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Research Article

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Existence and concentration behavior of solutions for a class of quasilinear elliptic equations with critical growth

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Abstract: In this paper, we study a class of quasilinear elliptic equations involving the Sobolev critical exponent

$$-\varepsilon^p\Delta_p u-\varepsilon^p\Delta_p(u^2)u+V(x)|u|^{p-2}u=h(u)+|u|^{2p^*-2}u \quad \text{in } \mathbb{R}^N,$$

where $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator, $p^*=\frac{Np}{N-p}$ $(N\geq 3,\ N>p\geq 2)$ is the usual Sobolev critical exponent, the potential $V(x)$ is a continuous function, and the nonlinearity $h(u)$ is a nonnegative function of C^1 class. Under some suitable assumptions on V and h, we establish the existence, multiplicity and concentration behavior of solutions by using combing variational methods and the theory of the Ljusternik–Schnirelman category.

Keywords: Quasilinear Schrödinger equation, mountain pass theorem, ground state solution, Ljusternik–Schnirelman category

MSC 2010: 35J20, 35J62

1 Introduction

In this paper, we are concerned with the existence, multiplicity and concentration behavior of solutions of the following quasilinear elliptic equations involving the Sobolev critical exponent:

$$-\varepsilon^p\Delta_p u-\varepsilon^p\Delta_p(u^2)u+V(x)|u|^{p-2}u=h(u)+|u|^{2p^*-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $N\geq 3$, $2\leq p<N$, $p^*=\frac{Np}{N-p}$ is the Sobolev critical exponent, $\varepsilon>0$ is a small parameter, the potential $V:\mathbb{R}^N\to\mathbb{R}$ is a continuous function, and the nonlinearity $h(u):\mathbb{R}\to\mathbb{R}$ is a nonnegative function of C^1 class. The reduction form of equation (1.1) appears in many branches of mathematical physics and has been studied extensively in recent years. In particular, when $p=2$, the solution of (1.1) is related to the following quasilinear Schrödinger equation:

$$i\varepsilon\frac{\partial\psi}{\partial t}=-\varepsilon^2\Delta\psi+W(x)\psi-\kappa\varepsilon^2\Delta(\rho(|\psi|^2))\rho'(|\psi|^2)\psi-\widetilde{l}(|\psi|^2)\psi, \tag{1.2}$$

where $\psi:\mathbb{R}\times\mathbb{R}^N\to\mathbb{C}$, $W:\mathbb{R}^N\to\mathbb{R}$ is a given potential, κ is a real constant, and ρ, l are real functions. Equation (1.2) arises in several models of different physical phenomena corresponding to various types of ρ.

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The case $\rho(s) = s$ is used for the superfluid film equation in plasma physics by Kurihura in [26]. In the case $\rho(s) = (1 + s)^{\frac{1}{2}}$, it models the self-channeling of a high-power ultra short laser in matter; see [8, 9, 12, 15]. For more physical motivations, we can refer the interested readers to [6, 23, 25] and the references therein. If the quasilinear term $\varepsilon^p \Delta_p(u^2)u$ is not appearing and $p > 2$, problem (1.1) arises in a lot of applications when $\varepsilon = 1$, such as image processing, non-Newtonian fluids and pseudo-plastic fluids; for more details see [5, 13, 19] and the references therein.

When κ=1 and $\rho(s) = s$, considering standing wave solutions of the form $\psi(x, t) = u(x)e^{-iEt/\varepsilon}$ in (1.2), then $u(x)$ verifies the following equation:

$$-\varepsilon^2 \Delta u - \varepsilon^2 \Delta(u^2)u + V(x)u = l(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $V(x) = W(x) - E$, $l(u) = \widetilde{l}(u^2)u$. It is clear that when $\varepsilon = 1$, equation (1.3) reduces to the following equation:

$$-\Delta u - \Delta(u^2)u + V(x)u = l(u) \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

When $\kappa = 0$ and $\rho(s) = s$, the standing wave solutions of equation (1.2) satisfies the classical Schrödinger equation of the form

$$-\varepsilon^2 \Delta u + V(x)u = l(u) \quad \text{in } \mathbb{R}^N.$$

The existence and concentration behavior of positive solutions of the above equation have been extensively investigated under various hypotheses on the potential $V(x)$ and the nonlinearity $l(u)$; see, for example, [4, 16, 17, 22, 32, 34] and the references therein.

In recent years, many scholars have been interested in the study of the existence and multiplicity of solutions for equation (1.4). For example: the existence of a positive ground state solution has been obtained in [33] by using a constrained minimization argument which gives a solution of (1.4) with an unknown Lagrange multiplier λ in front of the nonlinear term. In [30], the existence of both one-sign and nodal ground states of soliton type solutions were established by the Nehari manifold method. In [29], Liu, Wang and Wang developed the methods of change of variables such that the quasilinear problem reduces to a semilinear one. They used an Orlicz space framework to prove the existence of positive solutions of (1.4) by the mountain pass theorem. Meanwhile, Colin and Jeanjean [14] developed the dual methods to treat the quasilinear problem (1.5), and the usual Sobolev space $H^1(\mathbb{R}^N)$ was used to prove the existence of positive solutions. The other recently interesting works can be found in [3, 11, 18, 20, 31] and the references therein.

Regarding critical problems, there are also some important results appearing in the literature. For example, in [24], the authors established the existence, multiplicity and concentration behavior of ground states for quasilinear Schrödinger equation with critical growth

$$\begin{cases} -\varepsilon^2 \Delta u - \varepsilon^2 \Delta(u^2)u + V(x)u = h(u) + |u|^{22^*-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), & u(x) > 0, \end{cases} \tag{1.5}$$

by using the variational methods and combining them with the theory of the Ljusternik–Schnirelman category which was used by Alves, Figueiredo and Severo [2] to establish the existence and multiplicity of nontrivial weak solutions for quasilinear elliptic equations of the form

$$-\varepsilon^p \Delta_p u - \varepsilon^p \Delta_p(u^2)u + V(x)|u|^{p-2}u = h(u) \quad \text{in } \mathbb{R}^N.$$

Yang and Ding [40] applied the perturbed methods to consider the following critical quasilinear Schrödinger equation:

$$-\varepsilon^2 \Delta u - \varepsilon^2 \Delta(u^2)u + V(x)u = h(x, u)u + K(x)|u|^{22^*-2}u \quad \text{in } \mathbb{R}^N,$$

and showed the existence of positive solutions as $\varepsilon \le \varepsilon_0$ and for any $m \in \mathbb{N}$; it has at least m pairs of solutions if $\varepsilon \le \varepsilon_m$, where ε_0 and ε_m are sufficient small positive numbers.

Unlike [2, 24, 40], where the minimum of $V(x)$ is global, Wang and Zou [38] studied the quasilinear Schrödinger equation with critical exponent

$$-\varepsilon^2 \Delta u - \varepsilon^2 [\Delta(u^2)]u + V(x)u = g(u) + |u|^{22^*-2}u \quad \text{in } \mathbb{R}^N,$$

where the potential $V(x)$ satisfies the local minimum condition $\inf_{\Omega} V < \inf_{\partial\Omega} V$, Ω is a bounded domain of $\mathbb{R}^N$, and proved the existence of positive bound states which concentrate around the local minimum point of V.

Motivated by the above-cited papers, the main purpose of this paper is to establish the existence, multiplicity and concentration behavior of the ground states for the quasilinear elliptic equation with critical growth

$$\begin{cases} -\varepsilon^p \Delta_p u - \varepsilon^p \Delta_p(u^2)u + V(x)|u|^{p-2}u = h(u) + |u|^{2p^*-2}u & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N),\ u(x) > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{$\mathcal{P}_\varepsilon$}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying
(V) $0 < V_0 = \inf_{x\in\mathbb{R}^N} V(x) < \lim_{|x|\to\infty} \inf V(x) = V_\infty < \infty$.
Assume that the nonlinearity $h : \mathbb{R} \to \mathbb{R}$ is of class C^1 and satisfies the following conditions:
(H_0) $h(s) = 0$ for $s \le 0$, $h'(s) = o(|s|^{p-2})$ as $s \to 0$;
(H_1) $\lim_{|s|\to\infty} \frac{h'(s)}{|s|^{q-1}} = 0$ for some $q \in (2p-1, 2p^*-1)$;
(H_2) there exists $2p < \theta < 2p^*$ such that $0 < \theta H(s) = \theta \int_0^s h(\tau)\,d\tau \le sh(s)$ for all $s > 0$;
(H_3) the function $s \to h(s)/s^{2p-1}$ is increasing for $s > 0$;
(H_4) there exist $C > 0$, $\sigma \in (\max\{2pN/(N-p) - \frac{2N}{N-p}, 2p\}, 2p^*)$ such that $h(s) \ge Cs^{\sigma-1}$ for $s > 0$.

As far as we know, little work has been done for the existence and concentration behavior of positive solutions for the quasilinear problem ($\mathcal{P}_\varepsilon$) where the nonlinearity has a critical growth. Our main result complements the corresponding conclusion of [2] and extends the main result of [24]. Alves, Figueiredo and Severo [2], He, Qian and Zou [24] and Wang and Zou [38] chose the Sobolev space E which is defined by

$$E = \Big\{ v \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|f(v)|^p \, dx < \infty \Big\}$$

equipped with the norm

$$\|v\|_E = \|\nabla v\|_{L^p} + |v|_f := \|\nabla v\|_{L^p} + \inf_{\xi>0} \frac{1}{\xi}\Big[1 + \int_{\mathbb{R}^N} V(x)|f(\xi v)|^p \, dx\Big].$$

The Orlicz–Sobolev space E may not be reflexible for $p = 2$, and so the bounded sequence may have no convergent subsequence in E. Unlike the work of [2, 24, 38], we directly choose the usual Sobolev space $W^{1,p}(\mathbb{R}^N)$ to deal with the autonomous problem and the usual Sobolev space X which will be defined in Section 2 to treat the nonautonomous problem. On the other hand, we use the mountain pass theorem under $(C)_c$ condition (see [36]); this is different from [2, 24, 38].

For stating our main result, we set

$$M = \{x \in \mathbb{R}^N : V(x) = V_0\}$$

and

$$M_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \le \delta\} \quad \text{for } \delta > 0.$$

In view of (V), the set M is compact. We recall that, if Y is a closed subset of a topological space X, the Ljusternik–Schnirelman category $\operatorname{cat}_X(Y)$ is the least number of closed and contractible sets in X which cover Y. By means of the Ljusternik–Schnirelman theory, we arrive at the following result.

Theorem 1.1. *Suppose that conditions* (V) *and* (H_0)–(H_5) *are satisfied. Given $\delta > 0$, there exists $\overline{\varepsilon} = \overline{\varepsilon}(\delta) > 0$ such that for any $\varepsilon \in (0, \overline{\varepsilon})$, problem ($\mathcal{P}_\varepsilon$) has at least $cat_{M_\delta}(M)$ positive weak solutions in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$. Moreover, each solution decays to zero at infinity, and if u_ε denotes one of these positive solutions and $\eta_\varepsilon \in \mathbb{R}^N$ its global maximum, then*

$$\lim_{\varepsilon\to 0} V(\eta_\varepsilon) = V_0.$$

The paper is organized as follows: In Section 2, we present the abstract framework of the problem as well as some preliminary results. In Section 3, we show the existence of ground state solution for autonomous problem. Section 4 is devoted to the proof of Theorem 1.1.

2 Variational framework and preliminary results

Formally, the energy functional associated to $(\mathcal{P}_\varepsilon)$ is defined by

$$I(u) = \frac{\varepsilon^p}{p} \int_{\mathbb{R}^N} (1 + 2^{p-1}|u|^p)|\nabla u|^p \,\mathrm{d}x + \frac{1}{p} \int_{\mathbb{R}^N} V(x)|u|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} H(u)\,\mathrm{d}x - \frac{1}{2p^*} \int_{\mathbb{R}^N} |u^+|^{2p^*} \,\mathrm{d}x, \tag{2.1}$$

where $H(u) = \int_0^u h(s)\,\mathrm{d}s$, $u^+ = \max\{u, 0\}$. Observe that $\Delta_p(u^2)u$ is not always in $L^1(\mathbb{R}^N)$, therefore the functional $I(u)$ is not well defined on the whole Sobolev space $W^{1,p}(\mathbb{R}^N)$. In fact, let $u \in C_0^1(\mathbb{R}^N \setminus \{0\})$ and $u(x) = |x|^{p-N/2p}$, $x \in B_1 \setminus \{0\}$; then $u \in W^{1,p}(\mathbb{R}^N)$, while the function $|u|^p|\nabla u|^p \notin L^1(\mathbb{R}^N)$. To overcome this difficulty, we use the change of variable methods developed in [29], making the change of variables $u = f(v)$, where f is a C^∞ function and defined by

$$f'(t) = \frac{1}{(1 + 2^{p-1}|f(t)|^p)^{\frac{1}{p}}} \quad \text{for } t \in [0, +\infty)$$

and

$$f(-t) = -f(t) \quad \text{for } t \in (-\infty, 0].$$

The following properties were proved in [35].

Lemma 2.1. *The following properties involving f and its derivative hold:*
(1) *f is a uniquely defined C^∞ function and invertible;*
(2) *$0 < f'(t) \le 1$ for all $t \in \mathbb{R}$;*
(3) *$\frac{f(t)}{t} \to 1$ as $t \to 0$;*
(4) *$\frac{1}{2}f(t) \le tf'(t) \le f(t)$ for all $t \ge 0$, and $\frac{1}{2}f^2(t) \le tf(t)f'(t) \le f^2(t)$ for all $t \in \mathbb{R}$;*
(5) *the function $\frac{f(t)}{t}$ is decreasing for $t > 0$;*
(6) *$|f(t)| \le |t|$ for all $t \in \mathbb{R}$;*
(7) *$|f(t)f'(t)| \le 1/(2^{\frac{p-1}{p}}) < 1$ for all $t \in \mathbb{R}$;*
(8) *$f(t)/\sqrt{|t|}$ is nondecreasing for $t > 0$ and $\lim_{t\to+\infty} f(t)/\sqrt{t} = a > 0$;*
(9) *there exists a positive constant C such that*

$$|f(t)| \ge \begin{cases} C|t|, & |t| \le 1, \\ C|t|^{\frac{1}{2}}, & |t| \ge 1; \end{cases}$$

(10) *$f(t) \le 2^{1/2p}\sqrt{t}$ for all $t \in \mathbb{R}^+$.*

Proposition 2.2. *The following properties involving f and h hold:*
(1) *$(f(t))^{p-1}f'(t)t^{1-p}$ is decreasing for $t > 0$;*
(2) *$(f(t))^{2r-1}f'(t)t^{1-p}$ is increasing for $r \ge p$ and $t > 0$;*
(3) *$h(f(t))f'(t)t^{1-p}$ is increasing for $t > 0$;*
(4) *$F(t) := \frac{1}{p}(f(t))^{2p^*-1}f'(t)t - \frac{1}{2p^*}f^{2p^*}(t)$ is increasing for $t > 0$;*
(5) *$\frac{1}{p}h(f(t))f'(t)t - H(f(t))$ is increasing for $t > 0$.*

Proof. (1) By computation, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{(f(t))^{p-1}f'(t)}{t^{p-1}}\Big) &= (p-1)\Big(\frac{f(t)}{t}\Big)^{p-2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{f(t)}{t}\Big)f'(t) + \frac{(f(t))^{p-1}}{t^{p-1}}f''(t) \\ &= (p-1)\Big(\frac{f(t)}{t}\Big)^{p-2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{f(t)}{t}\Big)f'(t) - 2^{p-1}\frac{(f(t))^{2p-2}}{t^{p-1}}|f'(t)|^{p+2}. \end{aligned}$$

Since $\frac{f(t)}{t}$ is a decreasing function, we can obtain $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{f(t)}{t}) < 0$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{(f(t))^{p-1}f'(t)}{t^{p-1}}\Big) < 0 \quad \text{for } t > 0.$$

(2) By Lemma 2.1 (4) and (7), we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{(f(t))^{2r-1}f'(t)}{t^{p-1}}\Big) &= \frac{(2r-1)f(t)^{2r-2}(f'(t))^2t^{p-1} + (f(t))^{2r-1}f''(t)t^{p-1} - (p-1)(f(t))^{2r-1}f'(t)t^{p-2}}{t^{2p-2}}\\ &= \frac{(f(t))^{2r-2}f'(t)}{t^p}[(2r-1)f'(t)t - 2^{p-1}(f(t))^p|f'(t)|^{p+1}t - (p-1)f(t)]\\ &\geq \frac{(f(t))^{2r-2}f'(t)}{t^p}(2r-2p)f'(t)t \geq 0.\end{aligned}$$

(3) Since

$$\frac{h(f(t))f'(t)}{t^{p-1}} = \frac{h(f(t))}{f(t)^{2p-1}}\frac{f(t)^{2p-1}f'(t)}{t^{p-1}},$$

by using conclusion (2) and (H_3), property (3) is proved.

(4) By Lemma 2.1 (4) and (7), we have that

$$\begin{aligned}F'(t) &= \frac{2p^*-1}{p}(f(t))^{2p^*-2}(f'(t))^2t + \frac{1}{p}(f(t))^{2p^*-1}f''(t)t + \frac{1}{p}(f(t))^{2p^*-1}f'(t) - (f(t))^{2p^*-1}f'(t)\\ &\geq (f(t))^{2p^*-2}f'(t)\Big[\frac{2p^*-1}{p}f'(t)t - \frac{2^{p-1}}{p}(f'(t))^{p+1}(f(t))^pt + (\frac{1}{p}-1)f(t)\Big]\\ &\geq \frac{p^*-p}{p}(f(t))^{2p^*-1}f'(t) > 0.\end{aligned}$$

(5) From (H_3) we obtain that $h'(s)s > (2p-1)h(s)$. Setting $G(t) = \frac{1}{p}h(f(t))f'(t)t - H(f(t))$ and using Lemma 2.1 (4) and (7), we deduce that

$$\begin{aligned}G'(t) &= \frac{1}{p}h'(f(t))(f'(t))^2t - \frac{2^{p-1}}{p}h(f(t))(f(t))^{p-1}(f'(t))^{p+2}t + \Big(\frac{1}{p}-1\Big)h(f(t))f'(t)\\ &= \frac{1}{f(t)}\Big[\frac{1}{p}h'(f(t))f(t)(f'(t))^2t - \frac{2^{p-1}}{p}h(f(t))(f(t))^p(f'(t))^{p+2}t + \Big(\frac{1}{p}-1\Big)h(f(t))f(t)f'(t)\Big]\\ &> \frac{1}{f(t)}\Big[\frac{2p-1}{p}h(f(t))(f'(t))^2t - \frac{2^{p-1}}{p}h(f(t))(f(t))^p(f'(t))^{p+2}t + \Big(\frac{1}{p}-1\Big)h(f(t))f(t)f'(t)\Big]\\ &\geq \frac{h(f(t))f'(t)}{f(t)}\Big[\frac{2p-1}{p}f'(t)t - \frac{1}{p}f'(t)t + \Big(\frac{1}{p}-1\Big)f(t)\Big]\\ &\geq \frac{h(f(t))f'(t)}{f(t)}\Big(\frac{p-1}{p}f(t) + \frac{1-p}{p}f(t)\Big) = 0\end{aligned}$$

as desired. ☐

Under the change of variables, we can rewrite the functional I defined by (2.1) in the following form:

$$J(v) := I(f(v)) = \frac{\varepsilon^p}{p}\int_{\mathbb{R}^N}|\nabla v|^p\,\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N}V(x)|f(v)|^p\,\mathrm{d}x - \int_{\mathbb{R}^N}H(f(v))\,\mathrm{d}x - \frac{1}{2p^*}\int_{\mathbb{R}^N}|f(v^+)|^{2p^*}\,\mathrm{d}x,$$

which is well defined on the Banach space

$$X = \Big\{v \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N}V(x)|v|^p\,\mathrm{d}x < \infty\Big\}$$

endowed with the norm

$$\|v\|_X = \Big(\int_{\mathbb{R}^N}(|\nabla v|^p + V(x)|v|^p)\,\mathrm{d}x\Big)^{\frac{1}{p}}.$$

In view of conditions (H_0) and (H_1), by the standard arguments, we conclude that $J \in C^1(X, \mathbb{R})$ and

$$\begin{aligned}\langle J'(v), \varphi\rangle = \varepsilon^p\int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v\nabla\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N}V(x)|f(v)|^{p-2}f(v)f'(v)\varphi\,\mathrm{d}x \int_{\mathbb{R}^N}h(f(v))f'(v)\varphi\,\mathrm{d}x\\ - \int_{\mathbb{R}^N}|f(v^+)|^{2p^*-2}f(v^+)f'(v^+)\varphi\,\mathrm{d}x\end{aligned}$$

for all $v, \varphi \in X$. Moreover, the critical points of J are the weak solutions of the Euler–Lagrange equation associated with the functional J given by

$$-\varepsilon^p \Delta_p v + V(x)|f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v) + |f(v)|^{2p^*-1} f'(v).$$

We observe that if $v \in X \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is a critical point of the functional J, then $u = f(v) \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is a weak solution of problem (1.1), that is,

$$\begin{aligned}\varepsilon^p \int_{\mathbb{R}^N} [(1 + 2^{p-1}|u|^p)|\nabla u|^{p-2} \nabla u \nabla \varphi + 2^{p-1}|\nabla u|^p |u|^{p-2} u\varphi]\, dx + \int_{\mathbb{R}^N} V(x)|u|^{p-2} u\varphi\, dx \\ = \int_{\mathbb{R}^N} [h(u) + |u|^{2p^*-2} u]\varphi\, dx\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$.

3 Autonomous problem

In this section, we will study the existence of a positive ground state solution for the following equation:

$$\begin{cases} -\Delta_p v + \mu |f(v)|^{p-2} f(v) f'(v) = h(f(v)) f'(v) + |f(v)|^{2p^*-1} f'(v) & \text{in } \mathbb{R}^N, \\ v \in W^{1,p}(\mathbb{R}^N),\ v(x) > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{$\mathcal{Q}_\mu$}$$

where μ is an arbitrary positive constant and $2 \le p < N$. The functional $\mathcal{J}_\mu$ associated to problem ($\mathcal{Q}_\mu$) is given by

$$\mathcal{J}_\mu(v) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla v|^p\, dx + \frac{\mu}{p} \int_{\mathbb{R}^N} |f(v)|^p\, dx - \int_{\mathbb{R}^N} H(f(v))\, dx - \frac{1}{2p^*} \int_{\mathbb{R}^N} |f(v^+)|^{2p^*}\, dx,$$

which is well defined on the Banach space W_μ endowed with the norm

$$\|v\|_\mu = \Big(\int_{\mathbb{R}^N} (|\nabla v|^p + \mu |v|^p)\, dx \Big)^{\frac{1}{p}}.$$

From the hypotheses (H_0)–(H_1) it is easy to verify that $J_\mu \in C^1(W_\mu, \mathbb{R})$ and

$$\begin{aligned}\langle \mathcal{J}'_\mu(v), \varphi \rangle = \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \varphi\, dx + \int_{\mathbb{R}^N} \mu |f(v)|^{p-2} f(v) f'(v) \varphi\, dx - \int_{\mathbb{R}^N} h(f(v)) f'(v) \varphi\, dx \\ - \int_{\mathbb{R}^N} |f(v^+)|^{2p^*-2} f(v^+) f'(v^+) \varphi\, dx,\end{aligned}$$

for all $v, \varphi \in W_\mu$. Moreover, the weak solution v of ($\mathcal{Q}_\mu$) corresponds to the critical point of the functional $\mathcal{J}_\mu$.

Let us denote the Nehari manifold associated to ($\mathcal{Q}_\mu$) by $\mathcal{N}_\mu$, that is,

$$\mathcal{N}_\mu = \{ v \in W_\mu : v \neq 0 \text{ and } \langle \mathcal{J}'_\mu(v), v \rangle = 0 \}.$$

3.1 Mountain pass geometry

Theorem 3.1 ([36]). *Let E be a real Banach space and $J : E \to \mathbb{R}$ a functional of class C^1. Let S be a closed subset of E which disconnects E into distinct connected components E_1 and E_2. Suppose further that $J(0) = 0$ and that the following conditions hold:*

(a) *$0 \in E_1$ and there is $\alpha > 0$ such that $J|_S \ge \alpha > 0$;*

(b) *there is $e \in E_2$ such that $J(e) < 0$.*

Then J possesses a $(C)_c$ sequence with $c \geq \alpha > 0$ given by

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ J(\gamma(1)) < 0\}.$$

Consider the set $S_\mu(\rho) = \{v \in W_\mu : Q_\mu(v) = \rho^p\}$ and define

$$Q_\mu(v) = \int_{\mathbb{R}^N} (|\nabla v|^p + \mu|f(v)|^p)\,\mathrm{d}x.$$

Since $Q_\mu(v)$ is continuous, $S_\mu(\rho)$ is a closed subset of W_μ and it disconnects this space.

Lemma 3.2. *Suppose that conditions* (V) *and* (H_0)–(H_1) *are satisfied. Then there exist $\rho_0, \delta_0 > 0$ such that*

$$\mathcal{J}_\mu(v) \geq \delta_0 \quad \textit{for all } v \in S_\mu(\rho_0).$$

Proof. By the hypotheses (H_0)–(H_1), we have that

$$\int_{\mathbb{R}^N} H(f(v))\mathrm{d}x \leq \frac{\varepsilon}{p}\int_{\mathbb{R}^N} |f(v)|^p\,\mathrm{d}x + C_\varepsilon \int_{\mathbb{R}^N} |f(v)|^{2p^*}\,\mathrm{d}x. \tag{3.1}$$

By (3.1), Lemma 2.1 (10) and the Sobolev inequality, we have that

$$\begin{aligned}
\mathcal{J}_\mu(v) &= \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v|^p + \mu|f(v)|^p)\,\mathrm{d}x - \int_{\mathbb{R}^N} H(f(v))\mathrm{d}x - \frac{1}{2p^*}\int_{\mathbb{R}^N}|f(v)|^{2p^*}\,\mathrm{d}x \\
&\geq \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v|^p + (\mu-\varepsilon)|f(v)|^p)\,\mathrm{d}x - C_0\int_{\mathbb{R}^N}|v|^{p^*}\,\mathrm{d}x \\
&\geq C_1\int_{\mathbb{R}^N}(|\nabla v|^p + \mu|f(v)|^p)\,\mathrm{d}x - C_2\Big(\int_{\mathbb{R}^N}|\nabla v|^p\,\mathrm{d}x\Big)^{\frac{p^*}{p}} \\
&\geq C_1\rho^p - C_2\rho^{p^*},
\end{aligned}$$

where $C_0, C_1, C_2 > 0$ are positive constants. Thus we choose $\rho = \rho_0$ sufficiently small and there exists $\delta_0 > 0$ such that $\mathcal{J}_\mu(v) \geq \delta_0 > 0$ for all $v \in S_\mu(\rho_0)$. □

Similar to the proofs of [2, Lemma 3.2 and 3.3], we can deduce the following two Lemmas.

Lemma 3.3. *Suppose that* (V), (H_0)–(H_1) *and* (H_4) *hold. Then for each $v \in W_\mu \setminus \{0\}$ the following limits hold:*
(1) *if $v_+ \neq 0$, then $\mathcal{J}_\mu(tv) \to -\infty$ as $t \to +\infty$;*
(2) *if $v_+ = 0$, then $\mathcal{J}_\mu(tv) \to +\infty$ as $t \to +\infty$.*

Lemma 3.4. *For every $v \in W_\mu \setminus \{0\}$ with $v^+ \neq 0$, there exists a unique $t_v > 0$ such that $t_v v \in \mathcal{N}_\mu$. Moreover, $\mathcal{J}_\mu(t_v v) = \max_{t\geq 0} \mathcal{J}_\mu(tv)$.*

From Lemmas 3.2–3.4 it follows that $\mathcal{J}_\mu$ possesses the mountain pass geometry with

$$c_\mu = \inf_{\gamma\in\Gamma_\mu} \max_{t\in[0,1]} \mathcal{J}_\mu(\gamma(t)),$$

where

$$\Gamma_\mu = \{\gamma \in C([0,1], W_\mu) : \gamma(0) = 0,\ \mathcal{J}_\mu(\gamma(1)) < 0\},$$

and c_μ can be characterized by the following identity:

$$c_\mu = \inf_{v\in\mathcal{N}_\mu} \mathcal{J}_\mu(v) = \inf_{v\in W_\mu\setminus\{0\}} \max_{t\geq 0} \mathcal{J}_\mu(tv). \tag{3.2}$$

Therefore, by Theorem 3.1, there exists a $(C)_{c_\mu}$ sequence $\{v_n\} \subset W_\mu$ of $\mathcal{J}_\mu$, that is,

$$\mathcal{J}_\mu(v_n) \to c_\mu \quad \text{and} \quad (1 + \|v_n\|_\mu)\mathcal{J}'_\mu(v_n) \to 0 \qquad \text{as } n \to \infty. \tag{3.3}$$

Now, we will give the detailed properties of the above $(C)_{c_\mu}$ sequence in the following two Lemmas.

Lemma 3.5. *Let $\{v_n\}$ be a $(C)_{c_\mu}$ sequence of $\mathcal{J}_\mu$. Then the following holds:*
(1) *$\{v_n\}$ is bounded in W_μ and there exists a $v \in W_\mu$ such that $v_n \rightharpoonup v$ in W_μ;*
(2) *$J'_\mu(v) = 0$;*
(3) *$v_n \geq 0$ for $n \in \mathbb{N}$.*

Proof. (1) By (3.3), (H_2) and Lemma 2.1 (4), we have that

$$\begin{aligned} c_\mu + o_n(1) &= \mathcal{J}_\mu(v_n) - \frac{2}{\theta}\langle \mathcal{J}'_\mu(v_n), v_n\rangle \\ &\geq \Big(\frac{1}{p} - \frac{2}{\theta}\Big)\int_{\mathbb{R}^N} |\nabla v_n|^p \,\mathrm{d}x + \mu\Big(\frac{1}{p} - \frac{2}{\theta}\Big)\int_{\mathbb{R}^N} |f(v_n)|^p \,\mathrm{d}x + \Big(\frac{1}{\theta}\int_{\mathbb{R}^N} h(f(v_n))f(v_n)\,\mathrm{d}x - \int_{\mathbb{R}^N} H(f(v_n))\Big)\,\mathrm{d}x \\ &\quad + \Big(\frac{1}{\theta} - \frac{1}{2p^*}\Big)\int_{\mathbb{R}^N} |f(v_n^+)|^{2p^*}\,\mathrm{d}x \\ &\geq \Big(\frac{1}{p} - \frac{2}{\theta}\Big)\int_{\mathbb{R}^N} |\nabla v_n|^p \,\mathrm{d}x + \mu\Big(\frac{1}{p} - \frac{2}{\theta}\Big)\int_{\mathbb{R}^N} |f(v_n)|^p \,\mathrm{d}x, \end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N} |\nabla v_n|^p \,\mathrm{d}x + \mu \int_{\mathbb{R}^N} |f(v_n)|^p \mathrm{d}x \leq C_1.$$

On the other hand, using Lemma 2.1 (9) and the Sobolev inequality, we have that

$$\begin{aligned} \int_{\mathbb{R}^N} |v_n|^p \,\mathrm{d}x &= \int_{\{|v_n|\leq 1\}} |v_n|^p \,\mathrm{d}x + \int_{\{|v_n|\geq 1\}} |v_n|^p \,\mathrm{d}x \\ &\leq \int_{\{|v_n|\leq 1\}} |f(v_n)|^p \,\mathrm{d}x + \int_{\{|v_n|\geq 1\}} |v_n|^{p^*} \,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^N} |f(v_n)|^p \,\mathrm{d}x + \Big(\int_{\mathbb{R}^N} |\nabla v_n|^p \,\mathrm{d}x\Big)^{\frac{p^*}{p}}. \end{aligned}$$

Therefore, $\{v_n\}$ is bounded in W_μ, and there exists a $v \in W_\mu$ such that $v_n \rightharpoonup v$ in W_μ. Hence, up to a subsequence, there exists $v \in W_\mu$ such that

$$\begin{cases} v_n \rightharpoonup v & \text{in } W_\mu, \\ v_n \to v & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^N) \text{ for } 1 \leq s < p^*, \\ v_n(x) \to v(x) & \text{a.e. } x \in \mathbb{R}^N. \end{cases} \tag{3.4}$$

Moreover, using [27, Theorem 1.6], we can get

$$\nabla v_n \to \nabla v \quad \text{a.e. in } \mathbb{R}^N. \tag{3.5}$$

Indeed, by translation, equation $(\mathcal{Q}_\mu)$ is reduced to

$$-\Delta_p v + \mu|v|^{p-2}v = h(f(v))f'(v) + |f(v)|^{2p^*-1}f'(v) - \mu|f(v)|^{p-1}f'(v) + \mu|v|^{p-2}v.$$

Let $\tilde{f}(x,v) = h(f(v))f'(v) + |f(v)|^{2p^*-1}f'(v) - \mu|f(v)|^{p-1}f'(v) + \mu|v|^{p-2}v$. By hypotheses (H_0)–(H_1) and using Lemma 2.1 (2), (6), (7), and (10), we have that

$$\lim_{t\to 0} \frac{\tilde{f}(x,t)}{|t|^{p-2}t} = 0, \qquad \lim_{|t|\to+\infty} \frac{\tilde{f}(x,t)}{|t|^{p^*-2}t} = c_0 > 0,$$

where $c_0 > 0$ is a constant. Thus we have verified all conditions of [27, Theorem 1.6, Step 2], hence (3.5) follows.

(2) Since $C_0^\infty(\mathbb{R}^N)$ is dense in W_μ, we only need to show $\langle J_\mu'(v_n), \varphi\rangle = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. We observe that

$$\begin{aligned}
&\langle \mathcal{J}_\mu'(v_n), \varphi\rangle - \langle \mathcal{J}_\mu'(v), \varphi\rangle \\
&\quad= \int\limits_{\mathbb{R}^N} (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)\nabla\varphi \,\mathrm{d}x + \mu \int\limits_{\mathbb{R}^N} [|f(v_n)|^{p-2} f(v_n) f'(v_n) - |f(v)|^{p-2} f(v) f'(v)]\varphi \,\mathrm{d}x \\
&\qquad\quad + \int\limits_{\mathbb{R}^N} (h(f(v))f'(v) - h(f(v_n))f'(v_n))\varphi \,\mathrm{d}x + \int\limits_{\mathbb{R}^N} [|f(v^+)|^{2p^*-1} f'(v^+) - |f(v_n^+)|^{2p^*-1} f'(v_n^+)]\varphi \,\mathrm{d}x,
\end{aligned}$$

thus we need to show that the following limits hold:

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)\nabla\varphi \,\mathrm{d}x = 0, \tag{3.6}$$

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} [|f(v_n)|^{p-2} f(v_n) f'(v_n) - |f(v)|^{p-2} f(v) f'(v)]\varphi \,\mathrm{d}x = 0, \tag{3.7}$$

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} (h(f(v))f'(v) - h(f(v_n))f'(v_n))\varphi \,\mathrm{d}x = 0, \tag{3.8}$$

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} [|f(v^+)|^{2p^*-1} f'(v^+) - |f(v_n^+)|^{2p^*-1} f'(v_n^+)]\varphi \,\mathrm{d}x = 0 \tag{3.9}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$.

By (3.5), it is easy to show that (3.6) holds by the weak convergence argument.

Let $\tilde{v}_n = v_n - v$; next we show that (3.7)–(3.9) hold. Using Lemma 2.1 (2), (6), (7) and Young's inequality, we deduce that

$$\begin{aligned}
&\big||f(v_n)|^{m-2} f(v_n) f'(v_n) - |f(v)|^{m-2} f(v) f'(v)\big| \\
&\quad= \Big|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}[|f(v + t\tilde{v}_n)|^{m-2} f(v + t\tilde{v}_n) f'(v + t\tilde{v}_n)] \,\mathrm{d}t\Big| \\
&\quad\le \int_0^1 [(m-1)|\tilde{v}_n||f(v + t\tilde{v}_n)|^{m-2}(f'(v + t\tilde{v}_n))^2 + |\tilde{v}_n||f(v + t\tilde{v}_n)|^{m-1}|f''(v + t\tilde{v}_n)|] \,\mathrm{d}t \\
&\quad\le \int_0^1 [(m-1)|\tilde{v}_n||v + t\tilde{v}_n|^{m-2} + 2^{p-1}|\tilde{v}_n||f(v + t\tilde{v}_n)|^{p+m-2}|f'(v + t\tilde{v}_n)|^{p+2}] \,\mathrm{d}t \\
&\quad\le C_3(|\tilde{v}_n|^{m-1} + |\tilde{v}_n||v|^{m-2}) \\
&\quad\le C_4|\tilde{v}_n|^{m-1} + C_5|v|^{m-1},
\end{aligned} \tag{3.10}$$

where $p \le m < 2p^*$. By (3.4), we obtain

$$|f(v_n)|^{m-2} f(v_n) f'(v_n) - |f(v)|^{m-2} f(v) f'(v) - C_4|\tilde{v}_n|^{m-1} \to 0 \quad \text{a.e. in } \mathbb{R}^N.$$

By the Hölder inequality, we have

$$\begin{aligned}
\int\limits_{\mathbb{R}^N} [|f(v_n)|^{m-2} f(v_n) f'(v_n) - |f(v)|^{m-2} f(v) f'(v) - C_4|\tilde{v}_n|^{m-1}]\varphi \,\mathrm{d}x &\le C_5 \int\limits_{\mathbb{R}^N} |v|^{m-1}\varphi \,\mathrm{d}x \\
&\le C_5\Big(\int\limits_{\mathbb{R}^N} |v|^m \,\mathrm{d}x\Big)^{\frac{m-1}{m}} \Big(\int\limits_{\mathbb{R}^N} |\varphi|^m \,\mathrm{d}x\Big)^{\frac{1}{m}}.
\end{aligned}$$

From the Lebesgue dominated convergence theorem it follows that

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} [|f(v_n)|^{m-2} f(v_n) f'(v_n) - |f(v)|^{m-2} f(v) f'(v) - C_4|\tilde{v}_n|^{m-1}]\varphi \,\mathrm{d}x = 0.$$

By (3.4), we deduce that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}[|f(v_n)|^{m-2}f(v_n)f'(v_n)-|f(v)|^{m-2}f(v)f'(v)]\varphi\,dx = C_5\lim_{n\to\infty}\int_{\mathbb{R}^N}|\tilde{v}_n|^{m-1}\varphi\,dx = 0.$$

Take $m = p$ and $m = q$; then (3.7) and (3.8) follows. In the case $m = 2p^*$, using Lemma 2.1 (10), we can deduce that

$$\big||f(v_n^+)|^{2p^*-2}f(v_n^+)f'(v_n^+)-|f(v^+)|^{2p^*-2}f(v^+)f'(v^+)\big| \le C_6(\tilde{v}_n^+)^{p^*-1}+C_7(v^+)^{p^*-1}.$$

Since $(\tilde{v}_n^+)^{p^*-1}\rightharpoonup 0$ in $L^{p^*/(p^*-1)}(\operatorname{supp}\varphi)$, so we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}[|f(v_n^+)|^{2p^*-2}f(v_n^+)f'(v_n^+)-|f(v^+)|^{2p^*-2}f(v^+)f'(v^+)]\varphi\,dx = \lim_{n\to\infty}\int_{\mathbb{R}^N}(\tilde{v}_n^+)^{p^*-1}\varphi\,dx = 0.$$

Hence (3.9) is proven.

Consequently, we obtain

$$\langle \mathcal{J}_\mu'(v_n),\varphi\rangle-\langle \mathcal{J}_\mu'(v),\varphi\rangle \to 0 \quad \text{as } n\to\infty.$$

Meanwhile, if $\{v_n\}$ is a bounded $(C)_{c_\mu}$ sequence of J_μ, then

$$\|\mathcal{J}_\mu'(v_n)\|_{(W_\mu)^*} = (1+\|v_n\|_\mu)\frac{\sup_{\varphi\in W_\mu}\langle \mathcal{J}_\mu'(v_n),\varphi\rangle}{1+\|v_n\|_\mu} \le \|(1+\|v_n\|_\mu)\mathcal{J}_\mu'(v_n)\|_{(W_\mu)^*}\|\varphi\|_\mu \to 0$$

for any $\varphi\in W_\mu$. Thus $\langle \mathcal{J}_\mu'(v_n),\varphi\rangle = 0$, and as a result $\langle \mathcal{J}_\mu'(v),\varphi\rangle = 0$ for any $\varphi\in C_0^\infty(\mathbb{R}^N)$.

(3) Since $\{v_n\}$ is bounded in W_μ, we get that $\{v_n^-\}$ is bounded in W_μ, where $v_n^- = \max\{-v_n, 0\}$. Using (H_0), we have that

$$\begin{aligned} o_n(1) &= \langle \mathcal{J}_\mu'(v_n), -v_n^-\rangle \\ &\ge \int_{\mathbb{R}^N}|\nabla v_n^-|^p\,dx + \mu\int_{\mathbb{R}^N}|f(v_n^-)|^{p-2}f(v_n^-)f'(v_n^-)(v_n^-)\,dx \\ &\ge \int_{\mathbb{R}^N}|\nabla v_n^-|^p\,dx + \frac{\mu}{2}\int_{\mathbb{R}^N}|f(v_n^-)|^p\,dx \\ &\ge \frac{1}{2}\Big(\int_{\mathbb{R}^N}|\nabla v_n^-|^p\,dx + \mu\int_{\mathbb{R}^N}|f(v_n^-)|^p\,dx\Big). \end{aligned}$$

Thus $Q_\mu(v_n^-)\to 0$. Similar to the proof of [37, Proposition 2.4], we can deduce that

$$\int_{\mathbb{R}^N}|\nabla v_n^-|^p\,dx + \mu\int_{\mathbb{R}^N}|f(v_n^-)|^p\,dx \ge C_8\|v_n^-\|_\mu$$

for some $C_8 > 0$ independent of n. Therefore, we get $v_n^- = 0$ in W_μ. Hence, we get $v_n = v_n^+ + o_n(1)$ in W_μ. □

Lemma 3.6. *Let $\{v_n\}$ be a $(C)_{c_\mu}$ sequence of $\mathcal{J}_\mu$ with $c_\mu < \frac{1}{2N}S^{N/p}$. Then one of the following conclusions holds:*
(1) $Q_\mu(v_n)\to 0$;
(2) *there exist $\{y_n\}\subset\mathbb{R}^N$ and positive constants R, ξ such that*

$$\lim_{n\to\infty}\inf\int_{B_R(y_n)}|v_n|^p\,dx \ge \xi.$$

Proof. Assume that (2) dose not occur, that is, for all $R > 0$ there holds

$$\lim_{n\to\infty}\inf\int_{B_R(y_n)}|v_n|^p\,dx = 0.$$

By the vanishing Lemma in [28], we can assume $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for all $s \in (p, p^*)$. By (H_0)–(H_1) and Lemma 2.1 (6) and (10), we have

$$\begin{aligned}\int_{\mathbb{R}^N} H(f(v_n))\mathrm{d}x &= \int_{\mathbb{R}^N} h(f(v_n))f(v_n)\,\mathrm{d}x\\ &\le \int_{\mathbb{R}^N} (\varepsilon|f(v_n)|^p + C_\varepsilon|f(v_n)|^{q+1})\,\mathrm{d}x\\ &\le \varepsilon\int_{\mathbb{R}^N} |v_n|^p + C_\varepsilon\int_{\mathbb{R}^N} |v_n|^{\frac{q+1}{2}}\,\mathrm{d}x = o_n(1),\end{aligned}$$

Next, since

$$\Big\|\frac{f(v_n)}{f'(v_n)}\Big\|_\mu \le 2\|v_n\|_\mu$$

and thus

$$\Big\langle J'_\mu(v_n), \frac{f(v_n)}{f'(v_n)}\Big\rangle = o_n(1),$$

that is,

$$\begin{aligned}\int_{\mathbb{R}^N}\Big[1 + \frac{2^{p-1}f^p(v_n)}{1+2^{p-1}f^p(v_n)}\Big]|\nabla v_n|^p\,\mathrm{d}x + \mu\int_{\mathbb{R}^N}|f(v_n)|^p\,\mathrm{d}x &= \int_{\mathbb{R}^N} h(f(v_n))f(v_n)\,\mathrm{d}x + \int_{\mathbb{R}^N}|f(v_n^+)|^{2p^*}\,\mathrm{d}x + o_n(1)\\ &= \int_{\mathbb{R}^N}|f(v_n^+)|^{2p^*}\,\mathrm{d}x + o_n(1).\end{aligned} \tag{3.11}$$

Denote by $l \ge 0$ a number such that

$$\int_{\mathbb{R}^N}\Big[1 + \frac{2^{p-1}f^p(v_n)}{1+2^{p-1}f^p(v_n)}\Big]|\nabla v_n|^p\,\mathrm{d}x + \mu\int_{\mathbb{R}^N}|f(v_n)|^p\,\mathrm{d}x \to l \quad\text{and}\quad \int_{\mathbb{R}^N}|f(v_n^+)|^{2p^*}\,\mathrm{d}x \to l.$$

Assume that $l > 0$; by the definition of $S = \inf\{\int_{\mathbb{R}^N}|\nabla u|^p\,\mathrm{d}x : \|u\|_{p^*} = 1\}$, we have

$$\begin{aligned} S &\le \frac{\int_{\mathbb{R}^N}|\nabla f^2(v_n^+)|^p\,\mathrm{d}x}{(\int_{\mathbb{R}^N}|f^2(v_n^+)|^{p^*}\,\mathrm{d}x)^{\frac{p}{p^*}}} = \frac{\int_{\mathbb{R}^N}\frac{2^p|f(v_n^+)|^p}{1+2^{p-1}|f(v_n^+)|^p}|\nabla v_n^+|^p\,\mathrm{d}x}{(\int_{\mathbb{R}^N}|f^2(v_n^+)|^{p^*}\,\mathrm{d}x)^{\frac{p}{p^*}}}\\ &\le \frac{\int_{\mathbb{R}^N}(1+\frac{2^{p-1}|f(v_n^+)|^p}{1+2^{p-1}|f(v_n^+)|^p})|\nabla v_n^+|^p\,\mathrm{d}x}{(\int_{\mathbb{R}^N}|f(v_n^+)|^{2p^*}\,\mathrm{d}x)^{\frac{p}{p^*}}}\\ &\to l^{1-\frac{p}{p^*}}\quad \text{as } n\to\infty,\end{aligned}$$

thus, $l \ge S^{\frac{N}{p}}$. Combining this with (3.11), we obtain that

$$\begin{aligned} c_\mu &= \lim_{n\to\infty}\Big\{\frac{1}{p}\int_{\mathbb{R}^N}[|\nabla v_n|^p + \mu f^p(v_n)]\,\mathrm{d}x - \frac{1}{2p^*}\int_{\mathbb{R}^N}|f(v_n^+)|^{2p^*}\,\mathrm{d}x - \int_{\mathbb{R}^N}H(f(v_n))\,\mathrm{d}x\Big\}\\ &\ge \lim_{n\to\infty}\Big\{\frac{1}{2p}\int_{\mathbb{R}^N}[(1+\frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p})|\nabla v_n|^p + \mu f^p(v_n)]\,\mathrm{d}x - \frac{1}{2p^*}\int_{\mathbb{R}^N}|f(v_n^+)|^{2p^*}\,\mathrm{d}x\Big\}\\ &= \Big(\frac{1}{2p} - \frac{1}{2p^*}\Big)l\\ &\ge \frac{1}{2N}S^{\frac{N}{p}},\end{aligned}$$

which yields a contradiction because $c_\mu < \frac{1}{2N}S^{N/p}$. Thus $l = 0$. □

For the least energy level c_μ, we have the following estimate.

Lemma 3.7. *For any $\mu > 0$, there exists $v_0 \in W_\mu \setminus \{0\}$ such that*

$$\max_{t\geq 0} \mathcal{J}_\mu(tv_0) < \frac{1}{2N} S^{\frac{N}{p}} \quad \text{and} \quad c_\mu = \inf_{v\in\mathcal{N}_\mu} \mathcal{J}_\mu(v) < \frac{1}{2N} S^{\frac{N}{p}},$$

where S denotes the best constant for the embedding $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$.

Proof. Define a functional $I_\mu : W_\mu \cap L^\infty(\mathbb{R}^N) \to \mathbb{R}$ by

$$I_\mu(u) = \frac{1}{p} \int_{\mathbb{R}^N} (1 + 2^{p-1}|u|^p)|\nabla u|^p \,\mathrm{d}x + \frac{\mu}{p} \int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} H(u)\,\mathrm{d}x - \frac{1}{2p^*} \int_{\mathbb{R}^N} |u^+|^{2p^*} \,\mathrm{d}x.$$

By the equivalent characteristic of c_μ (see (3.2)), we only need to prove that there exists $0 \neq v_0 \in W_\mu \cap L^\infty(\mathbb{R}^N)$ such that

$$\sup_{t\geq 0} I_\mu(tv_0) < \frac{1}{2N} S^{\frac{N}{p}}.$$

From Lemma 3.3 we know that $I_\mu(tv_0) \to -\infty$ as $t \to +\infty$; then there exists some $t^* > 0$ such that $I_\mu(t^* v_0) < 0$. Define $\gamma^*(t) := f^{-1}(tt^* v_0)$; by the definition of c_μ, we have

$$c_\mu := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathcal{J}_\mu(\gamma(t)) \leq \sup_{t\in[0,1]} \mathcal{J}_\mu(\gamma^*(t)) \leq \sup_{t\geq 0} I_\mu(tv_0) < \frac{1}{2N} S^{\frac{N}{p}}.$$

Fix $\varepsilon > 0$ and define the function

$$u_\varepsilon(x) = \frac{\psi(x)}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{2p}}}, \qquad v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon^2\|_{L^{p^*}}^{\frac{1}{2}}},$$

where $\psi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ is such that $0 \leq \psi \leq 1$ if $|x| < 1$ and $\psi(x) = 0$ if $|x| \geq 2$. By [21, Lemma 4.1] we know that u_ε verifies the following estimates:

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon^2|^p \,\mathrm{d}x = K_1 \varepsilon^{\frac{p-N}{p}} + O(1), \qquad \Big(\int_{\mathbb{R}^N} |u_\varepsilon^2|^{p^*} \,\mathrm{d}x\Big)^{\frac{p}{p^*}} = K_2 \varepsilon^{\frac{p-N}{p}} + O(1),$$

and

$$\int_{\mathbb{R}^N} |u_\varepsilon^2|^t \,\mathrm{d}x = \begin{cases} K_3 \varepsilon^{\frac{N(p-1)-t(N-p)}{p}} + O(1) & \text{if } t > \dfrac{N(p-1)}{N-p}, \\ K_3 |\ln\varepsilon| + O(1) & \text{if } t = \dfrac{N(p-1)}{N-p}, \\ O(1) & \text{if } t < \dfrac{N(p-1)}{N-p}, \end{cases}$$

where K_1, K_2, K_3 are positive constants independent of ε and $S = \frac{K_1}{K_2}$.

By computations, v_ε verifies

$$\|v_\varepsilon\|_{L^{2p^*}} = 1, \quad \|\nabla v_\varepsilon^2\|_{L^p}^p = S + O(\varepsilon^{\frac{N-p}{p}}), \quad \|\nabla v_\varepsilon\|_{L^p}^p = O(\varepsilon^{\frac{N-p}{2p}}) \tag{3.12}$$

and

$$\int_{\mathbb{R}^N} |v_\varepsilon|^q \,\mathrm{d}x = \begin{cases} O\Big(\varepsilon^{\frac{(N-p)q}{2p^2}}\Big) & \text{if } q < 2p^* - \dfrac{2N}{N-p}, \\ O\Big(\varepsilon^{\frac{(N-p)q}{2p^2}} |\ln\varepsilon|\Big) & \text{if } q = 2p^* - \dfrac{2N}{N-p}, \\ O\Big(\varepsilon^{\frac{N(p-1)-\frac{q}{2}(N-p)}{p} + \frac{q(N-p)}{2p^2}}\Big) & \text{if } q > 2p^* - \dfrac{2N}{N-p}. \end{cases} \tag{3.13}$$

Obviously, $v_\varepsilon \in W_\mu \cap L^\infty(\mathbb{R}^N)$, and by (H_4) we have

$$\begin{aligned} I_\mu(tv_\varepsilon) &= \frac{t^p}{p} \int_{\mathbb{R}^N} \Big(|\nabla v_\varepsilon|^p + \frac{t^p}{2}|\nabla v_\varepsilon^2|^p\Big)\,\mathrm{d}x + \frac{t^p}{p} \int_{\mathbb{R}^N} \mu|v_\varepsilon|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} H(tv_\varepsilon)\,\mathrm{d}x - \frac{t^{2p^*}}{2p^*} \int_{\mathbb{R}^N} |v_\varepsilon|^{2p^*} \,\mathrm{d}x \\ &\leq \frac{t^p}{p} \int_{\mathbb{R}^N} (|\nabla v_\varepsilon|^p + \mu|v_\varepsilon|^p)\,\mathrm{d}x + \frac{t^{2p}}{2p} \int_{\mathbb{R}^N} |\nabla v_\varepsilon^2|^p \,\mathrm{d}x - \frac{Ct^\sigma}{\sigma} \int_{\mathbb{R}^N} |v_\varepsilon|^\sigma \,\mathrm{d}x - \frac{t^{2p^*}}{2p^*} =: g_\mu(t). \end{aligned}$$

It is clear that $\lim_{t\to\infty} g_\mu(t) = -\infty$ and $g_\mu(t) > 0$ when t is small; then $\sup_{t\ge 0} g_\mu(t)$ is attained at some $t_\varepsilon > 0$. It follows that

$$\begin{aligned}
0 &= g'_\mu(t_\varepsilon)\\
&= t_\varepsilon^{p-1} \int\limits_{\mathbb{R}^N} (|\nabla v_\varepsilon|^p + \mu |v_\varepsilon|^p)\,\mathrm{d}x + t_\varepsilon^{2p-1} \int\limits_{\mathbb{R}^N} |\nabla v_\varepsilon^2|^p\,\mathrm{d}x - C t_\varepsilon^{\sigma-1} \int\limits_{\mathbb{R}^N} |v_\varepsilon|^\sigma\,\mathrm{d}x - t_\varepsilon^{2p^*-1}\\
&= t_\varepsilon^{p-1}\Big[\int\limits_{\mathbb{R}^N} (|\nabla v_\varepsilon|^p + \mu |v_\varepsilon|^p)\,\mathrm{d}x + t_\varepsilon^{p} \int\limits_{\mathbb{R}^N} |\nabla v_\varepsilon^2|^p\,\mathrm{d}x - C t_\varepsilon^{\sigma-p} \int\limits_{\mathbb{R}^N} |v_\varepsilon|^\sigma\,\mathrm{d}x - t_\varepsilon^{2p^*-p}\Big].
\end{aligned}$$

We have

$$\int\limits_{\mathbb{R}^N} (|\nabla v_\varepsilon|^p + \mu |v_\varepsilon|^p)\,\mathrm{d}x + t_\varepsilon^{p} \int\limits_{\mathbb{R}^N} |\nabla v_\varepsilon^2|^p\,\mathrm{d}x = C t_\varepsilon^{\sigma-p} \int\limits_{\mathbb{R}^N} |v_\varepsilon|^\sigma\,\mathrm{d}x + t_\varepsilon^{2p^*-p} \ge t_\varepsilon^{2p^*-p},$$

so t_ε is bounded from above by some $T_1 > 0$. On the other hand,

$$\int\limits_{\mathbb{R}^N} |\nabla v_\varepsilon^2|^p\,\mathrm{d}x \le t_\varepsilon^{2p^*-2p} + C t_\varepsilon^{\sigma-2p} \int\limits_{\mathbb{R}^N} |v_\varepsilon|^\sigma\,\mathrm{d}x.$$

Since $\sigma > 2p$, combining (3.12) with (3.13) and choosing ε small enough, we have $t_\varepsilon^{2p^*-2p} \ge S/2$, so t_ε is bounded from below by some $T_2 > 0$ independent of ε. Next, we define

$$e_\mu(t) := \frac{t^{2p}}{2p} \int\limits_{\mathbb{R}^N} |\nabla v_\varepsilon^2|^p\,\mathrm{d}x - \frac{t^{2p^*}}{2p^*},$$

which attains its unique global maximum at

$$t_0 = \Big(\int\limits_{\mathbb{R}^N} |\nabla v_\varepsilon^2|^p\,\mathrm{d}x\Big)^{\frac{1}{2p^*-2p}}.$$

Thus, by (3.12) and (3.13), using the fact that $\sigma > 2pN/(N-p) - \frac{2N}{N-p}$, we have that

$$\begin{aligned}
\max_{t\ge 0} I_\mu(t v_\varepsilon) \le g_\mu(t_\varepsilon) &\le e_\mu(t_0) + \frac{t_\varepsilon^p}{p} \int\limits_{\mathbb{R}^N} (|\nabla v_\varepsilon|^p + \mu |v_\varepsilon|^p)\,\mathrm{d}x - \frac{C t_\varepsilon^\sigma}{\sigma} \int\limits_{\mathbb{R}^N} |v_\varepsilon|^\sigma\,\mathrm{d}x\\
&\le \Big(\frac{1}{2p} - \frac{1}{2p^*}\Big)\Big(\int\limits_{\mathbb{R}^N} |\nabla v_\varepsilon^2|^p\,\mathrm{d}x\Big)^{\frac{2p^*}{2p^*-2p}} + C_1 \int\limits_{\mathbb{R}^N} (|\nabla v_\varepsilon|^p + \mu |v_\varepsilon|^p)\,\mathrm{d}x - C_2 \int\limits_{\mathbb{R}^N} |v_\varepsilon|^\sigma\,\mathrm{d}x\\
&= \frac{1}{2N} S^{\frac{N}{p}} + O(\varepsilon^{\frac{N-p}{p}}) + O(\varepsilon^{\frac{N-p}{2p}}) - O\Big(\varepsilon^{\frac{N(p-1)-\frac{\sigma}{2}(N-p)}{p} + \frac{\sigma(N-p)}{2p^2}}\Big)\\
&< \frac{1}{2N} S^{\frac{N}{p}}
\end{aligned}$$

for $\varepsilon > 0$ small enough. The proof is completed. □

3.2 The existence of the ground state solution for $(\mathcal{Q}_\mu)$

Now, we are able to prove the existence of positive ground state solution for problem $(\mathcal{Q}_\mu)$.

Theorem 3.8. *Suppose that conditions* (H_0)–(H_5) *are satisfied. Problem* $(\mathcal{Q}_\mu)$ *has a positive ground state solution* $v \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$ *satisfying* $v(x) \to 0$ *as* $|x| \to \infty$.

Proof. From Section 3.1, we know that $\mathcal{J}_\mu$ satisfies the mountain pass geometry. There exists a $(C)_{c_\mu}$ sequence $\{v_n\} \subset W_\mu$ of $\mathcal{J}_\mu$, which satisfies

$$\mathcal{J}_\mu(v_n) \to c_\mu < \frac{1}{2N} S^{\frac{N}{p}} \quad \text{and} \quad (1 + \|v_n\|_\mu)\mathcal{J}'_\mu(v_n) \to 0 \qquad \text{as } n \to \infty.$$

From Lemma 3.5, up to a subsequence, there is a $v \in W_\mu$ such that $v_n \rightharpoonup v$ in W_μ with $\mathcal{J}'_\mu(v) = 0$. Without loss of generality, we can suppose that $v \neq 0$; otherwise, if $Q_\mu(v_n) \to 0$, using Lemma 2.1 (4), we have that

$$\int_{\mathbb{R}^N} |\nabla v_n|^p \,dx + \mu \int_{\mathbb{R}^N} |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n \,dx \le \int_{\mathbb{R}^N} |\nabla v_n|^p \,dx + \mu \int_{\mathbb{R}^N} |f(v_n)|^p \,dx \to 0.$$

From $\langle \mathcal{J}'_\mu(v_n), v_n \rangle \to 0$ we conclude that

$$\int_{\mathbb{R}^N} [h(f(v_n)) f'(v_n) v_n + (f(v_n^+))^{2p^*-1} f'(v_n^+) v_n^+] \,dx \to 0.$$

Under the assumptions (H_0), (H_2) and Lemma 2.1 (4), we have

$$\begin{aligned} \int_{\mathbb{R}^N} H(f(v_n)) \,dx + \frac{1}{2p^*} \int_{\mathbb{R}^N} |f(v_n^+)|^{2p^*} \,dx &\le \frac{1}{\theta} \int_{\{v_n \ge 0\}} h(f(v_n)) f(v_n) \,dx + \frac{1}{2p^*} \int_{\mathbb{R}^N} |f(v_n^+)|^{2p^*} \,dx \\ &\le C_1 \int_{\mathbb{R}^N} [h(f(v_n)) f'(v_n) v_n + |f(v_n^+)|^{2p^*-1} f'(v_n^+) v_n^+] \,dx \to 0. \end{aligned}$$

Therefore, we conclude that

$$\mathcal{J}_\mu(v_n) = \frac{1}{p} Q_\mu(v_n) - \int_{\mathbb{R}^N} \Big[H(f(v_n)) + \frac{1}{2p^*} |f(v_n^+)|^{2p^*} \Big] \,dx \to 0,$$

which is a contradiction with $\mathcal{J}_\mu(v_n) \to c_\mu > 0$. Therefore, $Q_\mu(v_n) \nrightarrow 0$. By Lemma 3.6, there exist $\{y_n\} \subset \mathbb{R}^N$ and positive constants R, ξ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |v_n|^p \,dx \ge \xi.$$

Define $\hat{v}_n(x) = v_n(x + y_n)$. Then $\{\hat{v}_n\}$ is also a $(C)_{c_\mu}$ sequence of $\mathcal{J}_\mu$ and satisfies

$$\hat{v}_n \rightharpoonup \hat{v} \text{ in } W_\mu, \quad \mathcal{J}'_\mu(\hat{v}) = 0, \quad \hat{v}_n \to \hat{v} \text{ in } L^p(B_R).$$

Then

$$\begin{aligned} \int_{B_R(x)} |\hat{v}|^p \,dx &= \lim_{n\to\infty} \int_{B_R(x)} |\hat{v}_n|^p \,dx \\ &= \lim_{n\to\infty} \int_{B_R(y_n)} |v_n|^p \,dx \ge \xi \Rightarrow \hat{v} \neq 0, \end{aligned}$$

so we can assume that $v \neq 0$. By Lemma 2.1 (4), we get

$$|f(v_n)|^p - |f(v_n)|^{p-2} f(v_n) f'(v_n) v_n \ge |f(v_n)|^p - |f(v_n)|^{p-2} f^2(v_n) = 0 \tag{3.14}$$

and

$$\frac{1}{p} |f(v_n^+)|^{2p^*-1} f'(v_n^+) v_n^+ - \frac{1}{2p^*} |f(v_n^+)|^{2p^*} \ge \Big(\frac{1}{2p} - \frac{1}{2p^*} \Big) |f(v_n^+)|^{2p^*} > 0.$$

From (H_2) it follows that

$$\begin{aligned} \frac{1}{p} h(f(v_n)) f'(v_n) v_n - H(f(v_n)) &\ge \frac{1}{p} h(f(v_n)) f'(v_n) v_n - \frac{1}{\theta} h(f(v_n)) f(v_n) \\ &\ge \Big(\frac{1}{2p} - \frac{1}{\theta} \Big) h(f(v_n)) f(v_n) > 0. \end{aligned} \tag{3.15}$$

Combining (3.14)–(3.15) with Fatou's Lemma, we obtain

$$\begin{aligned}
c_\mu &= \lim_{n\to\infty}\Big[\mathcal{J}_\mu(v_n) - \frac{1}{p}\langle \mathcal{J}_\mu'(v_n), v_n\rangle\Big]\\
&= \liminf_{n\to\infty} \frac{\mu}{p}\int_{\mathbb{R}^N} [|f(v_n)|^p - |f(v_n)|^{p-2}f(v_n)f'(v_n)v_n]\,\mathrm{d}x + \liminf_{n\to\infty}\int_{\mathbb{R}^N}\Big[\frac{1}{p}h(f(v_n))f'(v_n)v_n - H(f(v_n))\Big]\,\mathrm{d}x\\
&\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^N}\Big[\frac{1}{p}|f(v_n^+)|^{2p^*-1}f'(v_n^+)v_n^+ - \frac{1}{2p^*}|f(v_n^+)|^{2p^*}\Big]\,\mathrm{d}x\\
&\geq \frac{\mu}{p}\int_{\mathbb{R}^N} [|f(v)|^p - |f(v)|^{p-2}f(v)f'(v)v]\,\mathrm{d}x + \int_{\mathbb{R}^N}\Big[\frac{1}{p}h(f(v))f'(v)v - H(f(v))\Big]\,\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^N}\Big[\frac{1}{p}|f(v^+)|^{2p^*-1}f'(v^+)v^+ - \frac{1}{2p^*}|f(v^+)|^{2p^*}\Big]\,\mathrm{d}x\\
&= \mathcal{J}_\mu(v) - \frac{1}{p}\langle \mathcal{J}_\mu'(v), v\rangle = \mathcal{J}_\mu(v).
\end{aligned}$$

Then $v \neq 0$ is a critical point of $\mathcal{J}_\mu$ satisfying $\mathcal{J}_\mu(v) \leq c_\mu$. On the other hand, $v \in \mathcal{N}_\mu$ and $c_\mu = \inf_{\mathcal{N}_\mu}\mathcal{J}_\mu$ imply that $\mathcal{J}_\mu(v) \geq c_\mu$, therefore $\mathcal{J}_\mu(v) = c_\mu$.

Now we show that v is nonnegative since

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\nabla\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} \mu|f(v)|^{p-2}f(v)f'(v)\varphi\,\mathrm{d}x\\
&\qquad = \int_{\mathbb{R}^N} h(f(v))f'(v)\varphi\,\mathrm{d}x + \int_{\mathbb{R}^N} |f(v^+)|^{2p^*-2}f(v^+)f'(v^+)\varphi\,\mathrm{d}x
\end{aligned}\tag{3.16}$$

for all $v, \varphi \in W_\mu$. Let $\varphi = -v^-$; then

$$0 \leq \frac{1}{2}Q_\mu(v^-) \leq \int_{\mathbb{R}^N} |\nabla v^-|^p\,\mathrm{d}x + \int_{\mathbb{R}^N} \mu|f(v^-)|^{p-2}f(v^-)f'(v^-)(v^-)\,\mathrm{d}x = 0$$

implies that $v^- = 0$, and thus $v \geq 0$.

Next, we will prove the L^∞-estimate of v and that it decays to zero at infinity.

For any $R > 0$, $0 < r \leq R/2$, set $\eta \in C^\infty(\mathbb{R}^N)$, $0 \leq \eta \leq 1$, with $\eta(x) = 1$ if $|x| \geq R$ and $\eta(x) = 0$ if $|x| \leq R - r$ and $|\nabla\eta| \leq 2/r$. For $l > 0$, let

$$v_l(x) = \begin{cases} v(x), & v \leq l,\\ l, & v > l,\end{cases}$$

and

$$z_l = \eta^p v_l^{p(\beta-1)}v \quad\text{and}\quad \omega_l = \eta v_l^{\beta-1}v,$$

with $\beta > 1$ to be determined later. Taking z_l as a test function, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} \eta^p v_l^{p(\beta-1)}|\nabla v|^p\,\mathrm{d}x &= \int_{\mathbb{R}^N} h(f(v))f'(v)v\eta^p v_l^{p(\beta-1)}\,\mathrm{d}x + \int_{\mathbb{R}^N} |f(v)|^{2p^*-1}f'(v)v\eta^p v_l^{p(\beta-1)}\,\mathrm{d}x\\
&\quad - p\int_{\mathbb{R}^N} \eta^{p-1}v_l^{p(\beta-1)}v|\nabla v|^{p-2}\nabla v\nabla\eta\,\mathrm{d}x - p(\beta-1)\int_{\mathbb{R}^N} \eta^p v_l^{p(\beta-1)-1}v|\nabla v|^{p-2}\nabla v\nabla v_l\,\mathrm{d}x\\
&\quad - \int_{\mathbb{R}^N} \mu|f(v)|^{p-1}f'(v)v\eta^p v_l^{p(\beta-1)}\,\mathrm{d}x.
\end{aligned}$$

By (H_0) and (H_1), we see that for any $\tau > 0$ there exists $D_\tau > 0$ such that

$$h(f(t)) \leq \tau f(t)^{p-1} + D_\tau(f(t))^{2p^*-1} \quad\text{for all } t \geq 0.$$

Choose τ sufficiently small; by Lemma 2.1 (10), we have that

$$\begin{aligned}\int_{\mathbb{R}^N} \eta^p v_l^{p(\beta-1)}|\nabla v|^p\,dx &\le C_2 \int_{\mathbb{R}^N} |f(v)|^{2p^*-1} f'(v) v\eta^p v_l^{p(\beta-1)}\,dx - p\int_{\mathbb{R}^N} \eta^{p-1} v_l^{p(\beta-1)} v|\nabla v|^{p-2}\nabla v\nabla\eta\,dx\\ &\le C_3 \int_{\mathbb{R}^N} v^{p^*}\eta^p v_l^{p(\beta-1)}\,dx + p\int_{\mathbb{R}^N} \eta^{p-1} v_l^{p(\beta-1)} v|\nabla v|^{p-1}\nabla\eta\,dx.\end{aligned}$$

For every $\vartheta > 0$, by Young's inequality, we have that

$$\begin{aligned}\int_{\mathbb{R}^N} \eta^p v_l^{p(\beta-1)}|\nabla v|^p\,dx &\le C_3 \int_{\mathbb{R}^N} v^{p^*}\eta^p v_l^{p(\beta-1)}\,dx + p\vartheta \int_{\mathbb{R}^N} \eta^p v_l^{p(\beta-1)}|\nabla v|^p\,dx + pC_\vartheta \int_{\mathbb{R}^N} v^p|\nabla\eta|^p v_l^{p(\beta-1)}\,dx\\ &\le C_3 \int_{\mathbb{R}^N} v^{p^*}\eta^p v_l^{p(\beta-1)}\,dx + C_4 \int_{\mathbb{R}^N} v^p|\nabla\eta|^p v_l^{p(\beta-1)}\,dx, \end{aligned}\tag{3.17}$$

where we have chosen $\vartheta > 0$ sufficiently small.

On the other hand, by the Sobolev inequality and the Hölder inequality, we have

$$\begin{aligned}\|\omega_l\|_{p^*}^p &\le C_5 \int_{\mathbb{R}^N} |\nabla\omega_l|^p\,dx = C_5 \int_{\mathbb{R}^N} |\nabla(\eta v_l^{\beta-1} v)|^p\,dx\\ &\le C_5\beta^p\Big(\int_{\mathbb{R}^N} v^p|\nabla\eta|^p v_l^{p(\beta-1)}\,dx + \int_{\mathbb{R}^N} \eta^p v_l^{p(\beta-1)}|\nabla v|^p\,dx\Big).\end{aligned}\tag{3.18}$$

Combining (3.17) with (3.18), we obtain

$$\|\omega_l\|_{p^*}^p \le C_6\beta^p\Big(\int_{\mathbb{R}^N} v^p|\nabla\eta|^p v_l^{p(\beta-1)}\,dx + \int_{\mathbb{R}^N} \eta^p v_l^{p(\beta-1)} v^{p^*}\,dx\Big).\tag{3.19}$$

Let $\beta = \frac{p^*}{p}$; using the fact that $R - r \ge \frac{R}{2}$, we have that

$$\begin{aligned}\|\omega_l\|_{p^*}^p &\le C_6\beta^p\Big(\int_{\mathbb{R}^N} v^p|\nabla\eta|^p v_l^{p^*-p}\,dx + \int_{\{|x|\ge R-r\}} \eta^p v_l^{p^*-p} v^{p^*}\,dx\Big)\\ &\le C_6\beta^p \int_{\mathbb{R}^N} v^p|\nabla\eta|^p v_l^{p^*-p}\,dx + C_6\beta^p\Big(\int_{\mathbb{R}^N}\Big(v\eta v_l^{\frac{p^*-p}{p}}\Big)^{p^*}dx\Big)^{\frac{p}{p^*}}\Big(\int_{|x|>R/2} v^{p^*}\,dx\Big)^{\frac{p^*-p}{p^*}}.\end{aligned}$$

From the definition of ω_l it follows that

$$\Big(\int_{\mathbb{R}^N}\Big(v\eta v_l^{\frac{p^*-p}{p}}\Big)^{p^*}dx\Big)^{\frac{p}{p^*}} \le C_6\beta^p \int_{\mathbb{R}^N} v^p|\nabla\eta|^p v_l^{p^*-p}\,dx + C_6\beta^p\Big(\int_{\mathbb{R}^N}\Big(v\eta v_l^{\frac{p^*-p}{p}}\Big)^{p^*}dx\Big)^{\frac{p}{p^*}}\Big(\int_{|x|>R/2} v^{p^*}\,dx\Big)^{\frac{p^*-p}{p^*}}.$$

Since $v \in L^{p^*}(\mathbb{R}^N)$, for $R > 0$ sufficient large there holds

$$\int_{|x|\ge R/2} v^{p^*}\,dx \le \vartheta.$$

Therefore,

$$\begin{aligned}\Big(\int_{|x|\ge R}\Big(v v_l^{\frac{p^*-p}{p}}\Big)^{p^*}dx\Big)^{\frac{p}{p^*}} &\le \Big(\int_{|x|\ge R-r}\Big(\eta v v_l^{\frac{p^*-p}{p}}\Big)^{p^*}dx\Big)^{\frac{p}{p^*}}\\ &\le C_7\beta^p \int_{\mathbb{R}^N} v^p|\nabla\eta|^p v_l^{p^*-p}\,dx\\ &\le \frac{C_7\beta^p}{r^p}\int_{\mathbb{R}^N} v^{p^*}\,dx \le \frac{C_8}{r^p}.\end{aligned}$$

Using Fatou's Lemma in the variable l, we get

$$\int\limits_{|x|\geq R} v^{\frac{(p^*)^2}{p}}\,\mathrm{d}x < \int\limits_{|x|\geq R-r} \eta^{p^*} v^{\frac{(p^*)^2}{p}}\,\mathrm{d}x < \frac{C_9}{r^{p^*}} < +\infty. \tag{3.20}$$

Next, we note that if

$$\beta = \frac{p^*(t-1)}{pt} \quad \text{with} \quad t = \frac{(p^*)^2}{p(p^*-p)},$$

then

$$\beta > 1, \quad t > 1, \quad \frac{pt}{t-1} < p^*, \quad v \in L^{\frac{p\beta t}{t-1}}(|x| \geq R-r).$$

By (3.19) and the Hölder inequality, we have that

$$\begin{aligned}
\|\omega_l\|_{p^*}^p &\leq C_6\beta^p\Big(\int\limits_{|x|\geq R-r} v^{p\beta}|\nabla\eta|^p\,\mathrm{d}x + \int\limits_{|x|\geq R-r} \eta^p v^{p^*-p}|v|^{p\beta}\,\mathrm{d}x\Big)\\
&\leq C_6\beta^p\Big[\Big(\int\limits_{|x|\geq R-r} v^{\frac{p\beta t}{t-1}}\,\mathrm{d}x\Big)^{\frac{t-1}{t}}\Big(\int\limits_{R-r\leq|x|\leq R} |\nabla\eta|^{pt}\,\mathrm{d}x\Big)^{\frac{1}{t}} + \Big(\int\limits_{|x|\geq R-r} (\eta^p v^{p^*-p})^t\,\mathrm{d}x\Big)^{\frac{1}{t}}\Big(\int\limits_{|x|\geq R-r} v^{\frac{p\beta t}{t-1}}\,\mathrm{d}x\Big)^{\frac{t-1}{t}}\Big]\\
&\leq C_6\beta^p\Big\{\frac{[R^N-(R-r)^N]^{\frac{1}{t}}}{r^p}\Big(\int\limits_{|x|\geq R-r} v^{\frac{p\beta t}{t-1}}\,\mathrm{d}x\Big)^{\frac{t-1}{t}} + \Big(\int\limits_{|x|\geq R-r} (\eta^{\frac{p(p^*-p)}{p^*}} v^{p^*-p})^t\,\mathrm{d}x\Big)^{\frac{1}{t}}\Big(\int\limits_{|x|\geq R-r} v^{\frac{p\beta t}{t-1}}\,\mathrm{d}x\Big)^{\frac{t-1}{t}}\Big\},
\end{aligned}$$

where we have used $p \geq \frac{p(p^*-p)}{p^*}$. By (3.20), we deduce that

$$\|\omega_l\|_{p^*}^p \leq C_{10}\beta^p\Big(\frac{1}{r^{\frac{p^*}{t}}} + \frac{R^{\frac{N}{t}}}{r^p}\Big)\Big(\int\limits_{|x|\geq R-r} v^{\frac{p\beta t}{t-1}}\,\mathrm{d}x\Big)^{\frac{t-1}{t}}.$$

Using Fatou's Lemma, we obtain

$$\|v\|_{p^*\beta,(|x|\geq R)}^{p\beta} \leq C_{10}\beta^p\Big(\frac{1}{r^{\frac{p^*}{t}}} + \frac{R^{\frac{N}{t}}}{r^p}\Big)\|v\|_{\frac{p\beta t}{t-1}\beta,(|x|>R-r)}^{p\beta}.$$

If we take $\psi = p^*(t-1)/pt$, $s = pt/(t-1)$, then

$$\|v\|_{\beta\psi s,(|x|\geq R)} \leq C^{\frac{1}{p\beta}}\beta^{\frac{1}{\beta}}\Big(\frac{1}{r^{\frac{p^*}{t}}} + \frac{R^{\frac{N}{t}}}{r^p}\Big)^{\frac{1}{p\beta}}\|v\|_{\beta s,(|x|>R-r)}. \tag{3.21}$$

Setting $\beta = \psi^m (m = 1, 2, \ldots)$, we obtain

$$\|v\|_{\psi^{m+1}s,(|x|\geq R)} \leq C_{10}^{\frac{1}{p}\psi^{-m}}\psi^{m\psi^{-m}}\Big(\frac{1}{r^{\frac{p^*}{t}}} + \frac{R^{\frac{N}{t}}}{r^p}\Big)^{\frac{1}{p\psi^m}}\|v\|_{\psi^m s,(|x|>R-r)}. \tag{3.22}$$

Note that $p > p^*/t$, $p > N/t$. Therefore, if we choose $r_m := 2^{-(m+1)}R$, then it follows from inequalities (3.21) and (3.22) that

$$\begin{aligned}
\|v\|_{\psi^{m+1}s,(|x|\geq R)} \leq \|v\|_{\psi^{m+1}s,(|x|\geq R-r_{m+1})} &\leq C_{10}^{\frac{1}{p}\psi^{-m}}\psi^{m\psi^{-m}}\Big(\frac{2^{\frac{p^*(m+1)}{t}}}{R^{\frac{p^*}{t}}} + 2^{p(m+1)}\Big)^{\frac{1}{p\psi^m}}\|v\|_{\psi^m s,(|x|>R-r_m)}\\
&\leq C_{10}^{\frac{1}{p}\psi^{-m}}\psi^{m\psi^{-m}}(2\times 2^{p(m+1)})^{\frac{1}{p\psi^m}}\|v\|_{\psi^m s,(|x|>R-r_m)}\\
&\leq C_{10}^{\frac{1}{p}\sum_{i=1}^m\psi^{-i}}\psi^{\sum_{i=1}^m i\psi^{-i}}\exp\Big(\sum_{i=1}^m\frac{\ln(2\times 2^{p(i+1)})}{p\psi^i}\Big)\|v\|_{\psi s,(|x|>R-r_1)}\\
&\leq C_{11}\|v\|_{p^*,(|x|>R/2)}.
\end{aligned}$$

Letting $m \to \infty$ in the last inequality, we have

$$\|v\|_{\infty,(|x|>R)} \leq C_{11}\|v\|_{p^*,(|x|>R/2)}.$$

Therefore, for any $\vartheta > 0$ there exists an $R > 0$ such that $\|v\|_{\infty,(|x|>R)} \leq \vartheta$. Consequently, we conclude that $\lim_{|x|\to\infty} v(x) = 0$. □

4 The nonautonomous problem

In this section, we will study the following problem (which is equivalent to ($\mathcal{P}_\varepsilon$)), which can be obtained under the change of variable $\varepsilon z = x$:

$$\begin{cases} -\Delta_p v + V(\varepsilon x)|f(v)|^{p-2}f(v)f'(v) = h(f(v))f'(v) + |f(v)|^{2p^*-1}f'(v) & \text{in } \mathbb{R}^N, \\ v \in W^{1,p}(\mathbb{R}^N),\ v(x) > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{$\mathcal{P}_\varepsilon^*$}$$

The functional J_ε corresponding to problem ($\mathcal{P}_\varepsilon^*$) is given by

$$J_\varepsilon(v) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v|^p \,\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} V(\varepsilon x)|f(v)|^p \,\mathrm{d}x - \int_{\mathbb{R}^N} H(f(v))\,\mathrm{d}x - \frac{1}{2p^*}\int_{\mathbb{R}^N} |f(v^+)|^{2p^*}\,\mathrm{d}x,$$

which is well defined on the Banach space

$$X_\varepsilon = \Big\{v \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)|v|^p \,\mathrm{d}x < \infty\Big\}$$

endowed with the norm

$$\|v\|_{X_\varepsilon} = \Big(\int_{\mathbb{R}^N} (|\nabla v|^p + V(\varepsilon x)|v|^p)\,\mathrm{d}x\Big)^{\frac{1}{p}}.$$

Obviously, $J_\varepsilon \in C^1(X_\varepsilon, \mathbb{R})$ with

$$\begin{aligned} \langle J_\varepsilon'(v), \varphi\rangle = {} & \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla\varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x)|f(v)|^{p-2}f(v)f'(v)\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N} h(f(v))f'(v)\varphi\,\mathrm{d}x \\ & - \int_{\mathbb{R}^N} |f(v^+)|^{2p^*-2}f(v^+)f'(v^+)\varphi\,\mathrm{d}x \end{aligned}$$

for all $v, \varphi \in X_\varepsilon$. Moreover, the weak solution v of ($\mathcal{P}_\varepsilon^*$) corresponds to the critical point of the functional J_ε. We define the Nehari manifold associated to ($\mathcal{P}_\varepsilon^*$) by $\mathcal{M}_\varepsilon$, that is,

$$\mathcal{M}_\varepsilon = \{v \in X_\varepsilon : v \neq 0 \text{ and } \langle J_\varepsilon'(v), v\rangle = 0\}.$$

Set

$$Q(v) = \int_{\mathbb{R}^N} (|\nabla v|^p + V(\varepsilon x)|f(v)|^p)\,\mathrm{d}x.$$

We first show that the Nehari manifold $\mathcal{M}_\varepsilon$ is bounded from below.

Lemma 4.1. *There exists a constant $C > 0$ such that $\|v\|_{X_\varepsilon} \geq C > 0$ for all $v \in \mathcal{M}_\varepsilon$.*

Proof. By Lemma 2.1 (10), the Hölder inequality and the Sobolev inequality, we have that

$$\begin{aligned} \int_{\mathbb{R}^N} |f(v)|^{q+1}\,\mathrm{d}x & \leq \Big(\int_{\mathbb{R}^N} |f(v)|^p\,\mathrm{d}x\Big)^{\frac{\tau(q+1)}{p}} \Big(\int_{\mathbb{R}^N} |f(v)|^{2p^*}\,\mathrm{d}x\Big)^{\frac{(1-\tau)(q+1)}{2p^*}} \\ & \leq 2^{\frac{(1-\tau)(q+1)}{2p}} \Big(\int_{\mathbb{R}^N} |f(v)|^p\,\mathrm{d}x\Big)^{\frac{\tau(q+1)}{p}} \Big(\int_{\mathbb{R}^N} |v|^{p^*}\,\mathrm{d}x\Big)^{\frac{(1-\tau)(q+1)}{2p^*}} \\ & \leq 2^{\frac{(1-\tau)(q+1)}{2p}} V_0^{-\frac{\tau(q+1)}{p}} S^{\frac{(1-\tau)(q+1)}{2p}} \Big(\int_{\mathbb{R}^N} V(\varepsilon x)|f(v)|^p\,\mathrm{d}x\Big)^{\frac{\tau(q+1)}{p}} \Big(\int_{\mathbb{R}^N} |\nabla v|^p\,\mathrm{d}x\Big)^{\frac{(1-\tau)(q+1)}{2p}} \\ & \leq CQ(v)^{\frac{(1+\tau)(q+1)}{2p}}, \end{aligned} \tag{4.1}$$

where $\tau \in (0, 1)$ verifies

$$\frac{1}{q+1} = \frac{\tau}{p} + \frac{1-\tau}{2p^*}.$$

By Lemma 2.1 (7) and the Sobolev inequality, we have

$$\int_{\mathbb{R}^N} f^{2p^*}(v^+)\,\mathrm{d}x \le C\Big(\int_{\mathbb{R}^N} |\nabla f^2(v^+)|^p\,\mathrm{d}x\Big)^{\frac{p^*}{p}} \le C\Big(\int_{\mathbb{R}^N} |\nabla v|^p\,\mathrm{d}x\Big)^{\frac{p^*}{p}} \le CQ(v)^{\frac{p^*}{p}}. \tag{4.2}$$

Thus, for $v \in \mathcal{M}_\varepsilon$, by (4.1), (4.2), Lemma 2.1 (4), and hypotheses (H_0), (H_1) and (H_4), we deduce that

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N} (|\nabla v|^p + V(\varepsilon x)|f(v)|^{p-2}f(v)f'(v)v)\,\mathrm{d}x - \int_{\mathbb{R}^N} h(f(v))f'(v)v\,\mathrm{d}x - \int_{\mathbb{R}^N} |f(v^+)|^{2p^*-1}f'(v^+)v^+\,\mathrm{d}x\\
&\ge \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla v|^p + V(\varepsilon x)|f(v)|^p)\,\mathrm{d}x - \int_{\mathbb{R}^N} |f(v^+)|^{2p^*}\,\mathrm{d}x - \int_{\mathbb{R}^N} h(f(v))f(v)\,\mathrm{d}x\\
&\ge \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\nabla v|^p + \Big(1 - \frac{2\varepsilon}{V_0}\Big)V(\varepsilon x)|f(v)|^p\Big)\,\mathrm{d}x - \int_{\mathbb{R}^N} |f(v^+)|^{2p^*}\,\mathrm{d}x - C_\varepsilon\int_{\mathbb{R}^N} |f(v)|^{q+1}\,\mathrm{d}x\\
&\ge \frac{1}{4}Q(v) - CQ(v)^{\frac{p^*}{p}} - CQ(v)^{\frac{(1+\tau)(q+1)}{2p}}.
\end{aligned}$$

Since $\frac{(1+\tau)(q+1)}{2p} > 1$, we have $Q(v) \ge C > 0$ for some $C > 0$. This implies that

$$\|v\|_{X_\varepsilon} \ge Q(v)^{\frac{1}{p}} \ge C > 0$$

for all $v \in \mathcal{M}_\varepsilon$. □

It is easy to check, by arguing as in Section 3, that J_ε exhibits the mountain pass geometry (Theorem 3.1) and there exists a $(C)_{c_\varepsilon}$ sequence $\{v_n\} \subset X_\varepsilon$, for which we can assume $v_n \ge 0$, such that $v_n \rightharpoonup v$ in X_ε for some $v \in X_\varepsilon$ and $J'_\varepsilon(v) = 0$ (similar arguments as in Lemma 3.5, using hypothesis (V)). Moreover, from Lemma 3.7, there exists a $v_0 \in W_{V_\infty} \setminus \{0\}$ such that

$$\max_{t\ge 0} J_\varepsilon(tv_0) < \max_{t\ge 0} \mathcal{J}_{V_\infty}(tv_0) < \frac{1}{2N}S^{\frac{N}{p}},$$

and thus

$$c_\varepsilon = \inf_{\mathcal{M}_\varepsilon} J_\varepsilon = \inf_{v\in X_\varepsilon\setminus\{0\}} \max_{t\ge 0} J_\varepsilon(tv) < \frac{1}{2N}S^{\frac{N}{p}}.$$

Similar to the proof Lemma 3.4, there is a unique $t_v > 0$ such that $J_\varepsilon(t_v v) = \max_{t\ge 0} J_\varepsilon(tv)$.

Similar to the proof of Lemma 3.6, we can characterize the $(C)_{c_\varepsilon}$ sequence in the following Lemma.

Lemma 4.2. *Let $\{v_n\}$ be a $(C)_{c_\varepsilon}$ sequence of J_ε with $c_\varepsilon < \frac{1}{2N}S^{N/p}$. Then one of the following conclusions holds:*
(1) $Q(v_n) \to 0$;
(2) *there exist $\{y_n\} \subset \mathbb{R}^N$ and positive constants R, ξ such that*

$$\lim_{n\to\infty}\inf \int_{B_R(y_n)} |v_n|^p\,\mathrm{d}x \ge \xi.$$

Lemma 4.3. *Suppose that $\{v_n\}$ is a $(C)_{c_\varepsilon}$ sequence of J_ε in X_ε with $c_\varepsilon < \frac{1}{2N}S^{N/p}$ and $v_n \rightharpoonup 0$ in X_ε. If $Q(v_n) \nrightarrow 0$, then $c_\varepsilon \ge c_{V_\infty}$, where c_{V_∞} is the minimax level of J_{V_∞}.*

Proof. Let $\{t_n\} \subset (0, \infty)$ be a sequence such that $\{t_n v_n\} \subset \mathcal{N}_{V_\infty}$. We claim that $\lim_{n\to\infty}\sup t_n \le 1$. Assume by contradiction that there exist $\delta > 0$ and a subsequence still denoted by $\{t_n\}$ such that $t_n \ge 1 + \delta$ for all $n \in \mathbb{N}$. Since $\{v_n\}$ is bounded in X_ε, we may assume that $v_n \ge 0$ for all $n \in \mathbb{N}$. From $\langle J'_\varepsilon(v_n), v_n\rangle = o_n(1)$ we get

$$\int_{\mathbb{R}^N} [|\nabla v_n|^p + V(\varepsilon x)|f(v_n)|^{p-1}f'(v_n)v_n]\,\mathrm{d}x = \int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n\,\mathrm{d}x + \int_{\mathbb{R}^N} |f(v_n)|^{2p^*-1}f'(v_n)v_n\,\mathrm{d}x + o_n(1). \tag{4.3}$$

Also since $\{t_n v_n\} \subset \mathcal{N}_{V_\infty}$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} [t_n^p|\nabla v_n|^p + V_\infty|f(t_n v_n)|^{p-1}f'(t_n v_n)t_n v_n]\,\mathrm{d}x\\
&\qquad = \int_{\mathbb{R}^N} h(f(t_n v_n))f'(t_n v_n)t_n v_n\,\mathrm{d}x + \int_{\mathbb{R}^N} |f(t_n v_n)|^{2p^*-1}f'(t_n v_n)t_n v_n\,\mathrm{d}x.
\end{aligned} \tag{4.4}$$

From (4.3) and (4.4) we have

$$\begin{aligned}\int_{\mathbb{R}^N}\Big[\frac{h(f(t_nv_n))}{|f(t_nv_n)|^{2p-1}}\frac{|f(t_nv_n)|^{2p-1}f'(t_nv_n)}{(t_nv_n)^{p-1}}-\frac{h(f(v_n))}{|f(v_n)|^{2p-1}}\frac{|f(v_n)|^{2p-1}f'(v_n)}{v_n^{p-1}}\Big]v_n^p\,\mathrm{d}x&\\ +\int_{\mathbb{R}^N}\Big[\frac{|f(t_nv_n)|^{2p^*-1}f'(t_nv_n)}{(t_nv_n)^{p-1}}-\frac{|f(v_n)|^{2p^*-1}f'(v_n)}{v_n^{p-1}}\Big]v_n^p\,\mathrm{d}x&\\ =\int_{\mathbb{R}^N}(V_\infty-V(\varepsilon x))|f(v_n)|^{p-1}f'(v_n)v_n\,\mathrm{d}x&\\ +\int_{\mathbb{R}^N}V_\infty\Big[\frac{|f(t_nv_n)|^{p-1}f'(t_nv_n)}{(t_nv_n)^{p-1}}-\frac{|f(v_n)|^{p-1}f'(v_n)}{v_n^{p-1}}\Big]v_n^p\,\mathrm{d}x+o_n(1).&\end{aligned}$$

Given $\xi>0$, by (V) there exists $R=R(\xi)>0$ such that $V(\varepsilon x)\geq V_\infty-\xi$ for all $|\varepsilon x|\geq R$. Since $v_n\rightharpoonup 0$ in X_ε, we have $v_n\to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $s\in[1,p^*)$ and $v_n\to 0$ a.e. in $\mathbb{R}^N$. Hence, we get

$$\begin{aligned}\int_{\mathbb{R}^N}(V_\infty-V(\varepsilon x))|f(v_n)|^{p-1}f'(v_n)v_n\,\mathrm{d}x&=\int_{|\varepsilon x|<R}(V_\infty-V(\varepsilon x))|f(v_n)|^{p-1}f'(v_n)v_n\,\mathrm{d}x\\ &\quad+\int_{|\varepsilon x|\geq R}(V_\infty-V(\varepsilon x))|f(v_n)|^{p-1}f'(v_n)v_n\,\mathrm{d}x\\ &\leq \xi C+2V_\infty\int_{|\varepsilon x|<R}|v_n|^p\,\mathrm{d}x=\xi C+o_n(1).\end{aligned}$$

This together with Proposition 2.2 (1) and (2) and the boundedness of $\{v_n\}$ in X_ε leads to

$$\int_{\mathbb{R}^N}\Big[\frac{h(f(t_nv_n))}{|f(t_nv_n)|^{2p-1}}\frac{|f(t_nv_n)|^{2p-1}f'(t_nv_n)}{(t_nv_n)^{p-1}}-\frac{h(f(v_n))}{|f(v_n)|^{2p-1}}\frac{|f(v_n)|^{2p-1}f'(v_n)}{v_n^{p-1}}\Big]v_n^p\,\mathrm{d}x\leq\xi C+o_n(1).\tag{4.5}$$

If $Q(v_n)\nrightarrow 0$ in $\mathbb{R}$, by Lemma 4.2, there exist $\{y_n\}\subset\mathbb{R}^N$ and positive constants R^*, η such that

$$\liminf_{n\to\infty}\int_{B_{R^*}(y_n)}|v_n|^p\,\mathrm{d}x\geq\eta.\tag{4.6}$$

Define $\tilde v_n=v_n(x+y_n)$; then there is a $\tilde v$ such that, up to a subsequence, $\tilde v_n\rightharpoonup\tilde v$ in X_ε, $\tilde v_n\to\tilde v$ in $L^s(B_{R^*}(0))$, $s\in[1,p^*)$, and $\tilde v_n\to\tilde v$ a.e. in $\mathbb{R}^N$. By (4.6), there exists a subset $\Omega\subset B_{R^*}(0)$ with a positive measure such that $\tilde v>0$ a.e. in Ω. It follows from (4.5), (H$_3$), Proposition 2.2 (2), and $t_n\geq 1+\delta$ that

$$\begin{aligned}\int_{\mathbb{R}^N}\Big[\frac{h(f((1+\delta)\tilde v_n))}{|f((1+\delta)\tilde v_n)|^{2p-1}}\frac{|f((1+\delta)\tilde v_n)|^{2p-1}f'((1+\delta)\tilde v_n)}{((1+\delta)\tilde v_n)^{p-1}}-\frac{h(f(\tilde v_n))}{|f(\tilde v_n)|^{2p-1}}\frac{|f(\tilde v_n)|^{2p-1}f'(\tilde v_n)}{\tilde v_n^{p-1}}\Big]\tilde v_n^p\,\mathrm{d}x&\\ \leq\xi C+o_n(1).&\end{aligned}\tag{4.7}$$

Let $n\to\infty$ in (4.7); using Fatou's Lemma, we get

$$0<\int_{\Omega}\Big[\frac{h(f((1+\delta)\tilde v))}{|f((1+\delta)\tilde v)|^{2p-1}}\frac{|f((1+\delta)\tilde v)|^{2p-1}f'((1+\delta)\tilde v)}{((1+\delta)\tilde v)^{p-1}}-\frac{h(f(\tilde v))}{|f(\tilde v)|^{2p-1}}\frac{|f(\tilde v)|^{2p-1}f'(\tilde v)}{\tilde v^{p-1}}\Big]\tilde v^p\,\mathrm{d}x\leq\xi C$$

for any $\xi>0$, which leads to a contradiction. Thus, $\lim_{n\to\infty}\sup t_n\leq 1$.

Next, we distinguish the following two cases.

Case 1: $\lim_{n\to\infty}\sup t_n=1$. There exists a subsequence, still denoted by $\{t_n\}$, such that $t_n\to 1$ as $n\to\infty$. Hence

$$c_\varepsilon+o_n(1)=J_\varepsilon(v_n)\geq c_{V_\infty}+J_\varepsilon(v_n)-\mathcal{J}_{V_\infty}(t_nv_n).\tag{4.8}$$

By the boundedness of $\{v_n\}$ in X_ε and (V), similar to the argument of (4.5), we have

$$\begin{aligned} J_\varepsilon(v_n) - \mathcal{J}_{V_\infty}(t_n v_n) &= \frac{1-t_n^p}{p}\int_{\mathbb{R}^N} |\nabla v_n|^p \,\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} \big[V(\varepsilon x)|f(v_n)|^p - V_\infty |f(t_n v_n)|^p\big]\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} [H(f(t_n v_n)) - H(f(v_n))]\mathrm{d}x + \frac{1}{2p^*}\int_{\mathbb{R}^N} \big[|f(t_n v_n)|^{2p^*} - |f(v_n)|^{2p^*}\big]\,\mathrm{d}x \\ &\geq o_n(1) + \frac{1}{p}\int_{\mathbb{R}^N} V_\infty\big[|f(v_n)|^p - |f(t_n v_n)|^p\big]\,\mathrm{d}x + \int_{\mathbb{R}^N} [H(f(t_n v_n)) - H(f(v_n))]\mathrm{d}x \\ &\quad + \frac{1}{2p^*}\int_{\mathbb{R}^N} \big[|f(t_n v_n)|^{2p^*} - |f(v_n)|^{2p^*}\big]\,\mathrm{d}x - \xi C. \end{aligned} \tag{4.9}$$

Using the mean value theorem and (H_0)–(H_1), we have

$$\begin{aligned} \Big|\int_{\mathbb{R}^N} [H(f(t_n v_n)) - H(f(v_n))]\,\mathrm{d}x\Big| &= \Big|\int_{\mathbb{R}^N} h(f(\tau v_n))f'(\tau v_n)(t_n - 1)v_n \,\mathrm{d}x\Big| \\ &\leq \int_{\mathbb{R}^N} (C_1|v_n|^{p-1} + C_2|v_n|^{\frac{q+1}{2}})|t_n - 1|\,\mathrm{d}x, \end{aligned}$$

where τ is between 1 and t_n. Using the Hölder inequality and $\lim_{n\to\infty}(t_n - 1) = 0$, we obtain

$$\int_{\mathbb{R}^N} [H(f(t_n v_n)) - H(f(v_n))]\,\mathrm{d}x = o_n(1),$$

$$\int_{\mathbb{R}^N} V_\infty\big[|f(v_n)|^p - |f(t_n v_n)|^p\big]\,\mathrm{d}x = o_n(1)$$

and

$$\int_{\mathbb{R}^N} \big[|f(t_n v_n)|^{2p^*} - |f(v_n)|^{2p^*}\big]\,\mathrm{d}x = o_n(1).$$

Combining (4.8) with (4.9), we obtain the following inequality:

$$c_\varepsilon + o_n(1) \geq c_{V_\infty} - C\xi + o_n(1).$$

Letting $n \to \infty$ in the above inequality, we have $c_\varepsilon \geq c_{V_\infty} - C\xi$ for all $\xi > 0$, thus $c_\varepsilon \geq c_{V_\infty}$.

Case 2: $\lim_{n\to\infty}\sup t_n = t_0 < 1$. There exists a subsequence, still denoted by $\{t_n\}$, such that $t_n \to t_0$ as $n \to \infty$ and $t_n < 1$ for all $n \in \mathbb{N}$. By Proposition 2.2 (1), (4) and (5), we see that

$$|f(t)|^p - |f(t)|^{p-1}f'(t)t, \quad \frac{1}{p}h(f(t))f'(t)t - H(f(t)), \quad \frac{1}{p}|f(t)|^{2p^*-1}f'(t)t - \frac{1}{2p^*}|f(t)|^{2p^*}$$

for $t > 0$ are nondecreasing. Then

$$\begin{aligned} c_{V_\infty} &\leq \mathcal{J}_{V_\infty}(t_n v_n) - \frac{1}{p}\langle \mathcal{J}'_{V_\infty}(t_n v_n), t_n v_n\rangle \\ &= \frac{1}{p}\int_{\mathbb{R}^N} V_\infty\big[|f(t_n v_n)|^p - |f(t_n v_n)|^{p-1}f'(t_n v_n)t_n v_n\big]\,\mathrm{d}x + \int_{\mathbb{R}^N}\Big[\frac{1}{p}h(f(t_n v_n))f'(t_n v_n)t_n v_n - H(f(t_n v_n))\Big]\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N}\Big[\frac{1}{p}|f(t_n v_n)|^{2p^*-1}f'(t_n v_n)t_n v_n - \frac{1}{2p^*}|f(t_n v_n)|^{2p^*}\Big]\,\mathrm{d}x \\ &\leq \frac{1}{p}\int_{\{|\varepsilon x|>R\}} (V(\varepsilon x) + \xi)\big[|f(v_n)|^p - |f(v_n)|^{p-1}f'(v_n)v_n\big]\,\mathrm{d}x + \frac{1}{p}\int_{\{|\varepsilon x|\leq R\}} V_\infty\big[|f(v_n)|^p - |f(v_n)|^{p-1}f'(v_n)v_n\big]\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N}\Big[\frac{1}{p}h(f(v_n))f'(v_n)v_n - H(f(v_n))\Big]\,\mathrm{d}x + \int_{\mathbb{R}^N}\Big[\frac{1}{p}|f(v_n)|^{2p^*-1}f'(v_n)v_n - \frac{1}{2p^*}|f(v_n)|^{2p^*}\Big]\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{p}\int_{\mathbb{R}^N} V(\varepsilon x)[|f(v_n)|^p - |f(v_n)|^{p-1}f'(v_n)v_n]\,\mathrm{d}x + \int_{\mathbb{R}^N}\Big[\frac{1}{p}h(f(v_n))f'(v_n)v_n - H(f(v_n))\Big]\,\mathrm{d}x\\
&\qquad + \int_{\mathbb{R}^N}\Big[\frac{1}{p}|f(v_n)|^{2p^*-1}f'(v_n)v_n - \frac{1}{2p^*}|f(v_n)|^{2p^*}\Big]\,\mathrm{d}x + C\xi + o_n(1)\\
&= J_\varepsilon(v_n) - \frac{1}{p}\langle J'_\varepsilon(v_n), v_n\rangle + C\xi + o_n(1)\\
&= c_\varepsilon + C\xi + o_n(1).
\end{aligned}
$$

Letting $\xi \to 0$, $n \to 0$, we have $c_\varepsilon \geq c_{V_\infty}$. □

4.1 Compactness condition

Lemma 4.4. *Let $\{v_n\}$ be a $(C)_{c_\varepsilon}$ sequence of J_ε in X_ε and $v_n \rightharpoonup v$ in X_ε for some $v \in X_\varepsilon$. Then*

$$J_\varepsilon(\widetilde{v}_n) = J_\varepsilon(v_n) - J_\varepsilon(v) + o_n(1), \quad J'_\varepsilon(\widetilde{v}_n) = o_n(1),$$

where $\widetilde{v}_n = v_n - v$.

Proof. Firstly, we show that the following equalities hold:

$$\int_{\mathbb{R}^N} H(f(\widetilde{v}_n))\,\mathrm{d}x = \int_{\mathbb{R}^N} H(f(v_n))\,\mathrm{d}x + \int_{\mathbb{R}^N} H(f(v))\,\mathrm{d}x + o_n(1), \tag{4.10}$$

$$\int_{\mathbb{R}^N} V(\varepsilon x)|f(\widetilde{v}_n)|^p\,\mathrm{d}x = \int_{\mathbb{R}^N} V(\varepsilon x)|f(v_n)|^p\,\mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon x)|f(v)|^p\,\mathrm{d}x + o_n(1), \tag{4.11}$$

$$\int_{\mathbb{R}^N} V(\varepsilon x)[|f(\widetilde{v}_n)|^{p-1}f'(\widetilde{v}_n) - |f(v_n)|^{p-1}f'(v_n) + |f(v)|^{p-1}f'(v)]\varphi\,\mathrm{d}x = o_n(1), \tag{4.12}$$

$$\int_{\mathbb{R}^N} [h(f(\widetilde{v}_n))f'(\widetilde{v}_n) - h(f(v_n))f'(v_n) + h(f(v))f'(v)]\varphi\,\mathrm{d}x = o_n(1), \tag{4.13}$$

$$\int_{\mathbb{R}^N} f^{2p^*}(\widetilde{v}_n^+)\mathrm{d}x = \int_{\mathbb{R}^N} f^{2p^*}(v_n^+)\mathrm{d}x - \int_{\mathbb{R}^N} f^{2p^*}(v^+)\,\mathrm{d}x + o_n(1), \tag{4.14}$$

$$\int_{\mathbb{R}^N} (f^{2p^*-1}(\widetilde{v}_n^+)f'(\widetilde{v}_n^+) - f^{2p^*-1}(v_n^+)f'(v_n^+) + f^{2p^*-1}(v^+)f'(v^+))\varphi\,\mathrm{d}x = o_n(1) \tag{4.15}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, where $\alpha \in (p, p^*)$.

The proof of (4.10)–(4.15) is similar to the proof of Lemma 3.5 (2). Here we only show that (4.14) holds true. Observe that by Lemma 2.1 (6), (7) and (10), we have that

$$
\begin{aligned}
\big|f^{2p^*}(v_n^+) - f^{2p^*}(v^+)\big| &= \Big|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f^{2p^*}((\widetilde{v}_n + tv)^+)\,\mathrm{d}t\Big|\\
&= \Big|\int_0^1 [2p^* f^{2p^*-2}((\widetilde{v}_n + tv)^+) f((\widetilde{v}_n + tv)^+) f'((\widetilde{v}_n + tv)^+)v]\,\mathrm{d}t\Big|\\
&\leq C\int_0^1 (|\widetilde{v}_n| + |v|)^{p^*-1}|v|\,\mathrm{d}t \leq C(|\widetilde{v}_n|^{p^*-1}|v| + |v|^{p^*}).
\end{aligned}
$$

From this and Young's inequality, for each $\delta > 0$ there exists $C_\delta > 0$ such that

$$\big|f^{2p^*}(\widetilde{v}_n^+) - f^{2p^*}(v_n^+) + f^{2p^*}(v^+)\big| \leq \delta|\widetilde{v}_n|^{p^*} + C_\delta|v|^{p^*}.$$

Define

$$G_{\delta,n}(x) = \max\{|f^{2p^*}(\tilde{v}_n^+) - f^{2p^*}(v_n^+) + f^{2p^*}(v^+)| - \delta|\tilde{v}_n|^{p^*}, 0\},$$

which verifies that

$$G_{\delta,n}(x) \to 0 \quad \text{a.e. in } \mathbb{R}^N, \qquad 0 \le G_{\delta,n}(x) \le C_\delta |v|^{p^*} \in L^1(\mathbb{R}^N).$$

Hence, by Lebesgue's theorem, we have

$$\int_{\mathbb{R}^N} G_{\delta,n}(x)\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

By the definition of $G_{\delta,n}$, we see that

$$|f^{2p^*}(\tilde{v}_n^+) - f^{2p^*}(v_n^+) + f^{2p^*}(v^+)| \le \delta|\tilde{v}_n|^{p^*} + CG_{\delta,n}(x).$$

Thus, we get

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |f^{2p^*}(\tilde{v}_n^+) - f^{2p^*}(v_n^+) + f^{2p^*}(v^+)|\,\mathrm{d}x \le C\delta$$

which implies that

$$\int_{\mathbb{R}^N} |f^{2p^*}(\tilde{v}_n^+) - f^{2p^*}(v_n^+) + f^{2p^*}(v^+)|\,\mathrm{d}x = o_n(1)$$

and (4.14) follows.

Secondly, similar to the proof of (3.6) and from [10, Brezis–Lieb Lemma], we can deduce that

$$\int_{\mathbb{R}^N} |\nabla\tilde{v}_n|^p \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla v_n|^p \mathrm{d}x - \int_{\mathbb{R}^N} |\nabla v|^p \mathrm{d}x + o_n(1). \tag{4.16}$$

Finally, by (4.10)–(4.16) and (3.6)–(3.9), we obtain

$$J_\varepsilon(\tilde{v}_n) = J_\varepsilon(v_n) - J_\varepsilon(v) + o_n(1)$$

and

$$\langle J'_\varepsilon(\tilde{v}_n), \varphi\rangle = \langle J'_\varepsilon(v_n), \varphi\rangle - \langle J'_\varepsilon(v), \varphi\rangle + o_n(1) = o_n(1)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$. ☐

Lemma 4.5. *The functional J_ε satisfies the $(C)_{c_\varepsilon}$ condition at any level $c_\varepsilon < c_{V_\infty}$.*

Proof. Let $\{v_n\}$ be a $(C)_{c_\varepsilon}$ sequence of J_ε in X_ε; then

$$J_\varepsilon(v_n) \to c_\varepsilon, \quad (1 + \|v_n\|_{X_\varepsilon})J'_\varepsilon(v_n) \to 0.$$

By the boundedness of $\{v_n\}$ in X_ε, we know that there exists $v \in X_\varepsilon$ such that $v_n \rightharpoonup v$ in X_ε and $J'_\varepsilon(v) = 0$. Let $\tilde{v}_n = v_n - v$; by Lemma 4.4, we have $J'_\varepsilon(\tilde{v}_n) \to 0$ and

$$J_\varepsilon(\tilde{v}_n) = J_\varepsilon(v_n) - J_\varepsilon(v) + o_n(1) = c_\varepsilon - J_\varepsilon(v) + o_n(1) := d + o_n(1).$$

From (H_2) and Lemma 2.1 (4), we have

$$\begin{aligned} J_\varepsilon(v) &= J_\varepsilon(v) - \frac{1}{p}\langle J'_\varepsilon(v), v\rangle \\ &= \frac{1}{p}\int_{\mathbb{R}^N} V(\varepsilon x)[f^p(v) - f^{p-1}(v)f'(v)v]\,\mathrm{d}x + \int_{\mathbb{R}^N} \Big[\frac{1}{p}h(f(v))f'(v)v - H(f(v))\Big]\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \Big[\frac{1}{p}|f(v)|^{2p^*-1}f'(v)v - \frac{1}{2p^*}|f(v)|^{2p^*}\Big]\,\mathrm{d}x \ge 0. \end{aligned}$$

Since $V_\infty < \infty$, we have $d \le c_\varepsilon < c_{V_\infty}$. It follows from Lemma 4.3 that $Q(\tilde{v}_n) \to 0$. By [37, Proposition 2.4], we have $\tilde{v}_n \to 0$ in X_ε, that is, $v_n \to v$ in X_ε. ☐

In order to apply the Ljusternik–Schnirelman category theory, we need the functional J_ε to satisfy the compactness condition (such as $(PS)_c$ or $(C)_c$ condition) on the Nehari manifold. The following two Lemmas will explore this property.

Lemma 4.6. *The Nehari manifold $\mathcal{M}_\varepsilon$ is of C^1 class and $\langle \ell_\varepsilon'(v), v\rangle < 0$ for any $v \in \mathcal{M}_\varepsilon$, where $\ell_\varepsilon : X_\varepsilon \to \mathbb{R}$ is given by*

$$\ell_\varepsilon(v) = \int_{\mathbb{R}^N} [|\nabla v|^p + V(\varepsilon x)|f(v)|^{p-2} f(v) f'(v) v]\,dx - \int_{\mathbb{R}^N} h(f(v)) f'(v) v\,dx - \int_{\mathbb{R}^N} |f(v^+)|^{2p^*-1} f'(v^+) v\,dx.$$

Proof. Observe that

$$\begin{aligned}\langle \ell_\varepsilon'(v), v\rangle = p \int_{\mathbb{R}^N} |\nabla v|^p\,dx + (p-1) \int_{\mathbb{R}^N} V(\varepsilon x)|f(v)|^{p-2}|f'(v)|^2 v^2\,dx + \int_{\mathbb{R}^N} V(\varepsilon x)|f(v)|^{p-2} f(v) f''(v) v^2\,dx \\ + \int_{\mathbb{R}^N} V(\varepsilon x)|f(v)|^{p-2} f(v) f'(v) v\,dx - \int_{\mathbb{R}^N} h'(f(v))|f'(v)|^2 v^2\,dx - \int_{\mathbb{R}^N} h(f(v)) f''(v) v^2\,dx \\ - \int_{\mathbb{R}^N} h(f(v)) f'(v) v\,dx - (2p^*-1) \int_{\mathbb{R}^N} |f(v^+)|^{2p^*-2} |f'(v^+)|^2 v^2\,dx \\ - \int_{\mathbb{R}^N} |f(v^+)|^{2p^*-1} f''(v^+) v^2\,dx - \int_{\mathbb{R}^N} |f(v^+)|^{2p^*-1} f'(v^+) v\,dx.\end{aligned}$$

By $v \in \mathcal{M}_\varepsilon$, we deduce that

$$\begin{aligned}\langle \ell_\varepsilon'(v), v\rangle = \int_{\mathbb{R}^N} V(\varepsilon x)\Big((p-1)|f(v)|^{p-2}|f'(v)|^2 v^2 + V(\varepsilon x)|f(v)|^{p-2} f(v) f''(v) v^2 \\ - (p-1) V(\varepsilon x)|f(v)|^{p-2} f(v) f'(v) v\Big)\,dx - \int_{\mathbb{R}^N} h'(f(v))|f'(v)|^2 v^2\,dx - \int_{\mathbb{R}^N} h(f(v)) f''(v) v^2\,dx \\ + (p-1) \int_{\mathbb{R}^N} h(f(v)) f'(v) v\,dx - (2p^*-1) \int_{\mathbb{R}^N} |f(v^+)|^{2p^*-2} |f'(v^+)|^2 v^2\,dx \\ - \int_{\mathbb{R}^N} |f(v^+)|^{2p^*-1} f''(v^+) v^2\,dx + (p-1) \int_{\mathbb{R}^N} |f(v^+)|^{2p^*-1} f'(v^+) v\,dx.\end{aligned}$$

Let

$$\tilde{g}(t) = (p-1)|f(t)|^{p-2}|f'(t)|^2 t^2 - 2^{p-1}|f(t)|^{p-2}|f(t)|^p|f'(t)|^{p+2} t^2 - (p-1)|f(t)|^{p-2} f(t) f'(t) t$$

for $t \in \mathbb{R}$; according to the definition of f, we obtain that $\tilde{g}(t) = \tilde{g}(-t)$ for $t \in \mathbb{R}$. Note that by Lemma 2.1 (4), we have that $\tilde{g}(t) \le 0$ for $t \ge 0$. Thus $\tilde{g}(t) \le 0$ for all $t \in \mathbb{R}$. Hence, we get

$$\begin{aligned}\int_{\mathbb{R}^N} V(\varepsilon x)((p-1)|f(v)|^{p-2}|f'(v)|^2 v^2 + |f(v)|^{p-2} f(v) f''(v) v^2 - (p-1)|f(v)|^{p-2} f(v) f'(v) v)\,dx \\ = \int_{\mathbb{R}^N} V(\varepsilon x)((p-1)|f(v)|^{p-2}|f'(v)|^2 v^2 - 2^{p-1}|f(v)|^{2p-2}|f'(v)|^{p+2} v^2 - (p-1)|f(v)|^{p-2} f(v) f'(v) v)\,dx \\ = \int_{\mathbb{R}^N} V(\varepsilon x)\tilde{g}(v)\,dx \le 0. \qquad (4.17)\end{aligned}$$

From Proposition 2.2 (3) and $h(s) = 0$ for $s \le 0$, we have

$$h'(f(s))|f'(s)|^2 v^2 + h(f(s)) f''(s) s^2 - (p-1) h(f(s)) f'(s) s \ge 0 \quad \text{for all } s \in \mathbb{R}.$$

Thus

$$\int_{\mathbb{R}^N} [h'(f(v))|f'(v)|^2 v^2 + h(f(v)) f''(v) v^2 - (p-1) h(f(v)) f'(v) v]\,dx \ge 0. \qquad (4.18)$$

Therefore, by (4.17), (4.18) and Lemma 2.1 (4) and (7), we have that

$$\begin{aligned}\langle \ell_\varepsilon'(v), v\rangle &\le -(2p^*-1)\int_{\mathbb{R}^N}|f(v^+)|^{2p^*-2}|f'(v^+)|^2v^2\,\mathrm{d}x - \int_{\mathbb{R}^N}|f(v^+)|^{2p^*-1}f''(v^+)v^2\,\mathrm{d}x\\ &\quad + (p-1)\int_{\mathbb{R}^N}|f(v^+)|^{2p^*-1}f'(v^+)v\,\mathrm{d}x\\ &\le -(2p^*-1)\int_{\mathbb{R}^N}|f(v^+)|^{2p^*-2}|f'(v^+)|^2v^2\,\mathrm{d}x + 2^{p-1}\int_{\mathbb{R}^N}|f(v^+)|^{2p^*-1}|f(v^+)|^{p-1}|f'(v^+)|^{p+2}v^2\,\mathrm{d}x\\ &\quad + 2(p-1)\int_{\mathbb{R}^N}|f(v^+)|^{2p^*-2}|f'(v^+)|^2v^2\,\mathrm{d}x\\ &\le 2(p-p^*)\int_{\mathbb{R}^N}|f(v^+)|^{2p^*}\,\mathrm{d}x < 0. \end{aligned}\tag{4.19}$$

□

Lemma 4.7. *The functional J_ε restricted to $\mathcal{M}_\varepsilon$ satisfies the $(C)_{c_\varepsilon}$ condition at any level $c_\varepsilon < c_{V_\infty}$.*

Proof. Let $\tilde{J}_\varepsilon = J_\varepsilon|_{\mathcal{M}_\varepsilon}$. Let $\{v_n\} \subset \mathcal{M}_\varepsilon$ such that $\tilde{J}_\varepsilon(v_n) \to c_\varepsilon$, $(1+\|v_n\|_{X_\varepsilon})\tilde{J}_\varepsilon'(v_n) \to 0$. Thus, using $v_n \in \mathcal{M}_\varepsilon$ and $\tilde{J}_\varepsilon(v_n) = J_\varepsilon(v_n) \to c$, similar to the proof of Lemma 3.5, we conclude that $\{v_n\}$ is bounded in X_ε. By Lemma 4.5, the constrained gradient has the form

$$\tilde{J}_\varepsilon'(v) = J_\varepsilon'(v) - \frac{\langle J_\varepsilon'(v), \ell_\varepsilon'(v)\rangle}{\|\ell_\varepsilon'(v)\|^2}\ell_\varepsilon'(v).$$

For $v_n \in \mathcal{M}_\varepsilon$ being a $(C)_{c_\varepsilon}$ sequence, we denote

$$\lambda_n = \frac{\langle J_\varepsilon'(v_n), \ell_\varepsilon'(v_n)\rangle}{\|\ell_\varepsilon'(v_n)\|^2},$$

and then we have that

$$(1+\|v_n\|_{X_\varepsilon})J_\varepsilon'(v_n) = \lambda_n(1+\|v_n\|_{X_\varepsilon})\ell_\varepsilon'(v_n) + o_n(1).$$

By (4.19), we see that $\langle \ell_\varepsilon'(v_n), v_n\rangle \to \gamma \le 0$; if $\gamma = 0$, we have $f(v_n^+) \to 0$ in $L^{2p^*}(\mathbb{R}^N)$. Therefore, using an interpolation argument, the boundedness of $\{v_n\}$ in X_ε and (H_0)–(H_1), we deduce that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(\varepsilon x)|f(v_n)|^p)\,\mathrm{d}x &\le \int_{\mathbb{R}^N}(|\nabla v_n|^p + V(\varepsilon x)|f(v_n)|^{p-2}f(v_n)f'(v_n)v_n)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N} h(f(v_n^+))f'(v_n^+)v_n^+\,\mathrm{d}x + \int_{\mathbb{R}^N}|f(v_n)|^{2p^*-1}f'(v_n)v_n\,\mathrm{d}x\\ &\le \int_{\mathbb{R}^N} h(f(v_n^+))f(v_n^+)\,\mathrm{d}x + \int_{\mathbb{R}^N}|f(v_n^+)|^{2p^*}\,\mathrm{d}x\\ &\le \varepsilon\int_{\mathbb{R}^N}|f(v_n^+)|^p\mathrm{d}x + C_\varepsilon\int_{\mathbb{R}^N}|f(v_n^+)|^{q+1}\mathrm{d}x + \int_{\mathbb{R}^N}|f(v_n^+)|^{2p^*}\,\mathrm{d}x \to 0\end{aligned}$$

as $n \to \infty$, which contradicts Lemma 4.1. Thus $\gamma \ne 0$. This together with $v_n \in \mathcal{M}_\varepsilon$ leads to

$$0 = (1+\|v_n\|_{X_\varepsilon})\langle J_\varepsilon'(v_n), v_n\rangle = \lambda_n(1+\|v_n\|_{X_\varepsilon})\langle \ell_\varepsilon'(v_n), v_n\rangle + o_n(1)(1+\|v_n\|_{X_\varepsilon}),$$

and so $\lambda_n = o_n(1)$. Thus $(1+\|v_n\|_{X_\varepsilon})J_\varepsilon'(v_n) = o_n(1)$. We have proved that $\{v_n\}$ is a $(C)_{c_\varepsilon}$ sequence of J_ε in X_ε; the conclusion is obtained by Lemma 4.5. □

By a similar argument, or using Lemma 4.6, we get the following corollary.

Corollary 4.8. *The critical points of J_ε on $\mathcal{M}_\varepsilon$ are critical points of J_ε in X_ε.*

4.2 The existence of the ground state solution for $(\mathcal{P}_\varepsilon^*)$

Theorem 4.9. *Suppose that conditions* (V) *and* (H_0)–(H_5) *are satisfied. Then there exists* $\overline{\varepsilon} > 0$ *such that problem* $(\mathcal{P}_\varepsilon^*)$ *has a ground state solution* $u_\varepsilon \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$ *for all* $0 < \varepsilon < \overline{\varepsilon}$.

Proof. From the above statement, we know that J_ε satisfies the mountain pass geometry. By Theorem 3.1, there exists a $(C)_{c_\varepsilon}$ sequence $\{v_n\} \subset X_\varepsilon$ of J_ε satisfying

$$J_\varepsilon(v_n) \to c_\varepsilon \quad \text{and} \quad (1 + \|v_n\|_{X_\varepsilon})J'_\varepsilon(v_n) \to 0 \qquad \text{as } n \to \infty.$$

Without loss of generalization, we may assume that $V_0 = V(0) = \inf_{x \in \mathbb{R}^N} V(x)$. Let $\mu \in \mathbb{R}$ such that $V_0 < \mu < V_\infty$; we have that $c_{V_0} < c_\mu < c_{V_\infty}$. Let ω_μ be a nonnegative ground state of problem $(\mathcal{Q}_\mu)$ and let $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be such that $\phi(x) = 1$ for $|x| \le 1$ and $\phi(x) = 0$ for $|x| \ge 2$. For $R > 0$, set $\phi_R(x) = \phi(x/R)$, and let $u_R = \phi_R(x)\omega_\mu$. By Lemma 3.4, there exists $t_R > 0$ such that $v_R = t_R u_R \in \mathcal{N}_\mu$. Then there exists $R_0 > 0$ such that $v_{R_0} \in \mathcal{N}_\mu$ satisfies $\mathcal{I}_\mu(v_{R_0}) < c_{V_\infty}$. If not, $\mathcal{I}_\mu(v_R) \ge c_{V_\infty}$ for all $R > 0$; by $\omega_\mu \in \mathcal{N}_\mu$ and $u_R \to \omega_\mu$ in W_μ as $R \to \infty$, we obtain that $t_R \to 1$. Thus

$$c_{V_\infty} \le \liminf_{R \to \infty} \mathcal{I}_\mu(t_R u_R) = \mathcal{I}_\mu(\omega_\mu) = c_\mu < c_{V_\infty}.$$

This achieves a contradiction. Since $\operatorname{supp} v_{R_0}$ is a compact set, we may choose $\overline{\varepsilon} > 0$ such that $V(\varepsilon x) \le \mu$ for any $\varepsilon \in (0, \overline{\varepsilon})$ and $x \in \operatorname{supp} v_{R_0}$. Thus

$$\int_{\mathbb{R}^N} V(\varepsilon x)|f(v_{R_0})|^p \, \mathrm{d}x \le \int_{\mathbb{R}^N} \mu |f(v_{R_0})|^p \, \mathrm{d}x \quad \text{for all } \varepsilon \in (0, \overline{\varepsilon}).$$

Therefore, for all $\varepsilon \in (0, \overline{\varepsilon})$ and $t \ge 0$, we have that

$$J_\varepsilon(t v_{R_0}) \le \mathcal{I}_\mu(t v_{R_0}) \le \mathcal{I}_\mu(v_{R_0}) < c_{V_\infty},$$

which implies that $c_\varepsilon < c_{V_\infty}$ for all $\varepsilon \in (0, \overline{\varepsilon})$.

By Lemma 4.5, there exists a $v \in X_\varepsilon$ (the limit of $\{v_n\}$) such that

$$J_\varepsilon(v) = c_\varepsilon \quad \text{and} \quad J'_\varepsilon(v) = 0.$$

That is, $v \in X_\varepsilon$ is a solution of problem $(\mathcal{P}_\varepsilon^*)$. By a standard argument, we can obtain that $v \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ with $0 < \alpha < 1$ and $v \in L^\infty(\mathbb{R}^N)$. ☐

4.3 Multiplicity of solutions to $(\mathcal{P}_\varepsilon^*)$

In this subsection, we will study the multiplicity of solutions and study the behavior of its maximum points concentrating on the set M of global minima of V given in Section 1. The main result of this section is equivalent to Theorem 1.1 and it can be restated as follows.

Theorem 4.10. *Suppose that conditions* (V) *and* (H_0)–(H_5) *are satisfied. For a given* $\delta > 0$ *there exists* $\varepsilon_\delta > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_\delta)$ *problem* $(\mathcal{P}_\varepsilon^*)$ *has at least* $\operatorname{cat}_{M_\delta}(M)$ *positive weak solutions in* $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \bigcap L^\infty(\mathbb{R}^N)$. *Moreover, each solution decays to zero at infinity and if* u_ε *denotes one of these positive solutions and* $z_\varepsilon \in \mathbb{R}^N$ *its global maximum, then*

$$\lim_{\varepsilon \to 0} V(\varepsilon z_\varepsilon) = V_0.$$

To prove Theorem 4.10, we fix some notation and give some preliminary lemmas. Fix $\delta > 0$ and let ω be a ground state solution of problem $(\mathcal{Q}_{V_0})$. Let η be a smooth nonincreasing cut-off function defined in $[0, \infty)$ such that $\eta(s) = 1$ if $0 \le s \le \frac{\delta}{2}$ and $\eta(s) = 0$ if $s \ge \delta$. For any $\varepsilon > 0$ and $y \in M$, define a function $\psi_{\varepsilon,y}(x)$ by

$$\psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|)\omega\Big(\frac{\varepsilon x - y}{\varepsilon}\Big),$$

$t_\varepsilon > 0$ satisfying

$$\max_{t\geq 0} J_\varepsilon(t\psi_{\varepsilon,y}) = J_\varepsilon(t_\varepsilon \psi_{\varepsilon,y})$$

and $\phi_\varepsilon : M \to \mathcal{M}_\varepsilon$ by

$$\phi_\varepsilon(y) = t_\varepsilon \psi_{\varepsilon,y}.$$

By construction, $\phi_\varepsilon(y)$ has compact support for any $y \in M$.

For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Let $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be defined as $\chi(x) = x$ for $|x| \leq \rho$ and $\chi(x) = \frac{\rho x}{|x|}$ for $|x| \geq \rho$. Finally, let us consider $\beta : \mathcal{M}_\varepsilon \to \mathbb{R}^N$ given by

$$\beta(u) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)|u(x)|^p \,\mathrm{d}x}{\int_{\mathbb{R}^N} |u(x)|^p \,\mathrm{d}x}.$$

Lemma 4.11. *The function ϕ_ε satisfies the following limit:*

$$\lim_{\varepsilon\to 0} J_\varepsilon(\phi_\varepsilon(y)) = c_{V_0} \quad \textit{uniformly in } y \in M,$$

and

$$\lim_{\varepsilon\to 0} \beta(\phi_\varepsilon(y)) = y \quad \textit{uniformly in } y \in M.$$

Proof. The proof of this lemma can be found in [2]; we omit its proof. □

Lemma 4.12 (A compactness lemma). *Let $\{v_n\} \subset \mathcal{N}_\mu$ be a sequence satisfying $\mathcal{I}_\mu(v_n) \to c_\mu$. Then one of the following holds:*

1. *$\{v_n\}$ has a subsequence strongly convergent in W_μ;*
2. *there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that, up to a subsequence, $\widehat{v}_n = v_n(x + y_n)$ converges strongly in W_μ. In particular, there exists a minimizer for c_μ.*

Proof. Since $\{v_n\} \subset \mathcal{N}_\mu$ and $\mathcal{I}_\mu(v_n) \to c_\mu$, it is easy to check that $\{v_n\}$ is bounded in W_μ. Since $c_\mu = \inf_{v\in\mathcal{N}_\mu} \mathcal{I}_\mu$, we can use the Ekeland variational principle (see [39, p. 122, Theorem 8.5]): there exists $\omega_n \subset \mathcal{N}_\mu$ such that $\omega_n = v_n + o_n(1)$, $\mathcal{I}_\mu(\omega_n) \to c_\mu$ and

$$\mathcal{I}'_\mu(\omega_n) = \lambda_n \ell'_\mu(\omega_n) + o_n(1).$$

Using the boundedness of $\{v_n\}$, we obtain that

$$(1 + \|\omega_n\|_\mu)\mathcal{I}'_\mu(\omega_n) = \lambda_n(1 + \|\omega_n\|_\mu)\ell'_\mu(\omega_n) + o_n(1),$$

where λ_n is a real number and $\ell_\mu(\omega) = \langle \mathcal{I}'_\mu(\omega), \omega\rangle$ for any $\omega \in W_\mu$. We claim that there exists $\alpha_0 > 0$ such that $|\langle \ell'_\mu(\omega_n), \omega_n\rangle| \geq \alpha_0$ for all $n \in \mathbb{N}$. Indeed, using a similar argument as we have done in Lemmas 4.6 and 4.7, we have $\lambda_n = o_n(1)$, which yields

$$\omega_n = v_n + o_n(1), \quad \mathcal{I}_\mu(\omega_n) \to c_\mu, \quad (1 + \|\omega_n\|_\mu)\mathcal{I}'_\mu(\omega_n) \to 0.$$

So without loss of generality, we may suppose that $\{v_n\}$ is a $(C)_{c_\mu}$ for $\mathcal{I}_\mu$. Hence, up to a subsequence still denoted by $\{v_n\}$, we may assume that there exists $v \in W_\mu$ such that $v_n \rightharpoonup v$ in W_μ, $v_n \to v$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $s \in [1, p^*)$, $\nabla v_n(x) \to \nabla v(x)$ a.e. in $\mathbb{R}^N$ (see (3.5) in Lemma 3.5), and $v_n \to v$ a.e. in $\mathbb{R}^N$. Moreover, from Lemma 3.5, we see that $\mathcal{I}'_\mu(v) = 0$ and $v_n \geq 0$ for all $n \in \mathbb{N}$.

Case 1: $v \neq 0$. In this case, from the semi-continuity of the semi-norm, we have

$$\int_{\mathbb{R}^N} |\nabla v|^p \,\mathrm{d}x \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^p \,\mathrm{d}x.$$

We claim that equality holds in the last inequality. Otherwise, using the fact that

$$\frac{1}{p}|f(t)|^p - \frac{2}{\theta}|f(t)|^{p-1}f'(t)t, \quad \frac{2}{\theta}h(f(t))f'(t)t - H(f(t)), \quad \frac{2}{\theta}|f(t)|^{2p^*-1}f'(t)t - \frac{1}{2p^*}|f(t)|^{2p^*}$$

are nonnegative functions for $t \geq 0$, and by Fatou's Lemma, we have that

$$\begin{aligned}
c_\mu \leq \mathcal{J}_\mu(v) &= \mathcal{J}_\mu(v) - \frac{2}{\theta}\langle \mathcal{J}_\mu'(v), v\rangle \\
&= \Big(\frac{1}{p} - \frac{2}{\theta}\Big) \int_{\mathbb{R}^N} |\nabla v|^p \, dx + \int_{\mathbb{R}^N} \Big[\frac{\mu}{p}|f(v)|^p - \frac{2}{\theta}\mu|f(v)|^{p-1}f'(v)v\Big] \, dx + \int_{\mathbb{R}^N} \Big[\frac{2}{\theta}h(f(v))f'(v)v - H(f(v))\Big] \, dx \\
&\quad + \int_{\mathbb{R}^N} \Big[\frac{2}{\theta}|f(v)|^{2p^*-1}f'(v)v - \frac{1}{2p^*}|f(v)|^{2p^*}\Big] \, dx \\
&< \lim_{n\to\infty} \Big\{\Big(\frac{1}{p} - \frac{2}{\theta}\Big) \int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} \Big[\frac{\mu}{p}|f(v_n)|^p - \frac{2}{\theta}\mu|f(v_n)|^{p-1}f'(v_n)v_n\Big] \, dx \\
&\quad + \int_{\mathbb{R}^N} \Big[\frac{2}{\theta}h(f(v_n))f'(v_n)v_n - H(f(v_n))\Big] \, dx + \int_{\mathbb{R}^N} \Big[\frac{2}{\theta}|f(v_n)|^{2p^*-1}f'(v_n)v_n - \frac{1}{2p^*}|f(v_n)|^{2p^*}\Big] \, dx\Big\} \\
&= \lim_{n\to\infty} \Big(\mathcal{J}_\mu(v_n) - \frac{2}{\theta}\langle \mathcal{J}_\mu'(v_n), v_n\rangle\Big) \\
&\leq c_\mu,
\end{aligned}$$

which leads to a contradiction. So we obtain

$$\int_{\mathbb{R}^N} |\nabla v|^p \, dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^p \, dx.$$

Combining (4.20) with $\nabla v_n(x) \to \nabla v(x)$ a.e. in $\mathbb{R}^N$ and the Brezis–Lieb Lemma [10], we can conclude that

$$\int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p \, dx \to 0 \quad \text{as } n \to \infty. \tag{4.20}$$

By an argument similar to (4.20), we also have that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \Big[\frac{\mu}{p}|f(v_n)|^p - \frac{2}{\theta}\mu|f(v_n)|^{p-1}f'(v_n)v_n\Big] \, dx = \int_{\mathbb{R}^N} \Big[\frac{\mu}{p}|f(v)|^p - \frac{2}{\theta}\mu|f(v)|^{p-1}f'(v)v\Big] \, dx.$$

Hence, up to a subsequence still denoted by $\{\frac{\mu}{p}|f(v_n)|^p - \frac{2}{\theta}\mu|f(v_n)|^{p-1}f'(v_n)v_n\}$, there exists $k(x) \in L^1(\mathbb{R}^N)$ such that

$$\frac{\theta - 2p}{p\theta}\mu|f(v_n)|^p \leq \frac{\mu}{p}|f(v_n)|^p - \frac{2}{\theta}\mu|f(v_n)|^{p-1}f'(v_n)v_n \leq k(x) \quad \text{a.e. in } \mathbb{R}^N,$$

where we have used Lemma 2.1 (4) and the fact that $v_n \geq 0$. By the Lebesgue dominated convergence theorem, we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \mu|f(v_n)|^p \, dx = \int_{\mathbb{R}^N} \mu|f(v)|^p \, dx,$$

which implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \mu|f(v_n) - f(v)|^p \, dx = 0.$$

Hence, up to a subsequence, there exists $k_1(x) \in L^1(\mathbb{R}^N)$ such that

$$\mu(|f(v_n)|^p - |f(v)|^p) \leq k_1(x) \quad \text{a.e. in } \mathbb{R}^N.$$

Using the convexity of $|f|^p$ and the evenness of $|f|^p$, we have that

$$\begin{aligned}
V(x)|f(v_n - v)|^p &\leq \frac{V(x)}{2}|f(2v_n)|^p + \frac{V(x)}{2}|f(2v)|^p \\
&\leq 2^{p-1}(V(x)|f(v_n)|^p + V(x)|f(v)|^p) \\
&\leq 2^p(k_1(x) + V(x)|f(v)|^p).
\end{aligned}$$

Since $k_1(x) + V(x)|f(v)|^p \in L^1(\mathbb{R}^N)$, by the condition $v_n \to v$ a.e. in $\mathbb{R}^N$ and the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \mu|f(v_n - v)|^p \, \mathrm{d}x = 0. \tag{4.21}$$

Therefore, by Lemma 2.1 (9), the Sobolev inequality, (4.20), and (4.21), we deduce that

$$\begin{aligned} \int_{\mathbb{R}^N} \mu|v_n - v|^p \, \mathrm{d}x &= \int_{\{|v_n-v|\le 1\}} \mu|v_n - v|^p \, \mathrm{d}x + \int_{\{|v_n-v|\ge 1\}} \mu|v_n - v|^p \, \mathrm{d}x \\ &\le C \int_{\{|v_n-v|\le 1\}} \mu|f(v_n - v)|^p \, \mathrm{d}x + \int_{\{|v_n-v|\ge 1\}} \mu|v_n - v|^{p^*} \, \mathrm{d}x \\ &\le C \int_{\mathbb{R}^N} \mu|f(v_n - v)|^p \, \mathrm{d}x + \mu S^{\frac{1}{p^*}} \int_{\mathbb{R}^N} |\nabla v_n - \nabla v|^p \, \mathrm{d}x \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Consequently, we conclude that $v_n \to v$ in W_μ.

Case 2: $v \equiv 0$. By Lemma 3.6, there exist a sequence R, $a > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_R(y_n)} |v_n|^p \, \mathrm{d}x \ge a > 0.$$

Let $\widehat{v}_n = v_n(x + y_n)$; then $\mathcal{I}_\mu(\widehat{v}_n) \to c_\mu$ and $(1 + \|\widehat{v}_n\|_\mu)\mathcal{I}'_\mu(\widehat{v}_n) \to 0$ as $n \to \infty$. It is clear that there exists $\widehat{v} \in W_\mu$ such that $\widehat{v}_n \rightharpoonup \widehat{v}$ in W_μ. Then we use the discussion given in Case 1 to obtain the result. ☐

Lemma 4.13. *Let $\varepsilon_n \to 0^+$ and $(v_n) \subset \mathcal{M}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(v_n) \to c_{V_0}$. Then there exists a sequence $(\widetilde{y_n}) \subset \mathbb{R}^N$ such that $\widehat{v}_n = v_n(x + \widetilde{y_n})$ has a convergent subsequence in W_{V_0}. In particular, up to a subsequence, we have $\varepsilon_n\widetilde{y_n} \to y \in M$.*

Proof. In view of $\langle J'_{\varepsilon_n}(v_n), v_n\rangle = 0$ and $J_{\varepsilon_n}(v_n) \to c_{V_0}$, by an argument similar to Lemma 3.5 (1), we conclude that $\{v_n\}$ is bounded in W_{V_0}. Since $c_\mu > 0$, we have $Q_{V_0}(v_n) \ne 0$ for all $n \in \mathbb{N}$. If not, it is easy to check that $c_\mu \le 0$, a contradiction. By Lemma 3.6, there exist a sequence $\{\widetilde{y_n}\} \subset \mathbb{R}^N$ and positive constants $\widetilde{R}$, ξ such that

$$\liminf_{n\to\infty} \int_{B_R(\widetilde{y_n})} |v_n|^p \, \mathrm{d}x \ge \xi > 0.$$

Let $\widetilde{v}_n = v_n(x + \widetilde{y_n})$; up to a subsequence, there exists $\widetilde{v} \in W_{V_0}$ such that $\widetilde{v}_n \rightharpoonup \widetilde{v} \not\equiv 0$ in W_{V_0}. Let $t_n > 0$ be such that $\omega_n := t_n\widetilde{v}_n \in \mathcal{N}_{V_0}$. Using $v_n \in \mathcal{M}_{\varepsilon_n}$, we deduce that

$$\begin{aligned} c_{V_0} &\le \mathcal{I}_{V_0}(\omega_n) \\ &\le \frac{1}{p} \int_{\mathbb{R}^N} [|\nabla \omega_n|^p + V(\varepsilon_n x + \widetilde{y_n})|f(\omega_n)|^p] \, \mathrm{d}x - \frac{1}{2p^*} \int_{\mathbb{R}^N} |f(\omega_n^+)|^{2p^*} \, \mathrm{d}x - \int_{\mathbb{R}^N} H(f(\omega_n)) \, \mathrm{d}x \\ &= \frac{1}{p} \int_{\mathbb{R}^N} [|\nabla (t_n\widetilde{v}_n)|^p + V(\varepsilon_n x + \widetilde{y_n})|f(t_n\widetilde{v}_n)|^p] \, \mathrm{d}x - \frac{1}{2p^*} \int_{\mathbb{R}^N} |f(t_n\widetilde{v}_n^+)|^{2p^*} \, \mathrm{d}x - \int_{\mathbb{R}^N} H(f(t_n\widetilde{v}_n)) \, \mathrm{d}x \\ &= J_{\varepsilon_n}(t_n v_n) \le J_{\varepsilon_n}(v_n) \\ &= c_{V_0} + o_n(1). \end{aligned}$$

Hence, $\lim_{n\to\infty} \mathcal{I}_{V_0}(\omega_n) = c_{V_0}$. From $\omega_n \in \mathcal{N}_{V_0}$ it follows that $\{\omega_n\}$ is bounded in W_{V_0}. By the boundedness of $\{\widetilde{v}_n\}$, we know that $\{t_n\}$ is bounded. Thus, up to a subsequence still denoted by $\{t_n\}$, we may assume that $t_n \to t_0$. If $t_0 = 0$, from the boundedness of $\widetilde{v}_n$ we have that $\omega_n = t_n\widetilde{v_n} \to 0$. Hence, $\mathcal{I}_{V_0}(\omega_n) \to 0$, which contradicts $c_{V_0} > 0$. Thus $t_0 > 0$. From the boundedness of $\{\omega_n\}$ and $\widetilde{v}_n \rightharpoonup \widetilde{v}$, up to a subsequence, we have that $\omega_n \rightharpoonup \omega = t_0\widetilde{v}$ in W_{V_0}. By $t_0 > 0$ and $\widetilde{v} \ne 0$, we see that $\omega \ne 0$. From Lemma 4.12 we have $\omega_n \to \omega$ in W_{V_0}, which implies that $\widetilde{v}_n \to \widetilde{v}$ in W_{V_0}.

It remains to show that $\varepsilon_n\widetilde{y_n}$ is bounded. In fact, suppose by contradiction that $|\varepsilon_n\widetilde{y_n}| \to \infty$. Since $\omega_n \to \omega$ in W_{V_0} and $V_0 < V_\infty$, it follows that

$$\begin{aligned}
c_{V_0} &= \mathcal{I}_{V_0}(\omega) < \mathcal{I}_{V_\infty}(\omega)\\
&\le \liminf_{n\to\infty}\Big[\frac{1}{p}\int_{\mathbb{R}^N}|\nabla\omega_n|^p\,\mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N}V(\varepsilon_n x+\varepsilon_n\widetilde{y_n})|f(\omega_n)|^p\,\mathrm{d}x - \frac{1}{2p^*}\int_{\mathbb{R}^N}|f(\omega_n)|^{2p^*}\,\mathrm{d}x - \int_{\mathbb{R}^N}H(f(\omega_n))\,\mathrm{d}x\Big]\\
&= \liminf_{n\to\infty}J_{\varepsilon_n}(t_n v_n) \le \liminf_{n\to\infty}J_{\varepsilon_n}(v_n) = c_{V_0},
\end{aligned}$$

which gives a contradiction. Thus, $\varepsilon_n\widetilde{y_n}$ is bounded in $\mathbb{R}^N$ and, up to a subsequence, $\varepsilon_n\widetilde{y_n} \to y$ in $\mathbb{R}^N$. If $y \notin M$, then $V(y) > V_0$ and we can obtain a contradiction by arguing as above. Hence $y \in M$. ☐

Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a positive function such that $g(\varepsilon) \to 0$ as $\varepsilon \to 0$ and define the set

$$\widetilde{\mathcal{M}_\varepsilon} = \{v \in \mathcal{M}_\varepsilon : J_\varepsilon(v) \le c_{V_0} + g(\varepsilon)\}.$$

Lemma 4.14. *Let $\delta > 0$; there holds that*

$$\lim_{\varepsilon\to 0}\sup_{v\in\widetilde{\mathcal{M}_\varepsilon}}\inf_{y\in M_\delta}|\beta(v) - y| = 0.$$

Proof. Let $\{\varepsilon_n\} \subset \mathbb{R}^+$ be such that $\varepsilon_n \to 0$. For each $n \in \mathbb{N}$, there exists $\{v_n\} \subset \widetilde{\mathcal{M}_{\varepsilon_n}}$ satisfying

$$\inf_{y\in M_\delta}|\beta(v_n) - y| = \sup_{v\in\widetilde{\mathcal{N}}_{\varepsilon_n}}\inf_{y\in M_\delta}|\beta(v) - y| + o_n(1).$$

Thus it suffices to find a sequence $\{y_n\} \subset M_\delta$ such that

$$\lim_{n\to\infty}|\beta(v_n) - y_n| = 0.$$

To obtain this sequence, we note that $\mathcal{I}_{V_0}(tv_n) \le J_\varepsilon(tv_n)$ for $t \ge 0$ and $\{v_n\} \subset \widetilde{\mathcal{M}_{\varepsilon_n}} \subset \mathcal{M}_{\varepsilon_n}$, and so

$$c_{V_0} \le c_{\varepsilon_n} \le J_{\varepsilon_n}(v_n) \le c_{V_0} + g(\varepsilon_n).$$

This implies that $J_\varepsilon(v_n) \to c_{V_0}$. By Lemma 4.13, we obtain a sequence $\{\widetilde{y_n}\} \subset \mathbb{R}^N$ such that $\varepsilon_n\widetilde{y_n} \subset M_\delta$ for n sufficiently large. Then

$$\beta(v_n) = \varepsilon_n\widetilde{y_n} + \frac{\int_{\mathbb{R}^N}[\chi(\varepsilon_n z + \varepsilon_n\widetilde{y_n}) - \varepsilon_n\widetilde{y_n}]|v_n(z+\widetilde{y_n})|^p\,\mathrm{d}x}{\int_{\mathbb{R}^N}|v_n(z+\widetilde{y_n})|^p\,\mathrm{d}x}.$$

Recalling that $\varepsilon_n x + \varepsilon_n\widetilde{y_n} \to y \in M$, we have that $\beta(v_n) = \varepsilon_n\widetilde{y_n} + o_n(1)$, and therefore the sequence $\{y_n := \varepsilon_n\widetilde{y_n}\}$ is required. ☐

The following Lemma plays a fundamental role in the study of the behavior of the maximum points of the solutions.

Lemma 4.15. *Suppose that conditions* (V) *and* (H_0)–(H_5) *are satisfied. Let v_n be a solution of the following problem:*

$$\begin{cases}
-\Delta_p v_n + V_n(x)|f(v_n)|^{p-2}f(v_n)f'(v_n) = h(f(v_n))f'(v_n) + |f(v_n)|^{2p^*-1}f'(v_n) & \text{in } \mathbb{R}^N,\\
v_n \in W^{1,p}(\mathbb{R}^N),\ v_n(x) > 0 & \text{in } \mathbb{R}^N,
\end{cases}$$

where $V_n(x) = V(\varepsilon_n x + \varepsilon_n\widetilde{y_n})$. If $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$ with $v \not\equiv 0$, then $v_n \in L^\infty(\mathbb{R}^N)$ and $\|v_n\|_{L^\infty(\mathbb{R}^N)} \le C$ for all $n \in \mathbb{N}$. Moreover, $\lim_{|x|\to\infty} v_n(x) = 0$ uniformly in n.

Proof. We only replace v by v_n and apply the fact that $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$ in Theorem 3.8. ☐

Lemma 4.16. *Under the hypotheses of Lemma 4.15, there exists $\delta_0 > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \ge \delta_0$ for all $n \in \mathbb{N}$.*

Proof. Suppose by contradiction that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $n \to \infty$. Given $\varepsilon_0 = \frac{V_0}{4}$, by (H_1) there exists $n_0 \in \mathbb{N}$ such that

$$\frac{h(f(v_n(x)))}{(f(v_n(x)))^{p-1}} < \varepsilon_0 \quad \text{a.e. in } \mathbb{R}^N \text{ for } n \geq n_0.$$

Thus, by Lemma 2.1 (4) and (9), for n large enough, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \Big(|\nabla v_n|^p + \frac{V_0}{2}|v_n|^p\Big)\,\mathrm{d}x &\leq \int_{\mathbb{R}^N} \Big(|\nabla v_n|^p + \frac{V_0}{2}|f(v_n)|^p\Big)\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^N} (|\nabla v_n|^p + V_n(x)|f(v_n)|^{p-2}f(v_n)f'(v_n)v_n)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} |f(v_n)|^{2p^*-1}f'(v_n)v_n\,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{h(f(v_n(x)))}{(f(v_n(x)))^{p-1}}(f(v_n(x)))^{p-1}f'(v_n)v_n\,\mathrm{d}x \\ &\leq \varepsilon_0 \int_{\mathbb{R}^N} |f(v_n)|^p\,\mathrm{d}x + \int_{\mathbb{R}^N} |f(v_n)|^{2p^*}\,\mathrm{d}x,\end{aligned}$$

which implies that $v_n \to 0$ in $W^{1,p}(\mathbb{R}^N)$, which contradicts the hypothesis that $v_n \to v$ with $v \neq 0$. Hence, there exists $\delta_0 > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \geq \delta_0$ for all $n \in \mathbb{N}$. ☐

4.4 Proof of Theorem 4.10

Proof. For a fixed $\delta > 0$, by Lemma 4.11 and Lemma 4.14, there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta)$, $\beta \circ \phi_\varepsilon$ is well defined. Fix $\varepsilon > 0$ small enough, so $\beta \circ \phi_\varepsilon$ is homotopic to the inclusion map $\mathrm{id} : M \to M_\delta$ and by arguments similar to the ones contained in the proofs of [7, Lemmas 4.2 and 4.3], we obtain that

$$\mathrm{cat}_{\widetilde{\mathcal{M}_\varepsilon}}(\widetilde{\mathcal{M}_\varepsilon}) \geq \mathrm{cat}_{M_\delta}(M).$$

Since J_ε satisfies the $(C)_{c_\varepsilon}$ condition for $c_\varepsilon \in (c_{V_0}, c_{V_0} + g(\varepsilon))$, using the Ljusternik–Schnirelman theory of critical points in [39] (see [39, Theorem 5.19]; it can be true under the $(C)_c$ condition), we know that J_ε possesses at least $\mathrm{cat}_{M_\delta}(M)$ critical points on $\mathcal{M}_\varepsilon$. Consequently, by Corollary 4.8, J_ε has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in X_ε.

The remaining proof of concentration behavior can be deduced by using Lemma 4.15 and Lemma 4.16 and its proof is a standard argument; we refer the interested readers to [1, 2, 24]. We omit its details here. ☐

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