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Research article

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Liouville property of fractional Lane-Emden equation in general unbounded domain

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Abstract: Our purpose of this paper is to consider Liouville property for the fractional Lane-Emden equation

$$(-\Delta)^{\alpha} u = u^{p} \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{in} \quad \mathbb{R}^{N} \setminus \Omega,$$

where $\alpha \in (0,1)$, $N \geq 1$, $p > 0$ and $\Omega \subset \mathbb{R}^{N-1} \times [0,+\infty)$ is an unbounded domain satisfying that $\Omega_t := \{x' \in \mathbb{R}^{N-1} : (x',t) \in \Omega\}$ with $t \geq 0$ has increasing monotonicity, that is, $\Omega_t \subset \Omega_{t'}$ for $t' \geq t$. The shape of $\Omega_\infty := \lim_{t\to\infty} \Omega_t$ in $\mathbb{R}^{N-1}$ plays an important role to obtain the nonexistence of positive solutions for the fractional Lane-Emden equation.

Keywords: Fractional Laplacian; Lane-Emden equation; Nonexistence

MSC: 35J60; 35B53

1 Introduction

In this paper, we consider Liouville property for the fractional Lane-Emden equation

$$(-\Delta)^{\alpha} u = u^{p} \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{in} \quad \mathbb{R}^{N} \setminus \Omega, \tag{1.1}$$

where $\alpha \in (0,1)$, $p > 0$, Ω is an unbounded domain in $\mathbb{R}^N$ with $N \geq 1$, and $(-\Delta)^\alpha$ with $\alpha \in (0,1)$ is the fractional Laplacian defined in the principle value sense,

$$(-\Delta)^{\alpha} u(x) = c_{N,\alpha} \lim_{\epsilon \to 0^+} \int\limits_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{u(x) - u(x+z)}{|z|^{N+2\alpha}} dz,$$

here $B_\epsilon(0)$ is the ball with radius ϵ centered at the origin and $c_{N,\alpha} > 0$ is the normalized constant. We say that u is a bounded solution of (1.1) if $u \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and u satisfies (1.1) pointwisely.

As an important property, the Liouville theorem for Lane-Emden equation has attracted a lot of attentions by many mathematician by the application in the derivation of uniform bound via blowing-up analysis. Note that the nonexistence of stable solution is studied in [1] by finite morse index with restrictions on the boundary and at infinity. Without the zero Dirichlet boundary condition, Liouville results could be obtained by Hadamard property in [2, 3], by iterating the decaying rate at infinity in [4] and by Hardy estimates in [5].

It is known that the Leray-Schauder degree theory is a very useful method for deriving solutions of elliptic equations on bounded domains. The essential step is to obtain a uniform bound by considering a sequence of solutions $\{u_n\}_n$ such that $u_n(x_n) = \|u_n\|_{L^\infty} = M_n \to +\infty$ as $n \to +\infty$ for some $\{x_n\}_n \subset \Omega$. Then let $\Omega_n = \{x \in \mathbb{R}^N : \frac{1}{M_n^\kappa} x + x_n \in \Omega\}$ and $v_n(x) = \frac{1}{M_n} u_n(\frac{1}{M_n^\kappa} x + x_n)$ for some $\kappa > 0$, then $\{v_n\}_n$ is uniformly bounded,

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and the limit v_∞ of $\{v_n\}_n$ is a solution of related limit semilinear equation in the limit domain Ω^∞. Note that for C^2 domain Ω, the limit domain of Ω_n is either $\Omega^\infty = \mathbb{R}^N$ or $\Omega^\infty = \mathbb{R}^{N-1} \times (0, \infty)$. While if the original domain contains a cone point on the boundary, then the limit domain has the third possibility that Ω^∞ is a cone. As a consequence, the nonexistence of positive solutions to the limit equation in a cone has to be involved additionally. As far as we know, the nonexistence of elliptic equation depends on the shape of limit domain at infinity in some direction. Our concern in this article is to consider the non-existence of elliptic equations in one type of unbounded domain.

Recently, qualitative properties of solutions for nonlocal elliptic equations have been studied extensively, such as the existence of weak solutions or very weak solutions in [6, 7] by variational methods, a survey in [8] on variational methods, large solution [9] by Perron's method, the regularities in [10], Pohozaev's identity [11] and Liouville results [12–14]. In [15] it develops the method of moving plane in fractional setting to obtain the classification of critical elliptic equations in an integral form and then it is used to obtain nonexistence of bounded solutions for semilinear elliptic equations in half space [12, 13, 16] subject to various boundary type conditions. In particular, Fall and Weth in [13] obtained the nonexistence of positive bounded solution of (1.1) when $\Omega = \mathbb{R}^N_+$ and $p < \frac{N-1+2\alpha}{N-1-2\alpha}$, by applying the method of moving plane, via some interesting estimates of Green's kernel in $\mathbb{R}^N_+$. It is worth noting that star-shaped domain with respect to infinity is involved in obtaining the nonexistence of fractional Lane-Emden equation in [14] and it is an important notation in our derivation of nonexistence to (1.1).

Before stating our main result, we introduce the following notations.

$(\mathcal{D})$ Given $\mathcal{O} \subset \mathbb{R}^M \times [0, +\infty)$ and $t \in [0, +\infty)$, with $M \geq 1$ being an integer, denote

$$\mathcal{O}_t := \{x' \in \mathbb{R}^M : (x', t) \in \mathcal{O}\}.$$

For $t \geq 0$, $\mathcal{O}_t$ is nonempty, bounded and the mapping $t \mapsto \mathcal{O}_t$ is increasing in the following sense

$$\mathcal{O}_t \subset \mathcal{O}_{t'} \quad \text{if} \quad 0 \leq t \leq t' < +\infty.$$

Denote

$$\mathcal{O}_\infty = \bigcup_{t\in[0,\infty)} \mathcal{O}_t. \tag{1.2}$$

Now we introduce two types domains:

$(\mathcal{S}_0)$ *star-shaped domain $\mathcal{O} \subset \mathbb{R}^M$, if there exists $e \in \mathcal{O}$ such that for every point $x \in \partial\mathcal{O}$, the segment $\{te + (1-t)(x-e) : t \in [0, 1]\}$ is contained in $\mathcal{O}$;*

$(\mathcal{S}_\infty)$ *star-shaped domain $\mathcal{O} \subset \mathbb{R}^M$ with respect to infinity,* if there exists a point $e \in \mathbb{R}^M \setminus \bar{\mathcal{O}}$ such that for every point $x \in \mathcal{O}$, the half-line $\{e + t(x-e) : t \geq 1\}$ is contained in $\mathcal{O}$.

The main results state as follows.

Theorem 1.1. *Assume that $\alpha \in (0, 1)$, $\Omega \subset \mathbb{R}^{N-1} \times [0, +\infty)$ is a $C^{0,1}$ domain verifying $(\mathcal{D})$ and Ω_∞ is given as (1.2) with $\mathcal{O} = \Omega$. Then problem (1.1) has no nonnegative, nontrivial and bounded solutions if one of the following holds:*

(i) $\Omega_\infty = \mathbb{R}^{N-1}$ and $0 < p < \frac{N-1+2\alpha}{N-1-2\alpha}$ if $N > 1 + 2\alpha$, otherwise for $p > 0$, $1 \leq N \leq 1 + 2\alpha$;

(ii) $\Omega_\infty \subset \mathbb{R}^{N-1}$ is a star-shaped domain with respect to infinity and $1 \leq p \leq \frac{N-1+2\alpha}{N-1-2\alpha}$ with $N > 1 + 2\alpha$;

(iii) $\Omega_\infty = \mathbb{R}^{N-2} \times [0, +\infty)$ and $1 < p < \frac{N-2+2\alpha}{N-2-2\alpha}$ for $N > 2 + 2\alpha$, otherwise for $p > 1$, $1 \leq N \leq 2 + 2\alpha$;

(vi) Ω_∞ is bounded, C^2 star-shape in $\mathbb{R}^{N-1}$ and $N > 1 + 2\alpha$, $p \geq \frac{N-1+2\alpha}{N-1-2\alpha}$.

Our basic tool is the traditional method of moving plane, involved by [17] in the fractional setting, we develop this traditional method of moving planes to obtain the increasing monotonicity in the direction x_N and reduce problem (1.1) into

$$(-\Delta)^\alpha_{\mathbb{R}^{N-1}} u = u^p \quad \text{in} \quad \Omega_\infty, \tag{1.3}$$

subject to zero Dirichlet boundary condition when $\Omega_\infty \neq \mathbb{R}^{N-1}$, where $(-\Delta)^\alpha_{\mathbb{R}^{N-1}}$ is the fractional Laplacian in $\mathbb{R}^{N-1}$. Then the nonexistence results could be obtained for (1.3) as in [13, 14].

Remark 1.1. *In the particular case that Ω is a cone such as $\{x = (x', x_N) \in \mathbb{R}^N : x_N > \theta|x'|\}$ for some $\theta > 0$, problem (1.1) has no nonnegative, nontrivial and bounded solutions.*

If Ω_∞ also verifies the similar assumptions of Theorem 1.1, we can repeat our above procedure and derive the following corollary directly:

Corollary 1.1. *Under the assumptions of Theorem 1.1, we assume that the domain $\Omega_\infty \subset \mathbb{R}^{N-2} \times [0, +\infty)$ is a C^2 domain verifying $(\mathcal{D})$ and $\Omega_{\infty,\infty}$ is given by (1.2) with $\mathcal{O}_\infty = \Omega_{\infty,\infty}$. Then problem (1.1) does not admit nonnegative, nontrivial and bounded solutions if one of the following holds:*

(i) $\Omega_{\infty,\infty} = \mathbb{R}^{N-2}$ and $0 < p < \frac{N-2+2\alpha}{N-2-2\alpha}$ if $N > 2 + 2\alpha$, otherwise for $p > 0$, $1 \le N \le 1 + 2\alpha$;

(ii) $\Omega_{\infty,\infty} \subset \mathbb{R}^{N-2}$ is a star-shaped domain with respect to infinity and $1 \le p \le \frac{N-2+2\alpha}{N-2-2\alpha}$ with $N > 2 + 2\alpha$;

(iii) $\Omega_{\infty,\infty} = \mathbb{R}^{N-3} \times [0, +\infty)$ and $1 < p < \frac{N-3+2\alpha}{N-3-2\alpha}$ for $N > 3 + 2\alpha$, otherwise for $p > 1$, $1 \le N \le 3 + 2\alpha$;

(vi) $\Omega_{\infty,\infty}$ is bounded, star-shape C^2 domain in $\mathbb{R}^{N-2}$ and $N > 2 + 2\alpha$, $p \ge \frac{N-2+2\alpha}{N-2-2\alpha}$.

2 The proof of nonexistence results

For the domain Ω verifying $(\mathcal{D})$, we shall prove that the solution of (1.1) has the x_N-increasing property by using the method of moving planes. To this end, we introduce the following notations. For $\lambda > 0$, denote

$$\Sigma_\lambda = \{x = (x', x_N) \in \mathbb{R}^N \cap \Omega \mid x_N < \lambda\}, \tag{2.1}$$

$$T_\lambda = \{x = (x', x_N) \in \mathbb{R}^N \mid x_N = \lambda\}, \tag{2.2}$$

$$u_\lambda(x) = u(x_\lambda) \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x), \tag{2.3}$$

where $x_\lambda = (x', 2\lambda - x_N)$ for $x = (x', x_N) \in \mathbb{R}^N$. For any subset A of $\mathbb{R}^N$, we write $A_\lambda = \{x_\lambda : x \in A\}$ the reflection of A with regard to T_λ. Since Ω_t is bounded for $t \ge 0$, then the domain Σ_λ is always bounded for any $\lambda > 0$.

Proposition 2.1. *Under the hypotheses of Theorem 1.1, let u be a nonnegative nontrivial solution of (1.1), then*

$$u(x', t) \le u(x', s) \ \text{ for } \ s \ge t, \ \forall\, x' \in \mathbb{R}^{N-1}. \tag{2.4}$$

Proof. We divide the proof into two steps.
Step 1: By the assumption of $\Omega \subset \mathbb{R}^N_+$, we may assume

$$\lambda_0 = \inf\{\lambda \ge 0 : \ w_\lambda \ge 0 \ \text{ in } \ \Sigma_\lambda\}.$$

The purpose of this step is to show that if $\lambda > \lambda_0$ is close to λ_0, then $w_\lambda > 0$ in Σ_λ. To this end, let $\Sigma_\lambda^- = \{x \in \Sigma_\lambda \mid w_\lambda(x) < 0\}$, we first claim that if $\lambda > \lambda_0$ is close to λ_0, then

$$\Sigma_\lambda^- = \emptyset. \tag{2.5}$$

By contradiction, we assume (2.5) is not true, that is $\Sigma_\lambda^- \neq \emptyset$. We denote

$$w_\lambda^+(x) = w_\lambda(x), \ \ x \in \Sigma_\lambda^-, \qquad w_\lambda^+(x) = 0, \ \ x \in \mathbb{R}^N \setminus \Sigma_\lambda^-, \tag{2.6}$$

$$w_\lambda^-(x) = 0, \ \ x \in \Sigma_\lambda^-, \qquad w_\lambda^-(x) = w_\lambda(x), \ \ x \in \mathbb{R}^N \setminus \Sigma_\lambda^- \tag{2.7}$$

It is obvious that $w_\lambda^+(x) = w_\lambda(x) - w_\lambda^-(x)$ for all $x \in \mathbb{R}^N$. By direct computation, for $x \in \Sigma_\lambda^-$, we have

$$(-\Delta)^\alpha w_\lambda^-(x) \;=\; -\int\limits_{(\Omega\setminus\Omega_\lambda)\cup(\Omega_\lambda\setminus\Omega)} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}}\,dz$$

$$\begin{aligned} & - \int\limits_{(\Sigma_\lambda\setminus\Sigma_\lambda^-)\cup(\Sigma_\lambda\setminus\Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}}dz - \int\limits_{(\Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}}dz \\ = \; & -I_1 - I_2 - I_3. \end{aligned}$$

We look at each of these integrals separately. Since $u = 0$ in $\Omega_\lambda \setminus \Omega$ and $u_\lambda = 0$ in $\Omega \setminus \Omega_\lambda$, then

$$\begin{aligned} I_1 &= \int\limits_{\Omega_\lambda\setminus\Omega} \frac{u_\lambda(z)}{|x-z|^{N+2\alpha}}dz - \int\limits_{\Omega\setminus\Omega_\lambda} \frac{u(z)}{|x-z|^{N+2\alpha}}dz \\ &= \int\limits_{\Omega_\lambda\setminus\Omega} u_\lambda(z)(\frac{1}{|x-z|^{N+2\alpha}} - \frac{1}{|x-z_\lambda|^{N+2\alpha}})dz \ge 0, \end{aligned}$$

by the fact that $u_\lambda \ge 0$ and $|x - z_\lambda| > |x - z|$ for all $x \in \Sigma_\lambda^-$ and $z \in \Omega_\lambda \setminus \Omega$.

In order to fix the sign of I_2, note that $w_\lambda(z_\lambda) = -w_\lambda(z)$ for any $z \in \mathbb{R}^N$ and then

$$\begin{aligned} I_2 &= \int\limits_{\Sigma_\lambda\setminus\Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z|^{N+2\alpha}}dz + \int\limits_{\Sigma_\lambda\setminus\Sigma_\lambda^-} \frac{w_\lambda(z_\lambda)}{|x-z_\lambda|^{N+2\alpha}}dz \\ &= \int\limits_{\Sigma_\lambda\setminus\Sigma_\lambda^-} w_\lambda(z)(\frac{1}{|x-z|^{N+2\alpha}} - \frac{1}{|x-z_\lambda|^{N+2\alpha}})dz \ge 0, \end{aligned}$$

since $w_\lambda \ge 0$ in $\Sigma_\lambda \setminus \Sigma_\lambda^-$ and $|x - z_\lambda| > |x - z|$ for all $x \in \Sigma_\lambda^-$ and $z \in \Sigma_\lambda \setminus \Sigma_\lambda^-$. Finally, since $w_\lambda(z) < 0$ for $z \in \Sigma_\lambda^-$, we have

$$I_3 = \int\limits_{\Sigma_\lambda^-} \frac{w_\lambda(z_\lambda)}{|x-z_\lambda|^{N+2\alpha}}dz = -\int\limits_{\Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z_\lambda|^{N+2\alpha}}dz \ge 0.$$

Hence, we obtain that for all $\lambda > \lambda_0$,

$$(-\Delta)^\alpha w_\lambda^-(x) \le 0, \quad \forall\, x \in \Sigma_\lambda^- \tag{2.8}$$

and then for $x \in \Sigma_\lambda^-$,

$$(-\Delta)^\alpha w_\lambda^+(x) \ge (-\Delta)^\alpha w_\lambda(x) = (-\Delta)^\alpha u_\lambda(x) - (-\Delta)^\alpha u(x). \tag{2.9}$$

Combining (1.1) with (2.9) and (2.6), we have that

$$\begin{aligned} (-\Delta)^\alpha w_\lambda^+(x) &\ge (-\Delta)^\alpha u_\lambda(x) - (-\Delta)^\alpha u(x) \\ &= u_\lambda^p(x) - u^p(x) = -\varphi(x) w_\lambda^+(x), \;\; x \in \Sigma_\lambda^-, \end{aligned}$$

where

$$\varphi(x) = -\frac{u_\lambda^p(x) - u^p(x)}{u_\lambda(x) - u(x)}, \qquad |\varphi| \le 2^p[u_\lambda^{p-1} + u^{p-1}] \le 2^p \|u\|_{L^\infty}^{p-1}.$$

Hence, we have

$$\Delta^\alpha w_\lambda^+(x) \le \varphi(x) w_\lambda^+(x), \quad x \in \Sigma_\lambda^- \tag{2.10}$$

and observe that $w_\lambda^+ = 0$ in $(\Sigma_\lambda^-)^c$, then we have that

$$\sup_{x\in\Sigma_\lambda^-} w_\lambda^+(x) \le cR(\Sigma_\lambda^-)^{2\alpha} \|\varphi w_\lambda^+\|_{L^\infty(\Sigma_\lambda^-)} \le cR(\Sigma_\lambda^-)^{2\alpha} \|\varphi\|_{L^\infty(\Sigma_\lambda^-)} \sup_{x\in\Sigma_\lambda^-} w_\lambda^+(x),$$

that is,

$$1 \le c2^p R(\Sigma_\lambda^-)^{2\alpha} \|u\|_{L^\infty}^{p-1}, \tag{2.11}$$

where c is a positive constant independent of Σ_λ^- and

$$R(\Sigma_\lambda^-) = \inf\left\{ r > 0 : \; |B_r(x) \setminus \Sigma_\lambda^-| \ge \frac{1}{2}|B_r(x)|, \quad \forall\, x \in \Sigma_\lambda^- \right\}.$$

Choosing $\lambda > \lambda_0$ close enough to λ_0, $R(\Sigma_\lambda^-)$ is small, a contraction is derived from (2.11) and then

$$w_\lambda = w_\lambda^+ \geq 0 \quad \text{in } \Sigma_\lambda^-.$$

But this is a contradiction with the definition of Σ_λ^-, so we have that $w_\lambda \geq 0$ in Σ_λ.

We next claim that if $w_\lambda \geq 0$ and $w_\lambda \not\equiv 0$ in Σ_λ, then $w_\lambda > 0$ in Σ_λ. Assuming this claim is true, we complete the proof, since the function u is positive in Ω and $u = 0$ on $\partial\Omega$, so that w_λ is positive on $\partial\Omega \cap \partial\Sigma_\lambda$ and then, by continuity $w_\lambda \not\equiv 0$ in Σ_λ.

Now we prove above claim. In fact, assume there exists $x_0 \in \Sigma_\lambda$ such that $w_\lambda(x_0) = 0$, that is, $u_\lambda(x_0) = u(x_0)$. One hand we have that

$$(-\Delta)^\alpha w_\lambda(x_0) = (-\Delta)^\alpha u_\lambda(x_0) - (-\Delta)^\alpha u(x_0) = u_\lambda^p(x_0) - u^p(x_0) = 0. \tag{2.12}$$

On the other hand, let $A_\lambda = \{(x_1, x') \in \mathbb{R}^N \mid x_1 > \lambda\}$, since $w_\lambda(z_\lambda) = -w_\lambda(z)$ for any $z \in \mathbb{R}^N$ and $w_\lambda(x_0) = 0$, we find

$$\begin{aligned}(-\Delta)^\alpha w_\lambda(x_0) &= -\int\limits_{A_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2\alpha}} dz - \int\limits_{A_\lambda} \frac{w_\lambda(z_\lambda)}{|x_0 - z_\lambda|^{N+2\alpha}} dz \\ &= -\int\limits_{A_\lambda} w_\lambda(z)\Big(\frac{1}{|x_0 - z|^{N+2\alpha}} - \frac{1}{|x_0 - z_\lambda|^{N+2\alpha}}\Big) dz.\end{aligned}$$

Since $|x_0 - z_\lambda| > |x_0 - z|$ for $z \in A_\lambda$, $w_\lambda(z) \geq 0$ and $w_\lambda(z) \not\equiv 0$ in A_λ, then

$$(-\Delta)^\alpha w_\lambda(x_0) < 0, \tag{2.13}$$

which contradicts (2.12).

Step 2: Our purpose of this step is to move the planes forward for any $\lambda > \lambda_0$ up to

$$\lambda_1 := \sup\{\lambda > \lambda_0 \mid w_\mu > 0 \text{ in } \Sigma_\mu \text{ for any } \mu \in (\lambda_0, \lambda)\} = +\infty.$$

By contradiction, we assume that $\lambda_1 < +\infty$. Since for any $\lambda \in (0, \lambda_1)$, we have that $w_\lambda > 0$ in Σ_λ, by the continuity, we derive that $w_{\lambda_1} \geq 0$ in Σ_{λ_1} and $w_{\lambda_1} \not\equiv 0$ in Σ_{λ_1}. Thus, by the claim just proved above, we have $w_{\lambda_1} > 0$ in Σ_{λ_1}.

Now we claim that if $w_\lambda > 0$ in Σ_{λ_1}, then there exists $\epsilon > 0$ such that $w_\lambda > 0$ in Σ_λ for all $\lambda \in [\lambda_1, \lambda_1 + \epsilon)$. This claim directly implies that $\lambda_1 = +\infty$, completing Step 2.

In fact, if $\lambda_1 < +\infty$, then under our hypotheses, we have that $\overline{\Sigma}_\lambda$ is compact. Denote

$$D_\mu = \{x \in \Sigma_\lambda \mid dist(x, \partial\Sigma_\lambda) \geq \mu\}$$

for $\mu > 0$ small. Since $w_{\lambda_1} > 0$ in Σ_{λ_1} and D_μ is compact, then there exists $\mu_0 > 0$ such that $w_{\lambda_1} \geq \mu_0$ in D_μ. By continuity of $w_\lambda(x)$, for $\epsilon > 0$ small enough and any $\lambda \in (\lambda_1, \lambda_1 + \epsilon)$, we have that

$$w_\lambda \geq 0 \text{ in } \mathrm{D}_\mu.$$

As a consequence,

$$\Sigma_{\lambda_\epsilon}^- \subset \Sigma_{\lambda_\epsilon} \setminus D_\mu$$

and $R(\Sigma_{\lambda_\epsilon}^-)$ is small if ϵ and μ are small. Using (2.8) and proceeding as in Step 1, we have that for all $x \in \Sigma_{\lambda_\epsilon}^-$,

$$\begin{aligned}(-\Delta)^\alpha w_{\lambda_\epsilon}^+(x) &= (-\Delta)^\alpha u_{\lambda_\epsilon}(x) - (-\Delta)^\alpha u(x) - (-\Delta)^\alpha w_{\lambda_\epsilon}^-(x) \\ &\geq (-\Delta)^\alpha u_{\lambda_\epsilon}(x) - (-\Delta)^\alpha u(x) = u_{\lambda_\epsilon}^p(x) - u^p(x) \\ &= -\varphi(x) w_{\lambda_\epsilon}^+(x),\end{aligned}$$

where $\varphi(x) = -\frac{u_{\lambda_\epsilon}^p(x) - u^p(x)}{u_{\lambda_\epsilon}(x) - u(x)}$ is bounded. Since $w_{\lambda_\epsilon}^+ = 0$ in $(\Sigma_{\lambda_\epsilon}^-)^c$ and $|\Sigma_{\lambda_\epsilon}^-|$ is small, for ϵ and μ small, from the analysis Step 1, we obtain that $w_{\lambda_\epsilon} \geq 0$ in $\Sigma_{\lambda_\epsilon}$. Since $\lambda_\epsilon > 0$ and $w_{\lambda_\epsilon} \not\equiv 0$ in $\Sigma_{\lambda_\epsilon}$, as before we have $w_{\lambda_\epsilon} > 0$ in $\Sigma_{\lambda_\epsilon}$, completing the proof of the claim. The proof ends. ☐

Proof of Theorem 1.1. We prove this argument by contradiction. Assume that $u \ge 0$ is nontrivial solution of (1.1). When Ω verifies $(\mathcal{D})$, then by Proposition 2.1, we have that u satisfies (2.4). Let

$$u_m(x) = u(x', x_N + m), \quad \forall\, x \in \mathbb{R}^N,$$

then $\{u_m\}_m$ is an increasing and bounded sequence of functions and satisfies

$$(-\Delta)^\alpha u = u^p \quad \text{in} \quad \Omega - m.$$

Note that $\Omega_t \subset \Omega_{t'}$ for $0 \le t \le t' < +\infty$,

$$\Omega - m \subset \Omega - k \quad \text{if} \quad k > m, \qquad \lim_{m \to +\infty} \Omega - m = \mathbb{R}^N,$$

from the regularity results in [9, Theorem 2.1] and the stability property [9, Theorem 2.4], we obtain that

$$u_\infty := \lim_{m \to +\infty} u_m$$

is a bounded classical solution to

$$(-\Delta)^\alpha u = u^p \quad \text{in} \quad \Omega_\infty \times \mathbb{R}, \qquad u = 0 \;\text{ in } \mathbb{R}^N \setminus (\Omega_\infty \times \mathbb{R}) \;\text{ if } \Omega_\infty \ne \mathbb{R}^{N-1}.$$

By the increasing property of $\{u_m\}_m$, we have that

$$u_\infty(x', x_N) = u_\infty(x', y_N) \quad \text{for any } x_N, y_N \in \mathbb{R},$$

which implies that u_∞ is x_N-independent. Letting $v_\infty(x') = u_\infty(x', x_N)$, by the standard argument, we have that

$$(-\Delta)^\alpha_{\mathbb{R}^{N-1}} v_\infty(x') = c_N (-\Delta)^\alpha u_\infty(x), \quad \forall\, x \in \Omega_\infty \times \mathbb{R},$$

then v_∞ is a positive, bounded and classical solution for

$$(-\Delta)^\alpha_{\mathbb{R}^{N-1}} v_\infty = c_N v_\infty^p \quad \text{in} \quad \Omega_\infty, \tag{2.14}$$

subject to $v_\infty = 0$ in $\mathbb{R}^{N-1} \setminus \Omega_\infty$ if $\Omega_\infty \ne \mathbb{R}^{N-1}$.

A contradiction is derived by the fact that problem (2.14) has no positive bounded classical solution when

(i) $\Omega_\infty = \mathbb{R}^{N-1}$ and $0 < p < \frac{N-1+2\alpha}{N-1-2\alpha}$ if $N > 1 + 2\alpha$, otherwise for $p > 0$, $1 \le N \le 1 + 2\alpha$ by [13, Theorem 1.2];

(ii) $\Omega_\infty \subset \mathbb{R}^{N-1}$ is a star-shaped domain with respect to infinity and $1 \le p \le \frac{N-1+2\alpha}{N-1-2\alpha}$ with $N > 1 + 2\alpha$ by [14, Theorem 1.5];

(iii) $\Omega_\infty = \mathbb{R}^{N-2} \times [0, +\infty)$ and $1 < p < \frac{N-2+2\alpha}{N-2-2\alpha}$ for $N > 2 + 2\alpha$, otherwise for $p > 1$, $1 \le N \le 2 + 2\alpha$ by [13, Theorem 1.2];

(vi) Ω_∞ is bounded, C^2 and star-shape in $\mathbb{R}^{N-1}$, $N > 1 + 2\alpha$, and $p \ge \frac{N-1+2\alpha}{N-1-2\alpha}$. by [14, Corollary 1.2].

As a consequence, we obtain the nonexistence results of (1.1) in Theorem 1.1. □

Proof of Corollary 1.1. When problem (1.1) has a positive solution u, then Proposition 2.1 and regularity results guarantee that $v_\infty(x') = \lim_{m \to +\infty} u(x', x_N + m)$ is a positive, bounded and classical solution for problem

$$(-\Delta)^\alpha_{\mathbb{R}^{N-1}} v_\infty = c_N v_\infty^p \quad \text{in} \quad \Omega_\infty \qquad \text{and} \qquad v_\infty = 0 \quad \text{in} \quad \mathbb{R}^{N-1} \setminus \Omega_\infty. \tag{2.15}$$

Furthermore, we note that $v_{\infty,\infty}(x'') = \lim_{m \to +\infty} v_\infty(x'', x_{N-1} + m)$ is a positive, bounded and classical solution of

$$(-\Delta)^\alpha_{\mathbb{R}^{N-2}} v_{\infty,\infty} = c_N v_\infty^p \quad \text{in} \quad \Omega_{\infty,\infty} \qquad \text{and} \qquad v_{\infty,\infty} = 0 \quad \text{in} \quad \mathbb{R}^{N-2} \setminus \Omega_{\infty,\infty}, \tag{2.16}$$

subject to $v_{\infty,\infty} = 0$ in $\mathbb{R}^{N-2} \setminus \Omega_{\infty,\infty}$ if $\Omega_{\infty,\infty} \ne \mathbb{R}^{N-2}$. Then the same conclusion is obtained as the proof of Theorem 1.1. □

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