

## Research article

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# Concentration behavior of semiclassical solutions for Hamiltonian elliptic system

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**Abstract:** In this paper, we study the following nonlinear Hamiltonian elliptic system with gradient term

$$\begin{cases} -\epsilon^2 \Delta \psi + \epsilon \vec{b} \cdot \nabla \psi + \psi + V(x)\varphi = f(|\eta|)\varphi & \text{in } \mathbb{R}^N, \\ -\epsilon^2 \Delta \varphi - \epsilon \vec{b} \cdot \nabla \varphi + \varphi + V(x)\psi = f(|\eta|)\psi & \text{in } \mathbb{R}^N, \end{cases}$$

where  $\eta = (\psi, \varphi) : \mathbb{R}^N \rightarrow \mathbb{R}^2$ ,  $\epsilon$  is a small positive parameter and  $\vec{b}$  is a constant vector. We require that the potential  $V$  only satisfies certain local condition. Combining this with other suitable assumptions on  $f$ , we construct a family of semiclassical solutions. Moreover, the concentration phenomena around local minimum of  $V$ , convergence and exponential decay of semiclassical solutions are also explored. In the proofs we apply penalization method, linking argument and some analytical techniques since the local property of the potential and the strongly indefinite character of the energy functional.

**Keywords:** Hamiltonian elliptic system; semiclassical states; concentration; strongly indefinite functionals

**MSC:** 35J50, 58E05

## 1 Introduction and main results

In this paper, we will consider the following Hamiltonian elliptic system with gradient term

$$\begin{cases} -\epsilon^2 \Delta \psi + \epsilon \vec{b} \cdot \nabla \psi + \psi + V(x)\varphi = f(|\eta|)\varphi & \text{in } \mathbb{R}^N, \\ -\epsilon^2 \Delta \varphi - \epsilon \vec{b} \cdot \nabla \varphi + \varphi + V(x)\psi = f(|\eta|)\psi & \text{in } \mathbb{R}^N, \end{cases} \quad (\mathcal{P}_\epsilon)$$

where  $\eta = (\psi, \varphi) : \mathbb{R}^N \rightarrow \mathbb{R}^2$ ,  $\epsilon$  is small positive parameter,  $\vec{b}$  is constant vector,  $V$  is linear potential and  $f$  is continuous, superlinear and subcritical nonlinearity. We are interested in the existence, convergence and concentration phenomenon of semiclassical solutions of system  $(\mathcal{P}_\epsilon)$  when  $\epsilon \rightarrow 0$ .

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This type of systems arises when one is looking for the standing wave solutions to system of diffusion equations

$$\begin{cases} \partial_t \varphi - \Delta_x \varphi + \vec{b}(t, x) \cdot \nabla \varphi + V(x)\varphi = H_\psi(t, x, \varphi, \psi) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ -\partial_t \psi - \Delta_x \psi - \vec{b}(t, x) \cdot \nabla \psi + V(x)\psi = H_\varphi(t, x, \varphi, \psi) & \text{in } \mathbb{R} \times \mathbb{R}^N, \end{cases}$$

which comes from the time-space diffusion processes and is related to the Schrödinger equations. It appears in various fields, such as physics and chemistry, quantum mechanics, control theory and Brownian motions. For more details in the application backgrounds, we refer the readers to see the monographs [19] and [21].

In recent years, there has been increasing attention to Hamiltonian elliptic system on obtaining existence of solutions, ground state solutions, multiple solutions and semiclassical solutions by using variational methods. But most of them focused on the case  $\vec{b} = 0$ . More specifically, based on various hypotheses on the potential and nonlinearity, the existence and multiplicity of solutions have been established by many authors. For example, see [11, 12, 15] for the case of a bounded domain, and [2, 4, 7, 29, 37, 38] for the case of the whole space  $\mathbb{R}^N$ .

When  $\vec{b} \neq 0$  and  $\epsilon = 1$ , as we all know, there are a few works devoted to the existence and multiplicity of solutions of the following system under different assumption

$$\begin{cases} -\Delta \psi + \vec{b}(x) \cdot \nabla \psi + V(x)\psi = H_\varphi(x, \psi, \varphi) & \text{in } \mathbb{R}^N, \\ -\Delta \varphi - \vec{b}(x) \cdot \nabla \varphi + V(x)\varphi = H_\psi(x, \psi, \varphi) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

see [24, 35, 39, 40, 43]. For this case, since the appearance of the gradient term in system, system (1.1) has some differences and difficulties compared with the case  $\vec{b}(x) = 0$ . For example, the variational framework for the case  $\vec{b}(x) = 0$  cannot work any longer in this case, then the first problem is how to establish a suitable variational framework. To solve this problem, Zhao and Ding [39] handled (1.1) as a generalized Hamiltonian system, and established a strongly indefinite variational framework by studying the structure of essential spectrum of Hamiltonian operator. At the same time, the existence and multiplicity of solutions were obtained by using critical point theorems of strongly indefinite functional [10] and reduction method [1] for system (1.1) with periodic and non-periodic asymptotically quadratic growth condition. After that, Zhang et al.[40] studied the periodic super-quadratic case and proved the existence of ground state solutions by means of the linking and concentration compactness arguments. Later, this result has been extended to more general nonlinearity model by Liao et al.[24]. An asymptotically periodic case was considered in [43], and some properties of ground state solutions were obtained, by constructing linking levels and analyzing behavior of Cerami sequence. Besides, the existence of least energy solution for the non-periodic super-quadratic case was studied in [35]. Recently, the paper [45] studied the Hamiltonian elliptic system with inverse square potential of the form

$$\begin{cases} -\Delta u + \vec{b}(x) \cdot \nabla u + V(x)u - \frac{\mu}{|x|^2}v = H_v(x, u, v) & \text{in } \mathbb{R}^N, \\ -\Delta v - \vec{b}(x) \cdot \nabla v + V(x)v - \frac{\mu}{|x|^2}u = H_u(x, u, v) & \text{in } \mathbb{R}^N, \end{cases}$$

and the ground state solutions was obtained by using non-Nehari manifold developed by Tang [32]. Moreover, some asymptotic behaviors of ground state solutions, such as the monotonicity and convergence property of ground state energy, were also explored as parameter  $\mu$  tends to 0.

When  $\epsilon$  is small, the standing waves of system  $(\mathcal{P}_\epsilon)$  are referred to as semiclassical states. The concentration phenomenon of semiclassical states, when  $\epsilon$  goes to zero, reflects the transformation process between quantum mechanics and classical mechanics. So it possesses an important physical interest. For such case, the asymptotic behaviors of semiclassical states, such as concentration, convergence and exponential decay, etc., are very interesting problem in mathematics and physics. To put our result in perspective, we review briefly here the background and relate results. There have been intensive interests in studying the existence and qualitative properties of semiclassical states. In [41] the authors considered the singularly perturbed system

$$\begin{cases} -\epsilon^2 \Delta \psi + \epsilon \vec{b} \cdot \nabla \psi + \psi = K(x)|\eta|^{p-2} \varphi & \text{in } \mathbb{R}^N, \\ -\epsilon^2 \Delta \varphi - \epsilon \vec{b} \cdot \nabla \varphi + \varphi = K(x)|\eta|^{p-2} \psi & \text{in } \mathbb{R}^N, \end{cases}$$

with  $p \in (2, 2^*)$ , where  $2^*$  is the usual critical exponent. They proved that, by a global variational technique and reduction Nehari method, the semiclassical ground state solution concentrates around the maxima point

of the nonlinear potential  $K$  as  $\epsilon \rightarrow 0$ . This method and result were later generalized in [42] to the critical nonlinearities case. Further investigations to system with competing potentials

$$\begin{cases} -\epsilon^2 \Delta \psi + \epsilon \vec{b} \cdot \nabla \psi + \psi + V(x)\varphi = K(x)f(|\eta|)\varphi & \text{in } \mathbb{R}^N, \\ -\epsilon^2 \Delta \varphi - \epsilon \vec{b} \cdot \nabla \varphi + \varphi + V(x)\psi = K(x)f(|\eta|)\psi & \text{in } \mathbb{R}^N. \end{cases}$$

have also appeared in [44, 46]. For such a problem, the solutions depend not only on the linear potential but also on the nonlinear potential. As was shown in [44, 46], the semiclassical ground state solution concentrates around the global minimum points of linear potential  $V$  and the global maxima points of nonlinear potential  $K$ . Observe that, the method and results mentioned above basically depend on the global condition of the potential  $V$ , that is,

$$\inf_{x \in \mathbb{R}^N} V(x) < \liminf_{|x| \rightarrow \infty} V(x). \tag{1.2}$$

It is worth pointing out that the global condition used in [44, 46] plays an important role in proving the existence and concentration of semiclassical solutions. Indeed, the key point is that the property of the potential  $V$  at infinity can help us to restore the necessary compactness by comparing energy levels of original problem and limit problem. So, an interesting question, which motivates the present work, is whether one can find solutions which concentrate around local minima of the potential. As we will see, the answer is affirmative. Hence, based on the above facts, in this paper we will investigate the existence and localized concentration phenomenon of semiclassical states of system  $(\mathcal{P}_\epsilon)$  with potential satisfying local condition. More precisely, for the potential  $V$ , we assumed that the following local condition first introduced by del Pino and Felmer in [8]:

(V)  $V \in C(\mathbb{R}^N, \mathbb{R})$ ,  $\max |V| < 1$ , and there is a bounded domain  $\Omega$  in  $\mathbb{R}^N$  such that

$$v = \min_{x \in \Omega} V(x) < \min_{x \in \partial \Omega} V(x). \tag{1.3}$$

Compared with [44] and [46], the condition (V) is rather weak, without restriction on the global behavior of  $V$  is required, and the behavior of  $V$  outside  $\Omega$  is irrelevant. This fact shows that the limit problem at infinity and its properties are all unknown in this paper. So, from a variational point of view, one of the major differences between the global condition (1.2) and the local condition (1.3) is that the energy functional, under the local condition (1.3), does not satisfy the so-called compactness condition (such as (PS) or Cerami condition) in general.

Let us now describe the results of the present paper. For notational convenience, let

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{J}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and  $\mathcal{S}_\epsilon = -\epsilon^2 \Delta + 1$ . We denote

$$A_\epsilon := \mathcal{S}_\epsilon \mathcal{J}_0 + \epsilon \vec{b} \cdot \nabla \mathcal{J} = \begin{pmatrix} 0 & -\epsilon^2 \Delta - \epsilon \vec{b} \cdot \nabla + 1 \\ -\epsilon^2 \Delta + \epsilon \vec{b} \cdot \nabla + 1 & 0 \end{pmatrix}.$$

Then system  $(\mathcal{P}_\epsilon)$  can be rewritten as

$$A_\epsilon \eta + V(x)\eta = f(|\eta|)\eta. \tag{1.4}$$

Before stating our results, we make the following assumptions on the nonlinearity  $f$ :

- (F<sub>0</sub>)  $f \in C(\mathbb{R}^+, \mathbb{R})$ ,  $f(s) \rightarrow 0$  as  $s \rightarrow 0$  and  $f$  is increasing on  $\mathbb{R}^+ = [0, \infty)$ ;
- (F<sub>1</sub>) there exist  $c_0 > 0$  and  $p \in (2, 2^*)$  such that  $f(s) \leq c_0(1 + s^{p-2})$  for  $s \geq 0$ ;
- (F<sub>2</sub>) there exists  $\mu > 2$  such that  $0 < \mu F(s) \leq f(s)s^2$  for all  $s > 0$ , where  $F(s) = \int_0^s f(t)tdt$ .

For showing the concentration phenomenon, we denote by  $\mathcal{V}$  the set  $\mathcal{V} := \{x \in \Omega : V(x) = v\}$ . Without loss of generality, below we may assume that  $0 \in \mathcal{V}$  throughout the paper. Moreover, according to (V), we know that

$$\text{dist}(\mathcal{V}, \partial \Omega) > 0. \tag{1.5}$$

Now we are ready to state the main results of this paper as follows.

**Theorem 1.1.** Assume that  $|\vec{b}| < 2$ ,  $(V)$  and  $(F_0)$ - $(F_2)$  are satisfied. Then for all sufficiently small  $\epsilon > 0$ ,

- (a) system  $(\mathcal{P}_\epsilon)$  at least has a nontrivial solution  $\eta_\epsilon \in H^{2,q}$  for any  $q \geq 2$ ;
- (b)  $|\eta_\epsilon(x)|$  attains its maximum at  $p_\epsilon$ , moreover, up to a subsequence, there holds

$$\lim_{\epsilon \rightarrow 0} V(p_\epsilon) = v;$$

- (c)  $\eta_\epsilon(\epsilon x + p_\epsilon) \rightarrow \eta(x)$  in  $H^2(\mathbb{R}^N, \mathbb{R}^2)$  as  $\epsilon \rightarrow 0$ , and  $\eta$  is a ground state solution of the following system

$$\begin{cases} -\Delta\psi + \vec{b} \cdot \nabla\psi + \psi + v\varphi = f(|\eta|)\varphi & \text{in } \mathbb{R}^N, \\ -\Delta\varphi - \vec{b} \cdot \nabla\varphi + \varphi + v\psi = f(|\eta|)\psi & \text{in } \mathbb{R}^N, \end{cases}$$

- (d) there exist positive constants  $c, C$  such that

$$|\eta_\epsilon(x)| \leq C \exp\left(-\frac{c}{\epsilon}|x - p_\epsilon|\right).$$

Due to the above observations, we have an immediate consequence of our main results.

**Corollary 1.2.** Assume that  $|\vec{b}| < 2$  and  $(F_0)$ - $(F_2)$  are satisfied. If there exist mutually disjoint bounded domains  $\Omega_j, j = 1, \dots, k$  and constants  $v_1 < v_2 < \dots < v_k$  such that

$$v_j = \min_{x \in \Omega_j} V(x) < \min_{x \in \partial\Omega_j} V(x).$$

Then for all sufficiently small  $\epsilon > 0$ ,

- (a) system  $(\mathcal{P}_\epsilon)$  at least has  $k$  nontrivial solution  $\eta_\epsilon^j \in H^{2,q}$  for any  $q \geq 2, j = 1, \dots, k$ ;
- (b)  $|\eta_\epsilon^j(x)|$  attains its maximum at  $p_\epsilon^j$  in  $\Omega_j$ , moreover, up to a subsequence, there holds

$$\lim_{\epsilon \rightarrow 0} V(p_\epsilon^j) = v_j;$$

- (c)  $\eta_\epsilon^j(\epsilon x + p_\epsilon) \rightarrow \eta_j(x)$  in  $H^2(\mathbb{R}^N, \mathbb{R}^2)$  as  $\epsilon \rightarrow 0$ , and  $\eta_j$  is a ground state solution of the following system

$$\begin{cases} -\Delta\psi + \vec{b} \cdot \nabla\psi + \psi + v_j\varphi = f(|\eta|)\varphi & \text{in } \mathbb{R}^N, \\ -\Delta\varphi - \vec{b} \cdot \nabla\varphi + \varphi + v_j\psi = f(|\eta|)\psi & \text{in } \mathbb{R}^N, \end{cases}$$

- (d) there exist positive constants  $c, C$  such that

$$|\eta_\epsilon^j(x)| \leq C \exp\left(-\frac{c}{\epsilon}|x - p_\epsilon^j|\right).$$

We remark here that in Corollary 1.2, the solutions can be separated provided  $\epsilon > 0$  is small since  $\Omega_j$  are mutually disjoint. Furthermore, if  $v_1$  is a global minimum of  $V$ , then Corollary 1.2 describes a multiple concentrating phenomenon.

For the proof of our results, we do not handle the system  $(\mathcal{P}_\epsilon)$  directly, but instead we handle an equivalent system to  $(\mathcal{P}_\epsilon)$ . For this purpose, set  $z(x) = (u(x), v(x)) = (\psi(\epsilon x), \varphi(\epsilon x)) = \eta(\epsilon x)$ . Then the system  $(\mathcal{P}_\epsilon)$  is equivalent to the following:

$$\begin{cases} -\Delta u + \vec{b} \cdot \nabla u + u + V(\epsilon x)v = f(|z|)v & \text{in } \mathbb{R}^N, \\ -\Delta v - \vec{b} \cdot \nabla v + v + V(\epsilon x)u = f(|z|)u & \text{in } \mathbb{R}^N. \end{cases} \tag{\mathcal{P}'_\epsilon}$$

Moreover, system  $(\mathcal{P}'_\epsilon)$  can be expressed as

$$Az + V(\epsilon x)z = f(|z|)z, \tag{1.6}$$

where

$$A = \begin{pmatrix} 0 & -\Delta - \vec{b} \cdot \nabla + 1 \\ -\Delta + \vec{b} \cdot \nabla + 1 & 0 \end{pmatrix}.$$

Clearly, (1.4) is equivalent to (1.6). We will, in the sequel, focus on this equivalent problem.

As a motivation we recall that there are many enormous investigations concerning with the semiclassical states of Schrödinger equations

$$-\epsilon^2 \Delta u + V(x)u = f(x, u), \quad u \in H^1(\mathbb{R}^N). \quad (1.7)$$

In particular, initiated by Rabinowitz [27], the positive ground state solution of (1.7) for  $\epsilon > 0$  small under the global condition (1.2) was proved via mountain pass theorem. After that, Wang [34] showed that the positive solution obtained in [27] concentrates at global minimum points of  $V$ . It should be pointed out that, under the local condition (1.3), del Pino and Felmer [8] first succeeded in proving a localized version of concentration of single-peak solution by using a new penalization approach, moreover, the multi-peak bound state solution was obtained in [9, 18]. Based on a singular perturbation argument, the localized bound state solutions concentrating at an isolated component of the local minimum of  $V$  were also constructed in [5] and [6]. For further related topics including the Hamiltonian system and Dirac equation, we refer the reader to [3, 13, 14, 17, 26, 28, 33, 36] and their references.

From the commentaries above, it is quite natural to ask if the localized concentration results of semiclassical states can be obtained for the Hamiltonian elliptic system  $(\mathcal{P}_\epsilon)$  as in Schrödinger equation (1.7)? In the present paper, we shall give some answers for this system. However, compared with the Schrödinger equation (1.7), system  $(\mathcal{P}_\epsilon)$  becomes more complicated since system  $(\mathcal{P}_\epsilon)$  is strongly indefinite in the sense that both the negative and positive parts of the spectrum are unbounded and consist of essential spectrum, and the energy functional has complex geometric structure. Hence our problem poses more challenges in the calculus of variation.

Our argument is based on variational method, which can be outlined as follows. The solutions are obtained as critical points of the energy functional associated to system  $(\mathcal{P}_\epsilon)$ . We emphasize here that, since the energy functional is strongly indefinite, the classical critical point theory, such as mountain pass lemma and Nehari manifold arguments, cannot be applied directly. On the other hand, the reduction method [1] used in [42, 44, 46], which reduces the strongly indefinite case to the mountain case, also do not seem to be applicable to our problem. Because such method depends on the convexity of the nonlinearities, specifically, it requires that the second order derivative of the energy functional is negative definite on negative space. And by the anti-coercion and concavity properties of the energy functional, one can define a reduction functional such that critical points of original functional and reduction functional are in one-to-one correspondence via reduction map. So, along this line, the nonlinearity  $f$  requires the strong differentiability condition:  $f$  is of class  $C^1$ . However, we only assume that  $f$  satisfies continuous condition, and such a reduction method does not work. In addition, the main difficulty caused by the unboundedness of the domain is the lack of compactness of Sobolev embedding. Based on the above reasons, some new methods and techniques need to be introduced in the present paper.

More precisely, to prove our results, some arguments are in order. Firstly, since we have no global information on the potential  $V$ , we employ the truncation trick and make a slight modification of the energy functional corresponding to system  $(\mathcal{P}_\epsilon)$ . In such a way, the modified functional satisfies the so-called Cerami compactness condition. Here the modification of the energy functional corresponds to a penalization technique “outside  $\Omega$ ” (see [8, 9]). Secondly, to overcome the strongly indefiniteness of the functional, we utilize the generalized linking theorem and the diagonal method to construct a minimizing Cerami sequence for the modified functional, moreover, together with the generalized Nehari manifold, we prove the existence and relation of ground state solution for the modified problem and the limit problem. Lastly, the sub-solution estimates of  $|z|$  seem not work well since the effect of the gradient term, we establish the sub-solution estimate of  $|z|^2$ . Moreover, using this fact, we prove the uniformly exponential decay of ground state solution for the modified problem, which implies that the solution corresponding to the modified problem is indeed the solution of original problem  $(\mathcal{P}_\epsilon)$  for  $\epsilon$  sufficiently small. And then Theorem 1.1 follows naturally.

The remainder of this paper is organized as follows. In Section 2, we present the variational setting of the problem, introduce the modified functional, and give some useful preliminaries. In Section 3, we prove the modified problem has a ground state solution with ground state energy  $m_\epsilon$ . In Section 4, we show the limit problem possesses a ground state solution, and prove the upper limit of the ground state energy  $m_\epsilon$  is less

than or equal to the ground state energy of the limit problem as  $\epsilon \rightarrow 0$ . At last, we give the proof of Theorem 1.1 in Section 5.

## 2 Variational setting and preliminaries

Below by  $|\cdot|_q$  we denote the usual  $L^q$ - norm,  $(\cdot, \cdot)_2$  denotes the usual  $L^2$  inner product,  $c, c_i$  or  $C_i$  stand for different positive constants. Denote by  $\sigma(A)$  and  $\sigma_e(A)$  the spectrum and the essential spectrum of the operator  $A$ , respectively. In order to establish a suitable variational framework for system  $(\mathcal{P}'_\epsilon)$ , we need to analyze some properties of the spectrum of the associated Hamiltonian operator  $A$ . The proof can be seen [39], so we omit the details here.

**Lemma 2.1.** *The operator  $A$  is a selfadjoint operator on  $L^2 := L^2(\mathbb{R}^N, \mathbb{R}^2)$  with domain  $\mathcal{D}(A) := H^2(\mathbb{R}^N, \mathbb{R}^2)$ .*

**Lemma 2.2.** *The following two conclusions hold:*

- (1)  $\sigma(A) = \sigma_e(A)$ , i.e.,  $A$  has only essential spectrum;
- (2)  $\sigma(A) \subset \mathbb{R} \setminus (-1, 1)$  and  $\sigma(A)$  is symmetric with respect to origin.

It follows from Lemma 2.1 and Lemma 2.2 that the space  $L^2$  possesses the following orthogonal decomposition

$$L^2 = L^- \oplus L^+ \quad z = z^- + z^+$$

such that  $A$  is negative definite (resp. positive definite) in  $L^-$  (resp.  $L^+$ ). Let  $|A|$  denote the absolute value of  $A$  and  $|A|^{\frac{1}{2}}$  be the square root of  $|A|$ . Let  $E = \mathcal{D}(|A|^{\frac{1}{2}})$  be the Hilbert space with the inner product

$$\langle z, w \rangle = (|A|^{\frac{1}{2}}z, |A|^{\frac{1}{2}}w)_2$$

and norm  $\|z\| = \langle z, z \rangle^{\frac{1}{2}}$ . There is an induced decomposition

$$E = E^- \oplus E^+, \quad \text{where } E^\pm = E \cap L^\pm,$$

which is orthogonal with respect to the inner products  $(\cdot, \cdot)_2$  and  $\langle \cdot, \cdot \rangle$ . According to [39],  $\|\cdot\|$  and  $\|\cdot\|_{H^1}$  are equivalent norms, and thus  $E$  embeds continuously into  $L^p := L^p(\mathbb{R}^N, \mathbb{R}^2)$  for any  $p \in [2, 2^*]$  and compactly into  $L^p_{loc} := L^p_{loc}(\mathbb{R}^N, \mathbb{R}^2)$  for any  $p \in [1, 2^*)$ , and there exists positive constant  $\pi_p$  such that

$$\pi_p |z|_p \leq \|z\|, \quad \text{for all } z \in E, \quad p \in [2, 2^*]. \tag{2.1}$$

Additionally, the decomposition of  $E$  induces also a natural decomposition of  $L^q$ , hence there exists a positive constant  $d_q$  such that

$$d_q |z^\pm|_q \leq |z|_q, \quad \text{for all } z \in E. \tag{2.2}$$

Now we define the following functional on  $E$  as follows

$$\begin{aligned} I_\epsilon(z) &= \frac{1}{2} \int_{\mathbb{R}^N} Az \cdot z + \frac{1}{2} \int_{\mathbb{R}^N} V(\epsilon x) |z|^2 - \int_{\mathbb{R}^N} F(|z|) \\ &= \frac{1}{2} (\|z^+\|^2 - \|z^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V(\epsilon x) |z|^2 - \int_{\mathbb{R}^N} F(|z|) \end{aligned}$$

where  $\cdot$  denotes the usual inner product in  $\mathbb{R}^2$ . Lemma 2.2 implies that  $I_\epsilon$  is strongly indefinite. Moreover, our hypotheses imply that  $I_\epsilon \in C^1(E, \mathbb{R})$  and a standard argument shows that critical points of  $I_\epsilon$  are solutions of problem  $(\mathcal{P}'_\epsilon)$  (see [10]), and for  $z, \varphi \in E$ , there holds

$$\begin{aligned} I'_\epsilon(z)\varphi &= \int_{\mathbb{R}^N} Az \cdot \varphi + \int_{\mathbb{R}^N} V(\epsilon x) z\varphi - \int_{\mathbb{R}^N} f(|z|)z\varphi \\ &= (z^+, \varphi^+) - (z^-, \varphi^-) + \int_{\mathbb{R}^N} V(\epsilon x) z\varphi - \int_{\mathbb{R}^N} f(|z|)z\varphi. \end{aligned}$$

As we have mentioned in the introduction, the energy functional  $I_\epsilon$  does not satisfy compactness condition under local potential condition in general, we will not deal with  $I_\epsilon$  directly. Instead, we need make use of the penalization approach developed by del Pino and Felmer [8, 9] to modify the energy functional such that the modified functional satisfies the Cerami condition. After constructing solutions of the modified problems we will make these solutions localized, so they are solutions of the original problem for small  $\epsilon$ .

In virtue of the assumption (V), we can fix a small  $\delta > 0$  such that

$$v < V(x) \text{ for any } x \in \overline{\Omega^\delta} \setminus \Omega, \tag{2.3}$$

where  $\Omega^\delta = \{x \in \mathbb{R}^3 : \text{dist}(x, \Omega) := \inf_{y \in \Omega} |x - y| < \delta\}$  is the  $\delta$ -neighborhood of  $\Omega$  and  $\overline{\Omega^\delta}$  is the closure of  $\Omega^\delta$ . Let  $\zeta \in C(\mathbb{R}^N, \mathbb{R})$  be a function such that  $0 \leq \zeta(s) \leq 1$ ,  $\zeta(s) = 1$  if  $s \leq 0$  and  $\zeta(s) = 0$  if  $s \geq \delta$ . We choose a suitable constant  $a_0 > 0$  such that  $f(a_0) = \frac{1-|V|_\infty}{2}$ , and set  $\chi(x) = \zeta(\text{dist}(x, \Omega))$  and  $\tilde{f} \in C(\mathbb{R}^+, \mathbb{R})$ :

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \leq a_0, \\ \frac{1-|V|_\infty}{2} & \text{if } s > a_0. \end{cases}$$

We define

$$g(x, s) = \chi(x)f(s) + (1 - \chi(x))\tilde{f}(s), \quad G(x, s) = \int_0^s g(x, t)tdt,$$

then

$$G(x, s) = \chi(x)F(s) + (1 - \chi(x))\tilde{F}(s), \quad \tilde{F}(s) = \int_0^s \tilde{f}(t)tdt.$$

It is easy to check that  $(F_0)$ - $(F_2)$  implies that  $g$  is a Caratheodory function and it satisfies the following assumptions:

- $(g_1)$   $g(x, s) \rightarrow 0$  uniformly for  $x \in \mathbb{R}^N$ , and  $g(x, s)$  is nondecreasing in  $s \in \mathbb{R}^+$  for  $x \in \mathbb{R}^N$ ;
- $(g_2)$   $0 \leq g(x, s) \leq f(s)$  for every  $(x, s) \in \mathbb{R}^N \times \mathbb{R}^+$ ;
- $(g_3)$   $f(s)s^2 - 2F(s) \geq 0$  and  $\tilde{f}(s)s^2 - 2\tilde{F}(s) \geq 0$  for all  $s \geq 0$ .

From  $(F_0)$ ,  $(F_1)$  and the definition of  $\tilde{f}$ , it follows that there exists  $c_1 > 0$  such that

$$f(s) \leq \frac{1 - |V|_\infty}{2}, \quad s < a_0 \tag{2.4}$$

and  $f(s) \leq c_1 s^{p-2}$  for  $s \geq a_0$ . So for  $s \geq a_0$ , there holds  $(f(s)s)^{\frac{1}{p-1}} \leq c_1 s$  and

$$(f(s)s)^{\frac{p}{p-1}} = (f(s)s)^{\frac{1}{p-1}+1} \leq c_1 s(f(s)s) = c_1 f(s)s^2. \tag{2.5}$$

By  $(F_3)$ , we obtain

$$\frac{1}{2}f(s)s^2 - F(s) \geq \frac{1}{2}f(s)s^2 - \frac{1}{\mu}f(s)s^2 = \frac{\mu - 2}{2\mu}f(s)s^2, \quad s \geq 0. \tag{2.6}$$

According to (2.5) and (2.6), we have

$$(f(s)s)^{\frac{p}{p-1}} \leq c_2 \left( \frac{1}{2}f(s)s^2 - F(s) \right), \quad s \geq a_0.$$

This, together with (2.4), we get

$$f(s)s \leq \frac{1 - |V|_\infty}{2}s + c_3 \left( \frac{1}{2}f(s)s^2 - F(s) \right)^{\frac{p-1}{p}}, \quad s \geq 0.$$

By  $(F_0)$  and the definition of  $\tilde{f}$ , we can see that  $\tilde{f} \leq \frac{1-|V|_\infty}{2}$ , and

$$g(x, s) \leq \frac{1 - |V|_\infty}{2}s + c_3 \chi(x) \left( \frac{1}{2}f(s)s^2 - F(s) \right)^{\frac{p-1}{p}}, \quad s \geq 0. \tag{2.7}$$

Now we are ready to define the modified functional  $\Phi_\epsilon : E \rightarrow \mathbb{R}$ ,

$$\Phi_\epsilon(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V(\epsilon x)|z|^2 - \int_{\mathbb{R}^N} G(\epsilon x, |z|).$$

Similarly,  $\Phi_\epsilon$  is of class  $C^1$  and the critical points correspond to weak solutions of the following modified system

$$\begin{cases} -\Delta u + \vec{b} \cdot \nabla u + u + V(\epsilon x)v = g(\epsilon x, |z|)v \text{ in } \mathbb{R}^N, \\ -\Delta v - \vec{b} \cdot \nabla v + v + V(\epsilon x)u = g(\epsilon x, |z|)u \text{ in } \mathbb{R}^N. \end{cases} \quad (2.8)$$

For the sake of simplicity, in what follows, we denote by

$$V_\epsilon(x) = V(\epsilon x), \chi_\epsilon(x) = \chi(\epsilon x), g_\epsilon(x, s) = g(\epsilon x, s), G_\epsilon(x, s) = G(\epsilon x, s).$$

Recall that for a functional  $\Phi \in C^1(E, \mathbb{R})$ ,  $\Phi$  is said to be weakly sequentially lower semi-continuous if for any  $u_n \rightharpoonup u$  in  $E$  one has  $\Phi(u) \leq \liminf_{n \rightarrow \infty} \Phi(u_n)$ , and  $\Phi'$  is said to be weakly sequentially continuous if  $\lim_{n \rightarrow \infty} \Phi'(u_n)v = \Phi'(u)v$  for each  $v \in E$ . We recall that a sequence  $\{u_n\} \subset E$  is called Cerami sequence for  $\Phi$  at the level  $c$  ( $(C)_c$ -sequence in short) if

$$\Phi(u_n) \rightarrow c \text{ and } (1 + \|u_n\|)\|\Phi'(u_n)\| \rightarrow 0.$$

We say that  $\Phi$  satisfy  $(C)_c$ -condition if any  $(C)_c$ -sequence has a convergent subsequence in  $E$ .

To prove the main results, we need the generalized linking theorem due to [23].

**Lemma 2.3.** *Let  $X$  be a real Hilbert space with  $X = X^- \oplus X^+$ , and let  $\Phi \in C^1(X, \mathbb{R})$  be of the form*

$$\Phi(u) = \frac{1}{2} (\|u^+\|^2 - \|u^-\|^2) - \Psi(u), \quad u = u^- + u^+ \in X^- \oplus X^+.$$

Suppose that the following assumptions are satisfied:

( $\Psi_1$ )  $\Psi \in C^1(X, \mathbb{R})$  is bounded from below and weakly sequentially lower semi-continuous;

( $\Psi_2$ )  $\Psi'$  is weakly sequentially continuous;

( $\Psi_3$ ) there exist  $R > \rho > 0$  and  $e \in X^+$  with  $\|e\| = 1$  such that

$$\kappa := \inf \Phi(S_\rho^+) > \sup \Phi(\partial Q),$$

where

$$S_\rho^+ = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{v + se : v \in X^-, s \geq 0, \|v + se\| \leq R\}.$$

Then there exist a constant  $c \in [\kappa, \sup \Phi(Q)]$  and a sequence  $\{u_n\} \subset X$  satisfying

$$\Phi(u_n) \rightarrow c \text{ and } (1 + \|u_n\|)\|\Phi'(u_n)\| \rightarrow 0.$$

We introduce two technical results (see [22, 31]), which play an important role in the following proof.

**Lemma 2.4.** *Assume that a function  $h \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R})$  satisfies that*

( $h_1$ )  $h(x, s) \rightarrow 0$  uniformly in  $x$  as  $s \rightarrow 0^+$ ;

( $h_2$ )  $h(x, s)$  is increasing on  $\mathbb{R}^+$ ;

( $h_3$ )  $\frac{H(x, s)}{s^2} \rightarrow +\infty$  uniformly in  $x$  as  $s \rightarrow +\infty$ , where  $H(s) = \int_0^s h(x, t)tdt$ .

Then for  $t \geq 0, x \in \mathbb{R}^N$  and  $z, w \in \mathbb{R}^2$  such that  $z \neq tz + w$ , there holds

$$h(x, |z|)z \cdot \left( \frac{t^2 - 1}{2} z + tw \right) + H(x, |z|) - H(x, |tz + w|) < 0. \quad (2.9)$$

*Proof.* Observe that, from ( $h_1$ ) and ( $h_2$ ), it follows that

$$h(x, s)s^2 > 0, H(x, s) > 0, \text{ and } \frac{1}{2}h(x, s)s^2 - H(x, s) > 0, s \neq 0. \quad (2.10)$$

Let  $t, z$  and  $w$  as in the statement. Define

$$\mathcal{H}(x, t) := h(x, |z|)z \cdot \left( \frac{t^2 - 1}{2}z + tw \right) + H(x, |z|) - H(x, |tz + w|),$$

and note that we have to prove that  $\mathcal{H}(x, t) < 0$ . If  $z = 0$ , it follows from (2.10) that  $\mathcal{H}(x, t) = -H(x, |w|) < 0$ . If  $z \neq 0$ , using  $(h_3)$  we know that  $\mathcal{H}(x, t) \rightarrow -\infty$  as  $t \rightarrow \infty$ . Therefore,  $\mathcal{H}(x, t)$  attains its maximum at some point  $t_0 \in [0, \infty)$ . If  $t_0 = 0$ , the conclusion holds by (2.10). If  $t_0 > 0$ , then  $\partial_t \mathcal{H}(x, t_0) = 0$ . Thus, we have

$$h(x, |z|)z \cdot (t_0z + w) = h(x, |t_0z + w|)z \cdot (t_0z + w). \tag{2.11}$$

Let  $\phi = t_0z + w$ , there are two cases: (i)  $z \cdot \phi \neq 0$  and (ii)  $z \cdot \phi = 0$ .

When  $z \cdot \phi \neq 0$ , then  $z \neq 0$  and  $\phi \neq 0$ , from  $(h_2)$  and (2.11), it follows that  $|z| = |\phi|$  and  $H(x, |z|) = H(x, |\phi|)$ . Moreover, observe that  $f(x, |z|)z \cdot \phi < f(x, |z|)|z|^2$ . Then by (2.10) we get

$$\begin{aligned} \mathcal{H}(x, t_0) &= h(x, |z|)z \cdot \left( \frac{t_0^2 - 1}{2}z + t_0w \right) + H(x, |z|) - H(x, |t_0z + w|) \\ &= h(x, |z|)z \cdot \left( \frac{t_0^2 - 1}{2}z + t_0(\phi - t_0z) \right) + H(x, |z|) - H(x, |\phi|) \\ &= h(x, |z|)z \cdot \left( \left( \frac{t_0^2}{2} - t_0^2 + t_0 - \frac{1}{2} \right)z + t_0\phi - t_0z \right) \\ &= \left[ -\frac{1}{2}(t_0 - 1)^2 h(x, |z|)|z|^2 + t_0 \left( h(x, |z|)z \cdot \phi - h(x, |z|)|z|^2 \right) \right] < 0. \end{aligned}$$

When  $z \cdot \phi = 0$ , then  $f(x, |z|)z \cdot \phi = 0$ . By (2.10) again, we have

$$\begin{aligned} \mathcal{H}(x, t_0) &= h(x, |z|)z \cdot \left( \frac{t_0^2 - 1}{2}z + t_0(\phi - t_0z) \right) + H(x, |z|) - H(x, |\phi|) \\ &= -\frac{t_0^2}{2} h(x, |z|)|z|^2 - \frac{1}{2} h(x, |z|)|z|^2 + H(x, |z|) + t_0 h(x, |z|)z \cdot \phi - H(x, |\phi|) \\ &< -\frac{t_0^2}{2} h(x, |z|)|z|^2 - H(x, |\phi|) < 0. \end{aligned}$$

Therefore,  $\mathcal{H}(x, t) < 0$  for any  $t \geq 0$  and hence (2.9) holds. □

According to Lemma 2.4, we can prove a weaker version result than Lemma 2.4.

**Lemma 2.5.** Assume that  $h \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R})$  satisfies that

$(h'_1)$   $h(x, s) \rightarrow 0$  uniformly in  $x$  as  $s \rightarrow 0^+$ ;

$(h'_2)$   $h(x, s)$  is non-decreasing on  $\mathbb{R}^+$ ;

$(h'_3)$   $H(s) = \int_0^s h(x, t)tdt$ .

Then for  $t \geq 0, x \in \mathbb{R}^N$  and  $z, w \in \mathbb{R}^2$  such that  $z \neq tz + w$ , there holds

$$h(x, |z|)z \cdot \left( \frac{t^2 - 1}{2}z + tw \right) + H(x, |z|) - H(x, |tz + w|) \leq 0. \tag{2.12}$$

*Proof.* Applying the method in the proof of [22, Lemma 3.2], for  $\varepsilon > 0$ , we define  $h_\varepsilon : \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$h_\varepsilon(x, s) = h(x, s) + \varepsilon s.$$

It is easy to check that  $h_\varepsilon(x, s)$  satisfies the corresponding  $(h_1)$ - $(h_3)$  of Lemma 2.4. The desired result follows by applying Lemma 2.4 to  $h_\varepsilon$  and then letting  $\varepsilon \rightarrow 0$ . □

### 3 The modified problem

In this section, we will in the sequel focus on the modified problem (2.8) and study the existence of ground state solution. In order to seek for the ground state solutions of the modified problem (2.8), we consider the following set which is introduced in Pankov [25]

$$\mathcal{M}_\epsilon := \{z \in E \setminus E^- : \Phi'_\epsilon(z)z = 0 \text{ and } \Phi'_\epsilon(z)w = 0 \text{ for any } w \in E^-\}.$$

Following from Szulkin and Weth [31], we will call the set  $\mathcal{M}_\epsilon$  the generalized Nehari manifold. Obviously, the set  $\mathcal{M}_\epsilon$  is a natural constraint and it contains all nontrivial critical points of  $\Phi_\epsilon$ . Let

$$m_\epsilon := \inf_{z \in \mathcal{M}_\epsilon} \Phi_\epsilon.$$

If  $m_\epsilon$  is attained by  $z_\epsilon \in \mathcal{M}_\epsilon$ , then  $z_\epsilon$  is a critical point of  $\Phi_\epsilon$ . Since  $m_\epsilon$  is the lowest level for  $\Phi_\epsilon$ , then  $z_\epsilon$  is called a ground state solution of the modified problem (2.8).

Define

$$\Psi_\epsilon(z) = \int_{\mathbb{R}^N} G_\epsilon(x, |z|).$$

Then, using the fact that  $E$  embeds into  $L^q$  continuously for  $q \in [2, 2^*]$  and embeds into  $L^q_{loc}$  compactly for  $q \in [1, 2^*)$ , we can check easily the following lemma, and omit the details of proof.

**Lemma 3.1.**  $\Psi_\epsilon$  is weakly sequentially lower semi-continuous.  $\Psi'_\epsilon$  is weakly sequentially continuous.

**Lemma 3.2.** Let  $z \in E$ ,  $w \in E^-$  and  $t \geq 0$ , we have

$$\Phi_\epsilon(z) \geq \Phi_\epsilon(tz + w) - \Phi'_\epsilon(z) \left( \frac{t^2 - 1}{2} z + tw \right).$$

In particular, let  $z \in \mathcal{M}_\epsilon$ ,  $w \in E^-$  and  $t \geq 0$ , there holds

$$\Phi_\epsilon(z) \geq \Phi_\epsilon(tz + w).$$

*Proof.* Observe that

$$\Phi_\epsilon(tz + w) - \Phi_\epsilon(z) - \Phi'_\epsilon(z) \left( \frac{t^2 - 1}{2} z + tw \right) = -\frac{1}{2} \|w\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V_\epsilon(x) |w|^2 + \int_{\mathbb{R}^N} \mathcal{G}_\epsilon(x, t)$$

where

$$\mathcal{G}_\epsilon(x, t) := g_\epsilon(x, |z|)z \cdot \left( \frac{t^2 - 1}{2} z + tw \right) + G_\epsilon(x, |z|) - G_\epsilon(x, |tz + w|).$$

On the one hand, by (V) and (2.1) ( $\pi_2 = 1$  by Lemma 2.2) we deduce that

$$-\|w\|^2 + \int_{\mathbb{R}^N} V_\epsilon(x) |w|^2 < -\|w\|^2 + |w|_2^2 \leq 0.$$

On the other hand, from  $(g_1)$ , we know that for any  $x \in \mathbb{R}^N$ ,  $g_\epsilon(x, s)$  satisfies the assumptions  $(h'_1)$ - $(h'_3)$  in Lemma 2.5. So applying Lemma 2.5, we get the first conclusion. If  $z \in \mathcal{M}_\epsilon$ , then  $\Phi'_\epsilon(z)z = \Phi'_\epsilon(z)w = 0$ , then the second conclusion holds.  $\square$

For convenience of notation, we write  $E(z) := E^- \oplus \mathbb{R}^+ z = E^- \oplus \mathbb{R}^+ z^+$  for  $z \in E \setminus E^-$ . Let  $z \in \mathcal{M}_\epsilon$ , then Lemma 3.2 implies that  $z$  is the global maximum of  $\Phi_\epsilon|_{E(z)}$ . Next we shall verify that  $\Phi_\epsilon$  possesses the linking structure.

**Lemma 3.3.** There exist  $\rho > 0$  and  $\alpha > 0$  both independent of  $\epsilon$ . Then

(i) there holds:  $m_\epsilon = \inf_{\mathcal{M}_\epsilon} \Phi_\epsilon \geq \kappa := \inf_{S_\rho} \Phi_\epsilon \geq \alpha$ , where  $S_\rho := \{z \in E^+, \|z\| = \rho\}$ .

(ii)  $\|z^+\| \geq \max \left\{ \sqrt{\frac{1-|V|_\infty}{1+|V|_\infty}} \|z^-\|, \sqrt{\frac{2m_\epsilon}{1+|V|_\infty}} \right\}$  for all  $z \in \mathcal{M}_\epsilon$ .

*Proof.* (i) Observe that,  $\pi_2 = 1$  by Lemma 2.2. For  $z \in E^+$ , by  $(F_1)$ ,  $(g_1)$ ,  $(g_2)$  and (2.1), we obtain

$$\begin{aligned} \Phi_\epsilon(z) &= \frac{1}{2} \|z\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V_\epsilon(x) |z|^2 - \int_{\mathbb{R}^N} G_\epsilon(x, |z|) \\ &\geq \frac{1}{2} \|z\|^2 - \frac{1}{2} |V|_\infty |z|_2^2 - \epsilon |z|_2^2 - C_\epsilon |z|_p^p \\ &\geq \left( \frac{1}{2} - \frac{|V|_\infty}{2} - \epsilon \right) \|z\|^2 - \pi_p^{-p} C_\epsilon \|z\|^p. \end{aligned}$$

It is easy to see that there exist  $\rho > 0$  and  $\alpha > 0$  both independent of  $\epsilon$  such that  $\kappa := \inf_{S_\rho} \Phi \geq \alpha$  since  $|V|_\infty < 1$ . So the second inequality holds. Note that for every  $z \in \mathcal{M}_\epsilon$  there is  $s > 0$  such that  $sz^+ \in E(z) \cap S_\rho$ . Clearly, the first inequality follows from Lemma 3.2.

(ii) For  $z \in \mathcal{M}_\epsilon$ , by  $(V)$  and (2.1) we have

$$\begin{aligned} m_\epsilon &\leq \frac{1}{2} (\|z^+\|^2 - \|z^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_\epsilon(x) |z|^2 - \int_{\mathbb{R}^N} G_\epsilon(x, |z|) \\ &\leq \frac{1}{2} (\|z^+\|^2 - \|z^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_\epsilon(x) |z|^2 \\ &\leq \frac{1}{2} \left( (1 + |V|_\infty) \|z^+\|^2 - (1 - |V|_\infty) \|z^-\|^2 \right), \end{aligned}$$

hence  $\|z^+\| \geq \max \left\{ \sqrt{\frac{1-|V|_\infty}{1+|V|_\infty}} \|z^-\|, \sqrt{\frac{2m_\epsilon}{1+|V|_\infty}} \right\}$ . □

It follows from  $(F_2)$  and the definition of  $g$  that there exists a positive constant  $c_4$  such that

$$G(x, s) = F(s) \geq c_4 s^\mu - \frac{1 - |V|_\infty}{4} s^2, \text{ for all } s \geq 0 \text{ and } x \in \Omega. \quad (3.1)$$

For any fixed  $e \in E^+$  with  $\|e\| = 1$ , there exists  $R > 0$  such that

$$\frac{3 + |V|_\infty}{4} s^2 - c_4 d_\mu^\mu |e|_\mu^\mu s^\mu \leq -1, \quad s \geq R, \quad (3.2)$$

where  $d_\mu$  is given in (2.2). Setting

$$R_1 := \sqrt{\frac{2}{1 - |V|_\infty}} R \text{ and } Q_{R_1}(e) = \{se + w : w \in E^-, s \geq 0, \|se + w\| \leq R_1\}.$$

**Lemma 3.4.** *For  $\epsilon > 0$  small enough, there holds  $\sup \Phi_\epsilon(\partial Q_{R_1}(e)) \leq 0$ .*

*Proof.* It is sufficient to prove that

$$\overline{\lim}_{\epsilon \rightarrow 0} \sup_{z \in \partial Q_{R_1}(e)} \Phi_\epsilon(z) < 0.$$

Otherwise, there exist  $\epsilon_n \rightarrow 0$  and  $z_n \in \partial Q_{R_1}(e)$ , such that  $\Phi_{\epsilon_n}(z_n) \geq -\frac{1}{n}$ . Denote  $z_n = s_n e + z_n^-$  with  $s_n \geq 0$  and  $z_n^- \in E^-$  and  $\|z_n\| = R_1$ . Since  $\Phi_{\epsilon_n}(z_n) \geq -\frac{1}{n}$ , we get

$$\begin{aligned} -\frac{1}{n} &\leq \frac{1}{2} (s_n^2 - \|z_n^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_\epsilon(x) |z_n|^2 - \int_{\mathbb{R}^N} G_{\epsilon_n}(x, |z_n|) \\ &\leq \frac{1}{2} (s_n^2 - \|z_n^-\|^2) + \frac{|V|_\infty}{2} (s_n^2 |e|_2^2 + |z_n^-|_2^2) - \int_{\mathbb{R}^N} G_{\epsilon_n}(x, |z_n|), \end{aligned} \quad (3.3)$$

which implies that

$$\frac{1 - |V|_\infty}{2} \|z_n^-\|^2 \leq \frac{1 + |V|_\infty}{2} |s_n|^2 + \frac{1}{n} = \frac{1 + |V|_\infty}{2} (R_1^2 - \|z_n^-\|^2) + \frac{1}{n}.$$

Then

$$\|z_n^-\|^2 \leq \frac{1 + |V|_\infty}{2} R_1^2 \text{ and } R_1^2 \geq s_n^2 \geq \frac{1 - |V|_\infty}{2} R_1^2 + o_n(1).$$

Up to a subsequence,  $z_n^- \rightharpoonup z^-$  in  $E^-$  and  $s_n \rightarrow s_0$  with  $s_0 \in [\sqrt{\frac{1 - |V|_\infty}{2}} R_1, R_1]$ . Hence, we have  $z_n \rightarrow z = s_0 e + z^-$  in  $E$ ,  $z_n \rightarrow z$  in  $L^q_{loc}$  for  $q \in [1, 2^*)$  and  $z_n(x) \rightarrow z(x)$  a.e. on  $\mathbb{R}^N$ .

Since  $0 \in \mathcal{V}$ , from (1.5) we can assume that there exists  $\delta_1 > 0$  such that  $B_{\delta_1}(0) \subset \Omega$ . Define  $\xi(x) \in C^\infty_0(B_{\delta_1}, [0, 1])$  such that  $\xi(x) = 1$  for  $|x| \leq \delta_1/2$ ,  $\xi(x) = 0$  for  $|x| \geq \delta_1$  and  $|\nabla \xi| \leq 2/\delta_1$ . Then setting  $\chi_n(x) = \xi(\epsilon_n x)$ , we have  $\chi_n(x) z_n(x) \rightarrow z(x)$  a.e. on  $\mathbb{R}^N$  and

$$\text{supp} \chi_n \subset \Omega_{\epsilon_n} = \{x \in \mathbb{R}^N : \epsilon_n x \in \Omega\}.$$

Using Fatou's lemma, we obtain

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\chi_n z_n|^\mu \geq \int_{\mathbb{R}^N} |z|^\mu. \tag{3.4}$$

Since  $g(x, s) \geq 0$ ,  $G(x, s)$  is nondecreasing in  $s$ , we deduce from (2.2), and (3.1) that

$$\begin{aligned} \int_{\mathbb{R}^N} G_{\epsilon_n}(x, |z_n|) &\geq \int_{\Omega_{\epsilon_n}} G_{\epsilon_n}(x, |\chi_n z_n|) \\ &\geq c_4 \int_{\Omega_{\epsilon_n}} |\chi_n z_n|^\mu - \frac{1 - |V|_\infty}{4} \int_{\Omega_{\epsilon_n}} |\chi_n z_n|^2 \\ &= c_4 \int_{\mathbb{R}^N} |\chi_n z_n|^\mu - \frac{1 - |V|_\infty}{4} \int_{\mathbb{R}^N} |\chi_n z_n|^2 \\ &\geq c_4 \int_{\mathbb{R}^N} |z|^\mu - \frac{1 - |V|_\infty}{4} \int_{\mathbb{R}^N} |z_n|^2 + o_n(1) \\ &\geq c_4 d_\mu^\mu |e|_\mu^\mu s_0^\mu - \frac{1 - |V|_\infty}{4} \int_{\mathbb{R}^N} |z_n|^2 + o_n(1). \end{aligned} \tag{3.5}$$

Moreover, from (3.3) and (3.5) we obtain

$$-\frac{1}{n} \leq \frac{3 + |V|_\infty}{4} s_n^2 - \frac{1 - |V|_\infty}{4} \|z_n^-\|^2 - c_4 d_\mu^\mu |e|_\mu^\mu s_0^\mu + o_n(1).$$

Letting  $n \rightarrow \infty$ , then

$$\frac{3 + |V|_\infty}{4} s_0^2 - c_4 d_\mu^\mu |e|_\mu^\mu s_0^\mu \geq 0,$$

since  $s_0 \geq \sqrt{\frac{1 - |V|_\infty}{2}} R_1 = R$ , the above inequality implies that a contradiction to (3.2). The proof is completed.  $\square$

Applying Lemmas 2.3, 3.1, 3.3 and 3.4, we have

**Lemma 3.5.** *Suppose that (V), (F<sub>0</sub>)-(F<sub>2</sub>) and the conditions of g are satisfied. Then there exist a constant  $\hat{c}_\epsilon \in [\kappa, \sup \Phi_\epsilon(Q_{R_1}(e))]$  and a sequence  $\{z_n\} \subset E$  satisfying*

$$\Phi_\epsilon(z_n) \rightarrow \hat{c}_\epsilon \text{ and } \|\Phi'_\epsilon(z_n)\|(1 + \|z_n\|) \rightarrow 0.$$

In order to prove the existence of ground state solutions for the modified system (2.8), next we construct a  $(C)_{\tilde{c}_\epsilon}$ -sequence for some  $\tilde{c}_\epsilon \in [\kappa, m_\epsilon]$  via a diagonal method (see [32]), which is very important in our arguments.

**Lemma 3.6.** *Suppose that (V), (F<sub>0</sub>)-(F<sub>2</sub>) and the conditions of g are satisfied. Then there exist a constant  $\tilde{c}_\epsilon \in [\kappa, m_\epsilon]$  and a sequence  $\{z_n\} \subset E$  satisfying*

$$\Phi_\epsilon(z_n) \rightarrow \tilde{c}_\epsilon \text{ and } \|\Phi'_\epsilon(z_n)\|(1 + \|z_n\|) \rightarrow 0.$$

*Proof.* Choose  $\xi_k \in \mathcal{M}_\epsilon$  such that

$$m_\epsilon \leq \Phi_\epsilon(\xi_k) < m_\epsilon + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{3.6}$$

By Lemma 3.3-(ii),  $\|\xi_k^+\| \geq \sqrt{\frac{2m_\epsilon}{1+|V|_\infty}} > 0$ . Set  $e_k = \xi_k^+ / \|\xi_k^+\|$ . Then  $e_k \in E^+$  and  $\|e_k\| = 1$ . Form Lemma 3.4, it follows that there exists  $R_k$  such that  $\sup \Phi_\epsilon(\partial Q_k(e_k)) \leq 0$ , where

$$Q_k(e_k) = \{se_k + w : w \in E^-, s \geq 0, \|se_k + w\| \leq R_k\}, \quad k \in \mathbb{N}. \tag{3.7}$$

Hence, using Lemma 3.5 to the above set  $Q_k(e_k)$ , there exist a constant  $c_{\epsilon,k} \in [\kappa, \sup \Phi_\epsilon(Q_k(e_k))]$  and a sequence  $\{z_{k,n}\}_{n \in \mathbb{N}} \subset E$  satisfying

$$\Phi_\epsilon(z_{k,n}) \rightarrow c_{\epsilon,k} \quad \text{and} \quad \|\Phi'_\epsilon(z_{k,n})\|(1 + \|z_{k,n}\|) \rightarrow 0, \quad k \in \mathbb{N}. \tag{3.8}$$

By virtue of Lemma 3.2, one can get that

$$\Phi_\epsilon(\xi_k) \geq \Phi_\epsilon(t\xi_k + w), \quad \forall t \geq 0, w \in E^-. \tag{3.9}$$

Since  $\xi_k \in Q_k(e_k)$ , it follows from (3.7) and (3.9) that  $\Phi_\epsilon(\xi_k) = \sup \Phi_\epsilon(Q_k(e_k))$ . Hence, by (3.6) and (3.8), one has

$$\Phi_\epsilon(z_{k,n}) \rightarrow c_{\epsilon,k} < m_\epsilon + \frac{1}{k} \quad \text{and} \quad \|\Phi'_\epsilon(z_{k,n})\|(1 + \|z_{k,n}\|) \rightarrow 0, \quad k \in \mathbb{N}.$$

Now, we can choose a sequence  $\{n_k\} \subset \mathbb{N}$  such that

$$\Phi_\epsilon(z_{k,n_k}) < m_\epsilon + \frac{1}{k} \quad \text{and} \quad \|\Phi'_\epsilon(z_{k,n_k})\|(1 + \|z_{k,n_k}\|) < \frac{1}{k}, \quad k \in \mathbb{N}.$$

Let  $z_k = z_{k,n_k}, k \in \mathbb{N}$ . Then, going if necessary to a subsequence, we have

$$\Phi_\epsilon(z_k) \rightarrow \tilde{c}_\epsilon \in [\kappa, m_\epsilon] \quad \text{and} \quad \|\Phi'_\epsilon(z_k)\|(1 + \|z_k\|) \rightarrow 0.$$

□

**Lemma 3.7.** *For any  $z \in E \setminus E^-$ , then  $\mathcal{M}_\epsilon \cap E(z) \neq \emptyset$ , i.e., there exist  $t_\epsilon > 0$  and  $w_\epsilon \in E^-$  such that  $t_\epsilon z + w_\epsilon \in \mathcal{M}_\epsilon$ .*

*Proof.* Since  $E(z) = E^- \oplus \mathbb{R}^+z = E^- \oplus \mathbb{R}^+z^+ = E(z^+)$ , we may assume that  $z \in E^+$ . By Lemma 3.4, there exists  $R > 0$  such that  $\Phi_\epsilon(z) \leq 0$  for  $z \in E(z) \setminus B_R(0)$ . By Lemma 3.3-(i),  $\Phi_\epsilon(tz) > 0$  for small  $t > 0$ . Thus,  $0 < \sup \Phi_\epsilon(E(z)) < \infty$ . It is easy to see that  $\Phi_\epsilon$  is weakly upper semi-continuous on  $E(z)$ , therefore,  $\Phi_\epsilon(z_0) = \sup \Phi_\epsilon(E(z))$  for some  $z_0 \in E(z)$ . This shows that  $z_0$  is a critical point of  $\Phi_\epsilon|_{E(z)}$ , so  $\Phi'_\epsilon(z_0)z_0 = \Phi'_\epsilon(z_0)w = 0$  for all  $w \in E(z)$ . Consequently,  $z_0 \in \mathcal{M}_\epsilon \cap E(z)$ . □

**Lemma 3.8.** *For every  $\epsilon > 0$ , let  $\{z_n\}$  be a sequence such that  $\Phi_\epsilon(z_n)$  is bounded and  $(1 + \|z_n\|)\Phi'_\epsilon(z_n) \rightarrow 0$ . Then  $\{z_n\}$  has a convergent subsequence.*

*Proof.* We first show that the sequence  $\{z_n\}$  is bounded in  $E$ . In fact, suppose that  $\{z_n\}$  is a sequence such that  $\Phi_\epsilon(z_n)$  is bounded and  $(1 + \|z_n\|)\Phi'_\epsilon(z_n) \rightarrow 0$ . Then there exists a positive constant  $C > 0$ , there holds

$$C \geq \Phi_\epsilon(z_n) - \frac{1}{2}\Phi'_\epsilon(z_n)z_n = \int_{\mathbb{R}^N} \frac{1}{2}g_\epsilon(x, |z_n|)|z_n|^2 - G_\epsilon(x, |z_n|).$$

This, together with  $(g_3)$ , we obtain

$$\begin{aligned} C &\geq \int_{\mathbb{R}^N} \frac{1}{2}g_\epsilon(x, |z_n|)|z_n|^2 - G_\epsilon(x, |z_n|) \\ &= \int_{\mathbb{R}^N} \chi_\epsilon(x) \left( \frac{1}{2}f(|z_n|)|z_n|^2 - F(|z_n|) \right) \\ &\quad + \int_{\mathbb{R}^N} (1 - \chi_\epsilon(x)) \left( \frac{1}{2}\tilde{f}(|z_n|)|z_n|^2 - \tilde{F}(|z_n|) \right) \\ &\geq \int_{\mathbb{R}^N} \chi_\epsilon(x) \left( \frac{1}{2}f(|z_n|)|z_n|^2 - F(|z_n|) \right) \end{aligned} \tag{3.10}$$

On the other hand, by (V) and (2.7) we get

$$\begin{aligned} o_n(1) &= \Phi_\epsilon(z_n)(z_n^+ - z_n^-) \\ &= \|z_n\|^2 + \int_{\mathbb{R}^N} V_\epsilon(x)z_n(z_n^+ - z_n^-) - \int_{\mathbb{R}^N} g_\epsilon(x, |z_n|)z_n(z_n^+ - z_n^-) \\ &\geq \|z_n\|^2 - |V|_\infty \int_{\mathbb{R}^N} |z_n||z_n^+ - z_n^-| - \int_{\mathbb{R}^N} g_\epsilon(x, |z_n|)|z_n||z_n^+ - z_n^-| \\ &\geq \|z_n\|^2 - |V|_\infty |z_n|_2^2 - \frac{1 - |V|_\infty}{2} \int_{\mathbb{R}^N} |z_n||z_n^+ - z_n^-| \\ &\quad - \int_{\mathbb{R}^N} c_3 \chi_\epsilon(x) \left( \frac{1}{2} f(|z_n|)|z_n|^2 - F(|z_n|) \right)^{\frac{p-1}{p}} |z_n^+ - z_n^-|. \end{aligned}$$

By the Hölder inequality and the fact that  $\chi_\epsilon \in [0, 1]$ , we deduce that

$$\begin{aligned} o_n(1) &\geq \|z_n\|^2 - |V|_\infty |z_n|_2^2 - \frac{1 - |V|_\infty}{2} |z_n|_2^2 \\ &\quad - c_5 \left( \int_{\mathbb{R}^N} \chi_\epsilon(x) \left( \frac{1}{2} f(|z_n|)|z_n|^2 - F(|z_n|) \right)^{\frac{p-1}{p}} |z_n^+ - z_n^-| \right)^p. \end{aligned}$$

Taking the above inequality in (3.10), we obtain

$$\frac{1 - |V|_\infty}{2} \|z_n\|^2 \leq c_5 \|z_n\| + c_6,$$

which implies that  $\{z_n\}$  is bounded in  $E$ . Therefore, after passing to a subsequence, we may assume that  $z_n \rightharpoonup z$  in  $E$  and  $z_n \rightarrow z$  in  $L^q_{loc}$  for  $q \in [1, 2^*)$ .

Let  $w_n = z_n - z$ , then  $w_n \rightarrow 0$  in  $E$ , and  $w_n \rightarrow 0$  in  $L^q_{loc}$  for  $q \in [1, 2^*)$ . It easy to see that  $\Phi'_\epsilon(z_n)(w_n^+ - w_n^-) = o_n(1)$  and  $\Phi'_\epsilon(z)(w_n^+ - w_n^-) = 0$ . Then we have

$$\begin{aligned} \langle z_n^+, w_n^+ \rangle + \langle z_n^-, w_n^- \rangle + \int_{\mathbb{R}^N} V_\epsilon(x)z_n(w_n^+ - w_n^-) - \int_{\mathbb{R}^N} g_\epsilon(x, |z_n|)z_n(w_n^+ - w_n^-) &= o_n(1), \\ \langle z^+, w_n^+ \rangle + \langle z^-, w_n^- \rangle + \int_{\mathbb{R}^N} V_\epsilon(x)z(w_n^+ - w_n^-) - \int_{\mathbb{R}^N} g_\epsilon(x, |z|)z(w_n^+ - w_n^-) &= 0. \end{aligned} \tag{3.11}$$

By the exponential decay of  $z$  and the fact that  $w_n \rightarrow 0$  in  $L^q_{loc}$  for  $q \in [1, 2^*)$ , we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} g_\epsilon(x, |z|)z(w_n^+ - w_n^-) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} g_\epsilon(x, |z_n|)z(w_n^+ - w_n^-) = 0. \tag{3.12}$$

Subtracting the left and right sides of the two equations of (3.11) and using (3.12) we obtain

$$\|w_n\|^2 + \int_{\mathbb{R}^N} V_\epsilon(x)w_n(w_n^+ - w_n^-) = \int_{\mathbb{R}^N} g_\epsilon(x, |z_n|)w_n(w_n^+ - w_n^-) + o_n(1).$$

It follows from (V) and (2.1) that

$$(1 - |V|_\infty) \|w_n\|^2 \leq \int_{\mathbb{R}^N} \left( \chi_\epsilon(x)f(|z_n|) + (1 - \chi_\epsilon(x))\tilde{f}(|z_n|) \right) w_n(w_n^+ - w_n^-) + o_n(1).$$

Observe that  $\tilde{f} \leq \frac{1 - |V|_\infty}{2}$  and  $\chi_\epsilon \in [0, 1]$ , we get

$$\frac{1 - |V|_\infty}{2} \|w_n\|^2 \leq \int_{\mathbb{R}^N} \chi_\epsilon(x)f(|z_n|)w_n(w_n^+ - w_n^-) + o_n(1).$$

Since the support of  $\chi_\epsilon$  is bounded for every fixed  $\epsilon > 0$ , and  $w_n \rightarrow 0$  in  $L^q_{loc}$  for  $q \in [1, 2^*)$ , we know that  $w_n \rightarrow 0$  in  $E$ . So  $z_n \rightarrow z$  in  $E$ . □

Next we show the existence of ground state solutions of the modified problem (2.8).

**Lemma 3.9.** *The modified problem (2.8) possesses a ground state solution, and  $m_\epsilon$  is attained for all small  $\epsilon > 0$ .*

*Proof.* Applying Lemma 3.6, we deduce that there exists a  $(C)_{\tilde{c}_\epsilon}$ -sequence  $\{z_n\}$  of  $\Phi_\epsilon$  such that

$$\Phi_\epsilon(z_n) \rightarrow \tilde{c}_\epsilon \leq m_\epsilon \text{ and } \|\Phi'_\epsilon(z_n)\|(1 + \|z_n\|) \rightarrow 0.$$

By Lemma 3.8,  $\{z_n\}$  is bounded, then passing to a subsequence,  $z_n \rightarrow z$  in  $E$ ,  $z_n \rightarrow z$  in  $L^q$  for all  $q \in [2, 2^*)$  and  $z_n(x) \rightarrow z(x)$  a.e. on  $\mathbb{R}^N$ . Observe that, since  $z_n \rightarrow z$  in  $E$ , then  $z \neq 0$  and  $\Phi'_\epsilon(z) = 0$ . Hence,  $z$  is a nontrivial critical point of  $\Phi_\epsilon$ . Moreover, from  $(g_3)$  and Fatou's lemma, it follows that

$$\begin{aligned} m_\epsilon &\geq \tilde{c}_\epsilon = \lim_{n \rightarrow \infty} \left( \Phi_\epsilon(z_n) - \frac{1}{2} \Phi'_\epsilon(z_n) z_n \right) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left( \frac{1}{2} g_\epsilon(x, |z_n|) |z_n|^2 - G_\epsilon(x, |z_n|) \right) \\ &\geq \int_{\mathbb{R}^N} \lim_{n \rightarrow \infty} \left( \frac{1}{2} g_\epsilon(x, |z_n|) |z_n|^2 - G_\epsilon(x, |z_n|) \right) \\ &= \int_{\mathbb{R}^N} \left( \frac{1}{2} g_\epsilon(x, |z|) |z|^2 - G_\epsilon(x, |z|) \right) \\ &= \Phi_\epsilon(z) - \frac{1}{2} \Phi'_\epsilon(z) z = \Phi_\epsilon(z), \end{aligned}$$

which implies that  $\Phi_\epsilon(z) \leq m_\epsilon$ . On the other hand, by the definition of  $m_\epsilon$ , we know  $\Phi_\epsilon(z) = m_\epsilon$  and  $z$  is a ground state solution of the modified problem (2.8).  $\square$

## 4 The autonomous problem

In order to prove our main results, we need some results on related autonomous system. For the constant  $a \in (-1, 1)$ , we consider the autonomous system

$$\begin{cases} -\Delta u + \vec{b} \cdot \nabla u + u + av = f(|z|)v & \text{in } \mathbb{R}^N, \\ -\Delta v - \vec{b} \cdot \nabla v + v + au = f(|z|)u & \text{in } \mathbb{R}^N. \end{cases} \quad (\mathcal{P}_a)$$

It is well known that the solutions of system  $(\mathcal{P}_a)$  are critical points of the functional defined by

$$I_a(z) = \frac{1}{2} (\|z^+\|^2 - \|z^-\|^2) + \frac{a}{2} \int_{\mathbb{R}^N} |z|^2 - \int_{\mathbb{R}^N} F(|z|)$$

for  $z = z^+ + z^- \in E = E^+ \oplus E^-$ . We denote the generalized Nehari manifold and ground state energy as follows

$$\mathcal{M}_a := \{z \in E \setminus E^- : I'_a(z)z = 0 \text{ and } I'_a(z)w = 0 \text{ for any } w \in E^-\},$$

$$m_a := \inf_{z \in \mathcal{M}_a} I_a.$$

Similar to the proof of Lemma 3.2, by  $(F_0)$ ,  $(F_2)$  and Lemma 2.4 we can obtain

$$I_a(z) > I_a(tz + w) \text{ for } z \in \mathcal{M}_a, w \in E^- \text{ and } t \geq 0 \text{ with } z \neq tz + w. \quad (4.1)$$

It is obvious that, for each  $z \in \mathcal{M}_a$ ,  $I_a|_{E(z)}$  attains its unique maximum at  $z$ . Similarly, according to Lemma 3.7, we can prove that

$$\mathcal{M}_a \cap E(z) \neq \emptyset \text{ for any } z \in E \setminus E^-. \quad (4.2)$$

**Lemma 4.1.** *Assume that  $(F_0)$ - $(F_2)$  are satisfied, then system  $(\mathcal{P}_a)$  possesses a ground state solution  $\tilde{z}_a$  such that  $I_a(\tilde{z}_a) = m_a > 0$ .*

*Proof.* First, it is easy to check that the functional  $I_a$  satisfies all conditions of Lemma 2.3 by some standard arguments. Thus, using Lemma 2.3 and a diagonal method [32], we can construct a sequence  $\{z_n\}$  satisfies

$$I_a(z_n) \rightarrow c_a \leq m_a \text{ and } \|I'_a(z_n)\|(1 + \|z_n\|) \rightarrow 0.$$

Similar to the proof in Lemma 3.8,  $\{z_n\}$  is bounded. Moreover, there holds

$$\liminf_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |z_n|^2 > 0.$$

Otherwise, Lions' vanishing lemma [20] yields that  $z_n \rightarrow 0$  in  $L^q$  for any  $2 < q < 2^*$ . According to  $(F_0)$  and  $(F_1)$  we get

$$\int_{\mathbb{R}^N} \left( \frac{1}{2} f(|z_n|) |z_n|^2 - F(|z_n|) \right) = o_n(1),$$

and consequently

$$\begin{aligned} c_a + o_n(1) &= I_a(z_n) - \frac{1}{2} I'_a(z_n) z_n \\ &= \int_{\mathbb{R}^N} \left( \frac{1}{2} f(|z_n|) |z_n|^2 - F(|z_n|) \right) \\ &= o_n(1). \end{aligned}$$

However, this is impossible since  $c_a > \kappa > 0$ . Then, there exist  $\{y_n\} \subset \mathbb{Z}^N$  and  $\sigma > 0$  such that

$$\int_{B(y_n, 1 + \sqrt{N})} |z_n|^2 \geq \sigma.$$

Let us define  $\tilde{z}_n(x) = z_n(x + y_n)$  so that

$$\int_{B(0, 1 + \sqrt{N})} |\tilde{z}_n|^2 \geq \sigma. \tag{4.3}$$

Since system  $(\mathcal{P}_a)$  is autonomous, we have  $\|\tilde{z}_n\| = \|z_n\|$  and

$$I_a(\tilde{z}_n) \rightarrow c_a \leq m_a \text{ and } (1 + \|\tilde{z}_n\|) I'_a(\tilde{z}_n) \rightarrow 0. \tag{4.4}$$

Passing to a subsequence, we assume that  $\tilde{z}_n \rightharpoonup \tilde{z}$  in  $E$ ,  $\tilde{z}_n \rightarrow \tilde{z}$  in  $L^p_{loc}$  for  $2 \leq p < 2^*$ , and  $\tilde{z}_n(x) \rightarrow \tilde{z}(x)$  a.e. on  $\mathbb{R}^N$ . Hence it follows from (4.3) and (4.4) that  $\tilde{z} \neq 0$  and  $I'_a(\tilde{z}) = 0$ . This shows that  $\tilde{z} \in \mathcal{M}_a$  and  $I_a(\tilde{z}) \geq m_a$ . On the other hand, from Fatou's lemma, it follows that

$$\begin{aligned} m_a \geq c_a &= \lim_{n \rightarrow \infty} \left( I_a(\tilde{z}_n) - \frac{1}{2} I'_a(\tilde{z}_n) \tilde{z}_n \right) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \left( \frac{1}{2} f(|\tilde{z}_n|) |\tilde{z}_n|^2 - F(|\tilde{z}_n|) \right) \\ &\geq \int_{\mathbb{R}^N} \lim_{n \rightarrow \infty} \left( \frac{1}{2} f(|\tilde{z}_n|) |\tilde{z}_n|^2 - F(|\tilde{z}_n|) \right) \\ &= I_a(\tilde{z}) - \frac{1}{2} I'_a(\tilde{z}) \tilde{z} = I_a(\tilde{z}), \end{aligned}$$

which implies that  $I_a(\tilde{z}) \leq m_a$ . Hence  $I_a(\tilde{z}) = m_a = \inf_{z \in \mathcal{M}_a} I$  and  $\tilde{z}$  is a ground state solution of system  $(\mathcal{P}_a)$ . □

According to (V), we know that  $\nu \in (-1, 1)$ . Taking  $a = \nu$ , we replace  $m_a, I_a$  and  $\mathcal{M}_a$  by  $m_\nu, I_\nu$  and  $\mathcal{M}_\nu$ . Thus, as a special case of Lemma 4.1, the system  $(\mathcal{P}_\nu)$  has a ground state solution  $z_0$  such that  $I_\nu(z_0) = m_\nu = \inf_{z \in \mathcal{M}_\nu} I_\nu(z) > 0$ . Here, system  $(\mathcal{P}_\nu)$  is corresponding limit problem of system  $(\mathcal{P}'_\epsilon)$ . In the following, we give the relationship of the ground state energy between  $(\mathcal{P}'_\epsilon)$  and  $(\mathcal{P}_\nu)$ .

**Lemma 4.2.**  $\overline{\lim}_{\epsilon \rightarrow 0} m_\epsilon \leq m_\nu$ .

*Proof.* Let  $z$  be the ground state solution of  $(\mathcal{P}_\nu)$ . By Lemma 3.3 and Lemma 3.4, we know that  $z^+ \neq 0$ , and there exists  $R(z^+) > 0$  such that  $\Phi_\epsilon|_{\partial Q(z^+)} \leq 0$  for all small  $\epsilon > 0$ . Moreover, from Lemma 3.7, we deduce that there exist  $t_\epsilon > 0$  and  $w_\epsilon \in E^-$  such that  $z_\epsilon = t_\epsilon z + w_\epsilon \in \mathcal{M}_\epsilon \cap Q(z^+)$ , and hence

$$m_\epsilon \leq \Phi_\epsilon(z_\epsilon).$$

Observe that  $z_\epsilon \in Q(z^+)$ , then  $\|z_\epsilon\| \leq R(z^+)$ . Hence, up to a subsequence if necessary,  $t_\epsilon \rightarrow t$  and  $w_\epsilon \rightarrow w$  with  $t \geq 0$  and  $w \in E^-$ . From  $(F_2)$ , (2.1) and Lemma 3.3, we deduce that

$$\frac{1 + |V|_\infty}{2} \|t_\epsilon z^+\|^2 - \frac{1 - |V|_\infty}{2} \|t_\epsilon z^- + w_\epsilon^-\|^2 \geq \Phi_\epsilon(z_\epsilon) \geq \alpha,$$

this shows that  $t > 0$ . Noting that  $z_\epsilon \in \mathcal{M}_\epsilon$ , we get

$$\Phi'_\epsilon(z_\epsilon)z_\epsilon^+ = t_\epsilon \Phi'_\epsilon(z_\epsilon)z^+ = 0, \quad \Phi'_\epsilon(z_\epsilon)\varphi = 0 \text{ for any } \varphi \in E^-, \quad (4.5)$$

this, together with the fact  $z_\epsilon \rightarrow tz + w$  in  $E$ , we have

$$I'_\nu(tz + w)tz^+ = 0, \quad I'_\nu(tz + w)\varphi = 0 \text{ for any } \varphi \in E^-.$$

Thus by  $t > 0$ , we know  $tz + w \in \mathcal{M}_\nu \cap E(z)$ , moreover, according to (4.1), we know  $tz + w = z$  and hence  $t = 1$  and  $w = 0$ .

Next, we show that  $z_\epsilon \rightarrow z$  in  $E$ . Setting  $\phi_\epsilon = z - z_\epsilon$  and  $\tilde{h}_\epsilon(t) = \Phi_\epsilon(z_\epsilon + t\phi_\epsilon)$ , there holds

$$\begin{aligned} \Phi_\epsilon(z) - \Phi_\epsilon(z_\epsilon) &= \tilde{h}_\epsilon(1) - \tilde{h}_\epsilon(0) = \int_0^1 \tilde{h}'_\epsilon(s) ds = \int_0^1 \Phi'_\epsilon(z_\epsilon + s\phi_\epsilon)\phi_\epsilon ds \\ &= \int_0^1 \left( \int_{\mathbb{R}^N} (Az_\epsilon + V_\epsilon(x)z_\epsilon) \cdot \phi_\epsilon - \int_{\mathbb{R}^N} g_\epsilon(x, |z_\epsilon|)z_\epsilon \phi_\epsilon \right) ds \\ &\quad + \int_0^1 \left( \int_{\mathbb{R}^N} (As\phi_\epsilon + V_\epsilon(x)s\phi_\epsilon) \cdot \phi_\epsilon + \int_{\mathbb{R}^N} g_\epsilon(x, |z_\epsilon|)z_\epsilon \phi_\epsilon \right) ds \\ &\quad - \int_0^1 \left( \int_{\mathbb{R}^N} g_\epsilon(x, |z_\epsilon + s\phi_\epsilon|)(z_\epsilon + s\phi_\epsilon)\phi_\epsilon \right) ds. \end{aligned} \quad (4.6)$$

Since  $\phi_\epsilon = z - z_\epsilon = (1 - t_\epsilon)z^+ + ((1 - t_\epsilon)z^- - w_\epsilon)$ , by (4.5) we have

$$\begin{aligned} &\int_{\mathbb{R}^N} (Az_\epsilon + V_\epsilon(x)z_\epsilon) \cdot \phi_\epsilon - \int_{\mathbb{R}^N} g_\epsilon(x, |z_\epsilon|)z_\epsilon \phi_\epsilon \\ &= \Phi'_\epsilon(z_\epsilon)\phi_\epsilon = (1 - t_\epsilon)\Phi'_\epsilon(z_\epsilon)z^+ + \Phi'_\epsilon(z_\epsilon)((1 - t_\epsilon)z^- - w_\epsilon) = 0. \end{aligned} \quad (4.7)$$

Computing directly, we obtain

$$\int_0^1 \left( \int_{\mathbb{R}^N} (As\phi_\epsilon + V_\epsilon(x)s\phi_\epsilon) \cdot \phi_\epsilon \right) ds = \frac{1}{2} \int_{\mathbb{R}^N} A\phi_\epsilon \cdot \phi_\epsilon + V_\epsilon(x)|\phi_\epsilon|^2. \quad (4.8)$$

Thus, from (4.6), (4.7) and (4.8) we get

$$\begin{aligned} \int_{\mathbb{R}^N} (G_\epsilon(x, |z|) - G_\epsilon(x, |z_\epsilon|)) &= \int_0^1 \left( \int_{\mathbb{R}^N} g_\epsilon(x, |z_\epsilon + s\phi_\epsilon|)(z_\epsilon + s\phi_\epsilon)\phi_\epsilon \right) ds \\ &= \Phi_\epsilon(z_\epsilon) - \Phi_\epsilon(z) + \frac{1}{2} \int_{\mathbb{R}^N} A\phi_\epsilon \cdot \phi_\epsilon + V_\epsilon(x)|\phi_\epsilon|^2 + \int_{\mathbb{R}^N} g_\epsilon(x, |z_\epsilon|)z_\epsilon\phi_\epsilon. \end{aligned} \quad (4.9)$$

Similarly, we can also obtain

$$\int_{\mathbb{R}^N} (F(|z_\epsilon|) - F(|z|)) = I_\nu(z) - I_\nu(z_\epsilon) + \frac{1}{2} \int_{\mathbb{R}^N} A\phi_\epsilon \cdot \phi_\epsilon + \nu|\phi_\epsilon|^2 - \int_{\mathbb{R}^N} f(|z|)z\phi_\epsilon. \quad (4.10)$$

By (4.9) and (4.10) we have

$$\begin{aligned} &(\Phi_\epsilon(z_\epsilon) - I_\nu(z_\epsilon)) - (\Phi_\epsilon(z) - I_\nu(z)) \\ &= \int_{\mathbb{R}^N} (F(|z_\epsilon|) - G_\epsilon(x, |z_\epsilon|)) - \int_{\mathbb{R}^N} (F(|z|) - G_\epsilon(x, |z|)) \\ &\quad - \int_{\mathbb{R}^N} A\phi_\epsilon \cdot \phi_\epsilon - \frac{1}{2} \int_{\mathbb{R}^N} (V_\epsilon(x) + \nu)|\phi_\epsilon|^2 \\ &\quad + \int_{\mathbb{R}^N} f(|z|)z\phi_\epsilon - \int_{\mathbb{R}^N} g_\epsilon(x, |z_\epsilon|)z_\epsilon\phi_\epsilon. \end{aligned} \quad (4.11)$$

On the other hand, since

$$\Phi_\epsilon(z) = I_\nu(z) + \frac{1}{2} \int_{\mathbb{R}^N} (V_\epsilon(x) - \nu)|z|^2 + \int_{\mathbb{R}^N} F(|z|) - \int_{\mathbb{R}^N} G_\epsilon(x, |z|),$$

by a straightforward computation, we deduce that

$$\begin{aligned} &(\Phi_\epsilon(z_\epsilon) - I_\nu(z_\epsilon)) - (\Phi_\epsilon(z) - I_\nu(z)) \\ &= \int_{\mathbb{R}^N} (F(|z_\epsilon|) - G_\epsilon(x, |z_\epsilon|)) - \int_{\mathbb{R}^N} (F(|z|) - G_\epsilon(x, |z|)) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} (V_\epsilon(x) - \nu)(|z_\epsilon|^2 - |z|^2) \end{aligned} \quad (4.12)$$

According to (4.11) and (4.12), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} A\phi_\epsilon \cdot \phi_\epsilon + \frac{1}{2} \int_{\mathbb{R}^N} (V_\epsilon(x) + \nu)|\phi_\epsilon|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (V_\epsilon(x) - \nu)(|z_\epsilon|^2 - |z|^2) \\ &\quad + \int_{\mathbb{R}^N} g_\epsilon(x, |z_\epsilon|)z_\epsilon\phi_\epsilon - \int_{\mathbb{R}^N} f(|z|)z\phi_\epsilon = 0. \end{aligned} \quad (4.13)$$

Since  $\phi_\epsilon \rightarrow 0$  in  $E$  and the exponential decay of  $z$ , we know

$$\begin{aligned} \int_{\mathbb{R}^N} (V_\epsilon(x) - \nu)(|z_\epsilon|^2 - |z|^2) &= \int_{\mathbb{R}^N} (V_\epsilon(x) - \nu)|\phi_\epsilon|^2 - 2 \int_{\mathbb{R}^N} (V_\epsilon(x) - \nu)z\phi_\epsilon \\ &= \int_{\mathbb{R}^N} (V_\epsilon(x) - \nu)|\phi_\epsilon|^2 + o_\epsilon(1), \end{aligned}$$

and

$$\int_{\mathbb{R}^N} f(|z|)z\phi_\epsilon = o_\epsilon(1).$$

This, jointly with (4.13), implies that

$$\int_{\mathbb{R}^N} A\phi_\epsilon \cdot \phi_\epsilon + \int_{\mathbb{R}^N} V_\epsilon(x)|\phi_\epsilon|^2 + \int_{\mathbb{R}^N} g_\epsilon(x, |z_\epsilon|)z_\epsilon\phi_\epsilon = o_\epsilon(1). \tag{4.14}$$

On the one hand, by (2.1) and the fact that  $t_\epsilon \rightarrow 1$  we have

$$\begin{aligned} \int_{\mathbb{R}^N} A\phi_\epsilon \cdot \phi_\epsilon + \int_{\mathbb{R}^N} V_\epsilon(x)|\phi_\epsilon|^2 &\leq \|\phi_\epsilon^+\|^2 - \|\phi_\epsilon^-\|^2 + |V|_\infty(|\phi_\epsilon^+|^2 + |\phi_\epsilon^-|^2) \\ &\leq (1 + |V|_\infty)\|\phi_\epsilon^+\|^2 - (1 - |V|_\infty)\|\phi_\epsilon^-\|^2 \\ &= -(1 - |V|_\infty)\|\phi_\epsilon^-\|^2 + o_\epsilon(1). \end{aligned}$$

On the other hand, since  $g_\epsilon(x, |z_\epsilon|)|z_\epsilon|^2 \rightarrow f(|z||z|^2)$  a.e.  $x \in \mathbb{R}^N$ , by Fatou’s lemma we get

$$\begin{aligned} \int_{\mathbb{R}^N} g_\epsilon(x, |z_\epsilon|)z_\epsilon\phi_\epsilon &= \int_{\mathbb{R}^N} g_\epsilon(x, |z_\epsilon|)z_\epsilon(z - z_\epsilon) \\ &= \int_{\mathbb{R}^N} f(|z||z|^2) - \int_{\mathbb{R}^N} g_\epsilon(x, |z_\epsilon|)|z_\epsilon|^2 + o_\epsilon(1) \leq o_\epsilon(1). \end{aligned}$$

Consequently, from (4.14) we know that  $\|\phi_\epsilon^-\| \rightarrow 0$ , and together with  $t_\epsilon \rightarrow 1$ , then  $\phi_\epsilon \rightarrow 0$  in  $E$ . So we have  $z_\epsilon \rightarrow z$  in  $E$  and

$$\overline{\lim}_{\epsilon \rightarrow 0} m_\epsilon \leq \overline{\lim}_{\epsilon \rightarrow 0} \Phi_\epsilon(z_\epsilon) = I_V(z) = m_V.$$

The proof is completed. □

## 5 Proof of the main result

In this section we give the proof of the main results. Let

$$\mathcal{K}_\epsilon := \{z \in E \setminus \{0\} : \Phi'_\epsilon(z) = 0\}$$

denote the set of all critical points of  $\Phi_\epsilon$ . To describe some properties of ground state solutions, by using the standard bootstrap argument (see, e.g., [16, 29] for the iterative steps) we can obtain the following regularity result.

**Lemma 5.1.** *If  $z \in \mathcal{K}_\epsilon$  with  $|\Phi_\epsilon(z)| \leq C_1$  and  $|z|_2 \leq C_2$ , then, for any  $q \in [2, +\infty)$ ,  $z \in H^{2,q}(\mathbb{R}^N)$  with  $\|z\|_{H^{2,q}} \leq C_q$ , where  $C_q$  depends only on  $C_1, C_2$  and  $q$ .*

Let  $\mathcal{L}_\epsilon$  be the set of all ground state solutions of  $\Phi_\epsilon$ . If  $z \in \mathcal{L}_\epsilon$ , then by Lemma 4.2, we know  $\Phi_\epsilon(z) = m_\epsilon$  is uniformly bounded for all small  $\epsilon > 0$ . Moreover, by a similar argument as Lemma 3.8, we can also show that  $\mathcal{L}_\epsilon$  is bounded in  $E$ , hence,  $|z|_2 \leq C_2$  for all  $z \in \mathcal{L}_\epsilon$  and some  $C_2 > 0$ . Therefore, as a consequence of Lemma 5.1 we see that, for each  $q \in [2, +\infty)$ , there is  $C_q$  such that

$$\|z\|_{H^{2,q}} \leq C_q \text{ for all } z \in \mathcal{L}_\epsilon.$$

This, together with the Sobolev embedding theorem, implies that there is  $C_\infty > 0$  with

$$|z|_\infty \leq C_\infty \text{ for all } z \in \mathcal{L}_\epsilon. \tag{5.1}$$

**Lemma 5.2.** *Assume that  $(F_1)$  and  $|\vec{b}| < 2$  hold, then there is  $C_0 > 0$  independent of  $x$  and  $z \in \mathcal{L}_\epsilon$  such that*

$$|z(x)| \leq C_0 \left( \int_{B_1(x)} |z(y)|^2 dy \right)^{1/2}, \quad x \in \mathbb{R}^N, \tag{5.2}$$

where  $B_1(x) = \{y : |y - x| \leq 1\}$ .

The proof of Lemma 5.2 can be found in [43, 46], here we omit the details.

**Lemma 5.3.** *Assume that  $z_\epsilon \in \mathcal{L}_\epsilon$ . Then  $|z_\epsilon|$  attains its maximum at  $x_\epsilon$ . Moreover, if we set  $\tilde{z}_\epsilon = z_\epsilon(x + x_\epsilon)$ , up to a subsequence, as  $\epsilon \rightarrow 0$ , there hold*

$$\epsilon x_\epsilon \rightarrow \mathcal{V} \text{ and } \tilde{z}_\epsilon \rightarrow z_0 \text{ in } H^2(\mathbb{R}^N),$$

and  $z_0$  is a ground state solution of

$$\begin{cases} -\Delta u + \vec{b} \cdot \nabla u + u + v v = f(|z|)v & \text{in } \mathbb{R}^N, \\ -\Delta v - \vec{b} \cdot \nabla v + v + v u = f(|z|)u & \text{in } \mathbb{R}^N. \end{cases}$$

*Proof.* Let  $\epsilon \rightarrow 0$ ,  $z_\epsilon \in \mathcal{L}_\epsilon$ , we first claim that there exists  $\sigma > 0$  such that

$$\liminf_{\epsilon \rightarrow 0} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |z_\epsilon|^2 \geq \sigma.$$

Arguing indirectly, we assume

$$\liminf_{\epsilon \rightarrow 0} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |z_\epsilon|^2 = 0.$$

Similar to the proof of Lemma 3.8, we can obtain  $\{z_\epsilon\}$  is bounded in  $E$ . Then by Lions' vanishing lemma [20],  $z_\epsilon \rightarrow 0$  in  $L^q$  for  $q \in (2, 2^*)$ , and by  $(F_1)$  and  $(g_2)$  we have

$$\int_{\mathbb{R}^N} \left( \frac{1}{2} g_\epsilon(x, |z_\epsilon|) |z_\epsilon|^2 - G_\epsilon(x, |z_\epsilon|) \right) = o_\epsilon(1).$$

Thus, there holds

$$m_\epsilon = \Phi_\epsilon(z_\epsilon) - \frac{1}{2} \Phi'_\epsilon(z_\epsilon) z_\epsilon = \int_{\mathbb{R}^N} \left( \frac{1}{2} g_\epsilon(x, |z_\epsilon|) |z_\epsilon|^2 - G_\epsilon(x, |z_\epsilon|) \right) = o_\epsilon(1),$$

this implies that a contradiction to Lemma 3.3-(i). Therefore, there exists  $\{y_\epsilon\} \subset \mathbb{R}^N$  such that

$$\int_{B(y_\epsilon,1)} |z_\epsilon|^2 \geq \frac{\sigma}{2}. \tag{5.3}$$

Setting  $\hat{z}_\epsilon = z_\epsilon(x + y_\epsilon)$ , up to a subsequence,  $\hat{z}_\epsilon \rightarrow \hat{z}$  in  $E$ , and  $\hat{z} \neq 0$  by (5.3). It is obvious that  $\hat{z}_\epsilon = (\hat{u}_\epsilon, \hat{v}_\epsilon)$  satisfies

$$\begin{cases} -\Delta \hat{u}_\epsilon + \vec{b} \cdot \nabla \hat{u}_\epsilon + \hat{u}_\epsilon + V_\epsilon(x + y_\epsilon) \hat{v}_\epsilon = g_\epsilon(x + y_\epsilon, |\hat{z}_\epsilon|) \hat{v}_\epsilon & \text{in } \mathbb{R}^N, \\ -\Delta \hat{v}_\epsilon - \vec{b} \cdot \nabla \hat{v}_\epsilon + \hat{v}_\epsilon + V_\epsilon(x + y_\epsilon) \hat{u}_\epsilon = g_\epsilon(x + y_\epsilon, |\hat{z}_\epsilon|) \hat{u}_\epsilon & \text{in } \mathbb{R}^N, \end{cases} \tag{5.4}$$

and for any  $\varphi \in E$

$$\int_{\mathbb{R}^N} (A \hat{z}_\epsilon + V_\epsilon(x + y_\epsilon) \hat{z}_\epsilon - g_\epsilon(x + y_\epsilon, |\hat{z}_\epsilon|) \hat{z}_\epsilon) \cdot \varphi = 0. \tag{5.5}$$

Now, the rest of the proof is divided into several steps.

**Step 1.**  $\epsilon y_\epsilon \rightarrow \mathcal{V}$  as  $\epsilon \rightarrow 0$ . We assume by contradiction that  $\epsilon y_\epsilon \rightarrow \infty$  or  $\epsilon y_\epsilon \rightarrow y_0 \notin \Omega^\delta$  as  $\epsilon \rightarrow 0$ . Then  $V_\epsilon(x + y_\epsilon) \rightarrow V_\infty$  with  $|V_\infty| \leq |V|_\infty$  and  $\chi_\epsilon(x + y_\epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$  uniformly hold on bounded domain. Taking test function  $\varphi \in C_0^\infty(\mathbb{R}^N)$ , by (5.5) we have

$$\begin{aligned} 0 &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} (A \hat{z}_\epsilon + V_\epsilon(x + y_\epsilon) \hat{z}_\epsilon - g_\epsilon(x + y_\epsilon, |\hat{z}_\epsilon|) \hat{z}_\epsilon) \cdot \varphi \\ &= \int_{\mathbb{R}^N} (A \hat{z} + V_\infty \hat{z} - \tilde{f}(|\hat{z}|) \hat{z}) \cdot \varphi, \end{aligned}$$

and hence  $\hat{z} = (\hat{u}, \hat{v})$  satisfies

$$\begin{cases} -\Delta \hat{u} + \vec{b} \cdot \nabla \hat{u} + \hat{u} + V_\infty \hat{v} = \tilde{f}(|\hat{z}|)\hat{v} & \text{in } \mathbb{R}^N, \\ -\Delta \hat{v} - \vec{b} \cdot \nabla \hat{v} + \hat{v} + V_\infty \hat{u} = \tilde{f}(|\hat{z}|)\hat{u} & \text{in } \mathbb{R}^N. \end{cases}$$

Denoting by  $\hat{\Phi}$  the associate energy functional, then, using (2.1) and the fact that  $\tilde{f}(s) \leq \frac{1-|V|_\infty}{2}$  we obtain

$$\begin{aligned} 0 &= \hat{\Phi}'(\hat{z})(\hat{z}^+ - \hat{z}^-) \\ &= \|\hat{z}\|^2 + V_\infty \int_{\mathbb{R}^N} \hat{z}(\hat{z}^+ - \hat{z}^-) - \int_{\mathbb{R}^N} \tilde{f}(|\hat{z}|)\hat{z}(\hat{z}^+ - \hat{z}^-) \\ &\geq \|\hat{z}\|^2 - |V|_\infty |\hat{z}|_2^2 - \frac{1 - |V|_\infty}{2} |\hat{z}|_2^2 \\ &\geq \frac{1 - |V|_\infty}{2} \|\hat{z}\|^2. \end{aligned}$$

Obviously, this is a contradiction since  $\hat{z} \neq 0$ , and hence  $\epsilon y_\epsilon \rightarrow y_0 \in \Omega^\delta$  as  $\epsilon \rightarrow 0$ .

Let  $g_\infty(s) = \chi(y_0)f(s) + (1 - \chi(y_0))\tilde{f}(s)$ , where  $\chi$  is the cut-off function defined in Section 2. Then similarly,  $\hat{z} = (\hat{u}, \hat{v})$  satisfies

$$\begin{cases} -\Delta \hat{u} + \vec{b} \cdot \nabla \hat{u} + \hat{u} + V(y_0)\hat{v} = g_\infty(|\hat{z}|)\hat{v} & \text{in } \mathbb{R}^N, \\ -\Delta \hat{v} - \vec{b} \cdot \nabla \hat{v} + \hat{v} + V(y_0)\hat{u} = g_\infty(|\hat{z}|)\hat{u} & \text{in } \mathbb{R}^N, \end{cases}$$

and

$$\Phi_\infty(z) = \frac{1}{2} (\|z^+\|^2 - \|z^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V(y_0)|z|^2 - \int_{\mathbb{R}^N} G_\infty(|z|)$$

denotes the associate energy functional with  $G_\infty(t) = \int_0^t g_\infty(s)ds$  and setting

$$\mathcal{M}_\infty := \{z \in E \setminus E^- : \Phi'_\infty(z)z = 0 \text{ and } \Phi'_\infty(z)w = 0 \text{ for any } w \in E^-\}.$$

Observe that

$$\|\hat{z}^+\|^2 - \|\hat{z}^-\|^2 + V(y_0)|\hat{z}|_2^2 = \int_{\mathbb{R}^N} g_\infty(|\hat{z}|)|\hat{z}|^2 \geq 0,$$

which implies that

$$(1 + |V|_\infty)\|\hat{z}^+\|^2 - (1 - |V|_\infty)\|\hat{z}^-\|^2 \geq 0.$$

This, together with  $\hat{z} \neq 0$ , we have  $\hat{z}^+ \neq 0$ . By the definition of  $g_\infty$ ,  $g_\infty$  satisfies the assumptions of Lemma 2.5, and hence by the same procedure as the proof in Lemma 3.2, we know that if  $z \in \mathcal{M}_\infty$ , then  $\Phi_\infty|_{E(z)}$  attains its maximum at  $z$ . Thus by  $\hat{z}^+ \neq 0$  and  $\Phi'_\infty(\hat{z}) = 0$ , we find  $\hat{z} \in \mathcal{M}_\infty$ , and  $\Phi_\infty(\hat{z}) \geq \Phi_\infty(z)$  for any  $z \in E(\hat{z})$ . Furthermore, it follows from Lemma 3.7 that there exist  $t > 0$  and  $w \in E^-$  such that  $t\hat{z} + w \in \mathcal{M}_v$ . Note that, we deduce from  $(g_2)$  that  $G_\infty(s) \leq F(s)$  for all  $s \geq 0$ . Therefore, using the fact, we obtain

$$\Phi_\infty(\hat{z}) \geq \Phi_\infty(t\hat{z} + w) \geq I_v(t\hat{z} + w) + \frac{V(y_0) - v}{2}|t\hat{z} + w|_2^2 \geq m_v + \frac{V(y_0) - v}{2}|t\hat{z} + w|_2^2. \tag{5.6}$$

On the other hand, since  $g_\epsilon(x + y_\epsilon, |\hat{z}_\epsilon(x)|)|\hat{z}_\epsilon(x)|^2 \rightarrow g_\infty(|\hat{z}(x)|)|\hat{z}(x)|^2$  a.e. on  $\mathbb{R}^N$ , using Fatou's lemma we get

$$\begin{aligned} m_\epsilon &= \lim_{\epsilon \rightarrow 0} \left( \Phi_\epsilon(z_\epsilon) - \frac{1}{2} \Phi'_\epsilon(z_\epsilon)z_\epsilon \right) \\ &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N} \frac{1}{2} g_\epsilon(x + y_\epsilon, |\hat{z}_\epsilon|)|\hat{z}_\epsilon|^2 - G_\epsilon(x + y_\epsilon, |\hat{z}_\epsilon|) \\ &\geq \int_{\mathbb{R}^N} \lim_{\epsilon \rightarrow 0} \left( \frac{1}{2} g_\epsilon(x + y_\epsilon, |\hat{z}_\epsilon|)|\hat{z}_\epsilon|^2 - G_\epsilon(x + y_\epsilon, |\hat{z}_\epsilon|) \right) \\ &= \int_{\mathbb{R}^N} \frac{1}{2} g_\infty(|\hat{z}|)|\hat{z}|^2 - G_\infty(|\hat{z}|) \\ &= \Phi_\infty(\hat{z}) - \frac{1}{2} \Phi'_\infty(\hat{z})\hat{z} = \Phi_\infty(\hat{z}). \end{aligned} \tag{5.7}$$

This, jointly with (5.6), we obtain

$$\lim_{\epsilon \rightarrow 0} m_\epsilon \geq m_\nu + \frac{V(y_0) - \nu}{2} |t\hat{z} + w|_2^2.$$

Moreover, according to Lemma 4.2, we know that  $V(y_0) - \nu \leq 0$ . Since  $y_0 \in \Omega^\delta$ , it follows from (2.3) that  $V(y_0) = \nu$  and  $y_0 \in \mathcal{V}$ . So  $\epsilon y_\epsilon \rightarrow \mathcal{V}$  as  $\epsilon \rightarrow 0$ .

Step 2. Up to a subsequence, we show that  $\hat{z}_\epsilon \rightarrow \hat{z}$  in  $H^2(\mathbb{R}^N)$ . By Step 1, we have  $y_0 \in \mathcal{V}$  and  $\chi(y_0) = 1$ , then  $g_\infty(s) = f(s)$ ,  $\Phi_\infty(\hat{z}) = I_\nu(\hat{z})$  and hence  $\hat{z}$  is a nontrivial solution of  $(\mathcal{P}_\nu)$  with  $I_\nu(\hat{z}) \geq m_\nu$ . Moreover, from (5.7) and Lemma 4.2, it follows that  $I_\nu(\hat{z}) \leq m_\nu$ . Thus,  $I_\nu(\hat{z}) = m_\nu$  and  $\hat{z}$  is a ground state solution of  $(\mathcal{P}_\nu)$ .

Let  $\zeta : [0, +\infty) \rightarrow [0, 1]$  be a smooth function satisfying  $\zeta(t) = 1$  for  $t \leq 1$ ,  $\zeta(t) = 0$  for  $t \geq 2$ . At this point we make use of techniques in [10]. Define  $\hat{z}_j(x) = \zeta(2|x|/j)\hat{z}(x)$ , then we have

$$\|\hat{z}_j - \hat{z}\| \rightarrow 0 \text{ and } |\hat{z}_j - \hat{z}|_q \rightarrow 0 \text{ for } q \in [2, 2^*], \text{ as } j \rightarrow \infty. \tag{5.8}$$

From an argument in [10, Lemma 5.7], it follows that there exists a subsequence  $\hat{z}_{\epsilon_j}$  such that, for any  $\epsilon > 0$ , there exists  $r_\epsilon > 0$  satisfying

$$\overline{\lim}_{j \rightarrow \infty} \int_{B_j(0) \setminus B_r(0)} |\hat{z}_{\epsilon_j}|^p \leq \epsilon \text{ for } r \geq r_\epsilon \text{ and } p \in [2, 2^*].$$

Setting  $w_j = \hat{z}_{\epsilon_j} - \hat{z}_j$ , it is not difficult to verify that

$$\int_{\mathbb{R}^N} G_{\epsilon_j}(x + y_{\epsilon_j}, |\hat{z}_{\epsilon_j}|) - G_{\epsilon_j}(x + y_{\epsilon_j}, |w_j|) - G_{\epsilon_j}(x + y_{\epsilon_j}, |\hat{z}_j|) = o_j(1), \tag{5.9}$$

and

$$\int_{\mathbb{R}^N} (g_{\epsilon_j}(x + y_{\epsilon_j}, |\hat{z}_{\epsilon_j}|)\hat{z}_{\epsilon_j} - g_{\epsilon_j}(x + y_{\epsilon_j}, |w_j|)w_j - g_{\epsilon_j}(x + y_{\epsilon_j}, |\hat{z}_j|)\hat{z}_j) \varphi = o_j(1) \tag{5.10}$$

uniformly for  $\varphi \in E$  with  $\|\varphi\| \leq 1$  (see [10]). Moreover, by (5.8) and the fact  $\epsilon_j y_{\epsilon_j} \rightarrow \mathcal{V}$ , we get

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} V_{\epsilon_j}(x + y_{\epsilon_j})\hat{z}_{\epsilon_j}\hat{z}_j = \lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} V_{\epsilon_j}(x + y_{\epsilon_j})|\hat{z}_j|^2 = \nu|\hat{z}|_2^2, \tag{5.11}$$

and

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^N} G_{\epsilon_j}(x + y_{\epsilon_j}, |\hat{z}_j|) = \int_{\mathbb{R}^N} F(|\hat{z}|). \tag{5.12}$$

We denote by  $\tilde{\Phi}_\epsilon$  the energy functional corresponding to (5.4), it follows from (5.8), (5.9), (5.11), (5.12) and Lemma 4.2 that

$$\begin{aligned} \tilde{\Phi}_{\epsilon_j}(w_j) &= \frac{1}{2}(\|w_j^+\|^2 - \|w_j^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_{\epsilon_j}(x + y_{\epsilon_j})|w_j|^2 - \int_{\mathbb{R}^N} G_{\epsilon_j}(x + y_{\epsilon_j}, |w_j|) \\ &= \frac{1}{2}(\|\hat{z}_{\epsilon_j}^+\|^2 - \|\hat{z}_{\epsilon_j}^-\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_{\epsilon_j}(x + y_{\epsilon_j})|\hat{z}_{\epsilon_j}|^2 - \int_{\mathbb{R}^N} G_{\epsilon_j}(x + y_{\epsilon_j}, |\hat{z}_{\epsilon_j}|) \\ &\quad - \frac{1}{2}(\|\hat{z}_j^+\|^2 - \|\hat{z}_j^-\|^2) - \frac{1}{2} \int_{\mathbb{R}^N} V_{\epsilon_j}(x + y_{\epsilon_j})(2\hat{z}_{\epsilon_j}\hat{z}_j - |\hat{z}_j|^2) + \int_{\mathbb{R}^N} G_{\epsilon_j}(x + y_{\epsilon_j}, |\hat{z}_j|) \\ &\quad + \int_{\mathbb{R}^N} G_{\epsilon_j}(x + y_{\epsilon_j}, |\hat{z}_{\epsilon_j}|) - G_{\epsilon_j}(x + y_{\epsilon_j}, |w_j|) - G_{\epsilon_j}(x + y_{\epsilon_j}, |\hat{z}_j|) \\ &= \Phi_{\epsilon_j}(z_{\epsilon_j}) - \left( \frac{1}{2}(\|\hat{z}^+\|^2 - \|\hat{z}^-\|^2) - \frac{1}{2} \int_{\mathbb{R}^N} \nu|\hat{z}|^2 + \int_{\mathbb{R}^N} F(|\hat{z}|) \right) + o_j(1) \\ &= \Phi_{\epsilon_j}(z_{\epsilon_j}) - I_\nu(\hat{z}) + o_j(1) = m_{\epsilon_j} - m_\nu + o_j(1) \leq o_j(1). \end{aligned}$$

Similarly, using (5.10), we have

$$\tilde{\Phi}'_{\epsilon_j}(w_j)\varphi = \Phi'_{\epsilon_j}(z_{\epsilon_j})\varphi - I'_V(\hat{z})\varphi + o_j(1)\|\varphi\| = o_j(1)$$

uniformly holds for  $\varphi \in E$  with  $\|\varphi\| \leq 1$ . Then, from (g<sub>3</sub>), we deduce that

$$\begin{aligned} o_j(1) &\geq \tilde{\Phi}_{\epsilon_j}(w_j) - \frac{1}{2}\tilde{\Phi}'_{\epsilon_j}(w_j)w_j \\ &= \int_{\mathbb{R}^N} \frac{1}{2}g_{\epsilon_j}(x + y_{\epsilon_j}, |w_j|)|w_j|^2 - G_{\epsilon_j}(x + y_{\epsilon_j}, |w_j|) \\ &\geq \int_{\mathbb{R}^N} \chi_{\epsilon_j}(x) \left( \frac{1}{2}f(|w_j|)|w_j|^2 - F(|w_j|) \right). \end{aligned} \tag{5.13}$$

On the other hand, from (2.7), we deduce that

$$\begin{aligned} o_j(1) &\geq \tilde{\Phi}'_{\epsilon_j}(w_j)(w_j^+ - w_j^-) \\ &= \|w_j\|^2 + \int_{\mathbb{R}^N} V_{\epsilon_j}(x + y_{\epsilon_j})w_j(w_j^+ - w_j^-) - \int_{\mathbb{R}^N} g_{\epsilon_j}(x + y_{\epsilon_j}, |w_j|)w_j(w_j^+ - w_j^-) \\ &\geq \|w_j\|^2 - |V|_{\infty}|w_j|_2^2 - \frac{1 - |V|_{\infty}}{2}|w_j|_2^2 \\ &\quad - c_3 \int_{\mathbb{R}^N} \chi_{\epsilon_j}(x) \left( \frac{1}{2}f(|w_j|)|w_j|^2 - F(|w_j|) \right)^{\frac{p-1}{p}} |w_j^+ - w_j^-|. \end{aligned}$$

By (2.1), (5.13), the Hölder inequality and the fact  $\chi_{\epsilon_j} \in [0, 1]$ , we obtain

$$\begin{aligned} \frac{1 - |V|_{\infty}}{2}\|w_j\|^2 &\leq c_3 \left( \int_{\mathbb{R}^N} \chi_{\epsilon_j}(x) \left( \frac{1}{2}f(|w_j|)|w_j|^2 - F(|w_j|) \right)^{\frac{p-1}{p}} \right)^{\frac{p}{p-1}} |w_j^+ - w_j^-|_p + o_j(1) \\ &= o_j(1). \end{aligned}$$

This shows that  $w_j = \hat{z}_{\epsilon_j} - \hat{z}_j \rightarrow 0$  in  $E$ , moreover, by (5.8), we know that  $\hat{z}_{\epsilon_j} \rightarrow \hat{z}$  in  $E$ . Since  $\hat{z}_{\epsilon_j}$  and  $\hat{z}$  are solution of (2.8) and (P<sub>v</sub>), respectively. Using the notations as in (1.6), we have

$$A\hat{z}_{\epsilon_j} = (g_{\epsilon_j}(x + y_{\epsilon_j}, |\hat{z}_{\epsilon_j}|) - V_{\epsilon_j}(x + y_{\epsilon_j}))\hat{z}_{\epsilon_j} \text{ and } A\hat{z} = (f(|\hat{z}|) - v)\hat{z}.$$

Thus, by (5.1), the exponential decay of  $\hat{z}$  and the fact that  $\hat{z}_{\epsilon_j} \rightarrow \hat{z}$  in  $E$ , it is easy to show that

$$\begin{aligned} |A(\hat{z}_{\epsilon_j} - \hat{z})|_2^2 &= \int_{\mathbb{R}^N} |g_{\epsilon_j}(x + y_{\epsilon_j}, |\hat{z}_{\epsilon_j}|)\hat{z}_{\epsilon_j} - f(|\hat{z}|)\hat{z} + v\hat{z} - V_{\epsilon_j}(x + y_{\epsilon_j})\hat{z}_{\epsilon_j}|^2 \\ &\leq \int_{\mathbb{R}^N} |g_{\epsilon_j}(x + y_{\epsilon_j}, |\hat{z}_{\epsilon_j}|)\hat{z}_{\epsilon_j} - f(|\hat{z}|)\hat{z}|^2 \\ &\quad + \int_{\mathbb{R}^N} |v\hat{z} - V_{\epsilon_j}(x + y_{\epsilon_j})\hat{z}_{\epsilon_j}|^2 \\ &= o_j(1), \end{aligned}$$

which implies that  $\hat{z}_{\epsilon_j} \rightarrow \hat{z}$  in  $H^2(\mathbb{R}^N)$ .

Step 3. We claim that  $|z_{\epsilon}|$  attains its maximum at  $x_{\epsilon}$ , and up to a subsequence, as  $\epsilon \rightarrow 0$

$$\epsilon x_{\epsilon} \rightarrow \mathcal{V} \text{ and } \tilde{z}_{\epsilon} := z_{\epsilon}(x + x_{\epsilon}) \rightarrow z_0 \text{ in } H^2(\mathbb{R}^N),$$

and  $z_0$  is a ground state solution of (P<sub>v</sub>).

As mentioned previously, there exists  $\sigma > 0$  such that

$$\liminf_{\epsilon \rightarrow 0} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |z_\epsilon|^2 \geq \sigma,$$

then we know that  $|z_\epsilon|_\infty$  is uniformly bounded away from zero by  $c_\sigma > 0$  for all small  $\epsilon$ . While, for any fixed  $\epsilon > 0$ , it follows from Lemma 5.2 that there exists  $R_\epsilon > 0$  such that  $|z_\epsilon(x)| \leq \frac{1}{2}c_\sigma$  for  $|x| \geq R_\epsilon$ . So  $|z_\epsilon|$  can attain its maximum at point  $x_\epsilon$ . Setting  $k_\epsilon = x_\epsilon - y_\epsilon$ , then  $k_\epsilon$  is the maximum point of  $|\hat{z}_\epsilon|$  and hence  $|\hat{z}_\epsilon(k_\epsilon)| \geq c_\sigma > 0$  for all small  $\epsilon > 0$ . Next we show that  $\{k_\epsilon\}$  is bounded. Indeed, suppose to the contrary that there is a subsequence  $|k_\epsilon| \rightarrow \infty$  as  $\epsilon \rightarrow 0$ , then, by Lemma 5.2 and  $\hat{z}_\epsilon \rightarrow \hat{z}$  in  $E$ , we have

$$\begin{aligned} c_\sigma \leq |\hat{z}_\epsilon(k_\epsilon)| &\leq C_0 \left( \int_{B_1(k_\epsilon)} |\hat{z}_\epsilon|^2 \right)^{\frac{1}{2}} \\ &\leq C_0 \left( \int_{B_1(k_\epsilon)} |\hat{z}_\epsilon - \hat{z}|^2 \right)^{\frac{1}{2}} + C_0 \left( \int_{B_1(k_\epsilon)} |\hat{z}|^2 \right)^{\frac{1}{2}} \\ &\leq \tilde{C} \left( \int_{\mathbb{R}^N} |\hat{z}_\epsilon - \hat{z}|^2 \right)^{1/2} + C_0 \left( \int_{B_1(k_\epsilon)} |\hat{z}|^2 \right)^{\frac{1}{2}} \rightarrow 0, \end{aligned}$$

which implies a contradiction. Thus, for  $x_\epsilon = y_\epsilon + k_\epsilon$ , by Step 1, up to a subsequence, we get

$$\epsilon x_\epsilon = \epsilon y_\epsilon + \epsilon k_\epsilon \rightarrow \mathcal{V} \text{ as } \epsilon \rightarrow 0.$$

Moreover, by (5.3) we can choose  $r > 0$  such that

$$\int_{B_r(x_\epsilon)} |z_\epsilon|^2 = \int_{B_r(y_\epsilon+k_\epsilon)} |z_\epsilon|^2 \geq \int_{B_1(y_\epsilon)} |z_\epsilon|^2 \geq \frac{\sigma}{2}.$$

Note that we repeat the process of proof in Step 1, passing to a subsequence,  $\epsilon x_\epsilon \rightarrow \mathcal{V}$  as  $\epsilon \rightarrow 0$ . Therefore, similar to the process of proof in Step 2, we can prove that, passing to a subsequence,  $\tilde{z}_\epsilon = z_\epsilon(x + x_\epsilon) \rightarrow z_0$  in  $H^2(\mathbb{R}^N)$ , and  $z_0$  is a ground state solution of  $(\mathcal{P}_\nu)$ . The proof is completed.  $\square$

**Lemma 5.4.** *There are  $c, C > 0$  such that for all small  $\epsilon > 0$ , there holds*

$$|z_\epsilon(x)| \leq C \exp\left(-\frac{C}{2}|x - x_\epsilon|\right).$$

*Proof.* Firstly, we assume that  $z$  is a solution of the following system

$$\begin{cases} -\Delta u + \vec{b} \cdot \nabla u + u + V(x)v = f(|z|)v & \text{in } \mathbb{R}^N, \\ -\Delta v - \vec{b} \cdot \nabla v + v + V(x)u = f(|z|)u & \text{in } \mathbb{R}^N, \end{cases} \tag{5.14}$$

where  $|\vec{b}| < 2$ . Recall that  $z = (u, v)$  and  $|z|^2 = u^2 + v^2$ , then

$$D_i(|z|^2) = 2|z|D_i(|z|) = 2|z| \frac{uD_iu + vD_iv}{|z|} = 2(uD_iu + vD_iv)$$

and

$$D_{ii}(|z|^2) = 2D_i(uD_iu + vD_iv) = 2(D_iuD_iu + uD_{ii}u + D_ivD_iv + vD_{ii}v)$$

for  $i = 1, 2, \dots, N$ . This yields that

$$\begin{aligned} \Delta|z|^2 &= \sum_{i=1}^N D_{ii}(|z|^2) = 2 \sum_{i=1}^N (D_iuD_iu + uD_{ii}u + D_ivD_iv + vD_{ii}v) \\ &= 2(|\nabla u|^2 + |\nabla v|^2 + u\Delta u + v\Delta v) = 2(|\nabla z|^2 + z\Delta z), \end{aligned}$$

where  $\nabla z = (\nabla u, \nabla v)$ . Since  $z = (u, v)$  is a solution of system (5.14) and  $|\vec{b}| < 2$ , then

$$\begin{aligned} \Delta|z|^2 &= 2(|\nabla z|^2 + z\Delta z) \\ &= 2\left(|u|^2 + |v|^2 + 2V(x)uv - 2f(|z|)uv + \vec{b} \cdot \nabla uu - \vec{b} \cdot \nabla vv + |\nabla z|^2\right) \\ &\geq 2\left(|z|^2 - |V|_\infty(|u|^2 + |v|^2) - f(|z|)(|u|^2 + |v|^2) + \vec{b} \cdot \nabla uu - \vec{b} \cdot \nabla vv + |\nabla z|^2\right) \\ &\geq 2\left(|z|^2 - |V|_\infty|z|^2 - f(|z|)|z|^2 - |\vec{b}| \left(\frac{|\nabla u|^2 + |u|^2}{2} + \frac{|\nabla v|^2 + |v|^2}{2}\right) + |\nabla z|^2\right) \tag{5.15} \\ &= 2\left(|z|^2 - |V|_\infty|z|^2 - f(|z|)|z|^2 - \frac{|\vec{b}|}{2}|z|^2 - \frac{|\vec{b}|}{2}|\nabla z|^2 + |\nabla z|^2\right) \\ &\geq 2\left(|z|^2 - |V|_\infty|z|^2 - f(|z|)|z|^2 - \frac{|\vec{b}|}{2}|z|^2\right). \end{aligned}$$

Since  $\tilde{z}_\epsilon$  is a ground state solutions of (2.8), then it follows from Lemma 5.1 that  $\tilde{z}_\epsilon \in H^{2,q}$ . We replace  $V$  and  $f$  in (5.14) by  $V_\epsilon$  and  $g_\epsilon$  in (2.8), and by (5.15) we have, for all small  $\epsilon > 0$

$$\Delta|\tilde{z}_\epsilon|^2 \geq 2\left(|\tilde{z}_\epsilon|^2 - |V|_\infty|\tilde{z}_\epsilon|^2 - g_\epsilon(x, |\tilde{z}_\epsilon|)|\tilde{z}_\epsilon|^2 - \frac{|\vec{b}|}{2}|\tilde{z}_\epsilon|^2\right). \tag{5.16}$$

Now we claim that  $|\tilde{z}_\epsilon(x)| \rightarrow 0$  for all small  $\epsilon > 0$  as  $|x| \rightarrow \infty$ . If not, then there exist  $\varrho > 0$  and  $x_j \in \mathbb{R}^N$  with  $|x_j| \rightarrow \infty$  such that  $|\tilde{z}_\epsilon(x_j)| \geq \varrho$ . then, by Lemma 5.2 and the fact that  $\tilde{z}_\epsilon \rightarrow z_0$  in  $E$ , we obtain, for all small  $\epsilon > 0$

$$\begin{aligned} \varrho &\leq |\tilde{z}_\epsilon(x_j)| \leq C_0 \left(\int_{B_1(x_j)} |\tilde{z}_\epsilon|^2\right)^{\frac{1}{2}} \\ &\leq C_0 \left(\int_{B_1(x_j)} |\tilde{z}_\epsilon - z_0|^2\right)^{\frac{1}{2}} + C_0 \left(\int_{B_1(x_j)} |z_0|^2\right)^{\frac{1}{2}} \\ &\leq \tilde{C} \left(\int_{\mathbb{R}^N} |\tilde{z}_\epsilon - z_0|^2\right)^{1/2} + C_0 \left(\int_{B_1(x_j)} |z_0|^2\right)^{\frac{1}{2}} \rightarrow 0, \end{aligned}$$

this is a contradiction. Thus, it follows from  $(F_0)$  and  $(F_1)$  that, for any  $\epsilon > 0$ , there exists  $R > 0$  such that  $|g_\epsilon(x, |\tilde{z}_\epsilon|)| \leq \epsilon$  for  $|x| \geq R$ . Moreover, by (5.16), there exists  $\tau > 0$  such that

$$\Delta|\tilde{z}_\epsilon|^2 \geq \tau|\tilde{z}_\epsilon|^2$$

for all  $|x| \geq R$  and small  $\epsilon > 0$ . Let  $\Gamma(x)$  be a fundamental solution to  $-\Delta\Gamma + \tau\Gamma = 0$  (see, e.g., [30]). Using the uniform boundedness, we may choose  $\Gamma(x)$  so that  $|\tilde{z}_\epsilon(x)|^2 \leq \tau\Gamma(x)$  holds on  $|x| = R$  and for all small  $\epsilon > 0$ . Let  $w = |\tilde{z}_\epsilon|^2 - \tau\Gamma$ , then

$$\Delta w = \Delta|\tilde{z}_\epsilon|^2 - \tau\Delta\Gamma \geq \tau(|\tilde{z}_\epsilon|^2 - \tau\Gamma) = \tau w, \text{ for } |x| \geq R.$$

By the maximum principle we can conclude that  $w(x) \leq 0$  for  $|x| \geq R$ , i.e.,  $|\tilde{z}_\epsilon(x)|^2 \leq \tau\Gamma(x)$  for  $|x| \geq R$ . It is well known that there is  $\tilde{C} > 0$  such that

$$\Gamma(x) \leq \tilde{C} \exp(-\sqrt{\tau}|x|)$$

for  $|x| \geq 1$  (see [30]). Hence, we get

$$|\tilde{z}_\epsilon(x)|^2 \leq \tilde{C} \exp(-c|x|)$$

for all  $x \in \mathbb{R}^N$ , small  $\epsilon > 0$  and some  $c > 0$ , that is,

$$|\tilde{z}_\epsilon(x)| \leq \sqrt{\tilde{C}} \exp\left(-\frac{c}{2}|x|\right)$$

for all  $x \in \mathbb{R}^N$  and all small  $\epsilon > 0$ . Then

$$|z_\epsilon(x)| \leq C \exp\left(-\frac{C}{2}|x - x_\epsilon|\right)$$

with  $C = \sqrt{C}$ , for all  $x \in \mathbb{R}^N$  and all small  $\epsilon > 0$ . The proof is completed. □

Now we are in a position to finish the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Define

$$\eta_\epsilon(x) = z_\epsilon(x/\epsilon) \text{ and } p_\epsilon = \epsilon x_\epsilon.$$

It is clear that  $\eta_\epsilon$  is a solutions of

$$\begin{cases} -\epsilon^2 \Delta \psi + \epsilon \vec{b} \cdot \nabla \psi + \psi + V(x)\varphi = g(x, |\eta|)\varphi & \text{in } \mathbb{R}^N, \\ -\epsilon^2 \Delta \varphi - \epsilon \vec{b} \cdot \nabla \varphi + \varphi + V(x)\psi = g(x, |\eta|)\psi & \text{in } \mathbb{R}^N, \end{cases}$$

for all small  $\epsilon$ , and  $p_\epsilon$  is a maximum point of  $|\eta_\epsilon(x)|$ . Then by Lemma 5.3, up to a subsequence, we deduce that  $p_\epsilon \rightarrow \mathcal{V}$  and

$$\lim_{\epsilon \rightarrow 0} V(p_\epsilon) = v \text{ and } \eta_\epsilon(\epsilon x + p_\epsilon) = z_\epsilon(x + x_\epsilon) \rightarrow \eta(x) \text{ in } H^2(\mathbb{R}^N),$$

and  $\eta$  is a ground state solutions of

$$\begin{cases} -\Delta \psi + \vec{b} \cdot \nabla \psi + \psi + v\varphi = f(|\eta|)\varphi & \text{in } \mathbb{R}^N, \\ -\Delta \varphi - \vec{b} \cdot \nabla \varphi + \varphi + v\psi = f(|\eta|)\psi & \text{in } \mathbb{R}^N. \end{cases}$$

According to Lemma 5.4, it follows that

$$|\eta_\epsilon(x)| \leq C \exp\left(-\frac{C}{2\epsilon}|x - p_\epsilon|\right).$$

Moreover, from (1.5), we conclude that  $\pi := \text{dist}(\mathcal{V}, \partial\Omega) > 0$ . Since  $p_\epsilon \rightarrow \mathcal{V}$ , then we find that, if  $x \notin \Omega$ ,

$$|\eta_\epsilon(x)| \leq C \exp\left(-\frac{C}{2\epsilon}|x - p_\epsilon|\right) \leq C \exp\left(-\frac{C\pi}{4\epsilon}\right) < a_0$$

for  $\epsilon$  sufficiently small. Therefore, we have  $g(x, |\eta_\epsilon|) = f(|\eta_\epsilon|)$ , and  $\eta_\epsilon$  is a solution of original problem  $(\mathcal{P}_\epsilon)$ . The proof of Theorem 1.1 is completed. □

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