Regularity of Interfaces for an Inhomogeneous Filtration Equation

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Abstract

We study the regularity of the interfaces for solutions of the degenerate parabolic equation $\rho(x)u_t = (u^m)_{xx}\,, m > 1$. As opposed to the classical Porous Medium Equation, the interfaces in this equation may propagate with infinite speed and disappear within a finite time. We establish conditions on the initial function $u(x,0)$ and the function $\rho(x)$ sufficient to provide the local C^1-regularity of the interfaces until the moment of their disappearance. The interface equation (Darcy's law) is derived.

1. Introduction

In this article we study the regularity of interfaces for solutions of the Cauchy problem for the degenerate parabolic equation

$$\rho(x)u_t = (u^m)_{xx} \qquad \text{in } S_{T_0} := \mathbb{R} \times (0,T_0). \tag{1.1}$$

Equation (1.1) arises in plasma physics [20, ?] and in hydrology in the case of a slightly different nonlinear diffusion term [13]. We assume throughout this paper that $m > 1$ and that the initial datum $u_0(x)$ is a nonnegative function, finitely supported in $\mathbb{R}$, say, in an interval $(-a,a)$, $a > 0$. For the sake of simplicity, we assume that u_0 is an even function. Later on we shall also need certain assumptions on the regularity of u_0. The function ρ is supposed to be strictly positive in $\mathbb{R}$ and to satisfy suitable regularity assumptions which we shall specify later.

The number T_0 can be finite or infinite. We are interested in the regularity of interfaces due to finitely supported Cauchy data; that is to say in the regularity of the curves

$$\eta(t) = \sup\{x \in \mathbb{R} : u(x,t) > 0\}, \qquad \zeta(t) = \inf\{x \in \mathbb{R} : u(x,t) > 0\}.$$

Note that in the case where $\rho = 1$, Equation (1.1) becomes

$$u_t = (u^m)_{xx}, \tag{1.2}$$

which is called the *porous medium equation.* There is a large number of articles devoted to the porous medium equation (1.2) and to the regularity of the interfaces (e.g. Angenent [1], Aronson [2, 3, 4], Aronson and Vazquez [6], Caffarelli and Friedman [7], Knerr [16]). In particular its proved that the free boundaries are Lipschitz continuous functions on $[0, +\infty)$, and moreover that there exist times $t_i^\star \in [0, +\infty], i = 1, 2$, called the *waiting times* such that $\eta(t) = \eta(0)$ (resp. $\zeta(t) = \zeta(0)$) on $[0, t_1^\star]$ (resp. $[0, t_2^\star]$), and such that η and ζ are strictly monotone after the waiting times. Later on Caffarelli and Friedman [7] have shown that the free boundaries are C^1 functions after the waiting time. Aronson and Vazquez [6] have then proved that interfaces are indefinitely differentiable after the waiting time, and finally Angenent [1] has proved their analicity. Furthermore the solution of the Cauchy problem for the porous medium equation has a compact support for all times provided that it is so at time zero. One of the most striking features of Equation (1.1) is that on the contrary, if the function ρ decreases fast enough to zero as $|x|$ tends to ∞, namely if

$$\int_{-\infty}^{\infty} s\rho(s)ds < \infty,$$

the support of the solution becomes unbounded in finite time [9], [13], [17], [24], so that the functions η, ζ do not remain finite. Here we define $T_0 \leq \infty$ as the possible blow up time of the support.

In this paper, we are interested in the regularity properties of the interfaces. Due to the symmetry of Equation (1.1) we confine ourselves to considering the right interface $x = \eta(t)$. It seems to be a plausible conjecture that the regularity properties of the interfaces are locally the same as in the solutions of the porous medium equation. However it is difficult to apply the standard methods of proof for Equation (1.1) which is nonhomogeneous. This is our motivation for making use of Lagrangian coordinates and transform the problem into a problem for a nonlocal parabolic equation posed in a fixed bounded domain. This new problem is denoted by (P). The Lagrangian coordinate transformation is explained in Section 2.

In Section 3 we prove the existence of a solution to the auxiliary (Lagrangian) Problem (P) and prove that any solution generates a solution of the original problem, which is already known to be unique [19, 23, 13]. The advantage of this method of constructing a solution to the Cauchy problem for Equation (1.1) is that we automatically obtain a suitable representation for the interface. Relying on this representation, we show in Section 4 that the interface is a Lipschitz-continuous curve.

In Section 5 we derive the interface equation which holds for almost every t. We can then come back to the original coordinates (x, t) and to adapt methods for the study the regularity of the interface as proposed by [5] and [10]. In particular we prove that the function $\dfrac{m}{m-1}(u^{m-1})_x$ has a limit when $x \to \eta(t)-$ for every $t > 0$. If this limit is nonzero in a neighborhood of the time t_0, a suitable change of function with a scaling transformation permits to prove that the interfaces are locally C^1-curves after the waiting times and until the moment of disappearance, are Lipschitz-continuous, and obey an equation whose natural interpretation is that of Darcy's law in filtration theory. The proofs of these assertions are given in Section 6.

We remark that if the function ρ is smooth enough, one would expect that the interface should be quite smooth as well. Therefore this article leaves open the question of finding the optimal regularity of the interface and how it depends on the regularity of ρ.

2. Lagrangian coordinates

Given a solution $u(x,t) \in C^{2,1}(S_{T_0})$ of the Cauchy problem for Equation (1.1) , we introduce a system of Lagrangian coordinates which allows to replace the Cauchy problem by a problem posed in a bounded time-independent domain.

We present below a system of Lagrangian coordinates closely following [26, 27]. It is worth noting that Lagrangian type coordinates appear to be very useful in the study of the qualitative behaviour of solutions of various free boundary problems. The reader is referred to [25, 27] for more details on this method. The main idea – independently proposed by [22, 8, 15] – consists in viewing an evolution equation as the mass balance law for the motion of some fictitious continuum. The solutions of the Cauchy problem for Equation (1.1) possess the property, [23],

$$\forall\, t \in [0, T_0) \qquad \int_{-\infty}^{\infty} \rho(s)u(s,t)ds = 2M, \qquad M = const < \infty, \tag{2.1}$$

whose natural interpretation is that of *the mass conservation*. This leads us to set $d(x,t) = \rho(x)u(x,t)$. If we assume that the particles velocity is given by the expression

$$v(x,t) = -\frac{m}{m-1}\frac{1}{\rho(x)}\left(u^{m-1}\right)_x,$$

then Equation (1.1) takes the form of the mass conservation law

$$d_t + (d\,v)_x = 0$$

in the Euler variables.

In the Euler description of continua, the characteristics of the motion (velocity, density, pressure, etc.) are considered as functions of the time t and of some cartesian coordinate x not connected to the medium itself.

An alternative method of description is due to Lagrange. In the Lagrangian description the independent variables are the time t and the initial positions of the particles. If the total number of particles (e.g. the mass of the substance) doesn't change in time (cf. (2.1)), then in the plane of Lagrangian coordinates the boundary of the volume occupied by the continuum is time independent and hence the free boundary problem is now turned into a problem on a fixed domain.

Literally repeating the arguments of [27], it is easy to derive the following Lagrangian analog of the Cauchy problem for Equation (1.1):

$$X_t(z,t) = v\left[X(z,t),t\right] \quad t>0, \qquad X(z,0)=z\in(-a,a), \tag{2.2}$$

$$\rho(X(z,t))u\left[X(z,t),t\right]X_z(z,t) = \rho(z)u(z,0). \tag{2.3}$$

Equation (2.2) is *the trajectory equation* which controls the motion of particles along the trajectories $x = X(z,t)$ whereas (2.3) expresses *the mass balance law.*

The variable z is usually called *the geometric Lagrangian variable.* It is convenient to pass to the so-called *mass coordinate* which permits to eliminate the initial function u_0 from the mass balance law (2.3). Let us define ξ by the relation

$$\xi = \int_0^x \rho(s)u_0(s)ds : (0,a)\mapsto(0,M).$$

The physical meaning of ξ is transparent: it expresses the mass of substance initially contained in the interval $(0,x)$.

In what follows we use capitals to denote quantities which are viewed as functions of the mass Lagrangian coordinates (ξ,t). We set

$$x = X(\xi,t), \qquad u(X(\xi,t),t)=U(\xi,t), \qquad v(X(\xi,t),t)=V(\xi,t).$$

With these definitions, the mass balance law is given by the expression

$$\rho(X(\xi,t))U(\xi,t)X_\xi(\xi,t) = 1 \qquad \text{in } Q=(0,M)\times(0,T). \tag{2.4}$$

where $T < T_0$ is arbitrary. Equation (2.4) gives the rule to recalculate the derivative in the space variables,

$$\frac{\partial}{\partial x} = \rho(X(\xi,t))U(\xi,t)\frac{\partial}{\partial\xi},$$

which takes the trajectory equation to the form

$$X_t(\xi,t) = -\left(U^m\right)_\xi \quad t>0, \qquad \xi\in(0,M). \tag{2.5}$$

Eliminating $X(\xi,t)$ from (2.4)-(2.5), we arrive at the following initial-boundary value problem for the function $U=U(\xi,t)$:

$$U_t - \rho(X)U^2(U^m)_{\xi\xi} - \frac{\rho'(X)}{\rho(X)}U(U^m)_\xi = 0 \qquad \text{in } Q, \tag{2.6}$$

$$(U^m)_\xi(0,t)=0, \quad U(M,t)=0 \qquad t\in[0,T], \tag{2.7}$$

$$U(\xi,0)=u_0(X(\xi,0)) \qquad \xi\in[0,M], \tag{2.8}$$

and the function $X(\xi, 0)$ has to be recalculated from the definition of ξ.

The first boundary condition in (2.7) means that a particle initially located at the point $x = 0$ is motionless for all $t \geq 0$. By (2.5), this implies that the following condition is fulfilled at the symmetry axis:

$$X(0,t) = 0, \qquad t \geq 0.$$

The second boundary condition (2.7) expresses the fact that the stationary boundaries of the rectangle Q are the images of the free boundaries of the moving volume in the plane of the original Euler variables (x, t).

Given any $t \geq 0$, it follows from (2.4) that the function $X(\xi, t)$ can be obtained as the solution of the equation

$$\int_0^{X(\xi,t)} \rho(s)ds = \int_0^{\xi} \frac{ds}{U(s,t)}. \tag{2.9}$$

Note that Equation (2.9) is NONLOCAL: the function $\rho(X)$ is connected to $U(\xi, t)$ by an integral relation which can be derived either from (2.4) or from (2.5).

It is worth noting that our passage to the new (Lagrangian) coordinate was based so far on the formal resemblance between the original Equation (1.1) and the mass balance law in the Euler description of the motion of some continuum. The possibility of an alternative description of the same motion is due to the physical intuitions about the motions of continua, that is why the change of variables performed still lacks a rigorous mathematical justification. This can be done in the following way. Problem (2.6)-(2.8) will be treated as an independent mathematical object. We shall independently prove the existence of at least one classical solution to (2.6)-(2.8) and shall show thereafter that this solution corresponds to a weak solution of the Cauchy problem for Equation (1.1). Since the Cauchy problem is already known to have a unique weak solution, the connection between the two systems of coordinates will be thus established.

The functions ρ and u_0 are assumed to satisfy the following conditions.

A1
$$\begin{cases} \rho(x) > 0 & \text{for } x \geq 0, \qquad \rho(x) \in C^2_{loc}(\mathbb{R}), \\ \forall\, x \geq 0 & \left|\dfrac{\rho'(x)}{\rho(x)}\right| \leq \lambda, \qquad \lambda = \text{constant} \geq 0. \end{cases}$$

A2 For all $x \geq 0$ $\quad \left(\dfrac{\rho'(x)}{\rho(x)}\right)' \leq \mu \qquad \mu = \text{constant} \geq 0.$

B1 The functions ρ and u_0 are even: $u_0(-x) = u_0(x)$, $\rho(x) = \rho(-x)$ for all $x \in \mathbb{R}$,

B2

a) u_0 is bounded and continuous in $\mathbb{R}$ with $\sup_{\mathbb{R}} |u_0| = M_0 < \infty$;

b)
$$\begin{cases} u_0(x) > 0 & \text{in } [0,a),\ a > 0, \\ u_0 \equiv 0 & \text{in } \mathbb{R}^+ \setminus [0,a); \end{cases}$$

c) $$\left\| \frac{1}{\rho(x)} \left(u_0^{m-1}\right)_x \right\|_{L_\infty(\mathbb{R})} \leq M_1;$$

d) there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ the set $\{x \in (0,a) : u_0(x) > \varepsilon\}$ is an interval.

Our main interest is the interface regularity which is a *local* property. That is why all the further considerations will also be local in the following sense: given some finite number N, we shall consider the problem within the time interval $(0,T)$ such that the support of the solution is included in $(0,N)$ for all $t \in (0,T]$. This is not a serious limitation in the sense that at the time T the solution displays all the properties listed in **B2**. This allows us to repeat all the arguments taking T for the new initial time. On the other hand, if one of the items in **B2** is false at some T', this is the instant of disappearance of the interfaces.

The first step is to fix the correspondence between the solution $u(x,t)$ of the Cauchy problem for Equation (1.1) and the solution of its Lagrangian counterpart.

Theorem 2.1. *Let assumptions* **A1** – **A2** *and* **B1** – **B2** *be fulfilled, and let $u(x,t)$ be the solution of the Cauchy problem for Equation* (1.1). *Given some $N > a+1$, let $T_* < T_0$ be such that $\eta(t) + 1 \leq N$ for $t \in [0, T_*]$. Then:*

1) *Problem* (2.6) – (2.8) *has at least one classical solution $U(\xi,t)$ in the rectangle $Q = (0,M) \times (0,T)$ with $T \leq T^*$ and such that*

$$T \leq \frac{1}{2}\left(1 - e^{-m(N-a)}\right) \min\left\{ \frac{1}{m\lambda M_1}; \left(\frac{R_N}{m\lambda} \frac{(m\mu)^{(m-1)/m}}{M_0^{(m-1)/m}} \right)^m \right\},$$

where $R_N = \inf\{\rho(s) : s \in (0, N+1)\}$; the function U is strictly positive in $\overline{Q} \setminus \{\xi = M\}$ and $U(\xi,t) \in C^{2,1}(Q) \cap C^{\beta,\beta/2}(\overline{Q})$;

2) *The solution $u(x,t)$ to the Cauchy problem for Equation* (1.1) *admits the parameter representation*

$$u(x,t) = \begin{cases} U(\xi,t) & \text{if } x = X(\xi,t), \\ 0 & \text{elsewhere,} \end{cases}$$

$$X(\xi,t) = -\int_0^t (U^m)_\xi(\xi,\theta)d\theta + X(\xi,0), \quad (\xi,t) \in \overline{Q}.$$

Note that we need not prove the uniqueness of solution to problem (2.6)-(2.8). Any of the solutions of this problem generates a solution to the Cauchy problem for Equation (1.1) which is known to be unique.

The parameter representation given by Theorem 2.1 reduces the study of the interface $x = \eta(t)$ to the investigation of properties of the function $X(\xi,t)$ at the level $\xi = M$.

Let us assume that additionally to the above-formulated conditions the function ρ is such that

A3 $$\forall\, s \geq 0 \qquad \left| \left(\frac{\rho'(s)}{\rho(s)} \right)' \right| \leq \mu, \qquad \mu = const > 0.$$

Theorem 2.2. *Let the conditions of Theorem* 2.1 *and condition* **A3** *be fulfilled. Then:*

1) *the interface* $x = \eta(t)$ *is a locally Lipschitz-continuous curve in* $(0, T)$*;*

2) *for every* $t > 0$ *there exists the limit of* $(u^{m-1}(x,t))_x$ *as* $x \to \eta(t)$ *exists and the interface satisfies the equation*

$$\eta'(t) = -\frac{m}{m-1} \lim_{x \to \eta(t)} \frac{1}{\rho(x)} \left(u^{m-1}\right)_x (x,t) \qquad \textit{(Darcy's law)}, \qquad (2.10)$$

for almost every $t \in (0, T)$,

3) *for every* $t \in (0,T)$ *there exists* $D^+\eta(t)$ *and*

$$D^+\eta(t) = -\frac{m}{m-1} \frac{1}{\rho(\eta(t))} \left(u^{m-1}\right)_x (\eta(t), t), \qquad (2.11)$$

4) *if* $\limsup_{x \uparrow a} (u_0^{m-1})'(x) < 0$, *then the function* $\eta(t)$ *is strictly monotone increasing,* $(u^{m-1})_x(\eta(t), t) < 0$ *for all* $t \in (0, T_0)$, *furthermore for every* $t_0 \in (0, T)$ *the function* $\eta(t)$ *is a* C^1*-function in a neighborhood of* t_0 *and Equation* (2.10) *holds everywhere in this neighborhood.*

Remark 2.1. Condition **B2** *d*) will only be used in the proof of the strict positivity of U inside Q (subsection 3.3). One can drop this condition and verify the strict positivity of U assuming condition **A3** (Corollary 5.1 to Lemma 5.1).

3. Solvability of the nonlocal problem

It is convenient to introduce the function $W \equiv U^m$ which satisfies the equation

$$LW \equiv W_t - m\rho(X)W^{(m+1)/m}W_{\xi\xi} - m\frac{\rho'(X)}{\rho(X)}WW_\xi = 0 \qquad \text{in } Q,$$

subject to the initial and boundary conditions

$$W(M,t) = 0; \qquad W_\xi(0,t) = 0; \qquad W(\xi, 0) = W_0(\xi) \equiv u_0^m(X(\xi, 0)),$$

where the function $X(\xi, t)$ is defined by (2.9).

3.1. Regularization

The equation for W is a nonlocal degenerate parabolic equation. Its solution will be constructed as the pointwise limit of a sequence $\{W_n\}$ of solutions to the following regularized problems (problems (P_n)):

$$L_nW_n \equiv W_{nt} - m\rho_n(X_n)W_n^{(m+1)/m}W_{n\xi\xi} - m\frac{\rho_n'(X_n)}{\rho_n(X_n)}\left(W_n - \frac{1}{n}\right)W_{n\xi} = 0 \qquad \text{in } Q \quad , \tag{3.1}$$

$$W_n(M,t) = \frac{1}{n}, \qquad W_{n\xi}(0,t) = 0, \qquad W_n(\xi,0) = W_{0n}(\xi),$$

where the function X_n also defined as in (2.9):

$$\int_0^{X_n(\xi,t)} \rho_n(s)\,ds = \int_0^{\xi} \frac{ds}{W_n^{1/m}(s,t)}. \tag{3.2}$$

The sequence of initial data $\{W_{0n}\}$ is chosen in the following way:

$W_{0n} \in C^\infty[0,M], \qquad W_{0n} \to W_0$ in $C[0,M]$ as $n \to \infty$,

$W_{0n} \geq \frac{1}{n}$ in $[0,M] \qquad |W_{0n}'| \leq M_2$ in $[0,M]$,

$W_{0n} \equiv \frac{1}{n}$ in a neighborhood of the point $\xi = M$,

$W_{0n}=$ constant in a neighbourhood of the point $\xi = 0$,

if $\varepsilon_0 > \frac{1}{n}$, then every set $\{\xi \in (0,M) : W_{0n}(\xi) > \varepsilon\}$, $\varepsilon \in (1/n, \varepsilon_0)$, is an interval.

The functions $\rho_n(s)$ satisfy the conditions

$$\rho_n(s) \in C^2, \qquad \rho_n(s) = \begin{cases} \rho(s) & \text{if } s \leq N, \\ \rho(N+1) & \text{if } s \geq N+1. \end{cases}$$

Because of the nonlocal nature of Equation (3.1), the solvability of Problem (P_n) does not follow directly from any of the results of general parabolic theory. We shall establish the existence of a solution to (P_n) by adapting the Leray-Schauder principle [21, pp.449-455] in the form in which it was used in [27]. Let us associate with (3.1) the family of parabolic operators depending on the parameter $\tau \in [0,1]$

$$L_n^{(\tau)}W \equiv W_t - m\left[(1-\tau) + \tau\rho(X)V^{(m+1)/m}\right]W_{\xi\xi} - m\frac{\rho'(X)}{\rho(X)}\left(V - \frac{1}{n}\right)W_\xi = 0. \tag{3.3}$$

(To simplify the notation we omit the subindex "n"). Define the operator $W = \Phi(V,\tau)$ which assigns to the pair (V,τ) a solution W of the problem

$$(P_n^{(\tau)}) \qquad \begin{cases} L_n^{(\tau)}W = 0 & \text{in } Q, \\ W(M,t) = 1/n, \quad W_\xi(0,t) = 0, \quad W(\xi,0) = W_{0n}(\xi). \end{cases}$$

The operator Φ will be considered as a mapping from a ball

$$B_R \subset C^{1+\alpha,(1+\alpha)/2}(Q)$$

into itself. The radius R of the ball and the exponent α still have to be defined. To ensure that the mapping $\Phi(\cdot,1)$ has a fixed point, (that is, a solution of Problem (P_n)), it suffices to check the validity of the following assertions.

1) For $\tau = 0$ problem $(P_n^{(0)})$ has at least one nontrivial solution – this is evident and need not any special verification.

2) Each of the fixed points of the operators $\Phi(W,\tau)$ with $\tau \in [0,1]$ (e.g. each of the possible solutions of Problem $(P_n^{(\tau)})$ with $V = W$) satisfy the following uniform in τ estimates

a. $1/n \le W \le M_0$ in $\overline{Q}$;
b. $|W_\xi| \le M_1$ in $\overline{Q}$;
c. $\|W\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q})} \le M_2$ with some $\alpha \in (0,1]$.
d. $\Phi(V,\tau)$ is equicontinuous and uniformly compact.

In order to prove the solvability of (P_n) we derive below the above estimates.

3.2. A priori estimates. Existence of a solution to (P_n).

Let us note beforehand that the value of T will serve as a parameter in the derivation of the a priori estimates to follow. Each of the a priori estimates for the solutions of problems (P_n) will require a new restriction on T. Thus, if T is chosen in order to satisfy, say, the conditions of Lemma 3.1, we may use the estimate of Lemma 3.1 to get the estimate of Lemma 3.2, but the latter will hold in a narrower domain with some new value of T, etc.

Lemma 3.1. *Suppose that λ satisfies condition* **A1** *and let $T > 0$ satisfy the condition*

$$|W'_{0n}| \le M_1 \le \frac{1}{m\lambda T}.$$

If W be a fixed point of the mapping $\Phi(\cdot,\tau)$ for some $\tau \in [0,1]$, then

$$\frac{1}{n} \le W \le \frac{M-\xi}{m\lambda(T-t)} + \frac{1}{n} \qquad \text{in } \overline{Q}. \tag{3.4}$$

Proof. The lower estimate in (3.4) immediately follows from the maximum principle. Let us consider the function

$$\omega(\xi,t) = \frac{1}{n} + \frac{M-\xi}{m\lambda(T-t)}.$$

This function satisfies the relations

$$\omega(\xi,t) \ge W \quad \text{as } t = 0 \text{ and } \xi = M; \qquad (W-\omega)_\xi\Big|_{\xi=0} \ge 0.$$

Hence, the difference $W-\omega$ cannot attain a positive maximum at the parabolic boundary of Q, and since

$$L_n^{(\tau)}\omega \equiv \frac{1}{(m\lambda)^2}\frac{M-\xi}{(T-t)^2}\left[m\lambda + m\frac{\rho'(x)}{\rho(x)}\right] \geq 0 \qquad \text{in } Q$$

by virtue of **A1**, $W-\omega$ must be nonpositive in $\overline{Q}$. ■

Corollary 3.1.

$$\forall\, t \in [0,T] \qquad 0 \geq W_\xi(M,t) \geq -\frac{1}{m\lambda(T-t)}. \tag{3.5}$$

Proof. Each point of the vertical line $\xi = M$ corresponds to the minimum value of W in $\overline{Q}$, therefore $W_\xi(M,t) \leq 0$. To get the other estimate, rewrite (3.4) in the form

$$\frac{W(M,t)-W(\xi,t)}{M-\xi} \geq -\frac{1}{m\lambda(T-t)}$$

and then pass to the limit as $\xi \to M$. ■

Lemma 3.2. *Let T satisfy the hypotheses of Lemma 3.1 and let W be a fixed point of the mapping $\Phi(\cdot,\tau)$ for some $\tau\in[0,1]$. Additionally suppose that*

$$T < \left(\frac{R_N}{m\lambda}\frac{(m\mu)^{(m-1)/m}}{M^{(m-1)/m}}\right)^m$$

where the constants λ, μ satisfy conditions **A1** − **A2** *and R_N satisfies the conditions of Theorem 2.1. Then the function $Z = W_\xi$ is such that*

$$|Z| \leq \frac{1}{m\lambda(T-t)} \qquad \text{in } \overline{Q}. \tag{3.6}$$

Proof. Due to the new choice of T, estimate (3.5), and the boundary condition for W at $\xi = 0$, we have:

$$|Z| \leq \frac{1}{m\lambda(T-t)} \qquad \text{on the parabolic boundary of } Q.$$

The equation for Z follows from (3.3) and has the form

$$\begin{aligned}
\mathcal{A}Z &\equiv Z_t - m\left[(1-\tau)+\tau\rho(X)W^{(m+1)/m}\right]Z_{\xi\xi} - m\left(\frac{\rho'(X)}{\rho(X)}\right)'\frac{W-\frac{1}{n}}{W^{1/m}}\frac{1}{\rho(X)}Z\\
&- m\left[\tau\frac{m+1}{m}\rho(X)W^{1/m}Z + \tau W^{(m+1)/m}\frac{\rho'(X)}{\rho(X)}\frac{1}{W^{1/m}}\right]Z_\xi\\
&- m\frac{\rho'(X)}{\rho(X)}\left(W-\frac{1}{n}\right)Z_\xi = m\frac{\rho'(X)}{\rho(X)}Z^2,
\end{aligned}$$

whence it follows that Z satisfies the parabolic inequalities

$$\mathcal{M}_+ Z \equiv \mathcal{A}Z + m\lambda Z^2 \geq 0, \qquad \mathcal{M}_- Z \equiv \mathcal{A}Z - m\lambda Z^2 \leq 0.$$

Under assumptions **A1-A2**

$$\mathcal{M}_-\left(\frac{1}{2m\lambda(T-t)}\right) \geq 0, \qquad \mathcal{M}_+\left(\frac{-1}{2m\lambda(T-t)}\right) \leq 0 \qquad \text{in } Q,$$

and the assertion then follows from the maximum principle. ■

Lemma 3.3. *Problem (P_n) has in Q at least one classical solution W_n which is such that*

$$\|W_n\|_{C^{1+\alpha,1+\frac{\alpha}{2}}(\overline{Q})} \leq C,$$

for some positive constant C which does not depend on n.

Proof. The following estimates are stated by Lemmas 3.1-3.2:

$$\frac{1}{n} \leq W \leq M_2, \qquad |W_\xi| \leq M_2 \qquad \text{in } \overline{Q}$$

with an absolute positive constant M_3 independent of n and τ, but depending on T. The second of these estimates implies the uniform Hölder continuity in t of the function W [11, 20]. Thus, the equation $L_n^{(\tau)} W = 0$ can be viewed as a linear uniformly parabolic equation with the coefficients bounded in the norm of $C^{\alpha,\alpha/2}(\overline{Q})$, with some $\alpha \in (0,1)$. Such a problem has a unique solution $W \in C^{2+\alpha,(2+\alpha)/2}(\overline{Q})$, The estimate on the radius R of the ball B_R is given by the estimate on the norm of W in $C^{2+\alpha,(2+\alpha)/2}(\overline{Q})$.

Now let us verify that $\Phi(V,\tau)$ on $B_R \times [0,1]$ is equicontinuous in V and τ. Let V_1, $V_2 \in B_R$ be such that $\|V_1 - V_2\|_{C^{1+\alpha,(1+\alpha)/2}} < \varepsilon$. Set $\overline{W} = W_1 - W_2$, where $W_i = \Phi(V_i,\tau)$, $i = 1,2$. It follows from (3.3) that $\overline{W}$ satisfies the equation

$$\begin{aligned} &\overline{W}_t - m\left[1 + \tau\rho(X)V_1^{(m+1)/m}\right]\overline{W}_{\xi\xi} + \frac{1}{n}m\frac{\rho'(X)}{\rho(X)}\overline{W}_\xi \\ &\quad = m\tau\rho(X)\left\{V_2^{\frac{1+m}{m}} - V_1^{\frac{1+m}{m}}\right\}W_{2,\xi\xi} + \frac{\rho'(X)}{\rho(X)}\{V_1 - V_2\}W_{1,\xi}. \end{aligned} \tag{3.7}$$

Since the right-hand side of (3.7) is small uniformly in $\tau \in [0,1]$, $\|\overline{W}\|_{C^{1+\alpha,(1+\alpha)/2}}$ is also small. The uniform continuity with respect to τ is checked in the same way. ■

Let us derive one more a priori estimate which will be useful in the proof of convergence of the sequence $\{W_n\}$.

Lemma 3.4. *Let T be the value chosen in the proofs of Lemmas 3.1–3.2. There exists a positive constant A such that*

$$\forall\, t \in [0, T/2] \qquad \int_0^M \frac{ds}{W^{1/m}} \leq A.$$

Proof. Let us adopt the notation $U = W^{1/m}$ and rewrite Equation (3.1) in the form

$$\left(\frac{1}{U}\right)_t + \rho(X)\,(U^m)_{\xi\xi} = -\frac{\rho'(X)}{\rho(X)}\frac{W-\frac{1}{n}}{W}\frac{1}{U}W_\xi. \tag{3.8}$$

Integrating of (3.8) over the rectangle $(0, M) \times (0, t)$, $t \le T/2$, leads to the equality

$$\int_0^M \frac{ds}{U} = \int_0^M \frac{ds}{U_0(s)} - \int_0^t \rho(X(M,\tau))W_\xi(M,\tau)d\tau + \frac{1}{n}\int_0^t\int_0^M \frac{\rho'(X)}{\rho(X)}\frac{W_\xi}{W^{\frac{m+1}{m}}}dsd\tau. \tag{3.9}$$

To derive this equality we have used the relation

$$X_\xi = \frac{1}{\rho U}$$

which follows from (3.2) after differentiation. The second term on the right-hand side of (3.9) is uniformly in n bounded because

$$\int_0^t \rho(X(M,\tau))W_\xi(M,\tau)d\tau \le t\,\max_{[0,N]}\rho\,\max_{\overline{Q}}|W_\xi| \equiv C_2\,t.$$

Let us estimate the third term. Note that this term can be written as

$$I_n \overset{\text{def}}{=} -\frac{m}{n}\int_0^t\int_0^M \frac{\rho'(X)}{\rho(X)}(W^{-\frac{1}{m}})_\xi\,dsd\tau.$$

Integrating the right-hand side of I_n by parts, we have

$$I_n = -\frac{m}{n}\int_0^t \frac{\rho'(X)}{\rho(X)}W^{-\frac{1}{m}}(M,\tau)d\tau + \frac{m}{n}\int_0^t \frac{\rho'(X)}{\rho(X)}W^{-\frac{1}{m}}(0,\tau)d\tau$$

$$+\frac{m}{n}\int_0^t\int_0^M \left(\frac{\rho'(X)}{\rho(X)}\right)'\frac{1}{\rho W^{2/m}}dsd\tau.$$

Assumptions **A1**,**A2** and Lemma 3.2 imply the estimate

$$|I_n| \le \frac{2m\lambda}{n^{(m-1)/m}}t + \frac{m}{n^{(m-1)/m}}\frac{\mu}{\min_{[0,N]}\rho}\int_0^t\int_0^M \frac{dsd\tau}{U(s,\tau)}$$

$$\equiv \frac{2m\lambda}{n^{(m-1)/m}}t + C_3\,n^{(1-m)/m}\int_0^t\int_0^M \frac{dsd\tau}{U(s,\tau)}.$$

Let us introduce the function

$$Y(t)=\int_0^t\int_0^M\frac{d\tau ds}{U}.$$

It satisfies Gronwall's inequality

$$Y'(t)\le Y'(0)+\left(C_2+\frac{2m\lambda}{n^{(m-1)/m}}\right)t+C_3n^{\frac{1-m}{m}}Y(t)\qquad \text{for } t>0\quad,\qquad Y(0)=0, \tag{3.10}$$

which can be easily integrated. We have

$$Y(t)\le Kn^{\frac{m-1}{m}}\left(e^{C_3tn^{\frac{1-m}{m}}}-1\right)\qquad t>0,$$

with a constant K independent on t and n, whence

$$Y(t)\le\overline{K}t\left(1+\mathcal{O}\left(tn^{\frac{1-m}{m}}\right)\right)$$

for large n. The last inequality implies the estimate $Y(t)\le const.$ for $t\in[0,T/2]$ and, correspondingly,

$$Y'(t)\le const.+Cn^{\frac{1-m}{m}}.$$

■

Lemma 3.5. *Under the conditions of Lemma* 3.4

$$X_t+W_\xi=\mathcal{O}\left(n^{\frac{1-m}{m}}\right). \tag{3.11}$$

Proof. Equation (3.8) can be written as

$$\left(\frac{1}{\rho U}\right)_t+W_{\xi\xi}+\frac{\rho'}{\rho}\frac{1}{\rho U}(X_t+W_\xi)=\frac{1}{n}\frac{\rho'}{\rho^2}W^{-\frac{1+m}{m}}W_\xi. \tag{3.12}$$

Note that by virtue of (3.2)

$$X_{\xi t}=\left(\frac{1}{\rho U}\right)_t.$$

Define the function

$$G(\xi)\overset{\text{def}}{=}X_t+W_\xi$$

depending on t as a parameter. We have $G(0) = 0$ and

$$G'(\xi) + \frac{\rho'}{\rho}\frac{1}{\rho U}G(\xi) = -\frac{m}{n}\frac{\rho'}{\rho^2}(W^{-\frac{1}{m}})_\xi, \qquad \text{for } \xi \in (0, M).$$

Writing

$$\frac{d}{d\xi}\left\{Ge^{H(\xi)}\right\} = -\frac{m}{n}e^{H(\xi)}\frac{\rho'}{\rho^2}(W^{-\frac{1}{m}})_\xi \qquad \text{where } H(\xi) = \int_0^\xi \frac{\rho'}{\rho}\frac{ds}{\rho U},$$

and integrating in ξ, we arrive at the inequality

$$|G| \le Cn^{\frac{1-m}{m}}$$

with a constant C independent on n. ■

3.3. Passage to the limit. Proof of Theorem 2.1.

Let us preliminary fix some T such that the assertions of Lemmas 3.1–3.5 hold in the rectangle $Q = (0, M) \times (0, T)$.

It follows from Lemmas 3.1–3.2 that the sequence $\{W_n\}$ is such that, along a subsequence, the following lemma holds.

Lemma 3.6. *As* $n \to \infty$, *we have*

$$W_n \to W \in C^{\beta,\beta/2}(\overline{Q}), \; W_{n\xi} \to W_\xi \in C^{\beta,\beta/2}(\overline{Q}), \quad \beta < \alpha,$$

$$U_n \to U, \qquad X_n \to X,$$

uniformly in $\overline{Q}$.

Without loss of generality we may assume that $\{W_n\}$ coincides with this subsequence.

Integrating (3.11) in t and applying Lemma 3.2 we obtain the estimate

$$0 \le X_n(\xi, t) \le X_n(\xi, 0) + \frac{1}{m\lambda}\int_0^t \frac{d\tau}{T-\tau} + Cn^{\frac{1-m}{m}},$$

which means that for all n, beginning with some n_0, and every t sufficiently small

$$0 \le X_n(\xi, t) \le X_n(M, 0) - \frac{1}{m\lambda}\ln\left(1 - \frac{t}{T}\right) + Cn^{(1-m/m)} \le N. \qquad (3.13)$$

This inequality allows us to replace ρ_n by ρ in all further considerations.

Let $f(\xi, t)$ be a smooth function with compact support in Q. Multiplying Equation (3.12) for W_n by f and integrating by parts in Q yields the relation

$$\int_Q \left\{ \frac{f_t}{\rho(X_n)U_n} + f_\xi (U_n^m)_\xi \right\} ds d\tau = - \int_Q \frac{\rho'}{\rho} \frac{f}{\rho U_n} (X_{n_t} + W_{n_\xi})\, ds d\tau - \int_Q \frac{m}{n} \frac{f \rho'}{\rho^2} (W_n^{-\frac{1}{m}})_\xi \, ds d\tau. \tag{3.14}$$

The a priori estimates of the previous subsection allow us to pass to the limit as $n \to \infty$, which implies, by (3.10) and arguments from the proof of (3.10), the following identity for the limit function W:

$$\int_Q \left\{ \frac{f_t}{\rho(X)U} + f_\xi (U^m)_\xi \right\} ds d\tau = 0, \qquad U = W^{1/m}. \tag{3.15}$$

In view of (2.5), (3.15) implies that U is a weak solution of (2.6). Since the function W_n and W_{n_ξ} converge uniformly in Q to their limits, and by the construction of W_{0_n} the initial and boundary conditions for U are fulfilled. Hence, $U(\xi, t)$ is a continuous weak solution to problem (2.6)-(2.8).

To ensure that U satisfies Equation (2.6) in the classical sense amounts to proving that U is bounded away from zero on any compact subset of Q of the form $[\nu, M-\nu] \times [\nu, T]$ with $\nu > 0$ small enough, and applying "bootstrap" arguments. The positivity of U follows by a routine consideration of the level curves of the functions W_n. Assuming that $U(\xi_0, t_0) = 0$ and considering the sequence $\{W_n\}$ we arrive at the following situation: there exists a sequence of level curves $\{l_{n,\delta} : \ W_n = \delta$ along $l_{n,\delta}\}$ such that $\delta \to 0$ and $n \to \infty$. The maximum principle implies that each $l_{n,\delta}$ is monotone in time and connects $\overline{Q} \cap \{t = 0\}$ to the truncation $\overline{Q} \cap \{t = t_0\}$. Due to the strict positivity of W_0 the sequence of points $\{l_{n,\delta} \cap \{t = 0\}\}$ can only dense to the point $\xi = M$ where $W_0 = 0$, while the sequence $\{l_{n,\delta} \cap \{t = t_0\}\}$ converges to ξ_0 by the assumption. The maximum principle implies then that for any n sufficiently large $\rho(X_n)U_n(\xi, t_0) < \varepsilon_n$ for all $\xi \in (\xi_0, M)$ with $\varepsilon_n \to 0$ as $n \to \infty$, whence the contradiction to Lemma 3.4.

Note that an immediate byproduct of (3.11) and (3.15) is the following: the function $X(\xi, t)$, defined as the solution of Equation (2.9), satisfies Equation (2.5). The verification of assertion 2 of Theorem 2.1 is now straightforward. It consists in passing in Equation (2.6) to the original variables (x, t).

4. Lipschitz-continuity of the interface

Here we prove assertion 1 of Theorem 2.2. Now that Theorem 2.1 is proved, we can make the following observation: due to (2.9) (alias (2.4)) the function $X(\xi, t)$ is monotone in ξ, by (2.5) and the uniform in n estimate of Lemma 3.2 it is bounded in $\overline{Q}$, which means that $X(\xi, t)$ has a finite limit as $\xi \to M$ for all $t \in [0, T]$. Recalling now the representation via Lagrangian coordinates for the solution $u(x, t)$ of the Cauchy problem to (1.1) we have:

$$\eta(t) = \lim_{\xi \to M} X(\xi, t), \qquad t \in [0, T].$$

Integrating Equation (3.11) it t in the interval (t_0, t_1) and making use of the uniform in n estimate of Lemma 3.2, and then letting $n \to \infty$, we arrive at the inequality

$$|X(\xi,t_1)-X(\xi,t_0)| \le \frac{1}{m\lambda}\ln\frac{T-t_0}{T-t_1}, \qquad \forall\, \xi \in (0,M).$$

The local Lipschitz-continuity of $\eta(t)$ follows from this inequality after the limit passage $\xi \to M$.

5. The interface equation

In order to derive the interface equation a more delicate a priori estimate is needed. Let us return to the sequence $\{W_n\}$ of solutions to the approximating problems (P_n).

Lemma 5.1. *There exist a finite constant $C > 0$, depending on λ, μ, T, but independent of n, and an integer constant n_0 such that the functions $\sigma_n \equiv W_{nt}/W_n$ satisfy the estimate*

$$\sigma_n \ge -\frac{C}{t} \qquad \textit{in } \overline{Q} \quad \textit{for all } n \ge n_0. \tag{5.1}$$

Proof. We will use the notation $Q_s = (-M,M) \times (0,T]$. Denote by W_n the even continuation of the corresponding function on Q and by X_n its odd continuation.

A direct computation leads to the following equation for the function σ_n in the domain Q_s (for the simplicity we omit the index n) ,

$$\begin{aligned}
\mathcal{N}\sigma \equiv{}& \sigma_t - \frac{m+1}{m}\sigma^2 - m\rho(X)W^{(m+1)/m}\sigma_{\xi\xi} - m\left[2\rho(X)W^{1/m}W_\xi + \frac{\rho'(X)}{\rho(X)}(W-\frac{1}{n})\right]\sigma_\xi \\
&+ \left[(m+1)\frac{W-\frac{1}{n}}{W} - m\right]\frac{\rho'(X)}{\rho(X)}W_\xi\sigma - \frac{\rho'(X)}{\rho(X)}\left[-W_\xi + \mathcal{O}\left(n^{(1-m)/m}\right)\right]\sigma \\
={}& m\left[-\left(\frac{\rho'(X)}{\rho(X)}\right)^2 + \left(\frac{\rho'(X)}{\rho(X)}\right)'\right]\frac{W-\frac{1}{n}}{W}\left[-W_\xi + \mathcal{O}\left(n^{(1-m)/m}\right)\right]W_\xi, \\
={}& m\left[\left(\frac{\rho'}{\rho}\right)' - \left(\frac{\rho'}{\rho}\right)^2\right]\frac{W-\frac{1}{n}}{W}\left[W_\xi^2 - W_\xi\mathcal{O}\left(n^{(1-m)/m}\right)\right],
\end{aligned}$$

where we have used (3.11). Let us introduce the function $\omega = -C/t$, where the positive constant C still has to be defined. By the definition, $\sigma(\xi,t)$ is bounded for $t = 0$ and $\sigma = 0$ on the lateral boundaries of Q_s. Therefore

$$\omega < \sigma \qquad \text{on the parabolic boundary of } Q_s.$$

Next, it is easy to calculate that

$$\mathcal{N}\omega \equiv \frac{C}{t^2} - \frac{m+1}{m}\frac{C^2}{t^2} - \left\{\left[(m+1)\frac{W-\frac{1}{n}}{W} - (m-1)\right]\frac{\rho'(X)}{\rho(X)}W_\xi + \mathcal{O}\left(n^{(1-m)/m}\right)\right\}\frac{C}{t}.$$

By assumption **A1** and by virtue of Lemma 3.2 the coefficient of ω is bounded uniformly with respect to n. This allows us to take C so that the following inequality is fulfilled:

$$\mathcal{N}\omega < -\frac{m+1}{2m}\frac{C}{t} \qquad \text{in } Q_s \quad .$$

Increasing, if needed, C, we obtain the estimate $\mathcal{N}\sigma > \mathcal{N}\omega$ in Q_s, and the conclusion then follows from the maximum principle. ■

Corollary 5.1. *Under assumption* **A3** *the function $U(\xi,t)$ is strictly positive in $\overline{Q}\setminus\{\xi = M\}$.*

Proof. Let us take an arbitrary point $(\xi_0,t_0) \in \overline{Q}\setminus\{\xi = M\}$, $t_0 > 0$. Since $U(\xi,t)$ is Hölder-continuous in $\overline{Q}$ and $U_0(\xi_0) > 0$, there exists a time $t' \in (0,t_0)$ such that $U(\xi_0,t') \geq \alpha > 0$; moreover, the convergence $W_n \to U^m$ implies that $W_n(\xi_0,t') \geq (\alpha/2)^m$ for all n sufficiently large. Rewrite estimate (5.1) in the form

$$\left(\ln(t^C W_n)\right)_t \geq 0,$$

and integrate in t over the interval (t',t_0). This gives the inequality

$$W_n(\xi_0,t_0) \geq \left(\frac{t'}{t_0}\right)^C W_n(\xi_0,t') \geq \left(\frac{t'}{t_0}\right)^C \left(\frac{\alpha}{2}\right)^m,$$

which yields the positivity of W in the limit $n \to \infty$.

■

Corollary 5.2. *The limit function $U(\xi,t)$ satisfies the inequality*

$$\frac{U_t}{U} \geq -\frac{C}{m}\frac{1}{t} \qquad \text{in } Q \quad . \tag{5.2}$$

In order to derive the interface equation, we consider the auxiliary function

$$\Phi(\xi) = \delta X + (U^m)_\xi, \qquad \delta = const > 0,$$

depending on t as a parameter. The function $\Phi(\xi)$ is uniformly bounded from above. By (5.2) $\Phi(\xi)$ is monotone; indeed

$$\begin{aligned} \Phi'(\xi) &= \delta X_\xi + (U^m)_{\xi\xi} = \frac{\delta}{\rho(X)U} - \left(\frac{1}{\rho(X)U}\right)_t \\ &= \frac{\delta}{\rho(X)U} + \frac{\rho'(X)}{\rho(X)} X_t \frac{1}{\rho(X)U} + \frac{1}{\rho(X)U}\frac{U_t}{U} \\ &\geq \frac{1}{\rho(X)U}\left[\delta - \lambda \max|(U^m)_\xi| - \frac{C}{mt}\right] > 0, \end{aligned} \tag{5.3}$$

provided that $\delta = \delta(t) > 0$ is sufficiently large. Thus, for every $t > 0$, $\Phi(\xi)$ has a finite limit when $\xi \to M$. Since X also has a finite limit as $\xi \to M$, the same is true for the function

$$(U^m)_\xi = \Phi(\xi) - \delta X(\xi, t).$$

The interface equation follows now from (2.5) after passing to the limit as $\xi \to M$ and returning on the plane of the coordinates (x, t). Equation (2.11) will be proved in Section 6 below.

6. Local C^1-regularity of the interface

In the following study of the interface regularity we follow methods due to [5] and [10]. We fix an arbitrary point $t_0 \in (0, T_0)$ and set $v(x,t) = \frac{m}{m-1} u^{m-1}(x,t)$. It follows from the proofs in the previous sections that

$$v_x(\eta(t_0), t_0) = \lim_{x \to \eta(t_0)-0} v_x(x, t_0) \leq 0.$$

Next we introduce the function

$$v_\delta(x,t) \overset{\text{def}}{=} \frac{1}{\delta} v(\eta(t_0) + \delta x, t_0 + \delta t),$$

which depends on the positive parameter δ. For each $\delta > 0$ the function v_δ is a solution of the equation

$$\rho(\eta(t_0) + \delta x) v_{\delta,t} = (m-1) v_\delta v_{\delta,xx} + v_{\delta,x}^2.$$

According to estimates of the previous sections, on every interval $[0, T_0 - \epsilon]$ with $\epsilon > 0$ small enough,

$$|v_{\delta,x}| \leq C \tag{6.1}$$

uniformly with respect to δ. Estimate (6.1) means that on every compact subset of $\mathbb{R}^2$ the family $\{v_\delta\}$ is uniformly Lipschitz-continuous in x and, therefore, uniformly Hölder-continuous in the variable t with the exponent $\alpha = 1/2$ [20, 11]. This allows us to extract by a diagonal process from the family $\{v_\delta\}$ a subsequence which converges on every compact subset of $\mathbb{R}^2$ in the norm of C^α to a function $w(x,t)$:

$$w(x,t) = \lim_{\delta_n \to 0} v_{\delta_n}(x,t).$$

The function $\rho(\eta(t_0)) w(x,t)$ is a nonnegative continuous weak solution of the equation

$$w_t = (m-1) w w_{xx} + w_x^2 \qquad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{6.2}$$

and has the interface $x = s(t)$ which passes through the origin $(0,0)$.

We recall [5] that Equation (6.2) possesses a continuum of travelling wave solutions $\{f_\lambda\}$ which are given by

$$f_\lambda(x,t) = C\max\{t - \lambda x;\ 0\}, \qquad \lambda = \sqrt{\frac{1}{C}}.$$

The interfaces of the travelling wave solutions are the straight lines $\lambda x = t$ which pass through the origin $(0,0)$. At the time $t = 0$ the profile of w is given by

$$w(x,0) = q\ \max\{-x;\ 0\}, \qquad q = v_x(\eta(t_0), t_0),$$

which coincides with the profile of a travelling wave solution $f_{\lambda_1}(x,0)$. By uniqueness, we have that $w(x,t) \equiv f_{\lambda_1}(x,t)$ for all $t > 0$. This allows us to approximate the interface $x = \eta(t)$ by the interfaces of travelling wave solutions and to conclude, passing to the limit when $t \downarrow 0$, that there exists $D^+\eta(t_0)$ and that Equation (2.11) holds.

Finally we study the regularity of the moving part of the interface.

Lemma 6.1. *If* $\limsup_{x\uparrow a} v_{0x}(x) < 0$, *then* $v_x(\eta(t),t) < 0$ *for all* $t \in (0, T_0)$.

Proof. We return to the sequence $\{W_n\}$ approximating the solution of Problem (P). We have proved that its limit function W is strictly positive in Q. If $W_0 \in C[0,M]$ and $\limsup_{\xi\to M} W_{0\xi}(\xi) < 0$, the functions of the sequence $\{W_n\}$ satisfy the inequality

$$W_n \geq \omega \equiv C\frac{M-\xi}{T+t} \qquad \text{on the parabolic boundary of } Q$$

provided that the constant C is sufficiently small. Further, decreasing C we also obtain the inequality $L\omega < 0$ in Q, whence $W_n \geq \omega$ in $\overline{Q}$. Letting $n \to \infty$ we have $W(M,t) \geq \omega(M,t)$ and $W_\xi(M,t) \leq -C/(T+t)$. (see Corollary 3.1). ■

Let $x = s_\delta(t)$ be the interface of the function v_δ. Under the conditions of Lemma 6.1, there exists $\epsilon > 0$ such that for all δ sufficiently small $v_{\delta,x}(x,t) < -\epsilon$ if $x \uparrow s_\delta(t)$ and $t \approx 0$. This means that

$$\lim_{x\to s(t)} w_x(x,t) \leq -\epsilon \qquad \text{for } t \approx 0 \quad .$$

On the other hand, it is known [6] that the moving parts of the interface of $w(x,t)$, viewed as a solution of the Porous Media Equation, are infinitely differentiable. In the Porous Medium Equation the interface equation (along its smooth portion) has the form $s'(t) = -w_x(s(t),t)$, and the function $w_x(s(t),t)$ is continuous at the point $t = 0$. This yields the equality

$$\lim_{t\uparrow 0} w_x(s(t),t) = \lim_{t\downarrow 0} w_x(s(t),t), \qquad \text{whence} \qquad \lim_{t\uparrow t_0} v_x(\eta(t),t) = \lim_{t\downarrow t_0} v_x(\eta(t),t).$$

The right-hand side of the last equality is known to coincide with $D^+\eta(t_0)$. If the left-hand side coincides with $D^-\eta(t_0)$, the proof of Theorem 2.2 will be completed.

Let us make use of results of paper [10]. This paper is devoted to the study of generic regularity properties of solutions of degenerate parabolic equations of the type "weak diffusion/strong absorption" which admit a complete set of the travelling wave solutions. Equation (6.2) is a simplified version of such an equation. Applying [10, Th. 6.2, Th. 7.2], we have that at every instant $t \in \mathbb{R}$ the left derivative $D^- s(t)$ exists and satisfies the relation

$$\lim_{t \uparrow 0} D^- s(t) = \lim_{t \uparrow 0} w_x(s(t), t).$$

Since the right-hand side of this equality is continuous at the point $t = 0$ and, on the other hand, $D^- s(0) = D^- \eta(t_0)$, we conclude that $\eta'(t)$ exists and is continuous at the point t_0.

Note that although $D^- \eta(t)$ and $D^+ \eta(t)$ exist for every $t \in (0, T_0)$, they need not coincide if the t is the waiting time instant.

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