then we write

$$d = d^1 \oplus \ldots \oplus d^s$$

and call it the \textit{canonical decomposition} of $d$. This decomposition is unique up to ordering.

\textbf{Theorem 3.1.4:} The sum

$$d = d^1 \oplus \ldots \oplus d^s$$

is the canonical decomposition if and only if each $d^k$ is a Schur root and $\text{ext}_Q(d^k, d^l) = 0$ for all $k \neq l$.

In particular, as Kac showed in [Kac82, Proposition 1], a general representation of dimension vector $d$ is indecomposable if and only if $d$ is a Schur root.

For a detailed proof of the Theorem above, and for more details on the canonical decomposition, we also refer to [DW02] and to [DW17].

Now we will exhibit some basic results on general representations of quivers, which are due to Schofield.
Let \( \mathbf{e} \leq \mathbf{d} \) be a dimension vector such that \( e_i \leq d_i \) for all \( i \in Q_0 \). We say that \( \mathbf{e} \) is a \textit{general subdimension vector} of \( \mathbf{d} \) if a general representation of dimension vector \( \mathbf{d} \) has a subrepresentation of dimension vector \( \mathbf{e} \). A general subdimension vector \( \mathbf{e} \) of \( \mathbf{d} \) will be written as \( \mathbf{e} \rightarrow \mathbf{d} \).

As described in [BR22], the notion of general subdimension vectors translates geometrically into the following: consider the product of the representation space \( R_{\mathbf{d}}(Q) \) with the product of Grassmannians \( \text{Gr}_e(\mathbf{d}) \) as in (2.4), and consider the Zariski-closed subset \( \text{Gr}_e^Q(d) \subset R_{\mathbf{d}}(Q) \times \text{Gr}_e(\mathbf{d}) \) called the \textit{universal quiver Grassmannian} which is the variety

\[
\text{Gr}_e^Q(d) := \left\{ ((V_\alpha)_\alpha, (U_i)_i) \in R_{\mathbf{d}}(Q) \times \text{Gr}_e(\mathbf{d}) \mid V_\alpha(U_i) \subset U_j \forall (\alpha : i \rightarrow j) \in Q_1 \right\}.
\]

Then we have the induced projections

\[
\begin{array}{ccc}
R_{\mathbf{d}}(Q) & \xrightarrow{p_1} & \text{Gr}_e^Q(d) \\
p_2 \downarrow & & \downarrow \\
& & \text{Gr}_e(\mathbf{d}),
\end{array}
\]

where the map \( p_2 \) is a homogeneous vector bundle, which implies that the universal quiver Grassmannian \( \text{Gr}_e^Q(d) \) is a smooth and irreducible variety. On the other hand, the projection \( p_1 \) is proper, and therefore its image is a Zariski-closed subvariety of \( R_{\mathbf{d}}(Q) \); the details can be found in [CIFR12] or in [DW17, Section 10.12]. By definition of general subdimension vectors, \( \mathbf{e} \rightarrow \mathbf{d} \) is equivalent to \( p_1 : \text{Gr}_e^Q(d) \rightarrow R_{\mathbf{d}}(Q) \) having Zariski-dense image and, since \( p_1 \) has Zariski-closed image, this is the same as \( p_1 \) being surjective.

Schofield proved in [Sch92, Theorem 3.3] and in [Sch92, Theorem 5.4] that the condition of a dimension vector \( \mathbf{e} \) to be a general subdimension vector of \( \mathbf{d} \) as well as the value of \( \text{ext}_Q(\mathbf{d}, \mathbf{e}) \) can be determined combinatorially as follows:

\[
\text{ext}_Q(\mathbf{e}, \mathbf{d} - \mathbf{e}) = 0 \text{ if and only if } \mathbf{e} \rightarrow \mathbf{d}, \text{ and } \text{ext}_Q(\mathbf{d}, \mathbf{e}) = -\max\{\langle \mathbf{d}', \mathbf{e} \rangle \mid \mathbf{d}' \rightarrow \mathbf{d}\}.
\]

Before giving a characterization of Schur roots in terms of general subdimension vectors, we start by introducing the following stability condition.

**Definition 3.1.5:** For a given dimension vector \( \mathbf{d} \) of \( Q \), define the \textit{canonical stability} as the linear form

\[
\{ \mathbf{d}, \_ \} = \langle \mathbf{d}, \_ \rangle - \langle \_, \mathbf{d} \rangle.
\]

We denote by \( \Theta^\text{can} := \{ \mathbf{d}, \_ \} \) the canonical stability for \( \mathbf{d} \), and we observe that \( \{ \mathbf{d}, \mathbf{d} \} = 0 \) and that \( \{ \mathbf{d}, \mathbf{e} \} \) is nothing else than the antisymmetrized Euler form of \( Q \).

**Remark 3.1.6:** The canonical stability for \( \mathbf{d} \) is equivalently defined as the vector given by

\[
\Theta^\text{can} = \left( \sum_{i \rightarrow j} d_j - \sum_{j \rightarrow i} d_i \right)_{i \in Q_0}.
\]
Theorem 3.1.7: Let \( d \) be a dimension vector for \( Q \). Then \( d \) is a Schur root if and only if \( \{d, e\} < 0 \) for all \( e \hookrightarrow d \).

Proof. Assume that \( d \) is not a Schur root and that \( d = e \oplus f \) is a canonical decomposition. Then \( \text{ext}_Q(e, f) = 0 \) and \( \text{ext}_Q(f, e) = 0 \), and thus from (3.3) it follows that \( \langle e, f \rangle, \langle f, e \rangle \geq 0 \). Since
\[
\langle e, d \rangle - \langle d, e \rangle = \langle e, f \rangle - \langle f, e \rangle
\]
and
\[
\langle f, d \rangle - \langle d, f \rangle = \langle f, e \rangle - \langle e, f \rangle,
\]
we have either \( \{d, e\} \geq 0 \) or \( \{d, f\} \geq 0 \).

Conversely, if \( d \) is a Schur root and \( e \) is a non-zero dimension vector such that \( e \hookrightarrow d \). We write \( f = d - e \). Then, from (3.4), we have that \( \text{ext}_Q(e, f) = 0 \) and so \( \langle e, f \rangle \geq 0 \). Because \( d \) is a Schur root, this implies that \( \text{ext}_Q(f, e) > 0 \); otherwise we would have \( \text{ext}_Q(f, e) = \text{ext}_Q(e, f) = 0 \), and a general representation of dimension vector \( d \) would have a canonical decomposition \( d = e \oplus f \), yielding a contradiction to our assumption.

Moreover, since \( e \hookrightarrow d \), there exists a short exact sequence
\[
0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0,
\]
where \( U \) and \( W \) are representations of dimension vectors \( e \) and \( f \), respectively. In addition, assume that \( f : W \rightarrow U \) is a morphism, then we have a composition of morphisms
\[
V \rightarrow W \xrightarrow{f} U \xrightarrow{f} V
\]
which is a scalar multiple of the identity. Because the composition is not an isomorphism, we have \( f = 0 \) and hence \( \text{Hom}(W, U) = 0 \). It follows that \( \text{hom}_Q(f, e) = 0 \), and then from (3.3) we have \( \langle f, e \rangle < 0 \). Therefore, (3.6) gives us \( \{d, e\} < 0 \).

Note that every root in the fundamental domain \( F_Q \) is a Schur root; otherwise, if \( d \) is a dimension vector in \( F_Q \) that is not a Schur root, there exists a real Schur root \( e \) in the canonical decomposition of \( d \) such that \( s_e(d) < d \), see [Sch92, Theorem 6.2]. Since
\[
s_e(d) = d - \langle d, e \rangle e = d - \sum_{i \in Q_0} e_i (d, i) e,
\]
this implies that \( (d, i) > 0 \) at least for one vertex \( i \in Q_0 \). This contradicts the assumption of \( d \in F_Q \).

As a consequence of Theorem 3.1.7, for every Schur root \( d \) and every \( e \hookrightarrow d \) we have that
\[
\{d, e\} = \langle d, e \rangle - \langle e, d \rangle < 0 = \langle d, d \rangle - \langle d, d \rangle = \{d, d\}.
\]

Then, combining this fact with the statement that every root in \( F_Q \) is Schur, the next result immediately follows.

Corollary 3.1.8: If \( d \) is a fundamental dimension vector, then \( d \) is \( \Theta_{\text{can}} \)-stable.
3.2 Classifying fundamental dimension vectors

In view of Corollary 3.1.8 and to simplify the nas condition, in this section we will consider the class of fundamental dimension vectors and the canonical stability. This simplification helps us to verify “more easily” the nas condition, and leave a small class of exceptions, for example for the quiver with two vertices and $m$ arrows in one direction and $n$ arrows in the other direction, and for the quiver with three vertices and two arrows from the first and third vertices to the second vertex.

**Lemma 3.2.1:** For a dimension vector $d \in F_Q$, the following conditions are equivalent:

1. $d$ is $\Theta^\text{can}$-nas.
2. There is no decomposition $d = e + f$ such that $<e, f> = -1$ and $<f, e> \in \{-1, 0\}$.

This result makes it much easier to verify the $\Theta$-nas condition for the canonical stability. Before giving the proof, we need a key fact due to Kac (see [Kac82, Lemma 1]) which states the following:

**Lemma 3.2.2:** If $d$ belongs to the fundamental domain $F_Q$, then either $	ext{supp}(d)$ is an extended Dynkin quiver and $<d, d> = 0$, or for any decomposition $d = d^1 + \ldots + d^s$ with $s \geq 2$ we have

$$<d, d> \leq \sum_{1 \leq k \leq s} <d^k, d^k>.$$ 

**Proof.** Assume that $Q = \text{supp}(d)$, so $Q$ is connected, and let $d = d^1 + \ldots + d^s$ be a decomposition such that

$$<d, d> \geq \sum_{1 \leq k \leq s} <d^k, d^k>.$$ 

Since

$$(d, d) = \sum_{1 \leq k \leq s} (d - d^k, d^k) + \sum_{1 \leq k \leq s} (d^k, d^k),$$

it implies that

$$\sum_{1 \leq k \leq s} (d - d^k, d^k) \geq 0.$$ 

That is, $(d - d^k, d^k) \geq 0$ for some $k$. Thus, we write $e := d^k$, and we observe that $e \leq d$. Then the following identity holds, see [Kac80, Lemma 2.5]:

$$(d - e, e) = \sum_{i \in Q_0} \frac{e_i (d_i - e_i)}{d_i} (d, i) + \frac{1}{2} \sum_{i \neq j \in Q_0} d_i d_j \left(\frac{e_i}{d_i} - \frac{e_j}{d_j}\right)^2 (i, j).$$

By hypothesis $(d, i) \leq 0$ for all $i \in Q_0$, and $(i, j) \leq 0$ for all $i \neq j \in Q_0$. So both summands in the equality are non-positive, and it implies that $(d - e, e) \leq 0$; therefore $(d - e, e) = 0$. Hence, $e_i d_j = e_j d_i$ whenever $(i, j) < 0$; i.e. $d$ is a multiple of $e$. We must also have $(d, i) = 0$ for all $i \in Q_0$. This shows that $Q$ is an extended Dynkin quiver and $(d, d) = 0$. 

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3.2. Classifying fundamental dimension vectors

We note that, in the first case of this lemma, the moduli space $M_{d}^{\Theta-st}(Q)$ is one-dimensional, so it is isomorphic to either $\mathbb{A}^1$ or $\mathbb{P}^1$. Therefore, we are not interested in this case.

Proof of Lemma 3.2.1: Assume that $d \in F_Q$, and that $d = e + f$ is a decomposition such that $\{d, e\} \geq \{d, f\}$ (this is equivalent to the slope inequality in Definition 2.2.2 since $\{d, d\} = 0$) and such that $\langle e, f \rangle \geq -1$, that is, it contradicts $\Theta^{\text{can}}$-nas.

By the stability condition, we have

$$\langle d, e \rangle - \langle e, d \rangle \geq \langle d, f \rangle - \langle f, d \rangle,$$

and using $d = e + f$, this condition simplifies to

$$\langle f, e \rangle \geq \langle e, f \rangle \geq -1. \tag{3.7}$$

Furthermore, assuming that the second case of Lemma 3.2.2 holds, the decomposition $d = e + f$ gives us

$$\langle d, d \rangle < \langle e, e \rangle + \langle f, f \rangle;$$

thus we obtain

$$\langle e, f \rangle + \langle f, e \rangle < 0. \tag{3.8}$$

Combining (3.7) with condition (3.8) we only have two remaining cases, namely

$$\langle e, f \rangle = -1 \text{ and } \langle f, e \rangle = -1 \text{ or } 0. \tag{3.9}$$

The converse statement is trivial. □

Consider the $m$-arrow generalized Kronecker quiver $K_m$ with two vertices $i, j$ and $m \geq 1$ arrows from $i$ to $j$

$$i \xrightarrow{m} j.$$

In [RS17 Proposition 6.2], Reineke and Schröer proved that the $\Theta^{\text{can}}$-nas condition for $K_m$ is almost always fulfilled, with the exception of the dimension vector $d = (2, 2)$ for the quiver $K_3$. Their proof can be simplified using our new techniques, as we will see below.

**Proposition 3.2.3:** Let $d$ be a fundamental dimension vector for $K_m$. Then $d$ is $\Theta^{\text{can}}$-nas except for the case $m = 3$ and $d = (2, 2)$.

Proof. We can assume that $m \geq 3$, since in the case of $m = 1$ or $m = 2$, if non-empty, the moduli spaces of $\Theta^{\text{can}}$-stable representations are single points or a projective line (for $m = 2$ and $d = (1, 1)$).
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A dimension vector \( \mathbf{d} = (d_1, d_2) \) for \( K_m \) belongs to \( F_Q \) if \( 2d_1 \leq md_2 \) and \( 2d_2 \leq md_1 \). By symmetry, without loss of generality, we can also assume that \( d_1 \leq d_2 \). Let \( \mathbf{d} = \mathbf{e} + \mathbf{f} \) be a decomposition written as

\[
d_i = e_i + f_i, \quad e_i, f_i \geq 0 \text{ for } i \in \{1, 2\},
\]

such that we have

\[
\langle \mathbf{e}, \mathbf{f} \rangle = e_1f_1 + e_2f_2 - me_1f_2 = -1,
\]

\[
\langle \mathbf{f}, \mathbf{e} \rangle = e_1f_1 + e_2f_2 - me_2f_1 = -\varepsilon,
\]

for \( \varepsilon \in \{0, 1\} \). Taking the difference, we find

\[
\langle \mathbf{e}, \mathbf{f} \rangle - \langle \mathbf{f}, \mathbf{e} \rangle = m(e_2f_1 - e_1f_2) = \varepsilon - 1.
\]

Note that \( \varepsilon = 0 \) implies that \( m(e_2f_1 - e_1f_2) = -1 \) but, since \( m \geq 3 \) and \( e_i, f_i \in \mathbb{N} \) for \( i = \{1, 2\} \), this equality is not fulfilled. Consequently, \( \varepsilon = 1 \) and thus we obtain

\[e_1f_2 = e_2f_1.\]

Now write

\[
d_1 = ga \quad \text{and} \quad d_2 = gb,
\]

where \( g = \gcd(d_1, d_2) \) and \( a, b \) are coprime. Our assumption of \( \mathbf{d} \) being fundamental yields

\[
2a \leq mb, \quad 2b \leq ma \quad \text{and} \quad a \leq b.
\]

By adding \( e_1e_2 \) to both sides of \( e_1f_2 = e_2f_1 \), the coprimality of \( a \) and \( b \) gives us

\[
e_2 = hb \quad \text{and} \quad e_1 = ha,
\]

and similarly, by adding \( f_1f_2 \), we get

\[
f_2 = kb \quad \text{and} \quad f_1 = ka
\]

for \( h, k \geq 1 \) such that \( h + k = g \).

Hence, substituting these values into the initial condition, we find

\[
-1 = \langle \mathbf{e}, \mathbf{f} \rangle = hk(a^2 + b^2 - mab),
\]

which is only possible if \( h = 1 = k \) and \( a^2 + b^2 - mab = -1 \).

Note that the last equality can be rewritten as

\[
1 = mab - a^2 - b^2 = b(ma - 2b) + (b - a)(b + a).
\]

Since \( ma - 2b, b - a \geq 0 \), it is only possible to happen if \( a = 1 = b \) and \( m = 3 \).

This leads us to the exceptional case \( m = 3 \) and \( \mathbf{d} = (2, 2) \).

In the case of generalized Kronecker quivers, \( \mathbf{d} \) being a fundamental dimension vector can be assumed without loss of generality, since every moduli space is isomorphic to one with \( \mathbf{d} \) in the fundamental domain:

\[
M_{(d_1, d_2)}^{\Theta_{\text{st}}}(K_m) \cong M_{(d_1, md_1 - d_2)}^{\Theta_{\text{st}}}(K_m).
\]

This identification of moduli spaces was proved by Weist \[Wei13\] using reflection functors and duality, and it is indeed one of the reduction steps in the proof in \[RS17, Proposition 6.2\].
3.3 List of examples

Here we exhibit some interesting examples in which the numerical ample stability holds for dimension vectors in the fundamental domain and for $\Theta$ canonical. In particular, these examples illustrate Lemma 3.2.1. Also, we compute explicitly the moduli space for two special cases where the $\Theta$-nas condition does not hold.

3.3.1 A two vertex quiver

We first consider the quiver $K_{m,n}$ with two vertices $i, j$ and $m$ arrows from $i$ to $j$ and $n$ arrows from $j$ to $i$.

Given a dimension vector $\mathbf{d} = (d_i, d_j)$ for $K_{m,n}$, we assume that $\mathbf{d}$ belongs to the fundamental domain $F_Q$, which precisely means that

$$2d_1 \leq (m+n)d_2 \quad \text{and} \quad 2d_2 \leq (m+n)d_1.$$

We can assume without loss of generality that $d_1 \leq d_2$.

**Theorem 3.3.1:** $\mathbf{d}$ is $\Theta^\text{can}$-nas except in the following cases

- $m+n = 3$ and $\mathbf{d} = (2, 2)$,
- $m = n$ and $d_2 = md_1$.

**Proof.** Let $\mathbf{d} = \mathbf{e} + \mathbf{f} \in F_Q$ be a decomposition such that

$$d_i = e_i + f_i, \quad e_i, f_i \geq 0 \quad \text{for} \quad i \in \{1, 2\},$$

$$(\mathbf{e}, \mathbf{f}) = e_1(f_1 - mf_2) + e_2(f_2 - nf_1) = -1,$$

$$(\mathbf{f}, \mathbf{e}) = e_1(f_1 - nf_2) + e_2(f_2 - mf_1) = -\varepsilon$$

for $\varepsilon \in \{0, 1\}$, then $\langle \mathbf{e}, \mathbf{f} \rangle - \langle \mathbf{f}, \mathbf{e} \rangle$ gives us

$$(m-n)(e_2f_1 - e_1f_2) = \varepsilon - 1.$$

We start treating the case $\varepsilon = 1$: Here we have $(m-n)(e_2f_1 - e_1f_2) = 0$, then $m = n$ or $e_2f_1 = e_1f_2$.

First, we assume $e_2f_1 = e_1f_2$ and separate possible common divisors from $d_1$ and $d_2$ by writing

$$d_1 = ga \quad \text{and} \quad d_2 = gb,$$

where $g = \gcd(d_1, d_2)$ and $a, b$ are coprime. From the assumptions we get

$$2a \leq (m+n)b, \quad 2b \leq (m+n)a \quad \text{and} \quad a \leq b.$$
Let us now assume \( m \) and similarly, from \( f_1 f_2 \) we obtain
\[
f_2 = kb, \quad f_1 = ka
\]
for \( h, k \geq 1 \) such that \( h + k = g \). Putting all this together, we can then rewrite the initial condition as
\[
-1 = \langle e, f \rangle = hk(a^2 + b^2 - (m + n)ab).
\]
This happens if and only if \( h = 1 = k \) and \( a^2 + b^2 - (m + n)ab = -1 \).

From the last equality we thus have
\[
1 = (m + n)ab - a^2 - b^2 = b((m + n)a - 2b) + (b - a)(b + a),
\]
which is only possible when \( a = 1 = b \) and \( m + n = 3 \), since \((m + n)a - 2b, b - a \geq 0\).
This leads us to the exceptional case \( m + n = 3 \) and \( d = (2, 2) \).

Let us now assume \( m = n \), then from
\[
1 = e_1(mf_2 - f_1) + e_2(mf_1 - f_2)
\]
we can consider three different cases: either \( mf_2 - f_1, \ mf_1 - f_2 \geq 0 \); or \( mf_2 - f_1 \leq 0 \) and \( mf_1 - f_2 \geq 0 \); or \( mf_2 - f_1 \geq 0 \) and \( mf_1 - f_2 \leq 0 \).
First, we suppose that \( mf_2 - f_1, \ mf_1 - f_2 \geq 0 \). We thus find \( e_1 = 1 = mf_2 - f_1 \) and \( e_2 = 0 \), or \( e_2 = 1 = mf_1 - f_2 \) and \( e_1 = 0 \). Then, the conditions of \( d \) belonging to the fundamental domain are fulfilled only when \( m = 1 = n \) and \( d_1 = f_2 = d_2 \), or when \( d_2 = mf_1 = md_1 \), respectively. Bringing us back to an exceptional case.
If \( mf_2 - f_1 \leq 0 \) and \( mf_1 - f_2 \geq 0 \), we must restrict to \( mf_2 - f_1 \leq -1, \ mf_1 - f_2 \geq 1 \) and \( e_2 \geq 1 \); otherwise when \( mf_2 - f_1 = 0 \) we get that \( m \notin \mathbb{N} \), and when \( mf_1 - f_2 = 0 \) or \( e_2 = 0 \) we have that \( \langle e, f \rangle \neq -1 \). Moreover, the assumption \( f_2 \geq 0 \) yields \( f_1 \geq 1 \) and \( m \geq 1 \).

By substituting \( mf_1 - f_2 = 1 \) into \( mf_2 - f_1 \leq -1 \) we get
\[
(m^2 - 1)f_1 \leq m - 1,
\]
which happens exactly when \( m = 1 \). Thus, we obtain \( f_2 = f_1 - 1 \) and \( e_2 = e_1 + 1 \); the exceptional case \( m = n \) and \( d_1 = d_2 \).
We will now assume that \( mf_1 - f_2 = s \), where \( s \geq 2 \) and our aim is to prove that there are no dimension vectors in the fundamental domain in this case. If \( m = 1 \), the initial condition \( 1 = e_1(mf_2 - f_1) + e_2(mf_1 - f_2) \) reads
\[
1 = s(e_2 - e_1).
\]
This implies that \( s = 1 \) and \( e_2 = e_1 + 1 \), a contradiction to \( s \geq 2 \). So we have \( m \geq 2 \) and the initial condition can be rewritten as

\[
se_2 = 1 + e_1((1 - m^2)f_1 + ms).
\]

From \( mf_1 - s = f_2 \geq 0 \) and \( mf_2 - f_1 \leq -1 \) we find

\[
\frac{s}{m} \leq f_1 \leq \frac{ms - 1}{m^2 - 1}.
\]

Substituting the above condition into these inequalities we get

\[
\frac{1 + e_1}{s} \leq e_2 \leq \frac{1}{s} + \frac{1}{m} e_1.
\]

Since \( m \geq 2 \), we obtain

\[
e_2 \leq \frac{2 + se_1}{2s},
\]

and since \( s \geq 2 \) and \( e_1 \geq 1 \), we see that \( se_1 \geq 2 \), and thus

\[
e_2 \leq \frac{2 + se_1}{2s} \leq e_1.
\]

On the other hand, from the assumption \( mf_2 - f_1 \leq -1 \) we get \( f_2 < f_1 \). Therefore, we have that \( d_2 < d_1 \) which is a contradiction to the condition \( d_1 \leq d_2 \), or equivalently, to the hypothesis \( d \in F_Q \).

Lastly, we assume that \( mf_2 - f_1 \geq 0 \) and that \( mf_1 - f_2 \leq 0 \), then by a similar argument to the given in the case above, we have that \( e_1 \geq 1 \), \( mf_2 - f_1 \geq 1 \) and \( mf_1 - f_2 \leq -1 \). We also have \( f_2 \geq 1 \). We can write \( f_2 - mf_1 = s \) with \( s \geq 1 \) and thus, from the initial condition, it follows that

\[
e_2 = e_1((m^2 - 1)f_1 + ms) - 1.
\]

Since \( m = n \), the condition \( d \in F_Q \) is fulfilled if and only if the following holds:

\[
d_2 \leq md_1, \quad d_1 \leq md_2 \quad \text{and} \quad d_1 \leq d_2.
\]

Substituting \( e_2 \) and \( f_2 \) into the above inequalities we are left with

\[
(m^2 - 1)e_1f_1 + s^2 \leq 1,
\]

\[
m \leq (m^2 - 1)(me_1f_1 + s(e_1 + f_1)) + ms^2,
\]

\[
1 \leq (m^2 - 1)e_1f_1 + (m - 1)s(e_1 + f_1) + s^2,
\]

respectively.

Since \( s, m, e_1 \geq 1 \) and \( f_1 \geq 0 \), we conclude that the last two inequalities hold and that \( (m^2 - 1)e_1f_1 + s^2 \geq 1 \). Together with the first condition above, it implies

\[
(m^2 - 1)e_1f_1 + s^2 = 1,
\]
These cases translate to the exceptional case $m = n$ and $d_1 = d_2$.

Let us now deal with the case $\varepsilon = 0$: The condition $(m - n)(e_2f_1 - e_1f_2) = -1$ implies that $m - n = -1$ and $e_2f_1 - e_1f_2 = 1$, or $n - m = -1$ and $e_1f_2 - e_2f_1 = 1$.

It is enough to assume that $m - n = -1$ and $e_2f_1 - e_1f_2 = 1$, since the other case can be treated completely similar. Substituting these two equalities into

$$e_1(f_1 - nf_2) + e_2(f_2 - mf_1),$$

we see that the conditions $(e, f) = 1$ and $(f, e) = 0$ are equivalent to

$$1 = e_1(mf_2 - f_1) + e_2(nf_1 - f_2),$$

from which, as before, we can consider three cases: either $mf_2 - f_1, nf_1 - f_2 \geq 0$; or $mf_2 - f_1 \leq 0, nf_1 - f_2 \geq 0$; or $mf_2 - f_1 \geq 0$ and $nf_1 - f_2 \leq 0$. We will now verify that in none of these cases there are dimension vectors belonging to the fundamental domain.

We first assume that $mf_2 - f_1, nf_1 - f_2 \geq 0$. On the one hand, we find $e_1 = 1 = mf_2 - f_1$ and $nf_1 = f_2$; otherwise $f_2 = -1$ from $e_2f_1 - e_1f_2 = 1$. This leads us to

$$1 = mf_2 - f_1 = f_1(m(m + 1) - 1),$$

where the only possibility is $f_1 = 1$ and $m = 1$.

Thus we have that $n = 2, d_1 = 2$, and $d_2 = 5$ since $e_2f_1 - e_1f_2 = 1$. However, putting this together we get

$$2d_2 = 10 > 6 = (m + n)d_1,$$

a contradiction to the hypothesis $d \in F_Q$.

On the other hand, we obtain $e_2 = 1 = nf_1 - f_2$, and $e_1 = 0$ or $mf_2 = f_1$. It follows that $d_1 = 1$ and $d_2 = 1 + m$ or, by a similar argument to the case above, $n = m + 1 = 2, d_1 = 1$ and $d_2 = 2$, respectively.

But, since $2d_2 = 2(1 + m) > 1 + 2m = (m + n)d_1$ and $2d_2 = 2 > 6 = (m + n)d_1$, these are not fundamental dimension vectors.

Secondly, suppose that $mf_2 - f_1 \leq 0$ and $nf_1 - f_2 \geq 0$. In case $mf_2 - f_1 = 0$, we obtain $e_2 = 1 = nf_1 - f_2$ and thus, by substituting it into $e_2f_1 - e_1f_2 = 1$ and since $d_1 \leq d_2$, we are left with the dimension vector $d = (1, 2)$ and with $n = m + 1 = 2$. Note that $2d_2 \notin (m + n)d_1$; that is, $d$ is not fundamental. Hence, $mf_2 - f_1 \leq -1$.

In addition, we must restrict to $e_2, nf_1 - f_2 \geq 1$; otherwise $e_1(mf_2 - f_1) + e_2(nf_1 - f_2) = 1$ does not hold. Since $f_2 \geq 0$, it follows that $f_1 \geq 1$. Then, we can write $nf_1 - f_2 = s$, with $s \geq 1$. Observe that if $m = 0$, we find $f_2 = f_1 - s$ and

$$e_2 = \frac{1 + e_1f_1}{s},$$

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which follows from $1 = e_1(mf_2 - f_1) + e_2(nf_1 - f_2)$, and suppose for those $e_2$ and $f_2$ our assumption $e_2f_1 - e_1f_2 = 1$ is fulfilled. By substitution, it gives us

$$f_1 + e_1f_1^2 - se_1f_1 + s^2 = s,$$

where $f_1 \geq s$ yields $s + se_1f_1 - se_1f_1 + s^2 \leq s$. This leads us to $s^2 \leq 0$, a contradiction to $s \geq 1$. We thus have $m \geq 1$.

On the other hand, putting together $nf_1 - s = f_2 \geq 0$, $mf_2 - f_1 \leq -1$ and $m - n = -1$ we obtain

$$\frac{s}{m + 1} \leq f_1 \leq \frac{ms - 1}{m(m + 1) - 1},$$

and

$$se_2 = 1 + e_1((1 - m^2)f_1 + ms).$$

The above conditions yield

$$\frac{1 + e_1}{s} \leq e_2 \leq \frac{1}{s} + \frac{1}{m + 1}e_1,$$

which implies

$$e_2 \leq \frac{2 + e_1}{2} \leq e_1$$

for all $e_1 \geq 2$, since $m \geq 1$ and $s \geq 1$.

In case $e_1 = 1$, it follows that $e_2 \leq 1 + \frac{1}{2}$ which also implies $e_2 \leq 1 = e_1$, because $e_2 \in \mathbb{N}$.

We also have $f_2 < f_1$ from $mf_2 - f_1 \leq -1$. Hence, $d_2 < d_1$, contradicting the fact that $d_1 \leq d_2$. In conclusion, when $mf_2 - f_1 \leq -1$ and $nf_1 - f_2 \geq 1$ there are no dimension vectors $d$ belonging to the fundamental domain.

Finally, if $mf_2 - f_1 \geq 0$ and $nf_1 - f_2 \leq 0$, proceeding similarly to the above case we have that there are no dimension vectors in $FQ$ when $mf_2 - f_1 = 0$ and $nf_1 - f_2 = 0$. So we can assume $mf_2 - f_1 \geq 1$ and $nf_1 - f_2 \leq -1$. We will also assume $e_1$, $f_2$, $m \geq 1$ and, since $e_2f_1 - e_1f_2 = 1$, we have that $e_2$, $f_1 \geq 1$.

Thus, we can write $f_2 - nf_1 = s$ with $s \geq 1$ and so

$$e_2 = \frac{e_1((m(m + 1) - 1)f_1 + ms) - 1}{s}.$$

In addition, since $n = m + 1$, we have that

$$2d_2 \leq (2m + 1)d_1, \quad 2d_1 \leq (2m + 1)d_2 \text{ and } d_1 \leq d_2$$

if and only if $d = (d_1, d_2)$ is a fundamental dimension vector. More precisely, substituting $e_2$ and $f_2$ as given above, these inequalities are respectively equivalent to

$$[2(m(m + 1) - 1)f_1 - s|e_1 + 2s^2 \leq 2, \quad 2m + 1 \leq (2m + 1)(m(m + 1) - 1)e_1f_1 + s^2 + ((2m + 1)m - 2)se_1 + ((2m + 1)(m + 1) - 2)sf_1, \quad 1 \leq (m(m + 1) - 1)e_1f_1 + (m - 1)se_1 + msf_1 + s^2.$$
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Since $s$, $m$, $e_1$, $f_1 \geq 1$, the last two inequalities are in fact satisfied, and the first one is only fulfilled when $2(m(m + 1) - 1)f_1 - s \leq 0$.

Suppose there is a dimension vector $d = (d_1, d_2)$ which is fundamental. By the previous computations, this dimension vector satisfies

$$f_1 \leq \frac{s}{2(m(m + 1) - 1)}.$$

Then, we get

$$1 = e_2 f_1 - e_1 f_2 \leq e_2 \left( \frac{s}{2(m(m + 1) - 1)} \right) - e_1 f_2,$$

by substituting $e_2$ and $f_2$. This is possible if and only if

$$2(m(m + 1) - 1) + (m(2m + 1) - 2)se_1 + (m(m + 1) - 1)(2m + 1)e_1 f_1 \leq -1.$$

But note that

$$2(m(m + 1) - 1) \geq 2, \ (m(2m + 1) - 2)se_1 \geq 1 \ \text{and} \ (m(m + 1) - 1)(2m + 1)e_1 f_1 \geq 3,$$

which allows us to conclude that $d$ does not fulfill the condition $e_2 f_1 - e_1 f_2 = 1$. Hence, we conclude that no dimension vector is fundamental in this case. This completes the proof.

3.3.2 A three vertex quiver

Consider now the quiver $Q_{2,2}$ which has three vertices $i_1, i_2, i_3$, and two arrows from $i_1$ and from $i_3$ to $i_2$

\[
\begin{array}{c}
\bullet i_1 \\
\overrightarrow{\text{---}} & \overleftarrow{\text{---}} \\
\bullet i_2 & \bullet i_3
\end{array}
\]

A dimension vector $d = (d_1, d_2, d_3)$ for $Q_{2,2}$ is fundamental if

$$d_1 \leq d_2, \ d_3 \leq d_2 \ \text{and} \ d_2 \leq d_1 + d_3,$$

and if $d_2 \neq 0$ so that its support is connected.

**Theorem 3.3.2:** Let $d$ be a fundamental dimension vector for $Q_{2,2}$. Then $d$ is $\Theta^{\text{can}}$-nas if and only if it is not of the form $(1, a + 1, a)$, $(a, a + 1, 1)$ for some $a \geq 1$ or $(2, 2, 2)$.

**Proof.** Let $d$ be a dimension vector such that

$$0 \leq d_1, d_3 \leq d_2 \leq d_1 + d_3;$$

that is, $d \in F_Q$, and let us assume that

$$d_i = e_i + f_i, \ e_i, f_i \geq 0 \text{ for } i \in \{1, 2, 3\},$$

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\( \langle e, f \rangle = e_1 f_1 + e_2 f_2 + e_3 f_3 - 2f_2(e_1 + e_3) = -1, \)
\( \langle f, e \rangle = e_1 f_1 + e_2 f_2 + e_3 f_3 - 2e_2(f_1 + f_3) = -\varepsilon, \)
for \( \varepsilon \in \{0, 1\} \). Taking the difference of the two equalities, we find

\[
2e_2(f_1 + f_3) - 2(e_1 + e_3)f_2 = \varepsilon - 1.
\]

The left-hand side is even, so the right-hand side is also even, which implies that \( \varepsilon = 1 \).

Then we have

\[
2e_2(f_1 + f_3) - 2(e_1 + e_3)f_2 = 0,
\]

that is,

\[
e_2(f_1 + f_3) = (e_1 + e_3)f_2.
\]

We now separate possible common divisors from \( d_1 + d_3 \) and \( d_2 \). For this, we write

\[
d_2 = na \quad \text{and} \quad d_1 + d_3 = nb
\]

for \( n \geq 1 \) and \( a, b \) coprime. By the assumptions, we get that \( a \leq b \leq 2a \).

Adding \( e_2(e_1 + e_3) \) to both sides of \( e_2(f_1 + f_3) = (e_1 + e_3)f_2 \), we find

\[
nbe_2 = e_2(d_1 + d_3) = (e_1 + e_3)d_2 = na(e_1 + e_3).
\]

By coprimality of \( a \) and \( b \), we thus have

\[
e_2 = ka, \quad e_1 + e_3 = kb,
\]

and similarly we have

\[
f_2 = la, \quad f_1 + f_3 = lb,
\]

for \( k, l \geq 1 \) such that \( k + l = n \).

We can then rewrite the remaining identity as

\[
-1 = e_1 f_1 + e_2 f_2 + e_3 f_3 - 2(e_1 + e_3)f_2 = e_1 f_1 + e_3 f_3 + kla^2 - 2klab =
\]

\[
= e_1 f_1 + (kb - e_1)(lb - f_1) + kla(a - 2b) = 2e_1 f_1 + klb^2 - b(kf_1 + le_1) + kla(a - 2b),
\]

and writing \( x = e_1, y = f_1 \) we get

\[
2xy - b(lx + ky) = -1 - klb^2 - kla(a - 2b) = -1 - (a - b)^2 kl.
\]

We have

\[
x + y = e_1 + f_1 = d_1 \leq d_2 = (k + l)a
\]

and, since \( d_3 \leq d_2 \), we obtain

\[
x + y = e_1 + f_1 = d_1 \geq (k + l)b - d_2 = (k + l)(b - a).
\]
We have thus reformulated our setting as follows: We have \( a \) and \( b \) coprime such that \( a \leq b \leq 2a \), as well as \( k, l \geq 1 \), and \( x, y \geq 0 \) such that
\[
x \leq kb, \quad y \leq lb, \quad (k+l)(b-a) \leq x+y \leq (k+l)a \quad \text{and} \quad (b-a)^2kl+1 = b(lx+ky) - 2xy.
\]

We first treat two special cases:
If \( a = 1 \) and \( b = 1 \), the conditions above are equivalent to
\[
0 \leq x \leq k, \quad 0 \leq y \leq l \quad \text{and} \quad 1 = lx + ky - 2xy = (l - y)x + (k - x)y.
\]
This can only happen in the following four cases:

\[
\begin{align*}
k &= 1, & x &= 0, & y &= 1, \\
l &= 1, & x &= k - 1, & y &= 1, \\
l &= 1, & x &= 1, & y &= 0, \\
k &= 1, & x &= 1, & y &= l - 1.
\end{align*}
\]
These give us the exceptional cases \( d = (1, k + 1, k) \) or \( (k, k + 1, 1) \).

If \( a = 1 \) and \( b = 2 \), the above conditions are equal to
\[
0 \leq x \leq 2k, \quad 0 \leq y \leq 2l, \quad x + y = k + l \quad \text{and} \quad kl + 1 = 2(lx + ky) - 2xy.
\]
Substituting \( y = k + l - x \) into this, we find
\[
1 = 2(x - k)^2 + kl,
\]
which is only possible for \( k = 1, l = 1, x = 1 \). This gives us the exception case \( d = (2, 2, 2) \).

From now on we will assume that \( a \geq 2 \) and our aim is to arrive at a contradiction. The crucial step of the proof is to consider the value of \( x + y \) along the solutions of \((b-a)^2kl+1 = b(lx+ky) - 2xy\), in order to find out when the above conditions are fulfilled. This quadratic equation describes a hyperbola, and we want to find the two points on it where \( x + y \) is extremal. We will prove that they are outside the region given by the inequalities
\[
x \leq kb, \quad y \leq lb \quad \text{and} \quad (k+l)(b-a) \leq x+y \leq (k+l)a.
\]
For this, we will use the method of Lagrange multipliers. Given the Lagrange function
\[
L(x, y, \lambda) = (x+y) - \lambda(blx+bky - 2xy - (b-a)^2kl - 1),
\]
we can find the desired points by solving \( \text{grad} \ L(x, y, \lambda) = 0 \). This equation reads
\[
\begin{align*}
1 - bl\lambda + 2y\lambda &= 0, \\
1 - bk\lambda + 2x\lambda &= 0, \\
blx + bky - 2xy - (b-a)^2kl - 1 &= 0.
\end{align*}
\]
3.3. List of examples

From the first two equations we see that \( \lambda \neq 0 \), so we can rewrite

\[
x = \frac{b}{2} k - \frac{1}{2 \lambda}, \quad y = \frac{b}{2} l - \frac{1}{2 \lambda}.
\]

Substituting this into \( blx + bky - 2xy - (b - a)^2kl - 1 = 0 \), we find

\[
\frac{1}{\lambda^2} = b^2kl - 2(b - a)^2kl - 2.
\]

On the other hand, substituting the values of \( x \) and \( y \) into

\[
(k + l)(b - a) \leq x + y \leq (k + l)a,
\]

we find

\[
\left( \frac{b}{2} - a \right)(k + l) \leq \frac{1}{\lambda} \leq \left( a - \frac{b}{2} \right)(k + l),
\]

which, by squaring, is equivalent to

\[
\frac{1}{\lambda^2} \leq \left( a - \frac{b}{2} \right)^2 (k + l)^2.
\]

Moreover, substituting it into \( 0 \leq x \leq bk \) and \( 0 \leq y \leq bl \) we get

\[
\frac{1}{\lambda^2} \leq b^2k^2 \quad \text{and} \quad \frac{1}{\lambda^2} \leq b^2l^2.
\]

Putting this together, we find

\[
b^2kl - 2(b - a)^2kl - 2 \leq \left( a - \frac{b}{2} \right)^2 (k + l)^2, \quad b^2k^2, \quad b^2l^2.
\]

To solve this last inequality, we need to perform a further simplification. We note that the inequality is symmetric with respect to \( k \) and \( l \), so we can assume, without loss of generality, that \( k \leq l \). We can thus set \( p = l - k \), and we have \( k \geq 1 \) and \( p \geq 0 \). We can then omit the inequality

\[
b^2kl - 2(b - a)^2kl - 2 \leq b^2l^2.
\]

Since \( b \geq a \), we can write \( b = a + c \) for \( c \geq 0 \). Since \( a + c = b \leq 2a \), then \( c \leq a \) and thus we can write \( a = c + d \) for \( d \geq 0 \). Hence, we have \( a = c + d \) and \( b = 2c + d \). Our assumption \( a \geq 2 \) yields \( c, d \geq 1 \).

Substituting all this in the above inequality, we get

\[
-2c^2k^2 + (2c^2 + 4cd + d^2)kp \leq 2,
\]

\[
2c(c + 2d)k^2 + 2c(c + 2d)kp \geq \frac{d^2}{4} p^2 \leq 2.
\]

Solving for \( p \), we obtain

\[
0 \leq p \leq \frac{2c^2k^2 + 2}{(2c^2 + 4cd + d^2)k},
\]

\[
d^2p^2 - 8c(c + 2d)kp - (8c(c + 2d)k^2 - 8) \geq 0.
\]
Let us keep $k$ fixed and consider these conditions on $p$. The function 

$$f(p) = d^2p^2 - 8c(c + 2d)kp - (8c(c + 2d)k^2 - 8)$$

describes a parabola which assumes its minimum value at $p = 4c(c + 2d)/d^2 > 0$. The value of $f(p)$ at $p = 0$ is $8 - 8c(c + 2d)k^2 < 0$, and the value of $f(p)$ at

$$p = rac{2c^2k^2 + 2}{(2c^2 + 4cd + d^2)k}$$

can be written as

$$-\frac{4(2d^3k^2 + 10cd^2k^2 + 13c^2dk^2 + 4c^3k^2 + d)(2cd^2k^2 + 4c^3k^2 + 7c^2dk^2 - d)}{(2c^2 + 4cd + d^2)k^2}.$$  

Since $c, d \geq 1$, we see that $7c^2dk^2 - d > 0$, and thus all factors are positive, making the whole value negative.

By continuity of $f(p)$, we see that $f(p)$ is negative whenever

$$0 \leq p \leq \frac{2c^2k^2 + 2}{(2c^2 + 4cd + d^2)k},$$

a contradiction to $f(p) \geq 0$.

### 3.3.3 The m-subspace quiver

Let $S_m$ be the $m$-subspace quiver with vertices $i_1, \ldots, i_m, j$ and arrows $i_k \rightarrow j$ for $k \in \{1, \ldots, m\}$. Assuming $m \geq 2$, for a fixed $d \geq 2$, we consider the dimension vector

$$d = i_1 + \ldots + i_k + 2i_{k+1} + \ldots + 2i_m + dj,$$

and the canonical stability given by

$$\Theta^{\text{can}} = d \sum_k i_k - (2m - k)j.$$  

Assume now that the dimension vector $d$ of $S_m$ is $\Theta^{\text{can}}$-coprime in the following sense:

**Definition 3.3.3:** The dimension vector $d$ is called $\Theta^{\text{can}}$-coprime if $\{d, e\} \neq 0$ for all non-zero proper dimension vectors $e \leq d$.

Note that $d$ being $\Theta^{\text{can}}$-coprime implies that $d$ is indivisible, i.e. $\gcd(d_i \mid i \in Q_0) = 1$: Indeed, if $d$ is $\Theta^{\text{can}}$-coprime, and $k \in \mathbb{N}$ is a common divisor of $d_i$ for all $i \in Q_0$. Then, $\{d, \frac{1}{k}d\} = \{d, d\} = 0$ and thus by coprimality $k = 1$.

Then, $d$ being $\Theta^{\text{can}}$-coprime implies the coprimality of $d$ and $2m - k$. It is known by [FRS20] that the moduli space $M^{\text{st}}_{d}(S_m)$ is non-empty if and only if $2m - k > d$ and, in this case, it is of dimension

$$1 - 4m + 3k - d(d - (2m - k)).$$
3.3. List of examples

We will see when $\Theta^{\text{can}}$-coprime $\mathbf{d}$ is $\Theta^{\text{can}}$-nas: For this, consider $\mathbf{e} \leq \mathbf{d}$, which is given by

$$\mathbf{e} = \sum_s \mathbf{i}_s + \sum_t a_t \mathbf{i}_t + \mathbf{e}_j$$

for $S \subseteq \{1, \ldots, k\}$, $T \subseteq \{k+1, \ldots, m\}$, $0 \leq a_t \leq 2$ and $\mathbf{e} \leq \mathbf{d}$. We then get

$$\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle = \left( e - \left( |S| + \sum_t a_t \right) \right)(d - e) + |T'|,$$

where $T' = \{t \in T \mid a_t = 1\}$.

Moreover, since $\mathbf{d}$ is $\Theta^{\text{can}}$-coprime, for any $\mathbf{e} \leq \mathbf{d}$, we have that $\Theta(\mathbf{e}) \geq \Theta(\mathbf{d} - \mathbf{e})$ if and only if $\Theta(\mathbf{e}) > 0$, which holds if and only if $(|S| + \sum_t a_t)d > (2m - k)e$.

Therefore, we claim that $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle \leq 0$. Before showing this, we note that

- $2m - k > d > (2m - k)e/(|S| + \sum_t a_t)$ implies $e \leq (|S| + \sum_t a_t) - 1$.
- $T' \subseteq T \subseteq \{k+1, \ldots, m\}$ and thus $|T'| \leq m$.
- $2 \leq d \leq 2m - k - 1$ and $0 \leq k \leq m$ give us $2 \leq d \leq 2m - 1$.

When $e = (|S| + \sum_t a_t) - 1$, $|T'| = m$ and $d = 2m - 1$, we get $|S| = k = 0$ and $\sum_t a_t = |T'| = m$. Hence, in this case, we obtain $(e - (|S| + \sum_t a_t))(d - e) + |T'| = 0$ and, by maximality, this proves the claimed statement.

Observe that $|T'| = 0$ leads to

$$\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle < -\frac{(2m - k) - d}{d}e(d - e) < 0.$$

Assuming $|T'| \neq 0$, we will first treat the case $(e - (|S| + \sum_t a_t))(d - e) + |T'| = 0$ which happens if and only if

1.1. $e - (|S| + \sum_t a_t) = -1$ and $d - e = |T'|$ or,

1.2. $e - (|S| + \sum_t a_t) = -|T'|$ and $d - e = 1$ or,

1.3. $e - (|S| + \sum_t a_t) = -\alpha$ and $d - e = \beta$, with $\alpha, \beta \neq 1$ such that $\alpha \beta = |T'|$.

We will determine when all this holds in a combinatorial way. This is, we will study the possible dimension vectors and their proper decompositions that fulfill each one of the cases above, until we are able to obtain general formulas. Since we have

$$0 \leq |S| \leq k \leq m, \quad 1 \leq |T'| \leq m \quad \text{and} \quad 0 \leq \sum_t a_t \leq 2m,$$

we will do this by increasing $k$ and analyzing all the possibilities for $|T'|$, $\sum_t a_t$ and $|S|$.

1.1. $e - (|S| + \sum_t a_t) = -1$ and $d - e = |T'|$: 

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Note that $k \neq m$, otherwise $|T'| = 0 = \sum_t a_t$ and this contradicts our assumption. Hence, $k \leq m - 1$.

* If $k = m - 1$, then $|T'| = 1 = \sum_t a_t$, $e = |S|$ and $d = e+1$ (since $d \geq 2$ we have $|S| \neq 0$). Writing $|S| = m - r$, for $2 \leq r \leq m - 1$, and substituting the values given above, the inequality $(|S| + \sum_t a_t)d > (2m - k)e$ reads

$$(r - 1)^2 + r > (r - 1)m,$$

and this holds only when $r = 1$. Thus, for any $m \geq 2$, we obtain that $|S| = m - 1 = k$ and $d = e + 1 = m = 2m - k - 1$

fulfills 1.1.

* If $k = m - 2$, then $|T'| = 1$ or $|T'| = 2 = \sum_t a_t$. When $|T'| = 1$, either $\sum_t a_t = 1$ or $\sum_t a_t = 3$. Proceeding as above, we obtain that $|T'| = 1 = \sum_t a_t$ implies $(|S| + \sum_t a_t)d \neq (2m - k)e$ (in fact, rewriting $|S|$ as before, $(|S| + \sum_t a_t)d > (2m - k)e$ if and only if $m < 1 + \frac{1}{r}$ or, equivalently, $m \leq r$, a contradiction to $r \leq m - 1$).

Thus, we have $\sum_t a_t = 3$ if $|T'| = 1$, which implies that $e = |S| + 2$ and $d = |S| + 3$. Writing $|S| = m - r$, for $2 \leq r \leq m$, the condition $(|S| + \sum_t a_t)d > (2m - k)e$ is equivalent to

$$(r - 2)m > (r - 3)^2 + 2(r - 2),$$

and this is only fulfilled if $r = 2$. In this case, we obtain

$|S| = m - 2 = k$ and $d = e + 1 = m + 1 = 2m - k - 1$,

which satisfies 1.1.

Similarly, for $|T'| = 2 = \sum_t a_t$ we have that $e = |S| + 1$ and $d = |S| + 3$. Since $0 \leq |S| \leq k = m - 2$, by rewriting it, $(|S| + \sum_t a_t)d > (2m - k)e$ is equal to

$$(r - 2)m > (r - 2)(r - 3) + 2(r - 1),$$

where $2 \leq r \leq m$, and this holds only if $r = 2$, or $r = 3 = m$. Hence, in this case, 1.1. is fulfilled when

$|S| = m - 2 = k$ and $d = e + 2 = (m - 1) + 2 = m + 1 = 2m - k - 1$,

and also when

$m = 3, \; k = 1, \; |S| = 0$ and $d = e + 2 = 3 = 2m - k - 2$.

* If $k = m - 3$, then either $|T'| = 1$ or $|T'| = 2$ or $|T'| = 3 = \sum_t a_t$. When $|T'| = 1$, and $\sum_t a_t = 1$ or $\sum_t a_t = 3$ we get $(|S| + \sum_t a_t)d \neq (2m - k)e$. Moreover, if $|T'| = 2$ and $\sum_t a_t = 2$ we also have $(|S| + \sum_t a_t)d \neq (2m - k)e$. 

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We thus obtain $\sum t a_t = 5$ when $|T'| = 1$, and $\sum t a_t = 4$ when $|T'| = 2$. In these cases, 1.1. is satisfied if

$$|S| = m - 3 = k$$

and

$$|S| = m - 3 = k$$

respectively.

In a similar way, for $|T'| = 3 = \sum t a_t$ we get that 1.1. is fulfilled if $|S| = m - 3 = k$ and $d = e + 3 = (m - 1) + 3 = m + 2 = 2m - k - 1$, and also if $m = 4, k = 1, |S| = 0$ and $d = e + 3 = 5 = 2m - k - 2$.

If we continue the same process, increasing $k$, we can deduce from case 1.1. that $d$ is not $Θ^\text{can}$-nas if:

- $0 \leq k \leq m - 1$ and $d = 2m - k - 1$, for all $m \geq 2$ (with decompositions given by: $|T'| = n \neq 0$, with $n \leq m$, $e = d - n$, $|S| = k$ and $a_t = 2$, for all $t \notin T'$).

- $k = 1$ and $d = 2m - k - 2 = 2m - 3$, for all $m \geq 3$ (with decompositions as follows: $|T'| = m - 1 = \sum t a_t$, $e = |T'| - 1 = m - 2$ and $|S| = 0$).

1.2. $e - (|S| + \sum t a_t) = -|T'|$ and $d - e = 1$: 

As in case 1.1., since $|T'| \neq 0$, we have that $0 \leq k \leq m - 1$. Moreover, when $|T'| = 1$ and $a_t = 2$ for all $t \notin T'$, we obtain that $d$ is not $Θ^\text{can}$-nas if $0 \leq k \leq m - 1$ and $d = 2m - k - 1$ for all $m \geq 2$ (with decomposition: $|T'| = 1, e = d - 1, |S| = k$ and $a_t = 2$, for all $t \notin T'$).

This was already obtained in case 1.1. and thus, we assume that $|T'| \neq 1$.

Furthermore, in the same way as before, if $k = m - r'$, with $2 \leq r \leq m$, we have $(|S| + \sum t a_t) d \neq (2m - k) e$ for all $|T'| \neq r'$. It then remains $|T'| = r' = \sum t a_t$, and doing the exact same calculations as in case 1.1., it leads us to deduce that $d$ is not $Θ^\text{can}$-nas if:

- $k = 1$ and $d = 2$, for all $m \geq 3$ (with decomposition: $|T'| = m - 1 = \sum t a_t$, $e = d - 1 = 1$ and $|S| = 1 = k$).

For example, assuming $k = m - 3$ and $|T'| = 3 = \sum t a_t$ we have that $e = |S|$ and $d = |S| + 1$. Since $1 \leq |S| \leq k = m - 3$, we can also assume $m \geq 4$. Then, writing $|S| = m - r$ for $3 \leq r \leq m - 1$, $(|S| + \sum t a_t) d > (2m - k) e$ reads

$$r^2 - r + 3 > (r - 1)m,$$

which is fulfilled only if $r = 3$ and $m = 4$. In this case, 1.2. holds when $|S| = 1 = k$ and $d = e + 1 = 2$.

1.3. $e - (|S| + \sum t a_t) = -α$ and $d - e = β$, with $α, β \geq 1$ integers such that $α \cdot β = |T'|$: 

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Assume that there are $\alpha, \beta > 1$ fulfilling the conditions above. Then we have $0 \leq k \leq m-3$ (otherwise, if $k = m - 1$ or $m - 2$, we get a contradiction to $\alpha, \beta \neq 1$).

Writing $k = m - r'$ for $3 \leq r' \leq m$ integer, the condition $|S| + \sum t a_t d > (2m - k)e$ equals to $|S| + \sum t a_t d > (2m - k)e$ which, after renaming $x = |S| + \sum t a_t$, reads

$$x^2 - (\alpha - \beta + (m + r'))x + (m + r')\alpha > 0,$$

and $x \neq 0$ (otherwise $e < 0$).

However, there are no integers $m \geq 3$, $3 \leq r' \leq m$, $x \geq 1$ and $\alpha, \beta > 1$ fulfilling 1.3., such that the above inequality holds.

Let us now treat the second case: $(e - (|S| + \sum t a_t))(d - e) + |T'| = -1$ holds if and only if

2.1. $e - (|S| + \sum t a_t) = -1$ and $d - e = 1 + |T'|$ or,

2.2. $e - (|S| + \sum t a_t) = -(1 + |T'|)$ and $d - e = 1$ or,

2.3. $e - (|S| + \sum t a_t) = -\alpha$ and $d - e = \beta$, with $\alpha, \beta \neq 1$ such that $\alpha \cdot \beta = 1 + |T'|$.

The same calculations performed before (for $k \leq m$) lead us to deduce that, in this case, $d$ is not $\Theta^\text{can-nas}$ if:

2.1. $e - (|S| + \sum t a_t) = -1$ and $d - e = 1 + |T'|$:

- $k = m$ and $d = 2m - k - 1$, for all $m \geq 3$ (with decomposition: $|T'| = 0 = \sum t a_t$, $e = d - 1$ and $|S| = m - 1 = k - 1$),
- $1 \leq k \leq m - 1$ and $d = 2m - k - 1$, for all $m \geq 2$ (with decomposition: $|T'| = n \geq 0$, $e = d - n - 1$, $|S| = k - 1$ and $a_t = 2$, for all $t \notin T'$),
- $k$ is odd such that $1 \leq k \leq m - 1$ and $d = 2$, for all $m \geq 3$ (with decomposition: $|T'| = 1 = \sum t a_t$, $e = 0$, $|S| = 0$ and $a_t = 0$, for all $t \notin T'$),
- $k = 1$ and $d = 2m - k - 2 = 2m - 3$, for all $m \geq 3$ (with decomposition: $|T'| = m - 2 = \sum t a_t$, $e = |T'|$, $|S| = 1 = k$ and $a_t = 0$, for all $t \notin T'$),
- $k$ is odd such that $3 \leq k \leq m - 1$ and $d = 2m - k - 2$, for all $m \geq 4$ (with decomposition: $|T'| = m - k = \sum t a_t$, $e = m - 3$ and $|S| = k - 2$),
- $m = 5$, $k = 3$ and $d = 4$ (with decomposition: $|T'| = 2 = \sum t a_t$, $e = 1$ and $|S| = 0$),
- $m = 4$, $k = 1$ and $d = 4$ (with decomposition: $|T'| = 2 = \sum t a_t$, $e = 1$ and $|S| = 0$).

2.2. $e - (|S| + \sum t a_t) = -(1 + |T'|)$ and $d - e = 1$:

- $1 \leq k \leq m$ and $d = 2m - k - 1$, for all $m \geq 3$ (with decomposition: $|T'| = 0$, $e = d - 1$, $|S| = k - 1$ and $a_t = 2$, for all $t \notin T'$),

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k is odd such that $1 \leq k \leq m - 1$ and $d = 2m - k - 2$, for all $m \geq 3$ (with decomposition: $|T'| = 1$, $e = d - 1$, $|S| = k$ and $a_t = 2$, for all $t \not\in T'$),

• $k = 1$ and $d = 2$, for all $m \geq 3$ (with decomposition: $|T'| = m - k - 1$, $e = 1$, $|S| = 0 = k - 1$ and $a_t = 2$, for all $t \not\in T'$),

• $k = 3$ and $d = 2$, for all $m \geq 4$ (with decomposition: $|T'| = m - k = \sum_t a_t$, $e = 1$ and $|S| = 2 = k - 1$),

• $m = 4$, $k = 1$ and $d = 3$ (with decomposition: $|T'| = 2$, $e = 2$, $|S| = 1 = k$ and $a_t = 2$, for all $t \not\in T'$).

2.3. $e - (|S| + \sum_t a_t) = -\alpha$ and $d - e = \beta$, with $\alpha, \beta \neq 1$ such that $\alpha \cdot \beta = 1 + |T'|$:

• $m = 4$, $k = 1$ and $d = 3$ (with decomposition: $|T'| = 3 = \sum_t a_t$, $e = 1$ and $|S| = 0$),

• $m = 4$, $k = 1$ and $d = 4$ (with decomposition: $|T'| = 3 = \sum_t a_t$, $e = 2$ and $|S| = 1 = k$),

• $m = 4$, $k = 0$ and $d = \{3, 5\}$ (with decomposition: $|T'| = 3$, $e = d - 2$, $|S| = 0 = k$ and $\sum_t a_t = d$),

• $m = 5$, $k = 2$ and $d = \{3, 5\}$ (with decomposition: $|T'| = 3 = \sum_t a_t$, $e = d - 2$ and $|S| = d - 3$).

We summarize all the findings in the following result.

**Theorem 3.3.4:** Let $d = i_1 + \ldots + i_k + 2i_{k+1} + \ldots + 2i_m + d_j$ be a $\Theta^\text{can}$-coprime dimension vector for the $m$-subspace quiver $S_m$. Then $d$ is $\Theta^\text{can}$-nas except in the following cases:

• $0 \leq k \leq m$ and $d = 2m - k - 1$, for any $m \geq 3$,

• $k$ odd such that $1 \leq k \leq m - 1$ and $d = 2m - k - 2$, for any $m \geq 3$,

• $k$ odd such that $1 \leq k \leq m - 1$ and $d = 2$, for any $m \geq 2$,

• $m = 4$, $0 \leq k \leq 1$ and $d = 2m - k - 3$,

• $m = 5$, $2 \leq k \leq 3$ and $d = 2m - k - 3$,

• $m = 2$, $k = 0$ and $d = 3$,

• $m = 4$, $0 \leq k \leq 1$ and $d = 3$,

• $m = 5$, $k = 2$ and $d = 3$.

Since for none of the exceptional cases in Theorem 3.3.4 we have that $(d, i) \leq 0$ for all $i \in Q_0$, we immediately arrive at the following corollary:

**Corollary 3.3.5:** Let $d$ be as in Theorem 3.3.4, and let it also be a fundamental dimension vector for the $m$-subspace quiver $S_m$. Then $d$ is $\Theta^\text{can-nas}$. 

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Example 3.3.6: Let $S_2$ be the 2-subspace quiver and let $d$ be a dimension vector as in the Theorem 3.3.4.

* If $k = m = 2$, then $2m - k = 2$ and $d \leq 2m - k - 1 = 1$, a contradiction to our assumption $d \geq 2$.

* If $k = m - 1 = 1$, then $2m - k = 3$. This implies that $d = 2 = 2m - k - 1$. Therefore, we obtain the following dimension vector

$$d = \begin{array}{c} 2 \\ 1 \end{array} \begin{array}{c} 2 \end{array}.$$

When $|T'| = 0$ we have $\sum a_t = 0$ or $\sum a_t = 2$. Moreover, note that $|S| = 0$ implies $\sum a_t \neq 0$; otherwise from $(|S| + \sum a_t)d > (2m - k)e$ we would get $e < 0$. Hence, from $|S| = 0$ and $\sum a_t = 2$ we obtain that $e = 0$ or 1. For $e = 0$ we get $\langle e, d - e \rangle = (e - 2)(d - e) = -4$, and for $e = 1$ we have $e \leq d$ as follows:

$$e = \begin{array}{c} 1 \\ 0 \end{array} \begin{array}{c} 2 \end{array},$$

which satisfies $\langle e, d - e \rangle = (e - 2)(d - e) = -1$.

On the other hand, $|S| = 1$ and $\sum a_t = 0$ (resp. $\sum a_t = 2$) give us $e = 0$ (resp. $e = 0$ or 1), and then $\langle e, d - e \rangle = (e - 1)(d - e) = -2$ (resp. $\langle e, d - e \rangle = (e - (1 + \sum a_t))(d - e) = -6$ or $-2$).

If $|T'| = 1 = \sum a_t$ and $|S| = 0$, $(|S| + \sum a_t)d > (2m - k)e$ holds if and only if $e = 0$. We thus obtain

$$e = \begin{array}{c} 0 \\ 1 \end{array},$$

for which $\langle e, d - e \rangle = (e - 1)(d - e) + 1 = -1$.

Moreover, if $|T'| = 1 = \sum a_t$ and $|S| = 1$, then $e = 0$ or 1. Assuming that $e = 0$ we get $\langle e, d - e \rangle = (e - 2)(d - e) + 1 = -3$, and assuming that $e = 1$ we have the dimension vector

$$e = \begin{array}{c} 1 \\ 1 \end{array},$$

for which $\langle e, d - e \rangle = (e - 2)(d - e) + 1 = -1$.

Hence, $d = i_1 + 2i_2 + 2j$ is not $\Theta^{can}$-nas.
3.3. List of examples

* If $k = m - 2 = 0$, then $2m - k = 4$. This implies $2 \leq d \leq 2m - k - 1 = 3$ and thus, since $\mathbf{d}$ is $\Theta^\text{can}$-coprime, we have $d \neq 2$. Then, we obtain the dimension vector

\[
\mathbf{d} = \begin{array}{c}
3 \\
2 \\
2 \end{array}
\]

which is not $\Theta^\text{can}$-nas.

In fact, when $|T'| = 1$, $\sum_i a_i = 3$ and $e = 2$, we have the dimension vector

\[
\mathbf{e} = \begin{array}{c}
2 \\
1 \\
2 \end{array}
\]

which fulfills $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle = (d - e)(e - 3) + 1 = 0$.

Also, when $|T'| = 2 = \sum_i a_i$ and $e = 1$, the dimension vector $\mathbf{e}$ is given by

\[
\mathbf{e} = \begin{array}{c}
1 \\
1 \\
1 \end{array}
\]

and it satisfies $\langle \mathbf{e}, \mathbf{d} - \mathbf{e} \rangle = (d - e)(e - 2) + 2 = 0$.

3.3.4 Exceptional moduli spaces

Back in Section 3.3.1 and in Section 3.3.2, we verified the $\Theta$-nas condition for $\Theta$ canonical and for fundamental dimension vectors $\mathbf{d}$ of $K_{m,n}$ and $Q_2,2$, respectively. In the light of the results obtained in these sections: Theorem 3.3.1 and Theorem 3.3.2, we will now calculate some moduli spaces in the exceptional cases; that is, where the numerical ample stability condition does not hold.

First we consider the quiver $K_{m,n}$ as in Section 3.3.1 the dimension vector $\mathbf{d} = (d_1, d_2)$ and the stability vector

\[
\Theta^\text{can} = (m - n) \cdot (d_2, -d_1).
\]

We will treat the case $m = n$ and $d_2 = md_1$, in which the stability vector is identically zero. Recall that, in this case, the moduli space $M^\text{sst}_{(d_1, md_1)}(K_{m,m})$ is just the spectrum of the ring of invariant functions, see Remark 1.3.19.

**Lemma 3.3.7:** The moduli space $M^\text{sst}_{(d_1, md_1)}(K_{m,m})$ is isomorphic to the moduli space $M^\text{sst}_{d_1}(L_{m^*})$.

**Proof.** A representation $V$ of $K_{m,m}$ of dimension vector $\mathbf{d} = (d_1, md_1)$ is given by
where $V_1$ is a vector space of dimension vector $d_1$ and $f_i, g_i$ are linear maps for all $i \in \{1, \ldots, m\}$. We observe that the shortest cycles we can see in $V$ are the maps $g_i \circ f_j \in \text{End}(V_1)$ for $i, j \in \{1, \ldots, m\}$:

Assume that each $f_j$ is a block matrix of the form

\[
\begin{bmatrix}
A_{1j} \\
A_{2j} \\
\vdots \\
A_{mj}
\end{bmatrix},
\]

where $A_{ij}$ are $d_1 \times d_1$-matrices for $i, j \in \{1, \ldots, m\}$, and assume that each $g_i$ is a block matrix of the form

\[
\begin{bmatrix}
0 & \ldots & 0 & \text{Id}_{d_i} & 0 & \ldots & 0
\end{bmatrix},
\]

where $\text{Id}_{d_i}$ is the $i$-th component of $g_i$ for all $i \in \{1, \ldots, m\}$. Then, among others, we get the following cycles in $V$:

\[
g_i \circ f_j = [A_{ij}]_{d_1 \times d_1} \quad \text{and} \quad f_j \circ g_i = \begin{bmatrix}
0 & \ldots & 0 & A_{1j} & 0 & \ldots & 0 \\
0 & \ldots & 0 & A_{2j} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & A_{mj} & 0 & \ldots & 0
\end{bmatrix}_{md_1 \times md_1}
\]

with $A_{kj}$ in the $i$-th component of $g_i \circ f_j$ for all $k \in \{1, \ldots, m\}$.

We can then define the following morphism:

\[
R_{(d_1, md_1)}(K_{m,m}) \rightarrow R_{(d_1)}(L_{m^2}) \quad \text{and} \quad V \hookrightarrow (V_1, (g_i \circ f_j)_{i,j}).
\]

By Theorem 1.3.7, the rings of invariant functions

\[
\mathbb{C}[R_{(d_1, md_1)}(K_{m,m})] \quad \text{and} \quad \mathbb{C}[R_{(d_1)}(L_{m^2})]
\]

are generated by traces along oriented cycles, which are in fact sequences of the maps $g_i \circ f_j$ in both cases. In particular, all traces of these sequences generate the spectrum of the respective ring of invariants. This induces an isomorphism

\[
\text{Spec} \left( \mathbb{C}[R_{(d_1, md_1)}(K_{m,m})]^{\text{GL}_{d_1} \times \text{GL}_{md_1}} \right) \cong \text{Spec} \left( \mathbb{C}[R_{(d_1)}(L_{m^2})]^{\text{GL}_{d_1}} \right),
\]

and therefore an isomorphism of moduli spaces

\[
M^\text{stab}_{(d_1, md_1)}(K_{m,m}) \cong M^\text{stab}_{d_1}(L_{m^2}).
\]
Since the stability vector $\Theta^\text{can}$ is identically zero, we have that all representations are semistable and the stable representations correspond to the simple ones, refer to Example 3.3.18. So we will proceed to prove that the above isomorphism restricts to an isomorphism of the open sets of simples.

Thus we first assume that $V$ is a non-simple representation of $K_{m,m}$, then there exists a proper subrepresentation $U$ of $V$, i.e. we have a vector subspace $U_1 \subseteq V_1$ such that $f_j(U_1) \subseteq U_1^m$ and $g_i(U_1^m) \subseteq U_1$ for all $i, j \in \{1, \ldots, m\}$.

In particular, the restrictions $g'_i := g_i|_{U_1^m}$ and $f'_j := f_j|_{U_1}$ give us a proper subrepresentation $(U_1, (g'_i \circ f'_j)_{i,j})$ of $(V_1, (g_i \circ f_j)_{i,j})$. Hence, the representation $(V_1, (g_i \circ f_j)_{i,j})$ is not a simple representation of $L^2_m$.

Conversely, suppose that $V$ defines a simple representation of $K_{m,m}$, and consider the subspaces $\text{Im}f_1, \ldots, \text{Im}f_m$ of $V_1^m$ for which we write $U := \sum_{j=1}^m \text{Im}f_j$. We note that $\dim(\text{Im}f_j) \leq d_1$ for each $j \in \{1, \ldots, m\}$.

The representation $(V_1, U)$ defines a subrepresentation of $(V_1, V_1^m)$ and, by assumption, we require $\dim U = \dim V_1^m$. We thus get

$$md_1 = \dim V_1^m = \dim U = \dim \sum_{j=1}^m \text{Im}f_j \leq \sum_{j=1}^m \dim(\text{Im}f_j) \leq md_1,$$

and then $\dim(\text{Im}f_j) = d_1$ for all $j$. Consequently, every $f_j$ is an injective map and $V_1^m \cong \bigoplus_{j=1}^m \text{Im}f_j$.

Furthermore, after applying a change of basis, we can assume that each $f_j$ is the embedding of $V_1$ into the $j$-th component of $V_1^m$; that is, $f_j(V_1) = (0, \ldots, 0, V_1, 0, \ldots, 0)$ for all $j$.

Since $(g_i \circ f_j)(V_1) = \delta_{ij}V_1$, we have that $(V_1, (g_i \circ f_j)_{i,j})$ defines a simple representation of the $m^2$-loop quiver. With this we have proved the isomorphism of the moduli spaces of simples.

Consider now the quiver $Q_{2,2}$ as in Section 3.3.2, the dimension vector $d = (2, 2, 2)$ and the stability vector $\Theta^\text{can} = (4, -8, 4)$. Since the moduli space does not change if we replace the stability by a rational multiple, we can work with the stability $\Theta = (1, -2, 1)$ instead.

**Lemma 3.3.8:** $M_{(2,2,2)}^\Theta(Q_{2,2})$ is isomorphic to the locally closed subset of $\mathbb{P}^9$ with coordinates

$$(P_{0,0} : P_{0,1} : P_{0,2} : P_{1,0} : P_{1,1} : P_{1,2} : P_{2,0} : P_{2,1} : P_{2,2} : Q)$$

such that

$$P_{i,j}P_{k,l} = P_{i,k}P_{j,l} \quad \forall i, j, k, l \in \{0, 1, 2\},$$

$$Q^2 - (2P_{0,2} + P_{1,1} + 2P_{2,0})Q + (P_{0,2} + P_{2,0})^2 + (P_{0,1} + P_{1,0})(P_{1,2} + P_{2,1}) \neq 0.$$
Proof. A representation $V$ of $Q_{2,2}$ of dimension vector $d$ is given by four $2 \times 2$-matrices $A, B, C, D,$ on which the group $G_d = \text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})$ acts via conjugation:

$$(x, y, z) \cdot (A, B, C, D) = (yAx^{-1}, yBx^{-1}, yCz^{-1}, yDz^{-1}).$$

In order to reduce the problem of classifying the representations, we make the assumption that $\det(A) \neq 0 \neq \det(C)$. As it is defined by nonvanishing of polynomials, it is a Zariski-open set.

Under this assumption, we can use the group action to assume, without loss of generality, that the matrices $A$ and $C$ are the identity matrix $\text{Id}_2,$ and two representations given by tuples $(\text{Id}_2, B, \text{Id}_2, D)$ are equivalent if and only if there is $(x, y, z) \in G_d$ such that $x = y = z$. Then the group acting on it is only one copy of $\text{GL}_2(\mathbb{C})$:

$$x \cdot (\text{Id}_2, B, \text{Id}_2, D) = (\text{Id}_2, xBx^{-1}, \text{Id}_2, xDx^{-1}).$$

Consequently, we have reduced the classification problem to the classification of pairs of $2 \times 2$-matrices under simultaneous conjugation. It is well known that such pairs are classified by five invariants (see [KP96]), namely

$$\text{tr}(A), \det(A), \text{tr}(B), \det(B), \text{tr}(AB).$$

We consider now the polynomial function $P_1 : R_d(Q_{2,2}) \rightarrow \mathbb{C}$ defined by

$$P_1(aA, bB, cC, dD) := \det(aA + bB) \cdot \det(cC + dD),$$

for variables $a, b, c, d.$ Under the original group action, it transforms to

$$P_1((x, y, z) \cdot (aA, bB, cC, dD)) = \det(x)^{-1} \det(y)^2 \det(z)^{-1} \det(aA + bB) \det(cC + dD),$$

which, after writing $g = (x, y, z)$, reads

$$P_1(g \cdot (aA, bB, cC, dD)) = \prod_{i=1}^{3} \det g_i^{-\Theta_i} P_1(aA, bB, cC, dD);$$

that is, the map $P_1$ is $\Theta$-semiinvariant, and depends on four additional variables.

Since the function is polynomial, it can be written as follows:

$$\det(aA + bB) \cdot \det(cC + dD) = (a^2U_0 + abU_1 + b^2U_2) \cdot (c^2V_0 + cdV_1 + d^2V_2)$$

$$= a^2c^2P_{0,0} + a^2cdP_{0,1} + a^2d^2P_{0,2} + abc^2P_{1,0} + abcdP_{1,1} + abd^2P_{1,2} + b^2c^2P_{2,0} + b^2cdP_{2,1} + b^2d^2P_{2,2},$$

with $P_{i,j} = U_iV_j$ for $i, j \in \{0, 1, 2\},$ giving us nine semiinvariants.

In the case $A = \text{Id}_2 = C,$ we find

$$\det(aA + bB) \cdot \det(cC + dD) = (a^2 + ab\text{tr}(B) + b^2 \det(B)) \cdot (c^2 + cd\text{tr}(D) + d^2 \det(D)),$$

so we recover four of the five invariants distinguishing $2 \times 2$-matrices.
On the other hand, under the assumption that $A$ is an invertible matrix, we can consider the polynomial function $P_2 : \mathbb{R}^d(Q_{2,2}) \rightarrow \mathbb{C}$ given by

$$P_2(A, B, C, D) := \det \begin{bmatrix} A & C \\ B & D \end{bmatrix} = \det(A) \cdot \det(D - BA^{-1}C),$$

which is also semi-invariant for $\Theta$. Then we have

$$Q := \det \begin{bmatrix} \text{Id}_2 & \text{Id}_2 \\ B & D \end{bmatrix} = \det(B) + \det(D) - \text{tr}(B)\text{tr}(D) + \text{tr}(BD),$$

so we recover the fifth invariant that classifies $2 \times 2$-matrices.

Then, from Theorem 1.3.4, we get that the moduli space $M_{(2,2,2)}^{\Theta-\text{sst}}(Q_{2,2})$ equals the image of the map

$$P = (P_{0,0} : \ldots : P_{2,2} : Q) : \mathbb{R}^d_{\Theta-\text{sst}} \rightarrow \mathbb{P}^9$$

fulfilling the relations

$$P_{i,j}P_{k,l} = P_{i,l}P_{k,j}$$

for all $i, j, k, l \in \{0, 1, 2\}$.

The nonvanishing conditions characterizing stability are as follows: A representation $V$ is $\Theta$-semistable but not $\Theta$-stable if there is a non-zero subrepresentation $U \subseteq V$ of dimension vector $(2, 1, 0)$ or $(1, 1, 1)$. Since $A$ and $C$ are invertible matrices, it reduces to $V$ having a subrepresentation of dimension vector $(1, 1, 1)$.

We can then assume $A, B, C, D$ to be upper-triangular $2 \times 2$-matrices, with respective diagonal entries $a_{11}, a_{22}, b_{11}, b_{22}, c_{11}, c_{22}, d_{11}, d_{22}$. Then all our semi-invariants on such matrices read

$$U_0 = a_{11}a_{22}, \quad U_1 = a_{11}b_{22} + a_{22}b_{11}, \quad U_2 = b_{11}b_{22},$$

$$V_0 = c_{11}c_{22}, \quad V_1 = c_{11}d_{22} + c_{22}d_{11}, \quad V_2 = d_{11}d_{22},$$

$$Q = (a_{11}d_{11} - b_{11}c_{11})(a_{22}d_{22} - b_{22}c_{22}).$$

These fulfill the relation

$$Q^2 - (2P_{0,2} + P_{1,1} + 2P_{2,0})Q + (P_{0,2} - P_{2,0})^2 + (P_{0,1} + P_{1,0})(P_{1,2} + P_{2,1}) = 0.$$

Thus we arrive at the desired description of the moduli space $M_{(2,2,2)}^{\Theta-\text{sst}}(Q_{2,2})$.

### 3.3.5 Two potential further examples

We conclude this section by formulating two interesting conjectures which are based on calculations of explicit decompositions of fundamental dimension vectors which satisfy the $\Theta$-nas condition for the canonical stability vector $\Theta$. 

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Chapter 3. Approaches to ample stability

Let $Q$ be a quiver and let $d$ be a dimension vector in $F_Q$. From Lemma 3.2.1 we know that $d$ is $\Theta^\can$-nas if and only if there is no decomposition such that $\langle e, f \rangle = -1$ and $(f, e) \in \{-1, 0\}$.

Assume that $Q_{m,n}$ is a more general quiver with three vertices $i_1, i_2, i_3$, and $m \geq 1$ arrows from $i_1$ to $i_2$ and $n \geq 1$ arrows from $i_3$ to $i_2$:

$\bullet i_1 \overset{m}{\rightarrow} i_2 \overset{n}{\rightarrow} i_3$.

By symmetry, we can assume $m \leq n$. We can also assume $n \geq 2$; otherwise, if $m = 1 = n$, we have Dynkin quiver of type $A_3$ and thus all moduli spaces are empty or single points. We can also exclude the case $m = 2 = n$ since it was already treated back in Section 3.3.2.

**Conjecture 1:** Let $d$ be a fundamental dimension vector for $Q_{m,n}$. Then $d$ is $\Theta^\can$-nas if it is not one of the following cases:

- $m = 1, n = 2, d = (1, a, a)$ for some $a \geq 1$,
- $m = 1, n = 2, d = (2, a + 1, a)$ for some $a \geq 1$,
- $m = 1, n = 2, d = (2, 4, 4)$,
- $n = 3, d = (0, 2, 2)$,
- $m = 3, d = (2, 2, 0)$.

**Remark 3.3.9:** Conjecture 1 has a positive answer when $m = n$ are even. This can be done by generalizing the proof of Theorem 3.3.2, taking into account the following:

Assuming that $m = n = 2\alpha$, with $\alpha \geq 2$, a dimension vector $d = (d_1, d_2, d_3)$ for $Q_{m,n}$ is fundamental if:

$$0 \leq d_1, d_3 \leq \alpha d_2 \leq \alpha^2(d_1 + d_3),$$

and for every decomposition $d = e + f$, we have that $\langle e, f \rangle = \langle f, e \rangle = -1$.

Writing $d_2 = sa$ and $\alpha(d_1 + d_3) = sb$, for $s \geq 1$ and $a, b$ coprime such that $a \leq b \leq 2\alpha^2a$, we have that

$$e_2 = \alpha ka, f_2 = \alpha la, e_1 + e_3 = \frac{kb}{\alpha}, f_1 + f_3 = \frac{lb}{\alpha},$$

with $k, l \geq 1$ such that $k + l = s$.

Renaming $x = e_1$ and $y = f_1$ we find

$$x \leq \frac{kb}{\alpha}, y \leq \frac{lb}{\alpha}, (k + l) \left(\frac{b - \alpha^3a}{\alpha}\right) \leq x + y \leq (k + l)(\alpha^2a),$$

and

$$kl(b^2 + \alpha^3a(\alpha a - 2b)) + \alpha^2 = ab(lx + ky) - 2\alpha^2xy.$$ 

Then, proceeding by cases, treating different $a$ and $b$, it is possible to check that the above conditions are never fulfilled and thus there is no decomposition $d = e + f$ such that $\langle e, f \rangle = \langle f, e \rangle = -1$.
Now assume that $Q$ is the following symmetric quiver:

$$
\begin{array}{c}
\bullet_i \\
\downarrow \\
\bullet_j \\
\downarrow \\
\bullet_k
\end{array}
$$

So the Euler form $(\chi, \chi)_Q$ is symmetric and the canonical stability $\Theta^{\text{can}}$ is trivial. From Example 1.3.18 we know that in this case every representation is semistable, and the stable representations are the simples.

Note that, by symmetry of the Euler form, the condition that $d$ belongs to the fundamental domain just means that $(d, i) = (i, d) \leq 0$ for all $i \in Q_0$, and our numerical criterion for ample stability simplifies to:

$$d$$

is nas if and only if there is no decomposition $d = e + f$ such that $(e, f) = -1$.

A dimension vector $d = (d_1, d_2, d_3)$ is in the fundamental domain $F_Q$ if

$$d_1 \leq d_2, \quad d_3 \leq d_2 \quad \text{and} \quad d_2 \leq d_1 + d_3,$$

with $d_2 \neq 0$.

Assume that $d$ is a dimension vector in $F_Q$. Then, since $0 \leq d_1, d_3 \leq d_2 \leq d_1 + d_3$, we write

$$d_2 = d_1 + c \text{ for some } c \geq 0,$$

$$d_2 = d_3 + b \text{ for some } b \geq 0,$$

and we also write

$$d_1 + d_3 = d_2 + a \text{ for some } a \geq 0.$$

Putting this together we have

$$d_1 = a + b, \quad d_2 = a + b + c \quad \text{and} \quad d_3 = a + c,$$

and so we arrive at the following:

$$d \in F_Q \text{ if and only if there are } a, b, c \geq 0 \text{ not all zero such that } d = (a+b, a+b+c, a+c).$$

**Conjecture 2:** Let $Q$ be the symmetric quiver above and let $d$ be a dimension vector in $F_Q$. Then $d = (a + b, a + b + c, a + c)$ is nas if either $a = 0$ or $b = 0$ or $c = 0$.

### 3.4 Fano quiver moduli

The canonical stability appears naturally in [FRS20], since it is the only stability vector that implies the Fano property for quiver moduli spaces. In this section, we will review the main result of the mentioned paper, [FRS20 Theorem 4.3], which, in other words, states that for quivers without oriented cycles, if the ample stability condition for $\Theta^{\text{can}}$ is met, (under a mild assumption) the moduli space is a Fano variety of known dimension, Picard rank, and index. For a more in-depth exposition, we refer the reader to the original paper.
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For us, Fano varieties are smooth projective complex varieties whose anticanonical bundle, i.e. the determinant of the tangent bundle $T$, is ample. For a detailed exposition about these varieties, we refer to [Deb97].

Let $Q$ be an acyclic quiver and $\Theta$ a stability vector. We will first recall from Section 3.3.3 that a dimension vector $d$ is $\Theta$-coprime if, for all non-zero proper $e \leq d$, we have $\Theta(e) \neq 0$.

Recall also that a dimension vector $d$ is indivisible if $\gcd(d_i \mid i \in Q_0) = 1$ and that $\Theta$-coprimality implies indivisibility.

In the case of a $\Theta$-coprime dimension vector $d$, by definition of $\Theta$-stability, the $\Theta$-stable locus equals the $\Theta$-semistable locus in $R_d(Q)$, and thus $M^\Theta_{\text{sst}}(Q) = M^\Theta_{\text{st}}(Q)$ is a smooth projective complex variety, since $Q$ is a quiver without oriented cycles (see [Rei08]).

Let $d$ be an indivisible dimension vector for $Q$. Consider the abelian group of stabilities for $d$

$$\text{Stab}(d) = \{ \Theta \in (\mathbb{Z}Q_0)^* \mid \Theta(d) = 0 \},$$

and consider the evaluation at $d$ given by the map

$$\text{ev}_d : (\mathbb{Z}Q_0)^* \longrightarrow \mathbb{Z}$$

$$\Theta \longmapsto \Theta(d).$$

Then there is a short exact sequence of abelian groups

$$0 \longrightarrow \text{Stab}(d) \longrightarrow (\mathbb{Z}Q_0)^* \xrightarrow{d} \text{Im ev}_d \longrightarrow 0. \quad (3.10)$$

For a non-zero stability vector $\Theta \in \text{Stab}(d)$, we define

$$\gcd(\Theta) = \gcd(\Theta_i \mid i \in Q_0).$$

Inside the real vector space $\text{Stab}(d)_\mathbb{R} = \text{Stab}(d) \otimes \mathbb{Z} \mathbb{R}$ associated to $\text{Stab}(d)$, for every non-zero dimension vector $e \leq d$, we consider the hyperplane $W_e = \{ \Theta \in (\mathbb{Z}Q_0)^* \mid \Theta(e) = 0 \}$. It is called a wall in $\text{Stab}(d)_\mathbb{R}$.

In the complement

$$\text{Stab}^0(d)_\mathbb{R} = \text{Stab}(d)_\mathbb{R} \setminus \bigcup_e W_e,$$

we consider a connected component $C_\mathbb{R}$. Its closure $\overline{C_\mathbb{R}}$ and the set $\overline{C_\mathbb{R}} \cap \text{Stab}(d)$ are called a chamber in $\text{Stab}(d)_\mathbb{R}$ and in $\text{Stab}(d)$, respectively.

Note that $d$ is $\Theta$-coprime if and only if $\Theta$ belongs to the interior of a chamber. If $d$ is $\Theta^\text{can}$-coprime, the chamber whose interior contains $\Theta^\text{can}$ will be called the canonical chamber and will be denoted by $C^\text{can}$. 

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Furthermore, for $\Theta$-coprime and $\Theta$-amply stable dimension vectors $d$, we have a chain of isomorphisms of abelian groups $[FRS20$, Proposition 3.1$]$

$$\text{Stab}(d) \xrightarrow{\sim} X(PG_d) \xrightarrow{\sim} \text{Pic}(M^\Theta_{\text{st}}(Q))$$

where $L(\chi_{\Theta'})$ is a line bundle on $M^\Theta_{\text{st}}(Q)$.

Since $M^\Theta_{\text{st}}(Q)$ is a smooth projective variety over $\mathbb{C}$, the Picard group $\text{Pic}(M^\Theta_{\text{st}}(Q))$ is a free abelian group of finite rank, see [Har77]. In fact, from (3.10) and (3.11) we get that the Picard rank of $M^\Theta_{\text{st}}(Q)$ equals $|Q_0| - 1$.

If $\Theta$ and $\Theta'$ belong to the interior of the same chamber in $\text{Stab}(d)$, then the $\Theta$-stable locus $R^\Theta_{\text{st}}(Q)$ and the $\Theta'$-stable locus $R^\Theta'_{\text{st}}(Q)$ coincide, which induces an isomorphism

$$M^\Theta_{\text{st}}(Q) \cong M^\Theta'_{\text{st}}(Q).$$

This means that the line bundle $L(\chi_{\Theta'})$ on $M^\Theta_{\text{st}}(Q)$ is ample. In particular, $L(\chi_{\Theta})$ is ample.

Furthermore, in $\text{Pic}(M^\Theta_{\text{st}}(Q))$, we have that

$$\det(T) = L(\chi_{\Theta_{\text{can}}}),$$

where $T$ is the tangent bundle.

Combining all this, we arrive at the following result, which was proved in [FRS20, Theorem 4.3] by Franzen, Reineke, and Sabatini.

**Theorem 3.4.1:** Let $Q$ be an acyclic quiver and let $d$ be a dimension vector for $Q$. If $d$ is $\Theta_{\text{can}}$-coprime and $\Theta_{\text{can}}$-amply stable, then the moduli space $M^\Theta_{\text{can-st}}(Q)$ is a smooth irreducible projective complex Fano variety of dimension $1 - \langle d, d \rangle$. Its Picard rank is equal to $|Q_0| - 1$, and its index to $\gcd(\Theta_{\text{can}})$.

**Example 3.4.2:** Let $K_m$ be the generalized Kronecker quiver as in Section 3.2 and let $d = (d_1, d_2)$ be a fundamental dimension vector. Consider the canonical stability $\Theta_{\text{can}} = (me, -md)$.

then $d$ being $\Theta_{\text{can}}$-coprimality is the same as $d_1$ and $d_2$ being coprime.

From Proposition 3.2.3 we can deduce that for a given $m \geq 3$ and $d_1, d_2$ coprime, the moduli space $M^\Theta_{\text{can-st}}(d_1, d_2)(K_m)$, if non-empty, is a Fano variety of dimension $md_1d_2 - d_1^2 - d_2^2 + 1$, Picard rank 1, and index $m$.

This example holds not only for $\Theta_{\text{can}}$-coprime fundamental dimension vectors $d$, but for all $\Theta_{\text{can}}$-coprime dimension vectors for $K_m$, as seen in [FRS20].

**Remark 3.4.3:** By applying Theorem 3.4.1 to the (large) classification of $\Theta_{\text{can}}$-nas dimension vectors exhibited in Section 3.2 we obtain new classes of examples of Fano quiver moduli spaces of arbitrary high dimension.
Chapter 3. Approaches to ample stability
Chapter 4

Applying reflection functors to moduli spaces

As mentioned before in Section 3.2, there is an isomorphism of moduli spaces induced by reflection functors, which is special to moduli for generalized Kronecker quivers, see [Wei13, Proposition 4.3]. Therefore, in this case, the reduction to $\mathbf{d}$ in the fundamental domain is possible since moduli spaces do not change by applying a sequence of reflection functors to them.

The main goal of this chapter is to generalize this idea to any quiver, thus we need to study the behaviour of moduli spaces under the action of reflection functors. We will first give an explicit description of the reflection functors in the representation theory frame. Then we will introduce a duality of Grassmannians seeing them as geometric quotients, and it will be applied to obtain an isomorphism of varieties, which captures the definition of reflection functors. This will be restricted to the case of semistable representations in order to get the desired identification of moduli spaces.

4.1 Recollections on reflection functors

The reflection functors appeared as a tool to study representations of quivers in the work [BGA73]. They allow us to reflect representations of a quiver $Q$ at a given vertex $i \in Q_0$, which is a sink or a source, to transform it into a representation of a reflected quiver $s_i Q$. This reflected quiver is obtained by changing the orientations of the arrows connected to the vertex $i$. Here we introduce the action of reflecting a quiver, a dimension vector and a quiver representation and study the effect of reflection functors in the representations, following mainly [KJI6] and [Kra08]. In addition, we recollect some facts on these reflections that will be needed in Section 4.3.

Definition 4.1.1: Let $Q$ be a quiver. A vertex $i \in Q_0$ is called

- a sink for $Q$ if there are no arrows $\alpha: i \to j$,
- a source for $Q$ if there are no arrows $\alpha: j \to i$.
Chapter 4. Applying reflection functors to moduli spaces

Let $Q$ be a quiver and $i$ a sink (resp. a source) for $Q$. Define $s^+_i Q$ (resp. $s^-_i Q$) to be the quiver obtained from $Q$ by reversing all arrows ending at $i$ (resp. starting at $i$):

![Diagram of quiver with arrows reversed at i]

The reverse arrows $\alpha$ will be denoted by $\alpha^*$.

Note that if $i$ is a sink for $Q$, then $i$ becomes a source for $s^+_i Q$ and $s^-_i s^+_i Q = Q$.

**Notation 4.1.2:** Throughout this chapter, we will only be working with sinks $i$ in $Q$, thus we write $s_i Q$ instead of $s^+_i Q$.

Assume that the vertex $i$ is a sink in $Q$. Denote by $\text{rep}^i_Q(Q)$ the full subcategory of $\text{rep} Q$ consisting of representations $V$ such that the map

\[
\Phi_V : \bigoplus_{\alpha: j \to i} V_j \to V_i
\]

is surjective.

Dually, $\text{rep}^{i,+}(s_i Q)$ denotes the full subcategory of $\text{rep} s_i Q$ whose representations $W$ satisfy that the map

\[
\Phi^*_W : W_i \to \bigoplus_{\alpha^*: i \to j} W_j
\]

is injective.

**Remark 4.1.3:** This is equivalent to saying, respectively, that

\[\text{rep}^i_Q(Q) = \{V \in \text{rep} Q \mid \text{Hom}(V, S(i)) = 0\}\]

and

\[\text{rep}^{i,+}(s_i Q) = \{W \in \text{rep} s_i Q \mid \text{Hom}(S(i), W) = 0\}\].

Now we want to study the relation between representations on $\text{rep} Q$ and $\text{rep}^{i,+}(s_i Q)$. For this, we will define reflection functors acting on representations and morphisms of these categories.

**Definition 4.1.4:** Let $i$ be a sink in $Q$. The pair of reflection functors

\[
\text{rep}^i_Q(Q) \xrightarrow{s^+_i} \text{rep}^{i,+}(s_i Q) \xleftarrow{s^-_i}
\]

are constructed as follows: for $V \in \text{rep}^i_Q(Q)$, define $S_i^+(V)$ by setting
4.1. Recollections on reflection functors

\[(S_i^+(V))_j = \begin{cases} V_j, & \text{if } j \neq i, \\ \text{Ker} (\Phi_V = \bigoplus_{\alpha: j \to i} V_\alpha: \bigoplus_{\alpha: j \to i} V_j \to V_i), & \text{otherwise, and} \end{cases} \]

\[(S_i^+(V))_\alpha = \begin{cases} V_\alpha, & \text{if } t(\alpha) \neq i, \\ (S_i^+(V))_\alpha = \text{Ker} \Phi_V \to \bigoplus_{\alpha: j \to i} V_j \to V_j = (S_i^+(V))_j, & \text{otherwise.} \end{cases} \]

Moreover, given a morphism \(f : U \to V\) in \(\text{rep}^{i\to}(Q)\), set \((S_i^+(f))_j = f_j\) for all \(j \neq i\), and let \((S_i^+(f))_i : \text{Ker}(\Phi_V) \to \text{Ker}(\Phi_U)\) be the induced map \(f^i_\Phi\) in the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & \text{Ker} \Phi_U & \to & U_j & \overset{\Phi_U}{\to} & U_i & \to & 0 \\
& & \downarrow f_i & & \downarrow f_i & & \downarrow f_i & & \\
0 & \to & \text{Ker} \Phi_V & \to & V_j & \overset{\Phi_V}{\to} & V_i & \to & 0
\end{array}
\]

On the other hand, for \(W \in \text{rep}^{i\to}(s_iQ)\), define dually \(S_i^-(W)\) by setting

\[(S_i^-(W))_j = \begin{cases} W_j, & \text{if } j \neq i, \\ \text{Coker} (\Phi_W = \bigoplus_{\alpha^*: i \to j} W_\alpha^*: W_i \to \bigoplus_{\alpha^*: i \to j} W_j), & \text{otherwise, and} \end{cases} \]

\[(S_i^-(W))_\beta = \begin{cases} W_\beta, & \text{if } s(\beta) \neq i, \\ (S_i^-(W))_\beta = W_j \to \bigoplus_{\alpha^*: i \to j} W_j \to \text{Coker} \Phi_W = (S_i^-(W))_i, & \text{otherwise.} \end{cases} \]

The action of \(S_i^-\) on morphisms in \(\text{rep}^{i\to}(s_iQ)\) is defined in a similar way as the action of \(S_i^+\) on morphisms in \(\text{rep}^{i\to}(Q)\).

We summarize the description of reflection functors in other words: When we apply them to a representation \(V \in \text{rep}^{i\to}(Q)\) or \(W \in \text{rep}^{i\to}(s_iQ)\), they only modify the vector spaces \(V_i\) and \(W_i\) and leave the other vector spaces unaltered. Therefore, we recover them in the short exact sequences

\[
\begin{array}{ccccccccc}
0 & \to & (S_i^+(V))_i & \to & \bigoplus_{\alpha: j \to i} V_j & \to & V_i & \to & 0, \\
0 & \to & W_i & \to & \bigoplus_{\alpha^*: i \to j} W_j & \to & (S_i^-(W))_i & \to & 0,
\end{array}
\]

because they induce the maps for the reverse arrows \(\alpha^*\).
Example 4.1.5: Let $Q$ be the quiver

$$
\begin{array}{c}
\bullet \\
1 \\
\bullet
\end{array}
\xrightarrow{\bullet}

then we have that

$$
S_2^+ 
\begin{array}{c}
\bullet \\
\mathbb{C} \\
0
\end{array}
= 
\begin{array}{c}
\bullet \\
\mathbb{C} \\
1
\end{array}

S_2^- 
\begin{array}{c}
\bullet \\
1 \\
\mathbb{C}
\end{array}
= 
\begin{array}{c}
\bullet \\
\mathbb{C} \\
0
\end{array}
$$

Remark 4.1.6: If $i$ is a sink (resp. a source) in $Q$, then $S_i^+(S(i)) = 0$ (resp. $S_i^-(S(i)) = 0$), where $S(i)$ denotes the simple representation of $i \in Q_0$.

Now we will proceed to prove that $S_i^+$ and $S_i^-$ form a pair of inverse functors and thus, they determine an equivalence of categories. This result plays an important role in the development of this chapter, and therefore we will keep it in mind.

Theorem 4.1.7: Let $i$ be a sink in $Q$. The functors $S_i^+$ and $S_i^-$ are inverse to each other, inducing an equivalence of categories

$$
\text{rep}^{i,-}(Q) \simeq \text{rep}^{i,+}(s_iQ).
$$

Proof. Let $V \in \text{rep}^{i,-}(Q)$, then $S_i^- \circ S_i^+(V)$ is the representation with

$$(S_i^- \circ S_i^+(V))_j = \begin{cases} 
V_j, & \text{if } j \neq i,
Coker \left( \Psi_V : \text{Ker } \Phi_V \longrightarrow \bigoplus_{\alpha : i \rightarrow j} V_j \right), & \text{otherwise}.
\end{cases}$$

Thus there is a short exact sequence

$$
0 \longrightarrow \text{Ker } \Phi_V \xrightarrow{\Psi_V} \bigoplus_{\alpha : i \rightarrow j} V_j \xrightarrow{\Phi_V} V_i \longrightarrow 0
$$

which yields a unique isomorphism $V_i \xrightarrow{\sim} \text{Coker } \Psi_V$.

Therefore, $S_i^- \circ S_i^+(V) \cong V$.

On the other hand, let $f : U \longrightarrow V$ be a morphism in $\text{rep}^{i,-}(Q)$, then the composition $S_i^- \circ S_i^+(f)$ is the morphism defined by setting

$$(S_i^- \circ S_i^+(f))_j = \begin{cases} 
f_j, & \text{if } j \neq i,
S_i^-(f) : \text{Ker } \Phi_U \longrightarrow \text{Ker } \Phi_V, & \text{otherwise}.
\end{cases}$$
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where $S_i^-(\tilde{f}_i \colon \text{Ker } \Phi_U \to \text{Ker } \Phi_V)$ is the (unique) morphism $\tilde{f}_i \colon \text{Coker } \Psi_U^* \to \text{Coker } \Psi_V^*$ induced in the commutative diagram

$$
\begin{array}{cccccccccccc}
0 & \rightarrow & \text{Ker } \Phi_U & \rightarrow & \bigoplus_{\alpha \colon j \rightarrow i} U_j & \rightarrow & \text{Coker } \Psi_U^* & \rightarrow & 0 \\
\downarrow \tilde{f}_i & & \downarrow \tilde{f}_i & & & & \downarrow \tilde{f}_i & & \\
0 & \rightarrow & \text{Ker } \Phi_V & \rightarrow & \bigoplus_{\alpha \colon j \rightarrow i} V_j & \rightarrow & \text{Coker } \Psi_V^* & \rightarrow & 0
\end{array}
$$

Thus, since $U_i \cong \text{Coker } \Psi_U^*$ and $V_i \cong \text{Coker } \Psi_V^*$, we get $\tilde{f}_i \cong f_i$. Hence, $S_i^- \circ S_i^+ \cong \text{Id}_{\text{rep}^{-}(Q)}$, and $S_i^+ \circ S_i^- \cong \text{Id}_{\text{rep}^{+(Q)}}$ can be obtained in a similar way.

As a consequence, the functors $S_i^+$ and $S_i^-$ are adjoint to each other, which yields the following result.

**Corollary 4.1.8:** If $i$ is a sink in $Q$ and $U, V \in \text{rep}^{-}(Q)$, then $U \subset V$ if and only if $S_i^+(U) \subset S_i^+(V)$.

**Proposition 4.1.9:** If $i$ is a sink in $Q$ and $U, V \in \text{rep}^{-}(Q)$, then

$$
\begin{align*}
\text{Hom}_Q(U, V) &= \text{Hom}_{s_1} (S_i^+(U), S_i^+(V)), \\
\text{Ext}_Q^1(U, V) &= \text{Ext}_{s_1}^1 (S_i^+(U), S_i^+(V)).
\end{align*}
$$

Similarly, for $U'$ and $V'$ representations in $\text{rep}^{+(Q)}$ we have

$$
\begin{align*}
\text{Hom}_{s_1} (U', V') &= \text{Hom}_Q (S_i^-(U'), S_i^-(V')), \\
\text{Ext}_{s_1}^1 (U', V') &= \text{Ext}_Q^1 (S_i^-(U'), S_i^-(V')).
\end{align*}
$$

The first part of each statement in this proposition follows immediately from Theorem 4.1.7. For the proof of the second part of each statement, refer to [KJ16, Corollary 3.16].

Proposition 4.1.9 suggests that the Euler form, characterized in (1.4) as

$$
\langle \text{dim } V, \text{dim } W \rangle = \text{dim } \text{Hom}_Q(V, W) - \text{dim } \text{Ext}_Q^1(V, W),
$$

is preserved under reflection functors. This fact requires a notion of dimension for reflected representations, which is realized by the reflection of dimension vectors introduced in Section 3.1 and described below.

Assume that $Q$ is a quiver without oriented cycles. Let $s_i : \mathbb{Z}Q_0 \rightarrow \mathbb{Z}Q_0$ be the reflection with respect to a vertex $i \in Q_0$ as defined in (3.1) by

$$
s_i(d) = d - (d, i) i.
$$
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Remark 4.1.10: For any vertex $i \in Q_0$, we have $s_i^2 = \text{id}$. Indeed,

$$s_i^2 d = d - (d, i) i - (d, i) i + (d, i)(i, i) i = d.$$

Let $i$ be a sink in $Q$ and let $d \in \mathbb{N}Q_0$ be a dimension vector. Then, assuming that $\sum_{\alpha: j \to i} d_j \geq d_i$, the reflection of the dimension vector $d$ is given by

$$(s_i d)_j = \begin{cases} d_j, & \text{if } j \neq i, \\ -d_i + \sum_{\alpha: k \to i} d_k, & \text{otherwise}. \end{cases}$$

Theorem 4.1.11: Let $i$ be a sink in $Q$. If $V \in \text{rep}^i (Q)$, then $\dim S_i^+ (V) = s_i \dim V$.

Similarly, $\dim S_i^- (V') = s_i \dim V'$ for $V' \in \text{rep}^i (s_i Q)$.

Proof. Let $V \in \text{rep}^i (Q)$. By definition, we have that $\dim (S_i^+ (V))_j = \dim V_j = (s_i \dim V)_j$ for all $j \neq i$ in $Q_0$.

Furthermore, we have a short exact sequence

$$0 \to \text{Ker } \Phi_V \to \bigoplus_{\alpha: j \to i} V_j \overset{\Phi_V}{\to} V_i \to 0,$$

which leads to $\dim (\text{Ker } \Phi_V) = \sum_{\alpha: j \to i} \dim V_j - \dim V_i$. Thus, it follows that $\dim (S_i^+ (V))_i = (s_i \dim V)_i$, as desired.

The second part of this theorem is proved in an analogous way.

The next result is a direct consequence of this theorem and Corollary 4.1.8.

Corollary 4.1.12: If $i$ is a sink in $Q$ and $U, V \in \text{rep}^i (Q)$, then

$$\dim U \leq \dim V \text{ if and only if } s_i \dim U \leq s_i \dim V.$$

Furthermore, from Theorem 4.1.11, Proposition 4.1.9 and Remark 4.1.10 we get that reflections respect the Euler form.

Proposition 4.1.13: For $i$ a sink in $Q$ and $U, V$ representations in $\text{rep}^i (Q)$, we have

$$\langle s_i \dim U, s_i \dim V \rangle_{s_i Q} = \langle \dim U, \dim V \rangle_Q.$$

Similarly, for $U'$ and $V'$ representations in $\text{rep}^i (s_i Q)$, we have

$$\langle s_i \dim U', s_i \dim V' \rangle_Q = \langle \dim U', \dim V' \rangle_{s_i Q}.$$
4.2 Duality of Grassmannians as geometric quotients

In [CR08] it is shown that Grassmannians can be realized as geometric quotients and thus, an application of Lemma 1.3.5 induces the duality of Grassmannians, which will be recalled in this section.

Let $n > 0$ be a natural number and consider a complex vector space $V$ of dimension $n$. For $0 \leq k \leq n - 1$, the Grassmannian

$$\text{Gr}_k(V) = \{ U \subset V \mid \dim U = k \}$$

is a space that parametrizes all $k$-dimensional vector subspaces of $V$.

The Grassmannian has the structure of a smooth projective variety of dimension $k(n - k)$, which can be thought of as a partial flag variety:

Let $V$ be an $n$-dimensional complex vector space and let $k = (k_1, k_2, \ldots, k_r)$ be a vector of integers such that $1 \leq k_1 \leq k_2 \leq \ldots k_r \leq n - 1$. The partial flag variety is defined as

$${\mathcal{F}}l_k = \{ V_{k_1} \subset V_{k_2} \subset \ldots \subset V_{k_r} \subset V \mid \dim V_{k_i} = k_i \}.$$ 

In the extreme case that $r = 1$, we have that $k = (k)$ and

$${\mathcal{F}}l_k \cong \text{Gr}_k(V).$$

Let $X, Y$ and $Z$ be three complex vector spaces such that $\dim Y = \dim X + \dim Z$. Inside $\text{Hom}(X, Y)$, consider the open subset of injective maps $\text{Hom}^0(X, Y)$, on which the group $\text{GL}(X)$ acts via $g \cdot f = f \circ g^{-1}$.

Define the map

$$\pi_{\text{GL}(X)}: \text{Hom}^0(X, Y) \longrightarrow \text{Gr}_{\dim X}(Y)$$

$$f \longmapsto \text{Im} f \subset Y.$$ (4.1)

It is clearly surjective and, as in [CR08, Lemma 2], it is easy to check that every fiber of $\pi_X$ is a single $\text{GL}(X)$-orbit. This implies that the Grassmannian $\text{Gr}_{\dim X}(Y)$ is the geometric quotient $\text{Hom}^0(X, Y)/\text{GL}(X)$.

Dually, consider the action of the group $\text{GL}(Y)$ on the open subset of surjective maps $\text{Hom}^0(Y, Z) \subset \text{Hom}(Y, Z)$.

Let $\text{Gr}_{\dim Z}(Y)$ be the Grassmannian of $\dim Z$-codimensional subspaces of $Y$. As before, the surjective map

$$\pi_{\text{GL}(Z)}: \text{Hom}^0(Y, Z) \longrightarrow \text{Gr}_{\dim Z}(Y)$$

$$g \longmapsto \text{Ker} g \subset Y.$$ (4.2)
identifies the Grassmannian $\text{Gr}^{\dim Z}(Y)$ as a geometric quotient

$$\text{Gr}^{\dim Z}(Y) \cong \text{Hom}^0(Y, Z)/\text{GL}(Z).$$

Consider the variety of short exact sequences of the form $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ given by

$$W = \{(f, g) \in \text{Hom}^0(X, Y) \times \text{Hom}^0(Y, Z) \mid g \circ f = 0\},$$
on which the group $\text{GL}(X) \times \text{GL}(Z)$ acts naturally.

Since $g \circ f = 0$, we have that

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$
is a short exact sequence where $X \cong \text{Ker} g$ and $Z \cong \text{Coker} f$. Additionally, every injective map $f \in \text{Hom}^0(X, Y)$ induces a short exact sequence

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{\text{Coker} f} 0,$$and similarly, every surjective map $g \in \text{Hom}^0(Y, Z)$ induces a short exact sequence

$$0 \rightarrow \text{Ker} g \rightarrow Y \xrightarrow{g} Z \rightarrow 0.$$Therefore, the projections $p_2 : W \rightarrow \text{Hom}^0(Y, Z)$ and $p_1 : W \rightarrow \text{Hom}^0(X, Y)$, together with the facts above lead to the following isomorphisms of geometric quotients:

$$\text{Hom}^0(Y, Z) \cong W/\text{GL}(X) \quad \text{and} \quad \text{Hom}^0(X, Y) \cong W/\text{GL}(Z).$$

These identifications of geometric quotients allow us to apply Lemma 1.3.5, and thus we get a commutative diagram of quotient maps

$$\begin{array}{ccc}
\text{Hom}^0(Y, Z) & \xrightarrow{p_2} & W & \xrightarrow{p_1} & \text{Hom}^0(X, Y) \\
\pi_{\text{GL}(Z)} & \downarrow & & \downarrow & \pi_{\text{GL}(X)} \\
\text{Gr}^{\dim Z}(Y) & \cong & W/\text{GL}(X) \times \text{GL}(Z) & \cong & \text{Gr}^{\dim X}(Y).
\end{array}$$

We have thus proved that, for any short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ of vector spaces, we get a duality of Grassmannians

$$\text{Gr}^{\dim Z}(Y) \cong \text{Gr}^{\dim X}(Y)$$
realized as geometric quotients.
4.3 The effect of reflection functors on stability conditions

Kraft and Riedtmann in [KR86] proved that there is an isomorphism of $R_d'(Q)/G_d$ and $R_{s,d}'(Q)/G_{s,d}$ as topological spaces, where $R_d'(Q)$ is the open subset of $R_d(Q)$ consisting of representations in $\text{rep}^{b-}(Q)$ and $R_{s,d}'(Q)$ is the open set of reflected representations. This isomorphism of quotients captures the definition of a reflection functor.

Our aim in this section is to restrict these open subsets to the semistable representations for certain stability parameter, to describe an application of reflection functors to moduli spaces of those semistable representations and to show that reflection functors preserve the geometry of the aforementioned moduli spaces. In order to prove this, we need to start by checking that reflection functors preserve stability condition. Motivated by this, we will additionally show that (numerical) ample stability is also respected when applying reflection functors.

Let $Q$ be a quiver, let $d$ be a fixed dimension vector for $Q$, and let us fix a stability $\Theta$ such that $\Theta(d) = 0$.

Given $V$, a $\Theta$-semistable representation of $Q$ of dimension vector $d$, it follows from Lemma 2.1.7 that $\text{Hom}(V, S(i)) = 0$ if $0 = \Theta(d) > \Theta(\text{dim } S(i))$. Thus, by Remark 4.1.3, $\Theta_i < 0$ is necessary for $V$ to lie in $\text{rep}^{b-}(Q)$.

In the remainder of this thesis, we assume that $Q$ is a quiver without oriented cycles and that $\Theta_i < 0$.

We denote by $p_i = \text{dim } P(i)$ the dimension vector of the indecomposable projective representations for all $i \in Q_0$. Since $\langle p_i, e \rangle = e_i$ for all dimension vectors $e \in \mathbb{N}Q_0$ (see Example 1.1.12), we have

$$\Theta(e) = \sum_{i \in Q_0} \Theta_i e_i = \sum_{i \in Q_0} \Theta_i \langle p_i, e \rangle = \left( \sum_{i \in Q_0} \Theta_i p_i, e \right).$$

We may therefore rewrite the stability $\Theta$ as

$$\Theta = \langle \alpha, \_ \rangle_Q, \text{ where } \alpha = \sum_{i \in Q_0} \Theta_i p_i \in \mathbb{Z}Q_0.$$

In a natural manner, given a sink $i$ in $Q$, we define the stability $S_i^+(\Theta) = \langle s_i \alpha, \_ \rangle_{s_i Q}$ for the reflected quiver $s_i Q$ and we set $S_i^- \circ S_i^+(\Theta) = \langle s_i^2 \alpha, \_ \rangle_Q$.

Notice that $S_i^+(\Theta)(s_i d) = \Theta(d) = 0$ because of Proposition 4.1.13 and $S_i^- \circ S_i^+(\Theta) = \Theta$ because of Remark 4.1.10.

**Notation 4.3.1:** From now on we will use the notation $S_i$ instead of $S_i^+$, whenever this can be done without ambiguity.
4.3.1 Semistability under reflection functors

In this section we see that the reflection functor $S_i$ at a sink $i$ of $Q$ respects the semistability condition for any $\Theta$ such that $\Theta(d) = 0$, and we show that it yields a bijection between isomorphism classes of $\Theta$-semistable representations of $Q$ of dimension vector $d$ and isomorphism classes of $S_i\Theta$-semistable representations of $s_iQ$ of dimension vector $s_id$.

**Theorem 4.3.2:** Let $V$ be a $\Theta$-semistable representation of dimension vector $d$ in $\text{rep}^{i-}(Q)$. Then, for $i$ a sink in $Q$, we have that the representation $S_iV$ is $S_i\Theta$-semistable in $\text{rep}^{i+}(s_iQ)$.

**Proof.** Assume that $V$ is a $\Theta$-semistable representation of dimension vector $d$ in $\text{rep}^{i-}(Q)$ and that $0 \neq U \subset S_iV$ is a subrepresentation in $\text{rep}^{i+}(s_iQ)$. Then, from Theorem 4.1.7, Corollary 4.1.8 and Proposition 4.1.13 it follows that $S_i\Theta(\dim U) = S_i\Theta(s_i(\dim S_i^-(U))) = \langle \alpha, s_i(\dim S_i^-(U)) \rangle_Q \\ \leq \langle \alpha, d \rangle_Q$, since $0 \neq S_i^-(U) \subset V$ and $V$ is $\Theta$-semistable $= 0$.

Therefore, since $S_i\Theta(\dim S_iV) = 0$, it implies that $S_iV$ is a $S_i\Theta$-semistable representation in $\text{rep}^{i+}(s_iQ)$ as desired. \hfill $\Box$

As a consequence of this theorem and of the fact that $S_i^+ \circ S_i^- (\Theta) = \Theta$, we get the following result.

**Theorem 4.3.3:** Let $i$ be a sink in $Q$. Then the equivalence of categories $\text{rep}^{i-}(Q) \simeq \text{rep}^{i+}(s_iQ)$ restricts to an equivalence

$$
\left\{ V \in \text{rep}^{i-}(Q) \mid V \text{ is } \Theta\text{-semistable of dimension vector } d. \right\} \simeq \left\{ W \in \text{rep}^{i+}(s_iQ) \mid W \text{ is } S_i\Theta\text{-semistable of dimension vector } s_id. \right\}.
$$

**Proof.** Let $V \in \text{rep}^{i-}(Q)$ be a $\Theta$-semistable representation of dimension vector $d$ and let $V' \in \text{rep}^{i-}(s_iQ)$ be a $S_i\Theta$-semistable representation of dimension vector $s_id$. Then, because of Theorem 4.1.11 and $s_i^2 = id$, we have that $\dim S_i^- \circ S_i^+(V) = s_i^2(\dim V) = \dim V = d$. In a similar way, we get $\dim S_i^+ \circ S_i^-(V') = s_id$. \hfill $\Box$

This theorem shows that, for a fixed sink $i$ in $Q$, the reflection functors $S_i^+$ and $S_i^-$ induce mutually inverse bijections

$$
\left\{ \text{Isomorphism classes of } \Theta\text{-semistable representations of } Q \text{ of dimension vector } d. \right\} \leftrightarrow \left\{ \text{Isomorphism classes of } S_i\Theta\text{-semistable representations of } s_iQ \text{ of dimension vector } s_id. \right\}.
$$
4.3.2 Reflection functors on moduli spaces of semistable representations

Let $G_d$ be the reductive group acting via base change on the representation space $R_d(Q)$ as described in Section 1.2 Consider $R_d^{\Theta\text{-sst}}(Q) \subset R_d(Q)$, the open $\Theta$-semistable locus and the open subset $R'_d(Q) = \{ V \in \text{rep}^{\theta-}(Q) | \dim V = d \} \subset R_d(Q)$. Thus, we have a chain of open subsets

$$R_d^{\Theta\text{-sst}}(Q) \subset R'_d(Q) \subset R_d(Q).$$

Similarly, consider $G_{s,d}$ the reductive group acting via base change on $R_{s,d}(s_i Q)$, the space of representations of $s_i Q$ of dimension vector $s_i d$. Inside $R_{s,d}(s_i Q)$, consider $R_{s,d}^{S_i \Theta\text{-sst}}(s_i Q)$, the open $S_i \Theta$-semistable locus and the open subset $R'_{s,d}(s_i Q) = \{ W \in \text{rep}^{\theta-}(s_i Q) | \dim W = s_i d \}$, which gives us a chain of open subsets

$$R_{s,d}^{S_i \Theta\text{-sst}}(s_i Q) \subset R'_{s,d}(s_i Q) \subset R_{s,d}(s_i Q).$$

We observe that the equivalence in Theorem 4.3.3 does not induce an isomorphism of varieties $R_d^{\Theta\text{-sst}}(Q)$ and $R_{s,d}^{S_i \Theta\text{-sst}}(s_i Q)$, as we have natural actions of different groups in these spaces, namely $G_d$ and $G_{s,d}$ respectively. In fact, there is no existing $G_d$-equivariant map from $R_d^{\Theta\text{-sst}}(Q)$ to $R_{s,d}^{S_i \Theta\text{-sst}}(s_i Q)$.

Hence, we need to define new varieties that come from $R_d^{\Theta\text{-sst}}(Q)$ and $R_{s,d}^{S_i \Theta\text{-sst}}(s_i Q)$ such that they encode the definition of the reflection functor, and such that there is an equivariant map between them for certain group action. For this, we will apply the duality of Grassmannians of Section 4.2 to our setting.

Let $Q$ be a quiver with a fixed dimension vector $d$ and let $V \in R_d(Q)$. Define $Y = \bigoplus_{\alpha: j \to i} V_j$ and $Z = V_i$, and consider the map $\Phi_V: Y \to Z$. We will write $X = \text{Ker} \Phi_V$. Moreover, we define

$$\mathcal{R} = \bigoplus_{\alpha: j \to k \neq i} \text{Hom}_C(V_j, V_k) \quad \text{and} \quad \mathcal{R}^{\alpha} = \prod_{j \neq i} \text{GL}(V_j).$$

Then we have

$$R'_d(Q) = \mathcal{R} \times \text{Hom}^0(Y, Z) \quad \text{and} \quad R'_{s,d}(s_i Q) = \mathcal{R} \times \text{Hom}^0(X, Y)$$

and, by the duality (4.3), we get that the geometric quotients $R'_d(Q)/\text{GL}(Z)$ and $R'_{s,d}(s_i Q)/\text{GL}(X)$ produce an isomorphism

$$\mathcal{R} \times \text{Gr}^{\dim Z}(Y) \cong \mathcal{R} \times \text{Gr}^{\dim X}(Y), \quad (4.4)$$

which is $\mathcal{R}^{\alpha}$-equivariant on both sides. This precisely encodes the definition of the reflection functor $S_i^+$ because of (4.2).

We therefore consider the quotient map

$$\pi_{\text{GL}(Z)}: R'_d(Q) \longrightarrow \mathcal{R} \times \text{Gr}^{\dim Z}(Y)$$

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and restrict it to the \( \Theta \)-semistable locus \( R_{d}^{\Theta\text{-sst}}(Q) \). We denote its image by

\[
(\mathcal{R} \times \text{Gr}_{\dim Z}(Y))^{\Theta\text{-sst}} := \pi_{GL(Z)}(R_{d}^{\Theta\text{-sst}}(Q)).
\]

Dually, we consider the restriction

\[
\pi_{GL(Y)}: R_{s,d}^{S_{i}\Theta\text{-sst}}(s_{i}Q) \longrightarrow \pi_{GL(Y)}(R_{s,d}^{S_{i}\Theta\text{-sst}}(s_{i}Q)) =: (\mathcal{R} \times \text{Gr}_{\dim X}(Y))^{S_{i}\Theta\text{-sst}}
\]

of the quotient map

\[
\pi_{GL(Y)}: R_{s,d}(s_{i}Q) \longrightarrow \mathcal{R} \times \text{Gr}_{\dim X}(Y)
\]
to the \( S_{i}\Theta \)-semistable locus \( R_{s,d}^{S_{i}\Theta\text{-sst}}(s_{i}Q) \).

From Theorem 4.3.3, we know that the reflection functor \( S_{i} \) maps \( \Theta \)-semistable representations of dimension \( d \) to \( S_{i}\Theta \)-semistable representations of dimension \( s_{i}d \). Thus, since the isomorphism obtained in \( 4.4 \) encodes the reflection functor \( S_{i} \), we get an induced \( \mathcal{G} \)-equivariant isomorphism of open subsets

\[
(\mathcal{R} \times \text{Gr}_{\dim Z}(Y))^{\Theta\text{-sst}} \cong (\mathcal{R} \times \text{Gr}_{\dim X}(Y))^{S_{i}\Theta\text{-sst}}.
\]

Hence, applying Lemma 1.3.5 to both sides, we get the following commutative diagram of quotient maps:

\[
\begin{align*}
M_{d}^{\Theta\text{-sst}}(Q) \xrightarrow{\pi_{GL(Y)}} (\mathcal{R} \times \text{Gr}_{\dim Z}(Y))^{\Theta\text{-sst}} & \cong (\mathcal{R} \times \text{Gr}_{\dim X}(Y))^{S_{i}\Theta\text{-sst}} \xleftarrow{\pi_{GL(Z)}} R_{s,d}^{S_{i}\Theta\text{-sst}}(s_{i}Q) \\
M_{s,d}^{\Theta\text{-sst}}(s_{i}Q) \longrightarrow (\mathcal{R} \times \text{Gr}_{\dim Z}(Y))^{\Theta\text{-sst}} / \mathcal{G} & \cong (\mathcal{R} \times \text{Gr}_{\dim X}(Y))^{S_{i}\Theta\text{-sst}} / \mathcal{G} \xleftarrow{\sim} M_{s,d}^{S_{i}\Theta\text{-sst}}(s_{i}Q)
\end{align*}
\]

Therefore, the reflection functors determine an identification between the moduli spaces \( M_{d}^{\Theta\text{-sst}}(Q) \) and \( M_{s,d}^{S_{i}\Theta\text{-sst}}(s_{i}Q) \).

### 4.3.3 Numerical ample stability under reflection functors

We have already seen in Theorem 4.3.2 that reflection functors preserve \( \Theta \)-semistability. We will now prove that when applying reflections to \( \Theta \)-numerical amply stable dimension vectors, the numerical ample stability condition is also preserved.

**Lemma 4.3.4:** Let \( d \) be a \( \Theta \)-stable dimension vector. Then \( d \) is \( \Theta \)-nas if and only if \( s_{i}d \) is \( S_{i}\Theta \)-nas.

**Proof.** Assume that \( d \) is \( \Theta \)-nas and that \( e \leq s_{i}d \) is a non-zero dimension vector such that \( S_{i}\Theta(e) \geq S_{i}\Theta(s_{i}d - e) \). Then \( s_{i}e \leq d \) follows from Corollary 4.1.12, and we have that

\[
\Theta(s_{i}e) = \langle \alpha, s_{i}e \rangle = \langle s_{i}^{2}(\alpha), s_{i}e \rangle = \langle s_{i}\alpha, e \rangle = S_{i}\Theta(e)
\]

and

\[
\Theta(d - s_{i}e) = \langle \alpha, d - s_{i}e \rangle = \langle s_{i}\alpha, s_{i}d \rangle - \langle s_{i}\alpha, s_{i}^{2}(e) \rangle = \langle s_{i}\alpha, s_{i}d - e \rangle = S_{i}\Theta(s_{i}d - e).
\]

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Hence, the hypothesis that \( S_i \Theta(e) \geq S_i \Theta(s_i d - e) \) implies that \( \Theta(s_i e) \geq \Theta(d - s_i e) \) and thus, since \( d \) is \( \Theta \)-nas, it leads to \( \langle s_i e, d - s_i e \rangle \leq -2 \).

Therefore, since

\[
\langle e, s_i d - e \rangle = \langle e, s_i d \rangle - \langle e, e \rangle = \langle s_i e, d \rangle - \langle s_i e, s_i e \rangle = \langle s_i e, d - s_i e \rangle,
\]

we obtain that \( s_i d \) is \( S_i \Theta \)-nas.

The converse implication follows in a similar way. \( \square \)

Therefore, the \( \Theta \)-nas condition is compatible with reflection functors.

Given a dimension vector \( d \) for \( Q \), recall that in Section 3.1 we introduced the canonical stability for \( d \) as follows:

\[
\Theta_{\text{can}} := \{d, -\} = \langle d, - \rangle - \langle -, d \rangle.
\]

**Proposition 4.3.5:** Let \( i \) be a sink in \( Q \). Then \( (\Theta_{\text{can}})_i < 0 \) if and only if \( \sum_{j \to i} d_j \neq 0 \).

**Proof.** From Remark 3.1.6 we know that

\[
\Theta_{\text{can}} = \left( \sum_{i \to j} d_j - \sum_{j \to i} d_j \right)_{i \in Q_0}.
\]

Assuming that \( i \) is a sink in \( Q \), we have that \( \sum_{i \to j} d_j = 0 \). Then we obtain \( (\Theta_{\text{can}})_i = -\sum_{j \to i} d_j \leq 0 \).

Note that if \( d \) is fundamental we must have \( \sum_{j \to i} d_j \neq 0 \) for any sink \( i \) in \( Q \); otherwise \( (d, i) > 0 \). This gives the following result.

**Corollary 4.3.6:** Let \( d \) be a fundamental dimension vector and let \( i \) be a sink in \( Q \). Then \( (\Theta_{\text{can}})_i < 0 \).

Therefore, the theory developed in Section 4.3 works for the canonical stability when considering fundamental dimension vectors.

We will now see how we can write the canonical stability condition explicitly so that it agrees with the convention \( \Theta = \langle \alpha, - \rangle \) used throughout Section 4.3.

Let \( V \) and \( W \) be representations of \( Q \) of dimension vector \( d \) and \( e \), respectively, and assume that the matrix \( E \) representing the Euler form, given by (1.1), is invertible. Then, by the Auslander–Reiten formulas

\[
\text{Ext}_{Q}^1(V, W) \simeq \text{Hom}_{Q}(V, \tau W)^*,
\]

\[
\text{Hom}_{Q}(V, W) \simeq \text{Ext}_{Q}^1(V, \tau W)^*.
\]
and by the Euler form (1.4) it follows that

\[ \langle d, e \rangle = -\langle e, \tau d \rangle. \]

From (1.2) we know that \( \langle d, e \rangle = d^t E e \). Then by applying it to both sides of the above equality we can deduce that \( \tau = -E^{-1}E^T \).

More generally, \( \langle \omega, d \rangle = -\langle d, \tau \omega \rangle = -\langle \tau^{-1}d, \omega \rangle \) holds for any dimension vector \( d \) and thus, by substitution, we obtain

\[ \Theta_{\text{can}} = \langle d, \omega \rangle - \langle \omega, d \rangle = \langle d + \tau^{-1}d, \omega \rangle. \]
Bibliography


Bibliography


Bibliography


