

Research Article

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Limit profiles and uniqueness of ground states to the nonlinear Choquard equations

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Abstract: Consider nonlinear Choquard equations

$$\begin{cases} -\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ \lim_{x \rightarrow \infty} u(x) = 0, \end{cases}$$

where I_α denotes the Riesz potential and $\alpha \in (0, N)$. In this paper, we investigate limit profiles of ground states of nonlinear Choquard equations as $\alpha \rightarrow 0$ or $\alpha \rightarrow N$. This leads to the uniqueness and nondegeneracy of ground states when α is sufficiently close to 0 or close to N .

Keywords: Semilinear elliptic, Choquard, limit profile, uniqueness

MSC 2010: 35J10, 35J20, 35J61

1 Introduction

Let $N \geq 3$, $\alpha \in (0, N)$ and $p > 1$. We are concerned with the so-called nonlinear Choquard equations

$$\begin{cases} -\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ \lim_{x \rightarrow \infty} u(x) = 0, \end{cases} \quad (1.1)$$

where I_α is Riesz potential given by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha|x|^{N-\alpha}},$$

and Γ denotes the Gamma function. Equation (1.1) finds its physical origin especially when $N = 3$, $\alpha = 2$ and $p = 2$. In this case, a solution of the equation

$$-\Delta u + u = (I_2 * |u|^2)u \quad (1.2)$$

gives a solitary wave of the Schrödinger-type nonlinear evolution equation

$$i\partial_t \psi + \Delta \psi + (I_2 * |\psi|^2)\psi = 0,$$

which describes, through Hartree–Fock approximation, a dynamics of condensed states to a system of nonrelativistic bosonic particles with two-body attractive interaction potential I_2 that is Newtonian potential [2, 6]. Equation (1.2) also arises as a model of a polaron by Pekar [14] or in an approximation of Hartree–Fock theory for a one-component plasma [7].

Equation (1.1) enjoys a variational structure. It is the Euler–Lagrange equation of the functional

$$J_\alpha(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx.$$

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From the Hardy–Littlewood–Sobolev inequality (Proposition 2.1 below) one can see that J_α is well defined and is continuously differentiable on $H^1(\mathbb{R}^N)$ if $p \in [1 + \frac{\alpha}{N}, \frac{N+\alpha}{N-2}]$. We say a function $u \in H^1(\mathbb{R}^N)$ is a ground state solution to (1.1) if $J'_\alpha(u) = 0$ and

$$J_\alpha(u) = \inf\{J_\alpha(v) \mid v \in H^1(\mathbb{R}^N), J'_\alpha(v) = 0, v \neq 0\}.$$

When $N = 3, \alpha = 2$ and $p = 2$, the existence of a radial positive solution is proved in [7, 9, 11] by variational methods and in [1, 12, 18] by ODE approaches. In [13], Moroz and Van Schaftingen proved the existence of a ground state solution to (1.1) in the range of $p \in (1 + \frac{\alpha}{N}, \frac{N+\alpha}{N-2})$, and the nonexistence of a nontrivial finite energy solution of (1.1) for p outside of the above range. For qualitative properties of ground states to (1.1), we refer to [10, 13].

In this paper, we are interested in limit behaviors of ground state to (1.1) as either $\alpha \rightarrow 0$ or $\alpha \rightarrow N$. These shall play essential roles to prove the uniqueness and nondegeneracy of a positive radial ground state to (1.1) for α sufficiently close to 0 or N . From the existence results by Moroz and Van Schaftingen, we can see that a positive radial ground state of (1.1) exists for every $\alpha \in (0, N(p - 1))$ when $p \in (1, \frac{N}{N-2})$ is fixed. Also for given $p \in (2, \frac{2N}{N-2})$, a positive radial ground state of (1.1) exists for every $\alpha \in ((N - 2)p - N, N)$.

As $\alpha \rightarrow 0$, it is possible to see that the functional J_α formally approaches

$$J_0(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} |u|^{2p} \, dx \quad \text{on } H^1(\mathbb{R}^N)$$

because $I_\alpha * f$ approaches f as $\alpha \rightarrow 0$. It is well known that the Euler–Lagrange equation (equation (1.3) below) of J_0 admits a unique positive radial ground state solution. Thus it is reasonable to expect that the ground state of (1.3) is the limit profile of ground states of (1.1) as $\alpha \rightarrow 0$. Our first result is to confirm this.

Theorem 1.1. Fix $p \in (1, \frac{N}{N-2})$. Let $\{u_\alpha\}$ be a family of positive radial ground states to (1.1) for α close to 0 and let u_0 be a unique positive radial ground state of the equation

$$\begin{cases} -\Delta u + u = |u|^{2p-2}u & \text{in } \mathbb{R}^N, \\ \lim_{x \rightarrow \infty} u(x) = 0. \end{cases} \tag{1.3}$$

Then one has

$$\lim_{\alpha \rightarrow 0} \|u_\alpha - u_0\|_{H^1(\mathbb{R}^N)} = 0.$$

On the other hand, the functional J_α blows up when $\alpha \rightarrow N$ due to the term $\Gamma(\frac{N-\alpha}{2})$ in the coefficient of I_α . Thus we need to get rid of this by taking a scaling $v = s(N, \alpha, p)u$ where

$$s(N, \alpha, p) := \left(\frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}\right)^{\frac{1}{2p-2}} \sim \left(\frac{1}{N-\alpha}\right)^{\frac{1}{2p-2}} \quad \text{as } \alpha \rightarrow N.$$

With this scaling, J_α transforms into the following functional, which we still denote by J_α for simplicity:

$$J_\alpha(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + v^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha}} * |v|^p\right) |v|^p \, dx.$$

Then as $\alpha \rightarrow N$, the functional J_α approaches

$$J_N(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + v^2 \, dx - \frac{1}{2p} \left(\int_{\mathbb{R}^N} |v|^p \, dx\right)^2.$$

It is easy to see that for $p \in (2, \frac{2N}{N-2})$ the limit functional J_N is C^1 on $H^1(\mathbb{R}^N)$ and its Euler–Lagrange equation is

$$\begin{cases} -\Delta v + v = \left(\int_{\mathbb{R}^N} |v|^p \, dx\right) |v|^{p-2}v & \text{in } \mathbb{R}^N, \\ \lim_{x \rightarrow \infty} v(x) = 0. \end{cases} \tag{1.4}$$

The existence and properties of a ground state to (1.4) are studied in [15]. More precisely, it is shown in [15] that there exists a positive radial ground state v_0 of equation (1.4). Furthermore, the following properties for ground states to (1.4) are proved:

(i) The ground state energy level of (1.4) satisfies the mountain pass characterization, i.e.,

$$J_N(v_0) = \min_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} J_N(tv).$$

(ii) Any ground state of (1.4) is sign-definite, radially symmetric up to a translation and strictly decreasing in radial direction.

(iii) Any ground state of (1.4) decays exponentially as $|x| \rightarrow \infty$.

Our next result establishes uniqueness and linearized nondegeneracy of the ground state v_0 of (1.4).

Theorem 1.2. For $p \in (2, \frac{2N}{N-2})$, let v_0 be a positive radial ground state to (1.4). Then the following assertions hold:

(i) There is no other positive radial ground state to (1.4).

(ii) The linearized equation of (1.4) at v_0 , given by

$$-\Delta\phi + \phi - p\left(\int_{\mathbb{R}^N} v_0^{p-1}\phi dx\right)v_0^{p-1} - (p-1)\left(\int_{\mathbb{R}^N} v_0^p dx\right)v_0^{p-2}\phi = 0 \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

only admits solutions of the form

$$\phi = \sum_{i=1}^N c_i \partial_{x_i} v_0, \quad c_i \in \mathbb{R},$$

in the space $L^2(\mathbb{R}^N)$.

Using the uniqueness of v_0 , we can obtain an analogous result to Theorem 1.1.

Theorem 1.3. Fix $p \in (2, \frac{2N}{N-2})$. Let $\{u_\alpha\}$ be a family of positive radial ground states to (1.1) for α close to N and let $v_0 \in H^1(\mathbb{R}^N)$ be a unique positive radial ground state of (1.4). Then one has

$$\lim_{\alpha \rightarrow N} \|v_\alpha - v_0\|_{H^1(\mathbb{R}^N)} = 0,$$

where v_α is a family of rescaled functions given by $v_\alpha := s(N, \alpha, p)u_\alpha$.

Remark 1.4. By applying the standard comparison principle, it is also possible to see that there exist constants $C, c > 0$ which are independent of α close to N such that

$$u_\alpha(x) \leq C(N - \alpha)^{\frac{1}{2p-2}} e^{-c|x|},$$

which shows the vanishing profiles of u_α .

The limit profiles of ground states to (1.1) lead to the uniqueness and nondegeneracy of them for α either close to 0 or close to N . When $N = 3, \alpha = 2$ and $p = 2$, these were proved by Lenzmann [5] and Wei and Winter [19]. Xiang [20] extends this result to the case that $N = 3, \alpha = 2$ and $p > 2$ close to 2 by using perturbation arguments.

We say a positive radial ground state u_α of (1.1) is nondegenerate if the linearized equation of (1.1) at u_α , given by

$$-\Delta\phi + \phi - p(I_\alpha * (u_\alpha^{p-1}\phi))u_\alpha^{p-1} - (p-1)(I_\alpha * u_\alpha^p)u_\alpha^{p-2}\phi = 0 \quad \text{in } \mathbb{R}^N,$$

only admits solutions of the form

$$\phi = \sum_{i=1}^N c_i \partial_{x_i} u_\alpha, \quad c_i \in \mathbb{R},$$

in the space $L^2(\mathbb{R}^N)$. We should assume $p \geq 2$ for the well-definedness of the linearized equation.

Theorem 1.5 (Uniqueness and nondegeneracy). Fix $p \in [2, \frac{N}{N-2})$. Then a positive radial ground state of (1.1) is unique and nondegenerate for α sufficiently close to 0. Fix $p \in (2, \frac{2N}{N-2})$. Then the same conclusion holds true for α sufficiently close to N .

Remark 1.6. Here we note that in the case that α is close to 0, the uniqueness and nondegeneracy are proved only when $N = 3$, but in the case that α is close to N these are proved for every dimension $N \geq 3$.

It is worth mentioning that unlike the family of ground states u_α to (1.1), the family of least energy nodal solutions \tilde{u}_α to (1.1) (the minimal energy solution among all nodal solutions) does not converge to any nontrivial solution of the limit equations (1.3) or (1.4), even up to a translation and up to a subsequence. Actually, the asymptotic profile of \tilde{u}_α is shown to be

$$u_0(\cdot - \xi_\alpha^+) - u_0(\cdot - \xi_\alpha^-) \quad \text{as } \alpha \rightarrow 0$$

and

$$(N - \alpha)^{\frac{1}{2p-2}} (v_0(\cdot - \xi_\alpha^+) - v_0(\cdot - \xi_\alpha^-)) \quad \text{as } \alpha \rightarrow N$$

for some $\xi_\alpha^+, \xi_\alpha^- \in \mathbb{R}^N$ such that $\lim_{\alpha \rightarrow 0} |\xi_\alpha^+ - \xi_\alpha^-| = 0$; see [15] for the proof. By relying on this fact and the nondegeneracy of the ground state u_0 to (1.3), it is also proved in [15] that \tilde{u}_α is odd-symmetric with respect to the hyperplane normal to the vector $\xi_\alpha^+ - \xi_\alpha^-$ and through the point $(\xi_\alpha^+ - \xi_\alpha^-)/2$ when $\alpha \sim 0$ or $\alpha \sim N$.

The rest of this paper is organized as follows: In Section 2, we collect some useful auxiliary tools and technical results which are frequently invoked when proving the main theorems. Theorem 1.1 is proved in Section 3. Theorem 1.2 and 1.3 are proved in Section 4. In Sections 5 and 6, we prove our uniqueness and nondegeneracy results, respectively.

2 Auxiliary results

In this section, we provide some useful known results and auxiliary tools. We begin with giving sharp information on the best constant of the Hardy–Littlewood–Sobolev inequality. This plays an important role in our analysis.

Proposition 2.1 (Hardy–Littlewood–Sobolev inequality [3, 8]). *Let $p, r > 1$ and $0 < \alpha < N$ be such that*

$$\frac{1}{p} + \frac{1}{r} = 1 + \frac{\alpha}{N}.$$

Then for any $f \in L^p(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$ one has

$$\left| \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{f(x)g(y)}{|x - y|^{N-\alpha}} dx dy \right| \leq C(N, \alpha, p) \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^r(\mathbb{R}^N)}.$$

The sharp constant satisfies

$$C(N, \alpha, p) \leq \frac{N}{\alpha} (|\mathbb{S}^{N-1}|/N)^{\frac{N-\alpha}{N}} \frac{1}{pr} \left(\left(\frac{(N-\alpha)/N}{1-1/p} \right)^{\frac{N-\alpha}{N}} + \left(\frac{(N-\alpha)/N}{1-1/r} \right)^{\frac{N-\alpha}{N}} \right),$$

where $|\mathbb{S}^{N-1}|$ denotes the surface area of the $(N - 1)$ -dimensional unit sphere.

In addition, if $p = r = \frac{2N}{N+\alpha}$, then

$$C\left(N, \alpha, \frac{2N}{N+\alpha}\right) = \pi^{\frac{N-\alpha}{2}} \frac{\Gamma(\alpha/2)}{\Gamma((N+\alpha)/2)} \left(\frac{\Gamma(N)}{\Gamma(N/2)} \right)^{\frac{\alpha}{N}}.$$

The following Riesz potential estimate is equivalent to the Hardy–Littlewood–Sobolev inequality.

Proposition 2.2 ([3, 8]). *Let $1 \leq r < s < \infty$ and $0 < \alpha < N$ be such that*

$$\frac{1}{r} - \frac{1}{s} = \frac{\alpha}{N}.$$

Then for any $f \in L^r(\mathbb{R}^N)$ one has

$$\left\| \frac{1}{|\cdot|^{N-\alpha}} * f \right\|_{L^s(\mathbb{R}^N)} \leq K(N, \alpha, r) \|f\|_{L^r(\mathbb{R}^N)}.$$

Here, the sharp constant $K(N, \alpha, r)$ satisfies

$$\limsup_{\alpha \rightarrow 0} \alpha K(N, \alpha, r) \leq \frac{2}{r(r-1)} |\mathbb{S}^{N-1}|.$$

Corollary 2.3. *Let r, s satisfy the assumption in Proposition 2.2. Then for small $\alpha > 0$ there exists $C = C(N, r) > 0$ such that for any $f \in L^r(\mathbb{R}^N)$,*

$$\|I_\alpha * f\|_{L^s(\mathbb{R}^N)} \leq C \|f\|_{L^r(\mathbb{R}^N)}.$$

Proof. This immediately follows from Proposition 2.2 and the fact that $\Gamma(\frac{\alpha}{2}) \sim \frac{1}{\alpha}$ as $\alpha \rightarrow 0$. \square

We denote by $H_r^1(\mathbb{R}^N)$ the space of radial functions in $H^1(\mathbb{R}^N)$. The following compact embedding result is proved in [17].

Proposition 2.4. *The Sobolev embedding $H_r^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact if $2 < p < \frac{2N}{N-2}$.*

By combining the Hardy–Littlewood–Sobolev inequality (Proposition 2.1) and the compact Sobolev embedding, it is easy to see that the following convergence holds.

Proposition 2.5. *Let $\alpha \in (0, N)$ and $p \in (1 + \frac{\alpha}{N}, \frac{N+\alpha}{N-2})$ be given. Let $\{u_j\} \subset H_r^1(\mathbb{R}^N)$ be a sequence converging weakly to some $u_0 \in H_r^1(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$ as $j \rightarrow \infty$. Then*

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha}} * |u_j|^p \right) |u_j|^p dx \rightarrow \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha}} * |u_0|^p \right) |u_0|^p dx.$$

In addition, for any $\phi \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha}} * |u_j|^p \right) |u_j|^{p-2} u_j \phi dx \rightarrow \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha}} * |u_0|^p \right) |u_0|^{p-2} u_0 \phi dx.$$

It is useful to obtain estimates for $I_\alpha * (|u|^{p-1} u \phi)$ as $\alpha \rightarrow 0$ and $(1/|\cdot|^{N-\alpha}) * (|u|^{p-1} u \phi)$ as $\alpha \rightarrow N$ when $u, \phi \in H^1(\mathbb{R}^N)$.

Proposition 2.6. *Let $u, \phi \in H^1(\mathbb{R}^N)$. Then the following assertions hold:*

(i) *For every $1 < p < \frac{N}{N-2}$ and $0 < \alpha < \frac{p-1}{2}N$, there exists $C = C(N) > 0$ independent of α near 0 such that*

$$\|I_\alpha * (|u|^{p-2} u \phi)\|_{L^2(\mathbb{R}^N)} < C \|u\|_{H^1(\mathbb{R}^N)}^{p-1} \|\phi\|_{H^1(\mathbb{R}^N)}$$

and

$$\lim_{\alpha \rightarrow 0} \|I_\alpha * (|u|^{p-2} u \phi) - |u|^{p-2} u \phi\|_{L^2(\mathbb{R}^N)} = 0.$$

(ii) *For every $2 < p < \frac{2N}{N-2}$ and $\frac{(N-2)p}{2} < \alpha < N$, there exists $C = C(N, p) > 0$ independent of α near N such that*

$$\left\| \frac{1}{|\cdot|^{N-\alpha}} * (|u|^{p-2} u \phi) \right\|_{L^\infty(\mathbb{R}^N)} < C \|u\|_{H^1(\mathbb{R}^N)}^{p-1} \|\phi\|_{H^1(\mathbb{R}^N)}$$

and

$$\lim_{\alpha \rightarrow N} \left\| \left(\frac{1}{|\cdot|^{N-\alpha}} * (|u|^{p-2} u \phi) \right) - \int_{\mathbb{R}^N} |u|^{p-2} u \phi dx \right\|_{L^\infty(K)} = 0$$

for any compact set $K \subset \mathbb{R}^N$.

Proof. A proof for (i) can be found in [16]. We prove (ii). Observe from the Hölder inequality that

$$\begin{aligned} \left(\frac{1}{|\cdot|^{N-\alpha}} * |u|^{p-2} u \phi \right)(x) &\leq \int_{B_1(x)} \frac{1}{|x-y|^{N-\alpha}} (|u|^{p-2} u \phi)(y) dy + \int_{B_1^c(x)} \frac{1}{|x-y|^{N-\alpha}} (|u|^{p-2} u \phi)(y) dy \\ &\leq \left(\int_{B_1(0)} \frac{1}{|y|^{(N-\alpha)\frac{2^*}{2^*-p}}} dy \right)^{\frac{2^*-p}{2^*}} \|u\|_{L^{2^*}(\mathbb{R}^N)}^{p-1} \|\phi\|_{L^{2^*}(\mathbb{R}^N)} + \|u\|_{L^p(\mathbb{R}^N)}^{p-1} \|\phi\|_{L^p(\mathbb{R}^N)}, \end{aligned}$$

where 2^* denotes the critical Sobolev exponent $\frac{2N}{N-2}$. Note from the condition on α that $(N-\alpha)\frac{2^*}{2^*-p} < N$, so that the integral

$$\int_{B_1(0)} 1/|y|^{(N-\alpha)\frac{2^*}{2^*-p}} dy$$

is uniformly bounded for α sufficiently close to N . This proves the former assertion of (ii). To prove the latter, we suppose the contrary. Then there exist a compact set K and sequences $\alpha_j \rightarrow N$, $x_j \in K$ such that

$$\int_{\mathbb{R}^N} \frac{1}{|x_j - y|^{N-\alpha_j}} (|u|^{p-2} u \phi)(y) dy \not\rightarrow \int_{\mathbb{R}^N} |u|^{p-2} u \phi dy \quad \text{as } j \rightarrow \infty. \quad (2.1)$$

Define $f_j(y) = (|u|^{p-2} u \phi)(y) / |x_j - y|^{N-\alpha_j}$, so that $f_j(y) \rightarrow (|u|^{p-2} \phi)(y)$ almost everywhere as $j \rightarrow \infty$. We may assume $x_j \rightarrow x_0$ as $j \rightarrow \infty$ for some $x_0 \in K$. We claim that f_j is uniformly integrable and tight in \mathbb{R}^N , i.e., for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |f_j(y)| dy < \varepsilon$$

for every $E \subset \mathbb{R}^N$ satisfying $|E| < \delta$ and there exists $R > 0$ such that

$$\int_{B_R^c(0)} |f_j(y)| dy < \varepsilon.$$

Indeed, we have

$$\begin{aligned} \int_E |f_j(y)| dy &\leq \left(\int_E \frac{1}{|x_j - y|^{(N-\alpha_j) \frac{2^*}{2^*-p}}} dy \right)^{\frac{2^*-p}{2^*}} \|u\|_{L^{2^*}^p(E)}^{p-1} \|\phi\|_{L^{2^*}(E)} \\ &= \left(\left(\int_{E \cap B_1(x)} + \int_{E \cap B_1^c(x)} \right) \frac{1}{|x_j - y|^{(N-\alpha_j) \frac{2^*}{2^*-p}}} dy \right)^{\frac{2^*-p}{2^*}} \|u\|_{L^{2^*}^p(E)}^{p-1} \|\phi\|_{L^{2^*}(E)} \\ &\leq \left(\int_{B_1(0)} \frac{1}{|y|^{(N-\alpha_j) \frac{2^*}{2^*-p}}} dy + |E| \right)^{\frac{2^*-p}{2^*}} \|u\|_{L^{2^*}^p(E)}^{p-1} \|\phi\|_{L^{2^*}(E)}, \end{aligned}$$

which shows that f_j is uniformly integrable. Take also a large $R > 0$ such that $B_2(x_0) \subset B_R(0)$. Then since

$$\int_{B_R^c(0)} |f_j(y)| dy \leq \int_{B_R^c(0)} |u(y)|^{p-1} |\phi(y)| dy,$$

the tightness of f_j is proved. Now the Vitali convergence theorem says that $\int_{\mathbb{R}^N} f_j(y) dy \rightarrow \int_{\mathbb{R}^N} |u|^{p-2} u \phi dy$, which contradicts (2.1). This completes the proof. \square

Proposition 2.7. Fix $1 < p < \frac{N}{N-2}$. Let $\{\alpha_j\} > 0$ be a sequence converging to 0 and let $\{u_j\} \subset H_r^1(\mathbb{R}^N)$ be a sequence converging weakly in $H^1(\mathbb{R}^N)$ to some $u_0 \in H_r^1(\mathbb{R}^N)$. Then, as $j \rightarrow \infty$, the following holds:

$$\begin{aligned} \int_{\mathbb{R}^N} (I_{\alpha_j} * |u_j|^p) |u_j|^p dx &\rightarrow \int_{\mathbb{R}^N} |u_0|^{2p} dx, \\ \int_{\mathbb{R}^N} (I_{\alpha_j} * |u_j|^p) |u_j|^{p-2} u_j \phi dx &\rightarrow \int_{\mathbb{R}^N} |u_0|^{2p-2} u_0 \phi dx \quad \text{for any } \phi \in H^1(\mathbb{R}^N). \end{aligned}$$

Proof. For a proof of this proposition, we refer to [16]. \square

Proposition 2.8. Fix $2 < p < \frac{2N}{N-2}$. Let $\{\alpha_j\} > 0$ be a sequence converging to N and let $\{v_j\} \subset H_r^1(\mathbb{R}^N)$ be a sequence converging weakly in $H^1(\mathbb{R}^N)$ to some $v_0 \in H_r^1(\mathbb{R}^N)$. Then, as $j \rightarrow \infty$, the following holds:

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * |v_j|^p \right) |v_j|^p dx \rightarrow \left(\int_{\mathbb{R}^N} |v_0|^p dx \right)^2, \quad (2.2)$$

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * |v_j|^p \right) |v_j|^{p-2} v_j \phi dx \rightarrow \int_{\mathbb{R}^N} |v_0|^p dx \int_{\mathbb{R}^N} |v_0|^{p-2} v_0 \phi dx \quad \text{for any } \phi \in H^1(\mathbb{R}^N). \quad (2.3)$$

Proof. For (2.2), we decompose as

$$\begin{aligned} \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * |v_j|^p \right) |v_j|^p dx &= \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * |v_j|^p \right) (|v_j|^p - |v_0|^p) dx + \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * (|v_j|^p - |v_0|^p) \right) |v_0|^p dx \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * |v_0|^p \right) |v_0|^p dx \\ &=: A_j + B_j + C_j. \end{aligned}$$

Observe from Proposition 2.6 that

$$|A_j| \leq C \|v_j\|_{H^1(\mathbb{R}^N)}^p \| |v_j|^p - |v_0|^p \|_{L^1(\mathbb{R}^N)},$$

which goes to 0 as $j \rightarrow \infty$ by the compact Sobolev embedding $H_r^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$. The same argument in the proof of Proposition 2.6 (ii) also applies to show that there exists a constant $C > 0$ independent of α_j such that

$$|B_j| \leq C (\| |v_j|^p - |v_0|^p \|_{L^{2^*/p}(\mathbb{R}^N)} + \| |v_j|^p - |v_0|^p \|_{L^1(\mathbb{R}^N)}) \|v_0\|_{L^p(\mathbb{R}^N)}^p,$$

which also goes to 0 as $j \rightarrow \infty$. Finally, C_j goes to $(\int_{\mathbb{R}^N} |v_0|^p dx)^2$ as $j \rightarrow \infty$ by (2.2).

The idea of proof of (2.2) is equally applicable to prove (2.3). We omit it. □

3 Limit profile of ground states as $\alpha \rightarrow 0$

In this section, we prove Theorem 1.1. We choose an arbitrary $p \in (1, \frac{N}{N-2})$ and fix it throughout this section. We denote the ground state energy level of J_α by E_α . In other word, $E_\alpha = J_\alpha(u_\alpha)$, where u_α is a ground state solution to (1.1). The ground state energy level E_α of (1.1) satisfies the mountain pass characterization, i.e.,

$$E_\alpha := \min_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} J_\alpha(tu).$$

Recall that

$$J_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx - \frac{1}{2p} \int_{\mathbb{R}^N} |u|^{2p} dx \quad \text{on } H^1(\mathbb{R}^N),$$

whose Euler–Lagrange equation is (1.3). We define the mountain pass level of J_0 by

$$E_0 := \min_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} J_0(tu).$$

It is a well-known fact that E_0 is the ground state energy level of J_0 . Namely,

$$E_0 = \min \{ J_0(u) \mid u \in H^1(\mathbb{R}^N), J_0'(u) = 0, u \neq 0 \}.$$

The following lemma is proved in [15, Claim 1 of Proposition 4.1].

Lemma 3.1. *There holds*

$$\lim_{\alpha \rightarrow 0} E_\alpha = E_0.$$

Choose any sequences $\{\alpha_j\} > 0$ converging to 0 and $\{u_{\alpha_j}\}$ of positive radial ground states to (1.1).

Lemma 3.2. *There exists a positive radial solution $u_0 \in H^1(\mathbb{R}^N)$ to (1.3) such that $\{u_{\alpha_j}\}$ converges to u_0 in $H^1(\mathbb{R}^N)$ up to a subsequence.*

Proof. Multiplying equation (1.1) by u_{α_j} and integrating by parts, we get

$$\left(\frac{1}{2} - \frac{1}{2p} \right) \|u_{\alpha_j}\|_{H^1(\mathbb{R}^N)}^2 = J_{\alpha_j}(u_{\alpha_j}),$$

so $\|u_{\alpha_j}\|_{H^1(\mathbb{R}^N)}$ is uniformly bounded for j by Lemma 3.1. Then, up to a subsequence, $\{u_{\alpha_j}\}$ weakly converges in $H^1(\mathbb{R}^N)$ to some nonnegative radial function $u_0 \in H^1(\mathbb{R}^N)$. From Proposition 2.7 and the weak convergence of $\{u_{\alpha_j}\}$, one is able to deduce that u_0 is a weak solution of (1.3). In addition, we again multiply equation (1.1) by u_{α_j} , multiply equation (1.3) by u_0 and use Proposition 2.7 to get

$$\|u_{\alpha_j}\|_{H^1(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} (I_{\alpha_j} * |u_{\alpha_j}|^p) |u_{\alpha_j}|^p dx \rightarrow \int_{\mathbb{R}^N} |u_0|^{2p} dx = \|u_0\|_{H^1(\mathbb{R}^N)}^2 \quad \text{as } j \rightarrow \infty.$$

Combining this with the weak convergence of $\{u_{\alpha_j}\}$, we obtain the strong convergence of $\{u_{\alpha_j}\}$ to u_0 in $H^1(\mathbb{R}^N)$.

Now, it remains to prove that u_0 is positive. Observe from Corollary 2.3 and the Sobolev inequality that

$$\|u_{\alpha_j}\|_{H^1(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} (I_{\alpha_j} * |u_{\alpha_j}|^p) |u_{\alpha_j}|^p dx \leq C \|u_{\alpha_j}\|_{L^{2Np/(N+\alpha_j)}(\mathbb{R}^N)}^{2p} \leq C \|u_{\alpha_j}\|_{H^1(\mathbb{R}^N)}^{2p}. \quad (3.1)$$

Here, C is a universal constant independent of j . Then, dividing both sides of (3.1) by $\|u_{\alpha_j}\|_{H^1(\mathbb{R}^N)}^2$ and passing to a limit, we obtain a uniform lower bound for $\|u_{\alpha_j}\|_{H^1(\mathbb{R}^N)}$ which implies that u_0 is nontrivial due to the strong convergence of $\{u_{\alpha_j}\}$. Since u_0 is nonnegative, it is positive from the maximum principle. This completes the proof. \square

Then the next lemma follows.

Lemma 3.3. *There holds*

$$J_0(u_0) = E_0.$$

In other words, u_0 is a unique positive radial ground state to (1.3).

Proof. We see from Proposition 2.7, Lemma 3.1 and Lemma 3.2 that

$$\begin{aligned} E_0 &= \lim_{j \rightarrow \infty} E_{\alpha_j} = \lim_{j \rightarrow \infty} J_{\alpha_j}(u_{\alpha_j}) \\ &= \lim_{j \rightarrow \infty} \left(\frac{1}{2} \|u_{\alpha_j}\|_{H^1(\mathbb{R}^N)}^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_{\alpha_j} * |u_{\alpha_j}|^p) |u_{\alpha_j}|^p dx \right) \\ &= \frac{1}{2} \|u_0\|_{H^1(\mathbb{R}^N)}^2 - \frac{1}{2p} \int_{\mathbb{R}^N} |u_0|^{2p} dx = J_0(u_0), \end{aligned}$$

which proves the lemma. \square

Now, we are ready to complete the proof of Theorem 1.1. Let $\{u_\alpha\} \subset H_r^1(\mathbb{R}^N)$ be a family of positive radial ground states to (1.1) for α near 0. Suppose $\{u_\alpha\}$ does not converge in $H^1(\mathbb{R}^N)$ to the unique positive radial ground state u_0 of (1.3). Then there exists a positive number ε_0 and a sequence $\{\alpha_j\} \rightarrow 0$ such that $\|u_{\alpha_j} - u_0\|_{H^1(\mathbb{R}^N)} \geq \varepsilon_0$, which contradicts Lemma 3.2 and Lemma 3.3.

4 Limit profile of ground states as $\alpha \rightarrow N$

In this section, we prove Theorems 1.2 and 1.3. Choose and fix an arbitrary $p \in (2, \frac{2N}{N-2})$. By deleting the coefficient of the Riesz potential term from (1.1), we obtain the equation

$$\begin{cases} -\Delta v + v = \left(\frac{1}{|\cdot|^{N-\alpha}} * |v|^p \right) |v|^{p-2} v & \text{in } \mathbb{R}^N, \\ \lim_{x \rightarrow \infty} v(x) = 0. \end{cases} \quad (4.1)$$

For simplicity, we still denote by J_α the energy functional of (4.1). It is clear that the ground state energy level E_α of (4.1) also satisfies the mountain pass characterization:

$$E_\alpha := \min_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} J_\alpha(tv).$$

Recall that, as $\alpha \rightarrow N$, the functional J_α approaches a limit functional

$$J_N(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + v^2 \, dx - \frac{1}{2p} \left(\int_{\mathbb{R}^N} |v|^p \, dx \right)^2 \quad \text{on } H^1(\mathbb{R}^N),$$

whose Euler–Lagrange equation is (1.4).

4.1 Proof of Theorem 1.2

We first prove Theorem 1.2. To prove (i), we let v_1 and v_2 be two positive radial ground states to (1.4). By defining

$$a_1 = \int_{\mathbb{R}^N} |v_1|^p \, dx \quad \text{and} \quad a_2 = \int_{\mathbb{R}^N} |v_2|^p \, dx,$$

they are positive radial solutions of the equations $-\Delta w + w = a_1|w|^{p-2}w$ and $-\Delta w + w = a_2|w|^{p-2}w$, respectively. We note that $(a_1/a_2)^{1/(p-2)}v_1$ satisfies the latter equation. The classical result due to Kwong [4] says that a positive radial solution of the latter (and also the former) equation is unique, so one must have $(a_1/a_2)^{1/(p-2)}v_1 \equiv v_2$. Since both of v_1 and v_2 satisfy equation (1.4), we can conclude $(a_1/a_2)^{1/(p-2)} = 1$.

We next prove (ii). Let v_0 be the positive and radial ground state of (1.4). Let $a_0 = \int_{\mathbb{R}^N} v_0^p \, dx$. As discussed above, v_0 is a unique positive radial solution of

$$-\Delta w + w = a_0|w|^{p-2}w. \tag{4.2}$$

It is a well-known fact that the linearized operator of (4.2) at v_0 , given by

$$L(\phi) := -\Delta\phi + \phi - (p - 1)a_0v_0^{p-2}\phi,$$

admits only solutions of the form

$$\phi = \sum_{i=1}^N c_i \partial_{x_i} v_0, \quad c_i \in \mathbb{R}, \tag{4.3}$$

in the space $L^2(\mathbb{R}^N)$. To the contrary, suppose that (1.5) has a nontrivial solution $\phi \in L^2(\mathbb{R}^N)$, which is not of the form (4.3). Then we may assume that ϕ is L^2 orthogonal to $\partial_{x_i} v_0$ for every $i = 1, \dots, N$. By denoting $\lambda := p \int_{\mathbb{R}^N} v_0^{p-1} \phi \, dx$, we see $L(\phi) = \lambda v_0^{p-1}$, so λ should not be 0. Observe that

$$L\left(\frac{\lambda}{(2-p)a_0}v_0\right) = \frac{\lambda}{(2-p)a_0}L(v_0) = \frac{\lambda}{(2-p)a_0}(-\Delta v_0 + v_0 - (p-1)a_0v_0^{p-1}) = \lambda v_0^{p-1}.$$

This shows $L(\phi - \frac{\lambda}{(2-p)a_0}v_0) \equiv 0$, which implies that there are some $c_i \in \mathbb{R}$ such that

$$\phi - \frac{\lambda}{(2-p)a_0}v_0 = \sum_{i=1}^N c_i \partial_{x_i} v_0. \tag{4.4}$$

We claim that $c_i = 0$ for all i . Indeed, by multiplying the left-hand side of (4.4) by $\partial_{x_j} v_0$ and integrating, we get

$$\int_{\mathbb{R}^N} \phi \partial_{x_j} v_0 \, dx - \frac{\lambda}{(2-p)a_0} \int_{\mathbb{R}^N} v_0 \partial_{x_j} v_0 \, dx = -\frac{\lambda}{(2-p)a_0} \int_{\mathbb{R}^N} \frac{1}{2} \partial_{x_j} (v_0^2) \, dx = 0.$$

On the other hand, by multiplying (4.4) by $\partial_{x_j} v_0$ and integrating, we get

$$c_j \int_{\mathbb{R}^N} (\partial_{x_j} v_0)^2 \, dx + \sum_{i \neq j} c_i \int_{\mathbb{R}^N} \partial_{x_i} v_0 \partial_{x_j} v_0 \, dx = c_j \int_{\mathbb{R}^N} (\partial_{x_j} v_0)^2 \, dx + \sum_{i \neq j} c_i \int_{\mathbb{R}^N} \frac{x_i x_j}{r^2} v_0'(r) \, dx = c_j \int_{\mathbb{R}^N} (\partial_{x_j} v_0)^2 \, dx$$

since $\frac{x_i x_j}{r^2} v_0'(r)$ is odd in variables x_i and x_j . Combining these two integrals, the claim follows.

Now, observe that

$$\lambda = p \int_{\mathbb{R}^N} v_0^{p-1} \phi \, dx = p \frac{\lambda}{(2-p)a_0} \int_{\mathbb{R}^N} v_0^p \, dx = p \frac{\lambda}{2-p}.$$

This implies $p = 1$, which contradicts the hypothesis for p . This completes the proof of Theorem 1.2.

4.2 Proof of Theorem 1.3

Now it remains to prove Theorem 1.3. Choose an arbitrary positive sequence $\{\alpha_j\} \rightarrow N$ and an arbitrary sequence $\{v_{\alpha_j}\}$ of positive radial ground states to (4.1). Arguing similarly to the previous section, one can see that the following proposition also holds true.

Proposition 4.1. *By choosing a subsequence, $\{v_{\alpha_j}\}$ converges in $H^1(\mathbb{R}^N)$ to the unique positive radial ground state of (1.4).*

Proof. We follow the same lines as in Section 3. It is proved in [15, Claim 1 of Proposition 5.1] that

$$\lim_{\alpha \rightarrow N} E_\alpha = E_N,$$

where

$$E_N := \min_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} J_N(tv).$$

This implies that $\|v_{\alpha_j}\|_{H^1}$ is bounded and, consequently, has a weak subsequential limit $v_0 \in H^1_r(\mathbb{R}^N)$ which is radial and nonnegative. Proposition 2.8 says that v_0 is a solution of (1.4). Again using Proposition 2.8, we have

$$\|v_{\alpha_j}\|_{H^1}^2 = \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * v_{\alpha_j}^p \right) v_{\alpha_j}^p dx = \left(\int_{\mathbb{R}^N} v_0^p dx \right)^2 + o(1) = \|v_0\|_{H^1}^2 + o(1),$$

which implies the H^1 strong convergence of $\{v_{\alpha_j}\}$. We now invoke Proposition 2.6 (ii) to see

$$\|v_{\alpha_j}\|_{H^1}^2 = \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * v_{\alpha_j}^p \right) v_{\alpha_j}^p dx \leq C \|v_{\alpha_j}\|_{H^1}^p \|v_{\alpha_j}\|_{L^p}^p \leq C \|v_{\alpha_j}\|_{H^1}^{2p},$$

where C is independent of j . This shows that v_0 is nontrivial, so that it is positive by the strong maximum principle. Finally, as in Lemma 3.3, we can check $J_N(v_0) = E_N$, which completes the proof. \square

Now, we shall complete the proof of Theorem 1.3. Fix $p \in (2, \frac{2N}{N-2})$. Let $\{u_\alpha\}$ be a family of positive radial ground states to (1.1) for α close to N . Then it is clear that the rescaled functions $v_\alpha := s(N, \alpha)u_\alpha$ constitute a family of positive radial ground states of (4.1) by a direct computation. Therefore, as in the proof of Theorem 1.1, one may conclude $\lim_{\alpha \rightarrow N} \|v_\alpha - v_0\|_{H^1(\mathbb{R}^N)} = 0$, where we denote by v_0 a unique positive radial solution to (1.4).

5 Uniqueness of ground states

We begin this section with a simple elliptic estimate.

Lemma 5.1. *Let $\frac{2N}{N+2} \leq q \leq 2$. Then the operator $(-\Delta + I)^{-1}$ is bounded from $L^q(\mathbb{R}^N)$ into $H^1(\mathbb{R}^N)$.*

Proof. We multiply the equation $-\Delta u + u = f$ by u , integrate by parts and apply the Hölder inequality:

$$\|u\|_{H^1(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} fu dx \leq \|f\|_{L^q(\mathbb{R}^N)} \|u\|_{L^{q/(q-1)}(\mathbb{R}^N)}.$$

Since $2 \leq \frac{q}{q-1} \leq \frac{2N}{N-2}$, the Sobolev inequality applies to see

$$\|u\|_{H^1(\mathbb{R}^N)} \leq C \|f\|_{L^q(\mathbb{R}^N)}$$

for some C depending only on q and N . Then the density arguments complete the proof. \square

For $p \in (1, \frac{N}{N-2})$, choose and fix $\alpha_0 \in (0, \frac{N(p-1)}{2})$ and define an operator $A(\alpha, u)$ by

$$A(\alpha, u) := \begin{cases} u - (-\Delta + I)^{-1} [(I_\alpha * |u|^p)|u|^{p-2}u] & \text{if } \alpha \in (0, \alpha_0), \\ u - (-\Delta + I)^{-1} [|u|^{2p-2}u] & \text{if } \alpha = 0. \end{cases}$$

For $p \in (2, \frac{2N}{N-2})$, choose and fix $\alpha_N \in (\frac{(N-2)p}{2}, N)$ and define an operator $B(\alpha, v)$ by

$$B(\alpha, v) := \begin{cases} v - (-\Delta + I)^{-1} \left[\left(\frac{1}{|\cdot|^{N-\alpha}} * |v|^p \right) |v|^{p-2} v \right] & \text{if } \alpha \in (\alpha_N, N), \\ v - (-\Delta + I)^{-1} \left[\left(\int_{\mathbb{R}^N} |v|^p dx \right) |v|^{p-2} v \right] & \text{if } \alpha = N. \end{cases}$$

Lemma 5.2. *The operator A is a continuous map from $[0, \alpha_0] \times H_r^1(\mathbb{R}^N)$ into $H_r^1(\mathbb{R}^N)$ and is continuously differentiable with respect to u on $[0, \alpha_0] \times H_r^1(\mathbb{R}^N)$. The same conclusion holds true for B which is a map from $(\alpha_N, N] \times H_r^1(\mathbb{R}^N)$ into $H^1(\mathbb{R}^N)$.*

Proof. We first prove the continuity of A . Let $\{(\alpha_j, u_j)\}$ be a sequence in $[0, \alpha_0] \times H_r^1(\mathbb{R}^N)$ converging to some $(\alpha, u) \in [0, \alpha_0] \times H_r^1(\mathbb{R}^N)$. We only deal with the case $\alpha_j \neq 0$ and $\alpha = 0$. Then the remaining cases can be dealt with similarly as well as more easily. Lemma 5.1 shows that it is sufficient to prove that $(I_{\alpha_j} * |u_j|^p)|u_j|^{p-2}u_j$ converges to $|u|^{2p-2}u$ in $L^q(\mathbb{R}^N)$ for some $q \in [2N/(N+2), 2]$. We select $q = \frac{2p}{2p-1}$. Since $p \in (1, \frac{N}{N-2})$, one can easily see that q belongs to the above range. Then

$$\begin{aligned} & \| (I_{\alpha_j} * |u_j|^p)|u_j|^{p-2}u_j - |u|^{2p-2}u \|_{L^{2p/(2p-1)}(\mathbb{R}^N)} \\ & \leq \| (I_{\alpha_j} * |u_j|^p) - |u_j|^p \|_{L^{2p/(2p-1)}(\mathbb{R}^N)} \| |u_j|^{p-2}u_j \|_{L^{2p/(2p-1)}(\mathbb{R}^N)} + \| |u_j|^{2p-2}u_j - |u|^{2p-2}u \|_{L^{2p/(2p-1)}(\mathbb{R}^N)} \\ & \leq \| (I_{\alpha_j} * |u_j|^p) - |u_j|^p \|_{L^2(\mathbb{R}^N)} \| |u_j|^{p-1} \|_{L^{2p/(p-1)}(\mathbb{R}^N)} + o(1) \\ & \leq C \| (I_{\alpha_j} * |u_j|^p) - |u_j|^p + I_{\alpha_j} * (|u_j|^p - |u|^p) + |u|^p - |u_j|^p \|_{L^2(\mathbb{R}^N)} + o(1) \\ & \leq C \| I_{\alpha_j} * (|u_j|^p - |u|^p) \| + o(1) \leq C \| |u_j|^p - |u|^p \|_{L^{2N/(N+2\alpha)}(\mathbb{R}^N)} + o(1) = o(1), \end{aligned} \tag{5.1}$$

where we used the Hölder inequality, Sobolev inequality and sharp constant estimate in Proposition 2.2.

Differentiating A with respect to u , we get

$$\frac{\partial A}{\partial u}(\alpha, u)[\phi] = \begin{cases} \phi - (-\Delta + I)^{-1} [p(I_{\alpha} * |u|^{p-2}u\phi)|u|^{p-2}u + (p-1)(I_{\alpha} * |u|^p)|u|^{p-2}\phi] & \text{if } \alpha \in (0, \alpha_0), \\ \phi - (-\Delta + I)^{-1} [(2p-1)|u|^{2p-2}\phi] & \text{if } \alpha = 0. \end{cases}$$

Then one can apply essentially the same argument to (5.1) to see that $\frac{\partial A}{\partial u}$ is continuous on $[0, \alpha_0] \times H_r^1(\mathbb{R}^N)$.

We next address the operator B . Let $\{(\alpha_j, v_j)\}$ be a sequence in $(\alpha_N, N] \times H_r^1(\mathbb{R}^N)$ converging to some $(\alpha, v) \in (\alpha_N, N] \times H_r^1(\mathbb{R}^N)$. We only deal with the case $\alpha = N$ and $\alpha_j \neq N$. As above, it is sufficient to show that $(1/|\cdot|^{N-\alpha_j} * |v_j|^p)|v_j|^{p-2}v_j$ converges to $(\int_{\mathbb{R}^N} |v|^p dx)|v|^{p-2}v$ in $L^{p/(p-1)}(\mathbb{R}^N)$ for the continuity of B . This follows by arguing similarly to (5.1) with Proposition 2.6. \square

Lemma 5.3. *Suppose that u_0 is a unique positive radial ground state of (1.3). Then there exists a neighborhood $U_0 \subset [0, \alpha_0] \times H_r^1(\mathbb{R}^N)$ of a point $(0, u_0) \in [0, \alpha_0] \times H_r^1(\mathbb{R}^N)$ such that equation (1.1) admits a unique solution in U_0 . Suppose that v_0 is a unique positive radial ground state of (1.4). Then there exists a neighborhood $U_N \subset (\alpha_N, N] \times H_r^1(\mathbb{R}^N)$ of a point $(N, v_0) \in (\alpha_N, N] \times H_r^1(\mathbb{R}^N)$ such that equation (4.1) admits a unique solution in U_N .*

Proof. We only prove the former assertion. The latter assertion follows similarly. We claim that the linearized operator of A with respect to u at $(0, u_0)$, namely $\frac{\partial A}{\partial u}(0, u_0)$, is a linear isomorphism from $H_r^1(\mathbb{R}^N)$ into $H_r^1(\mathbb{R}^N)$. Observe that

$$\frac{\partial A}{\partial u}(0, u_0)[\phi] = \phi - (2p-1)(-\Delta + I)^{-1} [u_0^{2p-2}\phi].$$

Since u_0 decays exponentially, the map $\phi \mapsto u_0^{2p-2}\phi$ is compact from $H_r^1(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$, so the composite map $\phi \mapsto (-\Delta + I)^{-1} [u_0^{2p-2}\phi]$ is also compact from $H_r^1(\mathbb{R}^N)$ into $H_r^1(\mathbb{R}^N)$. This also shows that $\frac{\partial A}{\partial u}(0, u_0)$ is bounded. One can deduce from the radial linearized nondegeneracy of u_0 that the kernel of $\frac{\partial A}{\partial u}(0, u_0)$ is trivial. Then the Fredholm alternative applies to see that $\frac{\partial A}{\partial u}(0, u_0)$ is an onto map, so the claim is proved. We invoke the implicit function theorem to complete the proof. \square

Now, we claim that (1.1) admits a unique positive radial ground state for $p \in (1, \frac{N}{N-2})$ and α close to 0. Suppose the contrary. Then there exist sequences $\{\alpha_j\} > 0$, $\{u_{\alpha_j}^1\} \subset H_r^1(\mathbb{R}^N)$ and $\{u_{\alpha_j}^2\} \subset H_r^1(\mathbb{R}^N)$ such that $\alpha_j \rightarrow 0$ as $j \rightarrow \infty$, $\{u_{\alpha_j}^1\}$ and $\{u_{\alpha_j}^2\}$ are sequences of positive radial ground states of (1.1), and $u_{\alpha_j}^1 \neq u_{\alpha_j}^2$ for all j . Theo-

rem 1.1 tells us that both of $\{u_{\alpha_j}^1\}$ and $\{u_{\alpha_j}^2\}$ converge to a unique positive radial solution u_0 of (1.3) in $H^1(\mathbb{R}^N)$. This however contradicts Lemma 5.3, and thus shows the uniqueness of a positive radial ground state of (1.1) for $p \in (1, \frac{N}{N-2})$ and α close to 0. Note that the analogous conclusion holds for a family of ground states $\{v_\alpha\}$ of (4.1) when $p \in (2, \frac{2N}{N-2})$ and α close to N . By scaling back, this also shows the uniqueness of a positive radial ground state of (1.1) when $p \in (2, \frac{2N}{N-2})$ and α close to N .

6 Nondegeneracy of ground states

6.1 Nondegeneracy for α near 0

Throughout this subsection, we fix $N = 3$ due to the restriction $p \in [2, \frac{N}{N-2})$. We begin with proving a convergence lemma similar to Proposition 2.7, but slightly different.

Lemma 6.1. *For given $p \in [2, 3)$, let u_α be a family of the unique positive radial ground states of (1.1) and let u_0 be the positive radial ground state to (1.3). Then, for any $\{\alpha_j\} \rightarrow 0$ and $\{\psi_j\}, \{\phi_j\} \subset H^1$ weakly H^1 converging to ϕ_0 and ψ_0 , there holds*

$$\int_{\mathbb{R}^N} (I_{\alpha_j} * (u_{\alpha_j}^{p-1} \phi_j)) u_{\alpha_j}^{p-1} \psi_j \, dx \rightarrow \int_{\mathbb{R}^N} u_0^{2p-2} \phi_0 \psi_0 \, dx, \tag{6.1}$$

$$\int_{\mathbb{R}^N} (I_{\alpha_j} * u_{\alpha_j}^p) u_{\alpha_j}^{p-2} \phi_j \psi_j \, dx \rightarrow \int_{\mathbb{R}^N} u_0^{2p-2} \phi_0 \psi_0 \, dx. \tag{6.2}$$

Proof. We first note that $u_0^{p-1} \phi_j$ is compact in L^2 due to the uniform decaying property of u_0 . Then one has from the Hölder inequality that

$$\begin{aligned} \|u_{\alpha_j}^{p-1} \phi_j - u_0^{p-1} \phi_0\|_{L^2} &\leq \| (u_{\alpha_j}^{p-1} - u_0^{p-1}) \phi_j \|_{L^2} + \| u_0^{p-1} \phi_j - u_0^{p-1} \phi_0 \|_{L^2} \\ &\leq \| u_{\alpha_j}^{p-1} - u_0^{p-1} \|_{L^{2p/(p-1)}} \| \phi_j \|_{L^{2p}} + o(1) \\ &\leq C \| |u_{\alpha_j} - u_0| (|u_{\alpha_j}|^{p-2} + |u_0|^{p-2}) \|_{L^{2p/(p-1)}} \| \phi_j \|_{L^{2p}} + o(1) \\ &\leq \| u_{\alpha_j} - u_0 \|_{L^{2p}} (\| u_{\alpha_j} \|_{L^{2p}}^{p-2} + \| u_0 \|_{L^{2p}}^{p-2}) \| \phi_j \|_{L^{2p}} + o(1) \\ &= o(1), \end{aligned}$$

from which we deduce that $u_{\alpha_j}^{p-1} \phi_j$ is also compact in L^2 . We decompose the left-hand side of (6.1) as

$$\begin{aligned} \int_{\mathbb{R}^3} (I_{\alpha_j} * (u_{\alpha_j}^{p-1} \phi_j)) u_{\alpha_j}^{p-1} \psi_j \, dx &= \int_{\mathbb{R}^3} u_{\alpha_j}^{p-1} \phi_j u_{\alpha_j}^{p-1} \psi_j \, dx + \int_{\mathbb{R}^3} ((I_{\alpha_j} * (u_{\alpha_j}^{p-1} \phi_j)) - u_{\alpha_j}^{p-1} \phi_j) u_{\alpha_j}^{p-1} \psi_j \, dx \\ &= \int_{\mathbb{R}^3} u_{\alpha_j}^{p-1} \phi_j u_{\alpha_j}^{p-1} \psi_j \, dx + \int_{\mathbb{R}^3} (I_{\alpha_j} * (u_{\alpha_j}^{p-1} \phi_j - u_0^{p-1} \phi_0)) u_{\alpha_j}^{p-1} \psi_j \, dx \\ &\quad + \int_{\mathbb{R}^N} (I_{\alpha_j} * (u_0^{p-1} \phi_0) - u_{\alpha_j}^{p-1} \phi_j) u_{\alpha_j}^{p-1} \psi_j \, dx \\ &= \int_{\mathbb{R}^3} u_0^{2p-2} \phi_0 \psi_0 \, dx + \int_{\mathbb{R}^3} (I_{\alpha_j} * (u_{\alpha_j}^{p-1} \phi_j - u_0^{p-1} \phi_0)) u_{\alpha_j}^{p-1} \psi_j \, dx + o(1), \end{aligned}$$

where we used the Hölder inequality, Proposition 2.6 and the L^2 compactness of both $\{u_{\alpha_j}^{p-1} \phi_j\}$ and $\{u_{\alpha_j}^{p-1} \psi_j\}$. We now estimate by using Corollary 2.3 that

$$\begin{aligned} \int_{\mathbb{R}^N} (I_{\alpha_j} * (u_{\alpha_j}^{p-1} \phi_j - u_0^{p-1} \phi_0)) u_{\alpha_j}^{p-1} \psi_j \, dx &\leq \| I_{\alpha_j} * (u_{\alpha_j}^{p-1} \phi_j - u_0^{p-1} \phi_0) \|_{L^{2N/(N-2\alpha_j)}} \| u_{\alpha_j}^{p-1} \psi_j \|_{L^{2N/(N+2\alpha_j)}} \\ &\leq C \| u_{\alpha_j}^{p-1} \phi_j - u_0^{p-1} \phi_0 \|_{L^2} \| u_{\alpha_j}^{p-1} \psi_j \|_{L^1}^{2\alpha_j/N} \| u_{\alpha_j}^{p-1} \psi_j \|_{L^2}^{1-2\alpha_j/N} \\ &= o(1). \end{aligned}$$

This proves assertion (6.1). The proof of (6.2) follows exactly the same lines. □

Now we are ready to prove the nondegeneracy of ground states u_α to (1.1) near 0.

Proposition 6.2. *For given $p \in [2, 3)$, let u_α be a family of unique positive radial ground states of (1.1). Then for $\alpha > 0$ sufficiently close to 0 the linearized equation of (1.1) at u_α , given by*

$$-\Delta\phi + \phi - p(I_\alpha * (u_\alpha^{p-1}\phi))u_\alpha^{p-1} - (p-1)(I_\alpha * u_\alpha^p)u_\alpha^{p-2}\phi = 0 \quad \text{in } \mathbb{R}^3, \tag{6.3}$$

only admits solutions of the form

$$\phi = \sum_{i=1}^3 c_i \partial_{x_i} u_\alpha, \quad c_i \in \mathbb{R},$$

in the space $L^2(\mathbb{R}^3)$.

Proof. Differentiating (1.1) with respect to x_i , we see that $\partial_{x_i} u_\alpha \in L^2(\mathbb{R}^3)$ solves (6.3) for all $i = 1, \dots, N$. Define a finite-dimensional vector space

$$V_\alpha := \left\{ \sum_{i=1}^3 c_i \partial_{x_i} u_\alpha \mid c_i \in \mathbb{R} \right\}.$$

Arguing indirectly, we suppose there exists a sequence $\{\alpha_j\}$ converging to 0 such that for each j there exists a nontrivial solution $\phi_j \in L^2$ of (6.3) not belonging to V_{α_j} . We may assume that ϕ_j is L^2 orthogonal to V_{α_j} . We claim that any L^2 solution ϕ of (6.3) automatically belongs to $H^1(\mathbb{R}^3)$. Let us define

$$L[\phi] := p(I_\alpha * (u_\alpha^{p-1}\phi))u_\alpha^{p-1} + (p-1)(I_\alpha * u_\alpha^p)u_\alpha^{p-2}\phi.$$

By elliptic regularity theory, it is enough to show that $L[\phi] \in H^{-1}$. It is proved in [13] that $u_\alpha \in L^\infty$, and so $u_\alpha \in L^q$ for any $2 \leq q \leq \infty$ by interpolation. Then Proposition 2.2 and Proposition 2.6 imply that for any $\psi \in H^1$,

$$\begin{aligned} |L[\phi]\psi| &\leq \left| \int_{\mathbb{R}^3} p(I_\alpha * (u_\alpha^{p-1}\phi))u_\alpha^{p-1}\psi \, dx \right| + \left| \int_{\mathbb{R}^3} (p-1)(I_\alpha * u_\alpha^p)u_\alpha^{p-2}\phi\psi \, dx \right| \\ &\leq p \left| \int_{\mathbb{R}^3} (I_\alpha * (u_\alpha^{p-1}\psi))u_\alpha^{p-1}\phi \, dx \right| + (p-1) \left| \int_{\mathbb{R}^3} (I_\alpha * u_\alpha^p)u_\alpha^{p-2}\phi\psi \, dx \right| \\ &\leq p \|I_\alpha * (u_\alpha^{p-1}\psi)\|_{L^2} \|u_\alpha\|_{L^\infty}^{p-1} \|\phi\|_{L^2} + (p-1) \|I_\alpha * u_\alpha^p\|_{L^3} \|u_\alpha\|_{L^\infty}^{p-2} \|\phi\|_{L^2} \|\psi\|_{L^6} \\ &\leq C(\|u_\alpha\|_{H^1}^{p-1} \|u_\alpha\|_{L^\infty}^{p-1} \|\phi\|_{L^2} + \|u_\alpha\|_{L^{3p/(1+\alpha)}}^p \|u_\alpha\|_{L^\infty}^{p-2} \|\phi\|_{L^2}) \|\psi\|_{H^1}, \end{aligned}$$

which shows $L[\phi] \in H^{-1}$. We normalize ϕ_j as $\|\phi_j\|_{H^1} = 1$. As $j \rightarrow \infty$, it is possible to deduce from Lemma 6.1 that ϕ_j weakly converges in H^1 to some $\phi_0 \in H^1$ which satisfies

$$-\Delta\phi_0 + \phi_0 - (2p-1)u_0^{2p-2}\phi_0 = 0,$$

where u_0 is a unique positive radial solution of (1.4). Repeatedly applying Lemma 6.1, we also have

$$\begin{aligned} 1 &= \|\phi_j\|_{H^1}^2 \\ &= p \int_{\mathbb{R}^3} I_{\alpha_j} * (u_{\alpha_j}^{p-1}\phi_j)u_{\alpha_j}^{p-1}\phi_j \, dx + (p-1) \int_{\mathbb{R}^3} (I_{\alpha_j} * u_{\alpha_j}^p)u_{\alpha_j}^{p-2}\phi_j^2 \, dx \\ &= (2p-1) \int_{\mathbb{R}^3} u_0^{2p-2}\phi_0^2 \, dx + o(1) \quad \text{as } j \rightarrow \infty. \end{aligned}$$

This shows that ϕ_0 is nontrivial. Finally, we note that for all $i = 1, 2, 3$,

$$0 = \int_{\mathbb{R}^3} \partial_{x_i} u_{\alpha_j} \phi_j \, dx \rightarrow \int_{\mathbb{R}^3} \partial_{x_i} u_0 \phi_0 \, dx \quad \text{as } j \rightarrow \infty.$$

This means that ϕ_0 is not a linear combination of $\{\partial_{x_i} u_0 \mid i = 1, 2, 3\}$. This contradicts the linearized nondegeneracy of u_0 and completes the proof. □

6.2 Nondegeneracy for α near N

Arguing as in the proof of Proposition 6.2, we also obtain the nondegeneracy result of α near N . We need lemmas analogous to Lemma 6.1.

Lemma 6.3. *For given $p \in (2, \frac{2N}{N-2})$, let v_α be a family of unique positive radial ground states of (4.1) and let v_0 be the positive radial ground state to (1.4). Then, for any $\{\alpha_j\} \rightarrow 0$ and $\{\psi_j\}, \{\phi_j\} \subset H^1$ weakly H^1 converging to ϕ_0 and ψ_0 , there holds*

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * (v_{\alpha_j}^{p-1} \phi_j) \right) v_{\alpha_j}^{p-1} \psi_j \, dx \rightarrow \left(\int_{\mathbb{R}^N} v_0^{p-1} \phi_0 \, dx \right) \left(\int_{\mathbb{R}^N} v_0^{p-1} \psi_0 \, dx \right), \quad (6.4)$$

$$\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * u_{\alpha_j}^p \right) u_{\alpha_j}^{p-2} \phi_j \psi_j \, dx \rightarrow \left(\int_{\mathbb{R}^N} u_0^p \, dx \right) \left(\int_{\mathbb{R}^N} u_0^{p-2} \phi_0 \psi_0 \, dx \right). \quad (6.5)$$

Proof. From the same argument as in Proposition 6.1 one can see that $v_{\alpha_j}^{p-1} \phi_j$ is compact in $L^1 \cap L^{2^*/p}$, where $2^* = \frac{2N}{N-2}$. By following the same argument in the proof of Proposition 2.6 (ii), one is able to see

$$\left\| \frac{1}{|\cdot|^{N-\alpha}} * f \right\|_{L^\infty} \leq C \|f\|_{L^1 \cap L^{2^*/p}}, \quad (6.6)$$

where $f \in L^1 \cap L^{2^*/p}$, α is near N and C is independent of α near N . Using Proposition 2.6 (ii), we then compute

$$\begin{aligned} & \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * (v_{\alpha_j}^{p-1} \phi_j) \right) v_{\alpha_j}^{p-1} \psi_j \, dx \\ &= \underbrace{\int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * (v_{\alpha_j}^{p-1} \phi_j - v_0^{p-1} \phi_0) \right) v_{\alpha_j}^{p-1} \psi_j \, dx}_{(A)} + \int_{\mathbb{R}^N} \left(\frac{1}{|\cdot|^{N-\alpha_j}} * (v_0^{p-1} \phi_0) \right) v_{\alpha_j}^{p-1} \psi_j \, dx \\ &= (A) + \left(\int_{\mathbb{R}^N} v_0^{p-1} \phi_0 \, dx \right) \left(\int_{\mathbb{R}^N} v_0^{p-1} \psi_0 \, dx \right) + o(1), \end{aligned}$$

and estimate (6.6) implies

$$|(A)| \leq C \|v_{\alpha_j}^{p-1} \phi_j - v_0^{p-1} \phi_0\|_{L^1 \cap L^{2^*/p}} \|v_{\alpha_j}^{p-1} \psi_j\|_{L^1} = o(1),$$

which proves assertion (6.4).

To prove (6.5), we claim that $v_0^{p-2} \phi_j \psi_j$ is L^1 compact. Observe that

$$\int_{\mathbb{R}^N \setminus B_R} |v_0^{p-2} \phi_j \psi_j| \, dx \leq \|v_0\|_{L^p(\mathbb{R}^N \setminus B_R)}^{p-2} \|\phi_j\|_{L^p(\mathbb{R}^N)} \|\psi_j\|_{L^p(\mathbb{R}^N)},$$

so that $v_0^{p-2} \phi_j \psi_j$ is tight. Also we note that $\|v_0\|_{C^1(\mathbb{R}^N)}$ is finite by the elliptic regularity theory, and for every B_R there exists C_R such that $\|v_0^{p-3}\|_{B_R} \leq C_R$ because v_0 is continuous and positive everywhere. Then we can see that $v_0^{p-2} \phi_j \psi_j \in W^{1,1}(B_R)$ from the estimate

$$\begin{aligned} & \|v_0^{p-2} \phi_j \psi_j\|_{W^{1,1}(B_R)} \\ & \leq \|v_0^{p-2} \phi_j \psi_j\|_{L^1(B_R)} + \|v_0^{p-3} \nabla v_0 \phi_j \psi_j\|_{L^1(B_R)} + \|v_0^{p-2} \nabla \phi_j \psi_j\|_{L^1(B_R)} + \|v_0^{p-2} \phi_j \nabla \psi_j\|_{L^1(B_R)} \\ & \leq \|v_0\|_{L^p}^{p-2} \|\phi_j\|_{L^p} \|\psi_j\|_{L^p} + C_R \|\nabla v_0\|_{L^\infty} \|\phi_j\|_{L^2} \|\psi_j\|_{L^2} + \|v_0\|_{L^\infty}^{p-2} \|\nabla \phi_j\|_{L^2} \|\psi_j\|_{L^2} + \|v_0\|_{L^\infty}^{p-2} \|\phi_j\|_{L^2} \|\nabla \psi_j\|_{L^2} \\ & < \infty, \end{aligned}$$

so that $v_0^{p-2} \phi_j \psi_j$ is locally L^1 compact by the compact Sobolev embedding. By combining this with the tightness, we conclude that $v_0^{p-2} \phi_j \psi_j$ is L^1 compact, and consequently $v_{\alpha_j}^{p-2} \phi_j \psi_j$ is L^1 compact. Now, the remaining part of the proof follows the same lines as the previous one. \square

Proposition 6.4. Let $p \in (2, \frac{2N}{N-2})$ and let v_α be a family of unique positive radial ground states of (4.1). Then for $\alpha < N$ sufficiently close to N the linearized equation of (4.1) at v_α , given by

$$-\Delta\phi + \phi - p\left(\frac{1}{|\cdot|^{N-\alpha}} * (v_\alpha^{p-1}\phi)\right)v_\alpha^{p-1} - (p-1)\left(\frac{1}{|\cdot|^{N-\alpha}} * v_\alpha^p\right)v_\alpha^{p-2}\phi = 0 \quad \text{in } \mathbb{R}^N, \quad (6.7)$$

only admits solutions of the form

$$\phi = \sum_{i=1}^N c_i \partial_{x_i} v_\alpha, \quad c_i \in \mathbb{R},$$

in the space $L^2(\mathbb{R}^N)$.

Proof. The proof of Proposition 6.4 follows the same lines as the proof of Proposition 6.2. The only thing we need to show is that $L[\phi] \in H^{-1}$ when ϕ is a L^2 solution to (6.7) and

$$L[\phi] := p\left(\frac{1}{|\cdot|^{N-\alpha}} * (v_\alpha^{p-1}\phi)\right)v_\alpha^{p-1} + (p-1)\left(\frac{1}{|\cdot|^{N-\alpha}} * v_\alpha^p\right)v_\alpha^{p-2}\phi.$$

For $\psi \in H^1(\mathbb{R}^N)$, we compute from Proposition 2.6 that

$$\begin{aligned} L[\phi]\psi &= \int_{\mathbb{R}^N} p\left(\frac{1}{|\cdot|^{N-\alpha}} * (v_\alpha^{p-1}\psi)\right)v_\alpha^{p-1}\phi + (p-1)\left(\frac{1}{|\cdot|^{N-\alpha}} * v_\alpha^p\right)v_\alpha^{p-2}\phi\psi \, dx \\ &\leq pC\|v_\alpha\|_{H^1}^{p-1}\|\psi\|_{H^1}\|v_\alpha\|_{L^{2(p-1)}}^{p-1}\|\phi\|_{L^2} + (p-1)C\|v_\alpha\|_{H^1}^p\|v_\alpha\|_{L^\infty}^{p-2}\|\phi\|_{L^2}\|\psi\|_{L^2}, \end{aligned}$$

from which we deduce $L[\phi] \in H^{-1}$. □

Now, we shall end the proof of Theorem 1.5. For $2 < p < \frac{2N}{N-2}$ and $\alpha < N$ close to N , let $\{u_\alpha\}$ be a family of unique positive radial ground states of (1.1) and let $\phi_\alpha \in L^2(\mathbb{R}^N)$ be a solution of the linearized equation (6.3). Then ϕ_α is a solution of (6.7) with $v_\alpha = s(N, \alpha, p)u_\alpha$. Then Proposition 6.4 says that ϕ_α is a linear combination of the $\partial_{x_i} v_\alpha$, which is also a linear combination of the $\partial_{x_i} u_\alpha$. This completes the proof of Theorem 1.5.

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