Three Contributions on Irreversibilities in Economics and Finance

Dissertation zur Erlangung des Doktorgrades der Wirtschaftswissenschaften (Dr. rer. pol.) an der Fakultät für Wirtschaftswissenschaften der Universität Bielefeld

vorgelegt von

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Januar 2024

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Gedruckt auf alterungsbeständigem Papier °° ISO 9706

In any case, the mathematician
sees hundreds and thousands of
formidable new problems in dozens of
blossoming areas, puzzles galore, and
challenges to his heart's content.
He may never resolve some of these,
but he will never be bored.
What more can he ask?

– Richard Bellman, 1966

Acknowledgments

There are many people I want to thank for their support and encouragement during my doctoral studies.

First, I would like to thank my supervisor, Giorgio Ferrari. Giorgio has been a true mentor for me over the past three years and introduced me to the topics of mathematical finance and stochastic control theory, but also to the world of research in general. He showed me how precise and diligent work leads to fruitful results, but at the same time how important it is to maintain a good mood and derive enjoyment from one's work. I have benefited enormously from his contagious enthusiasm, ideas, and constant encouragement and working with him has been immensely enjoyable.

Second, I want to thank Frank Riedel. He was always very supportive during my time and with his critical questions and deep expertise across fields he provided me with great input and feedback for my projects. During my time at the Center for Mathematical Economics (IMW), he also gave me the chance to get an insight into the process of third party funding and academic management. Moreover, as the director of the IMW, he created an amazing learning and working environment, especially for young researchers.

I owe another special thanks to Stéphane Villeneuve, who invited me for a research stay at the Toulouse School of Economics. Stéphane was a great host and gave me the opportunity to talk about my ideas, attend seminars and workshops, and get in touch with other young researchers. During my stay, we exchanged many ideas that resulted in a joint project together with Neofytos Rodosthenous from the University College London. I would also like to thank him for the great experience I had working with him and the many helpful ideas and precise comments.

Furthermore, I want to thank all the members of the IMW. Each of them created a friendly and fun atmosphere, that made it a pleasure to get to work every day and made these three years fly by. I also thank the Bielefeld Graduate School of Economics and Management (BiGSEM) and the department of Business Administration and Economics for their administrative support. Moreover, my gratitude goes to the CRC 1283 for their financial support throughout these years as well as the Bielefeld Graduate School in Theoretical Sciences (BGTS) and the German Academic Exchange Service (DAAD) for financing my research stay in Toulouse.

This thesis would not have been possible without my family and friends. I want to thank my parents and siblings for their endless support, and Nils for being a great friend along the way. Finally, I want to thank Lea for her encouragement, support, and all the happiness she brought into my life.

Contents

Chapter 1

Introduction

Decision-making is an inherent part of our everyday lives. We seek to find optimal strategies for a variety of different decisions, that range from routine to life-altering. Given the complex world we live in, identifying an optimal decision is not always a trivial task. Moreover, it is essential to understand the key factors that influence our decisions: What characteristics impact our decision-making process? When is it advantageous to delay a decision? What criteria define an optimal decision?

Dixit and Pindyck (1994) identified three important features of decision-making processes, that were initially motivated by economic applications but are broadly applicable for decisions in other contexts: *timing*, *uncertainty/randomness*, and *irreversibility*.
As a matter of fact, many decisions are not now-or-never, but decision-makers are relatively free in timing their actions in the span of a given time horizon. This opportunity gives them the chance to obtain more information about the problem or to wait for better conditions that favor immediate action.
This becomes especially relevant when considering the degree of randomness of the environment in which these decisions are made. Foreseeing the future is usually not a tool at hand, and decision-makers resort to accessing the probabilities over future outcomes to base their action on these estimates. In the economic context, examples of unpredictable future evolutions include the price dynamics of a tradeable asset or the value of investment opportunities subject to changing market conditions.
Moreover, many decisions exhibit a degree of irreversibility. Actions, once taken, cannot be reversed without any cost, and important instances include investment decisions entailing a sunk cost that cannot be fully recovered or the liquidation of an asset without a feasible or desirable option to repurchase. In the presence of irreversibility, taking action thus represents a substantial commitment, since decision-makers give up the option to delay the decision or even refrain from taking the decision entirely. This “option value”, arising from the irreversibility of the decision, was overlooked for a long time in the (economic) literature, but is crucial when

aiming to obtain optimal strategies.

In this thesis, we consider three instances where decision-makers are faced with problems of optimization in the presence of irreversibility and the option of delaying decisions. We tackle these problems within the framework of stochastic control theory, in particular the classes of *optimal stopping* and *singular stochastic control*.
In optimal stopping problems, decision makers are usually restricted to act only once, such that the problem writes as an optimal timing problem. In contrast, singular stochastic control problems allow agents to dynamically intervene in the random environment and thus control the evolution of the underlying process. In both classes of problem, and thus independent of the number of interventions, each action has an irreversible impact on the system.
In Chapter 2 we consider the optimal execution problem of an investor, who aims at selling her finite amount of assets in the market. Selling is irreversible and causes an adverse price reaction, such that large orders could depress the market price. Moreover, the investor has only incomplete information on the return of the asset and monitors the evolution of the asset's price to learn its true value.
Chapter 3 deals with a problem of optimal debt management of a country, where we model a potential political game between two parties: a government and a legislative body. Both players are able to affect the debt-to-GDP ratio in an irreversible fashion. While the government aims at issuing new debt to finance its spending, the legislative body could impose a limiting mechanism to avoid a debt crisis. Since the objectives of the two players diverge, our formulation gives rise to strategic interaction.
In Chapter 4 we consider the optimal investment problem of a continuum of firms in the presence of a carbon pricing system. Each (polluting) firm has the opportunity to invest in a green technology that achieves carbon neutrality. The investment is irreversible, and the optimal timing depends on the imposed carbon price. Our stationary equilibrium framework allows us to determine an equilibrium carbon price that achieves a certain emission limit among the regulated firms.

Optimal policies in these problems usually take the form of barrier rules, thus prescribing to take action at a point when the state variable reaches a certain upper or lower threshold and doing nothing before this happens. For instance, the optimal liquidation rule prescribes to sell an amount of assets only whenever the market price is sufficiently large. In fact, optimal policies involve inaction for most of the time and exerting control only occasionally – again reflecting the value of waiting that arises in problems of irreversibility and the option to delay. For a broad discussion on this *economics of inaction*, we refer to the identically named book of Stokey (2008), and to Arrow (1968), McDonald and Siegel (1986), and Pindyck (1988, 1990) for early contributions.

In the following, we dive deeper into the topic of stochastic control theory, serving as a rigorous mathematical tool in order to tackle the complex problems we consider. First, we explore its historical roots and the usual methods of solving. We then discuss the particular classes of singular stochastic control and optimal stopping problems in more detail, as they find use in the three main contributions of this thesis. Afterwards, we give a comprehensive introduction to each of them.

The Origin of Stochastic Control

Stochastic Control Theory emerged in the 1950s, stimulated by the independent study of differential games in both the former Soviet Union as well as the United States. The typical structure of a (stochastic) control problem includes a state process and a number of agents with their control policy. This policy may influence the system's evolution as well as the criterion that measures its performance. Agents aim at identifying a strategy that optimizes their objective. Interestingly, the two foundational and most popular approaches in order to address problems of stochastic control trace back to the very beginning: Bellman's optimality principle and Pontryagin's maximum principle.

Bellman's optimality principle – also called dynamic programming principle – dates back to the contributions of Richard Bellman and his colleagues at the Rand Corporation. After some first publications (see Bellman, 1952, 1953), it was found that this approach has a natural application in optimal control problems (see Bellman, 1954). Firstly formulated in deterministic settings, the approach was quickly adapted to stochastic settings involving Itô-type stochastic differential equations as state equations (see Kushner, 1962) and then studied and refined in numerous contributions (see, e.g., Fleming and Pardoux, 1982; Krylov, 1980).
The fundamental concept underlying the dynamic programming approach in the context of optimal control is that if an (optimal) control strategy is selected and followed until an arbitrary time, then, given the information acquired, it remains optimal to continue the same strategy afterwards. In mathematical terms, this leads to considering a family of optimal control problems with varying initial data, encompassing different times, and states. Each problem is identified by a value function that represents the optimal payoff value as a function of this initial data. Bellman established a connection between these problems by introducing an evolution equation known as the Hamilton-Jacobi-Bellman (HJB) equation. Closely related to the Hamilton-Jacobi equation in mechanics, the HJB equation is the infinitesimal version of the dynamic programming principle and describes the local behaviour of the problem's value function. Given as a nonlinear first order (in the deterministic case) or second order (in the stochastic case) partial differential equation, it characterizes the value function as its unique solution in a well-defined sense. Consequently, to solve an optimal control problem using the dynamic programming approach, the objective is to identify a sufficiently smooth solution to the HJB equation and verify that this function identifies with the true value function under appropriate assumptions. In many formulations of the problem, this approach also facilitates the construction of an optimal feedback control. This procedure, mainly relying on Itô's formula, is the so-called verification technique. Notably, the dynamic programming approach yields a solution for the entire family of optimal control problems, each identified by their initial data, and thereby provides a solution for the original problem.

Another approach, developed around the same time as Bellman's principle, was initiated by Lev Pontryagin (see Pontryagin et al., 1962). The maximum principle states the existence of an adjoint equation, which is an ordinary differential equation (in the deterministic case) or a stochastic differential equation (in the stochastic case). The (extended) Hamiltonian system, consisting of the state equation, a maximum condition and this adjoint equation, again yields a solution to the optimal control problem. Like Bellman's principle, it was firstly formulated for the deterministic case, subsequently extended to the stochastic diffusion case (see Kushner, 1962, 1965; Kushner and Schweppe, 1964), and refined to allow for control dependent state coefficients (see Peng, 1990). For a comprehensive overview, we refer to Yong and Zhou (1999).

Since these early days of modern (stochastic) control theory, the research area has not only sparked interest among mathematicians, but also among researchers interested in their application in various fields such as Economics, Physics, Engineering and Biology. Here, given the impracticality of a thorough overview, we briefly mention the problem of production planning (see Bensoussan et al., 1983; Bensoussan et al., 1984), portfolio allocation and consumption models (see J. Cox et al., 1985; Merton, 1969, 1975) and optimal dividend payout schemes (see Asmussen and Taksar, 1997; Jeanblanc-Pique and Shiryaev, 1995; Leland and Toft, 1996) and refer the interested reader to Pham (2009) and Yong and Zhou (1999).

Singular Stochastic Control

A particular class of stochastic control problems that we consider in this thesis are the so-called singular stochastic control (SSC) problems. The motivation behind these problems stems from real-life applications where control strategies are able to significantly displace the state at some specific times, a feature that one is not able to capture in regular stochastic control models. A prime example was mentioned in the listed contributions above: the optimal dividend payout scheme. Since firms (usually) do not pay dividends at a continuous rate but rather distribute them as a lump sum payment, the controlled reserves of firms exhibit a shift on the dividend payday. We highlight that this characteristic feature was already discussed and considered in the mentioned contributions of Asmussen and Taksar (1997) and Jeanblanc-Pique and Shiryaev (1995). Mathematically, in singular control problems, the agent's actions produce an effect on the state and the performance that is (linearly) proportional to the action's size, thus influencing them instantaneously. Unlike regular control cases, the player's strategy at a specific point in time is not the action taken at that precise moment but rather the cumulative action up to that point. Consequently, the control is usually nondecreasing (highlighting the feature of irreversibility), singular with respect to the Lebesgue measure, and belongs to the set of processes with bounded variation.

Addressing singular stochastic control problems still involves techniques similar to those employed for regular control cases. Within a Markovian framework, the dynamic programming approach again leads to a Hamilton-Jacobi-Bellman equation that the corresponding value function is expected to solve in a suitable sense. However, this equation now takes the form of a second-order ordinary differential equation (ODE) in one dimension or a partial differential equation (PDE) in multiple dimensions, featuring a local gradient constraint. As such, the problem is closely linked to the theory of free-boundary problems. Here, the state space of the problem splits in two distinct regions: The inaction and the action region. Separated by an a priori unknown free boundary, that becomes part of the solution, these regions specify when it is optimal to exert control and when it is not. Usually, the optimal control is given as the solution to a *Skorokhod reflection problem* and prescribes to keep the controlled state process within the inaction region with minimal effort, thus – apart from a potential initial jump – reflecting it at the free-boundary in the direction of the inaction region.

The theory of singular stochastic control problems originated with Bather and Chernoff (1967), who considered an application of optimal spacecraft control. Beneš et al. (1980) proposed a one dimensional problem that was explicitly solved by the principle of smooth-fit, which conjectures a smoothness condition of the problem's value function at the free-boundary. We also mention Harrison and Taksar (1983), Karatzas (1983) and Chow et al. (1985) for early contributions, that provided an almost complete analysis of the one dimensional problem. For higher dimensions, the Hamilton-Jacobi-Bellman equation features a PDE and the problem becomes more intricate and is closely linked to the study of the smoothness of the value function and of the free-boundary, as well as the Skorokhod reflection problem (see, e.g., Evans, 1979; Soner and Shreve, 1989).
The particular feature of the singular stochastic control, causing an instantaneous effect on the state equation and the performance criterion, gave rise to a variety of different applications. Here, we mention the optimal liquidation problem (see, e.g., Becherer et al., 2018; Guo and Zervos, 2015), optimal inventory management (see, e.g., Federico et al., 2023; Guo et al., 2011; Harrison and Taksar, 1983), investment and portfolio selection problems (see, e.g., Baldursson and Karatzas, 1996; Bank and Riedel, 2001; Chiarolla and Ferrari, 2014; Davis and Norman, 1990; Ferrari, 2015; Riedel and Su, 2011) as well as the previously mentioned dividend payment problem (see, e.g., De Angelis, 2020; Løkka and Zervos, 2008; Radner and Shepp, 1996).

In this thesis, we consider two distinct problems of singular stochastic control, each motivated by variants of the previously mentioned applications.
In Chapter 2 we explore an optimal liquidation problem faced by a large investor. The investor aims at selling her finite amount of assets in the market, inducing an adverse price reaction during the selling process. This problem relates to the literature on optimal execution (see, e.g., Almgren, 2003; Almgren and Chriss, 2001), and we consider the feature of singular stochastic control in order to allow the investor to sell a large fraction of assets instantaneously (see also Guo and Zervos, 2015). Furthermore, we depart from the conventional assumption that the investor is able to completely observe the system state. Instead, we assume partial information on the future trend of the asset's price dynamics, which is encapsulated as a drift coefficient modelled as a Bernoulli random variable. The investor observes the price dynamics on the market and updates her belief regarding the true value of the drift. We find that the optimal execution rule prescribes to sell an amount of assets whenever the price process exceeds a

target price, that depends on the current belief of the investor. Due to the price impact of selling, the price process is reflected at this belief-dependent boundary. For more details, we refer to Section 1.1, where we introduce the problem and our solution method.
Chapter 3 relates to the literature on optimal stochastic control for public debt management. In contrast to the existing literature (see, e.g., Brachetta and Ceci, 2022; Cadenillas and Huamán-Aguilar, 2016, 2018; Ferrari, 2018), we consider a potential political game between two parties – a government and a legislative body – that are both able to affect the state dynamics, which is the controlled debt-to-GDP ratio. Since the objectives of the players diverge, our formulation leads to a non-zero-sum game of singular stochastic control, giving rise to strategic interaction. In our derived solution, we resort to the notion of Nash equilibria, in that we find (optimal) strategies for the two players and ensure that no player has an incentive to deviate unilaterally. We find that the optimal strategy of the government always prescribes a debt issuance policy, which is activated whenever the debt-to-GDP ratio falls below a certain threshold. In contrast, the legislative body's optimal policy strongly depends on their time preference rate. For large values, such that future costs are discounted at a high rate, we find that a laissez-faire policy is optimal. For a broader introduction to the problem, we refer to Section 1.2.

Optimal Stopping Problems and the Real Options Approach

As explained above, deriving solutions for problems involving singular stochastic control is not a straightforward task. An important solution method, that we adopt in Chapter 2 of this thesis, involves the link to *optimal stopping problems*. Unlike the problems discussed earlier, where the time horizon is either fixed or indirectly influenced by the control strategy, optimal stopping problems refer to a class of control problems wherein the terminal time can be directly controlled.

The link to optimal stopping problems dates back to the very beginnings of singular stochastic control problems, with Bather and Chernoff (1967) and Beneš et al. (1980) unexpectedly discovering this connection. Since then, numerous authors have contributed on this subject in order to make this link rigorous, and we mention Baldursson (1987), El Karoui and Karatzas (1988, 1991), Karatzas (1983, 1985), and Karatzas and Shreve (1984, 1985, 1986). The core concept is that the gradient of the SSC problem's value function – in the direction of the controlled variable – can be identified as the value function of an associated optimal stopping problem in which the underlying process remains uncontrolled. Furthermore, it relies on the shared structure of the state space in the two interrelated problems: the inaction region corresponds to a waiting region, while the action region is relabeled as a stopping region and includes all points in the state space at which the agent should stop the evolution of the underlying dynamics. Most importantly, the free-boundary separating the two regions remains the same. Consequently, in order to study the geometry of the state space and to derive properties of the free-boundary as well as the value function, the (potential) link allows to switch to an associated optimal stopping problem and use the established techniques and methodology in this field of research. Nevertheless, one should not take this link between the two problem classes for granted. Instances such as those highlighted by De Angelis et al. (2015, 2019) and the problem explored in Chapter 3 exemplify situations where this connection is not applicable.

Optimal stopping problems should however not be confused as solely a mathematical tool for studying problems involving singular stochastic control. In fact, its theory as well as their applications date even further back to the contributions of Wald (1947) within the context of sequential decision making. Snell (1952) was first to formulate a general optimal stopping problem for discrete-time stochastic processes, and subsequent authors followed by connecting continuous-time optimal stopping to free-boundary problems (see, e.g., Chernoff, 1961; Dynkin, 1963; Shiryaev, 1961a,b, 1963 and Peskir and Shiryaev (2006) and Shiryaev (1978) for an overview). A prime application of optimal stopping is the valuation and hedging of American-style options in finance (see, e.g., Duffie, 2010). Here, the option buyer possesses the right (but not the obligation) to exercise the option at any time before the expiration date, aiming to maximize expected gains. Interestingly, this also gave rise to a strand of literature dedicated to the study of so-called *real options*. A typical example is that of a firm that holds the right to invest in an opportunity yielding a random payoff. If the investment is irreversible, this problem is closely related with the American call option problem. Consequently, the term *real option* emphasizes the option-like characteristic of such investments. This methodology and approach has received much attention in the finance and economics literature with seminal papers of Myers (1977) and McDonald and Siegel (1986) as well as Alvarez (2001), Dixit (1989), Dixit and Pindyck (1994), Guo and Pham (2005), and Pindyck (1988, 1990).

In this thesis, we contribute to the existing literature by modelling the real option problem of a firm in the context of carbon pricing and green technology adoption. We assume that polluting firms are subject to a carbon pricing system and have the option to undergo an irreversible investment to become carbon neutral. The single-firm problem is solved by constructing a solution to the associated free-boundary problem, and its optimality is then verified through a verification theorem (see, e.g., Alvarez, 2001). We find that the firm's optimal strategy prescribes to invest when their technology shock process exceeds a certain threshold, which depends on the imposed carbon price. Furthermore, we embed the single-firm problem in a stationary equilibrium framework in the spirit of Hopenhayn (1992) and Hopenhayn and Rogerson (1993), where we derive a stationary distribution of incumbent firms. Here, the carbon price – that was fixed in the real option problem of the single firm – is endogenized by introducing two equilibrium conditions. For further details on the framework and our methodology, we refer to Section 1.3.

In the subsequent sections of this introduction, we delve into a more detailed examination of the problems considered in this thesis. Specifically, we elaborate on their motivation, present related literature, define the problem setting, and outline the method of solving.

1.1 Chapter 2: Optimal Execution with Permanent Price Impact and Incomplete Information on the Return

In this chapter, we consider an investor who possesses a fixed amount of assets and aims at selling them on the market.[1] We assume that the investor faces the issue of causing an adverse price reaction, so that fast selling depresses the stock price, while splitting the order over time may take too long. This problem – also known as the *optimal execution problem* in algorithmic trading – thus deals with the question of how to trade optimally in order to maximize a given profit, and therefore of how to determine the time as well as the size of the order.

Dating back to the early works of Bertsimas and Lo (1998), Almgren and Chriss (2001) and Almgren (2003), the study of optimal execution strategies has received much attention and resulted in a series of important contributions in various settings, which, amongst other modeling features, can be distinguished with respect to the considered type of price impact: Additive or multiplicative. A comprehensive discussion on the latter class of models can be found in Guo and Zervos (2015), who also point out that models with multiplicative price impact seem to be more natural since they ensure prices to remain positive. Amongst those works dealing with multiplicative price impact, let us mention Bertimas et al. (1999) for a discrete-time framework, Forsyth et al. (2012) for a continuous-time model à la Black-Scholes, Guo and Zervos (2015) and Becherer et al. (2018) for settings involving singular stochastic controls.

A common feature in the literature is the assumption that the investor has full information on the trend of the asset. This, however, can be a strong requirement. As pointed out by Ekström and Lu (2011), a statistical estimation of the drift is not an efficient procedure, and obtaining a reasonable precision would need data of decades or even centuries under the same market conditions – which is simply not feasible in reality (see also the discussion in Rogers, 2013, Section 4.2). In some cases, such as initial public offerings, this price history does not even exist.

To account for this fact, we propose a model of optimal execution with multiplicative price impact in which the drift of the stock price dynamics is a random variable, which is not directly observable by the investor. Through monitoring the evolution of the price on the market, the investor is able to update her belief regarding the drift value. However, such observation is noisy as the investor cannot perfectly distinguish whether price variations are caused by the drift or the stochastic driver of the underlying dynamics. From a mathematical point of view, our model leads to a finite-fuel singular stochastic control problem under partial observation, and we investigate how the presence of incomplete information influences the selling strategy of the investor. In particular, we show that the flow of incoming information

[1] This chapter is based on joint work with Giorgio Ferrari. Parts of this introduction and Chapter 2 have been published in Dammann and Ferrari (2023).

– through the observation of the asset's market price – has a direct effect on the optimal execution rule. Indeed, differently to the case of full information treated in Guo and Zervos (2015), the decision to sell is no longer triggered by a constant critical price, but the execution threshold changes dynamically depending on the investor's current belief on the future trend of the asset. Our results show that the optimal execution strategy is in fact determined by a boundary that is increasing in the belief in the larger drift value, corresponding to the intuition that the decision maker chooses to delay selling assets if future prices are expected to increase.

In this regard, our work relates to the bunch of economic and financial literature where questions of optimal decision-making under partial observation have been considered; amongst a large number of contributions, we refer to the seminal papers on portfolio selection by Detemple (1986) and Gennotte (1986); to Veronesi (1999) for an equilibrium model with uncertain dividend drift; to Sass and Haussmann (2004) for a terminal-wealth portfolio optimization problem, and to the more recent Colaneri et al. (2020) for an optimal liquidation problem with rate strategies and partial observation. Notably, the recent Drissi (2023) and Bismuth et al. (2019) incorporate Bayesian learning in a model of multi-asset optimal execution, although restricting the agent to absolutely continuous (regular) controls. Furthermore, we contribute to those models dealing with problems of optimal stopping and singular stochastic control. To name just a few recent works, Callegaro et al. (2020) for public debt control, De Angelis (2020) and Décamps and Villeneuve (2022) for dividend payments, Décamps et al. (2005) for investment timing, Ekström and Lu (2011) as well as Ekström and Vaicenavicius (2016) for asset liquidation, Federico et al. (2023) for inventory management, Johnson and Peskir (2017) for quickest detection, Gapeev (2022) for the pricing problem of perpetual commodity equities, and Gapeev and Rodosthenous (2021) for a zero-sum optimal stopping game associated with perpetual convertible bonds.

We now discuss the mathematical modeling and analysis. We consider an investor holding a fixed amount y of assets in her portfolio. In absence of the investor's actions, the stock price evolves according to a geometric Brownian motion $dS_t = \beta S_t dt + \sigma S_t dW_t$, where W is a standard Brownian motion and $\sigma > 0$ a constant volatility parameter. Furthermore, the price process exhibits a random future trend β, which is however unknown to the decision maker, and is assumed to be a random variable, independent of the Brownian noise, taking two values $\beta_0 < \beta_1$, for some $\beta_0, \beta_1 \in \mathbb{R}$ and $\beta_0 < 0$.
The decision maker is able to sell the assets on the market over an infinite time horizon, and we denote by ξ_t the cumulative amount of assets liquidated up to time t. Consequently, the remaining assets in the portfolio follow the dynamics $Y_t^{\xi} = y - \xi_t$. Clearly, it has to be $\xi_t \leq y$ at any time $t \geq 0$ (finite-fuel constraint), since no more than the initial amount of assets can be sold. As anticipated, we assume that the investor causes an adverse price reaction upon selling, which, following Guo and Zervos (2015), we assume to be of multiplicative type. Hence, the controlled asset's price evolves as

$$dS_t^{\xi} = \beta S_t^{\xi} dt + \sigma S_t^{\xi} dW_t - \alpha S_t^{\xi} \circ d\xi_t, \quad S_{0-}^{\xi} = s > 0,$$

where $\alpha > 0$ denotes the parameter of price impact, and the operator $\circ$ is defined as in

(2.2.3) below so to take care of the continuous and jump components of any admissible selling strategy ξ. As will be clear later, see 2.2.6, the multiplicative price impact structure allows to express the asset's price process as $S^{\xi} = \exp(X^{\xi})$. Here, X^{ξ} is then a linearly controlled drifted Brownian motion with volatility $\sigma > 0$ and drift value $\mu = \beta - \frac{1}{2}\sigma^2$.
The investor aims at maximizing the total expected discounted reward upon selling, net of transaction costs; that is,

$$\sup_{\xi} \mathbb{E}\bigg[\int_0^{\infty} e^{-rt}\big(e^{X_t^{\xi}} - \kappa\big) \circ d\xi_t\bigg],$$

where the optimization is taken over a suitable admissible class of selling strategies and the investor discounts her future revenues with a strictly positive factor $r > 0$, that can be interpreted as her subjective impatience. The latter is a finite-fuel singular stochastic control problem under partial observation.

By relying on classical filtering techniques (cf. Shiryaev, 1978, Section 4.2), we begin by determining an equivalent Markovian problem – the so-called *separated problem* – under full information (see Fleming and Pardoux, 1982 as a classical reference on the separated problem). To this end, we introduce the process Π, according to which the investor can update her belief regarding the true value of the drift. This is done by observing the evolution of the process X^0 (denoting the uncontrolled version of the process X^{ξ}), whose natural filtration $\mathcal{F}_t^{X^0}$ models the overall information available up to time t. More precisely, after forming a prior $\pi := \mathbb{P}[\mu = \mu_1] \in (0,1)$, the investor dynamically updates her belief upon the arrival of new information through observing the process X^{ξ}, so that the belief process is given by $\Pi_t = \mathbb{P}[\mu = \mu_1 \mid \mathcal{F}_t^{X^0}]$. Notice that a value of Π close to 1 indicates a strong belief in the larger value of the drift, while Π close to 0 displays a strong belief in the lower value. Hence, we expect the investor to change the liquidation strategy dynamically and not solely base it on the current price on the market, but also on the present belief at that time.
The separated problem turns out to be a *three-dimensional degenerate finite-fuel singular stochastic control problem*, so that obtaining explicit solutions through a traditional "guess-and-verify approach" is in general not feasible. We note that this would be applicable if we take $\beta_0 = -\beta_1$, which indeed allows for a dimension reduction; see, e.g., Décamps and Villeneuve (2022). In this chapter, however, we do not consider any relation amongst β_0 and β_1 other than $\beta_0 < \beta_1$.

In order to tame the multidimensional nature of the resulting optimal execution problem under full information, we then follow a direct approach which hinges on the study of a suitable optimal stopping problem with value v, that we expect to be associated to the singular stochastic control problem. This method was studied and refined by many authors such as Beneš et al. (1980), El Karoui and Karatzas (1989), and Karatzas and Shreve (1984), or De Angelis (2020), De Angelis et al. (2017, 2015), and Guo and Tomecek (2008) for more recent contributions. The optimal stopping problem, which involves the underlying two-dimensional diffusion (X^0, Π) taking values in $\mathbb{R} \times (0,1)$, can be interpreted as an optimal selling problem and exhibits a structure similar to that of the problem treated by Décamps et al. (2005)

(see also Ekström and Lu, 2011 for a parabolic version).[2] We then solve the optimal stopping problem by relying on techniques from free-boundary theory (as illustrated in the monography by Peskir and Shiryaev, 2006, Chapter 3) and first show that the optimal stopping rule is characterized through a belief-dependent free boundary $a(\pi)$ for $\pi \in (0,1)$.

However, the coupled dynamics of the underlying processes X^0 and Π, as well as the fact that they are driven by the same Brownian motion, makes a further study of the free boundary and the value function v not feasible. It is for that reason we proceed by deriving two equivalent representations of the optimal stopping problem, which allow for a thorough analysis. First, via a change of measure, the state process (X^0, Π) is transformed into (X^0, Φ) taking values in $\mathbb{R} \times (0, \infty)$ and with decoupled dynamics. Here, the process Φ is the so-called "likelihood ratio". Again, we can express the optimal stopping strategy in terms of a free boundary $\varphi \mapsto b(\varphi)$, which results from a simple transformation of the boundary $\pi \mapsto a(\pi)$. Second, we pass yet to another formulation by deriving the intrinsic parabolic formulation of the stopping problem in coordinates (X^0, Z), in which the process Z now follows purely deterministic dynamics and takes values in $\mathbb{R}$. Even though the monotonicity result of the associated free boundary $z \mapsto c(z)$ is certainly not trivial to derive and calls for a rigorous technical analysis, it is in this formulation that we are able to provide further regularity results of c and of the transformed optimal stopping value function $\widehat{v}$. In fact, borrowing arguments from De Angelis (2020), suitably adapted to the present setting, we achieve a global regularity of $\widehat{v}$, namely $\widehat{v} \in C^1(\mathbb{R}^2)$. The latter result also allows proving $\widehat{v}_{xx} \in L^{\infty}_{\text{loc}}(\mathbb{R}^2)$, and finally obtaining a nonlinear integral equation uniquely solved by the optimal stopping boundary c. It is worth mentioning that such a characterization can be translated back to both optimal stopping boundaries b and a and is thus tantamount to a complete specification of the optimal stopping rule in the original (x, π)-coordinates.

The thorough analysis developed for the optimal stopping problem is then exploited in order to identify an optimal execution strategy. In fact, the derived regularity results for $\widehat{v}$ permits us to prove a verification theorem, that identifies an optimal execution rule and shows that the optimal stopping value function v indeed coincides with a directional derivative of the separated problem's value function V. Namely, we show that

$$V(x, y, \pi) := \frac{1}{\alpha} \int_{x-\alpha y}^{x} v(x', \pi) dx', \quad (x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1).$$

Notice, that if $\alpha \downarrow 0$, one finds $V(x, y, \pi) = yv(x, \pi)$, which is the value of the problem in which the investor has no market impact.

The optimal execution rule can be thought of as a "myopic one". Indeed, it prescribes to sell assets as if the size of the investor's portfolio were infinite, and to stop selling once the asset's inventory is depleted (see also Karatzas, 1985 and El Karoui and Karatzas, 1989). The optimal

[2]However, the specific choice of possible drift values assumed in Décamps et al. (2005) allows for the explicit construction of a solution to the related variational inequality, which is instead not possible in our context.

selling rule involves lump-sum executions (whenever the asset's price is sufficiently large), that could eventually result into an immediate depletion of the portfolio (if the initial portfolio size is sufficiently small). However, for relatively large portfolios, an initial lump-sum selling is followed by a policy of oblique reflection type. This is triggered by the belief-dependent boundary $\varphi \mapsto b(\varphi)$ (equivalently, $\pi \mapsto a(\pi)$). Notably, given that all the transformations developed for the resolution of the optimal stopping problem are one-to-one and onto, the integral equation for the boundary $z \mapsto c(z)$ yields an integral equation for $\varphi \mapsto b(\varphi)$, and therefore a complete characterization of the optimal execution rule. In order to provide insights about the sensitivity of the optimal decision mechanism of the investor with respect to the model's parameters, we develop a recursive numerical scheme, which relies on an application of the Monte-Carlo method.

Overall, we believe that the contributions of this chapter are the following. Even though the literature on optimal execution problems is extensive (see, to name just a few, Almgren and Chriss, 2001, Almgren, 2003, Becherer et al., 2018, Bertsimas and Lo, 1998, Bertimas et al., 1999, Colaneri et al., 2020, Gatheral and Schied, 2011, Guo and Zervos, 2015, Moreau et al., 2017, Schied and Schöneborn, 2009), the combination of incomplete information on the future price trend while allowing for lump-sum as well as singularly continuous executions constitutes a novelty. Furthermore, the present study on the optimal execution strategy complements as well as extends the literature on problems with a similar structure under full information. As a matter of fact, the derived optimal execution rule exhibits a broader structure and prescribes to take actions depending on the current belief on the future trend of the asset.
From a mathematical point of view, to the best of our knowledge, ours is the first work providing a complete characterization of the value function and of the optimal control rule in a finite-fuel singular stochastic control problem under partial observation (which, in the present setting, is equivalent to a three-dimensional degenerate singular stochastic control problem). Furthermore, we believe that the optimal stopping (selling) problem, studied as a device to characterize the optimal solution of the optimal execution problem, is of interest of its own. By performing a thorough analysis on the regularity of (a transformed version of) its value function and free boundary, we are able to provide a complete characterization of the optimal selling rule through a nonlinear integral equation, thus extending the results of the related model studied by Décamps et al. (2005). Notice, that an integral equation for the free boundary has been obtained also in Ekström and Lu (2011) and Ekström and Vaicenavicius (2016), though in settings where the parabolic nature of the problem is arising because of an explicit time-dependency. Finally, the probabilistic numerical approach developed for the resolution of the free boundary's integral equation allows to understand the dependency of the investor's optimal execution strategy on relevant model's parameters such as volatility and trend. Moreover, based on the numerical evaluation of the boundary, we can compare the value of the control problem with partial information with that of an associated *average drift* problem under full information. This allows us to numerically evaluate the question on whether the introduction of uncertainty over the drift actually harms or benefits the investor.

1.2 Chapter 3: A Stochastic Non-Zero-Sum Game of controlling the Debt-to-GDP Ratio

There is probably no more topical issue in macroeconomics than the determination of the optimal level of debt that favours both its sustainability and the long-term growth of an economy.[3] Although paramount and extensively studied in the literature (see Barro, 1989 and Dornbusch and Draghi, 1990 for a general presentation), the question of the optimal debt level has not yet received clear theoretical foundations. This lack of a consensual theoretical framework has led to the implementation of exogenous mechanisms to monitor the level of debt. In the USA, one of these mechanisms is the statutory debt ceiling which restricts the amount of debt a government can be permitted to issue[4].
The traditional analysis of public debt has shown that high public debt has a negative effect on long-term economic growth, thus giving an argument to debt ceiling advocates. Indeed, a high level of debt generates high risk premiums that reflect creditors' doubts about the government's ability to refinance itself. Being unable not only to repay its debts but also to pay for the excess of its expenditures over its revenues, the government must then immediately balance its budget by taking exceptional measures, like increasing taxes and/or cutting its investments, which can have a dramatic impact on growth.

Motivated by this, the theoretical literature on debt management problems has focused on the stochastic control problem faced by a single decision maker to determine the optimal debt reduction policy and thus the debt ceiling, i.e. the level of debt-to-GDP ratio (also called "debt ratio") at which the government should intervene in order to reduce it. In Cadenillas and Huamán-Aguilar (2016, 2018), the debt ratio evolves as a controlled one-dimensional geometric Brownian motion that can be reduced via singular and bounded variation controls, respectively, in order to minimize the expected total costs resulting from the instantaneous cost of the debt ratio and intervention costs. Ferrari (2018) studies the optimal debt ratio reduction problem posed as a fully two-dimensional singular stochastic control problem, where the government takes into consideration the evolution of the inflation rate (evolving as an uncontrolled diffusion process) of the country as well. Callegaro et al. (2020) consider a model with partial observation, where the growth rate of GDP follows an unobservable Markov chain.

On the other hand, the positive effect of a high level of public debt on growth should not be overlooked, since public investments in social policies, education, healthcare, justice, research, and infrastructure help private initiatives to develop effectively. As Blanchard observed in his presidential address to the American Economic Association (see Blanchard, 2019), as long as the interest rate is lower than the growth rate, a large deficit can be allowed without

[3]This chapter is based on joint work with Stéphane Villeneuve and Neofytos Rodosthenous. Parts of this introduction and Chapter 3 have been published in Dammann et al. (2023).

[4]Within the European Community, a similar mechanism exists since the Maastricht treaty set 60% as the upper bound for the debt-to-GDP ratio for members of the European Union.

decreasing the debt ratio. Notable contributions that consider this ambivalent effect of public debt include Ferrari and Rodosthenous (2020), who model the growth rate of GDP by a continuous-time Markov chain and the government is allowed to both decrease and increase the debt ratio, as well as Brachetta and Ceci (2022), who consider a model of regular controls where interventions via fiscal policies affect the public debt and the GDP growth rate at the same time.

A common feature in all aforementioned papers is that an optimal level of debt ceiling is endogenously determined. However, the potential political game between a government and a legislative body (e.g. the US Congress), whose political interests may be divergent, has not been considered so far. In this chapter, we contribute to the literature on debt management by proposing a game that incorporates the opposing interests and strategic interaction between these two players. Our modeling framework results in a non-zero-sum game between a government, whose mandate is to manage its public debt issuance policy to finance its spending, while a legislative body is concerned with imposing a mechanism limiting the amount of debt to avoid a potential debt crisis. In mathematical terms, each player exerts a monotone control to set the path of the stochastically evolving debt ratio. The first player (government) can increase the level of debt ratio by exerting its control, while the second player (legislative body) can decrease the level of debt ratio by implementing exceptional measures. Each player aims at minimizing their own total cost functional and we allow the rate of increase/reduction of each player to be unbounded and have an instantaneous effect on the debt ratio. Consequently, this leads to the formulation of a stochastic non-zero-sum game of singular controls.

Even though there exists a considerable literature on one-player singular control problems (e.g. Bather and Chernoff, 1967, Beneš et al., 1980, Karatzas, 1981 and many others), the literature on non-zero-sum stochastic games with singular controls is still limited. Kwon and Zhang (2015) study a game of competitive market share control, where each player can make irreversible investment decisions via singular controls as well as decide to exit the market, and obtain and characterise Markov perfect equilibria. Our work is more closely related to De Angelis and Ferrari (2018), who prove the existence of a Nash equilibrium in the class of Skorokhod-reflection policies, by establishing a new connection of a non-zero-sum game of monotone controls with a non-zero-sum stopping game. In order to achieve this connection, it is necessary that both players have the same discount factor (or equivalently time preferences) and that the running cost is a differentiable function – same assumptions are imposed also in Kwon and Zhang (2015). In this chapter, we relax both of these assumptions, by considering different time preferences for each player and a non-differentiable running cost function. This results in the need for a different methodology to prove the existence of a Nash equilibrium. We first study separately two coupled constrained stochastic control problems faced by the two players, and then search for Nash equilibria in the game, i.e. where no player has an incentive to deviate unilaterally. From the government's perspective, assuming that the legislative body imposes a debt ceiling b (or equivalently a Skorokhod reflection policy for the debt ratio process at b), we investigate the optimal policy of the government for issuing new public debt. To this end, we establish a connection between this constrained stochastic control problem

and a free-boundary problem, that we solve via a guess-and-verify approach. As a result, we show that the best debt issuance policy is to reflect the debt ratio process upwards at a level $a(b)$, which clearly depends on the imposed debt ceiling mechanism by the legislator.
Consequently, considering that the government is going to use a debt issuance policy of reflecting the debt ratio process at a level a, the legislator should decide on whether to impose a debt ceiling. In particular, if there is already a statutory exogenous debt ceiling, is it optimal to raise it and by how much? Our results suggest that a debt ceiling $b(a)$ should indeed be imposed for a specific range of the legislative body's time preference rates. For larger values – implying that the legislator discounts future costs more heavily – the legislator's optimal strategy is in fact realised by a laissez-faire policy in which no debt ceiling mechanism is imposed.

The main contribution of this chapter is to eventually prove the existence and uniqueness of a Nash equilibrium in the game. We show that – depending on the legislator's time preference rate λ – two qualitatively different Nash equilibria exist in the game. More precisely, for specific values of λ the Nash equilibrium prescribes that the debt ratio is kept inside the interval $[a^*, b^*]$ with the minimal cost, associated to Skorokhod reflection policies. On the other hand, for large values of the legislative body's time preference rate, the legislative body should optimally not intervene, and an associated Nash equilibrium without debt ceiling is proved to hold. Interestingly, we prove that the optimality of adopting such a strategy relies solely on the legislative body's time preferences compared to the parameter constellation in the model – it does *not depend* on the actions of the opposing player – see specifically our results in Section 3.4.1. When the discount rate of future costs is high, the consideration of the risk of a debt crisis in the future is too low to make the implementation of a debt ceiling mechanism optimal.

Finally, our rigorous mathematical treatment of the problem is complemented by a comparative statics analysis of the equilibrium thresholds with respect to some of the model's parameters. For instance, we find that an increase in the interest rate on government debt leads to a lowered debt-issuance threshold, since holding debt becomes more costly for the government. At the same time, the legislative body decreases its implemented debt ceiling, since countries with larger cost of debt are more in danger of defaulting. It is also interesting to notice that a legislative body could switch from a laissez-faire policy to a debt ceiling mechanism for larger values of interest rate. An opposing effect can be observed for the growth rate of GDP, which implies that faster growing economies could allow for a larger deficit with less – or even no – interventions from a legislative body.

1.3 Chapter 4: A Stationary Equilibrium Model of Green Technology Adoption with Endogenous Carbon Price

In this chapter, we develop a model of carbon pricing within the framework of a general equilibrium model with strategic green technology adoption.[5] The analysis is conducted for a competitive economy that is in a steady state or stationary equilibrium, where equilibrium variables remain constant over time.

Climate change is one of the major topics in today's world and addressing the challenge of reducing global emissions, as a large contributor of global warming, is a broadly studied and thoroughly discussed topic across several disciplines. Within the economic literature, ever since the contributions of Nordhaus (1977), it is widely accepted that an efficient way to decrease emissions is by imposing a fee on the emissions of carbon dioxide and other greenhouse gases (cf. Stern, 2007). Its underlying idea is that by attaching a price to emissions, it creates financial incentives for those regulated to reduce their emissions, encourage them to adopt cleaner technologies, or to invest in renewable energy sources. As of today, approximately 30% of the world's total carbon emissions are subject to a carbon pricing system designed with the explicit goal of constraining emissions.

The study and development of (effective) mechanisms for attaching a price to emission of pollutants is an important topic in the literature on environmental economics, and essentially boils down to two fundamental concepts.

In the market-based or quantity approach, a regulator determines the maximum allowable level of overall emissions. Often implemented via a so-called cap-and-trade mechanism, it is considered to be one of the most promising and cost-effective market mechanisms in the effort to reduce carbon emissions. A prominent example includes the European Union's emission trading system, which was implemented in 2005 and includes around 10.000 installations that cover around 45% of the total emissions. By setting a legally binding target for the maximum allowed level of greenhouse gas (GHG) emissions, this approach allows regulators to commit to concrete emission reduction goals on international, national or even industry specific level. Emission allowances are distributed to firms, either through auctions or as free allocations, and firms buy and trade allowances among themselves in order to account for their emissions. Consequently, this market-driven approach naturally gives rise to a market price for emission permits.
In contrast to this approach, where the quantity of overall emissions is limited, the price approach to carbon emissions directly links the emission of one unit of pollutant to a fixed cost. Typically imposed by a regulator through a tax or penalty, the resulting level of overall emissions is a priori unclear, as market participants are not regulated in the amount of emissions they are allowed to emit. However, the carbon price may be determined by estimates of the

[5]This chapter is based on joint work with Giorgio Ferrari.

price required to limit emissions below some predetermined level.
In both cases, a carbon price – also referred to as *carbon tax* or *social cost of carbon* – emerges. The price encapsulates the cost associated with the right to emit one unit of pollutant, thus serving the purpose of correcting market distortions that result from market participants not weighing in the external effects of their harmful activities. As emphasized by Nordhaus (2007), "the key economic issue is how to balance the benefits and costs of global emissions reductions." Moreover, a transparent and comparable carbon price is essential to provide incentives to firms as well as stimulate research and development in carbon neutral technologies (cf. Stern, 2007). In particular, it should transmit the social cost of carbon emissions to the decisions of firms as well as individuals.

In the literature, there exist broad contributions addressing the topic of optimal taxes on carbon and fossil fuels in various climate economics settings. Nordhaus (2014) discussed the concept of social cost of carbon in order to mitigate losses that are caused by carbon emissions. Golosov et al. (2014) study a dynamic stochastic general equilibrium model, establishing a flexible optimal tax formula compatible with the climate impacts considered in Nordhaus (2014). Acemoglu et al. (2012) and Acemoglu et al. (2016) analyse directed technical change as well as optimal carbon taxes and subsidies. Other studies focused on the price formation of emission allowances, including works by Carmona et al. (2009, 2010) and Carmona and Hinz (2011), who establish market equilibria as well as design optimal emission trading mechanisms. We also mention Aïd and Biagini (2023), proposing a Stackelberg-type equilibrium model with a regulator optimally offestting shocks in the carbon price dynamics, Colla et al. (2012), who examine endogeneous prices of permits and study the social welfare optimizing policy of a regulator, as well as Hitzemann and Uhrig-Homburg (2018), studying trading rounds of a finite set of firms which maximize over abatement and trading.

In our model, we seek to embed the issue of carbon price formation into a stationary equilibrium model with strategic abatement investment.
In the first part of this chapter, we study the decision-making process of a single firms that is subject to a carbon pricing system and faces an irreversible investment opportunity of real-option type. We characterize the firm by its technology shock process, which evolves according to a general Itô-diffusion (cf. Hopenhayn, 1992; Miao, 2005), and assume that the firm's activities lead to carbon emissions. The latter are subject to a cost – the carbon price – that decreases the firm's profits. We assume that the firm is able to choose an abatement strategy, with the aim of lowering their emissions, that takes the shape of a real option problem of irreversible investment. Investment involves a sunk cost and results in the firm reducing their carbon emissions to zero. Hence, the firm can either continue paying the carbon price to account for their emissions, or decide to exercise the option. In accordance with the existing literature (see Aïd and Biagini, 2023; Flora and Vargiolu, 2020; Huang et al., 2021), such investment should be understood as a switch to a different underlying technology, and we account for this by allowing the technology shock process to follow different dynamics after the investment.
The methodology in the first part of the chapter employs a classical optimal stopping problem

that is solved using the guess-and-verify approach. We employ techniques presented in Alvarez (2001), that are tailored for dealing with general diffusions, in order to determine the firm's optimal investment time in a carbon neutral technology, thereby becoming independent of the imposed carbon price. Our general framework, in which we do not fix any explicit profit functions or particular underlying diffusions, is enriched by including the characteristic of the firm leaving market due to low levels of technology (absorption in zero) or non-observable reasons (Poisson death).
Under reasonable assumptions on the involved diffusions and functions, we are able to derive a complete characterization of the value function as well as the optimal investment time of the firm. The latter is given by the first time the technology shock process exceeds a threshold value b^*, reflecting the intuition that technologically advanced firms are those first to invest in carbon neutral technologies. Moreover, we show that the investment threshold is decreasing in the carbon price on the market: If a polluting firm faces an increased cost on their emissions, the cost-saving effect of installing an emission reduction technology becomes larger and leads to an incentive to become carbon neutral at an earlier stage (cf. Huang et al., 2021).
The first part of this work relates to the literature of optimal timing decisions in environmental economics, with the seminal papers of Pindyck (2000, 2002) studying irreversible policy adoptions to reduce emissions of a pollutant. Since then, the real option approach to environmental investments has received much attention (cf. Boomsma et al., 2012; Detemple and Kitapbayev, 2020b; Falbo et al., 2021). Let us mention Abadie and Chamorro (2008), Brauneis et al. (2013), and Flora and Vargiolu (2020) that study emission reduction investments in the presence of diffusive carbon prices, as well as Basei et al. (2023) and Flora and Tankov (2023) as recent contributions incorporating the feature of Bayesian learning. Closest to our formulation is the work of Huang et al. (2021), studying the problem of a company that chooses to invest in an emission reduction technology in the presence of a carbon tax. Here, the emission rate is assumed to be diffusive while the investment cost admits jumps, and the authors succeed in determining the optimal investment time to reduce its emissions. Nevertheless, it is crucial to notice that in all the aforementioned contributions of irreversible investment the carbon price is either nonexistent or exogenously given by an uncontrolled diffusion or a constant tax.

In the second part of this chapter, we aim to fill this theoretical gap. To this end, we move to an aggregate level and consider a continuum of firms that are subject to idiosyncratic shocks and solve the optimal investment problem laid out in the first part. We seek to derive a stationary equilibrium consisting of an equilibrium carbon price and a stationary distribution of incumbent, polluting firms. First, since all firms will eventually exit (due to Poisson death or their technology shock process falling below zero) or invest (to become carbon neutral), we introduce new firms to the market via the so-called *entry-condition.* Here, we equate the cost of entering and the expected benefit of entering. As usual for stationary equilibrium models, this condition guarantees the balance of the inflow and outflow of firms, such that the mass of incumbent firms is constant.
Second, we introduce an *equilibrium condition* that equates net emissions, resulting from firms' production activities, and a constant emission level. Formulated in the spirit of market-

clearing condition, the latter (fixed) parameter may be interpreted as an emission target or emission limit that is either imposed by a regulator or collectively decided on by firms (see also the discussion in Remark 4.3.4). We highlight that a similar condition is specified in the recent Anderson and Duanmu (2023), studying a general equilibrium model of a government setting either quotas or taxes on emissions, and then refraining from further actions. In the case of quotas – and in the same spirit as in this chapter – the equilibrium carbon price is determined by equating quota and total net emissions. The resulting carbon price should thus not be understood as a flat tax rate on emissions that is exogenously imposed by a legislative body. Instead, it is endogenously determined by the competitive actions of firms resulting from a fixed cumulative amount of emissions among them. Hence, it reflects the cost attached to carbon emissions that is necessary to achieve a given emission target and to remain in a steady state.

It is interesting to notice that, although each firm is subject to considerable change due to its idiosyncratic noise and strategic investment decisions, the resulting equilibrium values are constant. In this regard, our work closely relates to competitive equilibrium theory, that originated in Lucas and Prescott (1971), while dynamic models with entry and exit were introduced by Brock (1972) and Smith (1974). While these models did not contain any firm specific stochastic elements, Jovanovic (1982) first introduced a model including idiosyncratic productivity shocks. The notion of stationary equilibrium with entry and exit was then developed in the seminal papers of Hopenhayn (1992) and Hopenhayn and Rogerson (1993). Their approach serves as a tool to analyse long run behaviour of dynamic industries, resulting in equilibria with constant aggregate values. Since then, their techniques received much attention, resulting in numerous contributions, including Dixit and Pindyck (1994), Chapter 8, as well as Miao (2005), who study firms' investment choices and entry and exit behaviour in related frameworks.

Even though the literature on optimal carbon price mechanisms is extensive (see, for example, Acemoglu et al., 2012, 2016; Carmona et al., 2009; Golosov et al., 2014), our approach that combines the features of stationary equilibria, endogenous carbon pricing as well as abatement strategies presents, to our knowledge, a novelty.
Our modeling of abatement as a real option problem is motivated by the irreversibility in these choices. As highlighted by Abadie and Chamorro (2008), Brauneis et al. (2013), and Flora and Tankov (2023), among others, strategies or investments of reducing emissions are usually not decisions that firms are able to revise at a continuous rate, with typical examples including the switch to a different energy source or a different underlying technology. Notice that we are able to capture the effect of the latter policy by allowing for different dynamics of the underlying technology shock process after the investment of the firm. Our main contribution is then the study of the resulting stationary equilibrium and the derivation of an equilibrium carbon price. We are thus able to diverge from the assumption of an exogenously given carbon price, and characterize it as the cost that is consistent with a predetermined emission limit. As a result, we are able to examine the interplay between the irreversible investment decisions and the equilibrium carbon price.

For exposition, in Section 4.4, we establish a specific formulation of the problem. We consider the case of technology shock processes evolving according to drifted Brownian motions and an AK-structure for firms' production functions. Additionally, we assume a damage function of emissions on production, that admits a similar structure as the one considered in Golosov et al. (2014). Within this particular model, we conduct a comparative statics analysis to explore some of the parameters' effects on the equilibrium values. For instance, we study the effect of a shift in the tax rates of polluting or carbon neutral firms. Potential regulatory strategies might involve different tax rates for these installations to incentivise polluting firms to invest in a carbon-neutral technology. We find that a policy penalizing polluting firms through increased taxes may yield suboptimal results, since firms' reduced profits discourage potential firms from entering and thus causes a decreased market competition. This, in turn, leads to a smaller force for firms to become carbon-neutral, and we observe a decreasing carbon price that results in delayed investment. On the other hand, tax benefits for carbon neutral firms as well as subsidies on the investment cost lead to an opposing effect. Indeed, we observe that improving market conditions for carbon neutral firms prompts earlier investment by polluting firms (as exemplified by a lower investment threshold) and leads to a slightly higher equilibrium carbon price. Furthermore, we observe that reducing the cumulative emissions available to firms intensifies competition among them, which leads to an increased equilibrium carbon price. This effect also prompts firms to adopt a carbon-neutral technology earlier.

It is important to notice that our general formulation of the problem allows the implementation of features and parameters beyond those specified in Section 4.4. Our reasonable monotonicity and regularity assumptions on profit and cost functions as well as the general dynamics of the underlying processes enable the study of different instances of the problem, thus drawing the focus to other parameters not captured in our specific model.

Lastly, we want to emphasize that cumulative emissions are not part of the equilibrium values at this point, but are exogenously given (for discussions on this, we refer to Nordhaus (2014) and Stern (2007)). Clearly, this limit may be determined based on environmental considerations, such as the maximal allowed amount of emissions that can be released into the atmosphere while achieving a certain carbon concentration or limiting the global temperature increase to a certain degree. As such, the question of optimally setting the emission cap is far from being trivial. In our discussion on the specific model mentioned before, we explore the concept of a regulator aiming to maximize social welfare by optimally setting the emission cap. Inspired by related contributions (see, for example, Aïd and Biagini, 2023; Colla et al., 2012; Ulph, 1996), we investigate a regulator balancing the benefits from emission inducing production and their societal damages. Our comparative statics analysis also involves a comparison of the equilibrium values with exogenous and endogenous emission limits. We observe that the regulator's actions potentially counteract or mitigate effects on carbon price and investment decisions observed in the analysis with fixed overall emissions. For instance, an increased carbon intensity of production or elevated emission induced damages on production lead the regulator to set a lower emission target, encouraging earlier investment in carbon-neutral technology.

Chapter 2

Optimal Execution with Permanent Price Impact and Incomplete Information on the Return

2.1 Introduction

In this chapter, we study an optimal liquidation problem with multiplicative price impact in which the trend of the asset's price is an unobservable Bernoulli random variable. The investor aims at selling over an infinite time-horizon a fixed amount of assets in order to maximize a net expected profit functional, and lump-sum as well as singularly continuous actions are allowed. The mathematical modelling leads to a singular stochastic control problem featuring a finite-fuel constraint and partial observation. We provide the complete analysis of an equivalent three-dimensional degenerate problem under full information, whose state process is composed of the asset's price dynamics, the amount of available assets in the portfolio, and the investor's belief about the true value of the asset's trend. Its value function and the optimal execution rule are expressed in terms of the solution to a truly two-dimensional optimal stopping problem, whose associated belief-dependent free boundary b triggers the investor's optimal selling rule. The curve b is uniquely determined through a nonlinear integral equation, for which we derive a numerical solution through an application of the Monte-Carlo method. This allows us to understand the value of information in our model as well as the sensitivity of the problem's solution with respect to the relevant model's parameters.

The rest of the chapter is organized as follows. In Section 2.2 we present our setting and first preliminary results. In Section 2.3 we investigate the benchmark problem under full information, before we consider a corresponding optimal stopping problem and its optimal boundary in Section 2.4. In Section 2.5 and 2.6 we derive two equivalent formulations of this problem, which allow for a more thorough study. Eventually, in Section 2.7, we return to the optimal control problem and characterize the optimal selling rule of the investor. A numerical study based on the derived integral equation of the execution boundary in then carried out in Section 2.8.

2.2 Setting and Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, rich enough to accommodate a standard one-dimensional Brownian motion $(W_t)_{t\geq 0}$ and an independent random variable β taking two values β_0 and β_1. We denote by $\mathbb{F}^W := (\mathcal{F}^W_t)_{t\geq 0}$ the filtration generated by $(W_t)_{t\geq 0}$ augmented by $\mathbb{P}$-null sets of $\mathcal{F}$. We assume that, in absence of any actions of the investor, the asset's price on the stock market evolves stochastically according to a geometric Brownian motion

$$dS^0_t = \beta S^0_t dt + \sigma S^0_t dW_t, \quad S^0_0 = s > 0, \tag{2.2.1}$$

where $\sigma > 0$ is a constant volatility. The investor holds a finite amount $y \geq 0$ of assets, which she is able to sell. We identify the cumulative amount of assets sold up to time $t \geq 0$, which we denote by ξ_t, as the investor's control variable. We denote by $\mathbb{F}^Z := (\mathcal{F}^Z_t)_{t\geq 0}$ the natural filtration of any process Z, augmented by $\mathbb{P}$-null sets of $\mathcal{F}$, and hence, the set of admissible execution strategies in this context is given by

$$\mathcal{A}(y) := \big\{\xi : \Omega \times [0,\infty) \to \mathbb{R}_+ : (\xi_t)_{t\geq 0}\, \mathbb{F}^{S^0}\text{-adapted, increasing,} \\ \text{càdlàg, and } \xi_{0-} = 0, \xi_t \leq y \text{ a.s.}\big\},$$

where the last condition naturally arises from the fact that the investor cannot sell more than the initial amount of assets. Moreover, the remaining assets in the portfolio evolve according to the dynamics

$$Y^\xi_t = y - \xi_t, \quad Y^\xi_{0-} = y \geq 0,$$

where we stress the dependency on the selling strategy ξ. Following Guo and Zervos (2015), in our model we assume that the investor's transactions on the market have a proportional impact on the asset's price. More precisely, when selling a small amount $\varepsilon > 0$ of assets at time t, the price exhibits a jump of size

$$\Delta S_t = S_t - S_{t-} = -\alpha\varepsilon S_{t-},$$

for $\alpha > 0$ denoting the parameter of permanent price impact (see Almgren and Chriss, 2001, Almgren, 2003 for early works and Becherer et al., 2018, Ferrari and Koch, 2021, Guo and Zervos, 2015 for more recent contributions). Hence, a small transaction is such that $S_t = (1-\alpha\varepsilon)S_{t-} \simeq e^{-\alpha\varepsilon}S_{t-}$ and, by interpreting a lump-sum sale of $\Delta\xi_t$ shares as a sequence of N individual sales of size $\varepsilon = \Delta\xi_t/N$, we have

$$S_t = e^{-\alpha N\varepsilon}S_{t-} = e^{-\alpha\Delta\xi_t}S_{t-},$$

for N large enough. It follows that, for any $\xi \in \mathcal{A}(y)$, we can model the controlled asset's price process by

$$dS_t^\xi = \beta S_t^\xi dt + \sigma S_{t-}^\xi dW_t - \alpha S_t^\xi \circ d\xi_t, \quad S_{0-}^\xi = s, \tag{2.2.2}$$

where

$$\int_0^\cdot S_t^\xi \circ d\xi_t := \int_0^\cdot S_t^\xi d\xi_t^c + \sum_{t\le\cdot:\Delta\xi_t\neq 0} \frac{1}{\alpha} S_{t-}^\xi (1 - e^{-\alpha\Delta\xi_t}) = \int_0^\cdot S_t^\xi d\xi_t^c + \sum_{t\le\cdot:\Delta\xi_t\neq 0} S_{t-}^\xi \int_0^{\Delta\xi_t} e^{-\alpha u} du, \tag{2.2.3}$$

ξ^c denotes the continuous part of the process ξ, and $\Delta\xi_t := \xi_t - \xi_{t-}$. The solution to (2.2.2) can be explicitly determined via Itô's formula and it is given by

$$S_t^\xi = s \exp\Big(\big(\beta - \frac{1}{2}\sigma^2\big)t + \sigma W_t - \alpha\xi_t\Big) = S_t^0 \exp(-\alpha\xi_t), \tag{2.2.4}$$

where S^0 is the solution to (2.2.1) and we observe that the price impact of selling is additive to the logarithm of the asset's price.
We assume that the investor aims at maximizing the total expected (discounted) profits, net of the total cost of selling, and thus seeks to solve

$$\begin{aligned}\sup_{\xi\in\mathcal{A}(y)} \mathbb{E}\bigg[\int_0^\infty e^{-rt}(S_t^\xi - \kappa) \circ d\xi_t\bigg] \\ = \sup_{\xi\in\mathcal{A}(y)} \mathbb{E}\bigg[\int_0^\infty e^{-rt}(S_t^\xi - \kappa) d\xi_t^c + \sum_{t:\Delta\xi_t\neq 0} e^{-rt} \int_0^{\Delta\xi_t} (S_{t-}^\xi e^{-\alpha u} - \kappa) du\bigg].\end{aligned} \tag{2.2.5}$$

Here, $\kappa > 0$ is a proportional transaction cost, which, thinking of S_t^ξ as the ask-price of the stock at time t, can also be interpreted as a constant bid-ask spread. Notice that the structure of the expected net-profit functional in (2.2.5) can also be justified through stability results in the Skorokhod M_1-topology in probability (see Becherer et al., 2019). Moreover, problem (2.2.5) has finite value due to $\xi_t \le y$ a.s. Thanks to (2.2.4) we have $S_t^\xi = \exp(X_t^\xi)$, where

$$dX_t^\xi = \mu dt + \sigma dW_t - \alpha d\xi_t, \quad X_{0-}^\xi = x, \tag{2.2.6}$$

with $x := \ln(s)$ and $\mu := \beta - \frac{1}{2}\sigma^2$. In particular, the drift can take two values $\mu_i = \beta_i - \frac{1}{2}\sigma^2$, $i = 0, 1$. In the following, when needed, we let X^0 denote the solution to (2.2.6) with $\xi \equiv 0$, which is then an arithmetic Brownian motion. Furthermore, we state the following assumption.

Assumption 2.2.1. *We have $\beta_1 > \beta_0$ and $\beta_0 < 0$, which implies $\mu_0 < 0$.*

The maximization problem (2.2.5) thus can be rewritten in terms of (2.2.6) as

$$\sup_{\xi\in\mathcal{A}(y)} \mathbb{E}\bigg[\int_0^\infty e^{-rt}\big(e^{X_t^\xi} - \kappa\big) \circ d\xi_t\bigg]. \tag{2.2.7}$$

Notice that for a constant non-random drift coefficient, a close variant of this problem was considered and solved by Guo and Zervos, 2015, who also incorporate the option of buying shares of assets and the constraint that the whole inventory has to be depleted at terminal time. However - due to the presence of incomplete information on the drift of the asset - Problem (2.2.7) is not of Markovian nature and thus requires a thoroughly different analysis. In order to obtain an equivalent Markovian formulation of (2.2.7), we rely on classical results from filtering theory, dating back to the contribution of Shiryaev in the context of quickest detection models (see Shiryaev, 2010 for a survey). To this end, we introduce the *belief* process

$$\Pi_t := \mathbb{P}\Big[\mu = \mu_1 \big| \mathcal{F}_t^{X^0}\Big], \quad t \geq 0,$$

which reflects the probability at time t that $\mu = \mu_1$, conditional on the observations of the price process up to that time (indeed, $\mathbb{F}^{S^0} = \mathbb{F}^{X^0} = \mathbb{F}^{X^\xi}$). According to this process, the investor is able to update the belief regarding the true value of the drift, based on the arrival of new information by observing the asset's price evolution on the market. Notice that a large value of Π close to 1 implies a strong belief in the larger drift value μ_1, while a low value of Π implies the contrary. It follows (see, e.g., Shiryaev, 1978, Section 4.2) that the dynamics of X^ξ, Π and Y^ξ can be written as

$$\begin{cases} dX_t^\xi = \big(\mu_1 \Pi_t + \mu_0 (1 - \Pi_t)\big) dt + \sigma d\overline{W}_t - \alpha d\xi_t, & X_{0-}^\xi = x \in \mathbb{R}, \\ d\Pi_t = \gamma \Pi_t (1 - \Pi_t) d\overline{W}_t, & \Pi_0 = \pi \in (0,1), \\ Y_t^\xi = y - \xi_t, & Y_{0-}^\xi = y \geq 0, \end{cases} \tag{2.2.8}$$

where $\gamma = (\mu_1 - \mu_0)/\sigma$ is the *signal-to-noise ratio* and

$$d\overline{W}_t = \frac{dX_t^0}{\sigma} - \Big(\frac{\mu_0}{\sigma} + \gamma \Pi_t\Big) dt$$

denotes the *innovation process*, which is an $\mathbb{F}^{X^0}$-Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, notice that the value $\pi := \mathbb{P}[\mu = \mu_1]$ reflects the initial subjective belief of the investor regarding the true value of the drift. We do not question the origin of this initial belief, this can either be an instinctive decision or even the result of a constructive approach, for instance by observing the trends of similar assets over the past years. In the new formulation, the process (X^ξ, Y^ξ, Π) is an $\mathbb{F}^{X^0}$-adapted and time-homogeneous Markov process, as it is the unique and strong solution to the system of stochastic differential equations in (2.2.8). Furthermore, we observe that the drift μ is replaced by its conditional estimate and the process Π is a bounded martingale on $[0,1]$ with $\Pi_\infty \in \{0,1\}$, as all information will eventually get revealed. Denoting $\mathbb{E}_{x,y,\pi}[\cdot] = \mathbb{E}[\cdot | X_{0-}^\xi = x, Y_{0-}^\xi = y, \Pi_0 = \pi]$, we can thus reformulate the problem of incomplete information as a so-called *separated problem* (cf. Bensoussan, 1992, Chapter 7.1 and Fleming and Pardoux, 1982)

$$V(x,y,\pi) := \sup_{\xi \in \mathcal{A}(y)} J(x,y,\pi,\xi), \tag{2.2.9}$$

with

$$J(x,y,\pi,\xi) := \mathbb{E}_{x,y,\pi}\Big[\int_0^\infty e^{-rt}\big(e^{X_t^\xi}-\kappa\big)d\xi_t^c + \sum_{t:\Delta\xi_t\neq 0} e^{-rt}\int_0^{\Delta\xi_t}\big(e^{X_{t-}^\xi-\alpha u}-\kappa\big)du\Big],$$

for any $(x,y,\pi)\in\mathbb{R}\times(0,\infty)\times(0,1)$. Notice indeed that $\Pi_t\in(0,1)$ for all $t\geq 0$ a.s. if $\pi\in(0,1)$, while $\Pi_t\equiv\pi_0$ for all $t\geq 0$ a.s. if $\pi_0\in\{0,1\}$. Problem (2.2.9) is equivalent to (2.2.5): They share the same value and, because of the uniqueness of the strong solution to (2.2.8), a control is optimal for (2.2.5) if and only if it is optimal for (2.2.9).

The Hamilton-Jacobi-Bellman equation.

Problem (2.2.9) takes the form of a *three-dimensional singular stochastic control problem with finite-fuel constraint* (cf. Baldursson, 1987, Beneš et al., 1980, El Karoui and Karatzas, 1989, Karatzas, 1983 and Karatzas et al., 2000 for early contributions). We start our analysis by providing a heuristic derivation of the dynamic programming equation, that we expect the value function V to satisfy. To this end, we notice that the investor is faced with two possible actions at initial time. On the one hand, the investor could choose to wait for a short period of time Δt, not sell any fraction of the assets and then continue with an optimal execution strategy (supposing that one exists). Since this strategy is not necessarily optimal, we obtain

$$V(x,y,\pi)\geq \mathbb{E}_{x,y,\pi}\big[e^{-r\Delta t}V(X_{\Delta t},y,\Pi_{\Delta t})\big],\quad (x,y,\pi)\in\mathbb{R}\times(0,\infty)\times(0,1).$$

If we assume that the value function V has enough regularity, we can apply Itô's formula, divide by Δt and invoke the mean value theorem in order to let $t\to 0$, so to obtain

$$(\mathcal{L}_{X,\Pi}-r)V\leq 0.$$

Here, $\mathcal{L}_{X,\Pi}$ denotes the second-order differential operator, acting on twice-continuously differentiable functions,

$$\mathcal{L}_{X,\Pi} := \frac{1}{2}\gamma^2\pi^2(1-\pi)^2\partial_{\pi\pi} + \frac{1}{2}\sigma^2\partial_{xx} + \big(\pi\mu_1+(1-\pi)\mu_0\big)\partial_x + \sigma\gamma\pi(1-\pi)\partial_{x\pi}. \tag{2.2.10}$$

On the other hand, the investor can instantaneously sell an amount $\varepsilon>0$ of the assets and then proceed by following an optimal execution strategy. Again, this strategy is a priori suboptimal and, since this action is associated with the inequality

$$V(x,y,\pi)\geq V(x-\alpha\varepsilon,y-\varepsilon,\pi)+\frac{1}{\alpha}e^x(1-e^{-\alpha\varepsilon})-\kappa\varepsilon,$$

adding and subtracting $V(x-\alpha\varepsilon,y,\pi)$, and dividing by ε, yields

$$\frac{V(x,y,\pi)-V(x-\alpha\varepsilon,y,\pi)}{\varepsilon}\geq\frac{V(x-\alpha\varepsilon,y-\varepsilon,\pi)-V(x-\alpha\varepsilon,y,\pi)}{\varepsilon}+\frac{1}{\alpha}e^x\frac{(1-e^{-\alpha\varepsilon})}{\varepsilon}-\kappa.$$

Hence, by letting $\varepsilon \downarrow 0$, we obtain

$$\alpha V_x(x,y,\pi) \geq -V_y(x,y,\pi) + e^x - \kappa.$$

Since only one of these actions should be optimal, and given the Markovian setting of problem (2.2.9), we thus expect that the value function V should identify with an appropriate solution to the Hamilton-Jacobi-Bellman equation

$$\max\Big\{(\mathcal{L}_{X,\Pi} - r)u, -\alpha u_x - u_y + e^x - \kappa\Big\} = 0, \quad (x,y,\pi) \in \mathbb{R} \times (0,\infty) \times (0,1), \tag{2.2.11}$$

with boundary condition $u(x,0,\pi) = 0$, since $y = 0$ implies $\mathcal{A}(0) = \{\xi \equiv 0\}$ and $J(x,0,\pi,0) = 0$. It is worth noticing that the variable y plays the role of a parameter in (2.2.11), which is then a two-dimensional elliptic partial differential equation with a state-dependent directional derivative constraint, parametrized by $y > 0$. With reference to (2.2.11) and the reasoning above, we can introduce the *waiting region*

$$\mathbb{W}_1 := \big\{(x,y,\pi) \in \mathbb{R} \times (0,\infty) \times (0,1) : (\mathcal{L}_{X,\Pi} - r)V = 0, -\alpha V_x - V_y + e^x - \kappa < 0\big\},$$

in which it is expected to be suboptimal to sell any assets, and the *selling/execution region*, where it should be profitable for the investor to sell a fraction of the assets:

$$\mathbb{S}_1 := \big\{(x,y,\pi) \in \mathbb{R} \times (0,\infty) \times (0,1) : (\mathcal{L}_{X,\Pi} - r)V \leq 0, -\alpha V_x - V_y + e^x - \kappa = 0\big\}.$$

Due to the multi-dimensional structure of the problem, a traditional *guess-and-verify approach*, as seen for instance in Guo and Zervos (2015) and Ferrari and Koch (2021), is not effective. In fact, this would require the construction of an explicit solution to the second-order PDE with state dependent gradient constraint seen in (2.2.11) above, which is not feasible in general. Instead, we use a different approach and construct an optimal stopping problem connected to the stochastic control problem (2.2.9), which is then of a simpler structure. Before we do so, and in order to get insights from a benchmark problem, we briefly discuss the problem under *full information*, i.e. where the drift coefficient is constant and equal to either μ_0 or μ_1.

2.3 Benchmark Problem under Full Information

Suppose that the initial subjective belief $\pi = \mathbb{P}[\mu = \mu_1]$ is such that $\pi \in \{0,1\}$. Observe that there exists no uncertainty in the model other than the Brownian one and the belief process Π will remain constant, as the investor is already certain at initial time regarding the true value of the drift. Hence - in this formulation - we are in the case of *full information*. The problem we address in this section has a similar structure to the ones studied by Guo and Zervos (2015) as well as Koch (2020), Chapter 2, and we therefore do not provide full details. Let us assume $\pi = 0$, we thus obtain $\Pi_t = 0$ for all $t \geq 0$ and the dynamics of X^ξ and Y^ξ then write as

$$\underline{X}_t^\xi = x + \mu_0 t + \sigma W_t - \alpha \xi_t, \quad Y_t^\xi = y - \xi_t. \tag{2.3.1}$$

We denote the corresponding value function as

$$V_0(x,y) := \sup_{\xi\in\mathcal{A}(y)} \mathbb{E}_{x,y}\left[\int_0^\infty e^{-rt}\big(e^{\underline{X}^\xi_t}-\kappa\big)\circ d\xi_t\right], \quad (x,y)\in\mathbb{R}\times(0,\infty), \tag{2.3.2}$$

where $\mathbb{E}_{x,y}[\cdot] = \mathbb{E}[\cdot|\underline{X}^\xi_{0-} = x, Y^\xi_{0-} = y]$. By employing similar arguments as in the case of incomplete information, we can expect that V_0 should identify with an appropriate solution to the HJB equation

$$\max\Big\{(\mathcal{L}_{\underline{X}}-r)w, -\alpha w_x - w_y + e^x - \kappa\Big\} = 0, \quad \text{with } \mathcal{L}_{\underline{X}} = \frac{1}{2}\sigma^2\partial_{xx} + \mu_0\partial_x, \tag{2.3.3}$$

and $w(x,0)=0$. Defining the associated waiting and selling regions as

$$\mathbb{W}^{\mu_0} := \Big\{(x,y)\in\mathbb{R}\times[0,\infty) : (\mathcal{L}_{\underline{X}}-r)w(x,y) = 0, -\alpha w_x - w_y + e^x - \kappa < 0\Big\}, \tag{2.3.4}$$

$$\mathbb{S}^{\mu_0} := \Big\{(x,y)\in\mathbb{R}\times[0,\infty) : (\mathcal{L}_{\underline{X}}-r)w(x,y) \le 0, -\alpha w_x - w_y + e^x - \kappa = 0\Big\}, \tag{2.3.5}$$

we suppose that the investor is only willing to sell a share of assets when its price is sufficiently large. Hence, we guess that for every $y\ge 0$ there exists a critical price $G(y)$ such that (2.3.4)-(2.3.5) rewrite as

$$\mathbb{W}^{\mu_0} = \Big\{(x,y)\in\mathbb{R}\times[0,\infty) : y>0 \text{ and } x<G(y)\Big\}\cup(\mathbb{R}\times\{0\}),$$

$$\text{and} \qquad \mathbb{S}^{\mu_0} = \Big\{(x,y)\in\mathbb{R}\times[0,\infty) : y>0 \text{ and } x\ge G(y)\Big\}.$$

Notice that the candidate value function should then satisfy $(\mathcal{L}_{\underline{X}}-r)w(x,y) = 0$ for all $(x,y)\in\mathbb{W}^{\mu_0}$. It is well-known that the latter equation admits two fundamental strictly positive solutions; the only solution that remains bounded as $x\downarrow-\infty$ is then given by

$$w(x,y) = A(y)e^{nx},$$

for some functions $A:[0,\infty)\to\mathbb{R}$ and where n is the positive solution to $(\sigma^2/2)n^2+\mu_0 n - r = 0$. On the other hand, for $(x,y)\in\mathbb{S}^{\mu_0}$, we expect that the value function V_0 should instead satisfy

$$-\alpha w_x - w_y + e^x - \kappa = 0 \quad \text{and thus} \quad -\alpha w_{xx} - w_{yx} + e^x = 0.$$

In order to derive the solutions for $A(y)$ and $G(y)$, we evaluate the two previous formulas at $x=G(y)$, require that $A(0)=0$ and obtain

$$G(y) = \ln\Big(\frac{\kappa n}{n-1}\Big) =: x_0^* \quad \text{and} \quad A(y) = \frac{\kappa}{\alpha n(n-1)}\Big(\frac{\kappa n}{n-1}\Big)^{-n}(1-e^{-\alpha n y}). \tag{2.3.6}$$

Notice that the optimal execution threshold - determining the price at which the investor should sell - is independent of the current amount of assets in the portfolio. Moreover, the selling region is partitioned into

$$\mathbb{S}_1^{\mu_0} := \Big\{(x,y)\in\mathbb{R}\times(0,\infty) : x\ge x_0^*, y\le \frac{x-x_0^*}{\alpha}\Big\}$$

$$\text{and} \qquad \mathbb{S}_2^{\mu_0} := \Big\{(x,y)\in\mathbb{R}\times(0,\infty) : x\ge x_0^*, y> \frac{x-x_0^*}{\alpha}\Big\},$$

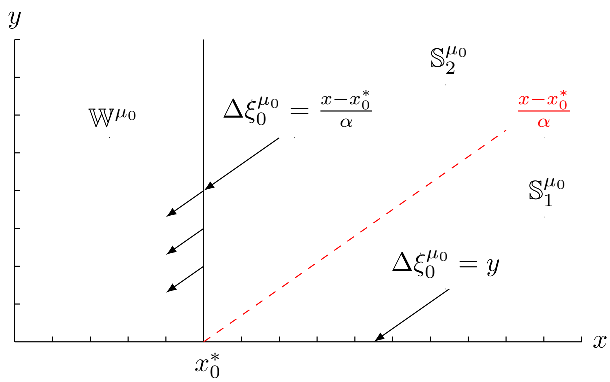


Figure 2.3.1: Illustrative drawing of the optimal execution strategy (2.3.8) under full information.

and we suppose that for $(x,y) \in \mathbb{S}_1^{\mu_0}$ it should be optimal to sell the complete amount of assets instantaneously, while for $(x,y) \in \mathbb{S}_2^{\mu_0}$ the investor is expected to make a lump-sum execution and then follow the strategy that keeps the process (X,Y) inside $\overline{\mathbb{W}}^{\mu_0}$ until all assets are sold. The candidate value function, according to our previous considerations, then takes the shape

$$w(x,y) = \begin{cases} A(y)e^{nx} & \text{for } (x,y) \in \mathbb{W}^{\mu_0}, \\ A\big(y - \frac{x-x_0^*}{\alpha}\big)e^{nx_0^*} + \frac{1}{\alpha}(e^x - e^{x_0^*}) - \frac{\kappa}{\alpha}(x - x_0^*) & \text{for } (x,y) \in \mathbb{S}_2^{\mu_0}, \\ \frac{1}{\alpha}e^x(1 - e^{-\alpha y}) - \kappa y & \text{for } (x,y) \in \mathbb{S}_1^{\mu_0}, \end{cases} \tag{2.3.7}$$

and via a verification theorem (cf. Guo and Zervos, 2015, Prop. 5.1, Koch, 2020, Prop. 2.4.1), one can indeed show that w is a $C^{2,1}$ solution to the HJB equation (2.3.3) and coincides with the value function V_0 of (2.3.2). Moreover, the process

$$\xi_t^{\mu_0} := y \wedge \sup_{0 \le s \le t} \frac{1}{\alpha}\Big(x - x_0^* + \mu_0 s + \sigma W_s\Big)^+, \quad t \ge 0, \quad \xi_{0-}^{\mu_0} = 0, \tag{2.3.8}$$

belongs to $\mathcal{A}(y)$ and provides an optimal execution strategy for problem (2.3.2) (cf. Guo and Zervos, 2015, Prop. 5.1; recall that here we are not assuming $\lim_{T\uparrow\infty} Y_T^\xi = 0$ as admissibility condition, see also Remark 2.7.6). Figure 2.3.1 sketches the optimal execution strategy (2.3.8) for problem (2.3.2) under full information. We observe that, for an initial price x strictly larger than x_0^*, the investor immediately does a lump-sum execution. The latter can already deplete the whole portfolio whenever $y \le \frac{1}{\alpha}(x - x_0^*)$, or bring it to the level $(x_0^*, y - \frac{1}{\alpha}(x - x_0^*))$ otherwise. Afterwards, the optimal strategy prescribes to keep the state process (X,Y) inside the waiting region $\overline{\mathbb{W}}^{\mu_0}$ with minimal effort, by reflecting it in the direction $(-\alpha, -1)$ according to a *Skorokhod reflection-type* policy (realized through the running supremum in (2.3.8)).

In light of our subsequent analysis, it is interesting to notice that the derivative $\alpha\partial_x V_0 + \partial_y V_0$ can be checked from (2.3.7) to identify with the value function of an optimal stopping problem. More precisely, for any $x \in \mathbb{R}$ one has

$$\alpha\partial_x V_0(x,y) + \partial_y V_0(x,y) =: v_0(x) = \sup_{\tau \ge 0} \mathbb{E}_x\Big[e^{-r\tau}\big(e^{X_\tau^0} - \kappa\big)\Big], \tag{2.3.9}$$

where $\underline{X}^0$ denotes the solution to (2.3.1) with $\xi_t \equiv 0$, the optimization is performed over all stopping times of the Brownian filtration and $\mathbb{E}_x$ is the expectation under $\mathbb{P}_x[\cdot] = \mathbb{P}[\cdot \mid \underline{X}_0^0 = x]$. Moreover, the stopping time

$$\tau_0^*(x) := \inf\{t \geq 0 : \underline{X}_t^0 \geq x_0^*\}, \quad \mathbb{P}_x\text{-a.s.}, x \in \mathbb{R}, \tag{2.3.10}$$

is optimal for (2.3.9). We can interpret (2.3.10) as the optimal time at which the investor should sell another unit of shares, and notice that it in fact characterizes the time at which the marginal expected profit $\alpha\partial_x V_0 + \partial_y V_0$ coincides with the marginal instantaneous net profit $e^x - \kappa$ from selling.

Remark 2.3.1. *It is easily checked that the results we obtained for the case $\mu \equiv \mu_0$ can be replicated for the case $\mu \equiv \mu_1$. More precisely, considering the dynamics*

$$\overline{X}_t^\xi = x + \mu_1 t + \sigma W_t - \alpha\xi_t, \quad t \geq 0, \tag{2.3.11}$$

and the value function

$$V_1(x, y) := \sup_{\xi \in \mathcal{A}(y)} \left[\int_0^\infty e^{-rt} \big(e^{\overline{X}_t^\xi} - \kappa\big) \circ d\xi_t \right], \quad (x, y) \in \mathbb{R} \times (0, \infty),$$

we can verify the existence of an optimal execution threshold x_1^*, which triggers the selling strategy of the investor through the optimal control ξ^{μ_1}, which is of similar structure as (2.3.8), *with μ_0 replaced by μ_1. Furthermore, we have*

$$\alpha\partial_x V_1 + \partial_y V_1 =: v_1(x) = \sup_{t \geq 0} \mathbb{E}_x \left[e^{-r\tau} \big(e^{\overline{X}_\tau^0} - \kappa\big) \right], \tag{2.3.12}$$

where $\overline{X}^0$ denotes the solution to (2.3.11) *with $\xi_t \equiv 0$, and $\tau_1^*(x) := \inf\{t \geq 0 : \overline{X}_t^0 \geq x_1^*\}$, $\mathbb{P}_x$-a.s., is the optimal stopping time for problem* (2.3.12).

2.4 A Related Optimal Stopping Problem

Motivated by the observed connection to an optimal stopping problem in the benchmark problem of Section 2.3 (see (2.3.12)), we pursue the following approach in the subsequent analysis: (i) we introduce and study an optimal stopping problem with value v, that we expect to be associated to the singular stochastic control problem (2.2.9); (ii) we provide a complete analysis of the optimal stopping problem, which is achieved by studying two equivalent formulations of it (cf. Sections 2.5 and 2.6). More precisely, we derive regularity results of the value function (cf. Proposition 2.6.9), as well as an integral equation for the free boundary (cf. Proposition 2.6.11); (iii) we verify the expected connection to the original problem of (2.2.9) by showing that (cf. Theorem 2.7.3)

$$V(x, y, \pi) = \frac{1}{\alpha} \int_{x-\alpha y}^{x} v(x', \pi) dx', \quad (x, y, \pi) \in \mathbb{R} \times (0, \infty) \times (0, 1),$$

and that the optimal execution strategy is triggered by the optimal stopping boundary studied in the previous step. In fact, as in the benchmark case, we can interpret the optimal stopping problem as the *marginal problem*, in the sense that its value coincides with the derivative of the value V of (2.2.9) in the direction of actions/execution and its optimal stopping strategy characterizes the time at which it is optimal to sell a unit of assets.

We recall that $(X_t^0, \Pi_t)_{t\geq 0}$ is the two-dimensional strong Markov process solving

$$\begin{cases} dX_t^0 = \big(\mu_1 \Pi_t + \mu_0(1-\Pi_t)\big)dt + \sigma d\overline{W}_t, & X_0^0 = x, \\ d\Pi_t = \gamma \Pi_t(1-\Pi_t) d\overline{W}_t, & \Pi_0 = \pi, \end{cases} \tag{2.4.1}$$

and in the following – in order to simplify notation – we write X instead of X^0. For a stopping time τ of the filtration $\mathbb{F}^X$, we then define

$$\Psi(x,\pi,\tau) := \mathbb{E}_{x,y}\Big[e^{-r\tau}\big(e^{X_\tau} - \kappa\big)\Big], \quad (x,\pi) \in \mathbb{R}\times(0,1),$$

and consider the optimal stopping problem

$$v(x,\pi) := \sup_{\tau} \Psi(x,\pi,\tau). \tag{2.4.2}$$

Above, and in the following, $\mathbb{E}_{x,\pi}[\cdot] = \mathbb{E}[\cdot|X_0 = x, \Pi_0 = \pi]$. Also, denoting $(X_t^{x,\pi}, \Pi_t^{\pi})_{t\geq 0}$ the unique strong solution to (2.4.1) we will often employ the equivalent notation $\mathbb{E}[f(X_t^{x,\pi}, \Pi_t^{\pi})] = \mathbb{E}_{x,\pi}[f(X_t, \Pi_t)]$, for any integrable measurable function $f : \mathbb{R}\times[0,1] \to \mathbb{R}$.

We make the next **standing assumption**.

Assumption 2.4.1. *We assume* $r > \big(\mu_1 + \frac{1}{2}\sigma^2\big) \vee \big(\mu_1 + \frac{1}{2}\sigma^2 + \frac{(2\mu_1+\sigma^2)(\mu_1-\mu_0)}{\sigma^2}\big) \vee \big(\frac{\gamma}{2\sigma}|\mu_0 + \mu_1|\big)$.

Remark 2.4.2. *(i) The different conditions we impose on the (subjective) discount factor* r *serve distinct purposes. Notice that the the first condition is equivalent to imposing* $r > \beta_1$ *and guarantees well-posedness of problem* (2.4.2).
(ii) Moreover, the forthcoming analysis (in particular Section 2.6) reveals that the other two terms are sufficient to ensure monotonicity of (a transformation of) the optimal stopping boundary of problem 2.4.2 (cf. Propositions 2.6.3 and 2.6.5). This result is crucial when deriving the smooth-fit property and thus, by relying on arguments developed in De Angelis and Peskir (2020), the global C^1*-regularity of (a transformation of) the value function* v *of* (2.4.2). *When* r *does not satisfy Assumption 2.4.1, the monotonicity of the (transformed version of the) boundary is not clear, and thus one needs an alternative route to achieve the needed regularity of* $\widehat{v}$*. A possible approach could be to prove directly the (locally) Lipschitz-regularity of the free-boundary (cf. De Angelis and Stabile, 2019), and then infer the* C^1*-property of* $\widehat{v}$ *from the continuity of the optimal stopping time with respect to the initial data. Since this is not straightforward to obtain in our formulation, we leave it for future research.*

In the following, we derive some preliminary results of the optimal stopping problem (2.4.2) and its associated free boundary. Noticing that $(x,\pi)\mapsto X_t^{x,\pi}$ as well as $\pi\mapsto\Pi_t^{\pi}$ are continuous and nondecreasing, due to classical comparison theorems for strong solutions to stochastic differential equations, the proof of the following lemma follows from standard arguments and it is therefore skipped.

Lemma 2.4.3. *The value function v of* (2.4.2) *is such that*

i) $x\mapsto v(x,\pi)$ *is nondecreasing;*

ii) $\pi\mapsto v(x,\pi)$ *is nondecreasing.*

Furthermore, using that $(x,\pi)\mapsto(X_t^{x,\pi},\Pi_t^{\pi})$ is continuous $\mathbb{P}$-a.s., by Assumption 2.4.1 and standard estimates using the fact that Π is bounded on $[0,1]$ we can invoke dominated convergence and obtain that

$$(x,\pi)\mapsto\mathbb{E}\Big[e^{-r\tau}\big(e^{X_\tau^{x,\pi}}-\kappa\big)\Big],$$

is continuous and hence, $(x,\pi)\mapsto v(x,\pi)$ is lower-semicontinuous. As it is customary in optimal stopping theory, we introduce the continuation and stopping regions associated to v as

$$\mathcal{C}_1:=\big\{(x,\pi)\in\mathbb{R}\times(0,1):v(x,\pi)>(e^x-\kappa)\big\},\tag{2.4.3}$$

$$\mathcal{S}_1:=\big\{(x,\pi)\in\mathbb{R}\times(0,1):v(x,\pi)=(e^x-\kappa)\big\}.\tag{2.4.4}$$

Then, the continuation region $\mathcal{C}_1$ is an open set, while the stopping region $\mathcal{S}_1$ in (2.4.4) is closed, and by Peskir and Shiryaev (2006), Chapter 1, Section 2, Corollary 2.9, the stopping time

$$\tau^*=\tau^*(x,\pi):=\inf\{t\geq 0:(X_t^{x,\pi},\Pi_t^{\pi})\in\mathcal{S}_1\},$$

is optimal whenever it is $\mathbb{P}$-a.s. finite, otherwise it is an optimal Markov time. We set

$$a(\pi):=\inf\{x\in\mathbb{R}:v(x,\pi)\leq(e^x-\kappa)\},\tag{2.4.5}$$

with the convention $\inf\emptyset=+\infty$, and state the following lemma.

Lemma 2.4.4. *It holds*

$$\mathcal{C}_1=\big\{(x,\pi)\in\mathbb{R}\times(0,1):x<a(\pi)\big\}\quad\textit{and}\quad\mathcal{S}_1=\big\{(x,\pi)\in\mathbb{R}\times(0,1):x\geq a(\pi)\big\}.$$

Proof. Recalling (2.2.10), an application of Dynkin's formula reveals

$$\begin{aligned}u(x,\pi)&:=v(x,\pi)-(e^x-\kappa)\\&=\sup_{\tau}\mathbb{E}_{x,\pi}\bigg[\int_0^{\tau}e^{-rt}\Big(e^{X_t}\big(\mu_1\Pi_t+(1-\Pi_t)\mu_0+\frac{1}{2}\sigma^2-r\big)+r\kappa\Big)dt\bigg].\end{aligned}$$

For $x_1 < x_2$ and τ^* optimal for $v(x_2, \pi)$ we have

$$\begin{aligned} u(x_1,\pi) - u(x_2,\pi) &\geq \mathbb{E}\Big[\int_0^{\tau^*} e^{-rt}\big(e^{X_t^{x_1,\pi}} - e^{X_t^{x_2,\pi}}\big)\big(\mu_1 \Pi_t^\pi + (1-\Pi_t^\pi)\mu_0 + \frac{1}{2}\sigma^2 - r\big)dt\Big] \\ &\geq \mathbb{E}\Big[\int_0^{\tau^*} e^{-rt}\big(e^{X_t^{x_2,\pi}} - e^{X_t^{x_1,\pi}}\big)\big(r - \mu_1 - \frac{1}{2}\sigma^2\big)dt\Big] \geq 0, \end{aligned}$$

due to Assumption 2.4.1. For $(x_1, \pi) \in \mathcal{S}_1$ and $x_2 > x_1$, we thus obtain $0 \leq u(x_2, \pi) \leq u(x_1, \pi) = 0$, so that $(x_2, \pi) \in \mathcal{S}_1$. □

The free boundary $a(\pi)$ thus splits $\mathbb{R} \times (0, 1)$ into the continuation and stopping region. In the following lemma we derive some preliminary properties.

Lemma 2.4.5. *One has:*

i) $\pi \mapsto a(\pi)$ *is nondecreasing on* $(0, 1)$*;*

ii) $\pi \mapsto a(\pi)$ *is left-continuous on* $(0, 1)$*;*

iii) There exist constants such that $x_0^* \leq a(\pi) \leq x_1^*$ *for all* $\pi \in (0, 1)$.

Proof. We prove the claims separately.

i) Let $\pi_2 > \pi_1$ and $(x, \pi_2) \in \mathcal{S}_1$. We thus have $x \geq a(\pi_2)$ and $v(x, \pi_2) = e^x - \kappa$. Since $\pi \mapsto v(x, \pi)$ is nondecreasing, $v(x, \pi_1) \leq v(x, \pi_2) = e^x - \kappa$, which, together with $v(x, \pi_1) \geq (e^x - \kappa)$, gives $(x, \pi_1) \in \mathcal{S}_1$. Therefore, $a(\pi_2) \geq a(\pi_1)$.

ii) Let $(\pi_n)_n$ be a sequence such that $\pi_n \uparrow \pi$. Due to i), the sequence $a(\pi_n)$ is increasing as $n \to \infty$ and $a(\pi_n) \leq a(\pi)$. Consequently, there exists $\lim_n a(\pi_n) =: a(\pi-)$ and $a(\pi-) \leq a(\pi)$. Because we have $v(a(\pi_n), \pi_n) = e^{a(\pi_n)} - \kappa$ for all $n \in \mathbb{N}$, by lower-semicontinuity of $(x, \pi) \mapsto v(x, \pi)$ we find that $v(a(\pi-), \pi) = e^{a(\pi-)} - \kappa$. Hence, $a(\pi) \leq a(\pi^-)$ and thus $\lim_n a(\pi_n) = a(\pi)$.

iii) Recall v_0 and v_1 of (2.3.9) and (2.3.12), the value functions in the optimal stopping problems with full information when either $\mu \equiv \mu_0$ or $\mu \equiv \mu_1$. The associated continuation regions are given by

$$\begin{aligned} &\{x \in \mathbb{R} : x \geq x_1^*\} = \{x \in \mathbb{R} : v_1(x) \leq e^x - \kappa\} \\ \text{and} \qquad &\{x \in \mathbb{R} : x \geq x_0^*\} = \{x \in \mathbb{R} : v_0(x) \leq e^x - \kappa\}, \end{aligned}$$

where x_0^* and x_1^* are the optimal execution thresholds (cf. (2.3.6) and Remark 2.3.1). Recalling $\mu_0 < \mu_1$ and $\Pi_t \in (0, 1)$ for $\pi \in (0, 1)$, we have $\underline{X}_t^0 \leq X_t \leq \overline{X}_t^0$ $\mathbb{P}$-a.s. for any $t \geq 0$, due to classical comparison arguments and where $\underline{X}^0$ and $\overline{X}^0$ denote the solutions to (2.3.1) and (2.3.11) with $\xi_t \equiv 0$. Thus, $v_0(x) \leq v(\pi, x) \leq v_1(x)$, which implies

$$\begin{aligned} \{x \in \mathbb{R} : v_1(x) \leq e^x - \kappa\} &\subset \{(x, \pi) \in \mathbb{R} \times (0, 1) : v(x, \pi) \leq e^x - \kappa\} \\ &\subset \{x \in \mathbb{R} : v_0(x) \leq e^x - \kappa\}, \end{aligned}$$

and the latter, combined with (2.4.5), allows to conclude that $x_0^* \leq a(\pi) \leq x_1^*$. □

2.5 Decoupling Change of Measure

We notice that the underlying dynamics in (2.4.1) are coupled. In order to derive further results about the properties of the optimal stopping problem (2.4.2) and its associated free boundary, it is useful to adress the problem under a different probability measure. With reference to related contributions (cf. De Angelis, 2020, Ekström and Lu, 2011, Johnson and Peskir, 2017 and Shiryaev, 2010 and references therein), we introduce the so-called *likelihood ratio process* via

$$\Phi_t := \frac{\Pi_t}{1-\Pi_t}, \quad t \geq 0.$$

Through an application of Itô's formula we can derive its associated dynamics, given by

$$d\Phi_t = \gamma\Phi_t(\gamma\Pi_t dt + d\overline{W}_t), \quad \Phi_0 = \varphi := \frac{\pi}{1-\pi},$$

and we aim to remove its dependency on the process Π through a change of measure. For a fixed $T > 0$, we define the measure $\mathbb{Q}_T \sim \mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ via the Radon-Nikodym derivative

$$\eta_T := \frac{dQ_T}{d\mathbb{P}} := \exp\bigg(-\int_0^T \gamma\Pi_s d\overline{W}_s - \frac{1}{2}\int_0^T \gamma^2\Pi_s^2 ds\bigg), \tag{2.5.1}$$

and notice that the process

$$dB_t = d\overline{W}_t + \gamma\Pi_t dt,$$

is a Brownian motion under $\mathbb{Q}_T$ on $[0, T]$. Rewriting the state process (X, Φ) under $\mathbb{Q}_T$ then yields

$$\begin{cases} dX_t = \mu_0 dt + \sigma dB_t, & t \in (0, T], \quad X_0 = x, \\ d\Phi_t = \gamma\Phi_t dB_t, & t \in (0, T], \quad \Phi_0 = \varphi, \end{cases} \tag{2.5.2}$$

and we notice that the processes decouple under this formulation. In the following, when needed, we will write $\mathbb{E}_{x,\varphi}^{\mathbb{Q}_T}$ to denote the expectation under $\mathbb{Q}_T$, conditioned on $X_0 = x, \Phi_0 = \varphi$. In order to rewrite problem (2.4.2) in terms of the new variables (X, Φ), we introduce

$$\Theta_t := \frac{1+\Phi_t}{1+\varphi}, \quad t \in [0, T],$$

and by an application of Itô's formula, it can be verified that Θ admits the representation

$$\Theta_t = \exp\bigg(\int_0^t \gamma\Pi_s d\overline{W}_s + \frac{1}{2}\int_0^t \gamma^2\Pi_s^2 ds\bigg) = \frac{1}{\eta_t}, \quad t \in [0, T]. \tag{2.5.3}$$

Upon using (2.5.1) and (2.5.3), we find

$$\begin{aligned}\mathbb{E}_{x,\pi}\Big[e^{-r(\tau\wedge T)}\big(e^{X_{\tau\wedge T}}-\kappa\big)\Big] &= \mathbb{E}_{x,\pi}\Big[e^{-r(\tau\wedge T)}\big(e^{X_{\tau\wedge T}}-\kappa\big)\eta_{\tau\wedge T}\Theta_{\tau\wedge T}\Big] \\ &= \mathbb{E}^{\mathbb{Q}_T}_{x,\varphi}\Big[e^{-r(\tau\wedge T)}\big(e^{X_{\tau\wedge T}}-\kappa\big)\frac{1+\Phi_{\tau\wedge T}}{1+\varphi}\Big] \\ &= (1+\varphi)^{-1}\mathbb{E}^{\mathbb{Q}_T}_{x,\varphi}\Big[e^{-r(\tau\wedge T)}\big(e^{X_{\tau\wedge T}}-\kappa\big)(1+\Phi_{\tau\wedge T})\Big], \end{aligned} \tag{2.5.4}$$

for any stopping time τ and $(x,\varphi)\in\mathbb{R}\times(0,\infty)$. With regard to (2.5.4) we introduce the stopping problems

$$\begin{aligned} v(x,\pi;T) &:= \sup_\tau \mathbb{E}_{x,\pi}\Big[e^{-r(\tau\wedge T)}\big(e^{X_{\tau\wedge T}}-\kappa\big)\Big], \\ \text{and}\quad v^{\mathbb{Q}_T}(x,\varphi;T) &:= \sup_\tau \mathbb{E}^{\mathbb{Q}_T}_{x,\varphi}\Big[e^{-r(\tau\wedge T)}\big(e^{X_{\tau\wedge T}}-\kappa\big)(1+\Phi_{\tau\wedge T})\Big], \end{aligned}$$

and notice that (2.5.4) implies $v^{\mathbb{Q}_T}(x,\varphi;T)=(1+\varphi)v(x,\varphi/(1+\varphi);T)$ for fixed $T>0$. However, since the measure $\mathbb{Q}_T$ changes with T, passing to the limit $T\to\infty$ in the latter expression (2.5.4) requires a bit of care. To this end, we define a probability space $(\widetilde{\Omega},\widetilde{\mathbb{F}},\widetilde{\mathbb{Q}})$ with a Brownian motion $\widetilde{B}$ and a filtration $\widetilde{\mathbb{F}}=(\widetilde{\mathcal{F}}_t)_{t\geq 0}$. Moreover, we let $(\widetilde{X},\widetilde{\Phi})$ be the strong solution to the stochastic differential equation (2.5.2) driven by the Brownian motion $\widetilde{B}$ instead of B. Let $\widetilde{\mathbb{E}}_{x,\varphi}[\cdot]$ denote the expectation under $\widetilde{\mathbb{Q}}$ and define the stopping problems

$$\begin{aligned} \overline{v}(x,\varphi;T) &:= \sup_\tau \widetilde{\mathbb{E}}_{x,\varphi}\Big[e^{-r(\tau\wedge T)}\big(e^{\widetilde{X}_{\tau\wedge T}}-\kappa\big)(1+\widetilde{\Phi}_{\tau\wedge T})\Big], \\ \overline{v}(x,\varphi) &:= \sup_\tau \widetilde{\mathbb{E}}_{x,\varphi}\Big[e^{-r\tau}\big(e^{\widetilde{X}_{\tau}}-\kappa\big)(1+\widetilde{\Phi}_{\tau})\Big]. \end{aligned}$$

Due to the equivalence in laws of the process $(\widetilde{X}_t,\widetilde{\Phi}_t,\widetilde{B}_t)_{t\geq 0}$ under $\widetilde{\mathbb{Q}}$ and the process $(X_t,\Phi_t,$ $B_t)_{t\geq 0}$ under $\mathbb{Q}_T$ on $[0,T]$, we have $v^{\mathbb{Q}_T}(x,\varphi;T)=\overline{v}(x,\varphi;T)$. Moreover, upon using Fatou's lemma and simple comparison arguments, one can show that

$$\lim_{T\to\infty} v(x,\pi;T)=v(x,\pi)\quad\text{as well as}\quad \lim_{T\to\infty}\overline{v}(x,\varphi;T)=\overline{v}(x,\varphi).$$

Hence, we finally obtain

$$\begin{aligned} \overline{v}(x,\varphi) &= \lim_{T\to\infty}\overline{v}(x,\varphi;T)=\lim_{T\to\infty}v^{\mathbb{Q}_T}(x,\varphi;T) \\ &= (1+\varphi)\lim_{T\to\infty}v(x,\varphi/(1+\varphi);T)=(1+\varphi)v(x,\varphi/(1+\varphi)). \end{aligned} \tag{2.5.5}$$

For the sake of clarity - and with a slight abuse of notation - from now on we simply write $(\Omega,\mathbb{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{Q},\mathbb{E}^{\mathbb{Q}},X,\Phi,B)$ instead of $(\widetilde{\Omega},\widetilde{\mathbb{F}},(\widetilde{\mathcal{F}}_t)_{t\geq 0},\widetilde{\mathbb{Q}},\widetilde{\mathbb{E}},\widetilde{X},\widetilde{\Phi},\widetilde{B})$. Henceforth, we thus study the optimal stopping problem

$$\overline{v}(x,\varphi)=\sup_\tau \mathbb{E}^{\mathbb{Q}}_{x,\varphi}\Big[e^{-r\tau}\big(e^{X_\tau}-\kappa\big)(1+\Phi_\tau)\Big]. \tag{2.5.6}$$

In the sequel, we will often write $\mathbb{E}^{\mathbb{Q}}_{x,\varphi}[f(X_t,\Phi_t)] = \mathbb{E}^{\mathbb{Q}}[f(X^x_t,\Phi^\varphi_t)]$, where $(X^x_t,\Phi^\varphi_t)_{t\geq 0}$ is the unique strong solution to (2.5.2). The continuation and stopping region associated to this problem are then given by

$$\mathcal{C}_2 := \big\{(x,\varphi)\in\mathbb{R}\times(0,\infty) : \overline{v}(x,\varphi) > (e^x-\kappa)(1+\varphi)\big\}, \tag{2.5.7}$$

$$\mathcal{S}_2 := \big\{(x,\varphi)\in\mathbb{R}\times(0,\infty) : \overline{v}(x,\varphi) = (e^x-\kappa)(1+\varphi)\big\}. \tag{2.5.8}$$

With regard to the lower-semicontinuity of v and (2.5.5), we find that $(x,\varphi)\mapsto\overline{v}(x,\varphi)$ is lower-semicontinuous as well. Hence, the stopping region $\mathcal{S}_2$ of (2.5.8) is a closed set, while the continuation region $\mathcal{C}_2$ of (2.5.7) is open. Also, $\tau^* := \tau^*(x,\varphi) := \inf\{t\geq 0 : (X^x_t,\Phi^\varphi_t)\in\mathcal{S}_2\}$ is optimal by Peskir and Shiryaev (2006), Chapter 1, whenever $\mathbb{Q}$-a.s. finite. Furthermore, we define

$$b(\varphi) := \inf\{x\in\mathbb{R} : \overline{v}(x,\varphi)\leq (e^x-\kappa)(1+\varphi)\}, \tag{2.5.9}$$

with $\inf\emptyset = \infty$. In the following lemma, we derive some preliminary properties of the value function (2.5.6). In light of the relation (2.5.5) we notice that some of the following results are a direct consequence of Lemma 2.4.3.

Lemma 2.5.1. *The value function $\overline{v}$ of* (2.5.6) *is such that*

i) $0\leq\overline{v}(x,\varphi)\leq K_1e^x(1+\varphi)$ *for all* $(x,\varphi)\in\mathbb{R}\times(0,\infty)$ *and some* $K_1>0$*;*

ii) $x\mapsto\overline{v}(x,\varphi)$ *is nondecreasing;*

iii) $\varphi\mapsto\overline{v}(x,\varphi)$ *is nondecreasing;*

iv) $(x,\varphi)\mapsto\overline{v}(x,\varphi)$ *is locally Lipschitz over* $\mathbb{R}\times(0,\infty)$*;*

v) $\varphi\mapsto\overline{v}(x,\varphi)$ *and* $x\mapsto\overline{v}(x,\varphi)$ *are convex.*

Proof. Property ii) follows from Lemma 2.4.3 i), upon using equality (2.5.5). We prove the remaining claims separately.

i) For the lower bound, we notice that $\{(x,\varphi)\in\mathbb{R}\times(0,\infty) : (e^x-\kappa)<0\}\subset\mathcal{C}_2$. Hence, since $\Phi^\varphi\geq 0$ a.s., we have $\overline{v}(x,\varphi)\geq 0$ for all $(x,\varphi)\in\mathbb{R}\times(0,\infty)$. For the upper bound, we observe that for any stopping time τ

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_{x,\varphi}\Big[e^{-r\tau}\big(e^{X_\tau}-\kappa\big)(1+\Phi_\tau)\Big] &= (1+\varphi)\mathbb{E}_{x,\pi}\Big[e^{-r\tau}\big(e^{X_\tau}-\kappa\big)\Big]\\ &\leq (1+\varphi)\mathbb{E}\Big[e^{-r\tau}e^{x+\mu_1\tau+\sigma W_\tau}\Big]\leq K_1e^x(1+\varphi),\end{aligned}$$

for $\pi=\varphi/(1+\varphi)$ and the last inequality follows from standard estimates upon using Assumption 2.4.1.

iii) Let $\varphi, \varphi' \in (0,\infty)$ with $\varphi' > \varphi$ and notice that $\Phi_t^{\varphi} = \varphi e^{-\frac{1}{2}\gamma^2 t + \gamma B_t}$. For $x \in \mathbb{R}$ and $\tau^* := \tau^*(x,\varphi)$ optimal for $\overline{v}(x,\varphi)$ we have

$$\begin{aligned}\overline{v}(x,\varphi') - \overline{v}(x,\varphi) &\geq \mathbb{E}^{\mathbb{Q}}_{x,\varphi'}\Big[e^{-r\tau^*}\big(e^{X_{\tau^*}} - \kappa\big)(1+\Phi_{\tau^*})\Big] - \mathbb{E}^{\mathbb{Q}}_{x,\varphi}\Big[e^{-r\tau^*}\big(e^{X_{\tau^*}} - \kappa\big)(1+\Phi_{\tau^*})\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[e^{-r\tau^*}\big(e^{X^x_{\tau^*}} - \kappa\big)(\varphi' - \varphi)e^{-\frac{1}{2}\gamma^2\tau^* + \gamma B_{\tau^*}}\Big] \geq 0,\end{aligned}$$

where the last inequality exploits that $\{(x,\varphi) \in \mathbb{R}\times(0,\infty) : (e^x - \kappa) < 0\} \subset \mathcal{C}_2$, and the claim follows.

iv) Let $x, x' \in \mathbb{R}$, $\pi \in (0,1)$ and $\varphi, \varphi' \in (0,\infty)$. Recall v of (2.4.2). Again, standard estimates yield

$$|v(x,\pi) - v(x',\pi)| \leq K_1|e^x - e^{x'}|, \quad \text{as well as} \quad |\overline{v}(x,\varphi) - \overline{v}(x,\varphi')| \leq K_2 e^x|\varphi - \varphi'|,$$

for some $K_1, K_2 > 0$. Hence, using (2.5.5), we obtain

$$\begin{aligned}|\overline{v}(x,\varphi) - \overline{v}(x',\varphi')| &\leq |\overline{v}(x,\varphi) - \overline{v}(x',\varphi)| + |\overline{v}(x',\varphi) - \overline{v}(x',\varphi')| \\ &\leq K_1(1+\varphi)|e^x - e^{x'}| + K_2 e^{x'}|\varphi - \varphi'|,\end{aligned} \tag{2.5.10}$$

and thus the locally-Lipschitz property follows.

v) We first prove convexity regarding $\varphi \in (0,\infty)$. For $\varphi_1, \varphi_2 \in (0,\infty)$, $x \in \mathbb{R}$ and $\lambda \in (0,1)$ we set $\overline{\varphi} := \lambda\varphi_1 + (1-\lambda)\varphi_2$ and obtain

$$\begin{aligned}\overline{v}(x,\overline{\varphi}) &= \sup_{\tau} \mathbb{E}^{\mathbb{Q}}_{x,\overline{\varphi}}\Big[e^{-r\tau}\big(e^{X_\tau} - \kappa\big)\big(1 + \overline{\varphi}e^{-\frac{1}{2}\gamma^2\tau + \gamma B_\tau}\big)\Big] \\ &\leq \sup_{\tau} \mathbb{E}^{\mathbb{Q}}\Big[e^{-r\tau}\big(e^{X^x_\tau} - \kappa\big)\lambda\big(1 + \varphi_1 e^{-\frac{1}{2}\gamma^2\tau + \gamma B_\tau}\big)\Big] \\ &\qquad + \sup_{\tau} \mathbb{E}^{\mathbb{Q}}\Big[e^{-r\tau}\big(e^{X^x_\tau} - \kappa\big)(1-\lambda)\big(1 + \varphi_2 e^{-\frac{1}{2}\gamma^2\tau + \gamma B_\tau}\big)\Big] \\ &= \lambda\overline{v}(x,\varphi_1) + (1-\lambda)\overline{v}(x,\varphi_2),\end{aligned}$$

and the claim follows. Analogously, upon exploiting the convexity of $x \mapsto e^x$, one can prove the convexity of $x \mapsto \overline{v}(x,\varphi)$. □

Lemma 2.5.2. *The continuation and stopping region regions as in* (2.5.7)-(2.5.8) *are such that*

$$\mathcal{C}_2 = \big\{(x,\varphi) \in \mathbb{R}\times(0,\infty) : x < b(\varphi)\big\}, \quad \mathcal{S}_2 = \big\{(x,\varphi) \in \mathbb{R}\times(0,\infty) : x \geq b(\varphi)\big\}.$$

Proof. We proceed similarly to Lemma 2.4.4. We first notice that the the second-order differential operator associated with the two-dimensional process (X,Φ) is such that

$$\mathcal{L}_{X,\Phi}f = \mu_0\partial_x f + \frac{1}{2}\sigma^2\partial_{xx}f + \frac{1}{2}\gamma^2\varphi^2\partial_{\varphi\varphi}f + \gamma\varphi\sigma\partial_{x\varphi}f, \quad \forall f \in C^2\big(\mathbb{R}\times(0,\infty)\big), \tag{2.5.11}$$

and apply Dynkin's formula to obtain

$$\begin{aligned}\overline{u}(x,\varphi) &:= \overline{v}(x,\varphi) - (e^x - \kappa)(1+\varphi)\\ &= \sup_{\tau} \mathbb{E}^{\mathbb{Q}}_{x,\varphi}\Big[\int_0^{\tau} e^{-rt}\Big(e^{X_t}(\mu_0 + \frac{1}{2}\sigma^2 - r) + r\kappa + \Phi_t\big(e^{X_t}(\mu_1 + \frac{1}{2}\sigma^2 - r) + r\kappa\big)\Big)dt\Big].\end{aligned} \tag{2.5.12}$$

For $x_2 > x_1$ and $\tau^* := \tau^*(x_2,\varphi)$ optimal for $\overline{v}(x_2,\varphi)$ we have

$$\begin{aligned}&\overline{u}(x_1,\varphi) - \overline{u}(x_2,\varphi)\\ &\quad\geq \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-rt}\Big(\big(e^{X_t^{x_2}} - e^{X_t^{x_1}}\big)\big(r - \mu_0 - \frac{1}{2}\sigma^2\big) + \Phi_t\big(e^{X_t^{x_2}} - e^{X_t^{x_1}}\big)\big(r - \mu_1 + \frac{1}{2}\sigma^2\big)\Big)dt\Big]\\ &\quad\geq 0,\end{aligned}$$

where the last inequality follows from $X^{x_2} \geq X^{x_1}$ $\mathbb{Q}$-a.s. and Assumption 2.4.1. Hence, for $(x_1,\varphi) \in \mathcal{S}_2$ and $x_2 > x_1$, we obtain $0 \leq \overline{u}(x_2,\varphi) \leq \overline{u}(x_1,\varphi) = 0$ and the claim follows. □

It is interesting to notice that there exists a one-to-one correspondence between the continuation regions $\mathcal{C}_1$ and $\mathcal{C}_2$ of (2.4.3) and (2.5.7) as well as the stopping regions $\mathcal{S}_1$ and $\mathcal{S}_2$ of (2.4.4) and (2.5.8). Indeed, introducing the diffeomorphism

$$T := (T_1, T_2) : \mathbb{R} \times (0,1) \to \mathbb{R} \times (0,\infty), \quad \big(T_1(x,\pi), T_2(x,\pi)\big) := \Big(x, \frac{\pi}{1-\pi}\Big), \tag{2.5.13}$$

with inverse

$$T^{-1}(x,\varphi) := \Big(x, \frac{\varphi}{1+\varphi}\Big), \quad (x,\varphi) \in \mathbb{R} \times (0,\infty),$$

one has

$$\mathcal{C}_2 = T(\mathcal{C}_1) \quad \text{as well as} \quad \mathcal{S}_2 = T(\mathcal{S}_1).$$

Furthermore, upon using Lemma 2.4.4 and Lemma 2.5.2, we find that

$$b(\varphi) = a\Big(\frac{\varphi}{1+\varphi}\Big). \tag{2.5.14}$$

Due to this explicit relationship between the optimal stopping boundaries, we obtain some first results on b thanks to Lemma 2.4.5.

Lemma 2.5.3. *The boundary $b(\varphi)$ of (2.5.9) is such that*

i) $\varphi \mapsto b(\varphi)$ is nondecreasing on $(0,\infty)$;

ii) $\varphi \mapsto b(\varphi)$ is left-continuous;

iii) b is bounded by $x_0^* \le b(\varphi) \le x_1^*$ for all $\varphi \in (0,\infty)$, with x_0^* and x_1^* as in Lemma 2.4.5.

The relationship (2.5.14) and the transformation (2.5.13) allow us to translate back our results from this section - as well as from the following section - to the initial optimal stopping problem (2.4.2). Moreover, (2.5.14) turns out to be valuable in the proof of Lemma 2.5.3, since proving the monotonicity result i) as well as the boundedness iii) is not straightforward without exploiting the relation between b and a and the results of Lemma 2.4.5.

2.6 A Parabolic Formulation

Observe that the dynamics of the processes X and Φ in (2.5.2) are driven by the same Brownian motion. In order to account for this degeneracy, we pass yet to another formulation of the optimal stopping problem. To this end, we rely on a transformation that reveals the true parabolic nature of the generator $\mathcal{L}_{X,\Phi}$ as in (2.5.11); i.e. that poses it in its canonical form (cf. Strauss, 2007, Chapter 1.6). Define

$$\overline{T} := (\overline{T}_1, \overline{T}_2) : \mathbb{R} \times (0,\infty) \to \mathbb{R}^2, \quad \big(\overline{T}_1(x,\varphi), \overline{T}_2(x,\varphi)\big) := \Big(x, \frac{\sigma}{\gamma}\ln(\varphi) - x\Big), \tag{2.6.1}$$

for any $(x,\varphi) \in \mathbb{R} \times (0,\infty)$, which is a diffeomorphism with inverse given by

$$\overline{T}^{-1}(x,z) := \Big(x, e^{\frac{\gamma}{\sigma}(x+z)}\Big), \quad (x,z) \in \mathbb{R}^2.$$

With regard to the transformation (2.6.1) we can introduce the process

$$Z_t = \frac{\sigma}{\gamma}\ln(\Phi_t) - X_t, \quad t \ge 0, \tag{2.6.2}$$

and an application of Itô's formula reveals that its dynamics are given by

$$dZ_t = -\frac{1}{2}(\mu_1 + \mu_0)dt, \quad Z_0 = z := \frac{\sigma}{\gamma}\ln(\varphi) - x. \tag{2.6.3}$$

Furthermore, we can define the transformed version of the value function $\overline{v}$ of (2.5.6) via

$$\widehat{v}(x,z) := \overline{v}\big(x, e^{\frac{\gamma}{\sigma}(x+z)}\big) = \sup_{\tau} \mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-r\tau}\big(e^{X_\tau} - \kappa\big)\big(1 + e^{\frac{\gamma}{\sigma}(X_\tau + Z_\tau)}\big)\Big], \tag{2.6.4}$$

for $(x,z) \in \mathbb{R}^2$ and where now $\mathbb{E}^{\mathbb{Q}}_{x,z}[\cdot] = \mathbb{E}^{\mathbb{Q}}[\cdot | X_0 = x, Z_0 = z]$. In light of this explicit relationship between the value functions $\overline{v}$ and $\widehat{v}$, we can conclude the following result from Lemma 2.5.1.

Lemma 2.6.1. *The value function $\widehat{v}(x,z)$ of (2.6.4) is locally Lipschitz continuous over $\mathbb{R}^2$.*

The associated continuation and stopping region are given by

$$\mathcal{C}_3 := \left\{(x,z) \in \mathbb{R}^2 : \widehat{v}(x,z) > (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)})\right\}, \tag{2.6.5}$$
$$\mathcal{S}_3 := \left\{(x,z) \in \mathbb{R}^2 : \widehat{v}(x,z) = (e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)})\right\}, \tag{2.6.6}$$

where $\mathcal{C}_3$ is open and $\mathcal{S}_3$ is closed. Furthermore, the global diffeomorphism (2.6.1) implies that $\mathcal{C}_3 = \overline{T}(\mathcal{C}_2)$ as well as $\mathcal{S}_3 = \overline{T}(\mathcal{S}_2)$, with $\mathcal{C}_2$ and $\mathcal{S}_2$ as in (2.5.7)-(2.5.8). Notice that the second-order infinitesimal generator associated to the process (X, Z) is now such that

$$\mathcal{L}_{X,Z} f = \mu_0 \partial_x f + \frac{1}{2}\sigma^2 \partial_{xx} f - \frac{1}{2}(\mu_1 + \mu_0)\partial_z f, \quad \forall f \in C^{2,1}(\mathbb{R}^2). \tag{2.6.7}$$

We can rely on standard arguments from classical PDE theory as well as optimal stopping theory (see, e.g., Karatzas and Shreve, 1991, Section 2.7, Th. 7.7) and obtain the following lemma.

Lemma 2.6.2. *The value function $\widehat{v}$ of* (2.5.5) *is the unique classical $C^{2,1}$-solution to the boundary value problem*

$$(\mathcal{L}_{X,Z} - r)w = 0 \quad \textit{in } \mathcal{R} \quad \textit{and} \quad w|_{\partial\mathcal{R}} = \widehat{v}|_{\partial\mathcal{R}},$$

for $\mathcal{L}_{X,Z}$ as in (2.6.7) *and any open set $\mathcal{R}$ such that its closure is contained in the continuation region $\mathcal{C}_3$ of* (2.6.5)*. In particular, $\widehat{v} \in C^{2,1}(\mathcal{C}_3)$.*

In the following, we aim at investigating the geometry of the state space in the coordinates (X, Z). To this end, we define the generalised inverse of the nondecreasing boundary b by

$$b^{-1}(x) := \inf\{\varphi \in (0,\infty) : b(\varphi) > x\}, \tag{2.6.8}$$

such that the continuation region $\mathcal{C}_2$ of (2.5.7) rewrites as

$$\mathcal{C}_2 = \left\{(x,\varphi) \in \mathbb{R} \times (0,\infty) : b^{-1}(x) < \varphi\right\}.$$

Since $\varphi \mapsto b(\varphi)$ is nondecreasing by Lemma 2.5.3, we observe that

$$(x,z) \in \mathcal{C}_3 \Leftrightarrow \left(x, e^{\frac{\gamma}{\sigma}(x+z)}\right) \in \mathcal{C}_2 \Leftrightarrow e^{\frac{\gamma}{\sigma}(x+z)} > b^{-1}(x) \Leftrightarrow z > \frac{\sigma}{\gamma}\log\left(b^{-1}(x)\right) - x,$$

and by setting

$$c^{-1}(x) := \frac{\sigma}{\gamma}\log\left(b^{-1}(x)\right) - x, \tag{2.6.9}$$

we can rewrite (2.6.5) and (2.6.6) as

$$\mathcal{C}_3 = \left\{(x,z) \in \mathbb{R}^2 : z > c^{-1}(x)\right\}, \quad \mathcal{S}_3 = \left\{(x,z) \in \mathbb{R}^2 : z \le c^{-1}(x)\right\}. \tag{2.6.10}$$

In contrast to the optimal stopping problems in the formulations (2.4.2) and (2.5.6), deriving the monotonicity of the boundary $x \mapsto c^{-1}(x)$ is not straightforward. Moreover - and differently to related contributions such as Federico et al. (2023) - we cannot translate it back to the monotonicity of the boundary b of (2.5.9), since its generalised inverse b^{-1} is nondecreasing as well, and this does not imply monotonicity of $x \mapsto c^{-1}(x)$. To this end, we follow and adapt arguments presented in Section 4.4 of De Angelis (2020), which studies separately the two cases in which the deterministic process Z as in (2.6.3) is either increasing ($\mu_0 + \mu_1 \geq 0$) or decreasing ($\mu_0 + \mu_1 < 0$).

For the following analysis, it is useful to define

$$\widehat{u}(x,z) := \widehat{v}(x,z) - \big(e^x - \kappa\big)\big(1 + e^{\frac{\gamma}{\sigma}(x+z)}\big), \tag{2.6.11}$$

as well as

$$\begin{aligned} g(x,z) &:= (\mathcal{L}_{X,Z} - r)\big((e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)})\big) \\ &= e^x\Big(\frac{1}{2}\sigma^2 + \mu_0 - r\Big) + r\kappa + e^{\frac{\gamma}{\sigma}(x+z)}\Big(e^x\big(\frac{1}{2}\sigma^2 + \mu_1 - r\big) + r\kappa\Big), \end{aligned} \tag{2.6.12}$$

and we observe that an application of Dynkin's formula implies

$$\widehat{u}(x,z) = \sup_{\tau} \mathbb{E}^{\mathbb{Q}}_{x,z}\bigg[\int_0^{\tau} e^{-rt} g(X_t, Z_t)dt\bigg], \quad (x,z) \in \mathbb{R}^2. \tag{2.6.13}$$

In the following Propositions 2.6.3 and 2.6.5, we establish the existence of a monotone boundary $c : \mathbb{R} \to \mathbb{R}$ that splits the state space into continuation and stopping region. As we will verify in Remark 2.6.6, the function c^{-1} of (2.6.9) is indeed the right-continuous inverse of this function.

Proposition 2.6.3. *Let $\mu_0 + \mu_1 \geq 0$. Then there exists a nondecreasing function $c : \mathbb{R} \to \mathbb{R}$ such that the continuation region $\mathcal{C}_3$ of* (2.6.5) *rewrites as*

$$\mathcal{C}_3 = \big\{(x,z) \in \mathbb{R}^2 : x < c(z)\big\}. \tag{2.6.14}$$

Proof. Let $(x_0, z_0) \in \mathcal{S}_3$, $x_1 > x_0$ and notice that (2.6.10) implies $(-\infty, z_0] \times \{x_0\} \in \mathcal{S}_3$. Furthermore, we have $x_0 > x_0^*$ and since the process Z is decreasing, we observe that the process (X^{x_1}, Z^{z_0}) crosses the half-line $(-\infty, z_0] \times \{x_0\}$ before reaching the level x_0^*. Hence, we have $\mathbb{Q}_{x_1,z_0}[\tau^* < \tau_{x_0^*}] = 1$, where $\tau_{x_0^*} := \inf\{t \geq 0 : X_t^{x_1} = x_0^*\}$ and $\mathbb{Q}_{x_1,z_0}[\cdot] = \mathbb{Q}[\cdot | X_0 = x_1, Z_0 = z_0]$. Moreover, it can be verified that the second condition of Assumption 2.4.1 implies $x_0^* > \widetilde{x}$, with the latter given by

$$\widetilde{x} := \log\bigg(\frac{r\kappa}{r - \frac{1}{2}\sigma^2 - \mu_1}\bigg). \tag{2.6.15}$$

Consequently, we have $\exp(X_s^{x_1})(r - \frac{1}{2}\sigma^2 - \mu_1) > r\kappa$ for all $s \in [0, \tau^*)$ and (2.6.12)-(2.6.13) imply $\widehat{u}(x_1, z_0) \le 0$ for all $x_1 > x_0$, and therefore $\{z_0\} \times [x_0, \infty) \in \mathcal{S}_3$. We can thus define

$$c(z) := \inf\{x \in \mathbb{R} : (x, z) \in \mathcal{S}_3\}. \tag{2.6.16}$$

and observe that (2.6.10) implies that $z \mapsto c(z)$ is nondecreasing. □

In order to establish the same result in the case when $\mu_0 + \mu_1 < 0$, we first state the following lemma.

Lemma 2.6.4. *We have*

$$\widehat{v}_z(x, z) = \mathbb{E}^{\mathbb{Q}}_{x,z}\left[\frac{\gamma}{\sigma} e^{-r\tau^*}\big(e^{X_{\tau^*}} - \kappa\big) e^{\frac{\gamma}{\sigma}(X_{\tau^*} + Z_{\tau^*})} \mathbb{1}_{\{\tau^* < \infty\}}\right], \tag{2.6.17}$$

for all $(x, z) \in \mathbb{R}^2 \setminus \partial\mathcal{C}_3$ *and* $\tau^* := \tau^*(x, z)$.

Proof. For $(x, z) \in \mathcal{S}_3$ the claim follows immediately, since $\mathbb{Q}_{x,z}[\tau^* = 0] = 1$. Hence, we let $(x, z) \in \mathcal{C}_3$ and for $\varepsilon > 0$ we obtain

$$\begin{aligned}
\widehat{v}(x, z + \varepsilon) - \widehat{v}(x, z) &\ge \mathbb{E}^{\mathbb{Q}}\Big[e^{-r(\tau^* \wedge t)}\big(\widehat{v}(X^x_{\tau^* \wedge t}, Z^{z+\varepsilon}_{\tau^* \wedge t}) - \widehat{v}(X^x_{\tau^* \wedge t}, Z^z_{\tau^* \wedge t})\big)\Big] \\
&\ge \mathbb{E}^{\mathbb{Q}}\Big[e^{-r\tau^*}\big(e^{X^x_{\tau^*}} - \kappa\big) e^{\frac{\gamma}{\sigma} X^x_{\tau^*}}\big(e^{\frac{\gamma}{\sigma} Z^{z+\varepsilon}_{\tau^*}} - e^{\frac{\gamma}{\sigma} Z^z_{\tau^*}}\big) \mathbb{1}_{\{\tau^* < t\}}\Big] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\Big[e^{-rt}\big(\widehat{v}(X^x_t, Z^{z+\varepsilon}_t) - \widehat{v}(X^x_t, Z^z_t)\big) \mathbb{1}_{\{\tau^* > t\}}\Big],
\end{aligned} \tag{2.6.18}$$

where the first inequality follows from the supermartingale property of $(e^{-r(\tau \wedge t)}\widehat{v}(X^x_{\tau \wedge t}, Z^{z+\varepsilon}_{\tau \wedge t}))_t$ and the martingale property of $\big(e^{-r(\tau^* \wedge t)}\widehat{v}(X^x_{\tau^* \wedge t}, Z^z_{\tau^* \wedge t})\big)_t$ for $\tau^* := \tau^*(x, z)$. Upon employing a change of measure as in Section 2.5, we find

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-rt}|\widehat{v}(X_t, Z_t)|\Big] &\le \mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-rt}\big|\overline{v}\big(x, e^{\frac{\gamma}{\sigma}(X_t + Z_t)}\big)\big|\Big] \le K_1 \mathbb{E}^{\mathbb{Q}}_{x, \exp(\frac{\gamma}{\sigma}(x+z))}\Big[e^{-rt} e^{X_t}(1 + \Phi_t)\Big] \\
&= K_1\big(1 + e^{\frac{\gamma}{\sigma}(x+z)}\big)\mathbb{E}_{x,\pi}\Big[e^{-rt} e^{X_t}\Big],
\end{aligned}$$

where $\pi = e^{\frac{\gamma}{\sigma}(x+z)}/(1 + e^{\frac{\gamma}{\sigma}(x+z)})$. It is then easy to verify that Assumption 2.4.1 implies

$$\lim_{t \uparrow \infty} \mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-rt}\widehat{v}(X_t, Z_t)\Big] = 0,$$

and hence, applying dominated convergence in (2.6.18) as $t \uparrow \infty$ yields

$$\widehat{v}(x, z + \varepsilon) - \widehat{v}(x, z) \ge \mathbb{E}^{\mathbb{Q}}\Big[e^{-r\tau^*}\big(e^{X^x_{\tau^*}} - \kappa\big) e^{\frac{\gamma}{\sigma} X^x_{\tau^*}}\big(e^{\frac{\gamma}{\sigma} Z^{z+\varepsilon}_{\tau^*}} - e^{\frac{\gamma}{\sigma} Z^z_{\tau^*}}\big) \mathbb{1}_{\{\tau^* < \infty\}}\Big]. \tag{2.6.19}$$

Similar arguments show

$$\widehat{v}(x, z) - \widehat{v}(x, z - \varepsilon) \le \mathbb{E}^{\mathbb{Q}}\Big[e^{-r\tau^*}\big(e^{X^x_{\tau^*}} - \kappa\big) e^{\frac{\gamma}{\sigma} X^x_{\tau^*}}\big(e^{\frac{\gamma}{\sigma} Z^{z+\varepsilon}_{\tau^*}} - e^{\frac{\gamma}{\sigma} Z^z_{\tau^*}}\big) \mathbb{1}_{\{\tau^* < \infty\}}\Big], \tag{2.6.20}$$

and since $\widehat{v} \in C^{2,1}(\mathcal{C}_3)$ (cf. Lemma 2.6.2), dividing (2.6.19) and (2.6.20) by ε and letting $\varepsilon \downarrow 0$, we obtain the desired result. □

Proposition 2.6.5. *Let $\mu_0 + \mu_1 < 0$. There exists a nondecreasing function $c : \mathbb{R} \to \mathbb{R}$ such that the continuation region of* (2.6.5) *can be written as*

$$\mathcal{C}_3 = \big\{(x, z) \in \mathbb{R}^2 : x < c(z)\big\}.$$

Proof. Let $(x, z) \in \mathbb{R}^2$. Notice that $x < x_0^*$ implies $(x', z) \in \mathcal{C}_3$ for all $x' < x$ and $z \in \mathbb{R}$, because of Lemma 2.4.5 and since the transformations T_1 and $\overline{T}_1$ of (2.5.13) and (2.6.1), respectively, are the identity; hence, $\{(x, z) : x < x_0^*\} \subset \mathcal{C}_3$. We can thus focus on the case that $x \geq x_0^*$ and distinguish two possibilities:

i) $\widehat{u}_x(x, z) \leq 0 \quad \forall x \in (x_0^*, \infty)$ such that $(x, z) \in \mathcal{C}_3$;

ii) $\exists x_0 \in \mathbb{R}$, $x_0 > x_0^*$ such that $(x_0, z) \in \mathcal{C}_3$ and $\widehat{u}_x(x_0, z) > 0$.

In case i), the map $x \mapsto \widehat{u}(x, z)$ is decreasing for $x \in (x_0^*, \infty)$ and $(x, z) \in \mathcal{C}_3$. Hence, for any (x, z) in the latter region we obtain $(-\infty, x] \times \{z\} \in \mathcal{C}_3$ and the claim follows in the same spirit as in Proposition 2.6.3. In case ii), we establish a contradiction scheme. As a first step, we show that ii) implies $[x_0, \infty) \times \{z\} \in \mathcal{C}_3$, which will then lead to a contradiction. We start by noticing that Lemma 2.6.2 and (2.6.13) imply

$$(\mathcal{L}_{X,Z} - r)\widehat{u}(x_0, z) = -g(x_0, z), \tag{2.6.21}$$

for (x_0, z) as given in ii) above. Since $\mu_0 < 0$ and $\widehat{u}_x(x_0, z) > 0$ we have $\mu_0 \widehat{u}(x_0, z) < 0$, and thus

$$\begin{aligned} \frac{1}{2}\sigma^2 \widehat{u}_{xx}(x_0, z) &= r\widehat{u}(x_0, z) - \mu_0 \widehat{u}_x(x_0, z) + \frac{1}{2}(\mu_0 + \mu_1)\widehat{u}_z(x_0, z) - g(x_0, z) \\ &> r\widehat{u}(x_0, z) + \frac{1}{2}(\mu_0 + \mu_1)\widehat{u}_z(x_0, z) - g(x_0, z). \end{aligned} \tag{2.6.22}$$

Next, we notice that we can rewrite (2.6.17) as

$$\widehat{v}_z(x, z) = \frac{\gamma}{\sigma}\Big(\widehat{v}(x, z) - \mathbb{E}^{\mathbb{Q}}\Big[e^{-r\tau^*}\big(e^{X^x_{\tau^*}} - \kappa\big)\Big]\Big), \tag{2.6.23}$$

and since

$$\widehat{v}_z(x, z) = \widehat{u}_z(x, z) + \frac{\gamma}{\sigma}\big(e^x - \kappa\big)e^{\frac{\gamma}{\sigma}(x+z)} \quad \text{and} \quad \widehat{v}(x, z) = \widehat{u}(x, z) + \big(e^x - \kappa\big)\big(1 + e^{\frac{\gamma}{\sigma}(x+z)}\big),$$

(2.6.23) gives

$$\widehat{u}_z(x, z) + \frac{\gamma}{\sigma}\big(e^x - \kappa\big)e^{\frac{\gamma}{\sigma}(x+z)} = \frac{\gamma}{\sigma}\Big(\widehat{u}(x, z) + \big(e^x - \kappa\big)\big(1 + e^{\frac{\gamma}{\sigma}(x+z)}\big) - \mathbb{E}^{\mathbb{Q}}\Big[e^{-r\tau^*}\big(e^{X^x_{\tau^*}} - \kappa\big)\Big]\Big),$$

which is equivalent to

$$\widehat{u}_z(x, z) = \frac{\gamma}{\sigma}\widehat{u}(x, z) + \frac{\gamma}{\sigma}\Big(e^x - \kappa - \mathbb{E}^{\mathbb{Q}}\Big[e^{-r\tau^*}\big(e^{X^x_{\tau^*}} - \kappa\big)\Big]\Big).$$

We can thus plug this last equality into (2.6.22) and obtain

$$
\begin{aligned}
&\frac{1}{2}\sigma^2\widehat{u}_{xx}(x_0,z)\\
&\quad > r\widehat{u}(x_0,z)+\frac{1}{2}(\mu_0+\mu_1)\Big(\frac{\gamma}{\sigma}\widehat{u}(x_0,z)+\frac{\gamma}{\sigma}\Big(e^{x_0}-\kappa-\mathbb{E}^{\mathbb{Q}}\Big[e^{-r\tau^*}\big(e^{X^{x_0}_{\tau^*}}-\kappa\big)\Big]\Big)\Big)-g(x_0,z)\\
&\quad =\Big(r+\frac{1}{2}(\mu_0+\mu_1)\frac{\gamma}{\sigma}\Big)\big(\widehat{u}(x_0,z)+e^{x_0}-\kappa\big)-\frac{1}{2}(\mu_0+\mu_1)\frac{\gamma}{\sigma}\mathbb{E}^{\mathbb{Q}}\Big[e^{-r\tau^*}\big(e^{X^{x_0}_{\tau^*}}-\kappa\big)\Big]-g(x_0,z)\\
&\quad > 0,
\end{aligned}
$$

where the last inequality follows precisely from $r > \frac{\gamma}{2\sigma}|\mu_0+\mu_1|$ in Assumption 2.4.1, upon noticing that $x_0 > x_0^*$. We deduce that $\widehat{u}_x(\cdot,z)$ increases in a right-neighbourhood of x_0 and repeating arguments for every $x > x_0$ yields $\widehat{u}_x(\cdot,z) > 0$ on $[x_0,\infty)$. It follows that $\widehat{u}(\cdot,z)$ is increasing on $[x_0,\infty)$ such that $[x_0,\infty)\times\{z\}\in\mathcal{C}_3$ and (combining the latter with (2.6.10)) we have $\mathcal{A} := [x_0,\infty)\times[z_0,\infty)\subset\mathcal{C}_3$. However, this leads to a contradiction. To see this, let $(x,z)\in\mathcal{A}$ and $\tau_{x_0} := \inf\{t>0 : X^x_t \le x_0\}$. Since $t\mapsto Z^z_t$ is increasing, the only possibility for the process (X^x,Z^z) to exit $\mathcal{A}$ and thus eventually the continuation region, is by passing through the horizontal line $[x_0,\infty)\times\{z_0\}$. We thus have $\tau_{x_0}\le\tau^*$ $\mathbb{Q}_{x,z}$-a.s. and moreover, since $\mu_0<0$, the stopping time τ_{x_0} is finite a.s. Upon using Lemma 2.5.1 i) and (2.6.4), it follows that

$$
\begin{aligned}
\big(e^x-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(x+z)}\big) < \widehat{v}(x,z) &= \mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-r\tau_{x_0}}\widehat{v}(X_{\tau_{x_0}},Z_{\tau_{x_0}})\Big]\\
&= \mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-r\tau_{x_0}}\widehat{v}\big(x_0, z-\frac{1}{2}(\mu_0+\mu_1)\tau_{x_0}\big)\Big]\\
&\le K_1e^{x_0}\mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-r\tau_{x_0}}\Big]+K_1e^{\frac{\gamma}{\sigma}(x_0+z)}e^{x_0}\mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-(r-\frac{1}{2}\frac{\gamma}{\sigma}|\mu_0+\mu_1|)\tau_{x_0}}\Big].
\end{aligned}
$$

Let now $\widehat{r} := r-\frac{\gamma}{2\sigma}|\mu_0+\mu_1|>0$, where the latter inequality follows from Assumption 2.4.1, and denote by ϕ_r (resp. $\phi_{\widehat{r}}$) the strictly decreasing solution to $\frac{1}{2}\sigma^2 f_{xx}+\mu_0 f_x - qf = 0$, for $q\in\{r,\widehat{r}\}$. Then, by results on hitting times for one-dimensional diffusions (see, e.g., Borodin and Salminen (2015), Ch.II, 10), the above inequality is equivalent to

$$
\big(e^x-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(x+z)}\big) \le K_1e^{x_0}\frac{\phi_r(x)}{\phi_r(x_0)}+K_1e^{x_0}e^{\frac{\gamma}{\sigma}(x_0+z)}\frac{\phi_{\widehat{r}}(x)}{\phi_{\widehat{r}}(x_0)}, \tag{2.6.24}
$$

which thus holds true for all $(x,z)\in\mathcal{A}$. Since $\mathcal{A}$ is right-connected, we can let $x\to\infty$ and notice that $(e^x-\kappa)(1+e^{\frac{\gamma}{\sigma}(x+z)})\to\infty$, while the right hand side of (2.6.24) decreases to 0 due to the decreasing property of $x\to\phi_q(x)$ for q positive. We thus obtain a contradiction, which concludes our proof. □

Remark 2.6.6. *Due to the implied geometry of the state space, as observed in Propositions 2.6.3 and 2.6.5, we notice that the function $x\mapsto c^{-1}(x)$ of* (2.6.9) *is nondecreasing as well. Moreover, we notice that*

$$
z > c^{-1}(x) \Longleftrightarrow c(z) > x,
$$

and hence, the function c^{-1} is the right-continuous inverse of c and thus admits the representation

$$c^{-1}(x) = \inf\{z \in \mathbb{R} : c(z) > x\}. \tag{2.6.25}$$

In light of the connection (2.6.9) *between c^{-1} and b^{-1} (the generalised inverse of the boundary b), equation* (2.6.25) *allows us to translate back our results to the formulation of Section 2.5 and then – through the representation* (2.5.14) *– to the original setting of Section 2.4.*

2.6.1 Regularity of the Value Function and of the Optimal Stopping Boundary

We established the existence of a nondecreasing boundary $z \mapsto c(z)$, such that $\mathbb{R}^2$ is split into the continuation region $\mathcal{C}_3$ of (2.6.5) and the stopping region $\mathcal{S}_3$ of (2.6.6). In the following, we derive some further properties of the optimal stopping boundary and of the value function $\widehat{v}$ of (2.6.4). We first state the following result, which will be helpful in the forthcoming analysis.

Lemma 2.6.7. *We have $\widehat{u}_z(x, z) \geq 0$ for $(x, z) \in \mathcal{C}_3$.*

Proof. Because of (2.6.4) and (2.6.1), we have that $\overline{v}(x, \varphi)$ of (2.5.6) is such that $\overline{v}(x, \varphi) = \widehat{v}(x, \frac{\sigma}{\gamma} \ln(\varphi) - x)$ for all $(x, \varphi) \in \mathbb{R} \times (0, \infty)$. Since $\widehat{v}_z \in C^0(\mathcal{C}_3)$ by Lemma 2.6.2, we then also have $\overline{v}_\varphi \in C^0(\mathcal{C}_2)$. Furthermore, $\varphi \mapsto \overline{v}(x, \varphi)$ is convex on $(0, \infty)$ by Lemma 2.5.1 iv) and thus also $\varphi \mapsto \overline{u}(x, \varphi)$ of (2.5.12). Then, for $(x, \varphi) \in \mathcal{C}_2$ and $\varphi' = b^{-1}(x)$ such that $(x, \varphi') \in \partial\mathcal{C}_2$, we obtain (as $\overline{u}_\varphi \in C^0(\mathcal{C}_2)$ as well)

$$0 \leq \overline{u}(x, \varphi) = \overline{u}(x, \varphi) - \overline{u}\big(x, b^{-1}(x)\big) \leq \overline{u}_\varphi(x, \varphi)\big(\varphi - b^{-1}(x)\big),$$

and $\varphi > b^{-1}(x)$ implies $\overline{u}_\varphi(x, \varphi) \geq 0$ for $(x, \varphi) \in \mathcal{C}_2$. In light of the relation (2.6.4) we then obtain $\widehat{u}_z(x, z) \geq 0$ on $\mathcal{C}_3$. □

Proposition 2.6.8. *The optimal stopping boundary $c(z)$ is such that $x_0^* \leq c(z) \leq x_1^*$ for all $z \in \mathbb{R}$ and with x_0^* and x_1^* as in Lemma 2.4.5. Furthermore, we have $c \in C(\mathbb{R})$.*

Proof. The first part of the claim follows from Lemma 2.5.3 iii) and by noticing that the transformation $\overline{T}_1$ of (2.6.1) is the identity. We derive the continuity of $z \mapsto c(z)$ in two steps.

1) *Left-Continuity:* Let $z_0 \in \mathbb{R}$ and $z_n \uparrow z_0$ as $n \to \infty$. Since $z \mapsto c(z)$ is nondecreasing and $\mathcal{S}_3$ is closed, we obtain $\lim_{n\to\infty}(c(z_n), z_n) = (c(z_0-), z_0) \in \mathcal{S}_3$, where $c(z_0-)$ denotes the left limit of c at z_0. The definition of c in (2.6.16) implies $c(z_0-) \geq c(z_0)$, but since c is nondecreasing, we must have $c(z_0-) = c(z_0)$ and the claim follows.

2) *Right-Continuity:* We argue by contradiction and assume there exists $z_0 \in \mathbb{R}$ s.t. $c(z_0) < c(z_0+)$. Using techniques developed in De Angelis (2015), we take $c(z_0) < x_1 < x_2 < c(z_0+)$ and a nonnegative function $\phi \in C_c^\infty(x_1, x_2)$ such that $\int_{x_1}^{x_2} \phi(x)dx = 1$. Recalling (2.6.21), we have

$$\mathcal{L}_{X,Z}\widehat{u}(x,z) - r\widehat{u}(x,z) = -g(x,z), \tag{2.6.26}$$

for $(x,z) \in (x_1,x_2) \times (z_0,\infty)$. In the following, it is helpful to treat the cases i) $\mu_0 + \mu_1 \geq 0$ and ii) $\mu_0 + \mu_1 < 0$ separately. Let us start with i) and recall that $\widehat{u}_z(x,z) \geq 0$ for x and z as above, due to Lemma 2.6.7. Integration by parts reveals

$$\begin{aligned}
0 &\geq -\frac{1}{2}(\mu_0+\mu_1)\int_{x_1}^{x_2} \widehat{u}_z(x,z)\phi(x)dx \\
&= \int_{x_1}^{x_2} \Big(r\widehat{u}(x,z) - \mu_0\widehat{u}_x(x,z) - \frac{1}{2}\sigma^2\widehat{u}_{xx}(x,z) - g(x,z)\Big)\phi(x)dx \\
&= \int_{x_1}^{x_2} \Big(r\widehat{u}(x,z)\phi(x) + \mu_0\widehat{u}(x,z)\phi'(x) - \frac{1}{2}\sigma^2\widehat{u}(x,z)\phi''(x) - g(x,z)\phi(x)\Big)dx.
\end{aligned}$$

Hence, employing dominated convergence as $z \downarrow z_0$ and using $\widehat{u}(x,z_0) = 0$, yields

$$0 \geq -\int_{x_1}^{x_2} g(x,z)\phi(x)dx > 0,$$

where the latter inequality follows from $x_1, x_2 \geq x_0^*$ and Assumption 2.4.1, which implies $x > \widetilde{x}$ for all $x \in [x_1, x_2]$ and $\widetilde{x}$ as in (2.6.15). We thus obtain a contradiction and $c(z_0) = c(z_0+)$.

In case ii), we rely on classical results of internal regularity of PDEs (cf. Th. 10 in Chapter 3 of Friedman, 1982), which allow to take derivatives in (2.6.26) with respect to x and have $\widehat{u}_x \in C^{2,1}(\mathcal{C}_3)$ solving

$$(\mathcal{L}_{X,Z} - r)\widehat{u}_x(x,z) = -g_x(x,z), \quad (x,z) \in (x_1,x_2) \times (z_0,\infty).$$

Then, for $z > z_0$ we obtain

$$\int_{x_1}^{x_2} \big((\mathcal{L}_{X,Z} - r)\widehat{u}_x(x,z) + g_x(x,z)\big)\phi(x)dx = 0. \tag{2.6.27}$$

Let $F_\phi(z) := \int_{x_1}^{x_2} \widehat{u}_{xz}(x,z)\phi(x)dx$. Integration by parts allows to rewrite (2.6.27) as

$$\begin{aligned}
&\frac{1}{2}|\mu_0+\mu_1|F_\phi(z) \\
&\quad= \int_{x_1}^{x_2} \Big(r\widehat{u}_x(x,z) - \frac{1}{2}\sigma^2\widehat{u}_{xxx}(x,z) - \mu_0\widehat{u}_{xx}(x,z) - g_x(x,z)\Big)\phi(x)dx \\
&\quad= \int_{x_1}^{x_2} \Big(- r\widehat{u}(x,z)\phi'(x) + \frac{1}{2}\sigma^2\widehat{u}(x,z)\phi'''(x) - \mu_0\widehat{u}(x,z)\phi''(x) - g_x(x,z)\phi(x)\Big)dx,
\end{aligned}$$

and using dominated convergence as $z \downarrow z_0$ as well as $\widehat{u}(x, z_0) = 0$ results in

$$F_\phi(z_0+) = \frac{2}{|\mu_0 + \mu_1|} \int_{x_1}^{x_2} -g_x(x, z_0)\phi(x)dx \geq p_0 > 0,$$

for some p_0, where the second to last inequality again follows from Assumption 2.4.1. Thus, there exists $\varepsilon > 0$ such that $F_\phi(z) \geq p_0/2$ for all $z \in (z_0, z_0 + \varepsilon)$ and we finally obtain

$$\begin{aligned} \frac{1}{2}p_0\varepsilon \leq \int_{z_0}^{z_0+\varepsilon} F_\phi(z)dz = \int_{z_0}^{z_0+\varepsilon}\int_{x_1}^{x_2} \widehat{u}_{xz}(x, z)\phi(x)dxdz = -\int_{x_1}^{x_2}\int_{z_0}^{z_0+\varepsilon} \widehat{u}_z(x, z)\phi'(x)dzdx \\ = -\int_{x_1}^{x_2} (\widehat{u}(x, z_0 + \varepsilon) - \widehat{u}(x, z_0))\phi'(x)dx = \int_{x_1}^{x_2} \widehat{u}_x(x, z_0 + \varepsilon)\phi(x)dx \leq 0, \end{aligned}$$

where we used $\widehat{u}(x, z_0) = 0$ as well as $\widehat{u}_x(x, z) \leq 0$ for $x \in [x_1, x_2]$ and $z > z_0$ (cf. Proposition 2.6.5). Hence, $c(z) = c(z+)$ for all $z \in \mathbb{R}$ and together with 1) we conclude that $z \mapsto c(z)$ is continuous. □

In the next step, we derive the regularity of the value function. Its proof can be found in Appendix B.1.

Proposition 2.6.9. *The value function* $\widehat{v}$ *of* (2.6.4) *satisfies* $\widehat{v} \in C^1(\mathbb{R}^2)$ *and* $\widehat{v}_{xx} \in L^\infty_{loc}(\mathbb{R}^2)$.

In light of Proposition 2.6.9, we are able to derive an integral equation for the free boundary c. Let us first recall that by standard arguments, based on the strong Markov property and Proposition 2.6.9, the value function $\widehat{v}$ and the free boundary c solve the free-boundary problem

$$\begin{cases} (\mathcal{L}_{X,Z} - r)\widehat{v}(x, z) \leq 0, & (x, z) \in \mathbb{R}^2, \\ (\mathcal{L}_{X,Z} - r)\widehat{v}(x, z) = 0, & x < c(z), z \in \mathbb{R}, \\ \widehat{v}(x, z) \geq \big(e^x - \kappa\big)\big(1 + e^{\frac{\gamma}{\sigma}(x+z)}\big), & (x, z) \in \mathbb{R}^2, \\ \widehat{v}(x, z) = \big(e^x - \kappa\big)\big(1 + e^{\frac{\gamma}{\sigma}(x+z)}\big), & x \geq c(z), z \in \mathbb{R}, \\ \widehat{v}_x(x, z) = e^x\big(1 + e^{\frac{\gamma}{\sigma}(x+z)}\big) + \frac{\gamma}{\sigma}\big(e^x - \kappa\big)e^{\frac{\gamma}{\sigma}(x+z)}, & x = c(z), z \in \mathbb{R}, \\ \widehat{v}_z(x, z) = \frac{\gamma}{\sigma}\big(e^x - \kappa\big)e^{\frac{\gamma}{\sigma}(x+z)}, & x = c(z), z \in \mathbb{R}. \end{cases} \tag{2.6.28}$$

In the next Proposition, upon using a suitable application of Itô's Lemma, we derive a probabilistic representation of the value function $\widehat{v}$. Its proof is postponed to Appendix B.2.

Proposition 2.6.10. *Recall the free boundary* c *of* (2.6.16) *and the function* g *of* (2.6.12). *For any* $(x, z) \in \mathbb{R}^2$, *the value function* $\widehat{v}$ *can be written as*

$$\widehat{v}(x, z) = \mathbb{E}^{\mathbb{Q}}_{x,z}\Big[-\int_0^\infty e^{-rs}g(X_s, Z_s)\mathbb{1}_{\{X_s \geq c(Z_s)\}}ds\Big]. \tag{2.6.29}$$

Denote now by

$$G(w;m,v) := \frac{1}{\sqrt{2\pi v^2}} e^{-\frac{(w-m)^2}{2v^2}}, \quad w \in \mathbb{R}, m \in \mathbb{R}, v > 0, \tag{2.6.30}$$

the density function of a Gaussian random variable with mean m and variance v^2. Then, from Proposition 2.6.10 we obtain the following result.

Proposition 2.6.11. *Let*

$$\mathcal{M} := \big\{ f : \mathbb{R} \mapsto \mathbb{R} : f \textit{ is nondecreasing, continuous and s.t. } x_0^* \leq f(z) \leq x_1^* \big\}.$$

Then, the free boundary c of (2.6.16) *is the unique solution in $\mathcal{M}$ to the integral equation*

$$\big(e^{c(z)} - \kappa\big)\big(1 + e^{\frac{\gamma}{\sigma}(c(z)+z)}\big) = \int_0^\infty e^{-rs} \bigg(\int_{\mathbb{R}} -g(w, Z_s) G\big(w; c(z) + \mu_0 s, \sigma\sqrt{s}\big) \mathbb{1}_{\{w \geq c(z)\}} dw \bigg) ds, \tag{2.6.31}$$

with g as in (2.6.12) *and G as in* (2.6.30)*.*

Proof. We take $x = c(z)$ in Proposition 2.6.10. Employing the continuity of the value function we find

$$\big(e^{c(z)} - \kappa\big)\big(1 + e^{\frac{\gamma}{\sigma}(c(z)+z)}\big) = \mathbb{E}^{\mathbb{Q}}\bigg[-\int_0^\infty e^{-rs} g(X_s^{c(z)}, Z_s^z) \mathbb{1}_{\{X_s^{c(z)} \geq c(Z_s^z)\}} ds \bigg], \quad z \in \mathbb{R}. \tag{2.6.32}$$

By noticing that Z^z is deterministic and $X_s^{c(z)}$ is Gaussian under $\mathbb{Q}$ with mean $c(z) + \mu_0 s$ and variance $\sigma^2 s$, we can reformulate (2.6.32) as (2.6.31), upon using (2.6.30). To show uniqueness one can employ a four-step-approach exploiting the superharmonic characterization of $\widehat{v}$, as originally developed in Th. 3.1 of Peskir (2005). Since the present setting does not exhibit additional challenges, we omit details for the sake of brevity. □

Remark 2.6.12. *As is turns out, the integral equation* (2.6.31) *allows to derive an integral equation for the boundary b^{-1} of* (2.6.8) *as well. Indeed, taking $z = c^{-1}(x)$ in* (2.6.31) *and using* (2.6.9) *yields*

$$\big(e^x - \kappa\big)\big(1 + b^{-1}(x)\big) = \mathbb{E}^{\mathbb{Q}}\bigg[-\int_0^\infty e^{-rs} g\Big(X_s^x, \frac{\sigma}{\gamma}\ln(\Phi_s^{b^{-1}(x)}) - X_s^x\Big) \mathbb{1}_{\{\Phi_s^{b^{-1}(x)} \leq b^{-1}(X_s^x)\}} ds \bigg],$$

for $x \in \mathbb{R}$. In particular, it follows from the latter

$$b^{-1}(x) = \frac{1}{e^x - \kappa} \mathbb{E}^{\mathbb{Q}}\bigg[-\int_0^\infty e^{-rs} g\Big(X_s^x, \frac{\sigma}{\gamma}\ln(\Phi_s^{b^{-1}(x)}) - X_s^x\Big) \mathbb{1}_{\{\Phi_s^{b^{-1}(x)} \leq b^{-1}(X_s^x)\}} ds \bigg] - 1, \quad x \in \mathbb{R}. \tag{2.6.33}$$

Notice that the domain of b^{-1} is given by the interval $[x_0^*, x_1^*]$ (cf. Lemma 2.5.3) and hence, we do not encounter any problems when dividing by $e^x - \kappa$ since Assumption 2.4.1 guarantees $e^x - \kappa > 0$ for $x \geq x_0^*$.

2.7 Solution of the Optimal Execution Problem

In this section, we finally return to the optimal execution problem of Section 2.4 and provide its solution. Before we do so, it is helpful to transform the singular stochastic control problem (2.2.9) by arguing as for the optimal stopping problem in Sections 2.5 and 2.6, respectively. Since the arguments are in the same spirit of those developed in Section 2.5, details are omitted (see also Section 4 in Federico et al., 2023). First, we make a change of measure as in Section 2.5, and for $\mathbb{Q}$ as introduced therein, we let

$$dX_t^\xi = \mu_0 dt + \sigma dB_t - \alpha d\xi_t, \quad X_{0-}^\xi = x,$$

denote the dynamics of the controlled process X^ξ under $\mathbb{Q}$. Hence, conditionally to $X_{0-}^\xi = x$, $Y_{0-}^\xi = y$ and $\Phi_0 = \varphi$, we introduce the transformed optimal control problem

$$\overline{V}(x,y,\varphi) := \sup_{\xi\in\mathcal{A}(y)} \mathbb{E}^{\mathbb{Q}}_{x,y,\varphi}\bigg[\int_0^\infty e^{-rt}\big(e^{X_t^\xi}-\kappa\big)(1+\Phi_t)\circ d\xi_t\bigg], \quad (x,y,\varphi)\in\mathbb{R}\times(0,\infty)\times(0,\infty), \tag{2.7.1}$$

and observe that $\overline{V}(x,y,\varphi) = (1+\varphi)V(x,y,\frac{\varphi}{1+\varphi})$. Furthermore, we set

$$Z_t^\xi := \frac{\sigma}{\gamma}\log(\Phi_t) - X_t^\xi, \quad z := \frac{\sigma}{\gamma}\log(\varphi) - x,$$

for any $(x,\varphi)\in\mathbb{R}\times(0,\infty)$, which, through an application of Meyer-Itô formula (see, e.g., Protter, 2004, Chapter 4.7), is easily shown to have dynamics

$$dZ_t^\xi = -\frac{1}{2}(\mu_0+\mu_1)dt + \alpha d\xi_t, \quad Z_{0-}^\xi = z. \tag{2.7.2}$$

Finally, analogously to (2.6.4), we define

$$\widehat{V}(x,y,z) := \overline{V}(x,y,e^{\frac{\gamma}{\sigma}(x+z)}) = \sup_{\xi\in\mathcal{A}(y)} \mathbb{E}^{\mathbb{Q}}_{x,y,z}\bigg[\int_0^\infty e^{-rt}\big(e^{X_t^\xi}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X_t^\xi+Z_t^\xi)}\big)\circ d\xi_t\bigg], \tag{2.7.3}$$

for $(x,y,z)\in\mathcal{O} := \mathbb{R}\times(0,\infty)\times\mathbb{R}$, where $\mathbb{E}^{\mathbb{Q}}_{x,y,z}$ denotes the expectation conditional on $X_{0-}^\xi = x$, $Y_{0-}^\xi = y$ and $Z_{0-}^\xi = z$.

In the following, we introduce a candidate for the value function V of (2.2.9) and – through the explicit relationships between the value functions $v, \overline{v}$ and $\widehat{v}$ – also for the value functions $\overline{V}$ and $\widehat{V}$ of (2.7.1) and (2.7.3). To this end, we set

$$U(x,y,\pi) := \frac{1}{\alpha}\int_{x-\alpha y}^x v(x',\pi)dx', \tag{2.7.4}$$

where v denotes the value function of (2.4.2). Upon using the explicit relationship (2.5.5) of v and $\overline{v}$ it follows that

$$\overline{U}(x,y,\varphi) := (1+\varphi)U\Big(x,y,\frac{\varphi}{1+\varphi}\Big) = (1+\varphi)\frac{1}{\alpha}\int_{x-\alpha y}^{x} v\Big(x',\frac{\varphi}{1+\varphi}\Big)dx' = \frac{1}{\alpha}\int_{x-\alpha y}^{x}\overline{v}(x',\varphi)dx', \tag{2.7.5}$$

as the candidate for the value function $\overline{V}$ of (2.7.1). Furthermore, we let $\widehat{U}(x,y,z) := \overline{U}(x,y,e^{\frac{\gamma}{\sigma}(x+z)})$ and exploit the relationship (2.6.4) we can derive

$$\widehat{U}(x,y,z) = \frac{1}{\alpha}\int_{x-\alpha y}^{x}\widehat{v}(x',x+z-x')dx' = \frac{1}{\alpha}\int_{z}^{z+\alpha y}\widehat{v}(x+z-q,q)dq, \tag{2.7.6}$$

where the last equality above follows from a simple change of variables. With regard to Proposition 2.6.9 we can state the following result, whose proof is based on direct computations.

Lemma 2.7.1. *The function $\widehat{U}$ of (2.7.6) is such that $\widehat{U}\in C^1(\mathcal{O})$. Moreover, $\widehat{U}_{xy},\widehat{U}_{yz}\in C(\mathcal{O})$ as well as $\widehat{U}_{xx},\widehat{U}_{xz}\in L^\infty_{loc}(\mathcal{O})$.*

Proof. Notice that (2.7.6) gives

$$\widehat{U}_x(x,y,z) = \frac{1}{\alpha}\int_{z}^{z+\alpha y}\widehat{v}_x(x+z-q,q)dq, \quad \widehat{U}_y(x,y,z) = \widehat{v}(x-\alpha y, z+\alpha y), \tag{2.7.7}$$

$$\begin{aligned}\widehat{U}_z(x,y,z) &= \frac{1}{\alpha}\int_{z}^{z+\alpha y}\widehat{v}_x(x+z-q,q)dq + \frac{1}{\alpha}\big(\widehat{v}(x-\alpha y,z+\alpha y)-\widehat{v}(x,z)\big)\\ &= \frac{1}{\alpha}\int_{z}^{z+\alpha y}\widehat{v}_z(x+z-q,q)dq,\end{aligned} \tag{2.7.8}$$

so that $\widehat{U}_x, \widehat{U}_y$ and $\widehat{U}_z$ are continuous due to Proposition 2.6.9. Moreover, Proposition 2.6.9 also implies that

$$\widehat{U}_{xx}(x,y,z) = \frac{1}{\alpha}\int_{z}^{z+\alpha y}\widehat{v}_{xx}(x+z-q,q)dq, \tag{2.7.9}$$

is locally bounded, and the mixed derivatives

$$\widehat{U}_{xy}(x,y,z) = \widehat{v}_x(x-\alpha y,z+\alpha y), \quad \widehat{U}_{yz}(x,y,z) = \widehat{v}_z(x-\alpha y,z+\alpha y),$$

are continuous, while

$$\widehat{U}_{xz}(x,y,z) = \frac{1}{\alpha}\int_{z}^{z+\alpha y}\widehat{v}_{xx}(x+z-q,q)dq + \frac{1}{\alpha}\big(\widehat{v}_x(x-\alpha y,z+\alpha y)-\widehat{v}_x(x,z)\big),$$

is locally bounded. Furthermore, it is easy to see that $\widehat{U}_{yx}=\widehat{U}_{xy}$, $\widehat{U}_{xz}=\widehat{U}_{zx}$ and $\widehat{U}_{yz}=\widehat{U}_{zy}$. □

The proof of the next corollary follows from (2.7.7)-(2.7.8) and direct computations.

Corollary 2.7.2. *One has*

$$\alpha \widehat{U}_x(x,y,z) - \alpha \widehat{U}_z(x,y,z) + \widehat{U}_y(x,y,z) = \widehat{v}(x,z) \geq \big(e^x - \kappa\big)\big(1 + e^{\frac{\gamma}{\sigma}(x+z)}\big), \tag{2.7.10}$$

so

$$\mathbb{W}_3 := \big\{(x,y,z) \in \mathcal{O} : \alpha \widehat{U}_x(x,y,z) - \alpha \widehat{U}_z(x,y,z) + \widehat{U}_y(x,y,z) > \big(e^x - \kappa\big)\big(1 + e^{\frac{\gamma}{\sigma}(x+z)}\big)\big\} = \mathcal{C}_3, \tag{2.7.11}$$

with $\mathcal{C}_3$ *as in* (2.6.5)*. Furthermore,* $\widehat{U}_x - \widehat{U}_z = \overline{U}_x$ *as well as* $\widehat{U}_y = \overline{U}_y$*, and we have*

$$\mathbb{W}_2 := \big\{(x,y,\varphi) \in \mathbb{R} \times (0,\infty) \times (0,\infty) : \alpha \overline{U}_x(x,y,\varphi) + \overline{U}_y(x,y,\varphi) > \big(e^x - \kappa\big)\big(1 + \varphi\big)\big\} = \mathcal{C}_2, \tag{2.7.12}$$

with $\mathcal{C}_2$ *of* (2.5.7)*.*

2.7.1 Construction of the Optimal Control for the State Space Process (X, Y, Φ).

Recall b as in (2.5.9), which is nondecreasing and left-continuous by Lemma 2.5.3. Then, we define the admissible control strategy

$$\widehat{\xi}_t := y \wedge \sup_{0 \leq s \leq t} \frac{1}{\alpha}\Big(x - b(\Phi^{\varphi}_s) + \mu_0 s + \sigma B_s\Big)^+, \quad t \geq 0, \quad \widehat{\xi}_{0-} = 0, \tag{2.7.13}$$

for any $(x,y,\varphi) \in \mathbb{R} \times (0,\infty) \times (0,\infty)$, according to which the investor should only execute a lump-sum amount of shares whenever the process X_{t-} is strictly inside the selling region and hence strictly above the boundary $b(\Phi_t)$. More precisely, if $y \leq \frac{1}{\alpha}(x - b(\varphi))$ it is optimal to sell the complete amount of shares instantaneously, while for $y > \frac{1}{\alpha}(x - b(\varphi))$ the system is brought immediately to the level $(X_0, Y_0, \Phi_0) = (b(\varphi), y - \frac{1}{\alpha}(x - b(\varphi)), \varphi)$. Afterwards, the strategy (2.7.13) prescribes to take action whenever the process X_t approaches the boundary $b(\Phi_t)$ from below and the process (X_t, Y_t) is obliquely reflected at the belief-dependent boundary $b(\Phi_t)$ in the direction $(-\alpha, -1)$. Hence, the process X_t is kept inside the interval $(-\infty, b(\Phi_t)]$ with "minimal effort". These actions are the so-called *Skorokhod reflection-type policies* and caused by the continuous part $\widehat{\xi}^c$ of the control $\widehat{\xi}$. Notice that the nondecreasing process $\widehat{\xi}$, and the induced random measure $d\widehat{\xi}$ on $[0,\infty)$, are such that (recall (2.7.12))

$$\begin{cases} (X^{\widehat{\xi}}_t, Y^{\widehat{\xi}}_t, \Phi_t) \in \overline{\mathbb{W}}_2, \quad \mathbb{Q} \otimes dt\text{-a.s.}; \\ d\widehat{\xi}_t \text{ has support on } \{t \geq 0 : (X^{\widehat{\xi}}_{t-}, Y^{\widehat{\xi}}_{t-}, \Phi_t) \notin \mathbb{W}_2\}; \\ \widehat{\xi}_t \leq y, \quad t \geq 0. \end{cases}$$

Furthermore, due to (2.7.2)-(2.7.3) and Corollary 2.7.2, we can express the control $\widehat{\xi}$ equivalently in terms of the state-process $(X^{\widehat{\xi}}, Y^{\widehat{\xi}}, Z^{\widehat{\xi}})$ by (cf. (2.7.11))

$$\begin{cases} (X_t^{\widehat{\xi}}, Y_t^{\widehat{\xi}}, Z_t^{\widehat{\xi}}) \in \overline{\mathbb{W}}_3, \quad \mathbb{Q} \otimes dt\text{-a.s.}; \\ d\widehat{\xi}_t \text{ has support on } \{t \geq 0 : (X_{t-}^{\widehat{\xi}}, Y_{t-}^{\widehat{\xi}}, Z_{t-}^{\widehat{\xi}}) \notin \mathbb{W}_3\}; \\ \widehat{\xi}_t \leq y, \quad t \geq 0. \end{cases} \tag{2.7.14}$$

In the following, we prove that in fact $\widehat{\xi}$ is an optimal control for problem (2.7.3) and $\widehat{U} = \widehat{V}$. As an immediate consequence we have that $\overline{U} = \overline{V}$ and $U = V$.

Theorem 2.7.3. *Let* $(x, y, z) \in \mathbb{R} \times [0, \infty) \times \mathbb{R}$ *and* $\widehat{U}(x, y, z)$ *as in* (2.7.6)*. Then,* $\widehat{U}(x, y, z) = \widehat{V}(x, y, z)$ *and* $\widehat{\xi}$ *as in* (2.7.13) *is optimal for the singular control problem* (2.7.3)*.*

Proof. First of all, for $y = 0$ we have $\widehat{U}(x, 0, z) = 0 = \widehat{V}(x, 0, z)$. Hence, in the following we assume $(x, y, z) \in \mathcal{O}$.

Step 1. We prove $\widehat{U} \geq \widehat{V}$. Take an arbitrary control $\xi \in \mathcal{A}(y)$ and for $R > 0$ and $N \in \mathbb{N}$ we define $\tau_{R,N} := \inf\{s \geq 0 : |(X_s^{\xi}, Z_s^{\xi})| > R\} \wedge N$. Due to Lemma 2.7.1 we can proceed as in Fleming and Soner (2006), Chapter 8, Th. 4.1 to obtain (after performing an approximation of $\widehat{U}$ via mollifiers and taking limits)

$$\begin{aligned} &\mathbb{E}_{x,y,z}^{\mathbb{Q}}\Big[e^{-r\tau_{R,N}}\widehat{U}(X_{\tau_{R,N}}^{\xi}, Y_{\tau_{R,N}}^{\xi}, Z_{\tau_{R,N}}^{\xi}) - \widehat{U}(x, y, z)\Big] \\ &= \mathbb{E}_{x,y,z}^{\mathbb{Q}}\bigg[\int_0^{\tau_{R,N}} e^{-rs}(\mathcal{L}_{X,Z} - r)\widehat{U}(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi})ds + \underbrace{\sigma\int_0^{\tau_{R,N}} e^{-rs}\widehat{U}_x(X_{\tau_{R,N}}^{\xi}, Y_{\tau_{R,N}}^{\xi}, Z_{\tau_{R,N}}^{\xi})dB_s}_{=:M_{R,N}} \\ &+ \sum_{0 \leq s \leq \tau_{R,N}} e^{-rs}\big(\widehat{U}(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi}) - \widehat{U}(X_{s-}^{\xi}, Y_{s-}^{\xi}, Z_{s-}^{\xi})\big) \\ &+ \int_0^{\tau_{R,N}} e^{-rs}\big(-\alpha\widehat{U}_x(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi}) - \widehat{U}_y(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi}) + \alpha\widehat{U}_z(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi})\big)d\xi_s^c\bigg]. \end{aligned} \tag{2.7.15}$$

Notice that

$$\begin{aligned} \widehat{U}(X_s^{\xi}, Y_s^{\xi}, Z_s^{\xi}) - \widehat{U}(X_{s-}^{\xi}, &Y_{s-}^{\xi}, Z_{s-}^{\xi}) \\ &= \widehat{U}(X_{s-}^{\xi} - \alpha\Delta\xi_s, Y_{s-}^{\xi} - \Delta\xi_s, Z_{s-}^{\xi} + \alpha\Delta\xi_s) - \widehat{U}(X_{s-}^{\xi}, Y_{s-}^{\xi}, Z_{s-}^{\xi}) \\ &= \int_0^{\Delta\xi_s} \frac{\partial\widehat{U}(X_{s-}^{\xi} - \alpha u, Y_{s-}^{\xi} - u, Z_{s-}^{\xi} + \alpha u)}{\partial u}du \\ &= \int_0^{\Delta\xi_s} (-\alpha\widehat{U}_x - \widehat{U}_y + \alpha\widehat{U}_z)(X_{s-}^{\xi} - \alpha u, Y_{s-}^{\xi} - u, Z_{s-}^{\xi} + \alpha u)du. \end{aligned} \tag{2.7.16}$$

Hence, combining (2.7.15) and (2.7.16), upon adding the term

$$\mathbb{E}^{\mathbb{Q}}_{x,y,z}\bigg[\int_0^{\tau_{R,N}} e^{-rs}\big(e^{X^\xi_s}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X^\xi_s+Z^\xi_s)}\big)d\xi^c_s + \sum_{0\le s\le\tau_{R,N}} e^{-rs}\int_0^{\Delta\xi_s}\big(e^{X^\xi_{s-}-\alpha u}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X^\xi_{s-}+Z^\xi_{s-})}\big)du\bigg],$$

on both sides, yields

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}_{x,y,z}\bigg[\int_0^{\tau_{R,N}} e^{-rs}\big(e^{X^\xi_s}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X^\xi_s+Z^\xi_s)}\big)d\xi^c_s \\
&\qquad + \sum_{0\le s\le\tau_{R,N}} e^{-rs}\int_0^{\Delta\xi_s}\big(e^{X^\xi_{s-}-\alpha u}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X^\xi_{s-}+Z^\xi_{s-})}\big)du - \widehat{U}(x,y,z)\bigg] \\
=&\mathbb{E}^{\mathbb{Q}}_{x,y,z}\bigg[\int_0^{\tau_{R,N}} e^{-rs}(\mathcal{L}_{X,Z}-r)\widehat{U}(X^\xi_s,Y^\xi_s,Z^\xi_s)ds + M_{R,N} - e^{-r\tau_{R,N}}\widehat{U}(X^\xi_{\tau_{R,N}},Y^\xi_{\tau_{R,N}},Z^\xi_{\tau_{R,N}}) \\
&\qquad + \sum_{0\le s\le\tau_{R,N}} e^{-rs}\int_0^{\Delta\xi_s}\Big((-\alpha\widehat{U}_x-\widehat{U}_y+\alpha\widehat{U}_z)(X^\xi_{s-}-\alpha u, Y^\xi_{s-}-u, Z^\xi_{s-}+\alpha u) \\
&\qquad\qquad + \big(e^{X^\xi_{s-}-\alpha u}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X^\xi_{s-}+Z^\xi_{s-})}\big)\Big)du \\
&\qquad + \int_0^{\tau_{R,N}} e^{-rs}\Big((-\alpha\widehat{U}_x-\widehat{U}_y+\alpha\widehat{U}_z)(X^\xi_s,Y^\xi_s,Z^\xi_s) + \big(e^{X^\xi_s}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X^\xi_s+Z^\xi_s)}\big)\Big)d\xi^c_s\bigg].
\end{aligned} \tag{2.7.17}$$

We observe that (2.7.7)-(2.7.9) imply

$$(\mathcal{L}_{X,Z}-r)\widehat{U}(x,y,z) = \frac{1}{\alpha}\int_{x-\alpha y}^{x}(\mathcal{L}_{X,Z}-r)\widehat{v}(x',x+z-x')dx' \le 0, \tag{2.7.18}$$

where the last inequality follows from the supermartingale property of $(e^{-rt}\widehat{v}(X_t,Z_t))_t$ combined with the regularity obtained in Proposition 2.6.9. Hence, due to (2.7.10), $\widehat{U}\ge 0$ and $\mathbb{E}^{\mathbb{Q}}_{x,y,z}[M_{R,N}]=0$, (2.7.17) writes as

$$\widehat{U}(x,y,z) \ge \mathbb{E}^{\mathbb{Q}}_{x,y,z}\bigg[\int_0^{\tau_{R,N}} e^{-rs}\big(e^{X^\xi_s}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X^\xi_s+Z^\xi_s)}\big)d\xi^c_s + \sum_{0\le s\le\tau_{R,N}} e^{-rs}\int_0^{\Delta\xi_s}\big(e^{X^\xi_{s-}-\alpha u}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X^\xi_{s-}+Z^\xi_{s-})}\big)du\bigg]. \tag{2.7.19}$$

Taking limits as $R\uparrow\infty$ as well as $N\uparrow\infty$, invoking the dominated convergence theorem due to Assumption 2.4.1, we obtain

$$\begin{aligned}\widehat{U}(x,y,z) \geq \mathbb{E}^{\mathbb{Q}}_{x,y,z}\bigg[\int_0^\infty & e^{-rs}\big(e^{X^\xi_s}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X^\xi_s+Z^\xi_s)}\big)d\xi^c_s \\ &+\sum_{s:\Delta\xi_s\neq 0} e^{-rs}\int_0^{\Delta\xi_s}\big(e^{X^\xi_{s-}-\alpha u}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X^\xi_{s-}+Z^\xi_{s-})}\big)du\bigg] \\ = J(x,y,z,\xi).&\end{aligned}$$

Since ξ was arbitrary, we have

$$\widehat{U}(x,y,z) \geq \widehat{V}(x,y,z), \tag{2.7.20}$$

for all $(x,y,z) \in \mathcal{O}$. That is, $\widehat{U} \geq \widehat{V}$ on $\mathcal{O}$.

Step 2. We prove that $\widehat{U} \leq \widehat{V}$. In order to accomplish that, let $\widehat{\xi}$ satisfy the conditions in (2.7.14) and define $\widehat{\tau}_{R,N} = \inf\{t \geq 0 : |(X^{\widehat{\xi}}_t, Z^{\widehat{\xi}}_t)| > R\} \wedge N$, again for $R > 0$ and $N \in \mathbb{N}$. Notice that the properties of $\widehat{\xi}$ imply equalities in (2.7.10) and (2.7.18), where the equality in (2.7.18) follows from the monotonicity of c and we can deduce that $(x', x+z-x') \in \mathbb{W}_3$ for $(x,y,z) \in \mathbb{W}_3$ and $x' \leq x$. Employing the same arguments as in the first part of the proof yields

$$\begin{aligned}\widehat{U}(x,y,z) = \mathbb{E}^{\mathbb{Q}}_{x,y,z}&\Big[e^{-r\widehat{\tau}_{R,N}}\widehat{U}\big(X^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}, Y^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}, Z^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}\big)\Big] \\ +\mathbb{E}^{\mathbb{Q}}_{x,y,z}&\bigg[\int_0^{\widehat{\tau}_{R,N}} e^{-rs}\big(e^{X^{\widehat{\xi}}_s}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X^{\widehat{\xi}}_s+Z^{\widehat{\xi}}_s)}\big)d\widehat{\xi}^c_s \\ &+\sum_{0\leq s\leq\widehat{\tau}_{R,N}} e^{-rs}\int_0^{\Delta\widehat{\xi}_s}\big(e^{X^{\widehat{\xi}}_{s-}-\alpha u}-\kappa\big)\big(1+e^{\frac{\gamma}{\sigma}(X^{\widehat{\xi}}_{s-}+Z^{\widehat{\xi}}_{s-})}\big)du\bigg].\end{aligned}$$

It is thus left to prove that

$$\lim_{N\uparrow\infty}\lim_{R\uparrow\infty}\mathbb{E}^{\mathbb{Q}}_{x,y,z}\Big[e^{-r\widehat{\tau}_{R,N}}\widehat{U}\big(X^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}, Y^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}, Z^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}\big)\Big] = 0, \tag{2.7.21}$$

since taking limits as $R \uparrow \infty$ and $N \uparrow \infty$ together with (2.7.19) implies $J(x,y,z,\widehat{\xi}) = \widehat{U}(x,y,z)$ and hence $\widehat{V}(x,y,z) \geq \widehat{U}(x,y,z)$ for all $(x,y,z) \in \mathcal{O}$. Combining the latter with (2.7.20) yields $\widehat{U} = \widehat{V}$ on $\mathcal{O}$.

In order to prove (2.7.21) we notice that Lemma 2.5.1 i), (2.6.4) and (2.7.6) imply

$$\widehat{U}(x,y,z) \leq \frac{1}{\alpha}\int_z^{z+\alpha y} K_1 e^{x+z-q}\big(1+e^{\frac{\gamma}{\sigma}(x+z-q+q)}\big)dq = \frac{1}{\alpha}K_1 e^x\big(1+e^{\frac{\gamma}{\sigma}(x+z)}\big)\big(1-e^{-\alpha y}\big),$$

and, since $y \mapsto \widehat{U}(x,y,z)$ is increasing, we obtain

$$\begin{aligned} 0 \le e^{-r\widehat{\tau}_{R,N}}\widehat{U}\big(X^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}, Y^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}, Z^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}\big) &\le e^{-r\widehat{\tau}_{R,N}}\widehat{U}\big(X^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}, y, Z^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}\big) \\ &\le \frac{K_1}{\alpha}(1-e^{-\alpha y})e^{-r\widehat{\tau}_{R,N}}e^{X^0_{\widehat{\tau}_{R,N}}}\big(1+e^{\frac{\gamma}{\sigma}(X^0_{\widehat{\tau}_{R,N}}+Z^0_{\widehat{\tau}_{R,N}})}\big), \end{aligned}$$

where we used that $X^{\xi}_t \le X^0_t$ as well as $X^{\xi}_t + Z^{\xi}_t = X^0_t + Z^0_t$ a.s. Hence, taking expectations yields

$$\begin{aligned} 0 &\le \mathbb{E}^{\mathbb{Q}}_{x,y,z}\Big[e^{-r\widehat{\tau}_{R,N}}\widehat{U}\big(X^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}, Y^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}, Z^{\widehat{\xi}}_{\widehat{\tau}_{R,N}}\big)\Big] \\ &\le \frac{K_1}{\alpha}\big(1-e^{-\alpha y}\big)\mathbb{E}^{\mathbb{Q}}_{x,y,z}\Big[e^{-r\widehat{\tau}_{R,N}}e^{X^0_{\widehat{\tau}_{R,N}}}\big(1+e^{\frac{\gamma}{\sigma}(X^0_{\widehat{\tau}_{R,N}}+Z^0_{\widehat{\tau}_{R,N}})}\big)\Big] \\ &= \frac{K_1}{\alpha}\big(1-e^{-\alpha y}\big)\big(1+e^{\frac{\gamma}{\sigma}(x+z)}\big)\mathbb{E}_{x,y,\pi}\Big[e^{-r\widehat{\tau}_{R,N}}e^{X^0_{\widehat{\tau}_{R,N}}}\Big], \end{aligned}$$

with $\pi := e^{\frac{\gamma}{\sigma}(x+z)}/(1+e^{\frac{\gamma}{\sigma}(x+z)})$ and the last equality follows from a change of measure as in Section 2.5. Upon using Assumption 2.4.1, it is easy to check that (2.7.21) holds true, thus completing the proof. □

Remark 2.7.4. *We can use the transformation* (2.6.1) *from* (x,z)*- to* (x,φ)*-coordinates in order to show that* $\widehat{\xi}$ *is an optimal control for problem* (2.7.1) *as well. Indeed, recall* (2.6.2) *and the equality* $\widehat{V}(x,y,z) = \widehat{V}(x,y,\frac{\sigma}{\gamma}\ln(\varphi)-x) = \overline{V}(x,y,\varphi)$ *to conclude*

$$\begin{aligned} &\overline{V}(x,y,\varphi) \\ &\quad = \mathbb{E}^{\mathbb{Q}}_{x,y,\varphi}\Big[\int_0^{\infty} e^{-rs}\big(e^{X^{\widehat{\xi}}_s}-\kappa\big)(1+\Phi_s)d\widehat{\xi}^c_s + \sum_{s:\Delta\widehat{\xi}_s\neq 0} e^{-rs}\int_0^{\Delta\widehat{\xi}_s}\big(e^{X^{\widehat{\xi}}_{s-}-\alpha u}-\kappa\big)(1+\Phi_s)du\Big]. \end{aligned}$$

Furthermore, the latter equation and (2.7.5) *imply* $U(x,y,\pi) = V(x,y,\pi)$ *for all* $(x,y,\pi) \in \mathbb{R}\times(0,\infty)\times(0,1)$.

Remark 2.7.5. *Letting* $\widetilde{\tau}(x,y,\varphi) := \inf\{t \ge 0 : x+\mu_0 t+\sigma B_t \ge b(\Phi^{\varphi}_s)\}$*, the optimal execution strategy* $\widehat{\xi}$ *as in* (2.7.13) *converges as* $\alpha \downarrow 0$ *to the execution strategy*

$$\widetilde{\xi}_t = \begin{cases} 0 & t < \widetilde{\tau}(x,y,\varphi), \\ y & t \ge \widetilde{\tau}(x,y,\varphi), \end{cases}$$

which prescribes to sell the total amount of shares instantaneously when the process X *reaches the optimal execution boundary* $b(\Phi)$*. It is interesting to notice that the optimal solution and the value function are robust w.r.t. the parameter* α*. Indeed, by L'Hôpital's rule, we see from* (2.7.5) *that* $\lim_{\alpha\downarrow 0} = y\overline{v}(x,\varphi)$*,* $(x,y,\varphi) \in \mathbb{R}\times(0,\infty)\times(0,\infty)$*. It is in fact easy to show via a verification theorem that* $y\overline{v}(x,\varphi)$ *and* $\widetilde{\xi}$ *are the value function and the optimal execution rule in the problem with no market impact.*

Remark 2.7.6. *Let $\widehat{\sigma} := \inf\{t \geq 0 : Y_t^{\xi} = 0\}$ denote the time at which the portfolio is fully depleted. Imposing the constraint that the investor has to sell all assets until terminal time (cf. Guo and Zervos, 2015), we notice that for $y \leq \frac{1}{\alpha}(x - b(\varphi))$ the control strategy $\widehat{\xi}$ of* (2.7.13) *still defines an optimal control, as the complete amount of shares is sold immediately at time $t = 0$. However, for $y > \frac{1}{\alpha}(x - b(\varphi))$, simple calculations yield*

$$\lim_{T\uparrow\infty} \mathbb{Q}[\widehat{\sigma} > T] \geq 1 - \exp\Big(\frac{2\mu_0}{\sigma^2}(\alpha y + x_0^* - x)\Big),$$

and we notice that for increasing y and decreasing x, the probability increases that the investor does not sell the entire amount of shares until terminal time. Hence, if we restrict the admissible strategies to all $\xi \in \mathcal{A}(y)$ such that $\lim_{T\to\infty} Y_T^{\xi} = 0$, the control strategy $\widehat{\xi}$ of (2.7.13) *does not provide an admissible execution strategy. In this case, arguing as in Guo and Zervos (2015), Proposition 5.1, we can use $\widehat{\xi}$ to construct a sequence of ε-optimal strategies.*

2.8 Numerical Study

In this section, we (i) perform a comparative statics analysis on the optimal execution boundaries a and b of (2.4.5) and (2.5.9), respectively, as well as (ii) investigate the *value of information* in our model, by comparing the value function V of (2.2.9) to the value of an *average drift problem*.

2.8.1 Comparative Statics Analysis

Based on the integral equation (2.6.31) we implement a recursive numerical scheme, which relies on an application of the Monte-Carlo method and is discussed in Appendix A in more detail. For the sake of exposition, we explain how the procedure is implemented in this framework. We let ζ denote an auxiliary exponentially distributed random variable with parameter r, that is independent of the Brownian motion B. Recalling that (2.6.31) can be reformulated as (2.6.33), we notice that the latter takes the shape of a fixed point problem

$$b^{-1}(x) = \Gamma(b^{-1}(x), x; b^{-1}), \tag{2.8.1}$$

for $x \in \mathbb{R}$ and b^{-1} being the generalized inverse of b as in (2.6.8). Here, the operator Γ is defined via

$$\Gamma(\varphi, x; f) := \frac{1}{e^x - \kappa}\frac{1}{r}\mathbb{E}^{\mathbb{Q}}\Big[- g\big(X_\zeta^x, \frac{\sigma}{\gamma}\ln(\Phi_\zeta^\varphi) - X_\zeta^x\big)\mathbb{1}_{\{\Phi_\zeta^\varphi \leq f(X_\zeta^x)\}}\Big] - 1, \tag{2.8.2}$$

for $(x, \varphi) \in \mathbb{R} \times (0, \infty)$ and a function $f : \mathbb{R} \to (0, \infty)$. By employing techniques seen in Christensen and Salminen (2018), Dammann and Ferrari (2022) and Detemple and Kitapbayev

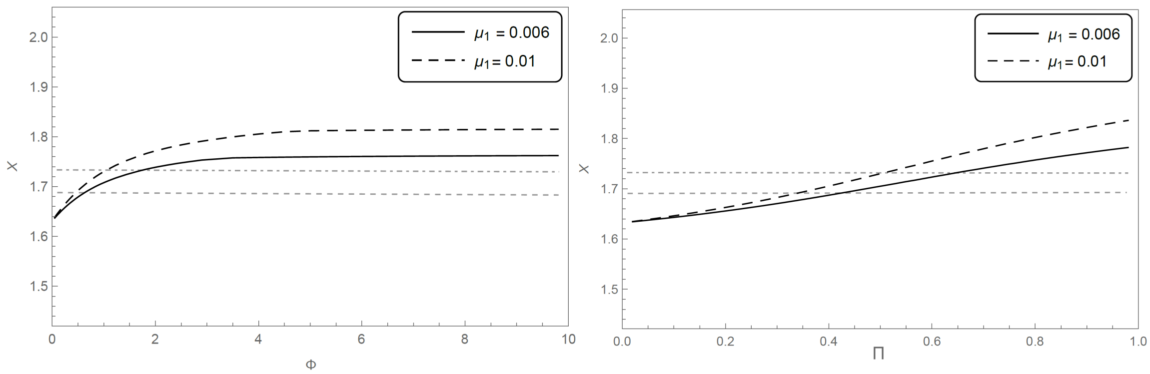


Figure 2.8.1: The optimal execution boundaries $b(\varphi)$ and $a(\pi)$ as well as the pre-committed strategies for different values of μ_1 and following parameters: $r = 0.07$, $\mu_0 = -0.01$, $\sigma = 0.17$, $\kappa = 3$, $\pi = 0.6$.

(2020a), we aim to solve (2.8.1) via an iterative scheme. To this end, we let

$$(b^{-1})^{[n]}(x) = \Gamma\Big((b^{-1})^{[n-1]}(x), x; (b^{-1})^{[n-1]}\Big), \quad x \in \mathbb{R}, n \geq 1, \tag{2.8.3}$$

define a sequence of boundaries and – for a given boundary $(b^{-1})^{[k]}$ – we estimate the expectation in (2.8.2) by

$$-\frac{1}{N}\sum_{i=1}^{N} g\Big(X^{i,x}_{\zeta_i}, \frac{\sigma}{\gamma}\ln\big(\Phi^{i,(b^{-1})^{[k]}(x)}_{\zeta_i}\big) - X^{i,x}_{\zeta_i}\Big)\mathbb{1}_{\big\{\Phi^{i,(b^{-1})^{[k]}(x)}_{\zeta_i} \leq (b^{-1})^{[k]}(X^{i,x}_{\zeta_i})\big\}},$$

where N denotes the total amount of realizations of the exponential random variable. The initial boundary $(b^{-1})^{[0]}$ can be chosen as a simple exponential function with $(b^{-1})^{[0]}(x^*_0) = 0$ and $(b^{-1})^{[0]}(x) \to \infty$ for $x \uparrow x^*_1$ with x^*_0 and x^*_1 as in (2.3.6) and Remark 2.3.1, respectively. The numerical scheme (2.8.3) is then iterated until the variation between steps drops below a predetermined level. Finally, we calculate b from its generalized inverse b^{-1} and can transform the resulting boundary according to the explicit relationship (2.5.14). We can thus study the sensitivity of $b(\varphi)$ as well as $a(\pi)$ with respect to some of the model's parameters. Furthermore, we can compare the belief-dependent boundaries to the strategy of a pre-committed agent, who – after forming an initial belief $\pi = \mathbb{P}[\mu = \mu_1]$ – restrains from updating her belief and thus acts as if the drift value was constant and equal to $\mu_1\pi + \mu_0(1-\pi)$. The resulting strategy is then triggered by a constant execution threshold, which is of similar structure as the one derived in Section 2.3. Consequently, we observe that such an agent cannot react to any price movements on the market and is thus not able to decrease or increase the target price at which she would like to sell the asset.

Sensitivity with respect to the drift. In Figure 2.8.1 we can observe the sensitivity of the optimal execution boundaries with respect to one of the possible drift values. Since an increase in μ_1 implies higher expected prices on the market, the investor delays her decision

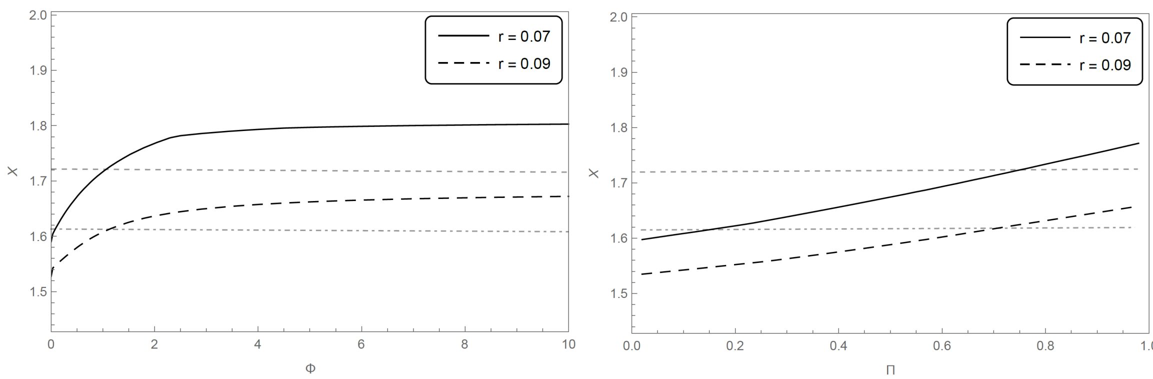


Figure 2.8.2: The optimal execution boundaries $b(\varphi)$ and $a(\pi)$ as well as the pre-committed strategies for different values of r and following parameters: $\mu_0 = -0.01$, $\mu_1 = 0.007$, $\sigma = 0.17$, $\kappa = 3.$, $\pi = 0.6$.

to sell a fraction of her shares and waits for larger prices to evolve. This effect is strongest for higher values of π, which reflect a stronger belief in the drift μ_1. On the other hand, we notice that the lower bound x_0^* remains untouched by a change in μ_1, since it results from the case of full information when $\mu = \mu_0$. Consequently, for a strong belief in the drift value μ_0, the investor does not significantly change her execution strategy.

Sensitivity with respect to the discount rate. Figure 2.8.2 shows the effect on the boundaries a and b for a change in r, the latter can be interpreted as the subjective impatience of the investor. For an increasing value of r the investor gets more impatient and discounts future revenues more heavily. Consequently, the investor is willing to liquidate her assets earlier, which is realized by decreasing the target price she aims at achieving on the market. This clear effect can be observed for every value of belief $\pi \in [0, 1]$.

Sensitivity with respect to the volatility. The sensitivity of the optimal execution boundaries a and b on the volatility of the underlying asset is more delicate. As pointed out by Décamps et al. (2005), who consider an optimal stopping problem of a structure similar to the one in (2.4.2), the effect of an increase in volatility is ambiguous and cannot always be predicted with the help of standard real option models (see for example Dixit and Pindyck, 1994, Chapter 6, McDonald and Siegel, 1986). In general, one expects an increasing value function with rising volatility, as this increases the spread of possible future values of the asset and thus the maximal possible profit, while the maximal possible loss remains unchanged. The investor exploits this upside potential by delaying her liquidation decision and increasing the target price she aims at realizing on the market. This effect, widely known and referred to as the "real option effect" in Décamps et al. (2005), can be observed in the benchmark case of (2.3.2) as well as in the problem (2.2.9) under partial information, as Figure 2.8.3 reveals.

However, this effect does not need to be robust. To understand how an increase in volatility might indeed harm the investor, we recall the dynamics of the belief process Π, given by

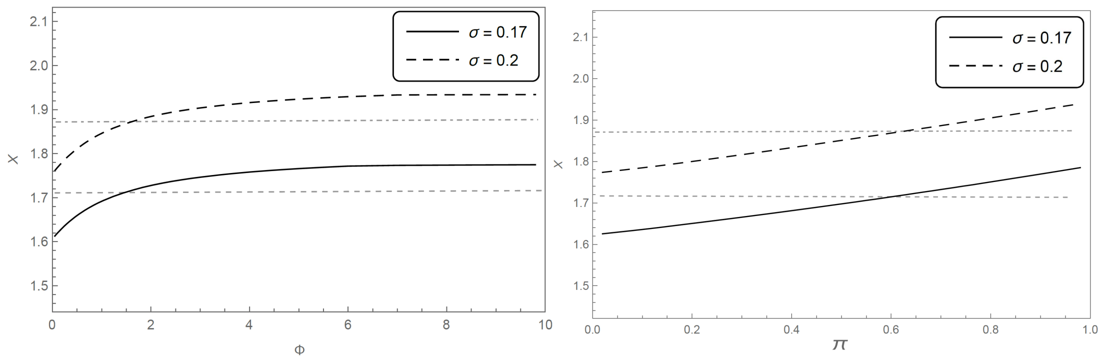


Figure 2.8.3: The optimal execution boundaries $b(\varphi)$ and $a(\pi)$ as well as the pre-committed strategies for different values of σ and following parameters: $r = 0.07$, $\mu_0 = -0.01$, $\mu_1 = 0.007$, $\kappa = 3$, $\pi = 0.6$.

(2.2.8). In particular, we observe that increasing volatility lowers the signal-to-noise ratio $\gamma = (\mu_1 - \mu_0)/\sigma$ (determining the variance of the process Π) and thus the *efficiency of learning*. The latter effect is in contrast to the mentioned real option effect, and the sensitivity of the value function with respect to an increase in volatility "depends on which of the real option and the inefficient learning effect dominates" (Décamps et al., 2005, p. 487). The overall impact of a change in volatility thus clearly depends on the parameters' constellation of the model, a division of the parameters' space is however not straightforward. For a broader discussion on this subject we refer to Décamps et al. (2005), Section 6.2.

2.8.2 The Value of Information.

Here, we want to address the question on whether incomplete information about the drift actually harms or benefits the investor. To this end, we introduce the "average drift problem", whose value is denoted by $V^A(x,y)$ and modelled as in (2.3.2), but with constant and known drift $\pi\mu_1 + (1-\pi)\mu_0$; i.e. the average of μ with respect to the prior Bernoulli distribution. We then investigate the preference of an investor faced with the decision of choosing between two portfolios containing assets with either an unknown drift coefficient, or with a constant and known *average* drift. An analytical attempt to answer this question is presented in Décamps et al. (2005), although the derived result does not hold true in general, as pointed out by Klein (2009).

Here, we are able to analyse this question with numerical methods based on the numerical evaluation of the optimal execution boundary (cf. Section 8.1) and the representation (2.6.29) of the optimal stopping value function $\widehat{v}$. In order to accomplish that, we plug in the numerical evaluation of b^{-1} into (2.6.29) and we transform the result according to (2.5.13). This yields the value function v of (2.4.2), which can be finally integrated via (2.7.4) to obtain a numerical

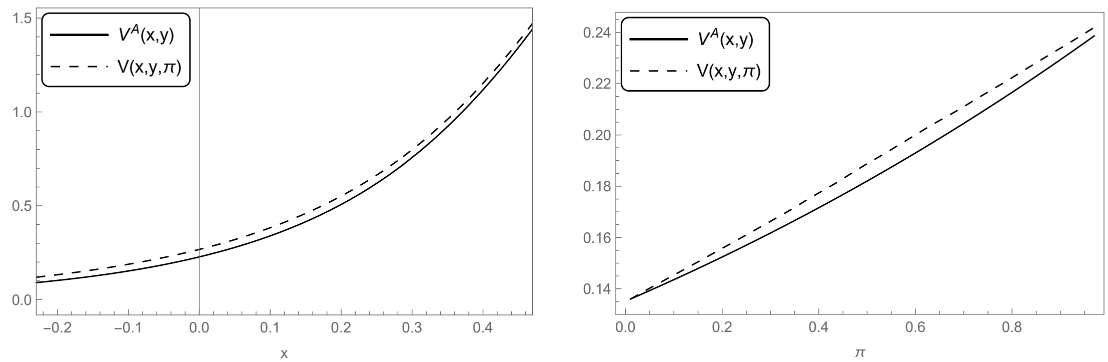


Figure 2.8.4: The value function V of (2.2.9) and the average drift value function V^A as functions of x and π, respectively. The parameters of the model have been specified as $r = 0.15$, $\sigma = 0.15$, $\mu_0 = -0.012$, $\mu_1 = 0.01$, $\kappa = 1$, $\pi = 0.3, x = -0.1$

approximation of the control problem's value function V. In general, the results derived in Décamps et al. (2005) and Klein (2009) suggest that the overall impact of introducing uncertainty over the drift is governed by two separate effects: The introduction of uncertainty in general and the impact of learning. If learning is efficient, which is achieved by – for example – specifying a small volatility coefficient σ, the latter effect seems to outweighs the former and the investor indeed prefers the problem with only incomplete information on the return. We observe this overall effect in Figure 2.8.4.

In their model, Décamps et al. (2005) give an analytical proof to this observation in an optimal stopping environment, although restricting the possible drift values to 0 and 1. For small values of σ, depending on the other parameters in the model, this result seems to hold true in our more generalized framework.

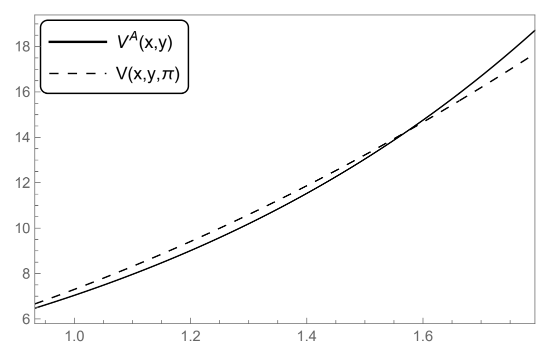


Figure 2.8.5: The value function V of (2.2.9) and the average drift value function V^A as functions of x. The parameters of the model have been specified as $r = 0.2$, $\sigma = 0.5$, $\mu_0 = -0.012$, $\mu_1 = 0.01$, $\kappa = 1$, $\pi = 0.5$

Nevertheless, this effect cannot be expected to be robust over the whole parameter space. In an example, where the parameter values are aligned such that $\beta_0+\beta_1 = \sigma^2$ (and thus $\mu_0 = -\mu_1$ in our model), Klein (2009) obtains an explicit solution to the optimal stopping problem and shows how the introduction of uncertainty might harm the decision maker. This effect appears to have the peculiarity of being, at least in some cases, dependent on the initial value of the price process, as it determines the distance to the target price at which the investor is willing to execute. We can observe an example of this in Figure 2.8.5. In particular, if the asset's price is close to the target value under the current belief and learning is inefficient, the investor will not choose a portfolio with drift uncertainty. This is due to the fact that the downside risk outweighs the upside potential, which could only be achieved if learning is efficient. On the other hand, we observe that for low prices the upside potential might still dominate and the investor is willing to choose the uncertain environment, even if learning is inefficient.

Chapter 3

A Stochastic Non-Zero-Sum Game of controlling the Debt-to-GDP Ratio

3.1 Introduction

In this chapter, we introduce a non-zero-sum game between a government and a legislative body to study the optimal level of debt. Each player, with different time preferences, can intervene on the stochastic dynamics of the debt-to-GDP ratio via singular stochastic controls, in view of minimizing non-continuously differentiable running costs. We completely characterise Nash equilibria in the class of Skorokhod-reflection-type policies. We highlight the importance of different time preferences resulting in qualitatively different type of equilibria. Regarding the optimal strategy of the government, we show that it is always optimal to devise an appropriate debt issuance policy whenever the debt ratio is sufficiently low. As for the legislative body, we show that the magnitude of their time preference rate plays a crucial role, which is a key result of our analysis. In particular, if it is relatively small, a debt ceiling mechanism is optimal, while the legislative body should follow a laissez-faire policy for high values.

The chapter is organized as follows. Section 3.2 describes the setting and the two problems faced by the two players. In Section 3.3, we solve the government constrained control problem by distinguishing two cases: the legislative body does not intervene or forces the government to keep its debt ratio below a debt ceiling $b > 0$. In Section 3.4, we solve the constrained control problem of the legislative body and find its best response strategy to the above governmental policy. We prove the existence and uniqueness of a Nash Equilibrium in the class of Skorokhod-reflection policies in Section 3.5. Finally, Section 3.6 is devoted to a comparative statics analysis; we explore how the optimal debt issuance policy and debt ceiling mechanism are affected by changes in the model parameters. In particular, we are able to quantify the transition between a legislative body's optimal intervention and non-intervention regimes.

3.2 Setting and Problem Formulation

3.2.1 Motivation for the Model

The model applies to a government that has to finance its expenditures through public debt, under the control of a legislative body. The nominal debt grows at rate r, i.e. it evolves according to

$$dD_t = rD_t dt, \qquad t \geq 0,$$

in the absence of any intervention, where we denote by $r \in \mathbb{R}$ the interest rate on government debt. When designing its economic policy, the government can choose to increase the current level of the debt by a new issuance. Denoting by ξ_t the cumulative percentage of debt increase up to time $t \geq 0$, the dynamics of the adjusted debt reads as

$$dD_t = rD_t dt + D_t d\xi_t, \qquad t \geq 0,$$

where the latter integral (and others of the same type in the following motivation) will be defined later in Section 3.2.2. The GDP follows the stochastic exogenous dynamics

$$dG_t = gG_t dt + \sigma G_t d\widehat{W}_t, \qquad t \geq 0,$$

in the absence of any intervention, where we denote by $g \in \mathbb{R}$ the growth rate of the GDP and by $\widehat{W}$ a standard one-dimensional Brownian motion. A legislative body can implement a liberalisation policy in order to boost the GDP by forcing the government to favor the job market, moderating social insurance programs, reducing burdensome regulations, lowering the marginal tax rate and privatizing businesses.

Denoting by η_t the cumulative percentage of GDP increase up to time $t \geq 0$, the dynamics of the adjusted GDP read as

$$dG_t = gG_t dt + \sigma G_t d\widehat{W}_t + G_t d\eta_t, \qquad t \geq 0,$$

Hence, we may conclude that the dynamics of the debt-to-GDP ratio, obtained via the use of Itô's formula on $X := D/G$, evolves according to

$$dX_t = (r - g)X_t dt + \sigma X_t(\sigma dt - d\widehat{W}_t) + X_t d\xi_t - X_t d\eta_t, \qquad t \geq 0,$$

Without loss of generality, a change of measure to one under which $W_t := \sigma t - \widehat{W}_t$ is a Brownian motion, will allow for the following problem formulation.

3.2.2 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space accommodating a one-dimensional Brownian motion $W := (W_t)_{t \geq 0}$. We denote by $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ the filtration generated by W augmented by $\mathbb{P}$-null sets. In absence of any interventions, the debt-to-GDP ratio (also called "debt ratio") evolves according to the stochastic differential equation (SDE)

$$dX_t^0 = (r-g)X_t^0 dt + \sigma X_t^0 dW_t, \qquad X_t^0 = x > 0. \tag{3.2.1}$$

The classical macroeconomic dynamics of the debt ratio, see e.g. Blanchard and Fischer, 1989, are simply the deterministic version of (3.2.1) with $\sigma = 0$. When increasing the current debt ratio level by $\varepsilon > 0$ percentage points, the debt ratio exhibits a jump

$$\Delta X_t = X_t - X_{t-} = \varepsilon X_{t-}.$$

Hence, for small $\varepsilon > 0$, we can associate a governmental intervention on the debt ratio with $X_t = (1+\varepsilon)X_{t-} \approx e^{\varepsilon} X_{t-}$. Furthermore, interpreting an intervention $\Delta\xi_t$ as a sequence of N individual interventions of size $\varepsilon = \Delta\xi_t/N$ we have $X_t = e^{N\varepsilon}X_{t-} = e^{\Delta\xi_t}X_{t-}$, for N large enough. We can thus model the controlled debt ratio dynamics (by arguing similarly for the interventions of the legislative body) via

$$dX_t^{\xi,\eta} = (r-g)X_t^{\xi,\eta}dt + \sigma X_t^{\xi,\eta}dW_t + X_t^{\xi,\eta} \circ_u d\xi_t - X_t^{\xi,\eta} \circ_d d\eta_t, \qquad t \geq 0, \tag{3.2.2}$$

where the operators $\circ_u$ and $\circ_d$ are defined as

$$\begin{aligned} X_t^{\xi,\eta} \circ_u d\xi_t &= X_t^{\xi,\eta} d\xi_t^c + X_{t-}^{\xi,\eta} \int_0^{\Delta\xi_t} e^u du = X_t^{\xi,\eta} d\xi_t^c + X_{t-}^{\xi,\eta}\big[e^{\Delta\xi_t} - 1\big], \\ X_t^{\xi,\eta} \circ_d d\eta_t &= X_t^{\xi,\eta} d\eta_t^c + X_{t-}^{\xi,\eta} \int_0^{\Delta\xi_t} e^{-u} du = X_t^{\xi,\eta} d\eta_t^c + X_{t-}^{\xi,\eta}\big[1 - e^{\Delta\xi_t}\big]. \end{aligned} \tag{3.2.3}$$

Here, ξ^c (resp., η^c) denotes the continuous part of the process ξ (resp., η).

Using Itô's formula we can verify that the solution to (3.2.2) starting at time zero from level $x > 0$ is given by

$$X_t^{\xi,\eta} = x \exp\Big(\Big(r - g - \frac{1}{2}\sigma^2\Big)t + \sigma W_t + \xi_t - \eta_t\Big) = X_t^0 \exp(\xi_t - \eta_t), \qquad t \geq 0, \tag{3.2.4}$$

where X_t^0 denotes the solution to (3.2.1). Notice that the impact of interventions by the government and legislative body are of multiplicative structure and additive to the logarithm of the debt ratio.

In accordance with our reasoning above, ξ_t denotes the cumulative percentage amount of debt increase by the government and η_t denotes the cumulative percentage amount of debt decrease by the legislative body, up to time $t \geq 0$. It is therefore natural to model them as nondecreasing stochastic processes, adapted with respect to the available flow of information $\mathbb{F}$. Hence we take ξ and η in the set

$$\mathcal{U} := \Big\{ \upsilon : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+ : \quad (\upsilon_t)_{t \geq 0} \ \mathbb{F}\text{-adapted, nondecreasing, càdlàg, and } \upsilon_{0-} = 0 \Big\}.$$

The Problem of the Government

In this framework, the government is facing a potential debt ceiling (or debt limit) as a hard constraint imposed by a legislative body, when the country's debt ratio is too high. In other words, the government has an exogenous factor, namely a debt ratio ceiling b, to take into consideration when designing its economic policy. This is the level at which a legislative body will demand the decrease of the debt ratio and the adoption of liberalisation policies by the government. In the following, we assume that having a debt level $X_t^{\xi,\eta}$ at time $t \geq 0$, the government incurs an instantaneous cost $h(X_t^{\xi,\eta})$. This may be interpreted as an opportunity cost resulting from having less room for financing public investments. We make the following standing assumption.

Assumption 3.2.1. *The instantaneous (running) cost function $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies:*

(i) *$x \mapsto h(x)$ is strictly convex, continuously differentiable and increasing on $[0, \infty)$;*

(ii) *the derivative h' of h satisfies $\lim_{x\to 0} h'(x) = 0$ and $\lim_{x\to\infty} h'(x) = +\infty$;*

(iii) *there exists $p > 1$, $K_1 > 0$ such that*

$$h(x) \leq K_1(1 + |x|^p), \quad x \in \mathbb{R}.$$

Remark 3.2.2. *It is worth noticing that a cost function of the form $h(x) = \frac{1}{2}x^2$ for $x > 0$ satisfies Assumption 3.2.1. Notice that $h(0) = 0$ together with $h'(0) = 0$ imply that any infinitesimal amount of debt does not generate holding costs for the country; indeed, $h(\varepsilon) \approx h'(0)\varepsilon = 0$. If one wishes to obtain closed-form solutions, a specific function h must be chosen according to Assumption 3.2.1; our choice will be precisely the above one.*

Moreover, whenever a legislative body decides to impose a debt ceiling mechanism, the government incurs a proportional cost to the amount of debt reduction (see also Cadenillas and Huamán-Aguilar, 2016, Ferrari, 2018 and Ferrari and Rodosthenous, 2020). This might be seen as a measure of the social and financial consequences, or repercussions for the financial stability of households and individuals, deriving from the enforcement of debt-reduction policies. The associated constant marginal cost $c_1 > 0$ allows to express it in monetary terms. Finally, the government's main aim is to increase the current level of debt ratio through public investments, e.g. investments in infrastructure, healthcare, education and research, etc. We assume that this has a positive political, social and financial effect, thus overall reduces the total expected "costs" of the government. The marginal benefit of increasing the debt ratio is a strictly positive constant $c_2 > 0$. From the point of view of the government, assuming that it discounts at a rate $\rho > 0$, the total expected cost functional, net of investment benefits, is thus given by

$$\mathcal{J}_{x,\eta}(\xi) := \mathbb{E}_x\Big[\int_0^\infty e^{-\rho t} h(X_t^{\xi,\eta})dt + c_1 \int_0^\infty e^{-\rho t} X_t^{\xi,\eta} \circ_d d\eta_t - c_2 \int_0^\infty e^{-\rho t} X_t^{\xi,\eta} \circ_u d\xi_t\Big], \quad (3.2.5)$$

where, for any $x \in \mathbb{R}_+$, $\mathbb{E}_x$ denotes the expectation under the measure $\mathbb{P}_x[\cdot] := \mathbb{P}[\cdot \mid X_{0_-}^{\xi,\eta} = x]$.

The Problem of the Legislative Body

On one hand, the legislative body (e.g. Congress) would like governments to ideally keep their country's debt ratio at low levels to maintain a low probability of default and a feasible borrowing from the markets. Even though countries that can print their own currency cannot default on their debts, there are many countries that do not control their own monetary policy, e.g. EU members who rely on the European Central Bank (ECB), or countries that hold large amounts of foreign denominated debts, e.g. Argentina (who defaulted on US government bonds). Several levels $m > 0$ defining the "healthy" region $[0, m]$ of relatively "low" debt ratio have been used in the last decades, e.g. $m = 60\%$ is the Maastricht Treaty's reference value of 1992 for all EU countries, or $m = 77\%$ is the threshold found by researchers at the World Bank (see Caner et al., 2010) for developed economies and $m = 64\%$ for emerging markets.[1]

When the debt ratio $X^{\xi,\eta}$ exceeds this pre-specified value $m > 0$, the legislative body would face social and political pressure, which may lead to the implementation of liberalisation policies in order to decrease the level of $X^{\xi,\eta}$ via a control strategy η. This could, for example, be done by setting a debt ceiling b. This debt ceiling b is expected to be bigger than m, since imposing structural adjustment programs on countries or restricting further borrowing by governments, is costly for the legislative body and the associated marginal cost is $\kappa > 0$. From the point of view of such a legislative body, assuming that it discounts at a rate $\lambda > 0$, we model the expected cost functional as[2]

$$\mathcal{I}_{x,\xi}(\eta) := \mathbb{E}_x\Big[\int_0^\infty e^{-\lambda t}\alpha(X_t^{\xi,\eta} - m)^+ dt + \kappa \int_0^\infty e^{-\lambda t} X_t^{\xi,\eta} \circ_d d\eta_t\Big]. \tag{3.2.6}$$

When a government wants to reduce its public deficit, it has, in simple terms, a choice between increasing tax revenues while keeping expenditures constant, or reducing public expenditures with stable tax revenues. The second choice is usually the more difficult to make: public spending is sometimes structural (for example, the payment of civil servants' salaries) and therefore incompressible in the short term. This is why, when seeking to reduce public deficits, one most frequently turns to taxation. Hence, $\alpha > 0$ can be interpreted as a country tax compliance factor, the smaller α is the bigger is the willingness to pay tax, if needed in the future. When this factor is low as it is in Denmark for instance, the legislative body has thus less social pressure to reduce the debt ratio.

[1] This study goes a step further to quantify the economic cost per percentage point the debt ratio exceeds m (see also Reinhart et al., 2012 for an empirical study on the effect of high debt towards private investments' crowding out and a low subsequent growth)

[2] We again highlight the fact that the legislative body discounts with a different discount rate than the government, which can be interpreted as different time preferences. Moreover, the running cost function inside the first integral is non-differentiable, which does not satisfy the assumptions in De Angelis and Ferrari (2018), thus the link between non-zero-sum games of singular controls and optimal stopping, developed therein, breaks down.

Debt Ceiling Mechanism as a Non-Zero-Sum Game of Singular Controls.

In our analysis, we restrict our attention to controls producing finite payoffs, which includes the realistic assumption that both players will not use an economic policy leading to infinite cost and/or benefit of interventions. Moreover, we note that the definition of the integrals with respect to the controls, as specified in (3.2.3), requires some attention since simultaneous jumps of ξ and η may be difficult to handle.

Given that the debt ratio is always a positive number, we therefore consider pairs $(\xi, \eta) \in \mathcal{U} \times \mathcal{U}$ such that

$$\begin{aligned} &\mathbb{E}_x\Big[\int_0^\infty e^{-\rho t} h(X_t^{\xi,\eta})dt + c_1 \int_0^\infty e^{-\rho t} X_t^{\xi,\eta} \circ_d d\eta_t + c_2 \int_0^\infty e^{-\rho t} X_t^{\xi,\eta} \circ_u d\xi_t\Big] < +\infty, \\ &\mathbb{E}_x\Big[\int_0^\infty e^{-\lambda t} \alpha(X_t^{\xi,\eta} - m)^+ dt + \kappa \int_0^\infty e^{-\lambda t} X_t^{\xi,\eta} \circ_d d\eta_t\Big] < +\infty, \\ &\mathbb{P}_x[\Delta\xi_t \cdot \Delta\eta_t > 0] = 0 \text{ for all } t \geq 0 \text{ and } x \in \mathbb{R}_+. \end{aligned} \tag{3.2.7}$$

To that end, we define the class of controls $(\xi, \eta) \in \mathcal{A} := \mathcal{A}_\eta \times \mathcal{A}_\xi$, where

$$\begin{aligned} &\mathcal{A}_\eta := \{\xi \in \mathcal{U} :\ (\xi, \eta) \in \mathcal{U} \times \mathcal{U} \text{ satisfies (3.2.7)}\} \\ \text{and} \qquad &\mathcal{A}_\xi := \{\eta \in \mathcal{U} :\ (\xi, \eta) \in \mathcal{U} \times \mathcal{U} \text{ satisfies (3.2.7)}\} \end{aligned}$$

The problem introduced partly in (3.2.5) and (3.2.6) is therefore formulated as a non-zero-sum game between two players: The government (player 1) which aims at solving

$$V_1(x; \eta) := \inf_{\xi \in \mathcal{A}_\eta} \mathcal{J}_{x,\eta}(\xi), \qquad x \in \mathbb{R}_+, \tag{3.2.8}$$

for any fixed control process $\eta \in \mathcal{U}$, and the legislative body (player 2) which aims at solving

$$V_2(x; \xi) := \inf_{\eta \in \mathcal{A}_\xi} \mathcal{I}_{x,\xi}(\eta), \qquad x \in \mathbb{R}_+, \tag{3.2.9}$$

for any fixed control process $\xi \in \mathcal{U}$.

Definition 3.2.3. *A couple $(\xi^*, \eta^*) \in \mathcal{A}$ forms a Nash equilibrium if and only if*

$$\begin{cases} \mathcal{J}_{x,\eta^*}(\xi^*) \leq \mathcal{J}_{x,\eta^*}(\xi) & \text{for any } \xi \in \mathcal{A}_{\eta^*}, \\ \mathcal{I}_{x,\xi^*}(\eta^*) \leq \mathcal{I}_{x,\xi^*}(\eta) & \text{for any } \eta \in \mathcal{A}_{\xi^*}. \end{cases}$$

Each player's value of the game is then given by $V_1(x; \eta^*) = \mathcal{J}_{x,\eta^*}(\xi^*)$ and $V_2(x; \xi^*) = \mathcal{I}_{x,\xi^*}(\eta^*)$.

The following assumptions on the model's parameters will hold true in the rest of this chapter.

Assumption 3.2.4. *The model's parameters satisfy:*

(i) $c_1 > c_2$;

(ii) $\rho > (p(r-g) + \frac{\sigma^2}{2}p(p-1))^+$, *where* p *is defined in Assumption 3.2.1;*

(iii) $\lambda > r - g$;

(iv) $m > 0$.

The condition in Assumption 3.2.4.(i) is typically assumed in the literature on bounded-variation stochastic control problems in order to ensure well-posedness of the optimisation problem (see, e.g., Guo and Tomecek, 2008, De Angelis and Ferrari, 2014 and Ferrari and Rodosthenous, 2020) and to avoid arbitrage opportunities. In economic terms, a possible interpretation is that the Keynesian multiplier is not high enough to offset the costs of liberalisation policies.

Assumption 3.2.4.(ii) reflects the fact that governments are more concerned about the present than the future, since they are in power for only a limited amount of years; hence discounting future costs and benefits at a sufficiently large rate. Moreover, combining this with Assumption (3.2.1).(iii), the trivial policy "never intervene on the debt ratio" is admissible, since it yields a finite expected cost. The latter is guaranteed also for the problem of the legislative body due to Assumption 3.2.4.(iii).

In this chapter, we will devote our attention to the existence of Nash equilibria of the game (3.2.8)–(3.2.9) in the class of strategies, where at least one of the players chooses a Skorokhod reflection type policy at a constant threshold. To this end, we first recall the following well known results on Skorokhod reflection.

Lemma 3.2.5. *Let* $a, b \in \mathbb{R}_+$ *with* $a < b$. *For any* $x \in [a,b]$ *there exists a unique couple* $(\xi(a), \eta(b)) \in \mathcal{A}$ *that solves the Skorokhod reflection problem*

$$\textit{Find } (\xi,\eta) \in \mathcal{A} \textit{ s.t. } \begin{cases} X_t^{\xi,\eta} \in [a,b], \mathbb{P}\text{-a.s. for } t > 0, \\ \int_0^T \mathbb{1}_{\{X_t^{\xi,\eta} > a\}} d\xi_t = 0, \mathbb{P}\text{-a.s. for any } T > 0, \\ \int_0^T \mathbb{1}_{\{X_t^{\xi,\eta} < b\}} d\eta_t = 0, \mathbb{P}\text{-a.s. for any } T > 0, \end{cases} \qquad (\mathbf{SP}(\text{a,b;x}))$$

and it follows that $supp\{d\xi_t(a)\} \cap supp\{d\eta_t(b)\} = \emptyset$.

Lemma 3.2.6. *For any* $\eta \in \mathcal{U}$, $a \in \mathbb{R}_+$ *and* $x \geq a$ *there exists a unique* $\xi(a) \in \mathcal{A}_\eta$ *solving the Skorokhod reflection problem*

$$\textit{Find } \xi \in \mathcal{A}_\eta \textit{ s.t. } \begin{cases} X_t^{\xi,\eta} \geq a, \mathbb{P}\text{-a.s. for } t > 0, \\ \int_0^T \mathbb{1}_{\{X_t^{\xi,\eta} > a\}} d\xi_t = 0, \mathbb{P}\text{-a.s. for any } T > 0. \end{cases} \qquad (\mathbf{SP}(\text{a;x}))$$

Analogously, for any $\xi \in \mathcal{U}, b \in \mathbb{R}_+$ and $x \leq b$ there exists a unique $\eta(b) \in \mathcal{A}_\xi$ solving

$$\text{Find } \eta \in \mathcal{A}_\xi \text{ s.t. } \begin{cases} X_t^{\xi,\eta} \leq b, \mathbb{P}\text{-a.s. for } t > 0, \\ \int_0^T \mathbb{1}_{\{X_t^{\xi,\eta} < b\}} d\eta_t = 0, \mathbb{P}\text{-a.s. for any } T > 0. \end{cases} \tag{$\mathbf{SP}$(b;x)}$$

Moreover, we define

$$\mathcal{M} := \{(\xi, \eta) \in \mathcal{A} : \ \xi \text{ solves } \mathbf{SP}(a; x) \text{ or } \eta \text{ solves } \mathbf{SP}(b; x) \text{ for some } a, b \in \mathbb{R}_+ \text{ and } x \in \mathbb{R}_+\} \tag{3.2.10}$$

and aim to prove the existence and uniqueness of a Nash equilibrium $(\xi, \eta) \in \mathcal{M}$, in different parameter configurations of the game. Indeed, we will show that if at least one player acts according to a Skorokhod reflection type policy as specified above, the game (3.2.8)–(3.2.9) admits a unique Nash equilibrium. Clearly, when not restricting at least one of the players to a Skorokhod reflection type policy there could also exist Nash equilibria outside of the set $\mathcal{M}$. However, as pointed out in previous contributions such as De Angelis and Ferrari (2018), it is impossible to rank different Nash equilibria without an additional optimality criterion.

3.3 The Optimal Governmental Debt Management Rule

In this section, we study the problem of the government choosing their investment economic policy ξ, taking into account that the legislative body (e.g. Congress) may or may not choose to intervene on the debt ratio. In the following, we distinguish between two cases, depending on the chosen control policy of the legislative body:

$$\text{(I)} \quad \eta_t = \overline{\eta}_t := 0, \qquad\qquad \text{(II)} \quad \eta_t = \eta_t^b := \mathbb{1}_{\{t>0\}}[(x - b)^+ + \eta_t(b)]. \tag{3.3.1}$$

In particular, the legislative body does not intervene in Case (I), while in Case (II) it imposes a debt ceiling mechanism, which forces the government to keep its debt ratio below a fixed level $b \in \mathbb{R}_+$ (via e.g. the adoption of liberalisation policies). In the latter definition, $\eta(b)$ uniquely solves the Skorokhod reflection problem $\mathbf{SP}(b; (x \wedge b))$. In the following, we aim at determining a best response (i.e. an optimal control strategy $\xi \in \mathcal{A}_\eta$) in both cases.

For simplicity of exposition, we assume the running cost function $h(x) = x^2/2$ in (3.2.5) (cf. Remark 3.2.2) in the rest of the chapter, which further yields that

$$\text{Assumption 3.2.4.(ii) with } p = 2 \quad \Leftrightarrow \quad \rho > 2(r - g) + \sigma^2. \tag{3.3.2}$$

It is clearer to present the two cases (I) and (II) separately.

3.3.1 The Government's Optimal Strategy under no Legislative Body Intervention: Case (I)

Let us assume that the legislative body does not intervene on the government's debt. The value function (3.2.8) thus rewrites as

$$\overline{V}_1(x) := \inf_{\xi \in \mathcal{A}_{\overline{\eta}}} \mathbb{E}_x \Big[\int_0^\infty e^{-\rho t} h(X_t^{\xi,\overline{\eta}}) dt - c_2 \int_0^\infty e^{-\rho t} X_t^{\xi,\overline{\eta}} \circ_u d\xi_t \Big]. \tag{3.3.3}$$

where we let $\overline{V}_1(x) := V_1(x;\overline{\eta})$. It follows from standard theory that we can associate the value function $\overline{V}_1$ of (3.3.3) with a suitable solution to the Hamilton-Jacobi-Bellman (HJB) equation

$$\min\big\{(\mathcal{L}-\rho)u(x) + \tfrac{1}{2}x^2,\ u'(x) - c_2\big\} = 0 \tag{3.3.4}$$

for all $x \in \mathbb{R}_+$, where the second order linear operator $\mathcal{L}$ defined by its action on functions $f \in C^2$ is defined by

$$\mathcal{L}f := \tfrac{1}{2}\sigma^2 x^2 f^{''} + (r-g)xf^{'}.$$

We guess that the government chooses to increase its debt ratio only when the current level is sufficiently small. Hence, we expect that there exists a critical x-level $\overline{a}$ at which the government increases their debt ratio via a Skorokhod reflection type policy. For any $x \in \mathbb{R}_+$, we thus consider the control

$$\xi_t^{\overline{a}} := \mathbb{1}_{\{t>0\}}[(\overline{a}-x)^+ + \xi_t(\overline{a})] \tag{3.3.5}$$

where $\xi(\overline{a})$ is the unique solution to the Skorokhod reflection problem **SP** $(\overline{a};(x \wedge \overline{a}))$. As a consequence, we can associate the given problem (3.3.3) with the free-boundary problem

$$\begin{cases} (\mathcal{L}-\rho)u(x) \geq \frac{1}{2}x^2, & x \in \mathbb{R}_+, \\ (\mathcal{L}-\rho)u(x) = \frac{1}{2}x^2, & \overline{a} < x, \\ u'(x) \geq c_2, & x \in \mathbb{R}_+, \\ u'(x) = c_2, & 0 < x \leq \overline{a}, \\ u''(\overline{a}) = 0, & \\ \lim\limits_{x\to+\infty}\Big(u(x) - \dfrac{x^2}{2(\rho - 2(r-g) - \sigma^2)}\Big) = 0, & \end{cases} \tag{3.3.6}$$

where we impose an additional smoothness condition at the boundary $\overline{a}$ and the latter one in order to guarantee uniqueness of the solution to the free-boundary problem. In the following theorem, we verify that the solution to the free-boundary problem (3.3.6) indeed coincides with the value function (3.3.3) and derive an optimal debt management policy for the government in Case (I). The proof can be found in Appendix C.1.

Theorem 3.3.1 (Verification Theorem: Case (I)). *Assume that the legislative body does not intervene on the debt ratio and thus acts according to the policy $\overline{\eta} \equiv 0$. Then, the value function $\overline{V}_1$ of* (3.3.3) *is given by*

$$\overline{V}_1(x) := \begin{cases} \overline{V}_1(\overline{a}) - c_2(\overline{a} - x), & 0 < x \leq \overline{a}, \\ \overline{D}_1(\overline{a})x^{\delta_2} + \frac{1}{2(\rho - 2(r-g) - \sigma^2)}x^2, & \overline{a} < x, \end{cases} \tag{3.3.7}$$

where

$$\overline{D}_1(a) := -\frac{1}{(\rho - 2(r-g) - \sigma^2)\delta_2(\delta_2 - 1)a^{\delta_2 - 2}}, \qquad \text{and} \qquad \overline{a} := \frac{(1-\delta_2)c_2(\rho - 2(r-g) - \sigma^2)}{(2 - \delta_2)}, \tag{3.3.8}$$

with δ_2 denoting the negative root to the equation $\frac{1}{2}\sigma^2\delta(\delta - 1) + (r-g)\delta - \rho = 0$. Moreover, the admissible $\xi_t^{\overline{a}}$ of (3.3.5)*, with $\overline{a}$ given by* (3.3.8)*, is optimal for problem* (3.3.3)*.*

3.3.2 The Government's Optimal Strategy under Legislative Body Interventions: Case (II)

We begin by fixing a constant $b \in \mathbb{R}_+$ and assume that the legislative body acts according to the control policy η^b of (3.3.1), thus keeping the debt ratio below the debt ceiling b according to a Skorokhod reflection type policy. From the government's point of view, we thus study the problem $V_1(x; \eta^b)$ defined in (3.2.8) and given by[3]

$$V_1(x; b) := \inf_{\xi \in \mathcal{A}_{\eta^b}} \mathbb{E}_x\Big[\int_0^\infty e^{-\rho t} h(X_t^{\xi,\eta^b})dt + c_1 \int_0^\infty e^{-\rho t} X_t^{\xi,\eta^b} \circ_d d\eta_t^b - c_2 \int_0^\infty e^{-\rho t} X_t^{\xi,\eta^b} \circ_u d\xi_t\Big]. \tag{3.3.9}$$

Again, we can associate the latter value function with the solution to the HJB equation

$$\min\big\{(\mathcal{L} - \rho)u(x; b) + \tfrac{1}{2}x^2,\ u'(x; b) - c_2\big\} = 0 \tag{3.3.10}$$

for all $x \in (0, b)$ with boundary condition $u(0; b) = 0$ and Neumann boundary condition $u'(b; b) = c_1$.

We guess that the government increases their debt ratio only when the current level is sufficiently small. Hence, we expect that for any given debt ceiling $b \in \mathbb{R}_+$, there exists a critical debt-issuance level $a(b)$ at which the government increases their debt ratio with *minimal effort*, via a Skorokhod reflection type policy, where we stress the (possible) dependency on the debt ceiling threshold $b \in \mathbb{R}_+$. For any $x \in \mathbb{R}_+$, we thus consider the control

$$\xi_t^{a(b)} := \mathbb{1}_{\{t>0\}}[(a(b) - x)^+ + \xi_t(a(b))], \tag{3.3.11}$$

[3]For ease of notation, we denote by $V_1(x; y)$ and $V_2(x; y)$, $x, y \in \mathbb{R}_+$, the control value functions $V_1(x; \eta^y)$ and $V_2(x; \xi^y)$, i.e. when their opponents choose the Skorokhod reflection type strategies η^y and ξ^y, respectively.

where $\xi(a(b))$ is the unique control such that the couple $(\xi(a(b)), \eta(b))$ solves the Skorokhod reflection problem $\mathbf{SP}(a(b), b; (x \vee a(b)) \wedge b)$. As a consequence, we can associate the given problem (3.3.9) with the free-boundary problem

$$\begin{cases} (\mathcal{L}-\rho)u(x;b) \geq -\frac{1}{2}x^2, & 0 < x < b, \\ (\mathcal{L}-\rho)u(x;b) = -\frac{1}{2}x^2, & a(b) < x < b, \\ u'(x;b) \geq c_2, & 0 < x < b, \\ u'(x;b) = c_2, & 0 < x \leq a(b), \\ u'(x;b) = c_1, & b \leq x, \\ u''(a(b);b) = 0, \end{cases} \tag{3.3.12}$$

where we impose an additional smoothness condition at the free boundary $a(b)$. The forthcoming analysis is dedicated to determining the optimal debt-issuance threshold $a(b)$ and proving the optimality of the control (3.3.11) for the original debt ratio management problem of the government (3.3.9), which corresponds to (3.2.8) with $\eta = \eta^b$ defined in (3.3.1).

We begin with solving the free-boundary problem (3.3.12) by constructing a solution to the ordinary differential equation and imposing the boundary conditions to obtain a *candidate* value function

$$U_1(x;b) = \begin{cases} U_1(a(b);b) - c_2(a(b)-x), & 0 < x \leq a(b), \\ D_1(a(b))x^{\delta_1} + D_2(a(b))x^{\delta_2} + \frac{1}{2(\rho-2(r-g)-\sigma^2)}x^2, & a(b) < x < b, \\ U_1(b;b) + c_1(x-b), & b \leq x, \end{cases} \tag{3.3.13}$$

with

$$D_i(a) = \frac{(\delta_{3-i}-2)a - c_2(\delta_{3-i}-1)(\rho-2(r-g)-\sigma^2)}{(-1)^{i+1}\delta_i(\delta_1-\delta_2)(\rho-2(r-g)-\sigma^2)a^{\delta_i-1}}, \qquad \text{for } i = 1,2,$$

and the constants δ_1, δ_2 denoting the positive and negative roots to the equation $\frac{1}{2}\sigma^2\delta(\delta-1)+(r-g)\delta-\rho = 0$, respectively. The optimal boundary is given by the solution $a(b) \in (0,b)$ to the equation

$$F(a(b), b) = 0, \tag{3.3.14}$$

where we define

$$\begin{aligned} F(a,b) :=& [(2-\delta_2)a - c_2(1-\delta_2)(\rho-2(r-g)-\sigma^2)]\left(\frac{b}{a}\right)^{\delta_1-1} \\ &+[(\delta_1-2)a - c_2(\delta_1-1)(\rho-2(r-g)-\sigma^2)]\left(\frac{b}{a}\right)^{\delta_2-1} \\ &- (\delta_1-\delta_2)[b - c_1(\rho-2(r-g)-\sigma^2)]. \end{aligned} \tag{3.3.15}$$

In the following result, we prove the existence and uniqueness of the optimal boundary, as well as some monotonicity and limit properties that will be useful in the search for a Nash equilibrium later in Section 3.5. Its proof can be found in Appendix C.2.

Lemma 3.3.2. *Let $b \in \mathbb{R}_+$ and recall $F(\cdot, b)$ defined by* (3.3.15) *on* $(0, b)$*. Then,*

(i) *There exists a unique $a(b) \in (0, b)$ solving $F(a(b), b) = 0$ in* (3.3.14)*, and it satisfies $\frac{\partial}{\partial a} F(a(b), b) > 0$ and $a(b) < \widetilde{a}$, where*

$$\widetilde{a} := c_2(\rho - (r - g)) > 0. \tag{3.3.16}$$

(ii) *We have*

$$a(\cdot) \text{ is} \begin{cases} \text{increasing on } (0, \widehat{b}), \\ \text{decreasing on } (\widehat{b}, \infty), \end{cases} \qquad \text{where } \widehat{b} \text{ is the unique solution to } \frac{\partial}{\partial b} F(a(\widehat{b}), \widehat{b}) = 0 \tag{3.3.17}$$

and $\lim_{b \to 0} a(b) = 0$ as well as $\lim_{b \to \infty} a(b) = \overline{a}$, where $\overline{a}$ is the optimal debt-issuance threshold defined by (3.3.8) *in Case (I) of non-intervention by a legislative body.*

(iii) *Furthermore, $b \mapsto a(b)$ is concave on the interval $(0, \widehat{b})$.*

The next result presents the solution to problem (3.3.9) and the optimality of the control $\xi^{a(b)}$. To this end, we first notice that the control policy $\xi^{a(b)}$ as in (3.3.11), combined with the policy η^b of (3.3.1), is indeed admissible. Clearly, the couple solves $\mathbf{SP}(a(b), b; x)$ and as such belongs to $\mathcal{A}$. Indeed, by arguing as in Lemma 4.1 in Shreve et al. (1984) one can easily show (3.2.7) and moreover, $\mathbb{P}_x[\Delta\xi^{a(b)} \cdot \Delta\eta^b > 0] = 0$ for all $t \geq 0$, by construction. The proof of the following result, which concludes this section, can be found in Appendix C.3.

Theorem 3.3.3 (Verification Theorem: Case (II))**.** *Assume that the legislative body acts according to the control policy η^b of* (3.3.1)*. Then, the function U_1 of* (3.3.13) *coincides with the government's value function V_1 in* (3.3.9) *and the admissible $\xi^{a(b)}$ of* (3.3.11) *is optimal for problem* (3.3.9)*.*

3.4 The Optimal Debt Ceiling

In this section, we study the control problem of the legislative body. As seen in Section 3.3, the best response of the government to either a legislative body non-intervention policy, or a debt ceiling mechanism (threshold-type policy) is given by a debt-issuance threshold-type policy. We now reverse the roles and assume that the government chooses to increase its debt ratio at a certain level $a \in \mathbb{R}_+$, i.e. to the debt-issuance control policy

$$\xi^a_t := \mathbb{1}_{\{t>0\}}[(a - x)^+ + \xi_t(a)], \tag{3.4.1}$$

where $\xi(a)$ uniquely solves the Skorokhod reflection problem $\mathbf{SP}(a; x \vee a)$. In the following, for any such level a, we study the problem (3.2.9) of finding a best response (i.e. an optimal control strategy $\eta \in \mathcal{A}_{\xi^a}$). We thus consider the problem[3]

$$V_2(x; a) := \inf_{\eta \in \mathcal{A}_{\xi^a}} \mathbb{E}_x\Big[\int_0^\infty e^{-\lambda t}\alpha(X^{\xi^a,\eta}_t - m)^+ dt + \kappa \int_0^\infty e^{-\lambda t} X^{\xi^a,\eta}_t \circ_d d\eta_t\Big]. \tag{3.4.2}$$

Via standard arguments, we can associate the value function V_2 of (3.4.2) with a suitable solution to the HJB equation

$$\min\left\{(\mathcal{L}-\lambda)u(x;a)+\alpha(x-m)^+,\kappa-u'(x;a)\right\}=0, \tag{3.4.3}$$

for all $x\in(a,\infty)$ with Neumann boundary condition $u'(a;a)=0$. We presume that the legislative body may only decrease the debt ratio when the current level is sufficiently large. Therefore, if the legislative body chooses to intervene, we expect that for any given debt-issuance threshold $a\in\mathbb{R}_+$, there exists a critical debt ceiling level $b(a)$ at which the legislative body forces a decrease in the debt ratio via a Skorokhod reflection type policy, where we stress the (possible) dependency on the debt-issuance threshold $a\in\mathbb{R}_+$. On the other hand, also a non-intervention policy is conceivable. As it turns out, it is crucial in our analysis to distinguish two different cases, depending on the legislative body's time preference rate λ:

$$\text{(I)}\quad \lambda>r-g+\frac{\alpha}{\kappa},\qquad\qquad \text{(II)}\quad \lambda<r-g+\frac{\alpha}{\kappa}.$$

In the forthcoming Sections 3.4.1 and 3.4.2 we study these cases separately, providing an optimal control strategy by the legislative body for each one of them.

3.4.1 The Legislative Body's Optimal Strategy under High Time Preference Rate: Case (I)

Notice that the legislative body discounts future events with a relatively large discount factor in this case, and it is therefore appropriate to assume that the legislative body disregards the risk of future government insolvency at a greater extent compared to Case (II). We verify this intuition by showing that indeed, the optimal control policy of the legislative body prescribes *not to intervene* on the debt ratio at all.

To this end, we prove that the value function V_2 of (3.4.2) coincides with a suitable solution to a fixed-boundary problem

$$\begin{cases}(\mathcal{L}-\lambda)u(x;a)=-\alpha(x-m)^+, & a<x,\\ u'(x;a)<\kappa, & a<x,\\ u'(x;a)=0, & 0<x\le a.\end{cases} \tag{3.4.4}$$

We can solve the latter problem by constructing a solution to the ordinary differential equation and imposing the stated boundary condition. In the following theorem, we verify that the solution to the fixed-boundary problem (3.4.4) indeed coincides with the value function V_2 of (3.4.2). Its proof can be found in Appendix C.4.

Theorem 3.4.1 (Verification Theorem: Case (I)). *Assume that the government acts according to the control policy ξ^a of (3.4.1). Then, the function V_2 of (3.4.2) is given by*

$$V_2(x;a) = \begin{cases} V_2(a;a), & 0 < x \le a; \\ \overline{D}_2(a)x^{\theta_2} + H(x), & a < x, \end{cases} \tag{3.4.5}$$

where

$$\begin{aligned} \overline{D}_2(a) &:= -\frac{\alpha}{\theta_2} a^{1-\theta_2} \int_0^\infty e^{-(\lambda-(r-g))t} \Phi(d_1(a,t)) dt, \\ H(x) &:= \alpha \int_0^\infty \Big(x e^{-(\lambda-(r-g))t} \Phi(d_1(x,t)) - m e^{-\lambda t} \Phi(d_2(x,t)) \Big) dt, \\ d_1(x,t) &:= \frac{\log\left(\frac{x}{m}\right) + (r-g+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}} \quad \text{and} \quad d_2(x,t) := d_1(x,t) - \sigma\sqrt{t}, \end{aligned} \tag{3.4.6}$$

with θ_2 denoting the negative root to the equation $\frac{1}{2}\sigma^2\theta(\theta-1) + (r-g)\theta - \lambda = 0$ and $\Phi(\cdot)$ denoting the cumulative distribution function of a standard normal random variable. The optimal policy for the legislative body prescribes not to act on the debt ratio, i.e. $\overline{\eta} := 0$.

Notice that the strategy of the legislative body not to intervene on the debt ratio is not triggered by some specific $a \in \mathbb{R}_+$. This comes solely from the fact that $\lambda > r - g + \frac{\alpha}{\kappa}$, independently of the governmental choice of a debt-issuance level $a \in \mathbb{R}_+$. On the other hand, the best response of the government to a legislative body non-intervention policy $\overline{\eta}$ from (3.3.1), has been treated in Section 3.3.1. In particular, the optimal control for problem $V_1(x;\overline{\eta})$ of (3.2.8) (cf. $\overline{V}_1(x)$ in (3.3.3)) is given by the Skorokhod reflection type policy $\xi^{\overline{a}}$ as in (3.3.11), with $\overline{a}$ defined in (3.3.8). Indeed, we prove in Section 3.5 that this pair of strategies leads to an equilibrium.

3.4.2 The Legislative Body's Optimal Strategy under Low Time Preference Rate: Case (II)

While it is optimal for the legislative body to never intervene in Case (I), we show that in this case the best response to a governmental policy (3.4.1) requires intervention. Indeed, for any given governmental debt-issuance threshold $a \in \mathbb{R}_+$, this will prescribe keeping the debt ratio below a certain debt ceiling $b(a)$ with *minimal effort*, via a Skorokhod reflection type policy, where we stress the (possible) dependency on the debt-issuance threshold $a \in \mathbb{R}_+$. For any $x \in \mathbb{R}_+$, we thus consider the control

$$\eta_t^{b(a)} := \mathbb{1}_{\{t>0\}}[(x - b(a))^+ + \eta_t(b(a))], \tag{3.4.7}$$

where $\eta(b(a))$ is the unique control such that the couple $(\xi(a), \eta(b(a)))$ solves the Skorokhod reflection problem $\mathbf{SP}(a, b(a); (x \vee a) \wedge b(a))$. As a consequence, we can associate the given problem (3.4.2) with the free-boundary problem

$$\begin{cases} (\mathcal{L} - \lambda)u(x;a) \geq -\alpha(x-m)^+, & a < x, \\ (\mathcal{L} - \lambda)u(x;a) = -\alpha(x-m)^+, & a < x < b(a), \\ u'(x;a) \leq \kappa, & a < x, \\ u'(x;a) = \kappa, & b(a) \leq x, \\ u'(x;a) = 0, & 0 < x \leq a, \\ u''(b(a);a) = 0, \end{cases} \tag{3.4.8}$$

where we imposed an additional smoothness condition at the free boundary $b(a)$. The forthcoming analysis is dedicated to determining the optimal threshold $b(a)$ and proving the optimality of the control (3.4.7) for the original debt ratio management problem of the government (3.4.2), which corresponds to (3.2.9) with $\xi = \xi^a$ defined in (3.4.1).

We begin with solving the free-boundary problem (3.4.8) by constructing a solution to the ordinary differential equation and imposing the boundary conditions to obtain a *candidate* value function

$$U_2(x;a) := \begin{cases} U_2(a;a), & 0 < x \leq a, \\ D_3(b(a))x^{\theta_1} + D_4(b(a))x^{\theta_2} + H(x), & a < x < b(a), \\ U_2(b(a));a) + \kappa(x - b(a)), & b(a) \leq x, \end{cases} \tag{3.4.9}$$

where

$$D_i(b) := \frac{b^{1-\theta_{i-2}}}{\theta_{i-2}(\theta_2 - \theta_1)} \Big[(\theta_{5-i} - 1)\Big(k - \alpha \int_0^\infty e^{-(\lambda-(r-g))t} \Phi(d_1(b,t))dt\Big) + \alpha \int_0^\infty e^{-(\lambda-(r-g))t} \frac{1}{\sqrt{2\pi t}\sigma} e^{-\frac{1}{2}d_1(b,t)^2} dt\Big], \qquad i = 3, 4,$$

and the constants θ_1, θ_2 are given by the positive and negative roots to the equation $\frac{1}{2}\sigma^2\theta(\theta - 1) + (r-g)\theta - \lambda = 0$, respectively. The optimal boundary is given by the solution $b(a) \in (a, \infty)$ to the equation

$$G(a, b(a)) = 0, \tag{3.4.10}$$

where $G(a, \cdot)$ is defined on (a, ∞) by

$$\begin{aligned} G(a,b) = {} & \Big[(\theta_1 - 1)\Big(\frac{b}{a}\Big)^{1-\theta_2} + (1-\theta_2)\Big(\frac{b}{a}\Big)^{1-\theta_1}\Big]\Big(\frac{\kappa}{\theta_1 - \theta_2} - \frac{\alpha}{(\theta_1 - \theta_2)(\lambda - (r-g))} \mathbb{1}_{\{b \geq m\}}\Big) \\ & + \frac{\alpha}{(\theta_1 - \theta_2)(\lambda - (r-g))} \Big[(1-\theta_2)\Big(\frac{m}{a}\Big)^{1-\theta_1} \mathbb{1}_{\{b > m > a\}} + (\theta_1 - 1)\Big(\frac{m}{a}\Big)^{1-\theta_2} \mathbb{1}_{\{b > m > a\}}\Big] \\ & + \frac{\alpha}{(\lambda - (r-g)} \mathbb{1}_{\{a \geq m\}}. \end{aligned} \tag{3.4.11}$$

In the following lemma we state our results on the existence and uniqueness of a solution $b(a) \in (a, \infty)$ solving (3.4.10), as well as some monotonicity and limit properties that will be useful in the search for a Nash equilibrium later in Section 3.5. Its proof can be found in Appendix C.5.

Lemma 3.4.2. *Let $a \in \mathbb{R}_+$ and recall $G(a, \cdot)$ defined by* (3.4.11) *on (a, ∞). Then,*
(i) *There exists a unique $b(a) \in (a, \infty)$ solving $G(a, b(a)) = 0$ in* (3.4.10)*, that satisfies $\frac{\partial}{\partial b} G(a, b(a)) < 0$.*
(ii) *We have*

$$b(a) \geq b_0 := \Big(\frac{\alpha}{\alpha - \kappa(\lambda - (r-g))} \Big)^{\frac{1}{1-\theta_2}} m > m. \tag{3.4.12}$$

(iii) *The function $a \mapsto b(a)$ is strictly increasing on $\mathbb{R}_+$. In particular, it takes a linear form $b(a) = (1/\widetilde{q})a$, for all $a > m$, where $\widetilde{q} \in (0, 1)$ is given by the solution to*

$$(1-\theta_2)(\kappa(\lambda - (r-g)) - \alpha)\widetilde{q}^{\theta_1 - 1} + (\theta_1 - 1)(\kappa(\lambda - (r-g)) - \alpha)\widetilde{q}^{\theta_2 - 1} + \alpha(\theta_1 - \theta_2) = 0. \tag{3.4.13}$$

(iv) *Moreover, $a \mapsto b(a)$ is convex on the interval $(0, m)$, with $\lim_{a \to 0} b(a) = b_0$, where b_0 is given by* (3.4.12)*, and $\lim_{a \to \infty} b(a) = \infty$.*

Before we present the optimality of the controls, we first notice that the control policy $\eta^{b(a)}$ as in (3.4.7), combined with the policy ξ^a of (3.4.1), is indeed admissible. Clearly, the couple solves $\mathbf{SP}(a, b(a); x)$ and as such belongs to $\mathcal{A}$. Indeed, by arguing as in Lemma 4.1 in Shreve et al. (1984) one can easily show (3.2.7) and moreover, $\mathbb{P}_x[\Delta\xi^a \cdot \Delta\eta^{b(a)} > 0] = 0$ for all $t \geq 0$, by construction. The proof of the next result can be found in Appendix C.6.

Theorem 3.4.3 (Verification Theorem: Case (II))**.** *Assume that the government acts according to the control policy ξ^a of* (3.4.1)*. Then, the function U_2 of* (3.4.9) *coincides with the value function V_2 of* (3.4.2)*. Furthermore, the policy $\eta^{b(a)}$ of* (3.4.7) *with the optimal threshold determined via* (3.4.10) *is optimal.*

3.5 Nash Equilibria in the Model

Our main results concern the existence of Nash equilibria in our model. These results stem from the analysis of the decision problems faced by the government and the legislative body developed in the previous Sections 3.3 and 3.4, respectively. We will focus on the two cases– each player's best response to a Skorokhod-reflection type strategy is either a Skorokhod-reflection type strategy or a no-intervention policy – which suggest that we should aim at determining a Nash equilibrium via its Definition 3.2.3 in the class $\mathcal{M}$ of (3.2.10), in which at least one player acts according to a Skorokhod-reflection type policy. The analysis in this section focuses on this direction.

We highlight the peculiarity arising from our results in Section 3.4, where we show that the legislative body may choose *not to intervene* on the debt ratio at all. Interestingly, we prove that the optimality of adopting such a strategy relies solely on their (individual) time preferences compared to the parameter constellation in the model – it does *not depend* on the actions of the opposing player (government) – see specifically our results in Section 3.4.1. We thus split our search for Nash equilibria in the forthcoming analysis based on the magnitude of the legislative body time preferences.

3.5.1 The Case of $\lambda > r - g + \alpha/\kappa$

Our results in Section 3.4.1, suggest that the legislative body should restrain themselves from reflecting the debt ratio at any threshold, when their time preference rate λ is relatively large. In light of this non-intervention policy, it is natural to then examine what should the governmental strategy be as a best response. The characterisation of such a strategy is in fact the main aim of Section 3.3.1, which studies the optimal control (debt issuance policy) of the government when they are the sole player (there is no opponent).

We present the resulting Nash equilibrium in the following theorem. Its proof is a simpler version of the one for Theorem 3.5.2, and it is thus omitted for brevity.

Theorem 3.5.1 (Existence and Uniqueness of Nash Equilibrium: Case (I))**.** *Suppose that the model's parameters satisfy Assumptions 3.2.4 as well as $\lambda > r - g + \alpha/\kappa$. A unique Nash equilibrium of the game* (3.2.8)–(3.2.9) *in the set $\mathcal{M}$ of* (3.2.10) *can be characterised by the couple of controls $(\xi^{\overline{a}}, \overline{\eta}) = (\xi^{\overline{a}}, 0)$, with the former component defined as in* (3.3.5) *and the threshold $\overline{a}$ is given explicitly by* (3.3.8).

3.5.2 The Case of $\lambda \in (r - g, r - g + \alpha/\kappa)$

Our results from Sections 3.3 and 3.4 on each player's best response suggest that a Nash equilibrium could be characterised by Skorokhod-reflection type policies at finite thresholds. More precisely, while the government increases the debt ratio at $a(b)$ (as a best response to a debt ceiling $b \in \mathbb{R}_+$), the legislative body forces a debt ratio reduction at a debt ceiling $b(a)$ (as a best response to a governmental debt-issuance threshold $a \in \mathbb{R}_+$).

The aim of the following theorem is to first prove that there always exists a pair (a^*, b^*) forming a fixed point of these best-response-maps, such that $a^* = a(b^*)$ and $b^* = b(a^*)$, and second, that this pair is unique in the set $\mathcal{M}$, in which at least one player plays a Skorokhod-reflection type policy.

Theorem 3.5.2 (Existence and Uniqueness of Nash Equilibrium: Case (II))**.** *Suppose that the model's parameters satisfy Assumptions 3.2.4 as well as* $\lambda \in (r-g, r-g+\alpha/\kappa)$*. A Nash equilibrium in the game* (3.2.8)–(3.2.9) *can be characterised by the couple of Skorokhod-reflection type policies* (ξ^{a^*}, η^{b^*})*, as defined in* (3.3.11) *and* (3.4.7)*, respectively. The pair of thresholds* $(a^*, b^*) \in \mathbb{R}^2_+$ *solves the coupled system of equations* $F(a^*, b^*) = 0 = G(a^*, b^*)$ *and satisfies* $a^* < b^*$, $a^* = a(b^*) < b^* = b(a^*)$ *according to Lemmata 3.3.2 and 3.4.2, for a unique* $b^* > b_0$ *where the latter is defined in* (3.4.12)*. Moreover, the Nash Equilibrium is unique in the class* $\mathcal{M}$ *specified in* (3.2.10)*.*

Proof. We prove separately the existence and uniqueness of the Nash Equilibrium.

Step 1. We prove the existence of a Nash Equilibrium. Recall the function $a(b)$ from Lemma 3.3.2 and define the function

$$a_1(x) := a(x), \quad \text{for all } x \in \mathbb{R}_+. \tag{3.5.1}$$

Then, we recall from Lemma 3.4.2 that the unique solution to $G(a, \cdot) = 0$ for any fixed $a \in \mathbb{R}_+$, is given in terms of a strictly increasing function $a \mapsto b(a)$. We can therefore invert this function and define

$$a_2(x) := b^{-1}(x), \quad \text{such that } x \mapsto a_2(x) \text{ is strictly increasing on } \mathbb{R}_+. \tag{3.5.2}$$

Thanks to Proposition 3.4.2, we also know that $b(a) = (1/\widetilde{q})a$, for all $a > m$, which yields that

$$a_2(x) = \widetilde{q}x, \quad \text{for } x > m/\widetilde{q}. \tag{3.5.3}$$

For illustration, Figure 3.5.1 sketches the maps for different parameter specifications. We can

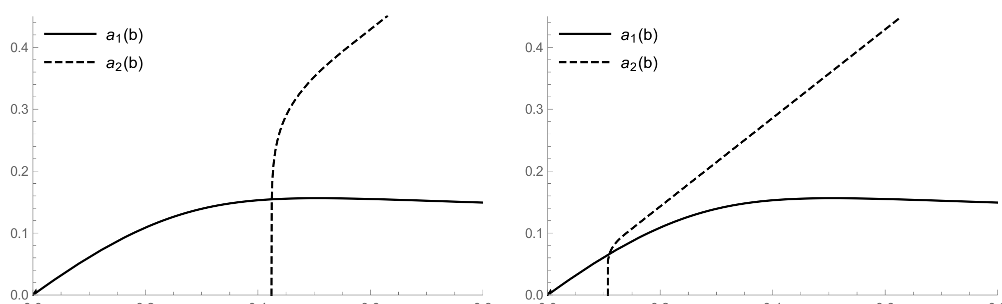


Figure 3.5.1: A Sketch of the maps $a_1(b)$ and $a_2(b)$ for different parameter specifications.

thus conclude that

$$\exists \text{ a Nash equilibrium} \quad \Leftrightarrow \quad \exists \text{ an intersection point } b^*, \text{ such that } a^* := a_1(b^*) = a_2(b^*). \tag{3.5.4}$$

In view of the definitions in (3.5.1)–(3.5.2), we have that $a^* = a(b^*)$ and $b^* = b(a^*)$. This would finally imply that (a^*, b^*) solves the system of equations $F(a^*, b^*) = 0 = G(a^*, b^*)$ and complete the proof.

In the remainder of the existence proof, we show that the b^* in (3.5.4) indeed exists. On one hand, it follows from (3.5.1) and (3.3.17) in Lemma 3.3.2 that $a_1(\cdot)$ is bounded from above by $\widehat{a} := a(\widehat{b})$. On the other hand, it follows from (3.5.2)–(3.5.3) that $a_2(\cdot)$ is strictly increasing with $\lim_{b\to\infty} a_2(b) = +\infty$. We further know from Lemmas 3.3.2 and 3.4.2 that the functions $a_1(\cdot)$ and $a_2(\cdot)$ have supports on $(0, \infty)$ and (b_0, ∞), respectively, with $b_0 > m > 0$. due to (3.4.12). Clearly, there exists at least one $b^* \in (b_0, \infty)$ such that $a_1(b^*) = a_2(b^*)$, therefore (3.5.4) implies that there exists a Nash equilibrium.

Step 2. We prove the uniqueness of the Nash Equilibrium. To this end, we must prove the uniqueness of the intersection point b^* established in the previous step.

We begin by defining the function

$$a \mapsto b_0(a), \quad \text{for } a \in \mathbb{R}_+, \quad \text{such that } b_0(a) = (1/\widetilde{q})a \quad \text{with } \widetilde{q} \in (0,1) \text{ as in Lemma 3.4.2,}$$

which can be inverted to define

$$a_0(b) := \widetilde{q}b, \quad \text{for } b \in \mathbb{R}_+.$$

Notice from Lemma 3.4.2 that $b(a) = b_0(a)$ for all $a > m$, hence in view of (3.5.2), we get $a_2(b) = a_0(b)$, for all $b > m/\widetilde{q}$. Moreover, Lemma 3.4.2 implies that $b \mapsto a_2(b)$ is strictly concave on $(b_0, m/\widetilde{q})$, where we notice that $m/\widetilde{q} > b_0$ due to the monotonicity of $b(a)$. Moreover, this implies $a_2(b) \le a_0(b)$ for all $b \ge b_0$.

As a first step, we prove that the curves $a_1(b)$ and $a_0(b)$ either admit no intersection or exactly one for $b > 0$. Since $a_0(b) = \widetilde{q}b$, any intersection clearly is of the form $(\widetilde{q}b, b)$. Plugging in points of this form into the function F of (3.3.15) we observe that

$$b \to F(\widetilde{q}b, b) = s(\widetilde{q})b + y(\widetilde{q}) \quad \text{is strictly increasing on } \mathbb{R}_+,$$

with

$$\begin{aligned} s(q) &= (2-\delta_2)q^{2-\delta_1} + (\delta_1 - 2)q^{2-\delta_2} - (\delta_1 - \delta_2) > 0, \\ y(q) &= (\rho - 2(r-g) - \sigma^2)\big[c_1(\delta_1 - \delta_2) - c_2(1-\delta_2)q^{1-\delta_1} - c_2(\delta_1 - 1)q^{1-\delta_2}\big]. \end{aligned}$$

Clearly, this implies (depending on the value $y(q)$) that either no intersection point exists or exactly one. Moreover, if no intersection of a_0 and a_1 exists, we conclude that $a_1'(0+) < a_0'(0+) = \widetilde{q}$.

Next, we come back to the uniqueness of an intersection of the maps $a_1(b)$ and $a_2(b)$. Recall that $b \mapsto a_1(b)$ is concave on $(0, \widehat{b})$, and $a \mapsto b(a)$ is convex, which implies that $b \mapsto a_2(b)$ is concave as well.

We now distinguish the following cases; an illustration for each one of them is given in Figure 3.5.2:

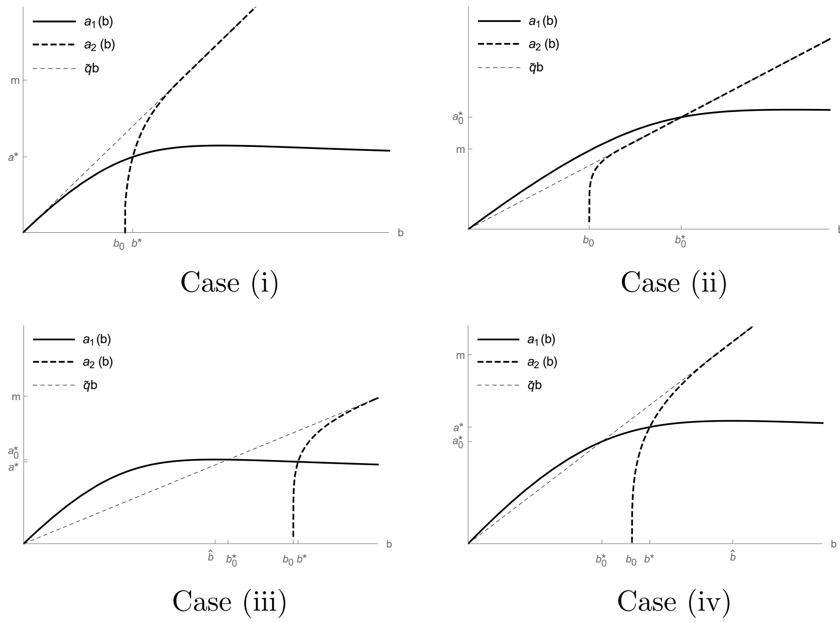


Figure 3.5.2: Case Study for the uniqueness of an intersection of $a_1(b)$ and $a_2(b)$

Case (i). No intersection exists of $a_1(b)$ and $a_0(b)$: In this case, we first notice that $a_2'(b) \geq a_0'(b) = \widetilde{q}$ for all b in the support of $a_2(b)$ due to the concavity of $a_2(b)$ and the fact that $a_2(b) \leq \widetilde{q}b$. Furthermore, we have $\widetilde{q} > a_1'(0+)$, and hence $\widetilde{q} > a_1'(b)$ for all $b > 0$. Combining these insights, it follows that $a_1(b) = a_2(b)$ for exactly one $b \in \mathbb{R}_+$.

For the following cases, we thus assume that there exists exactly one intersection of $a_1(b)$ and $a_0(b)$, which yields the point (a_0^*, b_0^*).

Case (ii). If $a_0^* \geq m$, the uniqueness of an intersection of $a_1(b)$ and $a_2(b)$ follows from the fact that $a_2(b) = a_0(b)$ for all $b \geq m/\widetilde{q}$. Hence, in this case, we obtain that the intersection of $a_1(b)$ and $a_2(b)$ is exactly given by $(a^*, b^*) = (a_0^*, b_0^*)$.

Case (iii). If $a_0^* < m$ and $b_0^* \geq \widehat{b}$, we again observe that $a_2(b) \leq \widetilde{q}b$, which implies that the intersection point is not realized on $b \leq b_0^*$. Therefore, since $a_1(b)$ is decreasing for $b \geq \widehat{b}$ and $a_2(b)$ is increasing, the intersection is unique and we denote it by (a^*, b^*).

Case (iv). If $a_0^* < m$ and $b_0^* < \widehat{b}$, the concavity of $a_1(b)$ implies $a_1'(b_0^*) < \widetilde{q}$ as well as $a_1'(b_0^*) > a_1'(b)$ for all $b \geq b_0^*$. Since $a_2'(b) > \widetilde{q}$ for all b, we must have $a_1(b) = a_2(b)$ for exactly one $b \in \mathbb{R}_+$ and we again denote the unique intersection by (a^*, b^*). □

The following corollary reveals the connection between the equilibria determined in Theorem 3.5.1-3.5.2 by considering the limit as $\lambda \uparrow r - g + \alpha/\kappa$. Its proof can be found in Appendix C.7.

Corollary 3.5.3. *Let $(a^*(\lambda), b^*(\lambda)) = (a^*, b^*)$ denote the pair of thresholds that characterises the unique Nash equilibrium derived in Theorem 3.5.2, where we stress the dependency on the parameter $\lambda \in (r - g, r - g + \alpha/\kappa)$. Then,*

$$\lim_{\lambda \uparrow r-g+\alpha/\kappa} a^*(\lambda) = \overline{a}, \quad \text{and} \quad \lim_{\lambda \uparrow r-g+\alpha/\kappa} b^*(\lambda) = \infty,$$

where $\overline{a}$ denotes the threshold characterising the optimal debt issuance policy of the government in equilibrium for large values of λ, as determined in Theorem 3.5.1.

3.6 Comparative Statics Analysis

In Section 5 we derived the existence and uniqueness of Nash equilibria under different parameter regimes in our model. It was revealed that the parameter λ, measuring the legislative body's time preferences, plays a crucial role on the characterisation of Nash equilibrium. While it is optimal not to intervene for large values of $\lambda > r - g + \alpha/\kappa$, the equilibrium strategies are characterised by a pair of thresholds (a^*, b^*) for intermediate values of $\lambda \in (r - g, r - g + \alpha/\kappa)$.

In this section, we study the sensitivity of these boundaries with respect to some of the model parameters. In order to highlight the transition from Nash equilibria that are characterised by thresholds (a^*, b^*) (cf. Theorem 3.5.2) to those that prescribe a non-intervention policy for the legislative body (cf. Theorem 3.5.1), as stated in Corollary 3.5.3, we plot the equilibrium values of (a^*, b^*) as functions of λ in the following comparative statics. Unless otherwise specified, we assume $\rho = 0.3$, $\sigma = 0.2$, $r = 0.025$, $g = 0.02$, $\alpha = 0.15$, $m = 0.6$, $c_1 = 2$, $c_2 = 1.25$, $\kappa = 0.6$.

Sensitivity with respect to λ. To begin with, it is interesting to study the dependency of the equilibrium values a^* and b^* on the discount factor λ. The numerical sensitivity analysis exhibited in Figure 3.6.1 depicts the optimal intervention thresholds as functions of the legislative body's time preference rate, which here takes values on the interval $\lambda \in (r-g, r-g+\alpha/\kappa)$. The latter guarantees, as shown in the previous analysis, the optimality of a finite debt ceiling mechanism. Some remarks are worth mentioning. Clearly, the equilibrium debt ceiling b^* exhibits a monotonically increasing behaviour as a function of λ, with the peculiarity of an exploding behaviour $b^* \uparrow +\infty$ for $\lambda \uparrow r - g + \alpha/\kappa$. This illustrates our finding of the smooth transition from a debt ceiling mechanism to a non-intervention policy by the legislative body, as stated in Corollary 3.5.3. It is interesting to notice that this monotonicity as well as limiting behaviour of b^* does not depend on the other parameters in the model, although some of them influence the interval bounds for the values of λ, for which the legislative body's optimal

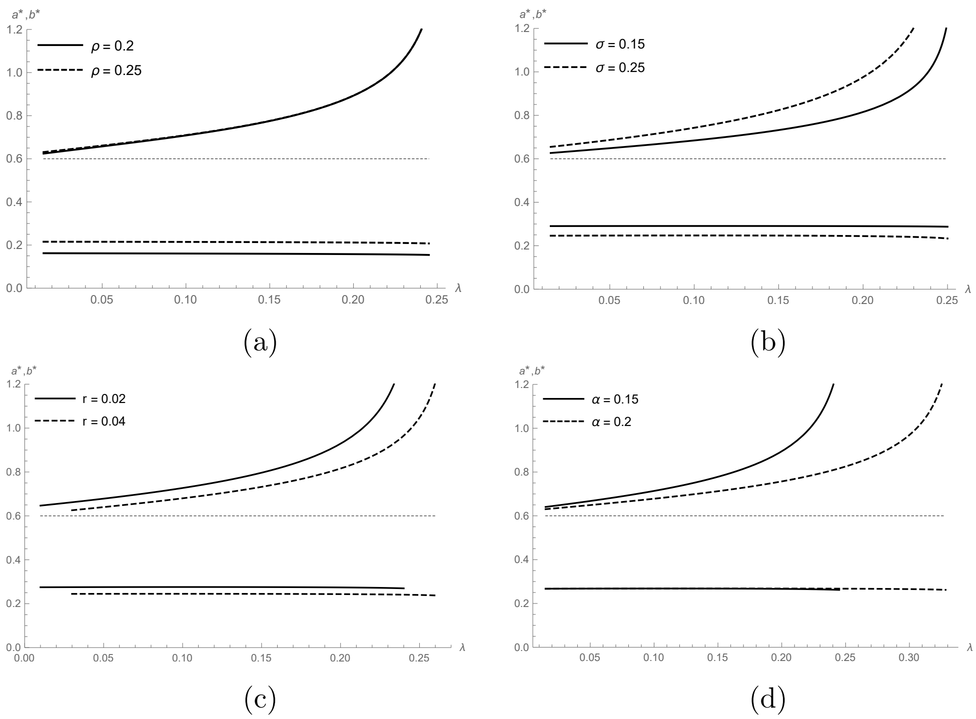


Figure 3.6.1: Sensitivity of the equilibrium values for a^* and b^* (as functions of λ) with respect to a change in some of the model parameters.

strategy is indeed a debt ceiling mechanism. More precisely, increasing (decreasing) the term $r - g$ shifts the interval to the right (left), while the fraction α/κ determines the length of the interval $(r - g, r - g + \alpha/\kappa)$. A short discussion on the implications of a shift in these parameters on the optimal strategy of the legislative body (depending on their time preference rate λ) is given in the subsequent sensitivity study. Last, we note that the intervention threshold a^*, characterising the optimal debt issuance policy by the government, again illustrates our finding of Corollary 3.5.3, in the sense that $a^* \to \overline{a}$ for $\lambda \uparrow r - g + \alpha/\kappa$.

Sensitivity with respect to governmental time preference rate ρ. The discount factor ρ serves as a measure on how myopic a government is regarding its debt. Increasing ρ has the effect that the government discounts future costs and revenues more heavily, and thus cares less and less about the future compared to the present. We observe the sensitivity regarding a change in ρ in Figure 3.6.1a. Clearly, the government aims at increasing its debt ratio earlier, at a higher debt-issuance threshold a^*, whenever its subjective discount rate increases. The legislative body reacts to such an increase by increasing the debt ceiling as well, although we observe Figure 3.6.1a that the equilibrium value b^* is relatively robust with respect to a change in ρ.

Sensitivity with respect to debt ratio volatility σ. Increasing volatility increases the fluctuations of the debt ratio. The government and the legislative body adapt by acting on the debt ratio later, which is achieved by the government decreasing its optimal debt-issuance threshold and by the legislative body increasing the debt ceiling. We can observe this in Figure 3.6.1b.

Sensitivity with respect to the interest rate r on government debt. Increasing interest rates on public debt result in holding debt getting more costly for the government, which in turn increases the drift of the debt ratio. Clearly, it is optimal for the government to increase its debt at a later stage, which is achieved by decreasing its debt-issuance threshold, as observed in Figure 3.6.1c.
In the equilibrium, the legislative body also decreases the debt ceiling, since countries with a higher cost of debt are more in danger of defaulting. Contrary, if interest rates decrease, the legislative body can be more flexible, since it is optimal to increase the debt ceiling. Intuitively, in such a case, the growth of GDP helps containing the debt ratio without interventions. Furthermore, we note that an increase in the interest rate r shifts the interval of λ-values, for which the optimal strategy of the legislative body prescribes to set a finite debt ceiling, to the right. It follows that a legislative body with a fixed time preference rate could change its optimal strategy from a non-intervention policy to a debt ceiling mechanism, if the government's interest rate on debt increases. Notice that an increase in the country's GDP growth rate has the contrary effect, thus implying that fast growing economies could allow a larger deficit without interventions from a legislative body.

Sensitivity with respect to the tax compliance factor α. The parameter α denotes the tax compliance factor, which measures the willingness to pay taxes within a country. This factor can be chosen freely, hence, the legislative body can account for the fact that some countries have a low probability of default, even though holding a lot of debt. In Figure 3.6.1d we observe the sensitivity of the equilibrium values a^* and b^* with respect to a change in α. Clearly, if the tax compliance factor increases (which implies decreasing willingness to pay tax), the legislative body faces stronger social and political pressure to act via the implementation of a debt ceiling mechanism. This has the consequence that a larger factor α (i) causes the legislative body to act earlier on the debt ratio (by decreasing the debt ceiling b^*) and (ii) enlarges the interval of time preference values $\lambda \in (r-g, r-g+\alpha/\kappa)$ for which the legislative body's optimal strategy is to impose a debt ceiling mechanism. The latter implies that, for a fixed time preference λ, a change in the factor α could incentivise the legislative body to switch from a laissez-faire policy to implementing a debt ceiling. On the other hand, if the assigned likelihood of the country's default decreases (in terms of a decrease in the parameter α), the legislative body is willing to postpone interventions by either implementing a larger debt ceiling b^* or even choosing a non-intervention policy.

Chapter 4

A Stationary Equilibrium Model of Green Technology Adoption with Endogenous Carbon Price

4.1 Introduction

In this chapter, we introduce a stationary equilibrium model in which a continuum of competing firms are subject to a carbon pricing system. Firms are identified by their level of technology, that is assumed to evolve according to an Itô-type stochastic differential equation, and their production decisions lead to emissions. Moreover, firms choose an abatement effort that takes the shape of an irreversible investment, and their strategy gives the time at which they switch to a carbon neutral technology. In equilibrium, we establish the existence and uniqueness of an endogenous carbon price, an optimal investment time and a stationary distribution of incumbent, polluting firms. In our notion of equilibrium we assume that cumulative emissions are specified – for example via a quota imposed by a regulator – and the equilibrium values are found by equating quota and the total net emissions, as well as through a suitable entry condition.

We proceed as follows. In Section 4.2 we introduce the model and analyse the problem of the single firm. Section 4.3 aggregates and discusses the notion of a competitive equilibrium with entry and exit. Its existence and uniqueness, in a general formulation of the problem, is proven in Section 4.3.3. In Section 4.4 we discuss an explicit model, in which we fix the profit functions of firms and their underlying technology shock processes. Moreover, we carry out a comparative statics analysis on the explicit model, studying some of the effects the model's parameters have on the equilibrium values. In Section 4.4.2 we finally discuss the introduction of a welfare maximizing regulator, that is able to strategically set a quota on emissions.

4.2 Setting and Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete and filtered probability space, rich enough to accommodate a standard one-dimensional Brownian motion $(W_t)_{t\geq 0}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by W, augmented by $\mathbb{P}$-null sets. We consider a continuum of firms that produce distinct goods and are subject to a carbon price and an emission limit. We assume that the production process leads to emissions of greenhouse gases, for example CO_2, and firms have to pay a fixed carbon price for every unit of pollution they emit. We assume that each firm faces idiosyncratic technology shocks (see for example Hopenhayn, 1992; Miao, 2005 and Dixit and Pindyck, 1994 for a general discussion on idiosyncratic and aggregate shocks), which evolve according to an Itô-diffusion

$$dZ_t = \mu_1(Z_t)dt + \sigma_1(Z_t)dW_t, \qquad Z_0 = z. \tag{4.2.1}$$

for some Borel-measurable functions μ_1, σ_1 to be specified. We assume that firms are able to restructure their business to become carbon neutral by paying a sunk cost $c(\cdot)$, which, in principle, could depend on the current level of technology of the firm (cf. Huang et al., 2021). Let us denote the time, at which the firm chooses to invest into becoming carbon neutral, as $\tau \geq 0$, which is an $\mathbb{F}$-stopping time. Since a firm's investment could influence their underlying technology shock process, we assume that the latter follows a (potentially different) SDE

$$d\overline{Z}_{t+\tau} = \mu_2(\overline{Z}_{t+\tau})dt + \sigma_2(\overline{Z}_{t+\tau})dW_t, \quad t \geq 0, \qquad \overline{Z}_\tau = Z_\tau \tag{4.2.2}$$

where μ_2, σ_2 again denote Borel-measurable functions to be specified. To account for the dependency of the processes (4.2.1) and (4.2.2) on their initial levels, we write Z^z as well as $\overline{Z}^z$ where appropriate. Moreover, we let $\mathbb{P}_z$ denote the probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}_z[\cdot] = \mathbb{P}[\cdot|Z_0 = z]$, and by $\mathbb{E}_z[\cdot] = \mathbb{E}[\cdot|Z_0 = z]$ the corresponding expectation under this measure. Regarding the coefficients in the SDEs (4.2.1)-(4.2.2) we state the following assumption.

Assumption 4.2.1. *We denote $\mathbb{R}_+ = (0, \infty)$. We assume that the state space of the diffusions Z^z and $\overline{Z}^z$ are given by $\mathcal{I} = (\underline{x}, \infty)$ and $\overline{\mathcal{I}} = (\underline{y}, \infty)$, where $\underline{x}, \underline{y} \leq 0$. The coefficients $\mu_i : \mathbb{R} \to \mathbb{R}$ and $\sigma_i : \mathbb{R} \to \mathbb{R}_+$, $i = 1, 2$ are such that $\mu_i \in C^1$, $\sigma_i \in C^2$ and*

$$|\mu_i(x) - \mu_i(y)| \leq K_i|x - y|, \qquad |\sigma_i(x) - \sigma_i(y)| \leq h_i(|x - y|)$$

for some $K_i > 0$, $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ strictly increasing, $h_i(0) = 0$ and

$$\int_{(0,\varepsilon)} \frac{dx}{h_i^2(x)} = \infty$$

for all $\varepsilon > 0$ and x, y in $\mathcal{I}$, or $\overline{\mathcal{I}}$, respectively. We highlight that, a priori, we do not specify whether zero is attainable or unattainable for the diffusions Z and $\overline{Z}$. In the forthcoming analysis, we will assume that the firms exit the market whenever their technology shock process falls below the level zero. Clearly, if zero is unattainable for the diffusion, no absorption takes place (consider, e.g., a geometric Brownian motion). On the other hand, it is straightforward to modify the assumption to include an absorption in a point $\varepsilon > 0$.

Under the previous assumption, it can be shown that if solutions to (4.2.1) and (4.2.2) exist, they are each pathwise unique by the Yamada-Watanabe's Theorem. Moreover, for all $x \in \mathcal{I}$, respectively $\overline{\mathcal{I}}$, there exists $\varepsilon > 0$ such that

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |\mu_i(y)|}{\sigma_i^2(y)} dy < +\infty, \qquad i = 1, 2,$$

such that (4.2.1)-(4.2.2) have a weak solution that is unique in the sense of probability law (see Karatzas and Shreve, 1991, Chapter 5.5). It follows that (4.2.1)-(4.2.2) have unique strong solutions (see Karatzas and Shreve, 1991, Corollary 5.3.23) that are regular in the sense that any point of the interior of their respective state space can be reached in finite time with positive probability. We assume that the boundary points $+\infty$ are not attainable for neither of the two processes, i.e cannot be reached in finite time with positive probability.
We assume that the profits of the firms are summarized by functions $\pi_1(\cdot)$ (emitting firms) and $\pi_2(\cdot)$ (carbon-neutral firms). Importantly, we assume that both profit functions may have a dependency on the overall emissions $E_{\max} > 0$, while only the emitting firms are subject to the carbon price $c_p > 0$ charged for each unit of pollutant that is emitted. To stress this dependency, we write $\pi_1(\,\cdot\,; E_{\max}, c_p)$ and $\pi_2(\,\cdot\,; E_{max})$ where appropriate. Moreover, we assume that the emissions of the single firm, which is polluting and has not invested into becoming carbon neutral yet, are given by a function $e(z; E_{\max}, c_p)$. This function will be of some importance in the later analysis, especially when considering the equilibrium model in Section 4.3. Here, we would like to highlight that one should expect a term similar to $-c_p e(z)$ within the profit function π_1 of a polluting firm. Although not needed in the analysis of our general model, it reflects the idea that polluting firms are obliged to pay a fixed cost that is proportional to their emissions.

In the following, we consider the optimal timing problem of a representative firm. We assume that firms are forced to exit the market whenever their technology shock process falls below a given level, which we, without loss of generality, assume to be given by 0. Hence, we denote the exit time by

$$\gamma_1 := \inf\{t \geq 0 : \; Z_t = 0\}, \qquad \gamma_2 := \inf\{t \geq \tau : \; \overline{Z}_t = 0\}. \tag{4.2.3}$$

Clearly, if 0 is unattainable for the diffusion Z or $\overline{Z}$, no absorption takes place and, as usual, we let $\inf \emptyset = +\infty$. On the other hand, the polluting firm can choose the time $\tau \in \mathcal{T}$ at which investing to become carbon neutral, in exchange to the sunk cost $c(\cdot)$. We assume that firms discount their profits with a factor $r > 0$. Additionally, with reference to Miao (2005), we assume that firms independently suffer exogenous death under the Poisson process with parameter $\eta > 0$. Although not needed in order to establish the equilibrium in our model – in contrast to Miao (2005) – including this element can help account for the possibility of firms exiting the market due to factors not captured in the model.
Then, the value function of the single firm takes the form

$$v(z) := \sup_{\tau \in \mathcal{T}} J(z, \tau), \tag{4.2.4}$$

where $\mathcal{T}$ denotes the set of all $\mathbb{F}$-stopping times and the objective is given by

$$\mathcal{J}(z,\tau) := \mathbb{E}_z\Big[\int_0^{\gamma_1\wedge\tau} e^{-(r+\eta)t}\pi_1(Z_t)dt + \Big(\int_\tau^{\gamma_2} e^{-(r+\eta)t}\pi_2(\overline{Z}_t)dt - e^{-(r+\eta)t}c(\overline{Z}_\tau)\Big)\mathbb{1}_{\{\tau<\gamma_1\}}\Big], \tag{4.2.5}$$

We begin by rewriting the value function and, for that purpose, let

$$\Phi_1(z) := \mathbb{E}\Big[\int_0^{\gamma_1} e^{-(r+\eta)t}\pi_1(Z_t^z)dt\Big], \qquad \text{and} \qquad \Phi_2(z) := \mathbb{E}\Big[\int_0^{\gamma_2} e^{-(r+\eta)t}\pi_2(\overline{Z}_t^z)dt\Big], \tag{4.2.6}$$

and are able to rewrite the objective (4.2.5) such that

$$\mathcal{J}(z,\tau) = \mathbb{E}\Big[\int_0^{\tau\wedge\gamma_1} e^{-(r+\eta)\tau}\pi_1(Z_t)dt + e^{-(r+\eta)t}\Big(\Phi_2(Z_\tau) - c(Z_\tau)\Big)\mathbb{1}_{\{\tau<\gamma_1\}}\Big], \tag{4.2.7}$$

where we used a simple application of the strong Markov and tower property. We observe that the second term in the expectation on the right hand side of (4.2.7) can be interpreted as a “real option” of the polluting firm. When investing, the firm obtains – after servicing the sunk cost – the stream of revenues π_2 until it eventually leaves the market due to an inefficient technology.

As we did for the profit functions π_1, π_2, we denote $v(z; E_{\max}, c_p) = v(z; c_p) = v(z)$ where it is appropriate to stress the dependency of the value function on the parameters $E_{\max}$ and c_p.

Remark 4.2.2. *Using arguments presented in Alvarez (1999), we are able to express the functions Φ_1 and Φ_2 in a purely analytical way. In the following, we let $\psi(z)$ and $\varphi(z)$ denote the increasing and decreasing, respectively, fundamental solutions to the ordinary differential equation $(\mathcal{L} - (r+\eta))u(z) = 0$, where $\mathcal{L}$ denotes the infinitesimal generator associated to the diffusion Z. We recall that the Green-kernel $G : \mathcal{I}\times\mathcal{I} \to \mathbb{R}_+$ of the linear diffusion Z is given by*

$$G_{r+\eta}(z,y) = \int_0^\infty e^{-rt}p(t;z,y)dt = \begin{cases} W^{-1}\psi(z)\varphi(y), & z < y \\ W^{-1}\psi(y)\varphi(z), & z \geq y, \end{cases}$$

where W denotes the constant Wronskian determinant of the fundamental solutions $\psi(z)$ and $\varphi(z)$, given by

$$W = \frac{\psi'(z)}{S'(z)}\varphi(z) - \frac{\varphi'(z)}{S'(z)}\psi(z) > 0,$$

and where $S'(z) = \exp(-\int(2\mu_1(z)/\sigma_1^2(z))dz$ denotes the scale density of the diffusion Z. It can then be shown (see Alvarez, 1999) that $\Phi_1(\cdot)$ as in (4.2.6) can be rewritten as

$$\Phi_1(z) = \int_0^\infty G_{r+\eta}^{(0,\infty)}(z,y)\pi_1(y)m'(y)dy,$$

where $m'(y) = 2/(\sigma_1^2(y)S'(y))$ *denotes the speed density of* Z *and* $G_{r+\eta}^{(0,\infty)}(z,y)$ *the Green-kernel of the constrained process* Z *that is killed at* 0*, given by*

$$G_{r+\eta}^{(0,\infty)}(z,y) = \begin{cases} W^{-1}\varphi(y)\psi(z,0), & z < y, \\ W^{-1}\varphi(z)\psi(y,0), & z \geq y, \end{cases}$$

where we let $\psi(z,0) = \psi(z) - (\psi(0)/\varphi(0))\varphi(z)$*. We note that*

$$\psi(z,0) = \begin{cases} \psi(z), & \text{if } 0 \text{ is natural or exit,} \\ \psi(z) - \frac{\psi(0)}{\varphi(0)}\varphi(z), & \text{if } 0 \text{ is regular or entrance.} \end{cases}$$

For a classification of boundary points we refer to Borodin and Salminen (2015), Chapter 2*. Analogously, we let* $\overline{\psi}(z)$ *and* $\overline{\varphi}(z)$ *denote the fundamental solutions to the ordinary differential equation* $(\overline{\mathcal{L}}-(r+\eta))u(z) = 0$*, where* $\overline{\mathcal{L}}$ *is the infinitesimal generator associated to the diffusion* $\overline{Z}$ *of* (4.2.2)*. We can proceed as above and obtain*

$$\Phi_2(z) = \int_0^\infty \overline{G}_{r+\eta}^{(0,\infty)}(z,y)\pi_2(y)\overline{m}'(y)dy,$$

where $\overline{W}, \overline{S}'(z), \overline{m}'(z), \overline{G}_{r+\eta}^{(0,\infty)}(x,y)$ *denote the Wronskian, scale density, speed density and Green-kernel of the constrained process* $\overline{Z}$*, respectively.*

Before we study the maximization problem (4.2.4), we state the following assumption regarding the profit of the polluting and carbon-neutral firms, as well as the sunk cost and the emissions of the polluting firms.

Assumption 4.2.3. *The functions* $\pi_1,\ \pi_2, e$ *and* c *are such that*

(i) $\pi_1(\,\cdot\,; E_{max},\,\cdot\,) \in C^{1,1}(\mathbb{R}_+ \times \mathbb{R}_+)$*,* $\pi_2(\,\cdot\,; E_{\max}) \in C^1(\mathbb{R}_+)$*,* $c(\cdot) \in C^1(\mathbb{R})$*,* $e(\cdot) \in C(\mathbb{R})$*;*

(ii) $0 \leq \pi_1(z; E_{\max}, c_p) \leq K_1(c_p), 0 \leq \pi_2(z; E_{\max}) \leq K_2$ *for some* $K_1(c_p), K_2 > 0$ *and all* $(z, E_{\max}) \in \mathbb{R}_+^2$*;*

(iii) $\frac{\partial}{\partial c_p}\pi_1(z; E_{max}, c_p) < 0$ *for all* $z \geq 0$*;*

(iv) $\lim_{c_p \to \infty} \pi_1(z, c_p) = 0$ *for all* $z \in \mathbb{R}_+$*;*

(v) $c(z), e(z) > 0$ *for all* $z \in \mathbb{R}_+$*, and* $|c'(0)| < \infty$*.*

Remark 4.2.4. *While Assumptions 4.2.3 (i), (v) deal with technical aspects, we can provide a theoretical foundation for assumptions (ii)-(iv). The second assumption posits that the profit function remains non-negative when considering positive values for technology* z*, overall emissions* $E_{\max}$*, and carbon price* c_p*. Furthermore, we assume that an increase in the carbon price, which firms must pay for each unit of emitted pollutants (which is indirectly linked to production* *output), adversely affects firm profits. Lastly, we presume that as the carbon price approaches infinity, firms' profits tend to zero. In simple terms, emitting firms cease production entirely when faced with an exorbitant carbon price, resulting in zero profits. This assumption implies the absence of fixed production costs in our model, although it is straightforward to extent the model in this regard.*

4.2.1 The Problem of the Single Firm

In this Section, we aim at solving the problem of a single firm that determines the optimal time to switch to a carbon-neutral technology. It is important to stress that each polluting firm, even though they contribute to the cumulative amount of pollution in the atmosphere, neglect their own impact. Hence, we assume that each firm, being a small player, acts as if their actions have no capability of affecting the overall level of pollution. A comparable concept is the one of a *price-taking agent* that assumes that their activity has no impact on the market price (see Maksimovic and Zechner, 1991; Miao, 2005).
Problem (4.2.4) takes the shape of an optimal stopping problem in the field of real options theory. Dating back to the contributions of Myers (1977) and McDonald and Siegel (1986), the real options approach to irreversible investment decisions has received much attention in various problems arising in economics and finance. Here, we mention Dixit (1989) as well as Pindyck (1988, 1990). In the case, where the underlying economic shock process is one-dimensional – as in our case – explicit solutions are often feasible (cf. Dixit and Pindyck, 1994). We build on their analysis and use the connection of optimal stopping and free boundary problems (cf. Peskir and Shiryaev, 2006) in order to identify the optimal investment rule of the single firm.
To begin with, and via standard techniques, we associate the value function with a variational inequality of the form

$$\max\left\{(\mathcal{L}-(r+\eta)w(z)+\pi_1(z),\ \Phi_2(z)-c(z)-w(z)\right\}=0, \tag{4.2.8}$$

with boundary condition $w(0) = 0$ and where $\mathcal{L} := \frac{1}{2}\sigma_1^2(z)\partial_{zz} + \mu_1(z)\partial_z$ denotes the infinitesimal generator associated to the diffusion Z of (4.2.1). In the following, we follow a guess-and-verify approach and conjecture that firms will restructure their business (and thus become carbon neutral) whenever their technology level is sufficiently large. Hence, we guess that there exists a threshold $b > 0$ such that the stopping time

$$\tau_b := \inf\{t \geq 0 :\ Z_t \geq b\} \tag{4.2.9}$$

is optimal. Accordingly, we can relate the above variational inequality to the following *free-boundary problem*

$$\begin{cases} (\mathcal{L}-(r+\eta)w(z)+\pi_1(z)=0, & 0<z<b, \\ (\mathcal{L}-(r+\eta)w(z)+\pi_1(z)\leq 0, & b\leq z, \\ w(z)=\Phi_2(z)-c(z), & b\leq z, \\ w(z)\geq \Phi_2(z)-c(z), & 0<z<b, \\ w(0)=0, \end{cases} \tag{4.2.10}$$

where we separated the values of technology into the so-called *waiting region* $\mathbb{W}=(0,b)$ and the *stopping region* $\mathbb{S}=[b,\infty)$. We solve the free-boundary problem (4.2.10) by first recalling that any solution to the ODE $(\mathcal{L}-(r+\eta))w(z)+\pi_1(z)=0$ is given by

$$w(z)=A\psi(z)+B\varphi(z)+\Phi_1(z),$$

where $\psi(\cdot)$ and $\varphi(\cdot)$ are as in Remark 4.2.2. The boundary condition implies $A\psi(0)+B\varphi(0)=0$, such that

$$w(z)=A\frac{\psi(z)\varphi(0)-\varphi(z)\psi(0)}{\varphi(0)}+\Phi_1(z)=:A\psi(z,0)+\Phi_1(z),$$

where $\psi(z,0)$ is defined as in Remark 4.2.2. For the ease of notation, we now let

$$G(z):=\Phi_2(z)-\Phi_1(z)-c(z),$$

and, by imposing smooth-fit and smooth-pasting conditions at the free boundary b, we obtain that the *candidate value function* takes the form

$$w(z):=\begin{cases} 0, & z\leq 0, \\ \Phi_1(z)+\frac{G(b)}{\psi(b,0)}\psi(z,0), & 0<z<b, \\ \Phi_2(z)-c(z), & z\geq b, \end{cases} \tag{4.2.11}$$

and the optimal stopping threshold b is given by the solution – provided that it exists – to the equation

$$G'(b)[\psi(0)\varphi(b)-\varphi(0)\psi(b)]-G(b)[\psi(0)\varphi'(b)-\varphi(0)\psi'(b)]=0.$$

We define

$$A(z):=\frac{G'(z)[\psi(0)\varphi(z)-\varphi(0)\psi(z)]-G(z)[\psi(0)\varphi'(z)-\varphi(0)\psi'(z)]}{S'(z)}, \tag{4.2.12}$$

where $S'(z)$ denotes the scale density of the diffusion Z. Furthermore, as in Remark 4.2.2, we let $m'(z)=2/(\sigma^2(z)S'(z))$ denote the speed measure density. We now search for a point b such that $A(b)=0$, and straightforward computations yield

$$\frac{\partial}{\partial z}A(z)=m'(z)\big(\psi(0)\varphi(z)-\varphi(0)\psi(z)\big)\big(\mathcal{L}-(r+\eta)\big)G(z). \tag{4.2.13}$$

We can write

$$A(b) = A(0) + \int_0^b \frac{\partial}{\partial z} A(z) dz,$$

and notice that, under Assumption 4.2.3, we have

$$\begin{aligned} A(0) &= G'(0) \frac{\varphi(0)\psi(0) - \varphi(0)\psi(0)}{S'(0)} + G(0) \frac{\varphi(0)\psi'(0) - \varphi'(0)\psi(0)}{S'(0)} \\ &= \big(\Phi_2(0) - \Phi_1(0) - c(0)\big) W = -c(0) W < 0, \end{aligned}$$

where the latter inequality follows from Assumption 4.2.3.

As a next step, we derive the existence and uniqueness of a solution to the equation $A(b) = 0$. To this end, we state the following **standing assumption**.

Assumption 4.2.5. *There exists a unique point* $\widetilde{z} \in (0, \infty)$*, such that*

$$\big(\mathcal{L} - (r + \eta)\big) G(z) = \pi_1(z) + \big(\mathcal{L} - (r + \eta)\big)[\Phi_2(z) - c(z)] \begin{cases} > 0, & 0 \le z \le \widetilde{z}, \\ = 0, & z = \widetilde{z}, \\ < 0, & z > \widetilde{z}. \end{cases} \tag{4.2.14}$$

Remark 4.2.6. *Assumption 4.2.5 is a well known criterion that is typical in optimal stopping problems in order to derive the existence and uniqueness of a point* $b \in \mathbb{R}_+$ *triggering the optimal stopping time (see, e.g., Alvarez (2001) and Falbo et al. (2021)). It is crucial to notice that, at this point, no qualitative statements are possible regarding sufficient conditions on the parameters* μ_i, σ_i *that imply the suggested shape of the function* (4.2.14) *and thus of the monotonicity of* $A(\cdot)$*, as derived in* (4.2.13)*. We remark that, under the assumption that* $\mu_1 = \mu_2$*,* $\sigma_1 = \sigma_2$ *and a constant investment cost* $c(z) = I$*, the condition* (4.2.14) *simplifies to the assumption* $\pi_1(z) - \pi_2(z) + (r - \eta)I < 0$ *for* $z > \widetilde{z}$*. Furthermore, we mention the fact that if Assumption 4.2.5 is not satisfied, firms do not invest into a carbon neutral technology. In the case in which* 0 *is natural for the diffusion* Z *(and firms do not exit the market when their technology level is low), our analysis may lead to a stationary equilibrium in which neither entry nor exit takes place. Since this is not the scope of this work, we refrain from studying this case and stick to the Assumption 4.2.5.*

The next theorem summarizes our findings in the single firm problem (4.2.4). Its proof can be found in Appendix D.1.

Theorem 4.2.7 (Verification Theorem)**.** *The value function* v *of* (4.2.4) *takes the form*

$$v(z) := \begin{cases} 0, & z \le 0, \\ \Phi_1(z) + \big(\Phi_2(b) - \Phi_1(b) - c(b)\big) \frac{\psi(z,0)}{\psi(b,0)}, & 0 < z < b, \\ \Phi_2(z) - c(z), & z \ge b, \end{cases} \tag{4.2.15}$$

where $b \in (0, \infty)$ *denotes the investment threshold triggering the optimal stopping time* τ_b *of* (4.2.9)*. It is given by the unique solution to the equation* $A(b) = 0$*, with* $A(\cdot)$ *given by* (4.2.12)*.*

4.2.2 Some Results regarding the Monotonicity with respect to the Carbon Price

Here, we develop some of the needed results in the later analysis of the existence and uniqueness of a market equilibrium. Without the risk of confusion, and where necessary, we let $b(c_p) = b$ denote the unique solution to $A(\cdot) = 0$, and denote $A(b(c_p), c_p) = A(b)$, $\Phi_1(z, c_p) = \Phi_1(z)$ as well as $G(z, c_p) = G(z)$, in order to highlight their dependency on $c_p > 0$. In particular, we now study the monotonicity of $b(c_p)$ as well as $v(z; c_p)$ with respect to the carbon price c_p. The proofs of Lemmas 4.2.8 and 4.2.9 can be found in Appendix D.2.

Lemma 4.2.8. *Under Assumption 4.2.3, the function $c_p \mapsto b(c_p)$ is strictly decreasing. Moreover, the limit $\lim_{c_p \to \infty} b(c_p) =: b_\infty > 0$ exists.*

Lemma 4.2.9. *Under Assumption 4.2.3, the map $c_p \mapsto v(z; c_p)$ is strictly decreasing on $z \in (0, b(c_p))$ and constant on $(b(c_p), \infty)$.*

4.3 A Continuum of Firms and the Market Equilibrium

In the latter section, we discussed the irreversible investment problem, posed as an optimal stopping problem (4.2.4) of a single firm. In this section, we move to an aggregate level and consider a continuum of emitting firms, that each face idiosyncratic shocks to their technology shock process and solve the investment problem introduced in Section 4.2. In our forthcoming analysis, we aim to derive the long-run stationary equilibrium of the regulated market. This involves a stationary distribution of incumbent firms, which are yet to either make the irreversible investment decision or to leave the market due to their technology shock process falling below zero. Clearly, since all incumbent firms will eventually leave the market over time, the derivation of a stationary distribution requires the existence of new entrants that strategically enter the market and take the spot of exiting firms. In the following two Sections 4.3.1 and 4.3.2, we develop the notion and needed assumptions for the market equilibrium. Its existence and uniqueness is then developed in Section 4.3.3.

4.3.1 New Entrants and the Entry Cost

We assume that there exists a continuum of potential new entrants, that – upon entry – incur a fixed sunk cost $c_e > 0$ that will be specified later. After entry, the firm's initial level of technology is drawn from the distribution ξ, that satisfies the following assumption.

Assumption 4.3.1. *The distribution ξ is supported on the interval $[\underline{z}, \overline{z}]$ and admits a density function $g(\cdot) \in C^1$. Moreover, we assume $\underline{z} > 0$, such that firms do not exit immediately after entering.*

We notice that, while firms are not directly exiting the market due to their initial technology being below zero, it is however possible that $\overline{z} > b$. The latter implies that firms are potentially entering the market as a polluting firm and immediately choose to invest in order to become carbon neutral.

In a competitive equilibrium, the expected benefit of entry must be equal to the entry cost, that is

$$\int_{\underline{z}}^{\overline{z}} v(z; c_p)\xi(dz) = c_e. \tag{4.3.1}$$

Condition (4.3.1), the so-called entry condition (see Hopenhayn, 1992; Hopenhayn and Rogerson, 1993; Miao, 2005), will be crucial when determining the equilibrium carbon price $c_p > 0$, that polluting firms are obliged to pay on their emissions. More precisely, its purpose is to balance the inflow and outflow of firms, which must be equal in a stationary equilibrium to keep the mass of incumbent firms constant. Hence, in order to guarantee the existence and uniqueness of a competitive equilibrium with entry and exit and a positive carbon price $c_p > 0$, we state the following assumption on the entry cost c_e.

Assumption 4.3.2. *We state the following assumptions.*

(i) The entry cost c_e satisfies

$$c_e \leq \int_{\underline{z}}^{\overline{z}} v(z; 0, E_{max})\xi(dz). \tag{4.3.2}$$

(ii) We distinguish two cases. Recall that $b_\infty = \lim_{c_p \to \infty} b(c_p) > 0$ exists due to Lemma 4.2.8. If $b_\infty > \underline{z}$, we assume

$$c_e > \lim_{c_p \to \infty} \int_{\underline{z}}^{\overline{z}} v(z; c_p, E_{max})\xi(dz). \tag{4.3.3}$$

Otherwise, if $b_\infty \leq \underline{z}$, we notice that, due to Lemma 4.2.8, there exists $\overline{c}_p > 0$ such that $b(\overline{c}_p) = \underline{z}$. We then assume

$$c_e > \int_{\underline{z}}^{\overline{z}} v(z; \overline{c}_p, E_{max})\xi(dz). \tag{4.3.4}$$

Assumption 4.3.2 is not only necessary when deriving the existence of an equilibrium in the competitive market, but also admits a clear economic interpretation. This is due to the fact that it enforces the idea that – in any meaningful model – the equilibrium carbon price should be such that $c_p \in (0, \infty)$, i.e. firms do not get a positive reward for emitting pollutants. Moreover, it guarantees that a positive number of firms are entering the market that do not immediately exercise their option to become carbon-neutral. For more details we refer to the proof of Proposition 4.3.5, here we concentrate on the intuition behind Assumptions 4.3.2 (*i*) and (*ii*).

Notice that condition (*i*) implies that the entry cost c_e is lower than the expected profit of entering firms in the absence of a carbon price c_p. If not fulfilled, the entry cost would thus exceed the highest possible expected benefit. This could either lead to a negative equilibrium carbon price (which is not desirable from neither a modelling nor a regulators perspective) or (if the carbon price is assumed to be positive) discourage firms from entering the market.

On the other hand, condition (*ii*) implies that the entry cost is larger than the expected benefit from entering a market with the largest carbon price that still guarantees a distribution of incumbent firms. An entry cost violating condition (*ii*) would thus lead to an imbalance, since no carbon price $c_p \in (0, \infty)$ could prevent new entrants from flooding the market. We notice that, if $b_\infty < \underline{z}$, condition (4.3.4) is not necessarily needed when deriving the stationary distribution. However, when violated, the entry condition (4.3.1) could lead to an equilibrium carbon price c_p such that $b(c_p) < \underline{z}$. This would imply that all entering firms are immediately switching to become carbon neutral. While this could be desirable from an environmental perspective, it implies a zero mass of emitting firms in the long run steady state. We exclude this trivial case here.

4.3.2 The Notion of Equilibria

In a long run steady state, there is a stationary distribution of polluting firms ν and a constant entry rate N of firms that enter the market via the mechanism we introduced in Section 4.3.1. As usual in general equilibrium models, we will observe that all equilibrium variables are constant over time. The underlying intuition lies in the fact that even though firms are subject to idiosyncratic shocks, these individual uncertainties aggregate into a stationary certainty (cf. Dixit and Pindyck, 1994, Chapter 8). Hence, using the equilibrium distribution of active polluting firms, we are able to compute aggregate values. In more explicit models, this includes variables like overall output supply, capital demand, but also the cumulative emissions of the firms. The latter plays a crucial role in our analysis, as it is used to derive the equilibrium condition in our model.
To be more precise, we impose that in a stationary equilibrium the overall emissions $E_{max} > 0$, that firms take as given in their profit functions π_1 and π_2, indeed equal the cumulative emissions of the emitting firms. Hence, the condition

$$E_{max} = \int_0^{b(c_p, E_{\max})} e(z; c_p, E_{max}) \nu(dz), \tag{4.3.5}$$

must be satisfied, where we noticed that, due to our results from Section 4.2, the support of ν is given by $(0, b(c_p, E_{\max}))$ and we highlight the dependency of the stopping threshold b on the parameters c_p as well as $E_{\max}$. We state the following definition, summarizing the notion of a stationary equilibrium in our model.

Definition 4.3.3. *A stationary equilibrium consists of an exit threshold $b > 0$, an entry rate N, a distribution ν and a carbon price c_p such that (i) τ_b is the optimal solution to the single firm's problem* (4.2.4)*, (ii) the entry condition* (4.3.1) *holds, (iii) the equilibrium condition* (4.3.5) *is satisfied and (iv) ν is an invariant distribution over* $(0, b)$.

Definition 4.3.3 reflects the concept that in a stationary equilibrium the aggregate variables are constant over time, even though individual firm are subject to considerable change over time. Depending on their technology shock process, polluting firms are constantly expanding, contracting, investing and even exiting, while other firms are entering.

Remark 4.3.4. *We examine the equilibrium condition described in* (4.3.5)*. It is interesting to note that this condition can be interpreted in two distinct ways.*
Firstly, it can be viewed as a "consistency condition." In this interpretation, it signifies that firms' expectations regarding overall emissions remain valid and accurate within the equilibrium context. More precisely, the emissions $E_{\max}$, that firms take as given in their maximization criterion (4.2.7)*, coincide with the true overall emissions that result from their cumulative production.*
Alternatively, the parameter $E_{\max}$ can be seen as a regulatory constraint imposed by external authorities. This dual interpretation becomes particularly intriguing when considering models where firms do not factor overall emissions into their optimization processes. In such cases, (4.3.5) *takes on the role of a "compliance condition." Here, a regulatory body establishes an upper limit for the cumulative emissions of polluting firms to ensure compliance with environmental standards, and the carbon price is determined endogenously among firms. We highlight that a similar condition was specified in the recent Anderson and Duanmu (2023). In their "quota equilibrium", a regulator refrains from further actions after setting an emission limit and the price of the quota (the carbon price) is determined by equating net pollution and the pre-specified level. In the subsequent section, we will demonstrate that the equilibrium entry rate is positively correlated with the maximum allowable emissions $E_{\max}$. Consequently, a regulator imposing emissions limits effectively deters potential market entrants, as their production activities could breach the stipulated emission target.*
At this point in our analysis, it is important to note that deriving qualitative insights regarding a potential regulator's decision criteria is challenging without further specifying the general model discussed in Section 4.2. In Section 4.4.2, we discuss the implementation of a welfare maximizing decision criterion within a more explicit model.

4.3.3 Existence and Uniqueness of an Equilibrium

Prior to presenting our central finding, which establishes the existence and uniqueness of a stationary equilibrium in our model, we offer a concise overview of the underlying rationale and the proof's methodology.

Our proof follows similar steps as those developed in Dixit and Pindyck (1994), Hopenhayn and Rogerson (1993), and Miao (2005). First, we derive the equilibrium carbon price using the entry condition (4.3.1). Via Assumption 4.3.2 and the derived monotonicity of the value function v with respect to c_p (see Lemma 4.2.9) it is straightforward to show that there exists a unique carbon price c_p^* that leads (4.3.1) to hold with equality. The equilibrium value c_p^* is then used to derive the technology level $b^* = b(c_p^*)$ at which firms choose to invest into becoming carbon neutral.
Next, we solve for the equilibrium distribution ν, which is, similarly as in related contributions, not a probability measure (see Hopenhayn, 1992; Miao, 2005). Instead, for a given Borel set $\mathcal{B}$ on the real line, $\nu(\mathcal{B})$ reflects the number of polluting firms whose technology shock process falls within the set $\mathcal{B}$. Its support is given by the interval $(0, b^*)$, since polluting firms exit when their technology either falls below zero (since they are assumed to be inefficient) or exceeds the threshold b (since they invest to become carbon neutral).
Furthermore, it is crucial to notice that we are able to scale the distribution ν by the entry rate N when solving for it. More precisely, one can show (see Dixit and Pindyck, 1994; Hopenhayn and Rogerson, 1993) that the stationary distribution is linearly homogeneous in the entry rate N, such that $\nu = Nf(z)$ for some function f to be found. The latter function is readily found using the *Kolmogorov-foward equation*, which is associated to the diffusion Z of (4.2.1) and complemented to include the two features entry and Poisson deaths (A derivation of the latter can be found in Appendix D.4). It follows that we can compute f via ordinary differential equations, that are, depending on the position of the exit threshold b^*, satisfied on particular intervals. More precisely, they are given by

$$-\frac{\partial}{\partial z}\big[\mu(z)f(z)\big] + \frac{\partial^2}{\partial z^2}\Big[\frac{\sigma^2(z)}{2}f(z)\Big] - \eta f(z) = 0, \qquad \text{for } z \in (0, \underline{z}) \cup (\min(\overline{z}, b), b), \tag{4.3.6}$$

$$-\frac{\partial}{\partial z}\big[\mu(z)f(z)\big] + \frac{\partial^2}{\partial z^2}\Big[\frac{\sigma^2(z)}{2}f(z)\Big] - \eta f(z) + g(z) = 0, \qquad \text{for } z \in (\underline{z}, \min(\overline{z}, b)). \tag{4.3.7}$$

Notice that, since b^* is endogenously determined, it is a priori unclear whether $b^* \geq \overline{z}$ or $b^* < \overline{z}$. The resulting stationary distribution of firms admits qualitatively different properties in these cases (notice that a fraction of entering firms is immediately switching to become carbon neutral in the latter case) and we thus distinguish them when deriving its scaled density f^* in the proof of Proposition 4.3.5 below.
In the last step, we use the equilibrium condition (4.3.5) to solve for the entry rate N^*. Here, it is crucial to notice that, due to the linearity of ν^* in the entry rate N^*, the overall emissions as determined in the integral on the right hand side of (4.3.5) are also linear in N^*. The equilibrium entry rate N^* as well as the stationary distribution $\nu^* = N^* f^*$ are thus derived using simple calculations.

We now state our main result. Its proof can be found in Appendix D.3.

Proposition 4.3.5. *There exists a unique stationary equilibrium, that includes a unique equilibrium carbon price* $c_p^* > 0$*, an exit threshold* $b^* = b(c_p^*)$*, an entry rate* N^* *and a stationary distribution* ν^* *such that the entry condition* (4.3.1) *and the equilibrium condition* (4.3.5) *are satisfied. Moreover,* b^* *is the threshold that triggers the optimal stopping time* τ_{b^*} *in the single firm problem* (4.2.4).

4.4 An Explicit Model

In this section we discuss an illustrative equilibrium model of firms switching to become carbon neutral. We begin by fixing a specific model and show that the model fulfills the assumptions we imposed throughout Sections 4.2 and 4.3. In Section 4.4.1 we perform a comparative statics analysis on the equilibrium parameters, analysing the sensitivity of the values on some of the underlying parameters. Furthermore, we discuss how a regulatory body could choose to constraint the overall emissions in a strategic way. Section 4.4.2 assumes a social welfare maximizing regulator, and studies the effect on the resulting equilibrium.
To begin with, we assume that the dynamics of Z and $\overline{Z}$, as in (4.2.1)-(4.2.2), are given by Brownian motions with a drift

$$\begin{aligned} dZ_t &= \mu_1 dt + \sigma_1 dW_t, \quad t \geq 0, \qquad Z_0 = z, \\ d\overline{Z}_{t+\tau} &= \mu_2 dt + \sigma_2 dW_t, \quad t \geq \tau, \qquad \overline{Z}_\tau = Z_\tau \end{aligned}$$

where $\mu_i \in \mathbb{R}$ and $\sigma_i > 0$, $i = 1, 2$. We assume that each firm produces an output via the function

$$y(z, k) = D(E_{\max})\theta z k \tag{4.4.1}$$

where $\theta > 0$ is a scale parameter and k denotes the capital stock of the firm, such that we are in a classical AK-model (see for example Wälde, 2011). Moreover, we assume that the externality, i.e. the atmospheric changes due to climate change caused by global emissions, may have a negative effect on production of each firm. Following Nordhaus (2014) and Golosov et al. (2014), we assume that damages are multiplicative and can be summarized by the damage function given by

$$D(E_{\max}) := \exp\big(-\rho(E_{\max} - \overline{E})\big). \tag{4.4.2}$$

Golosov et al. (2014) assume a similar structure that first translates emissions to carbon concentration, which is then translated to damages. Here, we simplify the structural form by assuming that damages are directly triggered by emissions, and $E_{\max} - \overline{E}$ measures the distance between current emissions and a benchmark level $\overline{E}$. The latter may be chosen in such a way that the corresponding carbon concentration is that of pre-industrial times (cf. Golosov et al., 2014). Notice that $\rho > 0$ results in a reduced productivity of production if the level of overall emissions is above the benchmark level $\overline{E}$.

Similar to Miao (2005), we assume a simple capital structure where firms rent capital from risk-neutral investors who discount future cash flows at a constant rate $r > 0$. Moreover, we assume that capital depreciates with rate $\delta > 0$, such that the rental rate for firms is given by $r + \delta$.

As in Section 4.2, we assume that firms production leads to pollution, and firms are subject to a carbon price that has to be paid according to their emissions. For simplicity, we assume that firms' emissions are proportional to their output (see for example Carmona et al., 2013; Ulph, 1996), i.e.

$$e(y(z,k)) = \lambda y(z,k), \tag{4.4.3}$$

for some $\lambda > 0$. The investment cost by the firm is assumed to be constant and equal to $I - \kappa$, where $\kappa \geq 0$ denotes a subsidy of the regulator given to firms in order to give incentives for an investment to become carbon neutral. We assume that firms face a constant elasticity demand function (see, for example Bertola, 1998), such that the market price is given by

$$p(y) = y(z,k)^{-\varepsilon}, \qquad \varepsilon \in (0,1).$$

Additionally to the restructering, the firm could also benefit from possible tax incentives the legislative body will implement in order to encourage firms to become carbon neutral. Hence, we assume that polluting and carbon neutral firms are taxed with potentially different tax rates before and after their restructure and denote them by $\tau_1 \geq \tau_2$. We note that Moyer et al. (2014) model the "carbon tax" as a production tax which differs by type of technology, whereas we model a carbon price alongside capital income taxes, and we refer to Barrage (2020) for a discussion on this topic. All in all, the firms' profit functions before and after their investment are given by

$$\begin{aligned}
\pi_1(z,k) &:= \max_k (1-\tau_1)\big[p(y)y(z,k) - \delta k - c_p e(y)\big] - rk, \\
\pi_2(z,k) &:= \max_k (1-\tau_2)\big[p(y)y(z,k) - \delta k\big] - rk.
\end{aligned}$$

We can optimize these functions and determine optimal capital levels

$$k_1^* = \Big(\frac{(1-\varepsilon)(D(E_{max})\theta z)^{1-\varepsilon}}{\delta + c_p\lambda D(E_{\max})\theta z + r/(1-\tau_1)}\Big)^{1/\varepsilon}, \qquad k_2^* = \Big(\frac{(1-\varepsilon)(D(E_{max})\theta z)^{1-\varepsilon)}}{\delta + r/(1-\tau_2)}\Big)^{1/\varepsilon}, \tag{4.4.4}$$

which, plugged into the value function, yield

$$\begin{aligned}
\pi_1(z) &:= \pi_1(z,k_1^*) = (1-\tau_1)\varepsilon\Big(\frac{(1-\varepsilon)D(E_{max})\theta z}{\delta + c_p\lambda D(E_{max})\theta z + r/(1-\tau_1)}\Big)^{\frac{1-\varepsilon}{\varepsilon}}, \\
\pi_2(z) &:= \pi_2(z,k_2^*) = (1-\tau_2)\varepsilon\Big(\frac{(1-\varepsilon)D(E_{max})\theta z}{\delta + r/(1-\tau_2)}\Big)^{\frac{1-\varepsilon}{\varepsilon}}, \\
e(z) &:= e(y(z,k_1^*) = \lambda\Big(\frac{(1-\varepsilon)D(E_{max})\theta z}{\delta + c_p\lambda D(E_{max})\theta z + r/(1-\tau_1)}\Big)^{\frac{1}{\varepsilon}},
\end{aligned}$$

Regarding the entry decision of firms, we assume that the firms' initial technology values after entry are uniformly distributed, i.e. $\xi \sim \mathcal{U}([\underline{z}, \overline{z}])$ (cf. Miao, 2005).
As a first step, we verify that the proposed model fulfills the imposed assumptions of Sections 4.2–4.3. We remark that, even in the explicit formulation of our model, a general verification of Assumption 4.2.5 is not possible. Clearly, this would lead to imposing specific assumptions on the involved parameters in our model, which is neither a straightforward task nor leading to any qualitative insights. Hence, we restrict our attention to the Assumptions 4.2.3, which states specific assumptions on the profit, cost and emission functions involved. The proof of Lemma 4.4.1 can be found in Appendix D.5.

Lemma 4.4.1. *The model proposed above satisfies Assumptions 4.2.3.*

4.4.1 Comparative Statics Analysis

In the following, we discuss some of the implications of our explicit model on the equilibrium values and, especially, their sensitivity with respect to changes in the model's parameters. To begin with, we fix the base case parameter values, which are summarized in Table 4.1.

	Parameter	Value
Polluting firm's shock drift	μ_1	0.02
Polluting firm's shock volatility	σ_1	0.15
Carbon Neutral firm's shock drift	μ_2	0.02
Carbon Neutral firm's shock volatility	σ_2	0.15
Polluting firm's tax rate	τ_1	0.3
Carbon Neutral firm's tax rate	τ_2	0.3
Depreciation Rate	δ	0.1
Riskless rate	r	0.5
Poisson death	η	0.04
Entry Cost	c_e	28.6
Entry Distribution Interval	$(\underline{z}, \overline{z})$	$(5, 30)$
Price Elasticity	ε	0.5
Scale Parameter Output	θ	0.3
Scale Parameter Emissions	λ	0.05
Scale Parameter Damage	ρ	0.02
Investment Cost	I	100
Subsidy	κ	0
Benchmark Emission Level	$\overline{E}$	100
Cumulative Emissions	$E_{\max}$	102

Table 4.1: Base Case Parameter Values

We want to emphasize that for all the parameters we have selected, both in the base

case model and in the upcoming sensitivity analysis, we have ensured that the condition (4.2.14) of Assumption 4.2.5 is consistently met. Moreover, it is essential to note that these parameter values, although chosen to align with the estimated data, serve merely as illustrative benchmarks.
Most of the data has been chosen to suit those assumed in related contribution as Golosov et al. (2014) and Miao (2005). To allow for a clear interpretation, the entry cost as well as the interval bounds for the entry distribution in the base case model have been calibrated such that the carbon price is $c_p^* = 1$.
In the first part of our comparative statics analysis, we maintain cumulative emissions as a fixed parameter. Here, we let $\overline{E} = 100$ and $E_{\max} = 102$, such that the model depicts a 2% breach of the benchmark level. Later, in the subsequent part (see Section 4.4.2), we optimally select this parameter using a welfare maximization criterion of a regulator.
In the following, we separately discuss the effects of a change in the underlying parameters on the equilibrium values. We focus on the effect on the carbon price c_p^*, the investment threshold b^*, as well as the overall output $Y(c_p^*, E_{\max})$ and the turnover rate T^* among firms. The former can be easily computed via

$$Y(c_p^*, E_{\max}) := \int_0^{b^*} y(z; c_p^*, E_{\max})\nu^*(dz) = \int_0^{b^*} D(E_{\max})\theta z k_1^*(z, c_p^*, E_{\max})\nu^*(dz), \qquad (4.4.5)$$

where k_1^* denotes the optimal capital demand as in (4.4.4). The turnover rate should be understood as the ratio between the entry rate to the mass of incumbent firms in equilibrium. We notice that the latter can be computed via

$$M^* := \int_0^{b^*} \nu^*(dz) = N^* \int_0^{b^*} f^*(z)dz,$$

such that the turnover rate writes as

$$T^* = \frac{N^*}{M^*} = \frac{1}{\int_0^{b^*} f^*(z)dz}.$$

Sensitivity with respect to the diffusion coefficients. We investigate the impact of a change in the coefficients governing the underlying diffusions. Our findings are summarized in Table 4.2.
We note that when the technology growth parameter μ_1 for polluting firms decreases, the exit threshold decreases correspondingly. This is a logical outcome since the profit function π_1 for polluting firms increases as technology levels improve. A smaller drift in this context leads to reduced expected operating profits, leading firms to invest earlier in achieving carbon neutrality. Consequently, the exit threshold b is lowered. Moreover, we observe that in equilibrium an increased technology growth among polluting firms results in a rising carbon price. This can be attributed to the fact that as firms experience greater technological advancements, their output and emissions also increase. To counteract this effect, in order to keep the cumulative

	Carbon Price	Investment Threshold	Turnover Rate	Overall Output
Base case	1.00	32.78	0.0403	2040
$\mu_1 = 0.01$	0.99	32.72	0.0401	2040
$\mu_1 = 0.03$	1.01	32.84	0.0406	2040
$\mu_2 = 0.01$	1.00	32.93	0.0403	2040
$\mu_2 = 0.03$	1.00	32.62	0.0403	2040
$\sigma_1 = 0.2$	1.00	32.91	0.0404	2040
$\sigma_1 = 0.25$	1.00	33.04	0.0406	2040
$\sigma_2 = 0.2$	1.00	32.77	0.0403	2040
$\sigma_2 = 0.25$	1.00	32.77	0.0403	2040

Table 4.2: Comparative Statics with respect to the diffusion coefficients

emissions in balance with the equilibrium condition, a higher equilibrium carbon price encourages firms to opt for lower output levels.

We also observe that a higher technology growth rate increases the turnover rate. It is worth noting that this increased turnover rate may have different causes: firms may exit the market due to decreased efficiency, as indicated by their technology shock process falling below zero, or they may become more efficient, leading to a higher number of firms adopting carbon-neutral technology and leaving the market while still operational. In this scenario, we find that, although the effect is minor, the latter factor seems to dominate.

Regarding changes in the technology growth parameter μ_2 for carbon-neutral firms, we find that this has a negligible effect on the carbon price, even though it influences the exit threshold as expected. An increase in the technology growth of carbon-neutral firms boosts the expected profit for firms following their investment. This incentive encourages firms to invest sooner, resulting in a decrease in the exit threshold.

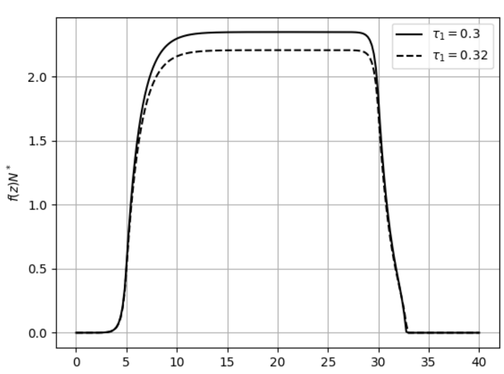
Sensitivity with respect to the tax rates τ_1 and τ_2. We now study the effect of a shift in the corporate tax rates of firms. A potential regulator or legislative body could install different tax rates for polluting and carbon neutral firms, in order to encourage firms to invest earlier into becoming carbon neutral. This targeted policy could be installed by either increasing taxes for polluting firms, or by installing tax benefits (and thus decreasing the corporate tax) for carbon neutral firms. The observed effects on the equilibrium parameters are summarized in Table 4.3.

Regarding an increase in the tax rates for polluting firms, we observe the following. Even though a higher tax rates should provide an incentive to adopt cleaner technology earlier, we observe a counter intuitive effect in our model. The reason lies in the fact that a tax increase has two separate effects. First, it decreases corporate profits and thus leads firms to invest earlier in alternative technologies. On the other hand, it decreases the expected profit that firms face when considering entering the market. The latter effect leads to less entry and a smaller number of incumbent firms, which can be observed in Figure 4.4.1, which plots the mass of polluting firms for different tax rates τ_1.

	Carbon Price	Investment Threshold	Turnover Rate	Overall Output
Base case	1.00	32.78	0.0403	2040
$\tau_1 = 0.32$	0.94	32.87	0.0403	2040
$\tau_1 = 0.35$	0.86	33.02	0.0403	2040
$\tau_2 = 0.27$	1.00	30.76	0.0413	2400
$\tau_2 = 0.25$	1.01	29.49	0.0433	2400
$\kappa = 10$	1.00	30.19	0.0421	2040
$\kappa = 20$	1.05	27.28	0.0479	2040

Table 4.3: Comparative Statics with respect to the tax rates and the investment subsidy

Since less firms are on the market, the competition among firms to achieve the given cumulative emissions decrease. Individual firms are able to produce and pollute more, at the same time the carbon price decreases, which increases firms' profits. This leads firms to stay longer in the market as a polluting firm, and the the exit threshold b increases. In the current parameter constellation, this latter effect seems to outweigh the former.
Concerning the effect on the carbon price, we notice the following. A tax increase on polluting firms is a targeted policy that aims to decrease firms profits and thus encourage them to invest into a carbon neutral technology. This, as explained above, decreases the mass of incumbent firms as entry gets less profitable. In equilibrium, the carbon price is chosen such that a given emission target is met. If taxes increase, less firms are in competition for the cumulative emissions and it follows that a lower carbon price is needed to keep firms emissions within the stipulated emission target.
As for a decrease in the tax rate for carbon neutral firms, we note the following. Similarly as the policy discussed above, which however targets the polluting firms, a tax reduction for carbon neutral firms aims at providing incentives to invest in cleaner technology and thus accelerate the process of firms becoming carbon neutral. The latter can be precisely observed



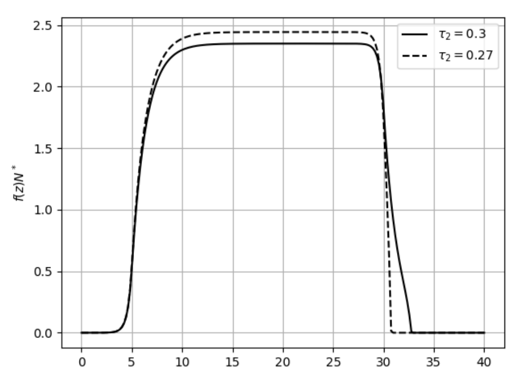


Figure 4.4.1: Equilibrium density of polluting firms with respect to a change in the tax rates τ_1 and τ_2, respectively.

in our model, as the exit threshold decreases. Furthermore, the turnover rate increases, which can be explained by the increased amount of firms switching to a carbon neutral technology, due to the tax policy making it financially untenable for highly polluting firms to continue operating without making significant changes to their practices.

Interestingly, the chosen parameter constellation leads to an exit threshold that is lower than the upper interval bound $\overline{z}$ up to which firms initial technology level is drawn. Hence, a lower tax rate for carbon neutral firms may lead some firms to enter the market and – if their initial technology is sufficiently high – immediately become carbon neutral. In this regard, a lower tax rate encourages a higher entry into the market, since firms observe that it is profitable to become carbon neutral and thus benefit from a tax cut for smaller values of the technology shock process.

The carbon price seems to be relatively robust with respect to a change in the tax rates τ_2. Intuitively, this results from the fact that this tax policy targets carbon neutral firms, while the carbon price in equilibrium is determined endogenously among the polluting firms. The small effect observed in Table 4.3 can be explained by the increased expected profit that potential entrants face when making their entry decision, especially when it becomes profitable to enter the market and directly switching to a carbon neutral technology. A slightly higher carbon price counteracts this in order to balance the entry condition and to prevent the market from getting flooded by new market entrants.

Sensitivity with respect to the investment subsidy. An alternative to a targeted tax increase/decrease in order to encourage firms to invest in a carbon neutral technology (as discussed previously) is by offering a subsidy to firms resulting in a lower investment cost. We observe that it is methodologically related to a tax cut for carbon neutral firms, as it rewards environmentally beneficial behaviour instead of punishing the polluting firms. Consequently, the observed effects on the equilibrium values for the carbon price, exit threshold and turnover rate are related to the ones discussed above, and the equilibrium density of polluting firms can be observed in Figure 4.4.2.

Sensitivity with respect to Poisson death. According to the the size of the parameter of Poisson death, firms are randomly subject to a shock that leads them to leave the market. We remark that, differently from related contributions as Miao (2005), the parameter is not needed in order to guarantee the existence of a stationary distribution in our model. This is due to the fact that the distribution of incumbent firms is supported on the bounded interval $(0, b^*)$, whereas an unbounded support coupled with a non-stationary process (such as the drifted Brownian motion we are considering) could lead to an exploding number of firms with large technology levels. We observe that an increasing parameter η that governs the rate at which firms randomly leave the market, leads – as expected – to a higher turnover rate on the market and to a lower mass of polluting firms, as observed in Figure 4.4.2. This reduces competition for firms, and consequently the carbon price decreases.

Sensitivity with respect to emission related parameters. Here, we discuss the sensitivity of the equilibrium parameters with respect to the emission related parameters, namely

	Carbon Price	Investment Threshold	Turnover Rate	Overall Output
Base case	1.00	32.78	0.0403	2040
$\eta = 0.035$	1.06	31.46	0.0359	2040
$\eta = 0.045$	0.94	34.11	0.0451	2040

Table 4.4: Comparative Statics with respect to the parameter of Poisson death

the emission target $E_{\max}$ and the scale parameters λ and ρ.
First, we notice that – differently from the comparative statics analysis so far – a shift in the values $E_{\max}$ or λ does influence the overall production in equilibrium. This is due to the fact that the firm's emissions are assumed to be proportional to their output with parameter λ, i.e. $e(z) = \lambda y(z)$, such that the equilibrium condition (4.3.5) rewrites as

$$\frac{E_{\max}}{\lambda} = \int_0^{b(c_p^*, E_{max})} y(z; c_p^*, E_{max}) \nu^*(dz),$$

where the latter integral thus denotes the aggregate production of the polluting firms. It follows that an increase in the overall permitted emissions leads to an increase in overall output, while it decreases if the production of firms leads to higher emissions. The latter can be observed in Table 4.5.

Regarding a shift in the overall emissions $E_{\max}$ we observe the following. Assuming a lower value of $E_{\max}$ translates to a more rigorous target for the cumulative emissions. First, as discussed above, it follows that the overall production decreases. Furthermore, the competition for firms to emit increases, which leads to an increase in the equilibrium carbon price. Consequently, firms are encouraged to invest into becoming carbon neutral, such that the exit threshold b decreases and the turnover rate increases accordingly. In Figure 4.4.3 we observe that the shift of overall emissions mostly affects firms with high values of technology. While

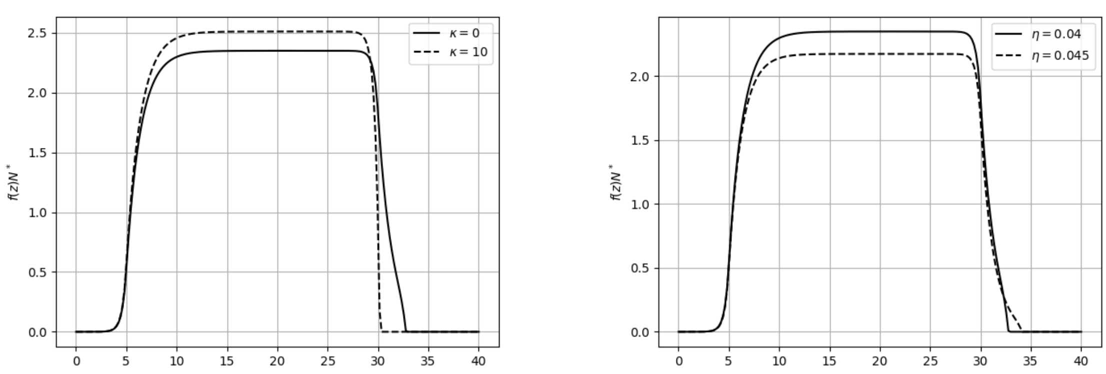


Figure 4.4.2: Equilibrium density of polluting firms with respect to a change in the subsidy κ and in the parameter of Poisson death η, respectively.

	Carbon Price	Investment Threshold	Turnover Rate	Overall Output
Base case	1.00	32.78	0.0403	2040
$E_{\max} = 100$	1.03	31.33	0.0409	2000
$E_{\max} = 105$	0.96	35.10	0.0401	2100
$\lambda = 0.049$	1.02	32.78	0.0403	2081
$\lambda = 0.051$	0.98	32.78	0.0403	2000
$\rho = 0.01$	1.01	32.04	0.0405	2040
$\rho = 0.03$	0.99	33.53	0.0402	2040

Table 4.5: Comparative Statics with respect to the emission target and the scale parameters of emissions and damage

the mass of firms with low or intermediate values of technology barely changes, the reduction of overall emissions is mainly achieved by firms with high values of technology, as they choose to invest into a carbon neutral technology at an earlier stage.
The parameter λ serves as a measure on how emission intensive the production of firms is – larger values of λ imply higher emissions per unit of output. Thus, as explained above, the equilibrium output is decreasing in λ. In Table 4.5 we observe that the carbon price is decreasing in the carbon intensity of production. In some sense, the carbon price acts as a counterpart to the carbon intensity, since $1.11 \times 0.045 \approx 0.91 \times 0.055 \approx 0.05$, where the latter is precisely the carbon intensity in the base case model. Hence, in order to achieve a cumulative emission target, the carbon price decreases in the carbon intensity in order to balance the cost of production of the firms. Interestingly, this leads to the same exit threshold and turnover rate as in the base case model. While the overall emissions stay constant, the decrease in output mainly results from the smaller mass of incumbent firms, as observed in Figure 4.4.3.

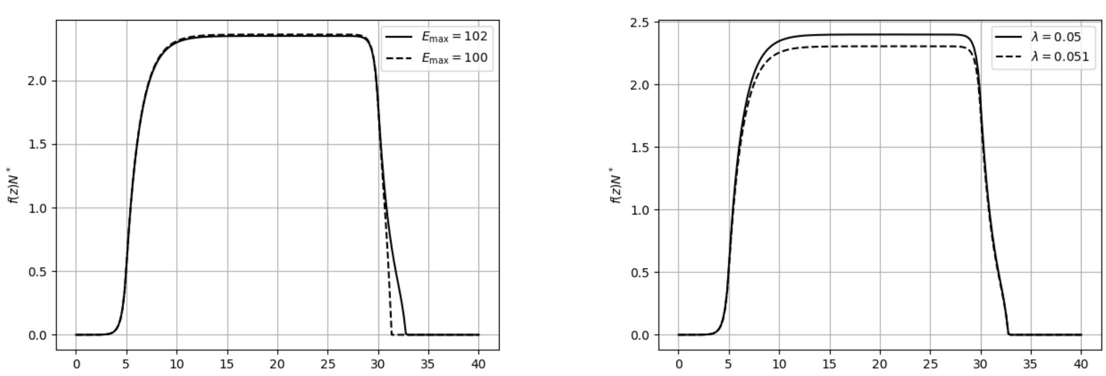


Figure 4.4.3: Equilibrium density of polluting firms with respect to a change in the overall emissions $E_{\max}$ and the scale parameter λ, respectively.

	Carbon Price	Investment Threshold	Turnover Rate	Overall Output
Base case	1.00	32.78	0.0403	2040
$c_e = 27$	1.10	32.14	0.0405	2040
$c_e = 30$	0.92	33.35	0.0402	2040
$\underline{z} = 3$	0.93	33.28	0.0402	2040
$\underline{z} = 7$	1.06	32.40	0.0404	2040

Table 4.6: Comparative Statics with respect to the entry cost and entry distribution

The parameter ρ, appearing in the damage function (4.4.2), scales the effect of the damage the breach of the benchmark level has on the production of the firms. It is crucial to notice that we assumed the damage function to appear in both the profit functions of the polluting as well as the carbon neutral firms, such that an increase in damage does not necessarily lead firms to invest earlier since they remain affected by the damage function. Here, we observe that the decrease in profit leads to less competition in the market, which leads to a lowered carbon price and an increased exit threshold.

Sensitivity with respect to the entry distribution and the entry cost. Table 4.6 summarizes the effect on the equilibrium parameters when subject to a change in the entry cost c_e as well as the lower interval bound $\underline{z}$ of the entry distribution. Notice that both parameters do not affect the running profit functions of any of the incumbent firms, both polluting and carbon neutral. However, they play a huge part in the entry decision of potential entrants. A decreasing entry cost thus leads to a lower barrier to entry and to more firms willing to enter, as reflected in the increased turnover rate and mass of incumbent firms, as indicated in Figure 4.4.4. Consequently, the enhanced competition leads to a larger carbon price, which results in incumbent firms aiming to become carbon neutral and thus invest at an earlier stage.
A similar effect can be seen for the increase in the lower interval bound of the entry distribution ξ, that is uniform on the interval $[\underline{z}, \overline{z}]$. Similar to a decrease in the entry cost, increasing $\underline{z}$ leads to a larger expected profit for potential entrants. As seen above, this increases the turnover rate and competition on the market, which results in a larger carbon price and a lower investment threshold.

4.4.2 A Welfare Maximizing Regulator

Here, we come back to the concept introduced in Remark 4.3.4, which posits that the overall emissions may be determined by deliberate decisions made by legislative bodies or environmental regulatory agencies. This concept is closely related with the principles underlying a cap-and-trade market framework, wherein a regulatory authority makes strategic decisions regarding the allocation of emission allowances to the market. In the context of our model, the cumulative emissions can be viewed as a regulatory constraint imposed on the collective

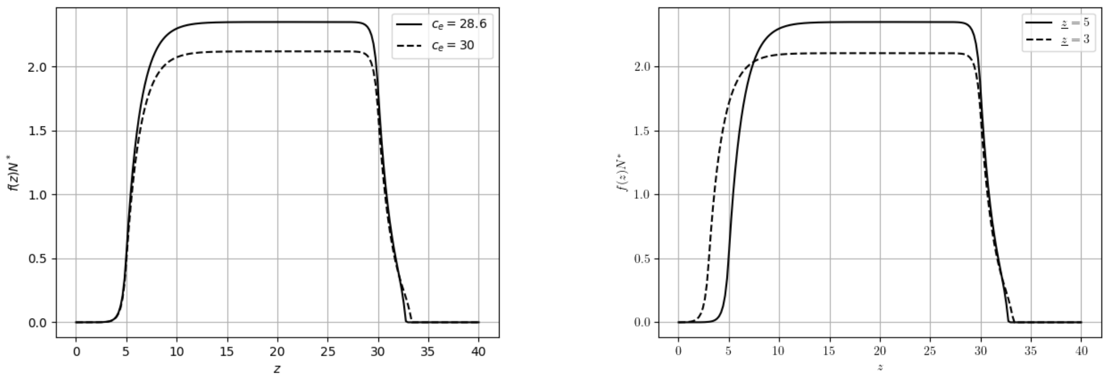


Figure 4.4.4: Equilibrium density of polluting firms with respect to a change in the entry cost and the entry dsitribution

emissions of firms.
Within the literature, a common approach entails the perspective of a "welfare-maximizing" regulator, as in Barrett (2001), Colla et al. (2012), Ulph (1996), and Germain et al. (2004). In this framework, the regulator, when setting a limit on overall emissions, aims to balance the societal benefits derived from emissions reduction and the economic losses incurred due to the constraints imposed. In the mentioned contributions, it is assumed that the regulator maximizes a function encompassing aggregate production, adjusted for capital consumption, and the "cost of pollution-induced damages", i.e.

$$\max_{E_{\max}} Y(E_{\max}) - rK(E_{\max}) - \Gamma E_{\max}^{1+w}, \tag{4.4.6}$$

where $\Gamma, w > 0$ weigh the cost of pollution-induced damages, Y denotes the aggregate supply of firms as in (4.4.5) and

$$K(E_{\max}) = \int_0^{b^*(c_p,E_{\max})} k_1^*(z, E_{\max})\nu^*(dz) = \int_0^{b^*(c_p,E_{\max})} k_1^*(z, E_{\max})N^* f^*(z)dz$$

denotes their consumption of capital. The proof of Proposition 4.3.5 reveals that the equilibrium entry rate N^* is determined via

$$N^* = \frac{E_{max}}{\int_0^b e(y(z; c_p, E_{max}))f^*(z)dz} = \frac{E_{max}}{\lambda \int_0^b y(z; c_p, E_{max})f^*(z)dz}$$

and hence, in equilibrium, we obtain

$$\begin{aligned} &Y(E_{\max}) - rK(E_{\max}) - \Gamma E_{\max}^{1+w} \\ &= \frac{E_{max} \int_0^{b^*(c_p,E_{\max})} y(z; c_p, E_{\max})f^*(z)dz}{\lambda \int_0^{b^*(c_p,E_{\max})} y(z; c_p, E_{max})f^*(z)dz} - \frac{E_{max} r \int_0^{b^*(c_p,E_{\max})} k_1^*(z; c_p, E_{\max})f^*(z)dz}{\lambda \int_0^{b^*(c_p,E_{\max})} y(z; c_p, E_{max})f^*(z)dz} - \Gamma E_{\max}^{1+w} \\ &= E_{\max}\Big(\frac{1}{\lambda} - \frac{r \int_0^{b^*(c_p,E_{\max})} k_1^*(z; c_p, E_{\max})f^*(z)dz}{\lambda \int_0^{b^*(c_p,E_{\max})} y(z; c_p, E_{max})f^*(z)dz} - \Gamma E_{\max}^{w}\Big). \end{aligned}$$

	Emissions	Carbon Price	Investment Threshold	Turnover Rate	Output
Base case	102.0	1.00	32.78	0.0403	2040
$\Gamma = 0.0985$	101.3	1.01	32.28	0.0404	2026
$\Gamma = 0.0990$	100.8	1.02	31.91	0.0406	2016

Table 4.7: Comparative Statics for the welfare maximizing equilibrium values with respect to the scale parameter Γ

Due to the dependencies of the firm's production y, their capital demand k_1^* (as in (4.4.4)), and the investment threshold b^* on the overall emissions $E_{\max}$, a closed form solution for the maximization problem (4.4.6) does not seem feasible. To this end, we focus solely on the numerical analysis here. In accordance with Pommeret and Prieur (2013), Pindyck (2002) and Colla et al. (2012), Section 4, we set $w = 1$ so to obtain a quadratic cost of damages. Our numerical analysis suggests that this indeed guarantees the existence of a maximum in the regulator's optimization problem (4.4.6), irrespective of the chosen parameter $\Gamma > 0$.
Note that the value of overall emissions $E_{\max}$ is now chosen endogenously as well, and thus becomes part of the equilibrium variables. We refer to the equilibrium, that involves the emission target set by the regulator via the optimality criterion (4.4.6), as the *welfare maximizing equilibrium*. In order to allow for a more comprehensible study of the latter, and to compare it with the equilibrium values we obtained in the previous comparative statics analysis for fixed overall emissions, we calibrate Γ such that the welfare maximizing equilibrium is such that $E^*_{\max} = 102$, as in our base case model. We do not repeat the extensive comparative statics analysis studied before, but mainly focus on the emission related parameters λ, ρ (as studied in Section 4.4.1) and the scale parameter Γ that measures the cost of damage induced by pollution.

Increasing the latter parameter clearly leads to an increased cost for the regulator. In Table 4.7 we observe that the regulator reacts by decreasing the emission target $E_{\max}$, even though this has the consequence of a lower overall output level. As observed in Section 4.4.1, decreasing the emission target leads to an increased carbon price, a lower investment threshold and a higher turnover rate. We conclude that a larger (negative) effect of pollution should be addressed by setting a stricter emission limit on the regulated firms. Moreover, we find that the higher the marginal damage, the lower the social welfare, as observed in Figure 4.4.5. This intuitive result complements the findings in Colla et al. (2012).

Next, we discuss the effect of a shift in the scale parameter λ, which measures the carbon intensity of production. Table 4.8 summarizes our findings, where we compare the equilibrium values from the previous sensitivity analysis (for fixed emission target) with those resulting from the welfare optimizing equilibrium. Recall that an increased carbon intensity lead to a decreased overall output level and a lower carbon price, where the latter results from firms endogenously balancing their cost of production.
Concerning the welfare-maximizing equilibrium, we note that as carbon intensity rises, the

	Emissions	Carbon Price	Investment Threshold	Turnover Rate	Output
Base case	102.0	1.00	32.78	0.0403	2040
$\lambda = 0.049$	102.0	1.02	32.78	0.0403	2081
	104.1	0.99	34.41	0.0401	2125
$\lambda = 0.051$	102.0	0.98	32.78	0.0403	2000
	100.0	1.01	31.33	0.0409	1961
$\rho = 0.01$	102.0	1.01	32.04	0.0405	2040
	102.1	1.01	32.08	0.0405	2042
$\rho = 0.03$	102.0	0.99	33.53	0.0402	2040
	101.9	0.99	33.42	0.0402	2038

Table 4.8: Comparative Statics with respect to the scale parameter λ and ρ. For each parameter, the upper row recalls the equilibrium values for fixed overall emissions, while the lower row states the values in the social welfare maximizing equilibrium.

regulator takes action by lowering the emission target. This action can be interpreted as a clear signal from the regulator to polluting firms, indicating consequences for inefficient production facilities. Consequently, the output of polluting firms drops, and as competition among firms intensifies, the regulator's action reverts the carbon price effect observed in the earlier sensitivity analysis. More precisely, we observe an increasing carbon price in response to the increasing carbon intensity. As a result, firms choose to invest in a cleaner technology at an earlier stage, such that the investment threshold b declines and the turnover rate increases.

Regarding the scale parameter ρ, we observe the following. Recall that an increase in ρ increases the damage of emissions on the production on both polluting as well as carbon neutral firms. While this decreased expected profits and thus lead to less competition in the equilibrium model with fixed overall emissions, we observe that the regulator's actions

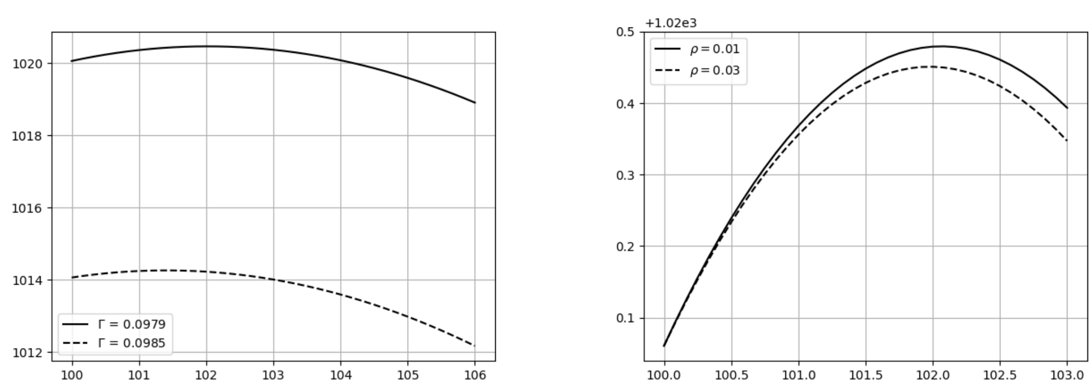


Figure 4.4.5: The social welfare for different values of the scale parameters Γ and ρ, respectively.

mitigates this effect by lowering the emission target. Hence, the regulator acknowledges that production becomes less efficient when increasing the carbon induced damages, and penalizes polluting firms by reducing their available emissions. We observe that the latter leads to a decreased overall output of polluting firms as well as, as seen in Figure 4.4.5, a lower social welfare.

Appendix

A A Numerical Scheme for the Evaluation of Free Boundaries

A.1 Introduction

In this chapter, we introduce a numerical method that makes it feasible to study optimal stopping boundaries which are characterized by an integral type equation.[1] For illustration we explain the procedure, as well as the needed preliminary results, at the example of a real options model that was first studied in Compernolle et al. (2021) and Dammann and Ferrari (2022). Here, a company is facing an irreversible investment decision in a two-dimensional framework, where, by paying a fixed sunk cost, a continuous stochastic cash-flow is generated. It is assumed that the company aims at maximizing its total expected profit arising from this investment and seeks to find a decision rule, which determines the optimal time to undertake this expenditure.

Our numerical method is inspired by the analysis in Detemple and Kitapbayev, 2020a and Christensen and Salminen, 2018, where similar methods have been employed in order to numerically determine the optimal stopping boundary through a derived integral equation. It relies on an application of the Monte-Carlo method and, as such, can be efficiently employed in problems with dimension larger than two as well, whenever an integral equation for the free boundary can be derived. Alternative numerical methods are clearly possible, and in fact employed in the related literature. A common approach is the finite difference scheme (see for example Compernolle et al., 2021). The method involves a direct numerical approximation of the variational inequality associated to the problem's value function and, consequently, does not account for an integral equation of the optimal boundary. However, as it is typical for analytical methods, this approach suffers from the curse of dimensionality so that an efficient approximation in larger dimensions can become problematic. Lange et al., 2020 propose a

[1]This chapter is based on joint work with Giorgio Ferrari. Parts of this have been published in Dammann and Ferrari (2022).

scheme in which the decision maker is only permitted to exercise the option at a set of Poisson arrival times that arrive at a finite rate. The project's value is then defined as the "fixed point" of an iterative scheme, where each iteration adds a single Poisson arrival time at which the decision maker is able to stop. This procedure defines a monotonically increasing sequence of lower bounds of the project's value, which can thus be found – if it is bounded – as the limit of this sequence. While this approach seems suitable also for problems in large dimensions, the exogenous Poisson process, however, adds a restriction, which can be seen as a liquidity constraint.

In the following, we first set up the model for the real option problem under consideration. We recall results that were derived in Compernolle et al. (2021) and Dammann and Ferrari (2022), but do not repeat the proofs therein for the sake of brevity. In Section A.4, we explain the iterative numerical scheme and illustrate it with a brief comparative statics analysis.

A.2 The Model

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space, with the filtration $\mathbb{F}$ generated by a two-dimensional Brownian motion $W = (W_t^X, W_t^Y)_{t\geq 0}$ and augmented with $\mathbb{P}$-null sets. We consider a profit-maximizing and risk-neutral company, which has the opportunity to invest into a production plant by paying a constant investment cost I. The production plant is capable of producing two goods in given quantities Q_1 and Q_2 and we assume that the prices of the two goods evolve stochastically according to the dynamics

$$\begin{cases} dX_t^x = \mu_1 X_t^x dt + \sigma_1 X_t^x dW_t^X, & X_0^x = x > 0, \\ dY_t^y = \mu_2 Y_t^y dt + \sigma_2 Y_t^y dW_t^Y, & Y_0^y = y > 0, \end{cases} \tag{A.1}$$

for some constants $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$. We assume that after the company has made the investment, it is able to sell the goods in their given quantities instantaneously and over an infinite time horizon on the market. If the investment is performed at initial time, its value for given price levels x and y is then obtained through the discounted perpetual revenue flow, net of the investment cost; that is,

$$\mathbb{E}\Big[\int_0^\infty e^{-rt}\pi(X_t^x, Y_t^y)dt - I\Big] =: F(x, y). \tag{A.2}$$

Here $\pi(x, y) := Q_1 x + Q_2 y$ denotes the profit function and $r > 0$ is a discount factor. In order to guarantee finite integrals, we make the following **standing assumption**.

Assumption A.1. *We have $r > \mu_1 \vee \mu_2$.*

Let $\mathcal{T}$ denote the set of all $\mathbb{F}$-stopping times. The company aims at determining the entry rule that maximizes its net total expected profits, and thus seeks to solve the two-dimensional optimal stopping problem

$$V(x,y) := \max_{\tau \in \mathcal{T}} \mathcal{J}(x,y,\tau), \tag{A.3}$$

where

$$\mathcal{J}(x,y,\tau) := \mathbb{E}\Big[e^{-r\tau}F(X^x_\tau, Y^y_\tau)\Big] = \mathbb{E}\Big[e^{-r\tau}\Big(\frac{Q_1 X^x_\tau}{\delta_1} + \frac{Q_2 Y^y_\tau}{\delta_2} - I\Big)\Big] \tag{A.4}$$

for $\delta_i = r - \mu_i$, $i = 1, 2$. The last equality in (A.4) follows by straightforward calculations upon using Assumption A.1.

A.3 Results

In this section, we summarize the most important findings in Compernolle et al. (2021) and Dammann and Ferrari (2022), that path the way towards a numerical study of the optimal investment strategy of the firm.

On the Value Function

As it is customary in optimal stopping, continuation and stopping regions of the optimal stopping problem (A.3) are given by

$$\mathcal{C} := \{(x,y) \in \mathbb{R}^2_+ :\ V(x,y) > F(x,y)\}, \quad \mathcal{S} := \{(x,y) \in \mathbb{R}^2_+ :\ V(x,y) = F(x,y)\}. \tag{A.5}$$

Notice that, since the value function V and the function F are continuous, the continuation region is open and the stopping region is closed (cf. Peskir and Shiryaev, 2006, p. 36). Moreover, the optimal stopping time is given by the first entry time of the process (X^x_t, Y^y_t) into the stopping region

$$\tau^* = \tau^*(x,y) := \inf\{t \geq 0 :\ (X^x_t, Y^y_t) \in \mathcal{S}\}, \tag{A.6}$$

whenever it is $\mathbb{P}$-a.s. finite (cf. Peskir and Shiryaev, 2006, p. 46).

The following proposition summarizes the findings with regard to the value function (A.3).

Proposition A.2. *Recall V from (A.3). The value function V is nondecreasing with respect to x and y. Moreover, $V \in C^1(\mathbb{R}^2_+)$, convex on $\mathbb{R}^2_+$ and admits the probabilistic representation*

$$V(x,y) = \mathbb{E}\Big[\int_0^\infty e^{-rt}(Q_1 X^x_t + Q_2 Y^y_t - rI)\mathbb{1}_{\{(X^x_t, Y^y_t) \in \mathcal{S}\}} dt\Big]. \tag{A.7}$$

for all $(x,y) \in \mathbb{R}^2_+$.

The proof of Proposition A.2 is mainly derived via straightforward techniques, while the probabilistic representation follows by adapting arguments presented in Section 3.1 in De Angelis et al. (2017). It involves a suitable approximation procedure whose analysis employs general results from PDE theory. We emphasize that representation (A.7) is essential for the forthcoming characterization of the optimal boundary being the solution to an integral equation.

Remark A.3. *The result concerning the probabilistic representation of the value function, as stated in Proposition A.2, can be generalized to cases involving more general dynamics and payoff functions. Indeed, the arguments of the proof of the representation* (A.7) *do not actually hinge on the particular form of the price processes.*

On The Optimal Boundary

Here, we discuss the most important findings of Dammann and Ferrari (2022) regarding the optimal price level triggering the investment in Problem (A.3). Most importantly, the optimal trigger is characterized as the unique solution to a nonlinear integral equation in a certain functional class.
Define

$$b(x) := \sup\{y \in \mathbb{R}_+ : \ V(x,y) > F(x,y)\}, \quad x \in \mathbb{R}_+, \tag{A.8}$$

with the convention $\sup \emptyset = 0$. We state the following proposition.

Proposition A.4. *The continuation region and stopping region of (A.5) can be written as*

$$\mathcal{C} = \{(x,y) \in \mathbb{R}_+^2 : \ y < b(x)\}, \quad \mathcal{S} = \{(x,y) \in \mathbb{R}_+^2 : \ y \geq b(x)\} \tag{A.9}$$

Furthermore, the optimal boundary b of (A.8) is continuous on $\mathbb{R}_+ \cup \{0\}$ and such that $b(0) = y^*$, $b(x) < y^*$ for all $x > 0$ and $b(x) = 0$ for all $x \geq x^*$, where

$$x^* = \frac{\beta_1}{(\beta_1 - 1)Q_1}\delta_1 I \quad \textit{and} \quad y^* = \frac{\beta_2}{(\beta_2 - 1)Q_2}\delta_2 I.$$

Here, β_i denotes the positive root to the equation $\frac{1}{2}\sigma_i^2\beta(\beta - 1) + \mu_i\beta - r = 0$ for $i = 1, 2$.

The results stated in Proposition A.4 guarantee that the continuation region $\mathcal{C}$ and the stopping region $\mathcal{S}$ are connected. Moreover, we can rewrite the optimal stopping time (A.6) due to (A.9) and obtain

$$\tau^* = \tau^*(x,y) := \inf\{t \geq 0 : \ Y_t^y \geq b(X_t^x)\} \tag{A.10}$$

for any $(x,y) \in \mathbb{R}_+^2$.

Next, we characterize the optimal boundary b as the unique solution to an integral equation in a certain functional class. For that purpose, we make use of the probabilistic representation of the value function V developed in Proposition A.2. Define

$$h(x) := \sup\{y \in \mathbb{R}_+ : \ (\mathcal{L} - r)F(x,y) > 0\} = \frac{1}{Q_2}(rI - Q_1 x). \tag{A.11}$$

and consider the class of functions

$$\mathcal{M} := \{f : \mathbb{R} \mapsto \mathbb{R}, \text{ continuous, decreasing and s.t. } f(x) \geq h(x)\}.$$

It can be shown that $\mathcal{M}$ is nonempty as $h \in \mathcal{M}$ as well as $b \in \mathcal{M}$ (see Dammann and Ferrari, 2022, Lemma 5.8 and Theorem 5.9). The following Proposition states the main result, identifying the optimal stopping boundary b as the unique solution to a nonlinear integral equation.

Proposition A.5. *The optimal boundary b of (A.8) is the unique function $y \in \mathcal{M}$ such that*

$$F(x, y(x)) = \mathbb{E}\Big[\int_0^\infty e^{-rt}\Big(Q_1 X_t^x + Q_2 Y_t^{y(x)} - rI\Big)\mathbb{1}_{\{Y_t^{y(x)} \geq y(X_t^x)\}}dt\Big], \quad x > 0. \tag{A.12}$$

The proof of Proposition relies on the previously derived representation of the value function V in Proposition A.2. Indeed, equation (A.12) results from evaluating (A.7) at points $y = b(x)$, upon using that $V(x, b(x)) = F(x, b(x))$. As for the uniqueness, one can adopt the four-step procedure in De Angelis et al. (2017), extending and refining the original probabilistic arguments from Peskir (2005). For further details, we refer to Dammann and Ferrari (2022), Theorem 5.9.

A.4 Numerical Scheme and Comparative Statics Analysis

In the following, we explain how to implement a numerical scheme in order to determine the optimal investment boundary b and to investigate its sensitivity with respect to some of the model's parameters.

Recall that b uniquely solves (A.12). We let

$$\lambda := \frac{\delta_2}{Q_2} \qquad \text{and} \qquad f(x) := \frac{\delta_2}{Q_2}\Big(I - \frac{Q_1 x}{\delta_1}\Big).$$

and, upon recalling the representation of F as in (A.4), rearrange (A.12) such that

$$b(x) = f(x) + \lambda\mathbb{E}\Big[\int_0^\infty e^{-rt}\Big(Q_1 X_t^x + Q_2 Y_t^{b(x)} - rI\Big)\mathbb{1}_{\{Y_t^{b(x)} \geq b(X_t^x)\}}dt\Big]. \tag{A.13}$$

Let ζ be an auxiliary exponentially distributed random variable with parameter r that is independent of (W^X, W^Y). It follows that (A.13) can be reformulated as

$$b(x) = f(x) + \lambda \frac{1}{r} \mathbb{E}\Big[\Big(Q_1 X_\zeta^x + Q_2 Y_\zeta^{b(x)} - rI\Big) \mathbb{1}_{\{Y_\zeta^{b(x)} \geq b(X_\zeta^x)\}}\Big]. \tag{A.14}$$

The latter representation is useful, as it allows for an application of Monte-Carlo methods in order to estimate expectations. For $(x, y) \in \mathbb{R}_+^2$ and a function $b : \mathbb{R}_+ \to \mathbb{R}_+$, we define the operator

$$\Psi(x, y; b) := f(x) + \lambda \frac{1}{r} \mathbb{E}\Big[\Big(Q_1 X_\zeta^x + Q_2 Y_\zeta^y - rI\Big) \mathbb{1}_{\{Y_\zeta^y \geq b(X_\zeta^x)\}}\Big]. \tag{A.15}$$

It follows that the equation (A.14) rewrites as a fixed point problem

$$b(x) = \Psi(x, b(x); b), \quad x \in \mathbb{R}_+, \tag{A.16}$$

which we aim to solve by an iterative scheme. In order to do so, we define the sequence of boundaries

$$b^{(n)}(x) = \Psi(x, b^{(n-1)}(x); b^{(n-1)}), \quad x \in \mathbb{R}_+, \tag{A.17}$$

for $n \geq 1$ and, with regard to Proposition A.2, choose the initial boundary $b^{(0)}$ such that $b^{(0)}(0) = y^*$, $b^{(0)}(x^*) = 0$, $b^{(0)}(x^*)$ is the vertex of a parabola and $b^{(0)}(x) = 0$ for all $x \geq x^*$.

Moreover, for a given boundary $b^{(k)}$ we estimate the expectation in (A.15) by

$$\frac{1}{N} \sum_{i=1}^{N} \big(Q_1 X_{\zeta_i}^{i,x} + Q_2 Y_{\zeta_i}^{i,b^{(k)}(x)} - rI\big) \mathbb{1}_{\left\{Y_{\zeta_i}^{i,b^{(k)}(x)} \geq b^{(k)}(X_{\zeta_i}^{i,x})\right\}}, \tag{A.18}$$

where N is the total amount of implemented realizations of an exponential random variable with parameter r. Consequently, for each $i = 1, ..., N$, ζ_i denotes the value of time, while $X_{\zeta_i}^{i,x}$ and $Y_{\zeta_i}^{i,y}$ are the prices of the two products. Under the described procedure, the scheme (A.17) is then iterated until the variation between steps falls below a predetermined level.

Remark A.6. *In principle, the suggested numerical approach is suitable for a general class of optimal stopping problems for which an integral equation for the free boundary can be derived (see, e.g. Cai et al., 2022; Christensen and Salminen, 2018). If an educated initial guess regarding the shape of the free boundary is possible, the algorithm seems to converge fast to an approximate solution of the integral equation. Notice also that the suggested method does not necessarily rely on the function $F(x, \cdot)$ to be linear or invertible. As a matter of fact, for a general payoff function F one can (trivially) rewrite* (A.12) *as*

$$F(x, b(x)) + b(x) = b(x) + \mathbb{E}\Big[\int_0^\infty e^{-rt}\Big(Q_1 X_t^x + Q_2 Y_t^{b(x)} - rI\Big) \mathbb{1}_{\{Y_t^{b(x)} \geq b(X_t^x)\}} dt\Big], \tag{A.19}$$

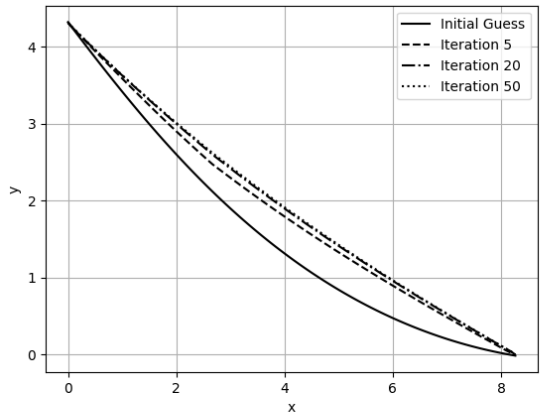


Figure A.1: Plot of different iterations of the numerical scheme

and then implement the iterative scheme

$$b^{(n)}(x) = \Psi(x, b^{(n-1)}(x), b^{(n-1)}), \quad x \in \mathbb{R},$$

where the operator now takes the slightly different form

$$\Psi(x, y; b) := y - F(x, y) + \frac{1}{r}\mathbb{E}\Big[\Big(Q_1 X^x_\zeta + Q_2 Y^y_\zeta - rI\Big)\mathbb{1}_{\{Y^y_\zeta \geq b(X^x_\zeta)\}}\Big].$$

Moreover, as we rely on an application of the Monte-Carlo method in order to evaluate the expected value on the right-hand side of (A.19)*, the proposed algorithm does not require the processes X and Y to have known densities ρ_1 and ρ_2, respectively. An Euler approximation of the dynamics of X and Y could indeed be used in order to simulate the random variable $(X^x_\zeta, Y^{b(x)}_\zeta)$.*

We end this section with a brief comparative statics analysis to illustrate the results of our numerical procedure. Unless otherwise specified, we fix the following parameters $r = 0.07$, $\mu_1 = 0.02$, $\mu_2 = 0.01$, $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $Q_1 = 5$, $Q_2 = 10$ and $I = 500$. To begin with, we illustrate the convergence of the iterative scheme. Here, we worked with a total of 300.000 realizations of the exponentially distributed random variable. In the example above, we observe that after roughly 20 iterations the derived boundary values resulting from the numerical procedure stabilize.

Next, we shortly discuss the results of the comparative statics analysis for the model established in Section A.2.
Regarding a shift in the volatility parameter σ_2, we observe that the boundary increases with larger values of σ_2. Notice that a larger volatility coefficient implies higher price fluctuation in our model. The price thus has larger distortions in the downward direction – but also

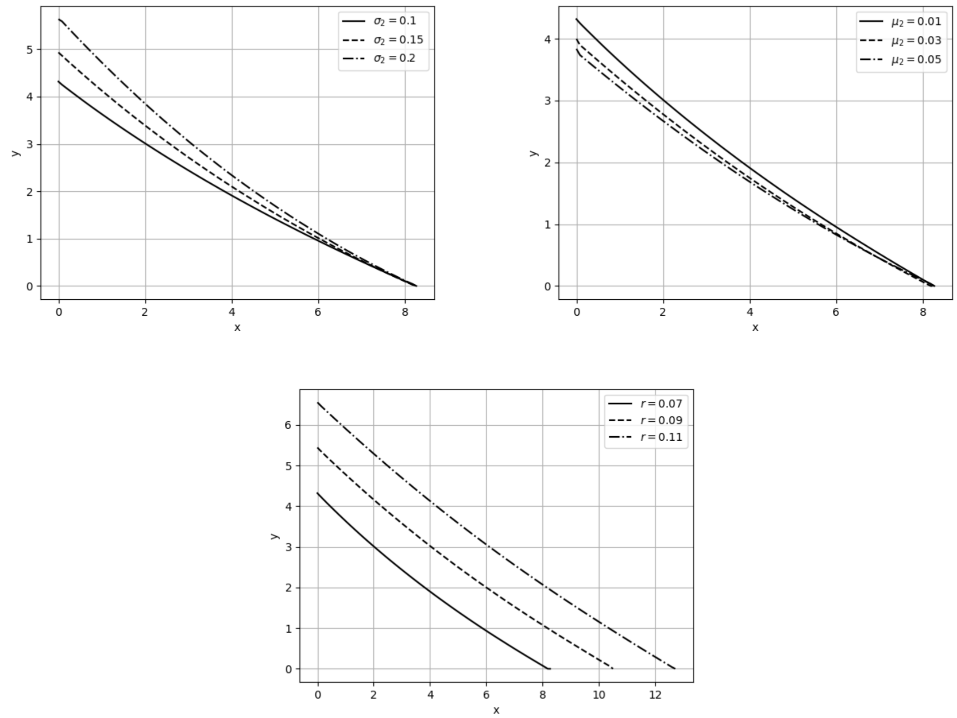


Figure A.2: Plot of the optimal stopping boundary b for different values of σ_2, μ_2 and r, respectively.

upwards. The firm reacts by waiting for higher prices to evolve. This effect also results in higher expected profits of the firm (see Dammann and Ferrari, 2022, Proposition 6.2). Notice that the threshold value x^*, that denotes the point at which $b(x^*) = 0$, does not change in figure A.2, as it depends exclusively on the parameters Q_1, r, μ_1 and σ_1. We can also study the sensitivity of the optimal boundary b with respect to the drift coefficient μ_2, and observe that b is decreasing in the latter parameter. We notice that a larger drift coefficient μ_2 implies higher expected prices of the second product on the market and, as a result, the value of the investment increases. Furthermore, we notice that the function F, which represents the value of exercising the investment option immediately, depends explicitly on μ_2. Notice that F increases for larger values of μ_2, which gives an incentive for the firm to invest earlier into the production plant. Consequently, the boundary decreases.
Regarding a change in the discount factor r, we again notice that the value to exercise immediately F depends explicitly on r. Since F decreases with r, the value of exercising the investment immediately decreases, so that the company prefers to delay the investment. As a result, the boundary b increases. Notice that, differently to what we can observe in the previous analysis, a change in r shifts the investment thresholds on both axes, as in fact r affects both x^* and y^*.

B Supplementary Material for Chapter 2

B.1 Proof of Proposition 2.6.9

The proof follows the lines of Section 4 in De Angelis (2020), suitably adapted to the present setting, and it is obtained through a series of intermediate results. Let $(x, z) \in \mathbb{R}^2$ be given and fixed and set

$$\sigma_* := \sigma_*(x,z) := \inf\{t \geq 0 : (X_t^x, Z_t^z) \in \mathcal{S}_3\}, \ \widehat{\sigma}_* := \widehat{\sigma}_*(x,z) := \inf\{t \geq 0 : (X_t^x, Z_t^z) \in \operatorname{int}(\mathcal{S}_3)\},$$

and observe that $\sigma_* = \tau_*$ $\mathbb{Q}$-a.s. on $\mathbb{R}^2 \setminus \partial\mathcal{C}_3$ due to the continuity of paths. It is crucial to show that this equality also holds for the boundary points $(x_0, z_0) \in \partial\mathcal{C}_3$. As it turns out, the cases i) $\mu_0 + \mu_1 \geq 0$ and ii) $\mu_0 + \mu_1 < 0$ should be treated in different fashions and the latter case exhibits some more technical difficulties than the first case. Let us start with case i), in which the needed result follows upon using the law of iterated logarithm.

Proposition B.1. *Assume that $\mu_0 + \mu_1 \geq 0$. Let $(x_n, z_n) \in \mathcal{C}_3$ be a sequence with $(x_n, z_n) \to (x_0, z_0) \in \partial\mathcal{C}_3$, such that $x_0 = c(z_0)$. We then have $\tau^*(x_n, z_n) \downarrow 0$ as well as $\widehat{\sigma}_*(x_n, z_n) \downarrow 0$ $\mathbb{Q}$-a.s.*

Proof. Fix $\omega \in \Omega$ and assume that $\limsup_{n\to\infty} \tau^*(x_n, z_n)(\omega) =: \delta > 0$. Hence, there exists a subsequence (still labelled by (x_n, z_n)) such that

$$X_t^{x_n}(\omega) < c(Z_t^{z_n}) \quad \forall n \in \mathbb{N}, \forall t \in [0, \delta/2], \tag{B.1}$$

which is equivalent to

$$x_n + \mu_0 t + \sigma B_t(\omega) < c\big(z_n - \frac{1}{2}(\mu_0 + \mu_1)t\big) \quad \forall n \in \mathbb{N}, \forall t \in [0, \delta/2].$$

Upon using that $z \mapsto c(z)$ is continuous, we let $n \to \infty$ and obtain

$$\sigma B_t(\omega) \leq c\big(z_0 - \frac{1}{2}(\mu_0 + \mu_1)t\big) - x_0 - \mu_0 t \leq c(z_0) - x_0 - \mu_0 t = -\mu_0 t \quad \forall t \in [0, \delta/2], \tag{B.2}$$

where the last inequality follows from $\mu_0 + \mu_1 \geq 0$ and Proposition 2.6.3. On the other hand, by the law of iterated logarithm, for all $\varepsilon > 0$, there exists a sequence $(t_n) \downarrow 0$ such that

$$B_{t_n} \geq (1-\varepsilon)\sqrt{2t_n \log\big(\log(1/t_n)\big)} \quad \forall n \in \mathbb{N}. \tag{B.3}$$

Combining (B.2) and (B.3) implies

$$\frac{1}{t}\sigma(1-\varepsilon)\sqrt{2t \log\big(\log(1/t)\big)} \leq -\mu_0,$$

but since $\sqrt{2t\log(\log(1/t))}/t \to \infty$ for $t \downarrow 0$, (B.1) can only happen on a $\mathbb{Q}$-null set. Thus, we have $\tau^*(x_n, z_n) \downarrow 0$ and by replacing the strict inequality in (B.1) by "$\leq$", we obtain that $\widehat{\sigma}_*(x_n, z_n) \downarrow 0$ as well. □

Notice that the proof of Proposition B.1 cannot be replicated for the case ii), in which $\mu_0 + \mu_1 < 0$, since the last inequality in (B.2) does not longer apply. As is turns out, in order to prove the same result for case ii), we have to take a longer route. The reason for this lies in the fact that the process (X, Z) is moving towards the right in the state space and hence – keeping in mind that the continuation region $\mathcal{C}_3$ of (2.6.14) lies below the increasing boundary c – could possibly evade from the stopping set. In the following, we show that this is not the case by adapting the procedure in of Section 4 in De Angelis (2020). As a first step, we state the following Lemma, whose proof follows the lines of A. Cox and Peskir (2015), Corollary 8, and is thus omitted for the sake of brevity.

Lemma B.2. *Assume that* $\mu_0 + \mu_1 < 0$ *and* $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$. *We have* $\mathbb{Q}[\sigma_* = \widehat{\sigma}_*] = 1$.

In the next step, we aim at proving *regularity* of the boundary points for the stopping set $\mathcal{S}_3$ in the sense of diffusions, that is, for $(x, z) \in \partial\mathcal{C}_3$ we have

$$\mathbb{Q}_{x,z}[\sigma_* > 0] = 0. \tag{B.4}$$

It is clear from Blumenthal's 0-1 law that if (B.4) does not hold, we have $\mathbb{Q}_{x,z}[\sigma_* > 0] = 1$. Due to the mentioned geometry of the problem, implying that the process (X, Z) could evade from the stopping set when $\mu_0 + \mu_1 < 0$, proving (B.4) is not a straightforward task, since we cannot apply an argument similar to the one on Proposition B.1. Instead, we establish (B.4) in two steps and begin by showing that the classical smooth-fit property holds at the free-boundary, i.e. continuity of $\widehat{v}_x(\cdot, z)$.

Lemma B.3. *Assume that* $\mu_0 + \mu_1 < 0$ *and* $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$. *For* $\widehat{v}$ *of* (2.6.4) *we have* $\widehat{v}_x(\cdot, z) \in C(\mathbb{R})$, *or, equivalently,* $\widehat{u}_x(\cdot, z) \in C(\mathbb{R})$ *for* $\widehat{u}$ *of* (2.6.11).

Proof. From (2.6.21) we obtain

$$\frac{1}{2}\sigma^2 \widehat{u}_{xx}(x, z) = r\widehat{u}(x, z) - \mu_0 \widehat{u}_x(x, z) + \frac{1}{2}(\mu_0 + \mu_1)\widehat{u}_z(x, z) - g(x, z),$$

for $(x, z) \in \mathcal{C}_3$, and due to (2.5.10) (which implies an analogous result for $\widehat{v}$) we deduce that for a bounded set B, we must have that $\widehat{u}_{xx}$ is bounded on the closure of $B \cap \mathcal{C}_3$. Moreover, we recall that $\widehat{u}_x \leq 0$ in $\mathcal{C}_3$, as verified in the proof of Proposition 2.6.5. Aiming for a contradiction we now assume that for $(x_0, z_0) \in \partial\mathcal{C}_3$, such that $x_0 = c(z_0)$, we have

$$\widehat{u}_x(x_0-, z_0) < -\delta_0, \tag{B.5}$$

for some $\delta_0 > 0$. We now take a bounded rectangular neighbourhood B of (x_0, z_0) and define the stopping time $\tau_B := \inf\{t > 0 : (X_t, Z_t) \notin B\}$. Notice that

$$\widehat{u}(x_0, z_0) \geq \mathbb{E}^{\mathbb{Q}}_{x_0,z_0}\left[e^{-r(\tau_B \wedge t)}\widehat{u}(X_{\tau_B \wedge t}, Z_{\tau_B \wedge t}) + \int_0^{\tau_B \wedge t} e^{-rs} g(X_s, Z_s) ds\right], \tag{B.6}$$

from the supermartingale property of $(e^{-rt}\widehat{v}(X_t, Z_t))_t$. Recall Lemma 2.6.7 and since $t \mapsto Z_{\tau_B \wedge t}$ is increasing, we have $\widehat{u}(X^{x_0}_{\tau_B\wedge t}, Z^{z_0}_{\tau_B\wedge t}) \geq \widehat{u}(X^{x_0}_{\tau_B\wedge t}, z_0)$ $\mathbb{Q}$-a.s. Moreover, since the integrand on the right-hand side of (B.6) is bounded on B, we obtain

$$\widehat{u}(x_0, z_0) \geq \mathbb{E}^{\mathbb{Q}}_{x_0,z_0}\Big[e^{-r(\tau_B\wedge t)}\widehat{u}(X_{\tau_B\wedge t}, z_0) - c_B(\tau_B \wedge t)\Big], \tag{B.7}$$

where c_B is a constant depending on B. Due to the previously discussed local boundedness of $\widehat{u}_{xx}$, we can apply Itô-Tanaka's formula to the first term in the expectation of (B.7). Let $\mathcal{L}_X := \frac{1}{2}\sigma^2\partial_{xx} + \mu_0\partial_x$ and denote the local time of X at x_0 by L^{x_0}. Moreover, noticing that $\widehat{u}_{xx}(\cdot, z_0) = 0$ for $x > x_0$, we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_{x_0,z_0}\Big[e^{-r(\tau_B\wedge t)}\widehat{u}(X_{\tau_B\wedge t}, z_0)\Big] = \widehat{u}(x_0, z_0) + \mathbb{E}^{\mathbb{Q}}_{x_0,z_0}\bigg[\int_0^{\tau_B\wedge t} e^{-rs}(\mathcal{L}_X - r)\widehat{u}(X_s, z_0)\mathbb{1}_{\{X_s\neq x_0\}}ds\bigg]\\ - \mathbb{E}^{\mathbb{Q}}_{x_0,z_0}\bigg[\int_0^{\tau_B\wedge t} e^{-rs}\widehat{u}_x(x_0-, z_0)dL^{x_0}_s\bigg],\end{aligned}$$

and, combining this with (B.7), as well as noticing that $(\mathcal{L}_X - r)\widehat{u}(X_s, Z_s)$ is bounded on B, we find

$$\begin{aligned}0 &\geq \mathbb{E}^{\mathbb{Q}}_{x_0,z_0}\bigg[\int_0^{\tau_B\wedge t} e^{-rs}(\mathcal{L}_X - r)\widehat{u}(X_s, z_0)\mathbb{1}_{\{X_s\neq x_0\}}ds - c_B(\tau_B\wedge t)\bigg]\\ &\quad - \mathbb{E}^{\mathbb{Q}}_{x_0,z_0}\bigg[\int_0^{\tau_B\wedge t} e^{-rs}\widehat{u}_x(x_0-, z_0)dL^{x_0}_s\bigg]\\ &\geq \delta_0 e^{-rt}\mathbb{E}^{\mathbb{Q}}_{x_0,z_0}[L^{x_0}_{\tau_B\wedge t}] - c_B\mathbb{E}^{\mathbb{Q}}_{x_0,z_0}[\tau_B\wedge t],\end{aligned}$$

where we used the assumption (B.5) in the last inequality. By rearranging terms, this is equivalent to $c_B\mathbb{E}^{\mathbb{Q}}_{x_0,z_0}[\tau_B\wedge t] \geq \delta_0 e^{-rt}\mathbb{E}^{\mathbb{Q}}_{x_0,z_0}[L^{x_0}_{\tau_B\wedge t}]$, and $\mathbb{E}^{\mathbb{Q}}_{x_0,z_0}[\tau_B\wedge t] \approx t$ while $\mathbb{E}^{\mathbb{Q}}_{x_0,z_0}[L^{x_0}_{\tau_B\wedge t}] \approx \sqrt{t}$ (see, e.g., Peskir, 2019, Lemma 15), we obtain the desired contradiction. Hence, $\widehat{u}_x(\cdot, z) \in C(\mathbb{R})$. ☐

We can now state the regularity of the boundary points.

Proposition B.4. *Assume that* $\mu_0 + \mu_1 < 0$ *and* $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$. *All points* $(x, z) \in \partial\mathcal{C}_3$ *are regular, i.e. we have* $\mathbb{Q}_{x,z}[\sigma_* > 0] = 0$.

Proof. We argue by contradiction and show that if $\mathbb{Q}_{x_0,z_0}[\sigma_* > 0] = 1$ for a boundary point $(x_0, z_0) \in \partial\mathcal{C}_3$ it follows that $\widehat{u}_x(x_0-, z_0) < 0$, which contradicts Lemma B.3. As a first step, we establish an upper bound for $\widehat{u}_x$. Fix $(x, z) \in \mathcal{C}_3$ such that $x > \widetilde{x}$, with the latter given by (2.6.15). Define the stopping time $\tau_\varepsilon := \tau_\varepsilon(x) := \inf\{t \geq 0 : X^x_t = \widetilde{x} + \varepsilon\}$ and observe that – by strong Markov property – we have

$$\widehat{u}(x, z) = \sup_\tau \mathbb{E}^{\mathbb{Q}}_{x,z}\bigg[e^{-r\tau_\varepsilon}\widehat{u}(\widetilde{x} + \varepsilon, Z_{\tau_\varepsilon})\mathbb{1}_{\{\tau>\tau_\varepsilon\}} + \int_0^{\tau_\varepsilon\wedge\tau} e^{-rt}g(X_t, Z_t)dt\bigg]. \tag{B.8}$$

Moreover, we let $\widetilde{\tau} := \widetilde{\tau}(x) := \inf\{t > 0 : X_t^x = \widetilde{x}\}$, and for $\tau' := \tau^*(x,z)$ we obtain

$$\widehat{u}(x-\varepsilon, z) \geq \mathbb{E}^{\mathbb{Q}}_{x-\varepsilon,z}\Big[e^{-r\widetilde{\tau}(x-\varepsilon)}\widehat{u}(\widetilde{x}, Z_{\widetilde{\tau}(x-\varepsilon)})\mathbb{1}_{\{\tau' > \widetilde{\tau}(x-\varepsilon)\}} + \int_0^{\tau' \wedge \widetilde{\tau}(x-\varepsilon)} e^{-rt} g(X_t, Z_t)dt\Big]. \quad \text{(B.9)}$$

Notice that $\tau_\varepsilon(x) = \widetilde{\tau}(x-\varepsilon)$. Hence, subtracting (B.9) from (B.8) yields

$$\begin{aligned}\widehat{u}(x,z) - \widehat{u}(x-\varepsilon,z) \leq\ & \mathbb{E}^{\mathbb{Q}}\Big[e^{-r\tau_\varepsilon}\big(\widehat{u}(\widetilde{x}+\varepsilon, Z^z_{\tau_\varepsilon}) - \widehat{u}(\widetilde{x}, Z^z_{\tau_\varepsilon})\big)\mathbb{1}_{\{\tau' > \tau_\varepsilon\}}\Big] \\ & + \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_\varepsilon \wedge \tau'} e^{-rt}\big(g(X_t^x, Z_t^z) - g(X_t^{x-\varepsilon}, Z_t^z)\big)dt\Big].\end{aligned}$$

Since $(\widetilde{x}+\varepsilon, Z^z_{\tau_\varepsilon}) \in \mathcal{C}_3$ on $\{\tau' > \tau_\varepsilon\}$ and $\widehat{u}_x \leq 0$ in $\mathcal{C}_3$ (see Proposition 2.6.5), we must have

$$\widehat{u}(\widetilde{x}, Z^z_{\tau_\varepsilon}) \geq \widehat{u}(\widetilde{x}+\varepsilon, Z^z_{\tau_\varepsilon}),$$

and we obtain

$$\widehat{u}(x,z) - \widehat{u}(x-\varepsilon,z) \leq \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_\varepsilon \wedge \tau'} e^{-rt}\big(g(X_t^x, Z_t^z) - g(X_t^{x-\varepsilon}, Z_t^z)\big)dt\Big].$$

If we now divide by $\varepsilon > 0$ and let $\varepsilon \downarrow 0$, we obtain (since $\tau_\varepsilon \downarrow \widetilde{\tau}$ and $\tau' = \tau^*(x,z)$)

$$\widehat{u}_x(x,z) \leq \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\widetilde{\tau} \wedge \tau'} e^{-rt} g_x(X_t^x, Z_t)dt\Big].$$

In the next step, we assume by contradiction that there exists $(x_0, z_0) \in \partial\mathcal{C}_3$ with $\mathbb{Q}_{x_0,z_0}[\sigma_* > 0] = 1$ and take an increasing sequence $x_n \uparrow x_0$ such that $x_n > \widetilde{x}$ for all $n \in \mathbb{N}$, which is possible due to Assumption 2.4.1. Let $\tau_n := \tau^*(x_n, z_n)$ and notice that $\tau_n = \sigma_n := \sigma_*(x_n, z_0)$ for all $n \in \mathbb{N}$ due to continuity of paths. Furthermore, σ_n decreases in n and $\sigma_n \geq \sigma_* := \sigma_*(x_0, z_0)$, since $x \mapsto X_t^x$ is increasing. Set $\widetilde{\tau}^n := \widetilde{\tau}(x_n)$ and notice that $\widetilde{\tau}^n \uparrow \widetilde{\tau}$. Moreover, we let $\sigma^\infty := \lim_{n\to\infty} \sigma_n$ and have

$$\sigma^\infty \wedge \widetilde{\tau} = \lim_{n\to\infty}(\sigma_n \wedge \widetilde{\tau}^n) \geq \sigma_* \wedge \widetilde{\tau} \quad \mathbb{Q}\text{-a.s.}$$

We then obtain

$$\begin{aligned}\widehat{u}_x(x_0-, z_0) = \lim_{n\to\infty} \widehat{u}_x(x_n, z_0) &\leq \lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\widetilde{\tau} \wedge \sigma_n} e^{-rt} g_x(X_t^{x_n}, Z_t^{z_0})dt\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\widetilde{\tau} \wedge \sigma^\infty} e^{-rt} g_x(X_t^{x_0}, Z_t^{z_0})dt\Big] < 0,\end{aligned}$$

where we used $x_0 > \widetilde{x}$ as well as $\widetilde{\tau} \wedge \sigma^\infty > 0$ due to our assumption $\mathbb{Q}_{x_0,z_0}[\sigma^\infty \geq \sigma^* > 0] = 1$. But this contradicts Lemma B.3 and the claim follows. □

As a corollary of Lemma B.2 and Proposition B.4 we obtain

Corollary B.5. *Assume that* $\mu_0 + \mu_1 < 0$ *and* $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$. *Then, for all* $(x, z) \in \mathbb{R}^2$ *we have*

$$\mathbb{Q}_{x,z}[\tau^* = \sigma_* = \widehat{\sigma}_*] = 1.$$

This result allows us to state the continuity result of the optimal stopping time with respect to the initial data.

Lemma B.6. *Assume that* $\mu_0 + \mu_1 < 0$ *and* $r > \frac{\gamma}{2\sigma}|\mu_0 + \mu_1|$. *We have* $\lim_{n\to\infty} \tau^*(x_n, z_n) = \tau^*(x, z)$ *for any* $(x, z) \in \mathbb{R}^2$ *and any sequence* $(x_n, z_n) \to (x, z)$. *In particular, if* $(x, z) \in \partial\mathcal{C}_3$, *the limit is zero.*

Proof. Let $(x, z) \in \mathbb{R}^2$ and denote $\tau_n := \tau^*(x_n, z_n)$ as well as $\tau := \tau^*(x, z)$ for simplicity. In order to show lower-semicontinuity, we fix $\omega \in \Omega$ ouside of a null-set. For $\tau(\omega) = 0$ we are finished and thus assume $\tau(\omega) > \delta > 0$. Due to Proposition 2.6.8 there exists $k_{\delta,\omega} > 0$ such that

$$c\big(Z_t(\omega)\big) - X_t(\omega) > k_{\delta,\omega},$$

for all $t \in [0, \delta]$. The map $(t, x, z) \mapsto c(Z^z_t(\omega)) - X^x_t(\omega)$ is uniformly continuous on any compact $[0, \delta] \times K$, hence we can find $N_\omega \geq 1$ such that for all $n \geq N_\omega$ and $t \in [0, \delta]$

$$c\big(Z^{z_n}_t(\omega)\big) - X^{x_n}_t(\omega) > k_{\delta,\omega},$$

and therefore $\liminf_n \tau_n(\omega) \geq \delta$. Since ω and δ were arbitrary, we obtain $\liminf_n \tau_n \geq \tau$ $\mathbb{Q}$-a.s. and thus lower-semicontinuity. By employing similar arguments we can show $\limsup_n \widehat{\sigma}_n \leq \widehat{\sigma}$ $\mathbb{Q}$-a.s. and the claim thus follows together with Corollary B.5. □

Before we finally state the proof of Proposition 2.6.9, we can derive a probabilistic representation of v_x by employing arguments similar to those employed in the proof of Lemma 2.6.4.

Lemma B.7. *For all* $(x, z) \in \mathbb{R}^2 \setminus \partial\mathcal{C}_3$, *we have*

$$\widehat{v}_x(x, z) = \mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-r\tau^*}\Big(e^{X_{\tau^*}}\big(1 + e^{\frac{\gamma}{\sigma}(X_{\tau^*}+Z_{\tau^*})}\big) + \frac{\gamma}{\sigma}\big(e^{X_{\tau^*}} - \kappa\big)e^{\frac{\gamma}{\sigma}(X_{\tau^*}+Z_{\tau^*})}\Big)\mathbb{1}_{\{\tau^*<\infty\}}\Big].$$

We are therefore ready to prove Proposition 2.6.9.

Proof of Proposition 2.6.9. The first statement trivially holds true for $(x, z) \in \mathrm{int}(\mathcal{S}_3)$ and $(x, z) \in \mathcal{C}_3$, due to the result in Lemma 2.6.2. It thus remains to prove that $\nabla_{x,z}\widehat{v}$ is continuous across the boundary $\partial\mathcal{C}_3$. Let $(x_0, z_0) \in \partial\mathcal{C}_3$ and take a sequence $(x_n, z_n) \to (x_0, z_0)$ with

$\tau_n := \tau^*(x_n, z_n)$. For a fixed $t > 0$, we notice $(X_t, Z_t) \in \mathcal{C}_3$ on $\{\tau_n > t\}$ and thus, upon using tower and Markov property, we obtain

$$\widehat{v}_x(x_n, z_n) = \mathbb{E}^{\mathbb{Q}}_{x_n,z_n}\Big[e^{-r\tau_n}\Big(e^{X_{\tau_n}}\big(1 + e^{\frac{\gamma}{\sigma}(X_{\tau_n}+Z_{\tau_n})}\big) + \frac{\gamma}{\sigma}\big(e^{X_{\tau_n}} - \kappa\big)e^{\frac{\gamma}{\sigma}(X_{\tau_n}+Z_{\tau_n})}\Big)\mathbb{1}_{\{\tau_n \le t\}}\Big] + \mathbb{E}^{\mathbb{Q}}_{x_n,z_n}\Big[e^{-rt}\widehat{v}_x(X_t, Z_t)\mathbb{1}_{\{\tau_n > t\}}\Big].$$

Due to Assumption 2.4.1 we can invoke dominated convergence as well as Lemma B.6 to obtain

$$\lim_{n\to\infty} \widehat{v}_x(x_n, z_n) = e^{x_0}\big(1 + e^{\frac{\gamma}{\sigma}(x_0+z_0)}\big) + \frac{\gamma}{\sigma}\big(e^{x_0} - \kappa\big)e^{\frac{\gamma}{\sigma}(x_0+z_0)} = \frac{\partial}{\partial x}\Big((e^x - \kappa)(1 + e^{\frac{\gamma}{\sigma}(x+z)})\Big)\Big|_{x_0,z_0},$$

and hence, the continuity of $\widehat{v}_x$ across the optimal boundary. The continuity of $\widehat{v}_z$ across the free boundary follows similarly. For the last claim we observe that Lemma 2.6.2 implies

$$\frac{1}{2}\sigma^2\widehat{v}_{xx}(x,z) = r\widehat{v}(x,z) - \mu_0\widehat{v}_x(x,z) + \frac{1}{2}(\mu_0 + \mu_1)\widehat{v}_z(x,z), \tag{B.10}$$

for all $(x, z) \in \mathcal{C}_3$. But the right-hand side of (B.10) only involves functions which are continuous on $\mathbb{R}^2$, hence we deduce that $\widehat{v}_{xx}$ admits a continuous extension on $\overline{\mathcal{C}}_3$ and is therefore bounded therein. It follows that $\widehat{v}_x(\cdot, z)$ is locally Lipschitz continuous on $\overline{\mathcal{C}}_3$, with a Lipschitz constant $K(z)$ that is locally bounded on $\mathbb{R}$. Now, because $\widehat{v}_x(\cdot, z)$ is infinitely many times continuously differentiable in the stopping region $\mathcal{S}_3$ (and hence locally bounded therein as well), we conclude that $\widehat{v}_{xx} \in L^\infty_{\text{loc}}(\mathbb{R}^2)$. $\square$

B.2 Proof of Proposition 2.6.10

Let $R > 0$ and define $\tau_R := \inf\{t \ge 0 : |X_t| \ge R \text{ or } |Z_t| \ge R\}$. Since $\widehat{v} \in C^1(\mathbb{R}^2)$ and $\widehat{v}_{xx} \in L^\infty_{\text{loc}}(\mathbb{R}^2)$, we can apply a weak version of Ito's Lemma (see, e.g., Bensoussan and Lions Bensoussan and Lions, 2011, Lemma 8.1 and Th. 8.5) up to the stopping time $\tau_R \wedge T$ for some $T > 0$, which results in

$$\widehat{v}(x,z) = \mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-r(\tau_R\wedge T)}\widehat{v}(X_{\tau_R\wedge T}, Z_{\tau_R\wedge T}) - \int_0^{\tau_R\wedge T} e^{-rs}(\mathcal{L}_{X,Z} - r)\widehat{v}(X_s, Z_s)ds\Big]. \tag{B.11}$$

The right-hand-side of (B.11) is well-defined, because Z is deterministic, X has an absolutely continuous transition density and $\mathcal{L}_{X,Z}\widehat{v}$ is defined up to a set of zero Lebesgue measure. Since $\widehat{v}$ solves the free-boundary problem (2.6.28), we have

$$(\mathcal{L}_{X,Z} - r)\widehat{v}(x,z) = (\mathcal{L}_{X,Z} - r)\widehat{v}(x,z)\mathbb{1}_{\{x<c(z)\}} + (\mathcal{L}_{X,Z} - r)\widehat{v}(x,z)\mathbb{1}_{\{x\ge c(z)\}} = g(x,z)\mathbb{1}_{\{x\ge c(z)\}},$$

for almost all $(x, z) \in \mathbb{R}^2$. Using again that the transition density of X is absolutely continuous with respect to the Lebesgue measure, equation (B.11) becomes

$$\widehat{v}(x,z) = \mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-r(\tau_R\wedge T)}\widehat{v}(X_{\tau_R\wedge T}, Z_{\tau_R\wedge T}) - \int_0^{\tau_R\wedge T} e^{-rs}g(X_s, Z_s)\mathbb{1}_{\{x\ge c(z)\}}ds\Big].$$

Now, upon employing a change of measure as in Section 2.5, we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-r(\tau_R\wedge T)}|\widehat{v}(X_{\tau_R\wedge T}, Z_{\tau_R\wedge T})|\Big] &= \mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-r(\tau_R\wedge T)}\big|\overline{v}\big(X_{\tau_R\wedge T}, e^{\frac{\gamma}{\sigma}(X_{\tau_R\wedge T}+Z_{\tau_R\wedge T})}\big)\big|\Big]\\ &\le K_1\mathbb{E}^{\mathbb{Q}}_{x,\exp(\frac{\gamma}{\sigma}(x+z)}\Big[e^{-r(\tau_R\wedge T)}e^{X_{\tau_R\wedge T}}(1+\Phi_{\tau_R\wedge T})\Big]\\ &= K_1\big(1+e^{\frac{\gamma}{\sigma}(x+z)}\big)\mathbb{E}_{x,\pi}\Big[e^{-r(\tau_R\wedge T)}e^{X_{\tau_R\wedge T}}\Big], \end{aligned} \tag{B.12}$$

where $\pi = e^{\frac{\gamma}{\sigma}(x+z)}/(1+e^{\frac{\gamma}{\sigma}(x+z)})$. Due to Assumption 2.4.1, it is easy to verify that taking limits in (B.12) yields

$$\lim_{T\uparrow\infty}\lim_{R\uparrow\infty}\mathbb{E}^{\mathbb{Q}}_{x,z}\Big[e^{-r(\tau_R\wedge T)}\widehat{v}(X_{\tau_R\wedge T}, Z_{\tau_R\wedge T})\Big] = 0. \tag{B.13}$$

Furthermore,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}_{x,z}\bigg[\int_0^{\tau_R\wedge T} e^{-rs}g(X_s,Z_s)\mathbb{1}_{\{x\ge c(z)\}}ds\bigg] &\le \mathbb{E}^{\mathbb{Q}}_{x,z}\bigg[\int_0^{\infty} e^{-rs}|g(X_s,Z_s)|ds\bigg]\\ \le \mathbb{E}^{\mathbb{Q}}_{x,\exp(\frac{\gamma}{\sigma}(x+z))}&\bigg[\int_0^\infty e^{-rs}\Big(e^{X_s}\big(r-\frac{1}{2}\sigma^2-\mu_0\big)+rk+\Phi_s\big(e^{X_s}(r-\frac{1}{2}\sigma^2-\mu_1)+rk\big)\Big)ds\bigg]\\ \le \mathbb{E}^{\mathbb{Q}}_{x,\exp(\frac{\gamma}{\sigma}(x+z))}&\bigg[\int_0^\infty e^{-rs}\Big(e^{X_s}\big(r-\frac{1}{2}\sigma^2-\mu_0\big)+rk\Big)ds\bigg]\\ &+\big(1+e^{\frac{\gamma}{\sigma}(x+z)}\big)\mathbb{E}_{x,\pi}\bigg[\int_0^\infty e^{-rs}\Big(e^{X_s}\big(r-\frac{1}{2}\sigma^2-\mu_1\big)+rk\Big)ds\bigg]<\infty,\end{aligned} \tag{B.14}$$

where $\pi = e^{\frac{\gamma}{\sigma}(x+z)}/(1+e^{\frac{\gamma}{\sigma}(x+z)})$ and the last inequality follows again from Assumption 2.4.1. Hence, given the finiteness of the expectation in (B.14), we can apply dominated convergence theorem in order to interchange expectation and limits as $R\uparrow\infty$ and $T\uparrow\infty$. Combining this result with (B.13) gives (2.6.29), which completes our proof. □

C Supplementary Material for Chapter 3

C.1 Proof of Theorem 3.3.1

We derive the result in a number of steps.

Step 1. We begin with solving the free-boundary problem (3.3.6), by constructing a solution to the ordinary differential equation and imposing the boundary conditions, to obtain a *candidate* value function

$$\overline{U}_1(x) := \begin{cases}\overline{U}_1(\overline{a}) - c_2(\overline{a}-x), & 0<x\le\overline{a},\\ \overline{D}_1(\overline{a})x^{\delta_2} + \frac{1}{2(\rho-2(r-g)-\sigma^2)}x^2, & \overline{a}<x,\end{cases} \tag{C.1}$$

with $\overline{D}_1(a)$ and $\overline{a}$ as in (3.3.8).

Step 2. We then aim at verifying that the function $\overline{U}_1$ of (C.1) solves the free-boundary problem (3.3.6) and satisfies the HJB equation (3.3.4). Notice that, in view of the construction of $\overline{U}_1$, it remains to check whether

$$(i)\ (\mathcal{L}-\rho)\overline{U}_1(x) \geq -\frac{1}{2}x^2 \quad \text{for } x \in (0,\overline{a}) \qquad \text{and} \qquad (ii)\ \overline{U}_1'(x) \geq c_2 \quad \text{for } x \geq \overline{a}.$$

Proof of (i). We firstly notice that by construction $(\mathcal{L}-\rho)\overline{U}_1(\overline{a}) = -\frac{1}{2}\overline{a}^2$. We then observe that $x \mapsto (\mathcal{L}-\rho)\overline{U}_1(x)$ decreases with slope $-c_2(\rho-(r-g))$, while $x \mapsto -\frac{1}{2}x^2$ decreases with slope $-x$. To conclude (i), it is thus sufficient to prove that $c_2(\rho-(r-g)) > x$ for all $x \in (0,\overline{a})$, or equivalently that $c_2(\rho-(r-g)) > \overline{a}$. The latter follows straightforwardly from the definition (3.3.8) of $\overline{a}$ and $\delta_2 < 0$.

Proof of (ii). We then show that $x \mapsto \overline{U}_1'(x)$ is increasing for $x \geq \overline{a}$, by computing

$$\overline{U}_1''(x) = \frac{1}{\rho-2(r-g)-\sigma^2}\left(1-\left(\frac{x}{\overline{a}}\right)^{\delta_2-1}\right) > 0,$$

where the latter inequality follows from (3.3.2). Hence, given that $\overline{U}_1'(\overline{a}) = c_2$ by construction, we conclude that (ii) holds true.

Step 3. Finally, we must verify that the obtained solution $\overline{U}_1$ of the HJB equation (3.3.4) identifies with the value function $\overline{V}_1$ of (3.3.3). The proof is similar to the one of Theorem 3.3.3 given in Appendix C.3 (*Steps 2–4*), thus it is omitted for brevity. □

C.2 Proof of Lemma 3.3.2

We prove each part separately.

Proof of (i). Regarding the existence and uniqueness of a solution $a(b) \in (0,b)$ solving (3.3.14), we straightforwardly calculate

$$\lim_{a \downarrow 0} F(a,b) = -\infty \quad \text{and} \quad F(b,b) = (\delta_1-\delta_2)(c_1-c_2)(\rho-2(r-g)-\sigma^2) > 0,$$

where the latter inequality follows from (3.3.2). Then, the first derivative of $F(a,b)$ with respect to a is given by

$$\begin{aligned}&\frac{\partial}{\partial a}F(a,b)\\ &\quad = a^{-1}\Big[\Big(\frac{b}{a}\Big)^{\delta_1-1} - \Big(\frac{b}{a}\Big)^{\delta_2-1}\Big]\big[c_2(\delta_1-1)(1-\delta_2)(\rho-2(r-g)-\sigma^2)-(\delta_1-2)(2-\delta_2)a\big],\end{aligned}$$

which implies

$$\frac{\partial}{\partial a}F(a,b) = \begin{cases} > 0, & 0 < a < \widetilde{a} \wedge b, \\ < 0, & \widetilde{a} \wedge b < a < b, \end{cases} \quad \text{and} \quad \frac{\partial}{\partial a}F(b,b) = 0, \tag{C.2}$$

for $\widetilde{a}$ defined in (3.3.16); note that, the positivity of $\widetilde{a}$ follows from (3.3.2). As a by-product from the above, F crosses zero only once on $(0, b)$ and we can further conclude that $0 < a(b) < \widetilde{a} \wedge b$ and $\frac{\partial}{\partial a}F(a(b), b) > 0$.

Proof of (ii). Regarding the monotonicity results for $a(\cdot)$, we first derive the following partial derivatives

$$\begin{aligned} b\frac{\partial}{\partial b}F(a,b) &= (\delta_1 - 1)[(2-\delta_2)a - c_2(1-\delta_2)(\rho - 2(r-g) - \sigma^2)]\Big(\frac{b}{a}\Big)^{\delta_1 - 1} \\ &\quad + (\delta_2 - 1)[(\delta_1 - 2)a - c_2(\delta_1 - 1)(\rho - 2(r-g) - \sigma^2)]\Big(\frac{b}{a}\Big)^{\delta_2 - 1} - (\delta_1 - \delta_2)b \end{aligned} \tag{C.3}$$

and

$$\begin{aligned} b^2\frac{\partial^2}{\partial b^2}F(a,b) &= (\delta_1 - 1)(\delta_1 - 2)\big[(2-\delta_2)a - c_2(\rho - 2(r-g) - \sigma^2)(1-\delta_2)\big]\Big(\frac{b}{a}\Big)^{\delta_1 - 1} \\ &\quad + (1-\delta_2)(2-\delta_2)\big[(\delta_1 - 2)a - c_2(\rho - 2(r-g) - \sigma^2)(\delta_1 - 1)\big]\Big(\frac{b}{a}\Big)^{\delta_2 - 1}. \end{aligned} \tag{C.4}$$

Furthermore, given that $\frac{\partial}{\partial a}F(a(b), b) > 0$ at the value $a(b)$ which satisfies (3.3.14) due to part (i), we can obtain the monotonicity of $a(b)$ on $(0, \infty)$ through

$$a'(b) = -\frac{\frac{\partial}{\partial b}F(a(b), b)}{\frac{\partial}{\partial a}F(a(b), b)} \geq 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial b}F(a(b), b) \leq 0. \tag{C.5}$$

In the following, we fix $b \in \mathbb{R}_+$ and consider the uniquely defined $a(b) \in (0, \widetilde{a} \wedge b)$ given by the solution to (3.3.14). We distinguish two cases depending on the location of the fixed $a(b)$ relative to $\overline{a}$ defined in (3.3.8).

Case (a): $a(b) \leq \overline{a}$. Given that $\overline{a} < \widetilde{a}$ due to the assumption in (3.3.2) and the definitions (3.3.8) of $\overline{a}$ and (3.3.16) of $\widetilde{a}$, we observe that

$$\frac{\partial^2}{\partial b^2}F(a(b), x) < 0, \text{ for all } x > a(b) \ \Rightarrow \ x \mapsto \frac{\partial}{\partial b}F(a(b), x) \text{ is strictly decreasing on } (a(b), \infty).$$

Since $\frac{\partial}{\partial b}F(a(b), a(b)) = 0$ due to (C.3), we have $\frac{\partial}{\partial b}F(a(b), x) < 0$ for all $x > a(b)$. Combining this with the fact that $b > a(b)$, we conclude from (C.5) that $a'(b) > 0$ for all $b \in \mathbb{R}_+$ s.t. $a(b) \leq \overline{a}$. This yields that

$$a(\cdot) \text{ is increasing on } (0, \overline{b}], \qquad \text{where } \overline{b} \text{ is such that } a(\overline{b}) = \overline{a}. \tag{C.6}$$

Also, we observe that $a'(\bar{b}) > 0$ and $a(b) > \bar{a}$ for all $b > \bar{b}$.

Case (b): $a(b) > \bar{a}$. We firstly note from Case (a) that this is realised when $b > \bar{b}$. We observe that $\frac{\partial^2}{\partial b^2} F(a(b), x) \le 0$ if and only if $x \le \widetilde{x}$, where

$$\widetilde{x} := \Big(\frac{(2-\delta_2)(1-\delta_2)(c_2(\delta_1-1)(\rho-2(r-g)-\sigma^2)-(\delta_1-2)a(b))a(b)^{\delta_1-\delta_2}}{(\delta_1-2)(\delta_1-1)((2-\delta_2)a(b)-c_2(1-\delta_2)(\rho-2(r-g)-\sigma^2)} \Big)^{\frac{1}{\delta_1-\delta_2}} > a(b),$$

which is well-defined since $a(b) \in (\bar{a}, \widetilde{a})$. To show the inequality via contradiction, assume that $\widetilde{x} \le a(b)$. Then, $\frac{\partial^2}{\partial b^2} F(a(b), x) \ge 0$ and hence $x \mapsto \frac{\partial}{\partial b} F(a(b), x)$ is increasing for all $x \ge a(b)$. But since $\frac{\partial}{\partial b} F(a(b), a(b)) = 0$, it would follow that $x \mapsto F(a(b), x)$ is increasing on $(a(b), \infty)$, which is a contradiction to $F(a(b), b) = 0$, given that $a(b) < b$. Therefore,

$$\frac{\partial^2}{\partial b^2} F(a(b), x) \begin{cases} \le 0, & a(b) \le x \le \widetilde{x}, \\ \ge 0, & x \ge \widetilde{x} \end{cases} \quad \Rightarrow \quad x \mapsto \frac{\partial}{\partial b} F(a(b), x) \text{ is } \begin{cases} \text{decreasing on } (a(b), \widetilde{x}), \\ \text{increasing on } (\widetilde{x}, \infty). \end{cases}$$

Combining this with the fact that $\frac{\partial}{\partial b} F(a(b), a(b)) = 0$ and $\lim_{x\to\infty} \frac{\partial}{\partial b} F(a(b), x) = +\infty$, we conclude that $\frac{\partial}{\partial b} F(a(b), x) = 0$ admits a unique solution on $(a(b), \infty)$, denoted by $x_m(b) \in (\widetilde{x}, \infty)$. Hence,

$$x \mapsto \frac{\partial}{\partial b} F(a(b), x) \begin{cases} < 0, & x \in (a(b), x_m(b)), \\ > 0, & x \in (x_m(b), \infty) \end{cases}$$
$$\text{and thus} \quad x \mapsto F(a(b), x) \text{ is } \begin{cases} \text{decreasing on } (a(b), x_m(b)), \\ \text{increasing on } (x_m(b), \infty). \end{cases}$$

Given that $F(a(b), a(b)) > 0$ (see the first part of the proof) and $\lim_{x\to\infty} F(a(b), x) = +\infty$, we conclude that there exist at most two solutions to $F(a(b), x) = 0$ and due to (C.5) that $a'(b)$ changes sign once. This implies – in view of the conclusion $a'(\bar{b}) > 0$ in Case (a) – that $a(\cdot)$ is either increasing on the whole $(\bar{b}, \infty)$, or it is increasing only on $(\bar{b}, \widehat{b})$ and then decreasing on $(\widehat{b}, \infty)$, where $\widehat{b} \in (\bar{b}, \infty)$ would be satisfying

$$a'(\widehat{b}) = 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial b} F(a(\widehat{b}), \widehat{b}) = 0 \quad \Leftrightarrow \quad \widehat{b} = x_m(\widehat{b}).$$

In order to show that such a $\widehat{b}$ always exists, we study the system of equations $F(\widehat{a}, \widehat{b}) = 0 = \frac{\partial}{\partial b} F(\widehat{a}, \widehat{b})$ (cf. equations above), which is equivalent to

$$\begin{cases} J_{1,2}(\widehat{a}) = J_{1,1}(\widehat{b}), \\ J_{2,2}(\widehat{a}) = J_{2,1}(\widehat{b}), \end{cases} \qquad \text{where } \; J_{i,j}(x) := \frac{(\delta_i - 2)x - c_j(\delta_i - 1)(\rho - 2(r-g) - \sigma^2)}{x^{\delta_{3-i}-1}} \tag{C.7}$$

It can be shown (see, e.g. Ferrari and Rodosthenous, 2020) that the system (C.7) admits a unique solution $(\widehat{a}, \widehat{b}) \in \mathbb{R}_+^2$, where

$$\widehat{a} = a(\widehat{b}), \text{ such that } a'(\widehat{b}) = 0 \text{ and } a(\cdot) \text{ is } \begin{cases} \text{increasing on } (\bar{b}, \widehat{b}), \\ \text{decreasing on } (\widehat{b}, \infty). \end{cases} \tag{C.8}$$

The monotonicity then follows by combining (C.6) and (C.8). Furthermore, this monotonicity together with the fact that for every choice of $b \in \mathbb{R}_+$, there always exists a best response $a(b)$ that satisfies (3.3.14)–(3.3.15), due to part (i), then yields $\lim_{b\to\infty} a(b) = \overline{a}$.

Proof of (iii). Regarding the concavity of $a(\cdot)$ on the interval $(0, \widehat{b})$ we investigate the term

$$a''(b) = \frac{2F_a(a(b),b)F_b(a(b),b)F_{ab}(a(b),b) - F_b^2(a(b),b)F_{aa}(a(b),b) - F_a^2(a(b),b)F_{bb}(a(b),b)}{F_a^3(a(b),b)},$$

where we set $F_a(a,b) := \partial_a F(a,b)$ (and the other terms analogously). We first notice that, due to (C.2), we have $F_a(a(b),b) > 0$. Direct computation yields

$$\begin{aligned}
&2F_a(a(b),b)F_b(a(b),b)F_{ab}(a(b),b) - F_b^2(a(b),b)F_{aa}(a(b),b) - F_a^2(a(b),b)F_{bb}(a(b),b)\\
&= \Big\{\frac{1}{a(b)b}(2-\delta_2)\Big(\frac{b}{a(b)}\Big)^{\delta_1-1} + \frac{1}{a(b)b}(\delta_1-2)\Big(\frac{b}{a(b)}\Big)^{\delta_2-1} - (\delta_1-\delta_2)\frac{1}{a(b)^2}\Big\}\\
&\times\Big\{(\delta_1-\delta_2)(\rho-2(r-g)-\sigma^2)\big[c_2(\delta_1-1)(1-\delta_2)(\rho-2(r-g)-\sigma^2) - (\delta_1-2)(2-\delta_2)a(b)\big]\\
&\qquad\times\frac{1}{b}\Big[c_2(\delta_1-\delta_2)\Big(\frac{b}{a(b)}\Big)^{\delta_1+\delta_2-2} - c_1(\delta_1-1)\Big(\frac{b}{a(b)}\Big)^{\delta_1-1} - c_1(1-\delta_2)\Big(\frac{b}{a(b)}\Big)^{\delta_2-1}\Big]\\
&\qquad + c_2(1-\delta_2)(\delta_1-1)(\rho-2(r-g)-\sigma^2)\Big[\Big(\frac{b}{a(b)}\Big)^{\delta_1-1} - \Big(\frac{b}{a(b)}\Big)^{\delta_2-1}\Big]F_b(a(b),b)\Big\}.
\end{aligned}$$

Some straightforward calculations reveal that the first term on the above right-hand side (second line) is strictly positive, while the term in the third line is strictly positive due to $a(b) \le \widetilde{a}$. Moreover, the term in the fourth line is strictly negative, which easily follows upon using $a(b) \le b$. Finally, we notice that for $b \le \widehat{b}$, we have $F_b(a(b),b) < 0$ (see (C.8)). Combining these facts, we conclude that indeed $a''(b) < 0$ for $b \in (0, \widehat{b})$ and the claim follows. □

C.3 Proof of Theorem 3.3.3

We derive the result in a number of steps. In particular, we show in *Step 1* that the candidate value function (3.3.13), with $a(b)$ solving (3.3.14) as in Lemma 3.3.2, indeed solves the HJB equation (3.3.10), and in *Steps 2–4* that the latter identifies with value function V_1 in (3.3.9).

Step 1. By construction, we have $(\mathcal{L}-\rho)U_1(x;b) = -\frac{1}{2}x^2$ for $x \in (a(b),b)$, $U_1'(x;b) = c_2$ for $x \in (0,a(b))$ and $U_1'(x;b) = c_1 > c_2$ for $x \ge a(b)$. Therefore, it remains to show that:

$$(i)\ (\mathcal{L}-\rho)U_1(x;b) \ge -\frac{1}{2}x^2 \quad \text{for } x \in (0,a(b)) \qquad \text{and} \qquad (ii)\ U_1'(x;b) \ge c_2 \quad \text{for } x \in (a(b),b).$$

In the following, we fix $b \in \mathbb{R}_+$, so that $a(b)$ is the (fixed) unique solution to (3.3.14) according to Lemma 3.3.2.

Proof of (i). For $x \in (0, a(b))$, we get

$$(\mathcal{L} - \rho)U_1(x; b) = (r - g)xc_2 - \rho U_1(a(b); b) + \rho c_2(a(b) - x).$$

Clearly, $x \mapsto (\mathcal{L} - \rho)U_1(x; b)$ decreases with slope $-c_2(\rho - (r - g))$, while $x \mapsto -\frac{1}{2}x^2$ decreases with slope $-x$. Since $(\mathcal{L} - \rho)U_1(a(b); b) = -\frac{1}{2}a(b)^2$, it is sufficient to then show that $c_2(\rho - (r - g)) > x$, for all $x \in (0, a(b))$. The latter is true due to (3.3.16), thus (i) holds true.

Proof of (ii). For $x \in (a(b), b)$, we can calculate

$$\begin{aligned} U_1'(x; b) = &\frac{(\delta_2 - 2)a(b) - c_2(\delta_2 - 1)(\rho - 2(r - g) - \sigma^2)}{(\delta_1 - \delta_2)(\rho - 2(r - g) - \sigma^2)} \Big(\frac{x}{a(b)}\Big)^{\delta_1 - 1} \\ &- \frac{(\delta_1 - 2)a(b) - c_2(\delta_1 - 1)(\rho - 2(r - g) - \sigma^2)}{(\delta_1 - \delta_2)(\rho - 2(r - g) - \sigma^2)} \Big(\frac{x}{a(b)}\Big)^{\delta_2 - 1} + \frac{x}{\rho - 2(r - g) - \sigma^2}. \end{aligned}$$

Combining this with the definition (3.3.15) of F, we notice that

$$U_1'(x; b) \geq c_2 \quad \Leftrightarrow \quad F(a(b), x) \leq (c_1 - c_2)(\delta_1 - \delta_2)(\rho - 2(r - g) - \sigma^2).$$

To prove (ii), given that

$$F(a(b), a(b)) = (c_1 - c_2)(\delta_1 - \delta_2)(\rho - 2(r - g) - \sigma^2) \qquad \text{and} \qquad F(a(b), b) = 0,$$

it is sufficient to have that $x \mapsto F(a(b), x)$ first decreases and changes sign at most once in $(a(b), b)$. The latter was shown in the proof of Lemma 3.3.2, thus (ii) holds true.

Step 2. Let $x \in \mathbb{R}_+$ and $\xi \in \mathcal{A}_{\eta^b}$ such that $(\xi, \eta^b) \in \mathcal{A}$. For $n \geq 1$, we let $\tau_n := \inf\{t \geq 0 : \ X_t^{0,\eta^b} \geq n\}$. Since $U_1 \in C^2(0, b)$, we can apply Itô-Meyer formula to the process $e^{-\rho\tau_n}U_1(X_{\tau_n}^{\xi,\eta^b}; b)$ on $[0, \tau_n]$ and obtain

$$\begin{aligned} e^{-\rho\tau_n}U_1(X_{\tau_n}^{\xi,\eta^b}; b) - U_1(x; b) = &\int_0^{\tau_n} e^{-\rho s}(\mathcal{L} - \rho)U_1(X_s^{\xi,\eta^b}; b)ds + \sigma \int_0^{\tau_n} e^{-\rho s}U_1'(X_s^{\xi,\eta^b}; b)dW_s \\ &- \int_0^{\tau_n} e^{-\rho s}X_s^{\xi,\eta^b}U_1'(X_s^{\xi,\eta^b}; b)d\eta_s^{c,b} + \int_0^{\tau_n} e^{-\rho s}X_s^{\xi,\eta^b}U_1'(X_s^{\xi,\eta^b}; b)d\xi_s^c \\ &+ \sum_{s \leq \tau_n} e^{-\rho s}\big(U_1(X_s^{\xi,\eta^b}; b) - U_1(X_{s-}^{\xi,\eta^b}; b)\big). \end{aligned} \tag{C.9}$$

Clearly, for any $s \in (0, \tau_n]$ we have $0 < X_s^{\xi,\eta^b} < n \wedge b$ and thus continuity of U_1' implies that the second term in (C.9) is a martingale. Furthermore, since $(\xi, \eta^b) \in \mathcal{A}$, the last term in (C.9) rewrites as

$$\begin{aligned} &\sum_{s < \tau_n} e^{-\rho s}\big(U_1(X_s^{\xi,\eta^b}; b) - U_1(X_{s-}^{\xi,\eta^b}; b)\big) \\ &\qquad = \sum_{s < \tau_n} e^{-\rho s}\big(U_1(X_s^{\xi,\eta^b}; b) - U_1(X_{s-}^{\xi,\eta^b}; b)\big)\big[\mathbb{1}_{\{\Delta\xi_s > 0\}} + \mathbb{1}_{\{\Delta\eta_s^b > 0\}}\big] \end{aligned}$$

and

$$\sum_{s<\tau_n}\big(U_1(X_s^{\xi,\eta^b};b)-U_1(X_{s-}^{\xi,\eta^b};b)\big)\mathbb{1}_{\{\Delta\xi_s>0\}}=\sum_{s<\tau_n}e^{-\rho s}\int_0^{\Delta\xi_s}e^{u}X_{s-}^{\xi,\eta^b}U_1'(e^{u}X_{s-}^{\xi,\eta^b};b)du$$
$$\sum_{s<\tau_n}\big(U_1(X_s^{\xi,\eta^b};b)-U_1(X_{s-}^{\xi,\eta^b};b)\big)\mathbb{1}_{\{\Delta\eta^b_s>0\}}=-\sum_{s<\tau_n}e^{-\rho s}\int_0^{\Delta\eta^b_s}e^{-u}X_{s-}^{\xi,\eta^b}U_1'(e^{-u}X_{s-}^{\xi,\eta^b};b)du \tag{C.10}$$

Taking expectations, rearranging terms and using the notation introduced in (3.2.3) we can thus write (C.9) as

$$\begin{aligned}
&U_1(x;b)\\
&\quad=\mathbb{E}_x\Big[e^{-\rho\tau_n}U_1(X_{\tau_n}^{\xi,\eta^b};b)-\int_0^{\tau_n}e^{-\rho s}(\mathcal{L}-\rho)U_1(X_s^{\xi,\eta^b};b)ds+\int_0^{\tau_n}e^{-\rho s}X_s^{\xi,\eta^b}U_1'(X_s^{\xi,\eta^b};b)\circ_d d\eta^b_s\\
&\qquad\qquad-\int_0^{\tau_n}e^{-\rho s}X_s^{\xi,\eta^b}U_1'(X_s^{\xi,\eta^b};b)\circ_u d\xi_s\Big]\\
&\quad\le\mathbb{E}_x\Big[e^{-\rho\tau_n}U_1(X_{\tau_n}^{\xi,\eta^b};b)+\int_0^{\tau_n}e^{-\rho s}h(X_s^{\xi,\eta^b})ds+c_1\int_0^{\tau_n}e^{-\rho s}X_s^{\xi,\eta^b}\circ_u d\eta^b_s\\
&\qquad\qquad-c_2\int_0^{\tau_n}e^{-\rho s}X_s^{\xi,\eta^b}\circ_u d\xi_s\Big],
\end{aligned}\tag{C.11}$$

where, in the latter inequality, we exploit the fact that U_1 solves the free-boundary problem (3.3.12), as stated above. By admissibility of ξ we have that the right-hand side of (C.11) is finite $\mathbb{P}$-a.s., and we notice that Assumption 3.2.4.(ii) guarantees

$$\lim_{n\to\infty}\mathbb{E}_x\big[e^{-\rho\tau_n}U_1(X_{\tau_n}^{\xi,\eta^b};b)\big]=0.$$

Then, noticing that $\tau_n\uparrow\infty$, $\mathbb{P}$-a.s., and taking limits in (C.11), we can use Assumptions 3.2.1.(iii) and 3.2.4.(ii), employ the dominated convergence theorem, and conclude that

$$U_1(x;b)\le\mathbb{E}_x\Big[\int_0^{\infty}e^{-\rho s}h(X_s^{\xi,\eta^b})ds+c_1\int_0^{\infty}e^{-\rho s}X_s^{\xi,\eta^b}\circ_d d\eta^b_s-c_2\int_0^{\infty}e^{-\rho s}X_s^{\xi,\eta^b}\circ_u d\xi_s\Big].$$

Since $\xi\in\mathcal{A}_{\eta^b}$ was arbitrary, we have $U_1(x;b)\le V_1(x;b)$ on $\mathbb{R}_+$.

Step 3. We can repeat the above arguments from *Step 2*, but now fix the control strategy $\xi^{a(b)}$ of (3.3.11). Since $X_t^{\xi^{a(b)},\eta^b}\in[a(b),b]$, $\mathbb{P}$-a.s., for all $t>0$, and $U_1(X_t^{\xi^{a(b)},\eta^b};b)=c_2$ on $\operatorname{supp}\{d\xi_t^{a(b)}\}$, the inequality in (C.11) becomes an equality and hence, employing dominated convergence arguments as before, we observe

$$\begin{aligned}
&U_1(x;b)\\
&\quad=\mathbb{E}_x\Big[\int_0^{\infty}e^{-\rho s}h(X_s^{\xi^{a(b)},\eta^b})ds+c_1\int_0^{\infty}e^{-\rho s}X_s^{\xi^{a(b)},\eta^b}\circ_d d\eta^b_s-c_2\int_0^{\infty}e^{-\rho s}X_s^{\xi^{a(b)},\eta^b}\circ_u d\xi_s^{a(b)}\Big]\\
&\quad\ge V_1(x;b).
\end{aligned}$$

Step 4. Combining the results from *Steps 2* and *3* then concludes that $U_1(x;b) = V_1(x;b)$ and $\xi^{a(b)}$ of (3.3.11) is an optimal control strategy for problem (3.3.9). □

C.4 Proof of Theorem 3.4.1

We derive the result in a number of steps.

Step 1. We begin by solving the fixed-boundary problem (3.4.4). To this end, we construct a solution to the ordinary differential equation an impose the stated boundary conditions. We then obtain a *candidate* value function

$$\overline{U}_2(x;a) = \begin{cases} \overline{U}_2(a;a), & 0 < x \leq a; \\ \overline{D}_2(a)x^{\theta_2} + H(x), & a < x, \end{cases} \tag{C.12}$$

where $\overline{D}_2(\cdot)$ and $H(\cdot)$ are as in (3.4.6).

Step 2. We then aim at verifying that $\overline{U}_2$ of (C.12) solves the free-boundary problem (3.4.4) and satisfies the HJB equation (3.4.3). In view of the construction of $\overline{U}_2$, it remains to check that $0 \leq \frac{\partial}{\partial x}\overline{U}_2(x;a) < \kappa$, for all $x > a$.

Straightforward calculations lead to

$$\frac{\partial}{\partial x}\overline{U}_2(x;a) = \alpha \int_0^\infty e^{-(\lambda-(r-g))t}\Big(\Phi(d_1(x,t)) - \Big(\frac{x}{a}\Big)^{\theta_2-1}\Phi(d_1(a,t))\Big)dt < \kappa,$$

where the inequality follows from the facts that $x > a$ and $\lambda > r - g + \frac{\alpha}{\kappa} > r - g$ under Case (I). Furthermore, Assumption 3.2.4.(iii) guarantees that $\frac{\partial}{\partial x}\overline{U}_2(x;a) \geq 0$, for all $x > a$.

Step 3. Finally, we must verify that indeed $V_2 = \overline{U}_2$ and that not intervening is an optimal debt management strategy. The proof follows the lines of the proof of Theorem 3.4.3 (*Steps 2–4*), and is thus omitted for brevity. □

C.5 Proof of Lemma 3.4.2

We prove each part separately.

Proof of (i). Regarding the existence and uniqueness of a solution $b(a) \in (a,\infty)$ solving (3.4.10), we first conclude from the representation of G in (3.4.11) that

$$G(a,a) = k > 0, \qquad \text{and} \qquad \lim_{b\to\infty} G(a,b) = -\infty,$$

where the latter follows precisely from the fact that $\lambda < r - g + \frac{\alpha}{\kappa}$ in Case (II). Hence, there exists a solution to the equation (3.4.10). Moreover, it follows from the following expression

$$\frac{\partial}{\partial b}G(a,b) = \Big[\Big(\frac{b}{a}\Big)^{1-\theta_2} - \Big(\frac{b}{a}\Big)^{1-\theta_1}\Big]\frac{(\theta_1 - 1)(1-\theta_2)}{(\theta_1-\theta_2)(\lambda-(r-g))b}\big[\kappa(\lambda-(r-g)) - \alpha\mathbb{1}_{\{b>m\}}\big], \tag{C.13}$$

that

$$\frac{\partial}{\partial b}G(a,b) = \begin{cases} > 0, & a < b < a \vee m, \\ < 0, & a \vee m < b. \end{cases} \tag{C.14}$$

Therefore, for any $a \in \mathbb{R}_+$, there exists a unique $b(a) \in (a \vee m, \infty)$ such that $G(a, b(a)) = 0$. Moreover, due to (C.13), we conclude that $\frac{\partial}{\partial b}G(a, b(a)) < 0$.

Proof of (ii). Let b_0 be defined as in (3.4.12) and fix $a \in \mathbb{R}_+$. In order to show (3.4.12), we examine two cases of a-values.

Case (a): $a \geq b_0$. In this case, we immediately have $b(a) > b_0$.

Case (b): $a < b_0$. Notice that simple comparison arguments yield that $b_0 > m$ and it can be shown that $G(a, b_0) \geq 0$. We then assume (aiming for contradiction) that $b(a) \in (a \vee m, b_0)$. Since $b \mapsto G(a, b)$ is strictly decreasing for $b > a \vee m$ (see (C.14)), it follows that

$$G(a, b(a)) > G(a, b_0) \geq 0,$$

which is a contradiction, and it thus follows that $b(a) \geq b_0$.

Proof of (iii). We are now in position to obtain the monotonicity of $b(\cdot)$ on $(0, \infty)$. Given that $\frac{\partial}{\partial b}G(a, b(a)) < 0$ at the value $b(a)$ which satisfies (3.4.10), due to the reasoning above, we have that

$$b'(a) = -\frac{\frac{\partial}{\partial a}G(a,b(a))}{\frac{\partial}{\partial b}G(a,b(a))} > 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial a}G(a,b(a)) > 0, \tag{C.15}$$

and we in order to obtain the desired results, we distinguish two cases.

Case (a). If $a > m$, then we notice that

$$\frac{\partial}{\partial a}G(a,b(a)) = \frac{(\theta_1-1)(1-\theta_2)(k(\lambda-(r-g))-\alpha)}{(\theta_1-\theta_2)(\lambda-(r-g))a}\Big[\Big(\frac{b}{a}\Big)^{1-\theta_1} - \Big(\frac{b}{a}\Big)^{1-\theta_2}\Big] > 0. \tag{C.16}$$

This implies, thanks to (C.15), that

$$a \mapsto b(a) \quad \text{is increasing on} \quad (m, \infty). \tag{C.17}$$

Furthermore, combining the expressions of the partial derivatives in (C.13) and (C.16) with the expression of $b'(\cdot)$ in (C.15) yields $b'(a) = \frac{b(a)}{a}$, which further implies that $b(a) = (1/\widetilde{q})a$, for some $\widetilde{q} \in (0,1)$. The latter can be specified as the unique equation to the solution $G(\widetilde{q}, 1) = 0$, which is equivalent to (3.4.13).

Case (b). If $a \leq m$, then we notice that

$$
\begin{aligned}
&\frac{\partial}{\partial a} G(a, b(a)) \\
&\quad = \frac{2(\alpha - \kappa(\lambda - (r-g))}{(\theta_1 - \theta_2) a \sigma^2} \Big[\Big(\frac{b(a)}{a} \Big)^{1-\theta_2} - \Big(\frac{b(a)}{a} \Big)^{1-\theta_1} \Big] + \frac{2\alpha}{(\theta_1 - \theta_2)\sigma^2 a} \Big[\Big(\frac{m}{a} \Big)^{1-\theta_1} - \Big(\frac{m}{a} \Big)^{1-\theta_2} \Big] \\
&\quad \geq \frac{2(\alpha - \kappa(\lambda - (r-g))}{(\theta_1 - \theta_2) a \sigma^2} \Big[\Big(\frac{b_0}{a} \Big)^{1-\theta_2} - \Big(\frac{b_0}{a} \Big)^{1-\theta_1} \Big] + \frac{2\alpha}{(\theta_1 - \theta_2)\sigma^2 a} \Big[\Big(\frac{m}{a} \Big)^{1-\theta_1} - \Big(\frac{m}{a} \Big)^{1-\theta_2} \Big] \\
&\quad = \Big[\frac{2\alpha}{(\theta_1 - \theta_2)\sigma^2 a} - \frac{2(\alpha - \kappa(\lambda - (r-g)) b_0^{1-\theta_1}}{(\theta_1 - \theta_2)\sigma^2 a} \Big] \geq 0,
\end{aligned}
$$

where the first inequality follows from $b(a) \geq b_0$ and the second one from its definition (3.4.12). This implies, thanks to (C.15), that

$$
a \mapsto b(a) \quad \text{is increasing on} \quad (0, m]. \tag{C.18}
$$

Proof of (iv). Regarding the convexity of $b(\cdot)$ on the interval $(0, m)$, we examine the term

$$
\begin{aligned}
&b''(a) \\
&\quad = \frac{2G_a(a,b(a))G_b(a,b(a))G_{ab}(a,b(a)) - G_b^2(a,b(a))G_{aa}(a,b(a)) - G_a^2(a,b(a))G_{bb}(a,b(a))}{G_b^3(a,b(a))},
\end{aligned}
$$

where $G_a(a,b) := \partial_a G(a,b)$ (and the other terms analogously). We first notice that, due to (C.14), we have $G_b(a, b(a)) < 0$. Upon using (3.4.11) and some direct calculation, we find

$$
\begin{aligned}
&2G_a(a,b(a))G_b(a,b(a))G_{ab}(a,b(a)) - G_b^2(a,b(a))G_{aa}(a,b(a)) - G_a^2(a,b(a))G_{bb}(a,b(a)) \\
&\quad = \frac{(\theta_1 - 1)^3 (1-\theta_2)^3 (\alpha - \kappa(\lambda - (r-g))\alpha}{(\lambda - (r-g))^3 (\theta_1 - \theta_2)^3 a^2 b(a)^2} \\
&\qquad \times \Big\{ (\alpha - \kappa(\lambda - (r-g)) \Big[\Big(\frac{b(a)}{a} \Big)^{1-\theta_1} - \Big(\frac{b(a)}{a} \Big)^{1-\theta_2} \Big]^2 \Big[\theta_2 \Big(\frac{m}{a} \Big)^{1-\theta_2} - \theta_1 \Big(\frac{m}{a} \Big)^{1-\theta_1} \Big] \\
&\qquad\quad + \alpha \Big[\Big(\frac{m}{a} \Big)^{1-\theta_2} - \Big(\frac{m}{a} \Big)^{1-\theta_1} \Big]^2 \Big[\theta_1 \Big(\frac{b(a)}{a} \Big)^{1-\theta_1} - \theta_2 \Big(\frac{b(a)}{a} \Big)^{1-\theta_2} \Big] \Big\}
\end{aligned}
$$

While the first term on the above right-hand side (second line) is clearly positive, one can employ the fact that $b(a) > b_0 > m$ for all $a > 0$, with b_0 as in (3.4.12), to show that the second term is strictly negative. Consequently, we obtain $b''(a) > 0$ and thus the strict convexity of $b(\cdot)$ on $(0, m)$.

Moreover, we straightforwardly calculate $\lim_{a \downarrow 0} G(a, b_0) = 0$, which due to the the reasoning above and the monotonicity of $b(a)$ implies $\lim_{a \downarrow 0} b(a) = b_0$. □

C.6 Proof of Theorem 3.4.3

We derive the result in a number of steps. In particular, we show in *Step 1* that the candidate value function $U_2(x;a)$ of (3.4.9) solves the free-boundary problem (3.4.8), with $b(a)$ solving (3.4.10) as in Lemma 3.4.2, and satisfies the HJB equation (3.4.3), and in *Steps 2–4* that the latter identifies with value function V_2 of (3.4.2).

Step 1. By construction, we have $(\mathcal{L}-\lambda)U_2(x;a)+\alpha(x-m)^+=0$ for $x\in(a,b(a))$ as well as $\frac{\partial}{\partial x}U_2(x;a)=\kappa$ for $x\geq b(a)$. It thus remains to show that:

(i) $(\mathcal{L}-\lambda)U_2(x;a)+\alpha(x-m)^+\geq 0$ for $x\geq b(a)$ and *(ii)* $U_2'(x;a)\leq\kappa$ for $x\in(a,b(a))$.

In the following, we fix $a\in\mathbb{R}_+$, so that $b(a)$ is the (fixed) unique solution to (3.4.10) according to Lemma 3.4.2.

Proof of (i). For $x\geq b(a)$, we notice from the expression (3.4.9) of U_2 that

$$\begin{aligned}(\mathcal{L}-\lambda)U_2(x;a)&=(\mathcal{L}-\lambda)\Big(U_2(b(a);a)+\kappa(x-b(a))\Big)\\&=(r-g)x\kappa-\lambda U_2(b(a);a)-\lambda\kappa(x-b(a))\end{aligned}$$

which is a decreasing, linear function in x, with slope $-\kappa(\lambda-(r-g))$. We then observe that $x\mapsto-\alpha(x-m)$ also decreases linearly for $x\geq b(a)>m$, with a slope $-\alpha$ that is considered to satisfy $-\alpha<-\kappa(\lambda-(r-g))$, according to the parameter regime under Case (II). The claim thus follows by noticing that for $x=b(a)$, we have $(\mathcal{L}-\lambda)U_2(b(a);a)=-\alpha(b(a)-m)^+=-\alpha(b(a)-m)$.

Proof of (ii). For $x\in(a,b(a))$, we notice that $U_2'(x;a)=G(x,b(a))$. Since $x\mapsto G(x,b(a))$ is increasing (recall (C.15) and Lemma 3.4.2), we obtain via the representation (3.4.11) of G that

$$U_2'(x;a)=G(x,b(a))\leq G(b(a),b(a))=\kappa,$$

which concludes our claim. Furthermore, we have $\frac{\partial}{\partial x}U_2(x;a)\geq G(a,b(a))=0$ for $x>a$.

Step 2. Let $x\geq 0$ and $\eta\in\mathcal{A}_{\xi^a}$ such that $(\xi^a,\eta)\in\mathcal{A}$. Since $U_2\in C^2(a,\infty)$, we can apply Itô-Meyer formula, up to a localising sequence of stopping times given by $\tau_n:=\inf\{t\geq 0: X_t^{\xi^a,0}\geq n\}$ $\mathbb{P}$-a.s., to the process $U_2(X^{\xi^a,\eta};a)$ and obtain

$$\begin{aligned}e^{-\lambda\tau_n}U_2(X_{\tau_n}^{\xi^a,\eta};a)-U_2(x;a)&=\int_0^{\tau_n}(\mathcal{L}-\lambda)U_2(X_s^{\xi^a,\eta};a)ds+\sigma\int_0^{\tau_n}U_2'(X_s^{\xi^a,\eta};b)dW_s\\&\quad-\int_0^{\tau_n}e^{-\lambda s}X_s^{\xi^a,\eta}U_2'(X_s^{\xi^a,\eta};a)d\eta_s^c+\int_0^{\tau_n}e^{-\lambda s}X_s^{\xi^a,\eta}U_2'(X_s^{\xi^a,\eta};a)d\xi_s^{c,a}\\&\quad+\sum_{s<\tau_n}e^{-\lambda s}\big(U_2(X_s^{\xi^a,\eta};a)-U_2(X_{s-}^{\xi^a,\eta};a)\big)\end{aligned}\tag{C.19}$$

The second term in (C.19) is a martingale due to the continuity of U_2' and the fact that $a \leq X_s^{\xi^a,\eta} < n$ for any $s \in (0, \tau_n]$. Furthermore, we can proceed similarly as in (C.10) in order to rewrite the last term in (C.19) and obtain, after taking expectations and rearranging terms, that

$$\begin{aligned} U_2(x;a) &= \mathbb{E}_x\Big[e^{-\lambda\tau_n}U_2(X_{\tau_n}^{\xi^a,\eta};a) - \int_0^{\tau_n} e^{-\lambda s}(\mathcal{L}-\lambda)U_2(X_s^{\xi^a,\eta};a)ds \\ &\qquad + \int_0^{\tau_n} e^{-\lambda s}X_s^{\xi^a,\eta}U_2'(X_s^{\xi^a,\eta};a) \circ_d d\eta_s - \int_0^{\tau_n} e^{-\lambda s}X_s^{\xi^a,\eta}U_2'(X_s^{\xi^a,\eta};a) \circ_u d\xi_s^a\Big] \\ &\leq \mathbb{E}_x\Big[e^{-\lambda\tau_n}U_2(X_{\tau_n}^{\xi^a,\eta};a) + \int_0^{\tau_n} e^{-\lambda s}\alpha(X_s^{\xi^a,\eta}-m)^+ds + \kappa\int_0^{\tau_n} e^{-\lambda s}X_s^{\xi^a,\eta} \circ_d d\eta_s\Big], \end{aligned} \tag{C.20}$$

where the latter inequality follows from the fact that U_2 solves the free-boundary problem (3.4.8) and $U_2'(X_s^{\xi^a,\eta};a) = 0$ for all s in the support of $d\xi^a$. By admissibility of η, the right-hand side of (C.20) is finite $\mathbb{P}$-a.s., and Assumption 3.2.4.(iii)

$$\lim_{n\to\infty} \mathbb{E}_x\big[e^{-\lambda\tau_n}U_2(X_{\tau_n}^{\xi^a,\eta};a)\big] = 0.$$

Then, taking limits in (C.20) upon using that $\tau_n \uparrow \infty$, we can employ dominated convergence due to Assumption 3.2.4.(iii) and obtain

$$U_2(x;a) \leq \mathbb{E}_x\Big[\int_0^\infty e^{-\lambda s}\alpha(X_s^{\xi^a,\eta}-m)^+ds + \kappa\int_0^\infty e^{-\lambda s}X_s^{\xi^a,\eta} \circ_d d\eta_s\Big].$$

We conclude that $U_2(x;a) \leq V_2(x;a)$ on $\mathbb{R}_+$.

Step 3. We can now repeat the arguments from *Step 2*, upon fixing the control strategy $\eta^{b(a)}$ of (3.4.7). Since $X_t^{\xi^a,\eta^{b(a)}} \in [a, b(a)]$ a.s. for all $t > 0$ and $U_2(X_t^{\xi^a,\eta^{b(a)}};a) = \kappa$ on $\mathrm{supp}\{d\eta^{b(a)}\}$, the inequality in (C.20) becomes an equality. Arguing as before, we thus obtain

$$U_2(x;a) = \mathbb{E}_x\Big[\int_0^\infty e^{-\lambda s}\alpha(X_s^{\xi^a,\eta^{b(a)}}-m)^+ds + \kappa\int_0^\infty e^{-\lambda s}X_s^{\xi^a,\eta^{b(a)}} \circ_d d\eta_s^{b(a)}\Big] \geq V_2(x;a).$$

Step 4. Combining the results from *Steps 2* and *3* then concludes that $U_2(x;a) = V_2(x;a)$ on $\mathbb{R}_+$ and $\eta^{b(a)}$ is an optimal control strategy in problem (3.4.2). □

C.7 Proof of Corollary 3.5.3

Recall that $a_2(b;\lambda) \leq a_0(b;\lambda) = \widetilde{q}(\lambda)b$ (see also Figure 3.5.2 for illustration), where $\widetilde{q} \equiv \widetilde{q}(\lambda) \in (0,1)$ is given by the solution to (3.4.13), which is equivalent to $G(\widetilde{q}(\lambda),1) = 0$. By defining

$$\widehat{G}(q) := (1-\theta_2)\kappa(\lambda-(r-g)) + (\theta_1-1)(\kappa(\lambda-(r-g))-\alpha)q^{\theta_2-1} + \alpha(\theta_1-1), \quad q \in (0,1),$$

we observe that $G(q,1) \geq \widehat{G}(q)$ for all $q \in (0,1)$, since $\lambda < r - g + \frac{\alpha}{\kappa}$. Moreover, we notice that

$$\lim_{q\downarrow 0} G(q,1) = -\infty, \quad \lim_{q\downarrow 0} \widehat{G}(q) = -\infty, \quad G(1,1) = \widehat{G}(1) = (\theta_1 - \theta_2)\kappa(\lambda - (r-g)) > 0$$

and that both $q \mapsto G(q,1)$ and $q \mapsto \widehat{G}(q)$ are monotonically increasing. Using these properties, we can denote by $\widehat{q} \equiv \widehat{q}(\lambda) \in (0,1)$ the unique solution to $\widehat{G}(q) = 0$ (which can be computed explicitly) and obtain

$$\widetilde{q}(\lambda) < \widehat{q}(\lambda) \quad \text{and} \quad \lim_{\lambda\uparrow r-g+\alpha/\kappa} \widehat{q}(\lambda) = 0 \quad \Rightarrow \quad \lim_{\lambda\uparrow r-g+\alpha/\kappa} \widetilde{q}(\lambda) = 0.$$

First of all, recall from Theorem 3.5.2 that for every $\lambda \in (r-g, r-g+\alpha/\kappa)$, there exists a unique equilibrium pair $(a^*(\lambda), b^*(\lambda))$, such that $a^*(\lambda) = a_1(b^*(\lambda)) = a_2(b^*(\lambda);\lambda)$. Since $a_2(b;\lambda) \leq \widetilde{q}(\lambda)b$, we notice that for any fixed $\widetilde{b} \in (0,\infty)$ we have $\lim_{\lambda\uparrow r-g+\alpha/\kappa} a_2(\widetilde{b};\lambda) = 0$. However, there is no such fixed $\widetilde{b} \in (0,\infty)$ that can give $a_1(\widetilde{b}) = \lim_{\lambda\uparrow r-g+\alpha/\kappa} a_2(\widetilde{b};\lambda) = 0$ to create an equilibrium pair $\lim_{\lambda\uparrow r-g+\alpha/\kappa}(a^*(\lambda), b^*(\lambda)) = (0, \widetilde{b})$. Hence, the only possibility for obtaining an equilibrium as $\lambda \uparrow r-g+\alpha/\kappa$, is for $b^*(\lambda) \to \infty$. Given that $b \mapsto a_1(b)$ is strictly decreasing on $(\widehat{b}, \infty)$ and $\lim_{b\to\infty} a_1(b) = \overline{a}$ (independently of λ), we conclude that

$$\lim_{\lambda\uparrow r-g+\alpha/\kappa} a^*(\lambda) = \lim_{\lambda\uparrow r-g+\alpha/\kappa} a_2(b^*(\lambda);\lambda) = \lim_{\lambda\uparrow r-g+\alpha/\kappa} a_1(b^*(\lambda)) = \overline{a} \quad \text{and} \quad \lim_{\lambda\uparrow r-g+\alpha/\kappa} b^*(\lambda) = \infty,$$

which completes the proof. □

D Supplementary Material for Chapter 4

D.1 Proof of Theorem 4.2.7

We divide the proof of Theorem (4.2.7) in three steps.

Step 1. First, we prove that there exists a unique point $b \in \mathbb{R}_+$ such that $A(b) = 0$. To this end, we recall that

$$A(b) = A(0) + \int_0^b \frac{\partial}{\partial z} A(z)dz = -c(0)W + \int_0^b \frac{\partial}{\partial z} A(z)dz.$$

Since $-c(0)W < 0$ and $A'(z) > 0$ for all $z > \widetilde{z}$ under Assumption 4.2.5, it is sufficient to prove that $\lim_{b\to\infty} A(b) = +\infty$. Straightforward calculations, upon employing the mean value theorem for some point $\xi \in (\widetilde{z}, b)$, yield

$$\begin{aligned}
&\lim_{b\to\infty} A(b) = -c(0)W + \lim_{b\to\infty}\int_0^b \frac{\partial}{\partial z}A(z)dz \\
&= -c(0)W + \int_0^{\widetilde{z}} \frac{\partial}{\partial z}A(z)dz + \lim_{b\to\infty}\int_{\widetilde{z}}^b m'(z)\big(\psi(0)\varphi(z)-\varphi(0)\psi(z)\big)\big(\mathcal{L}-(r+\eta)\big)G(z)dz \\
&= -c(0)W + \int_0^{\widetilde{z}} \frac{\partial}{\partial z}A(z)dz \\
&\qquad + \lim_{b\to\infty}\frac{\big(\mathcal{L}-(r+\eta)\big)G(\xi)}{r+\eta}\int_{\widetilde{z}}^b (r+\eta)m'(z)(\psi(0)\varphi(z)-\varphi(0)\psi(z))dz \\
&= -c(0)W + \int_0^{\widetilde{z}} \frac{\partial}{\partial z}A(z)dz \\
&\qquad + \lim_{b\to\infty}\frac{(\mathcal{L}-(r+\eta))G(\xi)}{r+\eta}\left(\psi(0)\Big(\frac{\varphi'(b)}{S'(b)}-\frac{\varphi'(\widetilde{z})}{S'(\widetilde{z})}\Big)-\varphi(0)\Big(\frac{\psi'(b)}{S'(b)}-\frac{\psi'(\widetilde{z})}{S'(\widetilde{z})}\Big)\right) \\
&\to +\infty,
\end{aligned}$$

and the latter follows from Assumption 4.2.3, $(\mathcal{L}-(r+\eta))G(\xi) < 0$ (since $\widetilde{z} < \xi$), and $\varphi'(b)/S'(b) \downarrow 0$ as well as $\psi'(b)/S'(b) \uparrow +\infty$ (see for example Borodin and Salminen, 2015, Chapter 2).

Step 2. Next, we show that the candidate value function $w(\cdot)$ of (4.2.11) solves the variational inequality (4.2.8). Let $b \in \mathbb{R}_+$ denote the unique solution to $A(\cdot) = 0$, as determined in Step 1. By construction, $(\mathcal{L}-(r+\eta))w(z)+\pi_1(z) = 0$ on $(0,b)$, while $w(z) = \Phi_2(z) - c(z)$ on (b,∞). In order to show that w solves the variational inequality on $\mathbb{R}_+$, it thus remains to show (i) $(\mathcal{L}-(r+\eta))w(z)+\pi_1(z) \le 0$ on (b,∞) and (ii) $w(z) \ge \Phi_2(z)-c(z)$ on $(0,b)$. Regarding (i), we recall that $w(z) = \Phi_2(z)-c(z) = G(z)+\Phi_1(z)$ on (b,∞). It follows that

$$(\mathcal{L}-(r+\eta))w(z)+\pi_1(z) = (\mathcal{L}-(r+\eta))G(z) < 0,$$

where the latter inequality follows from Assumption 4.2.5 and by construction, since $b > \widetilde{z}$. Regarding (ii), we recall that $w(z) = \Phi_1(z) + (G(b)/\psi(b,0))\psi(z,0)$ on $(0,b)$. Simple calculations reveal that $w(z) \ge \Phi_2(z)-c(z)$ is equivalent to

$$\frac{G(b)}{\psi(0)\varphi(b)-\varphi(0)\psi(b)} \le \frac{G(z)}{\psi(0)\varphi(z)-\varphi(0)\psi(z)},$$

and consequently, it is sufficient to prove that b is a local minimum of the function $G(z)/(\psi(0)\varphi(z)-\varphi(0)\psi(z))$ on $(0,b)$. We compute

$$\left(\frac{G(z)}{\psi(0)\varphi(z)-\varphi(0)\psi(z)}\right)' = A(z)\frac{S'(z)}{(\psi(0)\varphi(z)-\varphi(0)\psi(z))^2},$$

which is zero for $z = b$ by construction. Furthermore,

$$\left(\frac{G(z)}{\psi(0)\varphi(z) - \varphi(0)\psi(z)}\right)''\Big|_{z=b}$$
$$= \left(A'(b)\frac{S'(b)}{(\psi(0)\varphi(b) - \varphi(0)\psi(b))^2} + A(b)\Big(\frac{S'(b)}{\psi(0)\varphi(b) - \varphi(0)\psi(b)}\Big)'\right)$$
$$= A'(b)\frac{S'(b)}{(\psi(0)\varphi(b) - \varphi(0)\psi(b))^2} > 0,$$

where the latter inequality follows from Assumption 4.2.5 and $b > \widetilde{z}$.

Step 3. Last, we verify that the candidate value function w indeed coincides with the true value function v of (4.2.4) and that the threshold b triggers the optimal stopping time τ_b as in (4.2.9). Let $n \in \mathbb{N}$ and define $\sigma_n := \inf\{t \geq 0 : Z_t^z \geq n\}$. We let $\tau_n = \tau \wedge \sigma_n$ for any stopping time τ of the Brownian filtration. Due to our construction of the function w, we can employ Itô's formula to obtain

$$e^{-(r+\eta)(\tau_n\wedge\gamma_1)}w(Z^z_{\tau_n\wedge\gamma_1}) = w(z) + \int_0^{\tau_n\wedge\gamma_1} e^{-(r+\eta)s}(\mathcal{L} - (r+\eta))w(Z_s^z)\mathbb{1}_{\{Z_s^z\neq b\}}ds + M_{\tau_n\wedge\gamma_1}$$

where γ_1 is defined as in 4.2.3 and M_t denotes the stochastic integral

$$M_t = \int_0^t e^{-(r+\eta)s}\sigma_1(Z_s^z)w'(Z_s^z)dW_s.$$

Due to the regularity of $w(\cdot)$ and $\sigma_1(\cdot)$ we observe $\mathbb{E}[M_{\tau_n\wedge\gamma_1}] = 0$, such that taking expectations yields

$$\mathbb{E}\big[e^{-(r+\eta)(\tau_n\wedge\gamma_1)}w(Z^z_{\tau_n\wedge\gamma_1})\big] = w(z) + \mathbb{E}\Big[\int_0^{\tau_n\wedge\gamma_1} e^{-(r+\eta)s}(\mathcal{L} - (r+\eta))w(Z_s^z)ds\Big],$$

where we used $\mathbb{P}[Z_s^z = b] = 0$. Since w solves the variational inequality (4.2.8), as proven in Step 2., we notice that $w(z) \geq \Phi_2(z) - c(z)$ as well as $(\mathcal{L} - (r+\eta))w(z) \leq -\pi_1(z)$ a.e. It follows that

$$w(z) \geq \mathbb{E}\Big[\int_0^{\tau_n\wedge\gamma_1} e^{-(r+\eta)s}\pi_1(Z_s^z)ds + e^{-(r+\eta)\tau_n}\big(\Phi_2(Z^z_{\tau_n}) - c(Z^z_{\tau_n})\big)\mathbb{1}_{\{\tau_n<\gamma_1\}}\Big],$$

where we employed the equality $w(Z^z_{\gamma_1}) = w(0) = 0$, which follows by construction. Letting $n \to \infty$, invoking the dominated convergence theorem due to Assumption 4.2.3, we obtain

$$w(z) \geq \mathbb{E}\Big[\int_0^{\tau\wedge\gamma_1} e^{-(r+\eta)s}\pi_1(Z_s^z)ds + e^{-(r+\eta)\tau}\big(\Phi_2(Z^z_{\tau}) - c(Z^z_{\tau})\big)\mathbb{1}_{\{\tau<\gamma_1\}}\Big] = J(z,\tau),$$

for all $\tau \in \mathcal{T}$, and thus $w(z) \geq v(z)$ for all $z \in \mathbb{R}_+$. Repeating the previous steps, but choosing the stopping time τ_b of (4.2.9) with b determined via the equation $A(\cdot) = 0$, we obtain

$$w(z) = J(z,\tau_b) \leq \sup_\tau J(z,\tau) = v(z),$$

for all $z \in \mathbb{R}_+$. It follows that $w = v$ and τ_b is an optimal stopping time for problem (4.2.4). □

D.2 Proof of Lemma 4.2.8 and 4.2.9

We prove the claims separately.

Proof of Lemma 4.2.8. Since

$$b'(c_p) = -\frac{\frac{\partial}{\partial c_p} A(b(c_p), c_p)}{\frac{\partial}{\partial b} A(b(c_p), c_p)}, \tag{D.1}$$

and $\frac{\partial}{\partial b} A(b(c_p), c_p) > 0$ due to (4.2.13) and Assumption 4.2.5, it is thus left to study the sign of $\frac{\partial}{\partial c_p} A(b(c_p), c_p)$. Simple calculations yield

$$\frac{\partial}{\partial c_p} A(b, c_p) = \frac{G_{bc_p}(b, c_p)[\varphi(b)\psi(0) - \varphi(0)\psi(b)] - G_{c_p}(b, c_p)[\varphi'(b)\psi(0) - \varphi(0)\psi'(b)]}{S'(b)}.$$

Moreover, since $G(b, c_p) = \Phi_2(b) - \Phi_1(b, c_p) - c(b)$, where the first and third term are clearly independent of c_p, we have

$$\begin{aligned}
&G_{c_p}(b, c_p) \\
&\quad = -W^{-1}\Big(\varphi(b) \int_0^b \psi(y, 0) \frac{\partial}{\partial c_p} \pi_1(y, c_p) m'(y) dy + \psi(b, 0) \int_b^\infty \varphi(y) \frac{\partial}{\partial c_p} \pi_1(y, c_p) m'(y) dy\Big), \\
&G_{bc_p}(b, c_p) \\
&\quad = -W^{-1}\Big(\varphi'(b) \int_0^b \psi(y, 0) \frac{\partial}{\partial c_p} \pi_1(y, c_p) m'(y) dy + \varphi(b)\psi(b, 0) \frac{\partial}{\partial c_p} \pi_1(b, c_p) m'(b) \\
&\qquad + \psi'(b, 0) \int_b^\infty \varphi(y) \frac{\partial}{\partial c_p} \pi_1(y, c_p) m'(y) dy - \varphi(b)\psi(b, 0) \frac{\partial}{\partial c_p} \pi_1(b, c_p) m'(b)\Big) \\
&\quad = -W^{-1}\Big(\varphi'(b) \int_0^b \psi(y, 0) \frac{\partial}{\partial c_p} \pi_1(y, c_p) m'(y) dy + \psi'(b, 0) \int_b^\infty \varphi(y) \frac{\partial}{\partial c_p} \pi_1(y, c_p) m'(y) dy\Big).
\end{aligned}$$

The numerator of (D.1) then rewrites as

$$\begin{aligned}
G_{bc_p}(b, c_p)&[\varphi(b)\psi(0) - \varphi(0)\psi(b)] - G_{c_p}(b, c_p)[\varphi'(b)\psi(0) - \varphi(0)\psi'(b)] \\
&= W^{-1}\varphi(0)[\varphi'(b)\psi(b) - \varphi(b)\psi'(b)] \int_0^b \psi(y, 0) \frac{\partial}{\partial c_p} \pi_1(y, c_p) m'(y) dy \\
&= -S'(b)\varphi(0) \int_0^b \psi(y, 0) \frac{\partial}{\partial c_p} \pi_1(y, c_p) m'(y) dy,
\end{aligned}$$

and it follows that

$$\frac{\partial}{\partial c_p} A(b, c_p) = -\varphi(0) \int_0^b \psi(y, 0) \frac{\partial}{\partial c_p} \pi_1(y, c_p) m'(y) dy > 0,$$

where the latter inequality follows from $m'(\cdot) > 0$ and Assumption 4.2.3.

Regarding the limiting behaviour of $b(c_p)$ as $c_p \to \infty$, we recall Assumption 4.2.3 (iv), i.e. $\lim_{c_p \to \infty} \pi_1(z, c_p) = 0$. It follows that $b_\infty := \lim_{c_p \to \infty} b(c_p)$ is the solution to

$$A(b_\infty) = -c(0)W + \int_0^{b_\infty} m'(x)\big(\psi(0)\varphi(x) - \psi(x)\varphi(0)\big)\big(\mathcal{L} - (r+\eta)\big)\big(\Phi_2(x) - c(x)\big)dx. \tag{D.2}$$

Notice that Assumption 4.2.5 is still satisfied, such that b_∞ is uniquely determined via (D.2). Moreover, it follows that $b_\infty > 0$. □

Proof of Lemma 4.2.9. Clearly, $v_{c_p}(z; c_p) = 0$ for $z \geq b(c_p)$. For $z \in (0, b(c_p))$, we compute

$$\begin{aligned} v_{c_p}(z, c_p) &= \frac{\partial}{\partial c_p}\Big(\Phi_1(z, c_p) + G(b(c_p), c_p)\frac{\psi(z)\varphi(0) - \varphi(z)\psi(0)}{\psi(b(c_p))\varphi(0) - \varphi(b(c_p))\psi(0)}\Big) \\ &= \frac{\partial}{\partial c_p}\Phi_1(z, c_p) + \Big(G_{c_p}(b(c_p), c_p) + b'(c_p)G_b(b(c_p), c_p)\Big)\frac{\psi(z)\varphi(0) - \varphi(z) - \psi(0)}{\psi(b(c_p))\varphi(0) - \varphi(b(c_p))\psi(0)} \\ &\quad + G(b(c_p), c_p)b'(c_p)\big(\psi(0)\varphi'(b(c_p)) - \varphi(0)\psi'(b(c_p))\big)\frac{\psi(z)\varphi(0) - \varphi(z) - \psi(0)}{(\psi(b(c_p))\varphi(0) - \varphi(b(c_p))\psi(0))^2}. \end{aligned}$$

Recall that

$$G(b(c_p), c_p)\frac{\psi(0)\varphi'(b(c_p)) - \varphi(0)\psi'(b(c_p))}{\varphi(b(c_p))\psi(0) - \psi(b(c_p))\varphi(0)} = G_b(b(c_p), c_p),$$

and notice that $G_{c_p}(z, c_p) = -\frac{\partial}{\partial c_p}\Phi_1(z, c_p)$. It follows that

$$v_{c_p}(z, c_p) = \frac{\partial}{\partial c_p}\Phi_1(z, c_p) - \frac{\partial}{\partial c_p}\Phi_1(b(c_p), c_p)\frac{\psi(z)\varphi(0) - \varphi(z)\psi(0)}{\psi(b(c_p))\varphi(0) - \varphi(b(c_p))\psi(0)},$$

and the latter is negative if and only if

$$\frac{\frac{\partial}{\partial c_p}\Phi_1(b(c_p), c_p)}{\psi(b(c_p))\varphi(0) - \varphi(b(c_p))\psi(0)} > \frac{\frac{\partial}{\partial c_p}\Phi_1(z, c_p)}{\psi(z)\varphi(0) - \varphi(z)\psi(0)}.$$

It is thus sufficient to prove that

$$\begin{aligned} &\frac{\partial}{\partial z}\Big(\frac{\frac{\partial}{\partial c_p}\Phi_1(z, c_p)}{\psi(z)\varphi(0) - \varphi(z)\psi(0)}\Big) \\ &\quad = \frac{\frac{\partial^2}{\partial z \partial c_p}\Phi_1(z, c_p)[\psi(z)\varphi(0) - \varphi(z)\psi(0)] - \frac{\partial}{\partial c_p}\Phi_1(z, c_p)[\psi'(z)\varphi(0) - \varphi'(z)\psi(0)]}{(\psi(z)\varphi(0) - \varphi(z)\psi(0))^2} > 0, \end{aligned}$$

and the latter inequality follows from employing similar arguments as in the proof of Lemma 4.2.8. □

D.3 Proof of Proposition 4.3.5

We derive the desired result in a number of steps, as described in the paragraph stated before Proposition 4.3.5.

Step 1. We begin by determining the unique equilibrium carbon price via the entry condition (4.3.1). We recall that $c_p \to v(z; c_p, E_{max})$ is decreasing (see Lemma 4.2.9), and hence, so is $c_p \to \int_{\underline{z}}^{\overline{z}} v(z; c_p, E_{max})\xi(dz)$. We distinguish two cases.

(i) $b_\infty > \underline{z}$. Recall that Assumptions 4.3.2 implies

$$\int_{\underline{z}}^{\overline{z}} v(z; 0, E_{max})\xi(dz) > c_e > \int_{\underline{z}}^{\overline{z}} \lim_{c_p \to \infty} v(z; c_p, E_{max})\xi(dz).$$

Clearly, by the intermediate value theorem, there exists a unique $c_p^* \in (0, \infty)$ such that the entry condition is satisfied. Moreover, $b(c_p^*) \geq b_\infty > \underline{z}$ due to Lemma 4.2.8, which implies positive entry in the market.

(ii) $b_\infty \leq \underline{z}$. Assumption 4.3.2 guarantees

$$\int_{\underline{z}}^{\overline{z}} v(z; 0, E_{max})\xi(dz) > c_e > \int_{\underline{z}}^{\overline{z}} v(z; \overline{c}_p, E_{max})\xi(dz),$$

where $\overline{c}_p$ is chosen such that $b(\overline{c}_p) = \underline{z}$. Again, via the intermediate value theorem, there exists a unique $c_p^* \in (0, \overline{c}_p)$ such that the entry condition is satisfied. Moreover, $b(c_p^*) \geq b(\overline{c}_p) = \underline{z}$, and hence, there is positive entry.

Step 2. Next, we solve for the stationary distribution of polluting firms. As described, we start by deriving its scaled density f^* up to the scale factor N^*, i.e. the entry rate, which is derived in third step of our proof. In the following, we set $b^* = b(c_p^*)$. For the ease of notation, we let

$$\widetilde{\mathcal{L}} = b(z)\frac{\partial^2}{\partial z^2} + a(z)\frac{\partial}{\partial z}$$

with $b(z) := \sigma_1^2(z)/2$, $a(z) := (\sigma_1^2(z))_z - \mu_1(z)$ and $r(z) := \eta + (\mu_1(z))_z - (\sigma_1^2(z))_{zz}/2$. It follows that we can write (4.3.6) and (4.3.7) equivalently as $(\widetilde{\mathcal{L}} - r(z))f(z) = 0$ and $(\widetilde{\mathcal{L}} - r(z))f(z) + g(z) = 0$, respectively. It is then straightforward to see that the scaled density f should satisfy one of the following systems of equations

$$\text{(I)} \begin{cases} (\widetilde{\mathcal{L}} - r(z))f(z) = 0, & 0 < z < \underline{z}, \\ (\widetilde{\mathcal{L}} - r(z))f(z) + g(z) = 0, & \underline{z} < z < b^*, \end{cases}$$

$$\text{(II)} \begin{cases} (\widetilde{\mathcal{L}} - r(z))f(z) = 0, & 0 < z < \underline{z}, \\ (\widetilde{\mathcal{L}} - r(z))f(z) + g(z) = 0, & \underline{z} < z < \overline{z}, \\ (\widetilde{\mathcal{L}} - r(z))f(z) = 0, & \overline{z} < z < b^*, \end{cases}$$

where we distinguished the cases (I) $b^* \leq \overline{z}$ and (II) $b^* > \overline{z}$, respectively. We notice that the equation $(\widetilde{\mathcal{L}} - r(z))f(z) = 0$ admits two fundamental strictly positive solutions $\widetilde{\psi}(\cdot)$ and $\widetilde{\varphi}(\cdot)$, with $\widetilde{\psi}(\cdot)$ being strictly increasing and $\widetilde{\varphi}(\cdot)$ being strictly decreasing (see Borodin and Salminen, 2015, Chapter 2, Section 10). Moreover, Assumption 4.3.1 guarantees that the equation $(\widetilde{\mathcal{L}} - r(z))f(z) + g(z) = 0$ admits a particular solution, which we denote by $G(\cdot)$. Using these, we can write the general solution to the systems (I) and (II), and hence our scaled density f, as

$$\text{(I)} \qquad f(z) := \begin{cases} A_1\widetilde{\psi}(z) + A_2\widetilde{\varphi}(z), & 0 < z < \underline{z} \\ B_1\widetilde{\psi}(z) + B_2\widetilde{\varphi}(z) + G(z), & \underline{z} < z < b^* \end{cases} \tag{D.3}$$

$$\text{(II)} \qquad f(z) := \begin{cases} C_1\widetilde{\psi}(z) + C_2\widetilde{\varphi}(z), & 0 < z < \underline{z} \\ D_1\widetilde{\psi}(z) + D_2\widetilde{\varphi}(z) + G(z), & \underline{z} < z < \overline{z} \\ E_1\widetilde{\psi}(z) + E_2\widetilde{\varphi}(z), & \overline{z} < z < b^*, \end{cases} \tag{D.4}$$

for some constants $A_i, B_i, C_i, D_i, E_i, i = 1, 2$ to be determined. We can solve for the constants by imposing the usual boundary conditions to ensure smoothness of the function f, given by $f(0) = 0$, $f(b^*) = 0$, $f(\underline{z}-) = f(\underline{z}+)$, $f'(\underline{z}-) = f'(\underline{z}+)$ and, in the case $b^* > \overline{z}$, the additional conditions $f(\overline{z}-) = f(\overline{z}+)$, $f'(\overline{z}-) = f'(\overline{z}+)$. Here, the first two conditions reflect that firms either exit due to their technology shock process either falling below zero, or exceeding the threshold b^*. Hence, there do not exist any polluting firms with a technology level outside of the set $(0, b^*)$. In both cases, we are able to derive unique solutions to the system of equations. In case (I), the constants are given by

$$\begin{aligned}
A_1^* &= \frac{\widetilde{\varphi}(0)\big(G(\underline{z})[\widetilde{\psi}(b^*)\widetilde{\varphi}'(\underline{z}) - \widetilde{\psi}'(\underline{z})\widetilde{\varphi}(b^*)] + G(b^*)[\widetilde{\psi}'(\underline{z})\widetilde{\varphi}(\underline{z}) - \widetilde{\psi}(\underline{z})\widetilde{\varphi}'(\underline{z})]\big)}{[\widetilde{\psi}'(\underline{z})\widetilde{\varphi}(\underline{z}) - \widetilde{\psi}(\underline{z})\widetilde{\varphi}'(\underline{z})] \times [\widetilde{\psi}(0)\widetilde{\varphi}(b^*) - \widetilde{\psi}(b^*)\widetilde{\varphi}(0)]} \\
&\quad + \frac{\widetilde{\varphi}(0)G'(\underline{z})[\widetilde{\psi}(\underline{z})\widetilde{\varphi}(b^*) - \widetilde{\psi}(b^*)\widetilde{\varphi}(\underline{z})]}{[\widetilde{\psi}'(\underline{z})\widetilde{\varphi}(\underline{z}) - \widetilde{\psi}(\underline{z})\widetilde{\varphi}'(\underline{z})] \times [\widetilde{\psi}(0)\widetilde{\varphi}(b^*) - \widetilde{\psi}(b^*)\widetilde{\varphi}(0)]} \\
A_2^* &= -A_1^*\frac{\widetilde{\psi}(0)}{\widetilde{\varphi}(0)} \\
B_1^* &= A_1^*\frac{\widetilde{\varphi}(b)[\widetilde{\psi}(\underline{z})\widetilde{\varphi}(0) - \widetilde{\psi}(0)\widetilde{\varphi}(\underline{z})]}{\widetilde{\varphi}(0)[\widetilde{\psi}(\underline{z})\widetilde{\varphi}(b^*) - \widetilde{\psi}(b^*)\widetilde{\varphi}(\underline{z})]} + \frac{G(b^*)\widetilde{\varphi}(\underline{z}) - G(\underline{z})\widetilde{\varphi}(b^*)}{\widetilde{\psi}(\underline{z})\widetilde{\varphi}(b^*) - \widetilde{\psi}(b^*)\widetilde{\varphi}(\underline{z})} \\
B_2^* &= A_1^*\frac{\widetilde{\psi}(b)[\widetilde{\psi}(\underline{z})\widetilde{\varphi}(0) - \widetilde{\psi}(0)\widetilde{\varphi}(\underline{z})]}{\widetilde{\varphi}(0)[\widetilde{\psi}(\underline{z})\widetilde{\varphi}(b^*) - \widetilde{\psi}(b^*)\widetilde{\varphi}(\underline{z})]} - \frac{G(b^*)\widetilde{\psi}(\underline{z}) - G(\underline{z})\widetilde{\psi}(b^*)}{\widetilde{\psi}(\underline{z})\widetilde{\varphi}(b^*) - \widetilde{\psi}(b^*)\widetilde{\varphi}(\underline{z})},
\end{aligned}$$

and in case (II) by

$$\begin{aligned}
C_1^* &= \frac{\widetilde{\varphi}(0)\big(\widetilde{\psi}(b^*)[G'(\overline{z})\widetilde{\varphi}(\overline{z}) - G(\overline{z})\widetilde{\varphi}'(\overline{z})] + \widetilde{\varphi}(b^*)[G(\overline{z})\widetilde{\psi}'(\overline{z}) - G'(\overline{z})\widetilde{\psi}(\overline{z})]\big)}{[\widetilde{\psi}(b^*)\widetilde{\varphi}(0) - \widetilde{\psi}(0)\widetilde{\varphi}(b^*)] \times [\widetilde{\psi}(\overline{z})\widetilde{\varphi}'(\overline{z}) - \widetilde{\psi}'(\overline{z})\widetilde{\varphi}(\overline{z})]} \\
&\quad - \frac{\widetilde{\varphi}(0)\big(\widetilde{\psi}(b^*)[G'(\underline{z})\widetilde{\varphi}(\underline{z}) - G(\underline{z})\widetilde{\varphi}'(\underline{z})] + \widetilde{\varphi}(b^*)[G(\underline{z})\widetilde{\psi}'(\underline{z}) - G'(\underline{z})\widetilde{\psi}(\underline{z})]\big)}{[\widetilde{\psi}(b^*)\widetilde{\varphi}(0) - \widetilde{\psi}(0)\widetilde{\varphi}(b^*)] \times [\widetilde{\psi}(\underline{z})\widetilde{\varphi}'(\underline{z}) - \widetilde{\psi}'(\underline{z})\widetilde{\varphi}(\underline{z})]}
\end{aligned}$$

$$
\begin{aligned}
D_1^* &= C_1^* + \frac{G'(\underline{z})\widetilde{\varphi}(\underline{z}) - G(\underline{z})\widetilde{\varphi}'(\underline{z})}{\widetilde{\psi}(\underline{z})\widetilde{\varphi}'(\underline{z}) - \widetilde{\psi}'(\underline{z})\widetilde{\varphi}(\underline{z})}, \qquad D_2^* = -C_1^* \frac{\widetilde{\psi}(0)}{\widetilde{\varphi}(0)} - \frac{G'(\underline{z})\widetilde{\psi}(\underline{z}) - G(\underline{z})\widetilde{\psi}'(\underline{z})}{\widetilde{\psi}(\underline{z})\widetilde{\varphi}'(\underline{z}) - \widetilde{\psi}'(\underline{z})\widetilde{\varphi}(\underline{z})} \\
E_1^* &= C_1^* \frac{\widetilde{\varphi}(b^*)[\widetilde{\psi}(\overline{z})\widetilde{\varphi}(0) - \widetilde{\psi}(0)\widetilde{\varphi}(\overline{z})]}{\widetilde{\varphi}(0)[\widetilde{\psi}(\overline{z})\widetilde{\varphi}(b^*) - \widetilde{\psi}(b^*)\widetilde{\varphi}(\overline{z})]} + \frac{G(\overline{z})\widetilde{\varphi}(b^*)}{\widetilde{\psi}(\overline{z})\widetilde{\varphi}(b^*) - \widetilde{\psi}(b^*)\widetilde{\varphi}(\overline{z})} \\
&\quad + \frac{\widetilde{\varphi}(b^*)\big(\widetilde{\psi}(\overline{z})[G'(\underline{z})\widetilde{\varphi}(\underline{z}) - G(\underline{z})\widetilde{\varphi}'(\underline{z})] - \widetilde{\varphi}(\overline{z})[G'(\underline{z})\widetilde{\psi}((\underline{z}) - G(\underline{z})\widetilde{\psi}'(\underline{z})]\big)}{[\widetilde{\psi}(\overline{z})\widetilde{\varphi}(b^*) - \widetilde{\psi}(b^*)\widetilde{\varphi}(\overline{z})] \times [\widetilde{\psi}(\underline{z})\widetilde{\varphi}'(\underline{z}) - \widetilde{\psi}'(\underline{z})\widetilde{\varphi}(\underline{z})]} \\
C_2^* &= -C_1^* \frac{\widetilde{\psi}(0)}{\widetilde{\varphi}(0)}, \qquad E_2^* = -E_1^* \frac{\widetilde{\psi}(b^*)}{\widetilde{\varphi}(b^*)}.
\end{aligned}
$$

We denote the corresponding equilibrium density, that results from plugging in the solutions for the constants $A_i^*, B_i^*, C_i^*, D_i^*, E_i^*, i = 1, 2$ into (D.3) – (D.4), as $f^*(z)$.

Step 3. As a last step, we determine the entry rate N^* via the equilibrium condition (4.3.5). Since $\nu^*(z) = N^* f^*(z)$ is linear in the entry rate, we notice that (4.3.5) rewrites as

$$
E_{max} = \int_0^b e(z; c_p, E_{max})\nu(dz) = \int_0^b e(z; c_p, E_{max}) N^* f^*(z) dz,
$$

such that the equilibrium entry rate is given by

$$
N^* = \frac{E_{max}}{\int_0^b e(z; c_p, E_{max}) f^*(z) dz}. \tag{D.5}
$$

Notice that the integral in the denominator of (D.5) is finite due to Assumption 4.2.3.

D.4 Derivation of the Fokker Planck Equation with Entry and Poisson Death

Here, we derive the Fokker-Planck equation (Kolmogorov forward equation) for the stationary density of the polluting firms. We follow the standard derivation of the Fokker-Planck equation using the Chapman Kolmogorov equation and consider the stochastic process Z_t satisfying the stochastic differential equation

$$
dZ_t = \mu(t, Z_t)dt + \sigma(t, Z_t)dW_t, \qquad Z_0 = z,
$$

where W_t denotes a standard Brownian motion. Moreover, we assume that new entrants are introduced to the system via the density $g(z)$ and the firms suffer independently Poisson death at a rate $\eta > 0$. The Chapman Kolmogorov equation for $s < t$ and $\Delta t > 0$, extended by these two features, is thus given by (cf. Fleming and Soner, 2006, Chapter III)

$$
p(z, t + \Delta t | y, s) = \int p(z, t + \Delta t | x, t) p(x, t | y, s) dx + g(z)\Delta t - \eta p(x, t | y, s)\Delta t.
$$

We multiply the latter by a smooth test function $R(z)$ and integrate both sides with respect to z, which yields

$$\int R(z)p(z,t+\Delta t|y,s)dz = \int R(z)\int p(z,t+\Delta t|x,t)p(x,t|y,s)dxdz + \int R(z)\big(g(z)\Delta t - \eta p(z,t|y,s)\Delta t\big)dz. \quad \text{(D.6)}$$

In the first integral on the right hand side, we expand around x

$$R(z) = R(x) + R'(x)(z-x) + \frac{1}{2}R''(x)(z-x)^2 + \dots$$

and obtain

$$\begin{aligned}\int R(z)p(z,t+\Delta t|x,t)dz &= \int \big(R(x) + R'(x)(z-x) + \frac{1}{2}R''(x)(z-x)^2 + \dots\big)p(z,t+\Delta t|x,t)dz \\ &= R(x)\int p(z+\Delta t|x,t)dz + R'(x)\int (z-x)p(z,t+\Delta t|x,t)dz \\ &\quad + \frac{1}{2}R''(x)\int (z-x)^2 p(z,t+\Delta t|x,t)dz \\ &= R(x) + R'(x)\mu(t,x)\Delta t + \frac{1}{2}R''(x)\sigma(t,x)^2\Delta t + o(\Delta t),\end{aligned}$$

where the latter equality follows from straightforward calculations and the normalization $\int p(z+\Delta t|x,t)dz = 1$. On the left hand side of (D.6) we expand the short-time transition density

$$\int R(z)p(z,t+\Delta t|y,s)dz = \int R(z)\big(p(z,t|y,s) + \partial_t p(z,t|y,s)\Delta t + o(\Delta t)\big)dz.$$

Hence, we obtain

$$\begin{aligned}\int R(x)p(x,t|y,s)dx + \Delta t\int R(x)\big(\partial_t p(x,t|y,s) + o(\Delta t)\big)dx \\ = \int R(x)\big(p(x,t|y,s) + g(x)\Delta t - \eta p(x,t|y,s)\Delta t\big)dx \\ + \Delta t\int (R'(x)\mu(t,x) + \frac{1}{2}R''(x)\sigma^2(t,x)^2)p(t,x|y,s)dx.\end{aligned}$$

We collect all terms with Δt and ignore the error terms to obtain

$$\begin{aligned}\int R(x)\big(\partial_t p(x,t|y,s)\big)dx = \int R(x)\big(g(x) - \eta p(x,t|y,s)\big)dx \\ + \int (R'(x)\mu(t,x) + \frac{1}{2}R''(x)\sigma^2(t,x)^2)p(t,x|y,s)dx.\end{aligned}$$

Finally, we employ integration by parts, and obtain

$$0 = \int R(x)\Big(- \partial_t p(x,t|y,s) + g(x) - \eta p(x,t|y,s) \\ - \partial_x[\mu(t,x)p(x,t|y,s)] + \frac{1}{2}\partial_{xx}[\sigma^2(t,x)p(x,t|y,s)]\Big) dx.$$

Since this holds for any test function, the transition density thus satisfies the following Fokker-Planck equation with entry and Poisson death

$$\partial_t p(z,t|y,s) = g(z) - \eta p(z,t|y,s) - \partial_z[\mu(t,z)p(z,t|y,s)] + \frac{1}{2}\partial_{zz}[\sigma^2(t,z)p(z,t|y,s)].$$

D.5 Proof of Lemma 4.4.1

Clearly, the functions π_1, π_2 and e satisfy the regularity assumptions posed in Assumption 4.2.3 (i). Moreover, we observe $\pi_i(z) \geq 0$ for all $z \in \mathbb{R}_+$, since $\pi_i(0) = 0$ and $\partial_z \pi_i(z) > 0$. It is straightforward to see that $\partial_{c_p}\pi_1(z, c_p) < 0$, as well as $\lim_{c_p\to\infty} \pi_1(z, c_p) = 0$. Furthermore, we observe $0 \leq \pi_1(z, c_p) \leq K_1(c_p)$, since $\pi_1(0, c_p) = 0$, $\partial_z \pi_1(z, c_p) \geq 0$ and

$$\lim_{z\to\infty} \pi_1(z, c_p) = (1 - \tau_1)\varepsilon\Big(\frac{1-\varepsilon}{c_p\lambda}\Big)^{\frac{1-\varepsilon}{\varepsilon}} =: K_1(c_p).$$

Similarly, it follows that $0 \leq \pi_2(z) \leq K_2$ for some constant $K_2 > 0$. □

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On an Irreversible Investment Problem with Two-Factor Uncertainty,
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A Stochastic Non-Zero-Sum Game of controlling the Debt-To-GDP Ratio,
Preprint on ArXiv: arxiv.2311.17711. In collaboration with Stephané Villeneuve and Neofytos Rodosthenous.

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