DE GRUYTER Adv. Nonlinear Anal. 2019; 8: 1099–1110

Research Article

Józef Banaś* and Tomasz Zając

On a measure of noncompactness in the space of regulated functions and its applications

https://doi.org/10.1515/anona-2018-0024
Received January 26, 2018; accepted February 3, 2018

Abstract: In this paper we formulate a criterion for relative compactness in the space of functions regulated on a bounded and closed interval. We prove that the mentioned criterion is equivalent to a known criterion obtained earlier by D. Fraňkova, but it turns out to be very convenient in applications. Among others, it creates the basis to construct a regular measure of noncompactness in the space of regulated functions. We show the applicability of the constructed measure of noncompactness in proving the existence of solutions of a quadratic Hammerstein integral equation in the space of regulated functions.

Keywords: Space of regulated functions, criterion of relative compactness, measure of noncompactness, fixed point theorem of Darbo type, quadratic Hammerstein integral equation

MSC 2010: 26A45, 47H08

1 Introduction

Nonlinear integral equations play a significant role in describing numerous real-word events [6, 9, 13, 21]. In nonlinear analysis, we are looking for conditions guaranteeing the existence of solutions of integral equations in various function spaces [6, 13, 21]. The choice of a suitable function space generates the methods applied in the investigations of the solvability of the integral equations in question. On the other hand, we usually choose such a function space which admits a general form and which allows us to apply convenient tools of nonlinear analysis.

It is worthwhile mentioning that the fixed-point theory creates a powerful and convenient branch of nonlinear analysis which is very applicable in proving existence theorems for several types of operator equations (differential, integral, functional integral etc., cf. [3, 16, 19]). It seems that the use of fixed-point theorems, associated with the technique of measures of noncompactness, is very fruitful in the described investigations [1, 3, 4, 6]. It turns out that the application of the theory of measures of noncompactness depends strongly on the choice of a function space in which we are studying the solvability of a considered operator equation. Obviously, such a choice requires the application of a suitable measure of noncompactness, which makes our investigations more or less convenient.

The aim of the paper is to investigate an appropriate criterion for relative compactness in the space of the so-called regulated functions. To make our considerations transparent, we will here investigate the space of real functions defined and regulated on a bounded and closed interval $[a, b]$.

***Corresponding author: Józef Banaś,** Department of Nonlinear Analysis, Rzeszów University of Technology, al. Powstańców Warszawy 8, 35-959 Rzeszów, Poland, e-mail: jbanas@prz.edu.pl
Tomasz Zając, Department of Nonlinear Analysis, Rzeszów University of Technology, al. Powstańców Warszawy 8, 35-959 Rzeszów, Poland, e-mail: tzajac@prz.edu.pl

The concept of a regulated function (called sometimes a regular function) was introduced in the middle of the twentieth century [2]. Subsequently, some authors presented this concept from different points of view and indicated some of its applications [14, 15, 17, 18]. Especially the approach presented in [14] seems to be very clear, transparent and applicable.

A lot of essential results concerning the space of regulated functions were given in [15], where one can encounter also a criterion of relative compactness of bounded subsets in the space of regulated functions. This criterion depends on the use of one-sided limits of functions belonging to a given bounded subset of the space of regulated functions. As far as we know, it is the only criterion of relative compactness published up to now.

Unfortunately, the mentioned criterion is not convenient in practice, since its use requires to impose rather strong assumptions referring to one-sided limits. Consequently, a measure of noncompactness constructed on the basis of that criterion has also the indicated faults [11].

Our paper is dedicated to describe a criterion of relative compactness in the space of regulated functions based on the approach to the concept of one-sided limits associated with the classical Cauchy condition. Such an approach was discussed in [5], and in this paper, we are going to extend this direction of investigations. Namely, we formulate a criterion of relative compactness in the space of regulated functions based on the mentioned Cauchy condition. Subsequently, on the basis of that criterion we construct a measure of noncompactness in the space in question, and we prove that the constructed measure has properties handy in applications, i.e., the so-called regular measure of noncompactness. To show the applicability of the mentioned measure of noncompactness, we prove the existence of solutions of a quadratic Hammerstein integral equation in the space of regulated functions.

2 Regulated functions and auxiliary facts

In this section we collect auxiliary facts concerning regulated functions. First we establish the notation. By $\mathbb{R}$ we will denote the set of real numbers and the symbol $\mathbb{N}$ will stand for the set of natural numbers (positive integers). Moreover, we denote $\mathbb{R}_+ = [0, \infty)$.

We will consider real functions defined on the interval $[a, b]$. If $x\colon [a, b] \to \mathbb{R}$ is a given function, then for $t \in (a, b]$, we will write $\lim_{u\to t^-} x(u)$ or $x(t^-)$ to denote the left-hand limit of the function x at the point t. Similarly, if $t \in [a, b)$, then $\lim_{u\to t^+} x(u)$ or $x(t^+)$ stand for the right-hand limit of x at t.

We recall the classical concept of the one-sided Cauchy condition.

Definition 2.1. Let $x\colon [a, b] \to \mathbb{R}$ and let $t \in (a, b]$ (resp. $t \in [a, b)$). We say that the function x satisfies at the point t the left-hand Cauchy condition (resp. the right-hand Cauchy condition) if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $u, v \in (t - \delta, t) \cap [a, b]$ (resp. $u, v \in (t, t + \delta) \cap [a, b]$), we have $|x(u) - x(v)| \le \varepsilon$.

It is well known that the left-hand limit $x(t^-)$ exists and is finite if and only if the function f satisfies at the point t the left-hand Cauchy condition. A similar statement holds for the existence of the finite right-hand limit.

In what follows we will denote by $B([a, b])$ the Banach space of real functions bounded on the interval $[a, b]$, equipped with the classical supremum norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in [a, b]\}.$$

Obviously, the space $C([a, b])$, consisting of functions from the space $B([a, b])$ which are continuous on $[a, b]$, is a closed subspace of $B([a, b])$ under the norm $\|\cdot\|_\infty$.

Another important subspace of $B([a, b])$ is that consisting of the so-called step functions. Recall that the function $x\colon [a, b] \to \mathbb{R}$ is called a *step function* if there exists a finite sequence $\{t_0, t_1, \dots, t_n\} \subset [a, b]$, with $a = t_0 < t_1 < \cdots < t_n = b$, such that the function x is constant on each interval (t_{i-1}, t_i), $i = 1, 2, \dots, n$. The set of all step functions on the interval $[a, b]$ will be denoted by $S([a, b])$. Obviously, $S([a, b])$ is a linear space and $S([a, b]) \subset B([a, b])$. Notice also that $S([a, b])$ can be normed by the supremum norm but it is not a closed subspace of the space $B([a, b])$.

Now, we introduce the concept of a regulated function (see [2, 5, 14], for example).

Definition 2.2. A function $x \in B([a,b])$ is said to be a *regulated function* if it has one-sided limits at every point $t \in (a,b)$ and the limits $x(a^+)$ and $x(b^-)$ exist.

It can be shown, see [5], that the definition of a regulated function can be formulated equivalently in the following way.

Definition 2.3. A function $x\colon [a,b] \to \mathbb{R}$ is called regulated if for each $t \in (a,b)$, the limits $x(t^-)$ and $x(t^+)$ exist and are finite, and the limits $x(a^+)$ and $x(b^-)$ exist and are finite too.

In what follows we will denote by $R([a,b])$ the set of all functions regulated on $[a,b]$. Obviously, $R([a,b])$ is a linear subspace of the space $B([a,b])$. Moreover, it can be shown, see [14] (cf. also [5]), that $R([a,b])$ is a closed subspace of $B([a,b])$ with respect to the norm $\|\cdot\|_\infty$. This means that $R([a,b])$ is a Banach space with the norm $\|\cdot\|_\infty$. Moreover, the space $S([a,b])$ of step functions forms a dense subspace of the space $R([a,b])$, see [14]. For further properties of regulated functions, we refer to [5].

Now, we remind the criterion for relative compactness in the space $R([a,b])$ given in [15]. To this end, we introduce the concept of a equiregulated subset of the space $R([a,b])$ (cf. [15]).

Definition 2.4. Assume that X is a subset of the space $R([a,b])$. We will say that the set X is *equiregulated* on the interval $[a,b]$ if the following two conditions are satisfied:
(i) For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$, $t \in (a,b]$ and $u \in (t-\delta, t) \cap [a,b]$, we have $|x(u) - x(t^-)| \le \varepsilon$.
(ii) For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$, $t \in [a,b)$ and $u \in (t, t+\delta) \cap [a,b]$, we have $|x(u) - x(t^+)| \le \varepsilon$.

Now, we recall the result due to Fraňkova [15], which characterizes the relative compactness in the space $R([a,b])$.

Theorem 2.5. *Let X be a bounded subset of the space $R([a,b])$. The set X is relatively compact in $R([a,b])$ if and only if X is equiregulated on the interval $[a,b]$.*

Let us notice that the above result is formulated as [15, Corollary 2.4].

In what follows we present a result which turns out to be equivalent to that contained in Theorem 2.5, but being more handy in applications.

Theorem 2.6. *Let X be a bounded subset of the space $R([a,b])$. The set X is relatively compact in $R([a,b])$ if and only if the following two conditions are satisfied:*
(a) *For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$, $t \in (a,b]$ and $u, v \in (t-\delta, t) \cap [a,b]$, we have $|x(u) - x(v)| \le \varepsilon$.*
(b) *For all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$, $t \in [a,b)$ and $u, v \in (t, t+\delta) \cap [a,b]$, we have $|x(u) - x(v)| \le \varepsilon$.*

Proof. At first, let us assume that the set X is relatively compact. Then, according to Theorem 2.5, this means that X is equiregulated on $[a,b]$. Further, fix arbitrarily $\varepsilon > 0$ and choose a number $\delta > 0$ to the number $\frac{\varepsilon}{2}$ pursuant to conditions (i) and (ii) of Definition 2.4. Next, take a number $t \in (a,b]$ and choose arbitrary numbers $u, v \in (t-\delta, t) \cap [a,b]$. Then, in view of condition (i), we get

$$|x(u) - x(v)| \le |x(u) - x(t^-)| + |x(t^-) - x(v)| \le \varepsilon.$$

Hence, we infer that the set X satisfies condition (a). Similarly, we can show that the set X satisfies also condition (b).

Conversely, suppose that conditions (a) and (b) are satisfied. Fix a number $\varepsilon > 0$ and choose $\delta > 0$ pursuant to conditions (a) and (b). Next, take an arbitrary number $t \in (a,b]$ or $t \in [a,b)$. Assume, for example, that $t \in (a,b]$. Then, according to condition (a), for any function $x \in X$ and for arbitrary numbers $u, v \in (t-\delta, t) \cap [a,b]$, we have

$$|x(u) - x(v)| \le \varepsilon. \tag{2.1}$$

Now, let us take an arbitrary sequence (v_n) such that $(v_n) \subset (t-\delta, t) \cap [a, b]$ and $v_n \to t^-$. Then, in view of (2.1), we obtain

$$|x(u) - x(v_n)| \leq \varepsilon$$

for any $n = 1, 2, \ldots$. Hence, applying standard facts from classical analysis, we conclude that

$$|x(u) - x(t^-)| \leq \varepsilon$$

for an arbitrary $u \in (t-\delta, t) \cap [a, b]$. This shows that for functions belonging to the set X condition (i) of Definition 2.4 is satisfied.

Similarly, we can prove that the set X satisfies also condition (ii).

Combining the above established facts with Theorem 2.5, we complete the proof. □

3 A measure of noncompactness in the space of regulated functions

Now, we are going to construct a measure of noncompactness in the space of regulated functions $R([a, b])$. To our knowledge, the first attempt to construct a measure of noncompactness in $R([a, b])$ was made in [11]. This measure of noncompactness was based on Theorem 2.5. Unfortunately, since the formula expressing it involved one-sided limits of functions belonging to a set on which it was defined, this measure is not handy in practice.

Indeed, in order to apply such a measure of noncompactness to the theory of functional integral equations, K. Cichoń, M. Cichoń and Metwali [11] were forced to impose assumptions depending on one-sided limits of functions being components of the mentioned equations.

A measure of noncompactness which we are going to describe in this section is based on Theorem 2.6 and in its construction we will not utilize one-sided limits of functions involved. In this regard, this measure seems to be rather convenient and handy in applications. Our aim is to show its applicability in proving existence theorems for functional integral equations.

We begin by introducing some notation needed in our considerations. Let E be a Banach space with the norm $\|\cdot\|_E$ and the zero element θ. In our study, we will write $\|\cdot\|$ instead of $\|\cdot\|_E$ if this does not lead to misunderstanding. Next, by $B(x, r)$ we denote the closed ball centered at x with radius r and by B_r the ball $B(\theta, r)$. If X is a subset of the space E, we denote by $\overline{X}$ the closure of X and we write $\operatorname{Conv} X$ to denote the closed convex hull of X. Moreover, the symbols $X + Y$, λX ($\lambda \in \mathbb{R}$) stand for usual algebraic operations on sets.

In what follows, by $\mathfrak{M}_E$ we denote the family consisting of all nonempty and bounded subsets of E while $\mathfrak{N}_E$ denotes its subfamily consisting of all relatively compact sets.

The definition of the concept of a measure of noncompactness will be accepted according to [4].

Definition 3.1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ will be called a *measure of noncompactness* in the Banach space E if it satisfies the following conditions:

(1) The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(3) $\mu(X) = \mu(\overline{X}) = \mu(\operatorname{Conv} X)$.

(4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.

(5) If (X_n) is a sequence of closed sets belonging to $\mathfrak{M}_E$, with $X_n \supset X_{n+1}$ ($n = 1, 2, \ldots$) and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The set $\ker \mu$ described in axiom (1) is referred to as *the kernel* of the measure of noncompactness μ. Notice that if X_∞ is the set appearing in axiom (5), then $X_\infty \subset X_n$ for any $n = 1, 2, \ldots$. This implies that $\mu(X_\infty) \leq \mu(X_n)$ for $n = 1, 2, \ldots$. Hence, we infer that $X_\infty \in \ker \mu$. This simple observation plays a crucial role in our further considerations.

Further, let us recall that the measure of noncompactness μ is said to be *sublinear* if it satisfies the following additional conditions (cf. [4]):

(6) $\mu(X+Y) \le \mu(X) + \mu(Y)$,
(7) $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.
If the measure μ satisfies the condition
(8) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$,
then we say that it has the *maximum property*.

The measure of noncompactness μ such that $\ker \mu = \mathfrak{N}_E$ will be called the *full measure*.

Finally, let us remind (cf. [4]) that the measure of noncompactness μ will be called *regular* if it is sublinear, has the maximum property and is full .

Let us pay attention to the fact that every regular measure of noncompactness has also some additional useful properties and it is very convenient in applications (cf. [1, 3, 4, 6]). On the other hand, the most convenient regular measure of noncompactness seems to be the so-called *Hausdorff measure of noncompactness*, see [4], which is defined in the following way:

$$\chi(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in the space } E\}.$$

We will not discuss here a lot of questions associated with regular measures of noncompactness and the Hausdorff measure χ (cf. [1, 3, 4, 6]).

We recall now a lemma which will be useful in our investigations.

Lemma 3.2. *Let $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ be a function satisfying the following conditions:*
(i) $\mu(X) = 0 \Leftrightarrow X \in \mathfrak{N}_E$,
(ii) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$,
(iii) $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$.
Then μ satisfies axiom (5) *of Definition 3.1.*

The proof of this lemma may be found in [8]. Let us notice that this simple lemma is very important in practice, since the mentioned axiom (5) of Definition 3.1 forms a generalization of the well known Cantor intersection theorem and, in general, it is rather difficult to verify whether a set function satisfies it. Obviously, such a function does not satisfies the conditions listed in Lemma 3.2 (cf. [7]).

The above lemma will be essentially exploited in our further considerations.

Now, we recall the fixed-point theorem of Darbo type [4, 12], which is formulated with help of the concept of a measure of noncompactness. This theorem is often applied in problems associated with the solvability of operator equations (functional, differential, integral equations, etc., see [1, 3, 4, 6], for details).

Theorem 3.3. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E, and let μ be a measure of noncompactness defined on E. Assume that $F : \Omega \to \Omega$ is a continuous operator such that $\mu(FX) \le k\mu(X)$ for any nonempty subset X of Ω, where $k \in [0, 1)$ is a constant. Then the operator F has at least one fixed point in the set Ω.*

Remark 3.4. It can be shown that the set Fix F of all fixed points of the operator F in the set Ω is a member of the family $\ker \mu$.

Further on, we are going to present the construction of a regular measure of noncompactness in the space $R([a, b])$ of regular functions described in Section 2. To this end, let us take a set $X \in \mathfrak{M}_{R([a,b])}$. For the sake of simplicity, we will write $\mathfrak{M}_R$ instead of $\mathfrak{M}_{R([a,b])}$.

Next, fix a number $\varepsilon > 0$ and for an arbitrarily chosen $x \in X$ and for a number $t \in (a, b]$, let us define the following quantity:

$$\omega^-(x, t; \varepsilon) = \sup\{|x(u) - x(v)| : u, v \in (t - \varepsilon, t) \cap [a, b]\}.$$

Similarly, for a fixed $t \in [a, b)$, we define

$$\omega^+(x, t; \varepsilon) = \sup\{|x(u) - x(v)| : u, v \in (t, t + \varepsilon) \cap [a, b]\}.$$

The above defined quantities $\omega^-(x, t; \varepsilon)$, $\omega^+(x, t; \varepsilon)$ can be viewed as left-hand and right-hand-sided moduli of convergence of the function x at the point $t \in (a, b]$ or $t \in [a, b)$, respectively.

Now, let us put

$$\omega^-(X,t;\varepsilon)=\sup\{\omega^-(x,t;\varepsilon):x\in X\},\quad \omega^-(X,\varepsilon)=\sup\{\omega^-(X,t;\varepsilon):t\in(a,b]\},$$
$$\omega^+(X,t;\varepsilon)=\sup\{\omega^+(x,t;\varepsilon):x\in X\},\quad \omega^+(X,\varepsilon)=\sup\{\omega^+(X,t;\varepsilon):t\in(a,b]\}.$$

It is easily seen that the functions $\omega^-(X,\varepsilon)$ and $\omega^+(X,\varepsilon)$ are well defined, which is an immediate consequence of the definition of regulated functions and the relation between one-sided limits of a function and the Cauchy condition concerning the existence of one-sided limits (cf. Definition 2.1). Next, let us pay attention to the fact that the functions $\varepsilon\mapsto\omega^-(X,\varepsilon)$ and $\varepsilon\mapsto\omega^+(X,\varepsilon)$ are nondecreasing on the interval $(0,\infty)$. Thus, the following limits exist and are finite:

$$\omega_0^-(X)=\lim_{\varepsilon\to 0}\omega^-(X,\varepsilon),\quad \omega_0^+(X)=\lim_{\varepsilon\to 0}\omega^+(X,\varepsilon).$$

Finally, let us define the quantity

$$\mu(X)=\omega_0^-(X)+\omega_0^+(X). \tag{3.1}$$

We are ready to present the main result of the paper.

Theorem 3.5. *The function μ defined by formula* (3.1) *is a regular measure of noncompactness in the space $R([a,b])$.*

Proof. First let us observe that, in view of Theorem 2.6, we deduce that the function μ satisfies condition (i) of Lemma 3.2. This implies immediately that the function μ satisfies axiom (1) of Definition 3.1 with $\ker\mu=\mathfrak{N}_R$. Further, let us take into account the fact that the functions $\omega_0^-(X)$ and $\omega_0^+(X)$, being the components of the function μ defined by (3.1), are defined by using the supremum. Obviously this yields that the function μ satisfies axiom (2) of Definition 3.1 (or, equivalently, condition (ii) of Lemma 3.2). By the same reasoning, we conclude that the function μ has the maximum property, i.e., it satisfies condition (iii) of Lemma 3.2 (equivalently, axiom (8)).

In view of the above established facts and Lemma 3.2, we infer that the function $\mu=\mu(X)$ satisfies axiom (5) of Definition 3.1.

Next, let us observe that for arbitrary functions $x,y\in R([a,b])$ and for $\lambda\in\mathbb{R}$, we obtain

$$\omega^-(x+y,t;\varepsilon)\le\omega^-(x,t;\varepsilon)+\omega^-(y,t;\varepsilon),\quad \omega^-(\lambda x,t;\varepsilon)=|\lambda|\omega^-(x,t;\varepsilon),$$
$$\omega^+(x+y,t;\varepsilon)\le\omega^+(x,t;\varepsilon)+\omega^+(y,t;\varepsilon),\quad \omega^+(\lambda x,t;\varepsilon)=|\lambda|\omega^+(x,t;\varepsilon).$$

Hence, it is not difficult to deduce that for an arbitrary set $X\in\mathfrak{M}_R$, we have

$$\omega_0^-(X+Y)\le\omega_0^-(X)+\omega_0^-(Y),\quad \omega_0^-(\lambda X)=|\lambda|\omega_0^-(X),$$
$$\omega_0^+(X+Y)\le\omega_0^+(X)+\omega_0^+(Y),\quad \omega_0^+(\lambda X)=|\lambda|\omega_0^+(X).$$

Thus, we see that the function μ satisfies axioms (7) and (8), i.e., the function μ is sublinear.

Applying the same reasoning as above we can easily see that

$$\mu(\operatorname{conv}X)\le\mu(X)$$

for an arbitrary set $X\in\mathfrak{M}_R$, where the symbol $\operatorname{conv}X$ stands for the convex hull of the set X. Combining the above inequality with the fact that μ satisfies axiom (2), we infer that

$$\mu(\operatorname{conv}X)=\mu(X) \tag{3.2}$$

for an arbitrary set $X\in\mathfrak{M}_R$.

Finally, we show that

$$\mu(\overline{X})=\mu(X) \tag{3.3}$$

for $X\in\mathfrak{M}_R$. To this end, observe firstly that the inequality

$$\mu(X)\le\mu(\overline{X}) \tag{3.4}$$

is a consequence of axiom (2).

To show the converse inequality let us take an arbitrary function $x \in \overline{X}$. This means that x is a limit of a sequence (x_n) of functions belonging to the set X. Thus, we can write

$$x(t) = \lim_{n\to\infty} x_n(t)$$

uniformly on the interval $[a, b]$. Further, let us fix arbitrarily $\varepsilon > 0$. Then, for a fixed $t \in (a, b]$ and for $u, v \in (t - \varepsilon, t) \cap [a, b]$, we have

$$|x(u) - x(v)| = \lim_{n\to\infty} |x_n(u) - x_n(v)|,$$

since the sequence (x_n) is uniformly convergent to the function x on the interval $[a, b]$.

The above established facts allows us to infer that

$$\omega_0^-(\overline{X}) \le \omega_0^-(X).$$

In the same way we can show also that

$$\omega_0^+(\overline{X}) \le \omega_0^+(X).$$

Combining the above inequalities with (3.1), we get

$$\mu(\overline{X}) \le \mu(X).$$

The above estimate in conjunction with (3.4) implies (3.3). This means that the function μ satisfies the first part of axiom (3).

The second part of this axiom is a consequence of equality (3.2) and the fact that $\operatorname{Conv} X = \overline{\operatorname{conv} X}$ for an arbitrary set $X \in \mathfrak{M}_R$.

The proof is complete. □

Remark 3.6. Let us observe that in view of the result proved in [4], we have, for an arbitrary set $X \in \mathfrak{M}_R$, that the following inequality is satisfied:

$$\mu(X) \le \mu(B_1)\chi(X), \tag{3.5}$$

where χ denotes the Hausdorff measure of noncompactness in the space $R([a, b])$ and $B_1 = B(\theta, 1)$. It is not difficult to calculate that $\omega_0^-(B_1) = \omega_0^+(B_1) = 2$. Thus, from (3.5), we obtain the estimate

$$\mu(X) \le 4\chi(X)$$

for $X \in \mathfrak{M}_R$. It is an open question whether there exists a constant $q > 0$ such that

$$\chi(X) \le q\mu(X)$$

for $X \in \mathfrak{M}_R$ (cf. also [10, 20]).

Notice that the answer to the above question has no influence on our further considerations concerning the applicability of the measure of noncompactness μ given by (3.1).

4 An application

This section is dedicated to present an application of the measure of noncompactness μ introduced in the previous section. We will investigate the solvability of a quadratic Hammerstein integral equation of the form

$$x(t) = p(t) + g(t, x(t)) \int_a^b k(t, s) f(s, x(s))\, ds \tag{4.1}$$

for $t \in [a, b]$. Our considerations will focus on the space $R([a, b])$ of regulated functions on the interval $[a, b]$ (cf. Section 2).

We will consider equation (4.1) by imposing the following assumptions:

(i) $p \in R([a, b])$.

(ii) The function $g(t, x) = g\colon [a, b] \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition with respect to the variable x, with the constant $L > 0$. Moreover, the function $t \mapsto g(t, s)$ is regulated on the interval $[a, b]$, locally uniformly with respect to the variable $x \in \mathbb{R}$, i.e., for any $r > 0$, the function $t \mapsto g(t, x)$ is regulated on $[a, b]$ for $x \in [-r, r]$.

(iii) The function $k(t, s) = k\colon [a, b]^2 \to \mathbb{R}$ is continuous in s for any fixed $t \in [a, b]$. In addition, the function $t \mapsto k(t, s)$ is regulated on the interval $[a, b]$, uniformly with respect to $s \in [a, b]$.

(iv) The function $f\colon [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous on the set $[a, b] \times \mathbb{R}$. Moreover, there exist a nonnegative constant c and a positive constant d such that $|f(t, x)| \le c + d|x|$ for $t \in [a, b]$ and $x \in \mathbb{R}$.

Remark 4.1. Let us explain that the expression that the functions $g(t, x)$ and $k(t, s)$ are regulated with respect to t, (locally) uniformly with respect to the variables x and s, respectively (cf. the above formulated assumptions (ii) and (iii)), is understood in the sense of Definition 2.1.

To formulate our last assumption, let

$$\overline{g} = \sup\{|g(t, 0)| : t \in [a, b]\}.$$

In view of the fact the function $t \mapsto g(t, 0)$ is regulated on the interval $[a, b]$ (cf. assumption (iii)), we have that $\overline{g} < \infty$.

Moreover, we will denote by $\overline{k}$ the constant defined as follows:

$$\overline{k} = \sup\Bigg\{ \int_a^b |k(t, s)|\, ds : t \in [a, b]\Bigg\}.$$

The fact that $\overline{k} < \infty$ will be shown later on.

(v) The following inequalities are satisfied:

$$\overline{k}(cL + d\overline{g}) < 1, \quad \overline{p} + c\overline{g}\overline{k} < \frac{[1 - \overline{k}(cL + d\overline{g})]^2}{4d\overline{k}L},$$

where we set $\overline{p} = \|p\|_\infty$.

Now, we are prepared to state our existence result concerning equation (4.1).

Theorem 4.2. *Under assumptions* (i)–(v), *equation* (4.1) *has at least one solution in the space* $R([a, b])$.

Proof. Let us consider the operator T associated with equation (4.1). This means that T is defined on the space $R([a, b])$ by the formula

$$(Tx)(t) = p(t) + g(t, x(t)) \int_a^b k(t, s) f(s, x(s))\, ds. \tag{4.2}$$

For further purposes, let us consider the operators G, F and K defined in the following way:

$$(Gx)(t) = g(t, x(t)), \quad (Fx)(t) = f(t, x(t)), \quad (Kx)(t) = \int_a^b k(t, s) x(s)\, ds.$$

Then the operator T defined by (4.2) can be represented as

$$Tx = p + (Gx)(K \cdot F)(x), \tag{4.3}$$

where the symbol $K \cdot F$ is understood as the composition of the operators F and K.

Now, let us fix arbitrarily $\varepsilon > 0$ and $t \in (a, b]$. Then, for an arbitrary function $x \in R([a, b])$ and for arbitrary $u, v \in (t - \varepsilon, t)$, on the basis of assumption (ii), we obtain

$$\begin{aligned} |(Gx)(u) - (Gx)(v)| &= |g(u, x(u)) - g(v, x(v))| \\ &\le |g(u, x(u)) - g(u, x(v))| + |g(u, x(v)) - g(v, x(v))| \\ &\le L|x(u) - x(v)| + \omega^{-}_{1,\|x\|}(g, t; \varepsilon), \end{aligned} \tag{4.4}$$

where

$$\omega_{1,r}^{-}(g,t;\varepsilon)=\sup\{|g(u,x)-g(v,x)| : u,v\in(t-\varepsilon,t)\cap[a,b],\ x\in[-r,r]\}.$$

In view of assumption (ii), we infer that $\omega_{1,\|x\|}^{-}(g,t;\varepsilon)\to 0$ as $\varepsilon\to 0$. Particularly this implies that the operator G transforms the space $R([a,b])$ into itself.

In a similar way, taking into account assumption (iii), we derive

$$|(Kx)(u)-(Kx)(v)|\le\int_a^b|k(u,s)-k(v,s)||x(s)|\,ds\le\|x\|\int_a^b|k(u,s)-k(v,s)|\,ds\le\|x\|\omega_1^{-}(k,t;\varepsilon)(b-a),\tag{4.5}$$

where

$$\omega_1^{-}(k,t;\varepsilon)=\sup\{|k(u,s)-k(v,s)| : u,v\in(t-\varepsilon,t)\cap[a,b],\ s\in[a,b]\}.$$

Notice that, in view of assumption (iii), we have that $\omega_1^{-}(k,t;\varepsilon)\to 0$ (similarly, $\omega_1^{+}(k,t;\varepsilon)\to 0$) as $\varepsilon\to 0$ for each $t\in[a,b]$. This implies that the function $t\mapsto\int_a^b k(t,s)\,ds$ (or the function $t\mapsto\int_a^b|k(t,s)|\,ds$) is regulated on the interval $[a,b]$. Consequently, it follows that the function $t\mapsto\int_a^b|k(t,s)|\,ds$ is bounded on the interval $[a,b]$ and simultaneously justifies the fact that $\overline{k}<\infty$, which was stated before.

Now, from (4.4), (4.5), assumption (iv) and taking into account representation (4.3), we conclude that the operator T transforms the space $R([a,b])$ into itself. Obviously, the above reasoning can be repeated for any fixed $t\in[a,b)$ and for $u,v\in(t,t+\varepsilon)\cap[a,b]$, if we replace the quantity ω^- by ω^+.

Let us notice that by utilizing our assumptions, for an arbitrarily fixed $x\in R([a,b])$ and $t\in[a,b]$, we have

$$\begin{aligned}|(Tx)(t)|&\le|p(t)|+|g(t,x(t))|\int_a^b|k(t,s)||f(s,x(s))|\,ds\\&\le|p(t)|+\big[|g(t,x(t))-g(t,0)|+|g(t,0)|\big]\int_a^b|k(t,s)|[c+d|x(s)|]\,ds\\&\le|p(t)|+[L|x(t)|+|g(t,0)|](c+d\|x\|)\int_a^b|k(t,s)|\,ds\\&\le\overline{p}+(L\|x\|+\overline{g})(c+d\|x\|)\overline{k},\end{aligned}\tag{4.6}$$

where we write $\|x\|$ in place of $\|x\|_\infty$ and the constants $\overline{p}$, L, $\overline{g}$, a, b, $\overline{k}$ were defined previously or imposed in the assumptions.

The above inequality yields the estimate

$$\|Tx\|\le d\overline{k}L\|x\|^2+\overline{k}(cL+d\overline{g})\|x\|+\overline{p}+c\overline{g}\overline{k}.$$

Hence, keeping in mind assumption (v), we deduce that there exists a positive number r_0 such that for all functions $x\in B_{r_0}\subset R([a,b])$, we have that $Tx\in B_{r_0}$, i.e., the operator T transforms the ball B_{r_0} into itself.

Keeping in mind the convenience, we will further accept that

$$r_0=\frac{1-\overline{k}(cL+d\overline{g})}{2d\overline{k}L}.\tag{4.7}$$

To prove the continuity of the operator T defined by (4.3) on the ball B_{r_0} let us observe that, in view of assumption (ii), it is sufficient to prove the continuity of the operator $K\cdot F$ on B_{r_0}. Thus, fix arbitrarily $\varepsilon>0$ and take $x,y\in B_{r_0}$ such that $\|x-y\|\le\varepsilon$. Then, for a fixed $t\in[a,b]$, we obtain

$$|(K\cdot Fx)(t)-(K\cdot Fy)(t)|\le\int_a^b|k(t,s)||f(s,x(s))-f(s,y(s))|\,ds\le\int_a^b|k(t,s)|\omega(f,\varepsilon)\,ds\le\overline{k}\omega(f,\varepsilon),\tag{4.8}$$

where

$$\omega(f,\varepsilon)=\sup\{|f(t,x)-f(t,y)| : t\in[a,b],\ x,y\in[-r_0,r_0],\ |x-y|\le\varepsilon\}.$$

Observe that taking into account the uniform continuity of the function f on the set $[a,b]\times[-r_0,r_0]$ (cf. assumption (iv)), we deduce that $\omega(f,\varepsilon)\to 0$ as $\varepsilon\to 0$. Combining this fact with (4.8), we infer that the operator $K\cdot F$ is continuous on the ball B_{r_0}.

Further on, let us fix an arbitrary nonempty set $X\subset B_{r_0}$ and a number $\varepsilon>0$. Then, for $x\in X$ and for $t\in(a,b]$, let us choose arbitrary numbers $u,v\in(t-\varepsilon,t)\cap[a,b]$. Then we get the estimate

$$\begin{aligned}|(Tx)(u)-(Tx)(v)| &\le |p(u)-p(v)|+\big|(Gx)(u)(K\cdot Fx)(u)-(Gx)(v)(K\cdot Fx)(v)\big|\\ &\le \omega^-(p,t;\varepsilon)+\big|(Gx)(u)(K\cdot Fx)(u)-(Gx)(v)(K\cdot Fx)(u)\big|\\ &\qquad+\big|(Gx)(v)(K\cdot Fx)(u)-(Gx)(v)(K\cdot Fx)(v)\big|\\ &\le \omega^-(p,t;\varepsilon)+|(K\cdot Fx)(u)||(Gx)(u)-(Gx)(v)|+|(Gx)(v)||(K\cdot Fx)(u)-(K\cdot Fx)(v)|.\end{aligned}$$

Hence, in view of estimates (4.4) and (4.6), we obtain

$$|(Tx)(u)-(Tx)(v)|\le\omega^-(p,t;\varepsilon)+(a+br_0)\overline{k}\{L|x(u)-x(v)|+\omega^-_{1,r_0}(g,t;\varepsilon)\}+(Lr_0+\overline{g})|(K\cdot Fx)(u)-(K\cdot Fx)(v)|. \quad (4.9)$$

Further, utilizing estimates (4.5) and (4.6), we derive the following inequality:

$$\begin{aligned}|(K\cdot Fx)(u)-(K\cdot Fx)(v)| &\le \int_a^b\big|k(u,s)f(s,x(s))-k(v,s)f(s,x(s))\big|\,ds\\ &\le \int_a^b|k(u,s)-k(v,s)||f(s,x(s))|\,ds\\ &\le (b-a)(c+dr_0)\omega_1^-(k,t;\varepsilon),\end{aligned} \quad (4.10)$$

where $\omega_1^-(k,t;\varepsilon)$ was introduced previously. Now, linking estimates (4.9) and (4.10), we obtain

$$\omega^-(Tx,t;\varepsilon)\le\omega^-(p,t;\varepsilon)+(c+dr_0)\overline{k}\{L\omega^-(x,t;\varepsilon)+\omega^-_{1,r_0}(g,t;\varepsilon)\}+(Lr_0+\overline{g})(b-a)(c+dr_0)\omega_1^-(k,t;\varepsilon).$$

Next, keeping in mind assumption (i) and the properties of the functions $\varepsilon\mapsto\omega^-(p,t;\varepsilon)$, $\varepsilon\mapsto\omega^-_{1,r_0}(g,t;\varepsilon)$ and $\varepsilon\mapsto\omega_1^-(k,t;\varepsilon)$, and letting $\varepsilon\to 0$, we obtain the estimate

$$\omega_0^-(TX)\le(c+dr_0)\overline{k}L\omega_0^-(X). \quad (4.11)$$

In the same way, we can prove that

$$\omega_0^+(TX)\le(c+dr_0)\overline{k}L\omega_0^+(X). \quad (4.12)$$

Combining (4.11) and (4.12) and taking into account formula (3.1), expressing the measure of noncompactness μ , we get

$$\mu(TX)\le(c+dr_0)\overline{k}L\mu(X). \quad (4.13)$$

Observe that, in view of (4.7) and assumption (v), we obtain

$$(c+dr_0)\overline{k}L=\overline{k}(cL-d\overline{g})<\overline{k}(cL+d\overline{g})<1.$$

Hence, keeping in mind estimate (4.13) and Theorem 3.3, we conclude that the operator T has at least one fixed point x in the ball B_{r_0}. Obviously, the function $x=x(t)$ is a desired solution of equation (4.1) belonging to the space $R([a,b])$. The proof is complete. □

Now, we provide an example illustrating the result contained in Theorem 4.2.

Example 4.3. Let us fix a natural number $n\ge 2$ and consider the function $p=p(t)$ defined on the interval $I=[0,1]$ in the following way:

$$p(t)=\sum_{k=1}^{n}\frac{1}{n+k}\chi_k(t), \quad (4.14)$$

where χ_k stands for the characteristic function of the interval $[\frac{k-1}{n},\frac{k}{n}]$ for $k=1,2,\dots,n$. Obviously, the function $p=p(t)$, being the step function, is the regulated function on the interval $[0,1]$. Moreover, $\overline{p}=\|p\|_\infty=\frac{1}{n+1}$. Particularly, this means that the function p satisfies assumption (i) of Theorem 4.2.

Next, consider the function $g(t,x) = g\colon I \times \mathbb{R} \to \mathbb{R}$ defined by

$$g(t,x) = x\sum_{k=1}^{n} \frac{1}{nk+1}[k\sin \pi nt]\chi_k(t) + \sum_{k=1}^{n} \frac{1}{n^2+k}\chi_k(t), \tag{4.15}$$

where (similarly as above) the function χ_k denotes the characteristic function of the interval $[\frac{k-1}{n}, \frac{k}{n}]$ for $k = 1, 2, \dots, n$ and $[y]$ denotes the integer part of the number y. It is easily seen that the function $g(t,x)$ is Lipschitzian with respect to x. Indeed, for an arbitrary $t \in I$ and $x, y \in \mathbb{R}$, we get

$$|g(t,x) - g(t,y)| \le |x-y| \sum_{k=1}^{n} \frac{\chi_k(t)}{kn+1}[k\sin \pi nt] \le |x-y|\frac{n}{n^2+1} = \frac{n}{n^2+1}|x-y| \le \frac{1}{n}|x-y|.$$

This means that the function $g(t,x)$ satisfies the Lipschitz condition with the constant $L = \frac{1}{n}$.

Now, taking into account the fact that the functions $t \mapsto \sum_{k=1}^{n} \frac{1}{kn+1}\chi_k(t)[k\sin \pi nt]$ and $t \mapsto \sum_{k=1}^{n} \frac{1}{n^2+k}\chi_k(t)$ are step functions on the interval I and keeping in mind that for each fixed $r > 0$, the function

$$g_1(t,x) = x\sum_{k=1}^{n} \frac{1}{kn+1}\chi_k(t)[k\sin \pi nt]$$

is uniformly regulated for $|x| \le r$ on the interval $I = [0,1]$, in view of Remark 4.1, we conclude that the function $g(t,x)$ satisfies assumption (ii) of Theorem 4.2.

Further, let us take the function $k = k(t,s)$ defined on the set I^2 by

$$k(t,s) = s^2 + \alpha\sum_{n=1}^{\infty} \frac{1}{n}\chi'_n(t), \tag{4.16}$$

where χ'_n denotes the characteristic function of the interval $(\frac{1}{n+1}, \frac{1}{n}]$ for $n = 1, 2, \dots$ and $\alpha > 0$ is a constant.

Observe that the function $k = k(t,s)$ satisfies assumption (iii) of Theorem 4.2. Moreover, we have

$$\int_0^1 |k(t,s)|\,ds \le \frac{1}{3} + \alpha$$

for any $t \in I$, so we can accept that $\overline{k} = \frac{1}{3} + \alpha$, where the constant $\overline{k}$ was defined previously.

Further, let us consider the quadratic Hammerstein integral equation

$$x(t) = p(t) + g(t,x(t))\int_0^1 k(t,s)\frac{s}{s^2+1}x(s)\sin x(s)\,ds \tag{4.17}$$

for $t \in I = [0,1]$, where the functions $p(t)$, $g(t,x)$ and $k(t,s)$ are defined by formulas (4.14), (4.15) and (4.16), respectively.

Notice that equation (4.17) is a special case of equation (4.1), where

$$f(t,x) = \frac{t}{t^2+1}x\sin x.$$

Obviously, the function $f = f(t,x)$ is continuous on the set $I \times \mathbb{R}$ and

$$|f(t,x)| \le \frac{t}{t^2+1}|x| \le \frac{1}{2}|x|$$

for $t \in I$ and $x \in \mathbb{R}$. Thus, the function $f(t,x)$ satisfies assumption (iv) of Theorem 4.2 with $c = 0$ and $d = \frac{1}{2}$.

Summing up, we see that assumptions (i)–(iv) of Theorem 4.2 are satisfied. In addition, we have that

$$g(t,0) = \sum_{k=1}^{n} \frac{1}{n^2+k}\chi_k(t) \le \frac{1}{n^2+1}.$$

Thus, we can take $\overline{g} = \frac{1}{n^2+1}$.

To verify assumption (v), let us note that the first inequality in (v) has the form

$$\frac{1}{2}\left(\frac{1}{3}+\alpha\right)\frac{1}{n^2+1} < 1. \tag{4.18}$$

Hence, we see that for each fixed $\alpha > 0$, we can choose a number $n \in \mathbb{N}$ such that (4.18) is satisfied.

Further, let us take into account the second inequality in assumption (v). It has the form

$$\frac{1}{n+1} < \frac{1 - \frac{\frac{1}{3}+\alpha}{2(n^2+1)}}{2(\frac{1}{3}+\alpha)\cdot\frac{1}{n}}$$

or, equivalently,

$$\frac{1}{n(n+1)} < \frac{1 - \frac{\frac{1}{3}+\alpha}{2(n^2+1)}}{2(\frac{1}{3}+\alpha)}. \tag{4.19}$$

It is easily seen that for any $\alpha > 0$, we can choose $n \in \mathbb{N}$ so big that both inequalities (4.18) and (4.19) are satisfied.

Thus, on the basis of Theorem 4.2, we infer that equation (4.17) has at least one regulated solution on the interval $[0, 1]$, provided that we choose an arbitrary number $\alpha > 0$ and a natural number n big enough.

References

[1] R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina and B. N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators* (in Russian), "Nauka", Novosibirsk, 1986.
[2] G. Aumann, *Reelle Funktionen*, Grundlehren Math. Wiss. 68, Springer, Berlin, 1954.
[3] J. M. Ayerbe Toledano, T. Domínguez Benavides and G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Oper. Theory Adv. Appl. 99, Birkhäuser, Basel, 1997.
[4] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes Pure Appl. Math. 60, Marcel Dekker, New York, 1980.
[5] J. Banaś and M. Kot, On regulated functions, *J. Math. Appl.* **40** (2017), 21–36.
[6] J. Banaś and M. Mursaleen, *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations*, Springer, New Delhi, 2014.
[7] J. Banaś and R. Nalepa, On a measure of noncompactness in the space of functions with tempered increments, *J. Math. Anal. Appl.* **435** (2016), no. 2, 1634–1651.
[8] J. Banaś, D. Szynal and S. A. W'edrychowicz, On existence, asymptotic behaviour and stability of solutions of stochastic integral equations, *Stochastic Anal. Appl.* **9** (1991), no. 4, 363–385.
[9] S. Chandrasekhar, *Radiative Transfer*, Oxford University Press, London, 1950.
[10] L. Cheng, Q. Cheng, Q. Shen, K. Tu and W. Zhang, A new approach to measures of noncompactness of Banach spaces, *Studia Math.* **240** (2018), no. 1, 21–45.
[11] K. Cichoń, M. Cichoń and M. M. A. Metwali, On some parameters in the space of regulated functions and their applications, *Carpathian J. Math.* **34** (2018), no. 1, 17–30.
[12] G. Darbo, Punti uniti in trasformazioni a codominio non compatto, *Rend. Semin. Mat. Univ. Padova* **24** (1955), 84–92.
[13] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.
[14] J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, New York, 1969.
[15] D. Fraňková, Regulated functions, *Math. Bohem.* **116** (1991), no. 1, 20–59.
[16] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Stud. Adv. Math. 28, Cambridge University Press, Cambridge, 1990.
[17] C. S. Hönig, *Volterra Stieltjes-Integral Equations*, North-Holland, Amsterdam, 1975.
[18] C. S. Hönig, Équations intégrales généralisées et applications, *Publ. Math. Orsay* **83-01** (1983), Expose No. 5.
[19] A. Jeribi and B. Krichen, *Nonlinear Functional Analysis in Banach Spaces and Banach Algebras*, Monogr. Res. Notes Math., CRC Press, Boca Raton, 2016.
[20] J. Mallet-Paret and R. D. Nussbaum, Inequivalent measures of noncompactness, *Ann. Mat. Pura Appl. (4)* **190** (2011), no. 3, 453–488.
[21] P. P. Zabrejko, A. I. Koshelev, M. A. Krasnosel'skii, S. G. Mikhlin, L. S. Rakovschik and J. Stetsenko, *Integral Equations*, Nordhoff, Leyden, 1975.