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Research article

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Concentration results for a magnetic Schrödinger-Poisson system with critical growth

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Abstract: This paper is concerned with the following nonlinear magnetic Schrödinger-Poisson type equation

$$\begin{cases} \left(\frac{\epsilon}{i}\nabla - A(x)\right)^2 u + V(x)u + \epsilon^{-2}(|x|^{-1} \star |u|^2)u = f(|u|^2)u + |u|^4 u \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3, \mathbb{C}), \end{cases}$$

where $\epsilon > 0$, $V : \mathbb{R}^3 \to \mathbb{R}$ and $A : \mathbb{R}^3 \to \mathbb{R}^3$ are continuous potentials, $f : \mathbb{R} \to \mathbb{R}$ is a subcritical nonlinear term and is only continuous. Under a local assumption on the potential V, we use variational methods, penalization technique and Ljusternick-Schnirelmann theory to prove multiplicity and concentration of nontrivial solutions for $\epsilon > 0$ small.

Keywords: Schrödinger-Poisson system, Magnetic field, Critical growth, Variational methods

MSC: 35J60, 35J25

1 Introduction and main results

In this paper, we study multiplicity and concentration of the nontrivial solutions of the following Schrödinger-Poisson type equations with critical growth

$$\left(\frac{\epsilon}{i}\nabla - A(x)\right)^2 u + V(x)u + \epsilon^{-2}(|x|^{-1} \star |u|^2)u = f(|u|^2)u + |u|^4 u \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $u \in H^1(\mathbb{R}^3, \mathbb{C})$, $\epsilon > 0$ is a parameter, $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, $f \in C(\mathbb{R}, \mathbb{R})$ has a subcritical growth, the magnetic potential $A : \mathbb{R}^3 \to \mathbb{R}^3$ is Hölder continuous with exponent $\alpha \in (0, 1]$, and the convolution potential is defined by $|x|^{-1} \star |u|^2 = \int_{\mathbb{R}^3} |x - y|^{-1}|u(y)|^2 dy$.

In recent years a considerable amount of work has been devoted to investigating the existence and multiplicity of solutions for nonlinear Schrödinger-Poisson system without magnetic field. We notice that, by using minimax theorema and the Ljusternik-Schnirelmann theory, He [25] gave multiplicity and concentration of positive solutions of the following problem

$$\begin{cases} -\epsilon^2 \Delta u + V(x)u + \phi(x)u = f(u), & \text{in } \mathbb{R}^3, \\ -\epsilon^2 \Delta \phi = u^2, & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3),\ u(x) > 0, & \text{in } \mathbb{R}^3. \end{cases}$$

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where $f \in C^1(\mathbb{R})$ has the subcritical growth and the potential V satisfies a global condition introduced by Rabinowitz [31]. In [26], He and Zou studied the existence and concentration of ground state solutions for the following Schrödinger-Poisson system with the critical growth

$$\begin{cases} -\epsilon^2\Delta u + V(x)u + \phi(x)u = f(u) + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\epsilon^2\Delta\phi = u^2, \ u(x) > 0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $f \in C^1(\mathbb{R})$ and the potential V satisfies a global condition. Then, He [27] studied the multiplicity of concentrating positive solutions for Schrödinger-Poisson system (1.2) with nonlinear term $f \in C(\mathbb{R})$ under a local assumption introduced by del Pino and Felmer [17]. For further results on Schrödinger-Poisson system without magnetic field, we refer to[1, 4, 5, 14, 15, 22, 32, 33, 36, 40] and the references therein(see also [21] for the fractional case).

Concerning the magnetic nonlinear Schrödinger equation (1.1), we refer to [6–8, 10–13, 16, 19, 23, 24, 29, 38, 39] and references therein. It is well known that the first result involving the magnetic field was obtained by Esteban and Lions [19]. They used the concentration-compactness principle and minimization arguments to obtain solutions for $\varepsilon > 0$ fixed. In [39], the authors studied multiplicity and concentration of solutions for magnetic relativistic Schrödinger equations, Xia [38] studied a critical fractional Choquard-Kirchhoff problem with magnetic field. In particular, due to our scope, we want to mention [41] where the authors studied a Schrödinger-Poisson type equation with magnetic field by using the method of the Nehari manifold, the penalization method and Ljusternik-Schnirelmann category theory for subcritical nonlinearity $f \in C^1$. If f is only continuous, then the arguments in [41] failed. Recently, by variational methods, penalization technique, and Ljusternick-Schniremann theory, for the magnetic Schrödinger-Poisson system with subcritical growth nonlinearity f which is only continuous, in [29] we proved multiplicity and concentration properties of nontrivial solutions for $\epsilon > 0$ small. For the fractional Schrödinger-Poisson type equations with magnetic field, we refer to [2, 3].

Inspried by [27, 29], we intend to prove multiplicity and concentration of nontrivial solutions for problem (1.1) with critical growth. Since the probem we deal with has the critical growth, we need more refined estimates to overcome the lack of compactness. On the other hand, due to the appearance of magnetic field $A(x)$ and the nonlocal term $|x|^{-1} \star |u|^2$, problem (1.1) will be more difficult, and some estimates are also more complicated.

In this paper, we make the following assumptions on the potential V:

$(V1)$There exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^3$;

$(V2)$There exists a bounded open set $\Lambda \subset \mathbb{R}^3$ such that

$$V_0 = \min_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x).$$

Observe that

$$M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

On the nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$, we require that:

$(f1)$ $f(t) = 0$ if $t \leq 0$, and $\lim_{t\to 0^+} \frac{f(t)}{t} = 0$;

$(f2)$ There exist $\sigma, q \in (4, 6)$ and $\mu > 0$ such that

$$f(t) \geq \mu t^{\frac{\sigma-2}{2}} \quad \forall t > 0, \quad \text{and} \quad \lim_{t\to+\infty} \frac{f(t)}{t^{\frac{q-2}{2}}} = 0;$$

$(f3)$ there is a positive constant $\theta \in (4, 6)$ such that

$$0 < \frac{\theta}{2}F(t) \leq tf(t), \qquad \forall\, t > 0, \quad \text{where } F(t) = \int_0^t f(s)ds;$$

$(f4)$ $\frac{f(t)}{t}$ is strictly increasing in $(0, \infty)$.

The main result of this paper is listed as follows:

Theorem 1.1. *Assume that V satisfies $(V1)$, $(V2)$ and f satisfies $(f1)$–$(f4)$. Then, for any $\delta > 0$ such that*

$$M_\delta := \{x \in \mathbb{R}^3 : dist(x, M) < \delta\} \subset \Lambda,$$

there exists $\epsilon_\delta > 0$ such that, for any $0 < \epsilon < \epsilon_\delta$, problem (1.1) has at least $cat_{M_\delta}(M)$ nontrivial solutions. Moreover, for every sequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0^+$ as $n \to +\infty$, if we denote by u_{ϵ_n} one of these solutions of (1.1) for $\epsilon = \epsilon_n$ and $\eta_{\epsilon_n} \in \mathbb{R}^3$ the global maximum point of $|u_{\epsilon_n}|$, then

$$\lim_n V(\eta_{\epsilon_n}) = V_0.$$

The paper is organized as follows. In Section 2 we indicate the functional setting and give some preliminary results. In Section 3, we study the modified problem, and prove the Palais-Smale condition for the modified functional and provide some tools which are useful to establish a multiplicity result. In Section 4, we study the autonomous limit problem associated. It allows us to show the modified problem has the multiple solutions. Finally, the proof of Thereom 1.1 is derived in Section 5.

Notation

- $C, C_1, C_2, \ldots$ denote any positive constants, whose exact values are not relevant;
- $B_R(y)$ denotes the open disk centered at $y \in \mathbb{R}^3$ with radius $R > 0$ and $B_R^c(y)$ denotes the complement of $B_R(y)$ in $\mathbb{R}^3$;
- $\|\cdot\|$, $\|\cdot\|_q$, and $\|\cdot\|_{L^\infty(\Omega)}$ denote the usual norms of the spaces $H^1(\mathbb{R}^3, \mathbb{R})$, $L^q(\mathbb{R}^3, \mathbb{R})$, and $L^\infty(\Omega, \mathbb{R})$, respectively, where $\Omega \subset \mathbb{R}^3$. $\langle \cdot, \cdot \rangle_0$ denotes the inner product of the space $H^1(\mathbb{R}^3, \mathbb{R})$.

2 Abstract setting and preliminary results

For $u : \mathbb{R}^3 \to \mathbb{C}$, let us denote by

$$\nabla_A u := \Big(\frac{\nabla}{i} - A\Big)u,$$

and

$$D_A^1(\mathbb{R}^3, \mathbb{C}) := \{u \in L^6(\mathbb{R}^3, \mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^3, \mathbb{R})\},$$
$$H_A^1(\mathbb{R}^3, \mathbb{C}) := \{u \in D_A^1(\mathbb{R}^3, \mathbb{C}) : u \in L^2(\mathbb{R}^3, \mathbb{C}))\}.$$

The space $H_A^1(\mathbb{R}^3, \mathbb{C})$ is an Hilbert space endowed with the scalar product

$$\langle u, v \rangle := \mathrm{Re} \int_{\mathbb{R}^3} \Big(\nabla_A u \overline{\nabla_A v} + u\bar{v}\Big) dx, \quad \text{for any } u, v \in H_A^1(\mathbb{R}^3, \mathbb{C}),$$

where Re and the bar denote the real part of a complex number and the complex conjugation, respectively. Moreover we denote by $\|u\|_A$ the norm induced by this inner product.

On $H_A^1(\mathbb{R}^3, \mathbb{C})$, an important tool is the following diamagnetic inequality (see e.g. [28, Theorem 7.21])

$$|\nabla_A u(x)| \geq |\nabla |u(x)||. \tag{2.1}$$

Now, by a simple change of variables, we can see that (1.1) is equivalent to

$$\Big(\frac{1}{i}\nabla - A_\epsilon(x)\Big)^2 u + V_\epsilon(x)u + (|x|^{-1} \star |u|^2)u = f(|u|^2)u \quad \text{in } \mathbb{R}^3, \tag{2.2}$$

where $A_\epsilon(x) = A(\epsilon x)$ and $V_\epsilon(x) = V(\varepsilon x)$.

Let H_ϵ be the Hilbert space obtained as the closure of $C_c^\infty(\mathbb{R}^3, \mathbb{C})$ with respect to the scalar product

$$\langle u, v \rangle_\epsilon := \mathrm{Re} \int_{\mathbb{R}^3} \Big(\nabla_{A_\epsilon} u \overline{\nabla_{A_\epsilon} v} + V_\epsilon(x) u\bar{v}\Big) dx$$

and let us denote by $\|\cdot\|_\epsilon$ the norm induced by this inner product.

By the diamagnetic inequality (2.1), we have, if $u \in H^1_{A_\epsilon}(\mathbb{R}^3,\mathbb{C})$, then $|u| \in H^1(\mathbb{R}^3,\mathbb{R})$ and $\|u\| \le C\|u\|_\epsilon$. Therefore, the embedding $H_\epsilon \hookrightarrow L^r(\mathbb{R}^3,\mathbb{C})$ is continuous for $2 \le r \le 6$ and the embedding $H_\epsilon \hookrightarrow L^r_{\text{loc}}(\mathbb{R}^3,\mathbb{C})$ is compact for $1 \le r < 6$.

Arguing as in [29], by the Lax-Milgram Theorem there exists a unique $\phi_{|u|} \in D^{1,2}(\mathbb{R}^3,\mathbb{R})$ such that

$$-\Delta\phi_{|u|} = |u|^2, \quad \text{in } \mathbb{R}^3.$$

We obtain the following t-Riesz formula

$$\phi_{|u|}(x) = c\int_{\mathbb{R}^3} |x-y|^{-1}|u(y)|^2\,dy.$$

Arguing as in [14, 32, 40], the function $\phi_{|u|}$ possesses the following properties.

Lemma 2.1. *For any $u \in H_\varepsilon$, we have*
(i) $\phi_{|u|} : H^1(\mathbb{R}^3,\mathbb{R}) \to D^{1,2}(\mathbb{R}^3,\mathbb{R})$ *is continuous and maps bounded sets into bounded sets;*
(ii) *if $u_n \rightharpoonup u$ in H_ε, then $\phi_{|u_n|} \rightharpoonup \phi_{|u|}$ in $D^{1,2}(\mathbb{R}^3,\mathbb{R})$, and*

$$\liminf_n \int_{\mathbb{R}^3} \phi_{|u_n|}|u_n|^2\,dx \le \int_{\mathbb{R}^3} \phi_{|u|}|u|^2\,dx;$$

(iii) $\phi_{|ru|} = r^2\phi_{|u|}$ *for all $r \in \mathbb{R}$ and* $\phi_{|u(\cdot+y)|} = \phi_{|u|}(x+y)$;
(iv) $\phi_{|u|} \ge 0$ *for all $u \in H_\epsilon$ and we have*

$$\|\phi_{|u|}\|_{D^{1,2}} \le C\|u\|^2_{L^{\frac{12}{5}}(\mathbb{R}^3)} \le C\|u\|^2_\epsilon, \ \text{and} \ \int_{\mathbb{R}^3} \phi_{|u|}|u|^2\,dx \le C\|u\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)} \le C\|u\|^4_\epsilon.$$

For compact supported functions in $H^1(\mathbb{R}^3,\mathbb{R})$, the following result will be very useful for some estimates below.

Lemma 2.2. *If $u \in H^1(\mathbb{R}^3,\mathbb{R})$ and u has compact support, then $\omega := e^{iA(0)\cdot x}u \in H_\varepsilon$.*

Proof. Assume that $\text{supp}(u) \subset B_R(0)$. Since V is continuous, it is clear that

$$\int_{\mathbb{R}^3} V_\epsilon(x)|\omega|^2\,dx = \int_{B_R(0)} V_\epsilon(x)|\omega|^2\,dx \le C\|u\|_2^2 < +\infty.$$

Moreover, since V and A are continuous, we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla_{A_\epsilon}\omega|^2\,dx &= \int_{\mathbb{R}^3} |\nabla\omega|^2\,dx + \int_{\mathbb{R}^3} |A_\epsilon(x)|^2|\omega|^2\,dx + 2\text{Re}\int_{\mathbb{R}^3} iA_\epsilon(x)\overline{\omega}\nabla\omega\,dx \\ &\le 2\int_{\mathbb{R}^3} |\nabla\omega|^2\,dx + 2\int_{\mathbb{R}^3} |A_\epsilon(x)|^2|\omega|^2\,dx \\ &\le C\left[\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \int_{\mathbb{R}^3} |u|^2\,dx\right] < +\infty\end{aligned}$$

and we conclude. □

3 The modified problem

To study (1.1), we modify suitably the nonlinearity f so that, for $\epsilon > 0$ small enough, the solutions of such modified problem are also solutions of the original one. More precisely, we choose $K > 2$. By $(f4)$ there exists a unique number $a > 0$ verifying $f(a) + a^2 = V_0/K$, where V_0 is given in $(V1)$. Hence we consider the function

$$\tilde{f}(t) := \begin{cases} f(t) + (t^+)^2, & t \le a, \\ V_0/K, & t > a. \end{cases}$$

Now we introduce the penalized nonlinearity $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$

$$g(x, t) := \chi_\Lambda(x)(f(t) + (t^+)^2) + (1 - \chi_\Lambda(x))\tilde{f}(t), \tag{3.1}$$

where χ_Λ is the characteristic function on Λ and $G(x, t) := \int_0^t g(x, s)ds$.

From $(f1)$–$(f4)$, g is a Carathéodory function satisfying the following properties:

$(g1)$ $g(x, t) = 0$ for each $t \le 0$;

$(g2)$ $\lim_{t\to 0^+} \frac{g(x,t)}{t} = 0$ uniformly in $x \in \mathbb{R}^3$;

$(g3)$ $g(x, t) \le f(t) + t^2$ for all $t \ge 0$ and uniformly in $x \in \mathbb{R}^3$;

$(g4)$ $0 < \theta G(x, t) \le 2g(x, t)t$, for each $x \in \Lambda$, $t > 0$;

$(g5)$ $0 < G(x, t) \le g(x, t)t \le V_0 t/K$, for each $x \in \Lambda^c$, $t > 0$;

$(g6)$ for each $x \in \Lambda$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $t \in (0, +\infty)$ and for each $x \in \Lambda^c$, the function $t \mapsto \frac{g(x,t)}{t}$ is strictly increasing in $(0, a)$.

Then we consider the *modified* problem

$$\left(\frac{1}{i}\nabla - A_\epsilon(x)\right)^2 u + V_\epsilon(x)u + (|x|^{-1} * |u|^2)u = g(\epsilon x, |u|^2)u \quad \text{in } \mathbb{R}^3. \tag{3.2}$$

Note that, if u is a solution of problem (3.2) with

$$|u(x)|^2 \le a \quad \text{for all } x \in \Lambda_\epsilon^c, \quad \Lambda_\epsilon := \{x \in \mathbb{R}^3 : \epsilon x \in \Lambda\},$$

then u is a solution of problem (2.2).

We observe that the weak solutions of the modified problem (3.2) can be found as the critical points of the C^1 functional

$$J_\epsilon(u) := \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla_{A_\epsilon} u|^2 + V_\epsilon(x)|u|^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} (|x|^{-1} * |u|^2)|u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} G(\epsilon x, |u|^2)dx$$

defined in H_ϵ. Moreover, we denote by $\mathcal{N}_\epsilon$ the Nehari manifold of J_ϵ,

$$\mathcal{N}_\epsilon := \{u \in H_\epsilon \backslash \{0\} : J'_\epsilon(u)[u] = 0\},$$

and define the number c_ϵ by

$$c_\epsilon = \inf_{u \in \mathcal{N}_\epsilon} J_\epsilon(u).$$

Let H_ϵ^+ be open subset H_ϵ given by

$$H_\epsilon^+ = \{u \in H_\epsilon : |\mathrm{supp}(u) \cap \Lambda_\epsilon| > 0\},$$

and $S_\epsilon^+ = S_\epsilon \cap H_\epsilon^+$, where S_ϵ is the unit sphere of H_ϵ. Note that S_ϵ^+ is a non-complete $C^{1,1}$-manifold of codimension 1, modeled on H_ϵ and contained in H_ϵ^+. Therefore, $H_\epsilon = T_u S_\epsilon^+ \bigoplus \mathbb{R}u$ for each $u \in T_u S_\epsilon^+$, where $T_u S_\epsilon^+ = \{v \in H_\epsilon : \langle u, v\rangle_\epsilon = 0\}$.

Arguing as in [29, Lemma 3.1], we can show that the functional J_ϵ satisfies the Mountain Pass Geometry [37].

Lemma 3.1. *For any fixed $\epsilon > 0$, the functional J_ϵ satisfies the following properties:*

(i) *there exist $\beta, r > 0$ such that $J_\epsilon(u) \ge \beta$ if $\|u\|_\epsilon = r$;*

(ii) *there exists $e \in H_\epsilon$ with $\|e\|_\epsilon > r$ such that $J_\epsilon(e) < 0$.*

Due to f is only continuous, the next results are very important because they allow us to overcome the non-differentiability of $\mathcal{N}_\epsilon$ and the incompleteness of S_ϵ^+.

Lemma 3.2. *Assume that* $(V1)$–$(V2)$ *and* $(f1)$–$(f4)$ *are satisfied, then the following properties hold:*
$(A1)$*For any* $u \in H_\epsilon^+$, *let* $g_u : \mathbb{R}^+ \to \mathbb{R}$ *be given by* $g_u(t) = J_\epsilon(tu)$. *Then there exists a unique* $t_u > 0$ *such that* $g_u'(t) > 0$ *in* $(0, t_u)$ *and* $g_u'(t) < 0$ *in* (t_u, ∞);
$(A2)$*There is* $\tau > 0$ *independent on* u *such that* $t_u \geq \tau$ *for all* $u \in S_\epsilon^+$. *Moreover, for each compact* $\mathcal{W} \subset S_\epsilon^+$ *there is* $C_\mathcal{W}$ *such that* $t_u \leq C_\mathcal{W}$, *for all* $u \in \mathcal{W}$;
$(A3)$*The map* $\widehat{m}_\epsilon : H_\epsilon^+ \to \mathcal{N}_\epsilon$ *given by* $\widehat{m}_\epsilon(u) = t_u u$ *is continuous and* $m_\epsilon = \widehat{m}_\epsilon|_{S_\epsilon^+}$ *is a homeomorphism between* S_ϵ^+ *and* $\mathcal{N}_\epsilon$. *Moreover,* $m_\epsilon^{-1}(u) = \frac{u}{\|u\|_\epsilon}$;
$(A4)$*If there is a sequence* $\{u_n\} \subset S_\epsilon^+$ *such that* $dist(u_n, \partial S_\epsilon^+) \to 0$, *then* $\|m_\epsilon(u_n)\|_\epsilon \to \infty$ *and* $J_\epsilon(m_\epsilon(u_n)) \to \infty$.

Proof. $(A1)$ Arguing as in [29, Lemma 3.1], it follows that $g_u(0) = 0$, $g_u(t) > 0$ for $t > 0$ small and $g_u(t) < 0$ for $t > 0$ large. Thus, $\max_{t\geq 0} g_u(t)$ is achieved at a global maximum point $t = t_u$ satisfying $g_u'(t_u) = 0$ and $t_u u \in \mathcal{N}_\epsilon$. Now, we show that t_u is unique. Arguing by contradiction, suppose that there exist $t_1 > t_2 > 0$ such that $g_u'(t_1) = g_u'(t_2) = 0$. Then, for $i = 1, 2$,

$$t_i\|u\|_\epsilon^2 + t_i^3 \int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx = \int_{\mathbb{R}^3} g(\epsilon x, t_i^2|u|^2)t_i|u|^2 dx.$$

Hence,

$$\frac{\|u\|_\epsilon^2}{t_i^2} + \int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx = \int_{\mathbb{R}^3} \frac{g(\epsilon x, t_i^2|u|^2)|u|^2}{t_i^2} dx,$$

which implies that

$$\begin{aligned}
\Big(\frac{1}{t_1^2} - \frac{1}{t_2^2}\Big)\|u\|_\epsilon^2 &= \int_{\mathbb{R}^3} \Big(\frac{g(\epsilon x, t_1^2|u|^2)}{t_1^2|u|^2} - \frac{g(\epsilon x, t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4 dx \\
&\geq \int_{\Lambda_\epsilon^c \cap \{t_2^2|u|^2 \leq a_0 \leq t_1^2|u|^2\}} \Big(\frac{g(\epsilon x, t_1^2|u|^2)}{t_1^2|u|^2} - \frac{g(\epsilon x, t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4 dx \\
&\quad + \int_{\Lambda_\epsilon^c \cap \{a_0 \leq t_2^2|u|^2\}} \Big(\frac{g(\epsilon x, t_1^2|u|^2)}{t_1^2|u|^2} - \frac{g(\epsilon x, t_2^2|u|^2)}{t_2^2|u|^2}\Big)|u|^4 dx \\
&= \int_{\Lambda_\epsilon^c \cap \{t_2^2|u|^2 \leq a_0 \leq t_1^2|u|^2\}} \Big(\frac{V_0}{K}\frac{1}{t_1^2|u|^2} - \frac{f(t_2^2|u|^2) + t_2^4|u|^4}{t_2^2|u|^2}\Big)|u|^4 dx \\
&\quad + \frac{1}{K}\Big(\frac{1}{t_1^2} - \frac{1}{t_2^2}\Big) \int_{\Lambda_\epsilon^c \cap \{a_0 \leq t_2^2|u|^2\}} V_0|u|^2 dx.
\end{aligned}$$

Since $t_1 > t_2 > 0$, we have

$$\begin{aligned}
\|u\|_\epsilon^2 &\leq \frac{t_1^2 t_2^2}{t_2^2 - t_1^2} \int_{\Lambda_\epsilon^c \cap \{t_2^2|u|^2 \leq a_0 \leq t_1^2|u|^2\}} \Big(\frac{V_0}{K}\frac{1}{t_1^2|u|^2} - \frac{f(t_2^2|u|^2) + t_2^4|u|^4}{t_2^2|u|^2}\Big)|u|^4 dx \\
&\quad + \frac{1}{K} \int_{\Lambda_\epsilon^c \cap \{a_0 \leq t_2^2|u|^2\}} V_0|u|^2 dx \\
&\leq \frac{1}{K} \int_{\Lambda_\epsilon^c} V_0|u|^2 dx \leq \frac{1}{K}\|u\|_\epsilon^2,
\end{aligned}$$

which is a contradiction. Therefore, $\max_{t\geq 0} g_u(t)$ is achieved at a unique $t = t_u$ so that $g_u'(t) = 0$ and $t_u u \in \mathcal{N}_\epsilon$.

$(A2)$ For $\forall\, u \in S_\epsilon^+$, it follows that

$$t_u + t_u^3 \int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2\,dx = \int_{\mathbb{R}^3} g(\epsilon x, t_u^2|u|^2)t_u|u|^2\,dx.$$

From $(g2)$, the Sobolev embeddings and $4 < q < 6$, it is easy to obtain

$$t_u \le \zeta t_u^3 \int_{\mathbb{R}^3} |u|^4\,dx + C_\zeta t_u^{q-1} \int_{\mathbb{R}^3} |u|^q\,dx + t_u^5 \int_{\mathbb{R}^3} |u|^6\,dx \le C_1\zeta t_u^3 + C_2 C_\zeta t_u^{q-1} + C_3 t_u^5,$$

which implies $t_u \ge \tau$ for some $\tau > 0$. If $\mathcal{W} \subset S_\epsilon^+$ is compact, and suppose by contradiction that there is $\{u_n\} \subset \mathcal{W}$ with $t_n := t_{u_n} \to \infty$. Since $\mathcal{W}$ is compact, there exists $u \in \mathcal{W}$ such that $u_n \to u$ in H_ϵ. Using the proof of [29, Lemma 3.1(ii)], it follows that $J_\epsilon(t_n u_n) \to -\infty$.

On the other hand, let $v_n := t_n u_n \in \mathcal{N}_\epsilon$, from $(g4)$, $(g5)$, $(g6)$ and $\theta > 4$, it yields that

$$\begin{aligned}
J_\epsilon(v_n) =& J_\epsilon(v_n) - \frac{1}{\theta} J_\epsilon'(v_n)[v_n] \\
\ge& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|v_n\|_\epsilon^2 + (\frac{1}{4} - \frac{1}{\theta}) \int_{\mathbb{R}^3} (|x|^{-1} \star |v_n|^2)|v_n|^2\,dx \\
&+ \int_{\Lambda_\epsilon^c} \Big(\frac{1}{\theta} g(\epsilon x, |v_n|^2)|u_n|^2 - \frac{1}{2} G(\epsilon x, |v_n|^2)\Big)\,dx \\
\ge& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(\|v_n\|_\epsilon^2 - \frac{1}{K} \int_{\mathbb{R}^3} V(\epsilon x)|v_n|^2\,dx\Big) \\
\ge& \Big(\frac{1}{2} - \frac{1}{\theta}\Big)(1 - \frac{1}{K})\|v_n\|_\epsilon^2.
\end{aligned}$$

Thus, substituting $v_n := t_n u_n$ and $\|v_n\|_\epsilon = t_n$, we may obtain

$$0 < \Big(\frac{1}{2} - \frac{1}{\theta}\Big)(1 - \frac{1}{K}) \le \frac{J_\epsilon(v_n)}{t_n^2} \le 0$$

as $n \to \infty$, which yields a contradiction. This completes the proof of $(A2)$.

$(A3)$ We first show that $\widehat{m}_\epsilon$, m_ϵ and m_ϵ^{-1} are well defined. Indeed, by $(A2)$, for each $u \in H_\epsilon^+$, there is a unique $\widehat{m}_\epsilon(u) \in \mathcal{N}_\epsilon$. On the other hand, if $u \in \mathcal{N}_\epsilon$, then $u \in H_\epsilon^+$. Otherwise, we have $|\mathrm{supp}(u) \cap \Lambda_\epsilon| = 0$ and by $(g5)$ it follows that

$$\begin{aligned}
\|u\|_\epsilon^2 \le \|u\|_\epsilon^2 + \int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2\,dx =& \int_{\mathbb{R}^3} g(\epsilon x, |u|^2)|u|^2\,dx \\
=& \int_{\Lambda_\epsilon^c} g(\epsilon x, |u|^2)|u|^2\,dx \\
\le& \frac{1}{K} \int_{\mathbb{R}^3} V(\epsilon x)|u|^2\,dx \\
\le& \frac{1}{K}\|u\|_\epsilon^2
\end{aligned}$$

which is impossible since $K > 2$ and $u \ne 0$. Thus, $m_\epsilon^{-1}(u) = \frac{u}{\|u\|_\epsilon} \in S_\epsilon^+$ is well defined and continuous. From

$$m_\epsilon^{-1}\big(m_\epsilon(u)\big) = m_\epsilon^{-1}(t_u u) = \frac{t_u u}{t_u\|u\|_\epsilon} = u, \quad \forall u \in S_\epsilon^+,$$

we know that m_ϵ is a bijection. Now we prove $\widehat{m}_\epsilon : H_\epsilon^+ \to \mathcal{N}_\epsilon$ is continuous. Let $\{u_n\} \subset H_\epsilon^+$ and $u \in H_\epsilon^+$ such that $u_n \to u$ in H_ϵ. By $(A2)$, there is a $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. Using $t_n u_n \in \mathcal{N}_\epsilon$, i.e.,

$$t_n^2\|u_n\|_\epsilon^2 + t_n^2 \int_{\mathbb{R}^3} (|x|^{-1} \star |u_n|^2)|u_n|^2\,dx = \int_{\mathbb{R}^3} g(\epsilon x, t_n^2|u_n|^2)t_n^2|u_n|^2\,dx, \quad \forall n \in N,$$

and passing to the limit as $n \to \infty$ in the last inequality, it follows that

$$t_0^2\|u\|_\epsilon^2 + t_0^2 \int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx = \int_{\mathbb{R}^3} g(\epsilon x, t_0^2|u|^2)t_0^2|u|^2 dx,$$

which implies that $t_0 u \in \mathcal{N}_\epsilon$ and $t_u = t_0$. This proves $\widehat{m}_\epsilon(u_n) \to \widehat{m}_\epsilon(u)$ in H_ϵ^+. Thus, $\widehat{m}_\epsilon$ and m_ϵ are continuous and $(A3)$ is proved.

$(A4)$ Let $\{u_n\} \subset S_\epsilon^+$ be a subsequence such that $\operatorname{dist}(u_n, \partial S_\epsilon^+) \to 0$, then for each $v \in S_\epsilon^+$ and $n \in N$, we have $|u_n| = |u_n - v|$ a.e. in Λ_ϵ. Thus, by $(V1)$, $(V2)$ and the Sobolev embedding, for any $t \in [2, 6]$, there exists $C_t > 0$ such that

$$\begin{aligned} \|u_n\|_{L^t(\Lambda_\epsilon)} &\le \inf_{v \in \partial S_\epsilon^+} \|u_n - v\|_{L^t(\Lambda_\epsilon)} \\ &\le C_t \Big(\inf_{v \in \partial S_\epsilon^+} \int_{\Lambda_\epsilon} (|\nabla_{A_\epsilon} u_n - v|^2 + V_\epsilon(x)|u_n - v|^2) dx \Big)^{\frac{1}{2}} \\ &\le C_t \operatorname{dist}(u_n, \partial S_\epsilon^+) \end{aligned}$$

for all $n \in N$. From $(g2)$, $(g3)$ and $(g5)$, for each $t > 0$, it follows that

$$\begin{aligned} \int_{\mathbb{R}^3} G(\epsilon x, t^2|u_n|^2) dx &\le \int_{\Lambda_\epsilon} \Big(F(t^2|u_n|^2) + \frac{t^6|u_n|^6}{6} \Big) dx + \frac{t^2}{K} \int_{\Lambda_\epsilon^c} V(\epsilon x)|u_n|^2 dx \\ &\le C_1 t^4 \int_{\Lambda_\epsilon} |u_n|^4 dx + C_2 t^q \int_{\Lambda_\epsilon} |u_n|^q dx + \frac{t^6}{6} \int_{\Lambda_\epsilon} |u_n|^6 dx + \frac{t^2}{K}\|u_n\|_\epsilon^2 \\ &\le C_3 t^4 \operatorname{dist}(u_n, \partial S_\epsilon^+)^4 + C_4 t^q \operatorname{dist}(u_n, \partial S_\epsilon^+)^q + C_5 t^6 \operatorname{dist}(u_n, \partial S_\epsilon^+)^6 + \frac{t^2}{K}. \end{aligned}$$

Therefore,

$$\limsup_n \int_{\mathbb{R}^3} G(\epsilon x, t^2|u_n|^2) dx \le \frac{t^2}{K}, \ \forall t > 0.$$

On the other hand, from the definition of m_ϵ and the last inequality, for all $t > 0$, we have

$$\begin{aligned} \liminf_n J_\epsilon(m_\epsilon(u_n)) &\ge \liminf_n J_\epsilon(t u_n) \\ &\ge \liminf_n \frac{t^2}{2}\|u_n\|_\epsilon^2 - \frac{t^2}{K} \\ &= \frac{K-2}{2K} t^2 \end{aligned}$$

which implies that

$$\liminf_n \Big\{ \frac{1}{2}\|m_\epsilon(u_n)\|_\epsilon^2 + \frac{1}{4}\int_{\mathbb{R}^3} (|x|^{-1} \star |m_\epsilon(u_n)|^2)|m_\epsilon(u_n)|^2 dx \Big\} \ge \liminf_n J_\epsilon(m_\epsilon(u_n)) \ge \frac{K-2}{2K} t^2, \ \forall t > 0.$$

Since $t > 0$ is arbitrary, we can show that $\|m_\epsilon(u_n)\|_\epsilon \to \infty$ and $J_\epsilon(m_\epsilon(u_n)) \to \infty$ as $n \to \infty$. □

At this point we define the function

$$\widehat{\Psi}_\epsilon : H_\epsilon^+ \to \mathbb{R},$$

by $\widehat{\Psi}_\epsilon(u) = J_\epsilon(\widehat{m}_\epsilon(u))$ and denote by $\Psi_\epsilon := (\widehat{\Psi}_\epsilon)|_{S_\epsilon^+}$.

From Lemma 3.2, arguing as in [35, Corollary 10] we may obtain the following lemma.

Lemma 3.3. *Assume that* $(V1)$–$(V2)$ *and* $(f1)$–$(f4)$ *hold, then*

(B1) $\widehat{\Psi}_\epsilon \in C^1(H_\epsilon^+, \mathbb{R})$ *and*

$$\widehat{\Psi}_\epsilon'(u)v = \frac{\|\widehat{m}_\epsilon(u)\|_\epsilon}{\|u\|_\epsilon} J_\epsilon'(\widehat{m}_\epsilon(u))[v], \quad \forall\, u \in H_\epsilon^+ \text{ and } \forall\, v \in H_\epsilon;$$

(B2) $\Psi_\epsilon \in C^1(S_\epsilon^+, \mathbb{R})$ *and*

$$\Psi_\epsilon'(u)v = \|m_\epsilon(u)\|_\epsilon J_\epsilon'(\widehat{m}_\epsilon(u))[v], \quad \forall v \in T_u S_\epsilon^+;$$

(B3) *If* $\{u_n\}$ *is a* $(PS)_c$ *sequence of* Ψ_ϵ, *then* $\{m_\epsilon(u_n)\}$ *is a* $(PS)_c$ *sequence of* J_ϵ. *If* $\{u_n\} \subset \mathcal{N}_\epsilon$ *is a bounded* $(PS)_c$ *sequence of* J_ϵ, *then* $\{m_\epsilon^{-1}(u_n)\}$ *is a* $(PS)_c$ *sequence of* Ψ_ϵ;

(B4) u *is a critical point of* Ψ_ϵ *if and only if* $m_\epsilon(u)$ *is a critical point of* J_ϵ. *Moreover, the corresponding critical values coincide and*

$$\inf_{S_\epsilon^+} \Psi_\epsilon. = \inf_{\mathcal{N}_\epsilon} J_\epsilon.$$

As in [35], we have the variational characterization of the infimum of J_ϵ over $\mathcal{N}_\epsilon$:

$$c_\epsilon = \inf_{u\in\mathcal{N}_\epsilon} J_\epsilon(u) = \inf_{u\in H_\epsilon^+} \sup_{t>0} J_\epsilon(tu) = \inf_{u\in S_\epsilon^+} \sup_{t>0} J_\epsilon(tu). \tag{3.3}$$

Lemma 3.4. *Let* $\{u_n\}$ *be a* $(PS)_c$ *sequence for* J_ϵ *where* $c > 0$, *then* $\{u_n\}$ *is bounded in* H_ϵ.

Proof. Assume that $\{u_n\} \subset H_\epsilon$ be a $(PS)_c$ sequence for J_ϵ, that is, $J_\epsilon(u_n) \to c > 0$ and $J_\epsilon'(u_n) \to 0$. From $(g4)$, $(g5)$ and $4 < \theta < 6$, it follows that

$$\begin{aligned}
d + o_n(1) + o_n(1)\|u_n\|_\epsilon \geq & J_\epsilon(u_n) - \frac{1}{\theta} J_\epsilon'(u_n)[u_n] \\
= & \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\epsilon^2 + \Big(\frac{1}{4} - \frac{1}{\theta}\Big) \int_{\mathbb{R}^3} (|x|^{-1} * |u_n|^2)|u_n|^2 dx \\
& + \int_{\mathbb{R}^3} \Big(\frac{1}{\theta} g(\epsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\epsilon x, |u_n|^2)\Big) dx \\
\geq & \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\epsilon^2 + \int_{\Lambda_\epsilon^c} \Big(\frac{1}{\theta} g(\epsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\epsilon x, |u_n|^2)\Big) dx \\
\geq & \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\epsilon^2 + \Big(\frac{1}{\theta} - \frac{1}{2}\Big) \int_{\Lambda_\epsilon^c} G(\epsilon x, |u_n|^2) dx \\
\geq & \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\epsilon^2 + \Big(\frac{1}{\theta} - \frac{1}{2}\Big)\frac{1}{K} \int_{\mathbb{R}^3} V(\epsilon x)|u_n|^2 dx \\
\geq & \Big(\frac{1}{2} - \frac{1}{\theta}\Big)(1 - \frac{1}{K})\|u_n\|_\epsilon^2.
\end{aligned}$$

Since $K > 2$, from the above inequalities we know that $\{u_n\}$ is bounded in H_ϵ. □

The following lemma provides a range of levels in which the functional J_ϵ verifies the Palais-Smale condition.

Lemma 3.5. *The functional* J_ϵ *satisfies the* $(PS)_c$ *condition at any level* $c \in (0, \frac{1}{3}S^{\frac{3}{2}})$, *where* S *is the best constant for the Sobolev inequality*

$$S\Big(\int_{\mathbb{R}^3} |v|^6 dx\Big)^{1/3} \leq \int_{\mathbb{R}^3} (|\nabla v|^2 + |v|^2) dx, \quad \text{for } v \in H^1(\mathbb{R}^3, \mathbb{R}).$$

Proof. Let $(u_n)_n \subset H_\epsilon$ be a $(PS)_c$ for J_ϵ. By Lemma 3.4, $(u_n)_n$ is bounded in H_ϵ. Thus, up to a subsequence, $u_n \rightharpoonup u$ in H_ϵ and $u_n \to u$ in $L^r_{\text{loc}}(\mathbb{R}^3, \mathbb{C})$ for all $1 \leq r < 6$ as $n \to +\infty$.

Step 1: We show that for any given $\zeta > 0$, for R large enough,

$$\limsup_n \int_{B_R^c(0)} (|\nabla_{A_\epsilon} u_n|^2 + V_\epsilon(x)|u_n|^2) dx \leq \zeta. \tag{3.4}$$

Let $R > 0$ such that $\Lambda_\epsilon \subset B_{R/2}(0)$ and let $\phi_R \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be a cut-off function such that

$$\phi_R = 0 \quad x \in B_{R/2}(0), \quad \phi_R = 1 \quad x \in B_R^c(0), \quad 0 \le \phi_R \le 1, \quad \text{and} \quad |\nabla \phi_R| \le C/R$$

where $C > 0$ is a constant independent of R. Since the sequence $(\phi_R u_n)_n$ is bounded in H_ϵ, we have

$$J'_\epsilon(u_n)[\phi_R u_n] = o_n(1),$$

that is

$$\begin{aligned}&\operatorname{Re}\int_{\mathbb{R}^3} \nabla_{A_\epsilon} u_n \overline{\nabla_{A_\epsilon}(\phi_R u_n)} dx + \int_{\mathbb{R}^3} V_\epsilon(x)|u_n|^2 \phi_R dx + \int_{\mathbb{R}^3} (|x|^{-1} \star |u_n|^2)|u_n|^2 \phi_R dx\\ &= \int_{\mathbb{R}^3} g(\epsilon x, |u_n|^2)|u_n|^2 \phi_R dx + o_n(1).\end{aligned}$$

Since $\overline{\nabla_{A_\epsilon}(u_n \phi_R)} = i\overline{u_n}\nabla\phi_R + \phi_R \overline{\nabla_{A_\epsilon} u_n}$, using $(g5)$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} (|\nabla_{A_\epsilon} u_n|^2 + V_\epsilon(x)|u_n|^2)\phi_R dx &\le \int_{\mathbb{R}^3} g(\epsilon x, |u_n|^2)|u_n|^2 \phi_R dx - \operatorname{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_{A_\epsilon} u_n \nabla\phi_R dx + o_n(1)\\ &\le \frac{1}{K}\int_{\mathbb{R}^3} V_\epsilon(x)|u_n|^2\phi_R dx - \operatorname{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_{A_\epsilon} u_n \nabla\phi_R dx + o_n(1).\end{aligned}$$

By the definition of ϕ_R, the Hölder inequality and the boundedness of $(u_n)_n$ in H_ϵ, we obtain

$$\left(1 - \frac{1}{K}\right)\int_{\mathbb{R}^3} (|\nabla_{A_\epsilon} u_n|^2 + V_\epsilon(x)|u_n|^2)\phi_R dx \le \frac{C}{R}\|u_n\|_2 \|\nabla_{A_\epsilon} u_n\|_2 + o_n(1) \le \frac{C_1}{R} + o_n(1)$$

and so (3.4) holds.

From the Sobolev embedding and (3.4), we have that for any $\zeta > 0$, there exists $R = R(\zeta) > 0$ such that,

$$\begin{aligned}\|u_n - u\|_r &\le \|u_n - u\|_{L^r(B_R(0))} + \|u_n - u\|_{L^r(B_R^c(0))}\\ &\le \zeta + C\Big(\|u_n\|_{H_\epsilon(B_R^c(0))} + \|u\|_{H_\epsilon(B_R^c(0))}\Big)\\ &\le C_1\zeta\end{aligned}$$

where $r \in [2, 6)$ and n large enough. From this, we can obtain that

$$u_n \to u \quad \text{in } L^r(\mathbb{R}^3, \mathbb{C}), \quad \text{for any } r \in [2, 6). \tag{3.5}$$

By (2), since $\phi_{|u|} : L^{12/5}(\mathbb{R}^3, \mathbb{R}) \to D^{1,2}(\mathbb{R}^3, \mathbb{R})$ is continuous, from (3.5) we can get

$$\phi_{|u_n|} \to \phi_{|u|} \text{ in } D^{1,2}(\mathbb{R}^3, \mathbb{R}), \tag{3.6}$$

$$\int_{\mathbb{R}^3} \phi_{|u_n|}|u_n|^2 dx \to \int_{\mathbb{R}^3} \phi_{|u|}|u|^2 dx. \tag{3.7}$$

Using the boundedness of sequence $(u_n)_n$ and the Sobolev embedding again, for any $\varphi \in C_c^\infty(\mathbb{R}^3, \mathbb{C})$, we have

$$\operatorname{Re}\int_{\mathbb{R}^3} \Big(\nabla_{A_\epsilon} u_n \overline{\nabla_{A_\epsilon}\varphi} dx + V_\epsilon(x) u_n \overline{\varphi}\Big) dx \to \operatorname{Re}\int_{\mathbb{R}^3} \Big(\nabla_{A_\epsilon} u \overline{\nabla_{A_\epsilon}\varphi} dx + V_\epsilon(x) u \overline{\varphi}\Big) dx, \tag{3.8}$$

$$\operatorname{Re}\int_{\mathbb{R}^3} g(\epsilon x, |u_n|^2) u_n \overline{\varphi} dx \to \operatorname{Re}\int_{\mathbb{R}^3} g(\epsilon x, |u|^2) u \overline{\varphi} dx. \tag{3.9}$$

By (3.7)-(3.9), the Hölder inequality and the Sobolev embeddings, we have

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^3}\phi_{|u_n|}u_n\overline{\varphi}dx-\mathrm{Re}\int_{\mathbb{R}^3}\phi_{|u|}u\overline{\varphi}dx&=\mathrm{Re}\int_{\mathbb{R}^3}(\phi_{|u_n|}u_n-\phi_{|u|}u)\overline{\varphi}dx\\&=\mathrm{Re}\int_{\mathbb{R}^3}\phi_{|u_n|}(u_n-u)\overline{\varphi}dx+\mathrm{Re}\int_{\mathbb{R}^3}(\phi_{|u_n|}-\phi_{|u|})u\overline{\varphi}dx\\&\le C\|\nabla\phi_{|u_n|}\|_{D^{1,2}(\mathbb{R}^3,\mathbb{R})}\|u_n-u\|_{L^{12/5}(\mathbb{R}^3,\mathbb{C})}\|\varphi\|_{L^{12/5}((\mathbb{R}^3,\mathbb{C}))}\\&\quad+C\|\nabla(\phi_{|u_n|}-\phi_{|u|})\|_{D^{1,2}(\mathbb{R}^3,\mathbb{R})}\|u\|_{L^{12/5}(\mathbb{R}^3,\mathbb{C})}\|\varphi\|_{L^{12/5}(\mathbb{R}^3,\mathbb{C})}\\&\to 0,\quad \text{as } n\to\infty.\end{aligned}\tag{3.10}$$

By (3.8)-(3.10) and $J'_\epsilon(u_n)\to 0$, we have $J'_\epsilon(u)=0$ and

$$\|u\|_\epsilon^2+\int_{\mathbb{R}^3}(|x|^{-1}\star|u|^2)|u|^2dx=\int_{\mathbb{R}^3}g(\epsilon x,|u|^2)|u|^2dx.$$

Step 2:

$$\lim_n\int_{\mathbb{R}^3}g(\epsilon x,|u_n|^2)|u_n|^2dx=\int_{\mathbb{R}^3}g(\epsilon x,|u|^2)|u|^2dx.\tag{3.11}$$

Using $u_n\to u$ in $L^r_{\mathrm{loc}}(\mathbb{R}^3,\mathbb{C})$, for all $1\le r<6$ again, up to a subsequence, we have that

$$|u_n|\to|u| \text{ a.e. in } \mathbb{R}^3 \text{ as } n\to+\infty,$$

then

$$g(\epsilon x,|u_n|^2)|u_n|^2\to g(\epsilon x,|u|^2)|u|^2 \text{ a.e. in } \mathbb{R}^3 \text{ as } n\to+\infty.$$

By $(g5)$ and (3.4), for any $\zeta>0$, there exists $R>0$ large enough, we have

$$\int_{B_R^c(0)}\left|g(\epsilon x,|u_n|^2)|u_n|^2-g(\epsilon x,|u|^2)|u|^2\right|dx\le\frac{2}{K}\int_{B_R^c(0)}(|\nabla_{A_\epsilon}u_n|^2+V(\epsilon x)|u_n|^2)dx<\frac{2\zeta}{K}.$$

Thus,

$$\lim_n\int_{B_R^c(0)}g(\epsilon x,|u_n|^2)|u_n|^2dx=\int_{B_R^c(0)}g(\epsilon x,|u|^2)|u|^2dx.$$

Now, we show that

$$\lim_n\int_{B_R(0)}g(\epsilon x,|u_n|^2)|u_n|^2=\int_{B_R(0)}g(\epsilon x,|u|^2)|u|^2dx.$$

From the definition of g, we have that

$$g(\epsilon x,|u_n|^2)|u_n|^2\le f(|u_n|^2)|u_n|^2+|u_n|^6+\frac{V_0}{K}|u_n|^2,\quad \text{for any } x\in\mathbb{R}^3\backslash\Lambda_\epsilon.\tag{3.12}$$

Since $B_R(0)\cap(\mathbb{R}^3\backslash\Lambda_\epsilon)$ is bounded, from the above estimate, $(f1)$, $(f2)$, the Sobolev embedding and the Lebesgue Dominated Convergence Theorem, we can infer

$$\lim_n\int_{B_R(0)\cap(\mathbb{R}^3\backslash\Lambda_\epsilon)}g(\epsilon x,|u_n|^2)|u_n|^2=\int_{B_R(0)\cap(\mathbb{R}^3\backslash\Lambda_\epsilon)}g(\epsilon x,|u|^2)|u|^2dx.\tag{3.13}$$

If we can prove that

$$\lim_n \int_{B_R(0)\cap\Lambda_\epsilon} g(\epsilon x, |u_n|^2)|u_n|^2 = \int_{B_R(0)} g(\epsilon x, |u|^2)|u|^2 dx, \tag{3.14}$$

from (3.13) and (3.14), it yields (3.11). Now, in order to show that (3.14) holds, we only need to prove the following limit holds

$$\lim_n \int_{\Lambda_\epsilon} |u_n|^6 dx = \int_{\Lambda_\epsilon} |u|^6 dx. \tag{3.15}$$

Using the boundedness of $(u_n)_n$ in H_ϵ and the diamagnetic inequality (2.1), we may assume that

$$|\nabla|u_n||^2 \rightharpoonup \mu \quad \text{and} \quad |u_n|^6 \rightharpoonup \nu \tag{3.16}$$

in the sense of measures. Moreover, by the diamagnetic inequality (2.1) and (3.4), $(u_n)_n$ is a tight sequence in $H^1(\mathbb{R}^3,\mathbb{R})$, thus, using the concentration-compactness principle in [37], we can find an at most countable index I, sequences $(x_i)\subset\mathbb{R}^3$, $(\mu_i), (\nu_i)\subset(0,\infty)$ such that

$$\begin{aligned} &\mu \geq |\nabla|u||^2 dx + \sum_{i\in I}\mu_i\delta_{x_i},\\ &\nu = |u|^6 + \sum_{i\in I}\nu_i\delta_{x_i} \quad \text{and} \quad S\nu_i^{1/3} \leq \mu_i \end{aligned} \tag{3.17}$$

for any $i \in I$, where δ_{x_i} is the Dirac mass at the point x_i. Let us show that $(x_i)_{i\in I}\cap\Lambda_\epsilon = \emptyset$. Assume, by contradiction, that $x_i \in \Lambda_\epsilon$ for some $i \in I$. For any $\rho > 0$, we define $\psi_\rho(x) = \psi(\frac{x-x_i}{\rho})$ where $\psi \in C_0^\infty(\mathbb{R}^3,[0,1])$ such that $\psi = 1$ in B_1, $\psi = 0$ in $\mathbb{R}^3\setminus B_2$ and $\|\nabla\psi\|_{L^\infty(\mathbb{R}^3,\mathbb{R})} \leq 2$. We suppose that $\rho > 0$ such that $\operatorname{supp}(\psi_\rho)\subset\Lambda_\epsilon$. Since $(\psi_\rho u_n)$ is bounded in H_ϵ, we can see that $J_\epsilon'(u_n)[\psi_\rho u_n] = o_n(1)$, that is

$$\begin{aligned} &\operatorname{Re}\int_{\mathbb{R}^3}\nabla_{A_\epsilon}u_n\overline{\nabla_{A_\epsilon}(\psi_\rho u_n)}dx + \int_{\mathbb{R}^3}V_\epsilon(x)|u_n|^2\psi_\rho dx + \int_{\mathbb{R}^3}(|x|^{-1}\star|u_n|^2)|u_n|^2\psi_\rho dx\\ &= \int_{\mathbb{R}^3} g(\epsilon x,|u_n|^2)|u_n|^2\psi_\rho dx + o_n(1) = \int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2\psi_\rho dx + \int_{\mathbb{R}^3}|u_n|^6\psi_\rho dx + o_n(1). \end{aligned}$$

Since $\overline{\nabla_{A_\epsilon}(u_n\psi_\rho)} = i\overline{u_n}\nabla\psi_\rho + \psi_\rho\overline{\nabla_{A_\epsilon}u_n}$, using $(g5)$, we have

$$\int_{\mathbb{R}^3}|\nabla_{A_\epsilon}u_n|^2\psi_\rho dx \leq \int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2\psi_\rho dx + \int_{\mathbb{R}^3}|u_n|^6\psi_\rho dx - \operatorname{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_{A_\epsilon}u_n\nabla\phi_R dx + o_n(1). \tag{3.18}$$

Using the diamagnetic inequality (2.1) again, it follows that

$$\int_{\mathbb{R}^3}|\nabla|u_n||^2\psi_\rho dx \leq \int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2\psi_\rho dx + \int_{\mathbb{R}^3}|u_n|^6\psi_\rho dx - \operatorname{Re}\int_{\mathbb{R}^3} i\overline{u_n}\nabla_{A_\epsilon}u_n\nabla\phi_R dx + o_n(1).$$

Due to the fact that f has the subcritical growth and ψ_ρ has the compact support, we have that

$$\lim_{\rho\to0}\lim_{n\to\infty}\int_{\mathbb{R}^3} f(|u_n|^2)|u_n|^2\psi_\rho dx = \lim_{\rho\to0}\int_{\mathbb{R}^3} f(|u|^2)|u|^2\psi_\rho dx = 0. \tag{3.19}$$

It's also easy to show that

$$\lim_{\rho\to0}\limsup_{n\to\infty}\left|\int_{\mathbb{R}^3} i\overline{u_n}\nabla_{A_\epsilon}u_n\nabla\psi_\rho dx\right| = 0. \tag{3.20}$$

Then, taking into account (3.16), (3.18), (3.19) and (3.20), we can conclude that $\nu_i \geq \mu_i$. Together with the inequality $S\nu_i^{1/3} \leq \mu_i$ in (3.17), we have $\nu_i \geq S^{3/2}$. Now, from $(f3)$, $(g4)$ and $(g5)$, we have

$$c = J_\epsilon(u_n) - \frac{1}{4}J_\epsilon'(u_n)[u_n] + o_n(1)$$

$$
\begin{aligned}
&= \frac{1}{4}\|u_n\|_\epsilon^2 + \int_{\mathbb{R}^3} \Big(\frac{1}{4}g(\epsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2}G(\epsilon x, |u_n|^2)\Big)dx + o_n(1) \\
&\geq \frac{1}{4}\|u_n\|_\epsilon^2 + \int_{\Lambda_\epsilon^c} \Big(\frac{1}{4}g(\epsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2}G(\epsilon x, |u_n|^2)\Big)dx \\
&\quad + \frac{1}{12}\int_{\Lambda_\epsilon} |u_n|^6 dx + o_n(1) \\
&\geq \frac{1}{4}\Big(\int_{\Lambda_\epsilon} \psi_\rho|\nabla|u_n||^2 dx + \int_{\Lambda_\epsilon^c} V_\epsilon(x)|u_n|^2\Big) - \frac{1}{2}\int_{\Lambda_\epsilon^c} G(\epsilon x, |u_n|^2)dx \\
&\quad + \frac{1}{12}\int_{\Lambda_\epsilon} |u_n|^6 dx + o_n(1) \\
&\geq \frac{1}{4}\int_{\Lambda_\epsilon} \psi_\rho|\nabla|u_n||^2 dx + (\frac{1}{4} - \frac{1}{2K})\int_{\Lambda_\epsilon^c} V_\epsilon(x)|u_n|^2 dx + \frac{1}{12}\int_{\Lambda_\epsilon} \psi_\rho|u_n|^6 dx + o_n(1) \\
&\geq \frac{1}{4}\int_{\Lambda_\epsilon} \psi_\rho|\nabla|u_n||^2 dx + \frac{1}{12}\int_{\Lambda_\epsilon} \psi_\rho|u_n|^6 dx + o_n(1).
\end{aligned}
$$

From the above arguments and (3.17), we have

$$
\begin{aligned}
c &\geq \frac{1}{4}\sum_{\{i\in I: x_i\in\Lambda_\epsilon\}} \psi_\rho(x_i)\mu_i + \frac{1}{12}\sum_{\{i\in I: x_i\in\Lambda_\epsilon\}} \psi_\rho(x_i)\nu_i \\
&\geq \frac{1}{4}\mu_i + \frac{1}{12}\nu_i \\
&\geq \frac{1}{3}S^{3/2}
\end{aligned}
$$

which gives a contradiction. This means that (3.15) holds.

Step 3: From $J_\epsilon'(u_n)[u_n] \to 0$, $J_\epsilon'(u) = 0$, (3.7) and (3.15), we have

$$\lim_n \|u_n\|_\epsilon^2 = \|u\|_\epsilon^2,$$

and the proof is completed. □

Since f is only assumed to be continuous, the following result is required for multiplicity result in the next section.

Corollary 3.1. *The functional Ψ_ϵ satisfies the $(PS)_c$ condition on S_ϵ^+ at any level $c \in (0, \frac{1}{3}S^{\frac{3}{2}})$.*

Proof. Let $\{u_n\} \subset S_\epsilon^+$ be a $(PS)_c$ sequence for Ψ_ϵ where $c \in (0, \frac{1}{3}S^{\frac{3}{2}})$. Then $\Psi_\epsilon(u_n) \to c$ and $\|\Psi_\epsilon'(u_n)\|_\star \to 0$, where $\|\cdot\|_\star$ is the norm in the dual space $(T_{u_n}S_\epsilon^+)^\star$. By Lemma 3.3($B3$), we know that $\{m_\epsilon(u_n)\}$ is a $(PS)_c$ sequence for J_ϵ in H_ϵ. From Lemma 3.5, we know that there exists a $u \in S_\epsilon^+$ such that, up to a subsequence, $m_\epsilon(u_n) \to m_\epsilon(u)$ in H_ϵ. By Lemma 3.2($A3$), we obtain

$$u_n \to u \text{ in } S_\epsilon^+,$$

and the proof is completed. □

4 Multiple solutions for the modified problem

4.1 The autonomous problem

Now, we study the following *limit* problem

$$\begin{cases} -\Delta u + V_0 u + (|x|^{-1} \star |u|^2)u = f(u^2)u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3, \mathbb{R}), \ u(x) > 0, & \text{in } \mathbb{R}^3. \end{cases} \tag{4.1}$$

The solutions of problem (4.1) are the critical points of the C^1-functional defined by

$$I_0(u) := \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V_0 u^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^3} F(u^2)dx - \frac{1}{6}\int_{\mathbb{R}^3} (u^+)^6 dx.$$

Let

$$\mathcal{N}_0 := \{u \in H^1(\mathbb{R}^3, \mathbb{R}) \setminus \{0\} : I_0'(u)[u] = 0\}$$

and

$$c_{V_0} := \inf_{u \in \mathcal{N}_0} I_0(u).$$

Let $H_0 := H^1(\mathbb{R}^3, \mathbb{R})$ and define by H_0^+ the open set of H_0 given by

$$H_0^+ = \{u \in H_0 : |\text{supp}(u^+)| > 0\},$$

and $S_0^+ = S_0 \cap H_0^+$, where S_0 be the unit sphere of H_0.

As in Section 3, S_0^+ is a non-complete $C^{1,1}$-manifold of codimension 1, modeled on H_0 and contained in H_0^+. Therefore, $H_0 = T_u S_0^+ \bigoplus \mathbb{R}u$ for each $u \in T_u S_0^+$, where $T_u S_0^+ = \{v \in H_0 : \langle u, v\rangle_0 = 0\}$.

Now, arguing as in Lemma 3.2, we have the following important property.

Lemma 4.1. *Let V_0 be given in $(V1)$ and suppose that $(f1)$–$(f4)$ are satisfied, then the following properties hold:*

$(a1)$*For any $u \in H_0^+$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = I_0(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in (t_u, ∞);*

$(a2)$*There is a $\tau > 0$ independent on u such that $t_u > \tau$ for all $u \in S_0^+$. Moreover, for each compact $\mathcal{W} \subset S_0^+$ there is $C_{\mathcal{W}}$ such that $t_u \leq C_{\mathcal{W}}$, for all $u \in \mathcal{W}$;*

$(a3)$*The map $\widehat{m} : H_0^+ \to \mathcal{N}_0$ given by $\widehat{m}(u) = t_u u$ is continuous and $m_0 = \widehat{m}_0|_{S_0^+}$ is a homeomorphism between S_0^+ and $\mathcal{N}_0$. Moreover, $m^{-1}(u) = \frac{u}{\|u\|_0}$;*

$(a4)$*If there is a sequence $\{u_n\} \subset S_0^+$ such that $dist(u_n, \partial S_0^+) \to 0$, then $\|m(u_n)\|_0 \to \infty$ and $I_0(m(u_n)) \to \infty$.*

We shall consider the functional defined by

$$\widehat{\Psi}_0(u) = I_0(\widehat{m}(u)) \quad \text{and} \quad \Psi_0 := \widehat{\Psi}_0|_{S_0^+},$$

arguing as in [35, Proposition 9 and Corollary 10], the following result holds.

Lemma 4.2. *Let V_0 be given in $(V1)$ and suppose that $(f1)$–$(f4)$ are satisfied, then*

$(b1)$$\widehat{\Psi}_0^+ \in C^1(H_0^+, \mathbb{R})$ *and*

$$\widehat{\Psi}_0'(u)v = \frac{\|\widehat{m}(u)\|_0}{\|u\|_0} I_0'(\widehat{m}(u))[v], \quad \forall\, u \in H_0^+ \text{ and } \forall v \in H_0;$$

(b2) $\Psi_0 \in C^1(S_0^+, \mathbb{R})$ *and*

$$\Psi_0'(u)v = \|m(u)\|_0 I_0'(\widehat{m}(u))[v], \quad \forall v \in T_u S_0^+;$$

(b3)If $\{u_n\}$ *is a* $(PS)_c$ *sequence of* Ψ_0*, then* $\{m(u_n)\}$ *is a* $(PS)_c$ *sequence of* I_0*. If* $\{u_n\} \subset \mathcal{N}_0$ *is a bounded* $(PS)_c$ *sequence of* I_0*, then* $\{m^{-1}(u_n)\}$ *is a* $(PS)_c$ *sequence of* Ψ_0*;*

(b4)u is a critical point of Ψ_0 *if and only if* $m(u)$ *is a critical point of* I_0*. Moreover, the corresponding critical values coincide and*

$$\inf_{S_0^+} \Psi_0 = \inf_{\mathcal{N}_0} I_0.$$

Similar to the previous argument, we also have the following variational characterization of the infimum of I_0 over $\mathcal{N}_0$:

$$c_{V_0} = \inf_{u\in\mathcal{N}_0} I_0(u) = \inf_{u\in H_0^+\setminus\{0\}} \sup_{t>0} I_0(tu) = \inf_{u\in S_0^+} \sup_{t>0} I_0(tu). \tag{4.2}$$

From [26, Lemma 2.6], we have $0 < c_{V_0} < \frac{1}{3}S^{\frac{3}{2}}$.

Arguing as in [26, Lemma 2.8], the following important result holds.

Lemma 4.3. *Let* $\{u_n\} \subset H_0$ *be a* $(PS)_c$ *sequence for* I_0 *with* $c \in (0, \frac{1}{3}S^{\frac{3}{2}})$ *such that* $u_n \rightharpoonup 0$*. Then, one of the following alternatives occurs:*

(i) $u_n \to 0$ *in* H_0 *as* $n \to +\infty$*;*

(ii) *there is a sequence* $\{y_n\} \subset \mathbb{R}^3$ *and constants* $R, \beta > 0$ *such that*

$$\liminf_n \int_{B_R(y_n)} |u_n|^2 dx \geq \beta.$$

Remark 4.1. *From Lemma 4.3 we see that if u is the weak limit of* $(PS)_{c_{V_0}}$ *sequence* $\{u_n\}$ *of the functional* I_0*, then we have* $u \neq 0$*. Otherwise we have that* $u_n \rightharpoonup 0$ *and if* $u_n \not\to 0$*, from Lemma 4.3 it follows that there are a sequence* $\{y_n\} \subset \mathbb{R}^3$ *and constants* $R, \beta > 0$ *such that*

$$\liminf_n \int_{B_R(y_n)} |u_n|^2 dx \geq \beta > 0.$$

Then set $v_n(x) = u_n(x + z_n)$*, it is easy to see that* $\{v_n\}$ *is also a* $(PS)_{c_{V_0}}$ *sequence for the functional* I_0*, it is bounded, and there exists* $v \in H_0$ *such that* $v_n \rightharpoonup v$ *in* H_0 *with* $v \neq 0$*.*

Lemma 4.4. *Assume that* V *satisfies* $(V1)$*,* $(V2)$ *and* f *satisfies* $(f1)$*–*$(f4)$*, then problem* (4.1) *has a positive ground state solution.*

Proof. First of all, it is easy to show that $c_{V_0} > 0$. Moreover, if $u_0 \in \mathcal{N}_0$ satisfies $I_0(u_0) = c_{V_0}$, then $m^{-1}(u_0) \in S_0$ is a minimizer of Ψ_0, so that u_0 is a critical point of I_0 by Lemma 4.2. Now, we show that there exists a minimizer $u \in \mathcal{N}_0$ of $I_0|_{\mathcal{N}_0}$. Since $\inf_{S_0} \Psi_0 = \inf_{\mathcal{N}_0} I_0 = c_{V_0}$ and S_0 is a C^1 manifold, by Ekeland's variational principle, there exists a sequence $\omega_n \subset S_0$ with $\Psi_0(\omega_n) \to c_{V_0}$ and $\Psi_0'(\omega_n) \to 0$ as $n \to \infty$. Put $u_n = m(\omega_n) \in \mathcal{N}_0$ for $n \in N$. Then $I_0(u_n) \to c_{V_0}$ and $I_0'(u_n) \to 0$ as $n \to \infty$ by Lemma 4.2(*b*3). Similar to the proof of Lemma 3.4, it is easy to know that $\{u_n\}$ is bounded in H_0. Thus, we have $u_n \rightharpoonup u$ in H_0, $u_n \to u$ in $L^r_{\text{loc}}(\mathbb{R}^3)$, $1 \leq r < 6$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, thus $I_0'(u) = 0$. From [26, Lemma 2.6], we know that $c_{V_0} < \frac{1}{3}S^{\frac{3}{2}}$. Moreover, from Remark 4.1, we have that $u \neq 0$. Now, by Lemma 2.1,

$$\begin{aligned}
c_{V_0} &\leq I_0(u) = I_0(u) - \frac{1}{\theta} I_0'(u)[u] \\
&= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u\|_0^2 + \left(\frac{1}{4} - \frac{1}{\theta}\right)\int_{\mathbb{R}^3} (|x|^{-1} \star |u|^2)|u|^2 dx + \int_{\mathbb{R}^3} \left(\frac{1}{\theta}f(u^2)u^2 - \frac{1}{2}F(u^2)\right)dx + \frac{1}{12}\int_{\mathbb{R}^3} (u^+)^6 dx \\
&\leq \liminf_n \left\{ \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_0^2 + \left(\frac{1}{4} - \frac{1}{\theta}\right)\int_{\mathbb{R}^3} (|x|^{-1} \star |u_n|^2)|u_n|^2 dx + \int_{\mathbb{R}^3} \left(\frac{1}{\theta}f(u_n)u_n^2 - \frac{1}{2}F(u_n^2)\right)dx \right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{12}\int_{\mathbb{R}^3}(u_n^+)^6dx\Big\}\\
&=\liminf_n\Big\{I_0(u_n)-\frac{1}{\theta}I_0'(u_n)[u_n]\Big\}\\
&=c_{V_0},
\end{aligned}
$$

thus, u is a ground state solution. From the assumption of f, $u \geq 0$. Moreover, using the standard argument, we may prove that $u(x) > 0$ for $x \in \mathbb{R}^3$. The proof is complete. □

Note that, arguing as in [26, Proposition 3.3, Proposition 3.4 and Lemma 3.11], the ground state solution of problem decays exponentially at infinity with its gradient, and is $C^2(\mathbb{R}^3,\mathbb{R}) \cap L^\infty(\mathbb{R}^3,\mathbb{R})$. This result is very important for the proof of Lemma 4.6 later.

Lemma 4.5. *Let $(u_n)_n \subset \mathcal{N}_0$ such that $I_0(u_n) \to c_{V_0}$. Then $(u_n)_n$ has a convergent subsequence in H_0.*

Proof. Since $(u_n)_n \subset \mathcal{N}_0$, from Lemma 4.1($a3$), Lemma 4.2($b4$) and the definition of c_{V_0}, we have

$$v_n = m^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in S_0^+, \quad \forall n \in N,$$

and

$$\Psi_0(v_n) = I_0(u_n) \to c_{V_0} = \inf_{u\in S_0^+}\Psi_0(u).$$

Although S_0^+ is not a complete C^1 manifold, we still can use the Ekeland's variational principle [18] to the functional $\mathcal{E}_0 : H \to \mathbb{R}\cup\{\infty\}$ defined by $\mathcal{E}_0(u) := \widehat{\Psi}_0(u)$ if $u \in S_0^+$ and $\mathcal{E}_0(u) := \infty$ if $u \in \partial S_0^+$, where $H = \overline{S_0^+}$ is the complete metric space equipped with the metric $d(u, v) := \|u - v\|_0$. In fact, by Lemma 4.1(a4), $\mathcal{E}_0 \in C(H, \mathbb{R}\cup\{\infty\})$, and from Lemma 4.2(b4), $\mathcal{E}_0$ is bounded below. Therefore, there exists a sequence $\{\tilde{v}_n\} \subset S_0^+$ such that $\{\tilde{v}_n\}$ is a $(PS)_{c_{V_0}}$ sequence for Ψ_0 on S_0^+ and

$$\|\tilde{v}_n - v_n\|_0 = o_n(1).$$

Similar to the proof of Lemma 4.4, we may obtain the conclusion of this lemma. □

Now, we show the relationship between c_ϵ and c_{V_0}.

Lemma 4.6. *The numbers c_ϵ and c_{V_0} satisfy the following inequality*

$$\lim_{\epsilon\to 0} c_\epsilon = c_{V_0} < \frac{1}{3}S^{\frac{3}{2}}.$$

Proof. Let $\eta \in C_c^\infty(\mathbb{R}^3, [0, 1])$ be a cut-off function such that $\eta = 1$ in $B_{\rho/2}$ and $\mathrm{supp}(\eta) = B_\rho \subset \Lambda$ for some $\rho > 0$. Let us define $\omega_\epsilon(x) := \eta_\epsilon(x)\omega(x)e^{iA(0)\cdot x}$, where $\eta_\epsilon(x) = \eta(\epsilon x)$ for $\epsilon > 0$, ω is a positive and radial ground state solution of problem (4.1). We observe that $|\omega_\epsilon| = \eta_\epsilon\omega$ and $\omega_\epsilon \in H_\epsilon$ in view of Lemma 2.2. Arguing as in [16, Lemma 4.1] or [24, Lemma 4.6], we obtain

$$\lim_{\epsilon\to 0}\|\omega_\epsilon\|_\epsilon^2 = \|\omega\|_{V_0}^2 \tag{4.3}$$

and

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3}(|x|^{-1} * |\omega_\epsilon|^2)|\omega_\epsilon|^2dx = \int_{\mathbb{R}^3}(|x|^{-1} * |\omega|^2)|\omega|^2dx. \tag{4.4}$$

It is also easy to check that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3}|\omega_\epsilon|^6dx = \int_{\mathbb{R}^3}|\omega|^6dx. \tag{4.5}$$

Let $t_\epsilon > 0$ be the unique number such that

$$J_\epsilon(t_\epsilon\omega_\epsilon) = \max_{t\geq 0} J_\epsilon(t\omega_\epsilon).$$

Then t_ϵ satisfies

$$t_\epsilon^2\|\omega_\epsilon\|_\epsilon^2 + t_\epsilon^4\int_{\mathbb{R}^3}(|x|^{-1}\star|\omega_\epsilon|^2)|\omega_\epsilon|^2dx = \int_{\mathbb{R}^3} g(\epsilon x, t_\epsilon^2|\omega_\epsilon|^2)t_\epsilon^2|\omega_\epsilon|^2dx$$
$$= \int_{\mathbb{R}^3} f(t_\epsilon^2|\omega_\epsilon|^2)t_\epsilon^2|\omega_\epsilon|^2dx + \int_{\mathbb{R}^3} t_\epsilon^6|\omega_\epsilon|^6dx,$$

where we use $\mathrm{supp}(\eta)\subset\Lambda$ and the definition of $g(x,t)$. Moreover, combining the facts that $\eta = 1$ in $B_{\rho/2}$, u is a positive continuous function and hypothesis $(f4)$, we have

$$\frac{1}{t_\epsilon^2}\|\omega_\epsilon\|_\epsilon^2 + \int_{\mathbb{R}^3}(|x|^{-1}\star|\omega_\epsilon|^2)|\omega_\epsilon|^2dx = \frac{1}{t_\epsilon^2}\int_{\mathbb{R}^3} f(t_\epsilon^2|\omega_\epsilon|^2)|\omega_\epsilon|^2dx + \int_{\mathbb{R}^3} t_\epsilon^2|\omega_\epsilon|^6dx$$
$$\geq \frac{1}{t_\epsilon^2}\int_{\mathbb{R}^3} f(t_\epsilon^2\eta^2(|\epsilon x|)\omega^2(x))\eta^2(|\epsilon x|)\omega^2(x)dz$$
$$\geq \frac{1}{t_\epsilon^2}\int_{B_{\rho/(2\epsilon)}(0)} f(t_\epsilon^2\omega^2(z))\omega^2(z)dz$$
$$\geq \frac{1}{t_\epsilon^2}\int_{B_{\rho/2}(0)} f(t_\epsilon^2\omega^2(z))\omega^2(z)dz$$
$$\geq \frac{f(t_\epsilon^2 y^2)}{t_\epsilon^2}\int_{B_{\rho/2}(0)} \omega^2(z)dz$$

for all $0<\epsilon<1$ and where $y=\min\{\omega(z): |z|\leq\rho/2\}$.

If $t_\epsilon\to+\infty$ as $\epsilon\to 0$, by (f_4), we deduce that $\int_{\mathbb{R}^3}(|x|^{-1}\star|\omega_\epsilon|^2)|\omega_\epsilon|^2dx\to+\infty$ which contradicts (4.5).

Therefore, up to a subsequence, we may assume that $t_\epsilon\to t_0\geq 0$ as $\epsilon\to 0$.

If $t_\epsilon\to 0$, using the fact that f is increasing, the Lebesgue dominated convergence theorem and relation (4.5), we obtain

$$\|\omega_\epsilon\|_\epsilon^2 + t_\epsilon^4\int_{\mathbb{R}^3}(|x|^{-1}\star|\omega_\epsilon|^2)|\omega_\epsilon|^2dx = \int_{\mathbb{R}^3} f(t_\epsilon^2|\omega_\epsilon|^2)|\omega_\epsilon|^2dx + \int_{\mathbb{R}^3} t_\epsilon^4|\omega_\epsilon|^6dx \to 0, \text{ as } \epsilon\to 0$$

which contradicts (4.3). Thus, we have $t_0>0$ and

$$t_0^2\int_{\mathbb{R}^3}(|\nabla\omega|^2+V_0\omega^2)dx + t_0^4\int_{\mathbb{R}^3}(|x|^{-1}\star|\omega|^2)|\omega|^2dx = \int_{\mathbb{R}^3} f(t_0^2\omega^2)t_0^2\omega^2dx + \int_{\mathbb{R}^3} t_0^6|\omega|^6dx,$$

so that $t_0\omega\in\mathcal{N}_{V_0}$. Since $\omega\in\mathcal{N}_{V_0}$, we obtain that $t_0=1$ and so, using the Lebesgue dominated convergence theorem, we get

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3} F(|t_\epsilon\omega_\epsilon|^2)dx = \int_{\mathbb{R}^3} F(\omega^2)dx.$$

Hence

$$\lim_{\epsilon\to 0} J_\epsilon(t_\epsilon\omega_\epsilon) = I_{V_0}(u) = c_{V_0}.$$

Since $c_\epsilon\leq\max_{t\geq 0}J_\epsilon(t\omega_\epsilon) = J_\epsilon(t_\epsilon\omega_\epsilon)$, we can conclude that $\limsup_{\epsilon\to 0}c_\epsilon\leq c_{V_0}$. Moreover, by (3.3), (4.2) and $I_{V_0}(|u|)\leq J_\epsilon(u)$ for any $u\in H_\epsilon$, we have $c_{V_0}\leq c_\epsilon$. Then $c_{V_0}\leq\liminf_{\epsilon\to 0}c_\epsilon$. Combining with the previous arguments, we conclude that $\lim_{\epsilon\to 0}c_\epsilon = c_{V_0} < \frac{1}{3}S^{\frac{3}{2}}$. □

Remark 4.2. *From Lemma 4.1 and Lemma 3.5, we see that for $\epsilon>0$ small, problem* (3.2) *has a ground state solution u_ϵ such that $J_\epsilon(u_\epsilon)=c_\epsilon$ and $J_\epsilon'(u_\epsilon)=0$.*

4.2 The technical results

By the Ljusternik-Schnirelmann category theory, in this subsection we prove a multiplicity result for the modified problem (3.2). We first provide some useful preliminary results.

Let $\delta > 0$ such that $M_\delta \subset \Lambda$, $\omega \in H^1(\mathbb{R}^3, \mathbb{R})$ is a positive ground state solution of the limit problem (4.1), and $\eta \in C^\infty(\mathbb{R}^+, [0, 1])$ is a nonincreasing cut-off function defined in $[0, +\infty)$ such that $\eta(t) = 1$ if $0 \le t \le \delta/2$ and $\eta(t) = 0$ if $t \ge \delta$.

For any $y \in M$, let us introduce the function

$$\Psi_{\epsilon,y}(x) := \eta(|\epsilon x - y|)\omega\Big(\frac{\epsilon x - y}{\epsilon}\Big) \exp\Big(i\tau_y\Big(\frac{\epsilon x - y}{\epsilon}\Big)\Big),$$

where

$$\tau_y(x) := \sum_i^3 A_i(y)x_i.$$

Let $t_\epsilon > 0$ be the unique positive number such that

$$\max_{t\ge 0} J_\epsilon(t\Psi_{\epsilon,y}) = J_\epsilon(t_\epsilon\Psi_{\epsilon,y}).$$

Note that $t_\epsilon\Psi_{\epsilon,y} \in \mathcal{N}_\epsilon$.

Let us define $\Phi_\epsilon : M \to \mathcal{N}_\epsilon$ as

$$\Phi_\epsilon(y) := t_\epsilon\Psi_{\epsilon,y}.$$

By construction, $\Phi_\epsilon(y)$ has compact support for any $y \in M$.

Moreover, arguing as in Lemma 4.1, the energy of above function has the following behavior as $\epsilon \to 0^+$.

Lemma 4.7. *The limit*

$$\lim_{\epsilon\to 0^+} J_\epsilon(\Phi_\epsilon(y)) = c_{V_0}$$

holds uniformly in $y \in M$.

Now we define the barycenter map.

Let $\rho > 0$ be such that $M_\delta \subset B_\rho$ and consider $Y : \mathbb{R}^3 \to \mathbb{R}^3$ defined by setting

$$Y(x) := \begin{cases} x, & \text{if } |x| < \rho, \\ \rho x/|x|, & \text{if } |x| \ge \rho. \end{cases}$$

The barycenter map $\beta_\epsilon : \mathcal{N}_\epsilon \to \mathbb{R}^3$ is defined by

$$\beta_\epsilon(u) := \frac{1}{\|u\|_4^4} \int_{\mathbb{R}^3} Y(\epsilon x)|u(x)|^4 dx.$$

Lemma 4.8. *The limit*

$$\lim_{\epsilon\to 0^+} \beta_\epsilon(\Phi_\epsilon(y)) = y$$

holds uniformly in $y \in M$.

Proof. Assume by contradiction that there exists $\kappa > 0$, $(y_n) \subset M$ and $\epsilon_n \to 0$ such that

$$|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - y_n| \ge \kappa. \tag{4.6}$$

Using the change of variable $z = (\epsilon_n x - y_n)/\epsilon_n$, we can see that

$$\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^3} (Y(\epsilon_n z + y_n) - y_n)\eta^4(|\epsilon_n z|)\omega^4(z)dz}{\int_{\mathbb{R}^3} \eta^4(|\epsilon_n z|)\omega^4(z)dz}.$$

Taking into account $(y_n) \subset M \subset M_\delta \subset B_\rho$ and the Lebesgue Dominated Convergence Theorem, we can obtain that

$$|\beta_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - y_n| = o_n(1),$$

which contradicts (4.6). □

Now, we prove the following useful compactness result.

Proposition 4.1. *Let $\epsilon_n \to 0^+$ and $(u_n) \subset \mathcal{N}_{\epsilon_n}$ be such that $J_{\epsilon_n}(u_n) \to c_{V_0}$. Then there exists $(\tilde{y}_n) \subset \mathbb{R}^3$ such that the sequence $(|v_n|)_n \subset H^1(\mathbb{R}^3, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$, has a convergent subsequence in $H^1(\mathbb{R}^3, \mathbb{R})$. Moreover, up to a subsequence, $y_n := \epsilon_n \tilde{y}_n \to y \in M$ as $n \to +\infty$.*

Proof. Since $J'_{\epsilon_n}(u_n)[u_n] = 0$ and $J_{\epsilon_n}(u_n) \to c_{V_0}$, arguing as in the proof of Lemma 3.4, we can prove that there exists $C > 0$ such that $\|u_n\|_{\epsilon_n} \le C$ for all $n \in \mathbb{N}$.
Arguing as in the proof of Lemma 3.2 and recalling that $c_{V_0} > 0$, we have that there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that

$$\liminf_n \int_{B_R(\tilde{y}_n)} |u_n|^2 dx \ge \beta. \tag{4.7}$$

Now, let us consider the sequence $\{|v_n|\} \subset H^1(\mathbb{R}^3, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$. By the diamagnetic inequality (2.1), we get that $\{|v_n|\}$ is bounded in $H^1(\mathbb{R}^3, \mathbb{R})$, and using (4.7), we may assume that $|v_n| \rightharpoonup v$ in $H^1(\mathbb{R}^3, \mathbb{R})$ for some $v \neq 0$.
Let now $t_n > 0$ be such that $\tilde{v}_n := t_n|v_n| \in \mathcal{N}_{V_0}$, and set $y_n := \epsilon_n \tilde{y}_n$.
By the diamagnetic inequality (2.1), we have

$$c_{V_0} \le I_0(\tilde{v}_n) \le \max_{t \ge 0} J_{\epsilon_n}(tu_n) = J_{\epsilon_n}(u_n) = c_{V_0} + o_n(1),$$

which yields $I_0(\tilde{v}_n) \to c_{V_0}$ as $n \to +\infty$.
Since the sequences $\{|v_n|\}$ and $\{\tilde{v}_n\}$ are bounded in $H^1(\mathbb{R}^3, \mathbb{R})$ and $|v_n| \not\to 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, then (t_n) is also bounded and so, up to a subsequence, we may assume that $t_n \to t_0 \ge 0$.
We claim that $t_0 > 0$. Indeed, if $t_0 = 0$, then, since $(|v_n|)_n$ is bounded, we have $\tilde{v}_n \to 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, that is $I_0(\tilde{v}_n) \to 0$, which contradicts $c_{V_0} > 0$.
Thus, up to a subsequence, we may assume that $\tilde{v}_n \rightharpoonup \tilde{v} := t_0 v \neq 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, and, by Lemma 4.5, we can deduce that $\tilde{v}_n \to \tilde{v}$ in $H^1(\mathbb{R}^3, \mathbb{R})$, which gives $|v_n| \to v$ in $H^1(\mathbb{R}^3, \mathbb{R})$.
Now we show the final part, namely that $\{y_n\}$ has a subsequence such that $y_n \to y \in M$. Assume by contradiction that $\{y_n\}$ is not bounded and so, up to a subsequence, $|y_n| \to +\infty$ as $n \to +\infty$. Choose $R > 0$ such that $\Lambda \subset B_R(0)$. Then for n large enough, we have $|y_n| > 2R$, and, for any $x \in B_{R/\epsilon_n}(0)$,

$$|\epsilon_n x + y_n| \ge |y_n| - \epsilon_n |x| > R.$$

Since $u_n \in \mathcal{N}_{\epsilon_n}$, using $(V1)$ and the diamagnetic inequality (2.1), we get that

$$\begin{aligned} \int_{\mathbb{R}^3} (|\nabla |v_n||^2 + V_0|v_n|^2) dx &\le \int_{\mathbb{R}^3} g(\epsilon_n x + y_n, |v_n|^2)|v_n|^2 dx \\ &\le \int_{B_{R/\epsilon_n}(0)} \tilde{f}(|v_n|^2)|v_n|^2 dx + \int_{B^c_{R/\epsilon_n}(0)} f(|v_n|^2)|v_n|^2 dx + \int_{B^c_{R/\epsilon_n}(0)} |v_n|^6 dx. \end{aligned} \tag{4.8}$$

Since $|v_n| \to v$ in $H^1(\mathbb{R}^3, \mathbb{R})$ and $\tilde{f}(t) \le V_0/K$, we can see that (4.8) yields

$$\min\left\{1, V_0\left(1 - \frac{1}{K}\right)\right\} \int_{\mathbb{R}^3} (|\nabla |v_n||^2 + |v_n|^2) dx = o_n(1),$$

that is $|v_n| \to 0$ in $H^1(\mathbb{R}^3, \mathbb{R})$, which contradicts to $v \not\equiv 0$.
Therefore, we may assume that $y_n \to y_0 \in \mathbb{R}^3$. Assume by contradiction that $y_0 \notin \overline{\Lambda}$. Then there exists $r > 0$

such that for every n large enough we have that $|y_n - y_0| < r$ and $B_{2r}(y_0) \subset \overline{\Lambda}^c$. Then, if $x \in B_{r/\epsilon_n}(0)$, we have that $|\epsilon_n x + y_n - y_0| < 2r$ so that $\epsilon_n x + y_n \in \overline{\Lambda}^c$ and so, arguing as before, we reach a contradiction. Thus, $y_0 \in \overline{\Lambda}$.
To prove that $V(y_0) = V_0$, we suppose by contradiction that $V(y_0) > V_0$. Using the Fatou's lemma, the change of variable $z = x + \tilde{y}_n$ and $\max_{t\geq 0} J_{\epsilon_n}(tu_n) = J_{\epsilon_n}(u_n)$, we obtain

$$\begin{aligned} c_{V_0} = I_0(\tilde{v}) &< \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla\tilde{v}|^2 + V(y_0)|\tilde{v}|^2)dx + \frac{1}{4}\int_{\mathbb{R}^3}(|x|^{-1} \star |\tilde{v}|^2)|\tilde{v}|^2dx - \frac{1}{2}\int_{\mathbb{R}^3}F(|\tilde{v}|^2)dx - \frac{1}{6}\int_{\mathbb{R}^3}|\tilde{v}|^6dx \\ &\leq \liminf_n\Big(\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla\tilde{v}_n|^2 + V(\epsilon_n x + y_n)|\tilde{v}_n|^2)dx + \frac{1}{4}\int_{\mathbb{R}^3}(|x|^{-1} \star |\tilde{v}_n|^2)|\tilde{v}_n|^2dx - \frac{1}{2}\int_{\mathbb{R}^3}F(|\tilde{v}_n|^2)dx \\ &\quad - \frac{1}{6}\int_{\mathbb{R}^3}|\tilde{v}_n|^6dx\Big) \\ &= \liminf_n\Big(\frac{t_n^2}{2}\int_{\mathbb{R}^3}(|\nabla|u_n||^2 + V(\epsilon_n z)|u_n|^2)dz + \frac{t_n^4}{4}\int_{\mathbb{R}^3}(|x|^{-1} \star |u_n|^2)|u_n|^2dx - \frac{1}{2}\int_{\mathbb{R}^3}F(|t_n u_n|^2)dz \\ &\quad - \frac{1}{6}\int_{\mathbb{R}^3}|t_n u_n|^6dx\Big) \\ &\leq \liminf_n J_{\epsilon_n}(t_n u_n) \leq \liminf_n J_{\epsilon_n}(u_n) = c_{V_0} \end{aligned}$$

which is impossible and the proof is complete. □

Let now

$$\tilde{\mathcal{N}}_\epsilon := \{u \in \mathcal{N}_\epsilon : J_\epsilon(u) \leq c_{V_0} + h(\epsilon)\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(\epsilon) \to 0$ as $\epsilon \to 0^+$.
Fixed $y \in M$, since, by Lemma 4.7, $|J_\epsilon(\Phi_\epsilon(y)) - c_{V_0}| \to 0$ as $\epsilon \to 0^+$, we get that $\tilde{\mathcal{N}}_\epsilon \neq \emptyset$ for any $\epsilon > 0$ small enough.

The relation between $\tilde{\mathcal{N}}_\epsilon$ and the barycenter map is as follows.

Lemma 4.9. *We have*

$$\lim_{\epsilon\to 0^+} \sup_{u\in\tilde{\mathcal{N}}_\epsilon} \operatorname{dist}(\beta_\epsilon(u), M_\delta) = 0.$$

Proof. Let $\epsilon_n \to 0^+$ as $n \to +\infty$. For any $n \in \mathbb{N}$, there exists $u_n \in \tilde{\mathcal{N}}_{\epsilon_n}$ such that

$$\sup_{u\in\tilde{\mathcal{N}}_{\epsilon_n}} \inf_{y\in M_\delta} |\beta_{\epsilon_n}(u) - y| = \inf_{y\in M_\delta} |\beta_{\epsilon_n}(u_n) - y| + o_n(1).$$

Therefore, it is enough to prove that there exists $(y_n) \subset M_\delta$ such that

$$\lim_n |\beta_{\epsilon_n}(u_n) - y_n| = 0.$$

By the diamagnetic inequality (2.1), we can see that $I_0(t|u_n|) \leq J_{\epsilon_n}(tu_n)$ for any $t \geq 0$. Therefore, recalling that $\{u_n\} \subset \tilde{\mathcal{N}}_{\epsilon_n} \subset \mathcal{N}_{\epsilon_n}$, we can deduce that

$$c_{V_0} \leq \max_{t\geq 0} I_0(t|u_n|) \leq \max_{t\geq 0} J_{\epsilon_n}(tu_n) = J_{\epsilon_n}(u_n) \leq c_{V_0} + h(\epsilon_n) \tag{4.9}$$

which implies that $J_{\epsilon_n}(u_n) \to c_{V_0}$ as $n \to +\infty$.
Then, Proposition 4.1 implies that there exists $\{\tilde{y}_n\} \subset \mathbb{R}^3$ such that $y_n = \epsilon_n\tilde{y}_n \in M_\delta$ for n large enough.
Thus, making the change of variable $z = x - \tilde{y}_n$, we get

$$\beta_{\epsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^3}(Y(\epsilon_n z + y_n) - y_n)|u_n(z + \tilde{y}_n)|^4dz}{\int_{\mathbb{R}^3}|u_n(z + \tilde{y}_n)|^4dz}.$$

Since, up to a subsequence, $|u_n|(\cdot + \tilde{y}_n)$ converges strongly in $H^1(\mathbb{R}^3, \mathbb{R})$ and $\epsilon_n z + y_n \to y \in M$ for any $z \in \mathbb{R}^3$, we conclude. □

4.3 Multiplicity of solutions for problem (3.2)

Finally, we present a relation between the topology of M and the number of nontrivial solutions of the modified problem (3.2).

Theorem 4.1. *For any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\tilde{\epsilon}_\delta > 0$ such that, for any $\epsilon \in (0, \tilde{\epsilon}_\delta)$, problem (3.2) has at least $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions.*

Proof. For any $\epsilon > 0$, we define the function $\pi_\epsilon : M \to S_\epsilon^+$ by

$$\pi_\epsilon(y) = m_\epsilon^{-1}(\Phi_\epsilon(y)), \ \ \forall y \in M.$$

By Lemma 4.7 and Lemma 3.3($B4$), it follows that

$$\lim_{\epsilon\to 0} \Psi_\epsilon(\pi_\epsilon(y)) = \lim_{\epsilon\to 0} J_\epsilon(\Phi_\epsilon(y)) = c_{V_0}, \ \text{uniformly in } y \in M.$$

Therefore, there is a number $\hat{\epsilon} > 0$ such that the set $\tilde{S}_\epsilon^+ := \{u \in S_\epsilon^+ : \Psi_\epsilon(u) \leq c_{V_0} + h(\epsilon)\}$ is nonempty, for all $\epsilon \in (0, \hat{\epsilon})$, since $\pi_\epsilon(M) \subset \tilde{S}_\epsilon^+$. Here h is given in the definition of $\tilde{\mathcal{N}}_\epsilon$.

Given $\delta > 0$, by Lemma 4.7, Lemma 3.2($A3$), Lemma 4.8, and Lemma 4.9, we can find $\tilde{\epsilon}_\delta > 0$ such that for any $\epsilon \in (0, \tilde{\epsilon}_\delta)$, the following diagram

$$M \xrightarrow{\Phi_\epsilon} \Phi_\epsilon(M) \xrightarrow{m_\epsilon^{-1}} \pi_\epsilon(M) \xrightarrow{m_\epsilon} \Phi_\epsilon(M) \xrightarrow{\beta_\epsilon} M_\delta$$

is well defined and continuous. From Lemma 4.8, we can choose a function $\Theta(\epsilon, z)$ with $|\Theta(\epsilon, z)| < \frac{\delta}{2}$ uniformly in $z \in M$, for all $\epsilon \in (0, \hat{\epsilon})$ such that $\beta_\epsilon(\Phi_\epsilon(z)) = z + \Theta(\epsilon, z)$ for all $z \in M$. Define $H(t, z) = z + (1-t)\Theta(\epsilon, z)$. Then $H : [0, 1] \times M \to M_\delta$ is continuous. Clearly, $H(0, z) = \beta_\epsilon(\Phi_\epsilon(z))$, $H(1, z) = z$ for all $z \in M$. That is, $H(t, z)$ is a homotopy between $\beta_\epsilon \circ \Phi_\epsilon = (\beta_\epsilon \circ m_\epsilon) \circ \pi_\epsilon$ and the embedding $\iota : M \to M_\delta$. This fact implies that

$$\mathrm{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M)) \geq \mathrm{cat}_{M_\delta}(M). \tag{4.10}$$

By Corollary 3.1 and the abstract category theorem [35], Ψ_ϵ has at least $\mathrm{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M))$ critical points on S_ϵ^+. Therefore, from Lemma 3.3($B4$) and (4.10), we have that J_ϵ has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $\tilde{\mathcal{N}}_\epsilon$ which implies that problem (3.2) has at least $\mathrm{cat}_{M_\delta}(M)$ solutions. □

5 Proof of Theorem 1.1

In this section we shall show that the solutions u_ϵ obtained in Theorem 4.1 satisfy

$$|u_\epsilon(x)|^2 \leq a \text{ for } x \in \Lambda_\epsilon^c$$

for ϵ small and prove the main result of this paper.

Arguing as in [29] or [41], the following uniform result holds.

Lemma 5.1. *Let $\epsilon_n \to 0^+$ and $u_n \in \tilde{\mathcal{N}}_{\epsilon_n}$ be a solution of problem (3.2) for $\epsilon = \epsilon_n$. Then $J_{\epsilon_n}(u_n) \to c_{V_0}$. Moreover, there exists $\{\tilde{y}_n\} \subset \mathbb{R}^3$ such that, if $v_n(x) := u_n(x + \tilde{y}_n)$, we have that $\{|v_n|\}$ is bounded in $L^\infty(\mathbb{R}^3, \mathbb{R})$ and*

$$\lim_{|x|\to+\infty} |v_n(x)| = 0 \quad \textit{uniformly in } n \in \mathbb{N}.$$

Now it's the position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\delta > 0$ be such that $M_\delta \subset \Lambda$. We want to show that there exists $\hat{\epsilon}_\delta > 0$ such that for any $\epsilon \in (0, \hat{\epsilon}_\delta)$ and any $u_\epsilon \in \tilde{\mathcal{N}}_\epsilon$ solution of problem (3.2), it holds

$$\|u_\epsilon\|^2_{L^\infty(\Lambda_\epsilon^c)} \leq a. \tag{5.1}$$

We argue by contradiction and assume that there is a sequence $\epsilon_n \to 0$ such that for every n there exists $u_n \in \tilde{\mathcal{N}}_{\epsilon_n}$ which satisfies $J'_{\epsilon_n}(u_n) = 0$ and

$$\|u_n\|^2_{L^\infty(\Lambda^c_{\epsilon_n})} > a. \tag{5.2}$$

As in Lemma 5.1, we have that $J_{\epsilon_n}(u_n) \to c_{V_0}$, and therefore we can use Proposition 4.1 to obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^3$ such that $y_n := \epsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in M$. Then, we can find $r > 0$, such that $B_r(y_n) \subset \Lambda$, and so $B_{r/\epsilon_n}(\tilde{y}_n) \subset \Lambda_{\epsilon_n}$ for all n large enough.

Using Lemma 5.1, there exists $R > 0$ such that $|v_n|^2 \le a$ in $B^c_R(0)$ and n large enough, where $v_n = u_n(\cdot + \tilde{y}_n)$. Hence $|u_n|^2 \le a$ in $B^c_R(\tilde{y}_n)$ and n large enough. Moreover, if n is so large that $r/\epsilon_n > R$, then $\Lambda^c_{\epsilon_n} \subset B^c_{r/\epsilon_n}(\tilde{y}_n) \subset B^c_R(\tilde{y}_n)$, which gives $|u_n|^2 \le a$ for any $x \in \Lambda^c_{\epsilon_n}$. This contradicts (5.2) and proves the claim.

Let now $\epsilon_\delta := \min\{\hat{\epsilon}_\delta, \tilde{\epsilon}_\delta\}$, where $\tilde{\epsilon}_\delta > 0$ is given by Theorem 4.1. Then we have $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions to problem (3.2). If $u_\epsilon \in \tilde{\mathcal{N}}_\epsilon$ is one of these solutions, then, by (5.1) and the definition of g, we conclude that u_ϵ is also a solution to problem (2.2).

Finally, we study the behavior of the maximum points of $|\hat{u}_\epsilon|$, where $\hat{u}_\epsilon(x) := u_\epsilon(x/\epsilon)$ is a solution to problem (1.1), as $\epsilon \to 0^+$.

Take $\epsilon_n \to 0^+$ and the sequence (u_n) where each u_n is a solution of (3.2) for $\epsilon = \epsilon_n$. From the definition of g, there exists $y \in (0, a)$ such that

$$g(\epsilon x, t^2)t^2 \le \frac{V_0}{K} t^2, \quad \text{for all } x \in \mathbb{R}^3,\ |t| \le y.$$

Arguing as above we can take $R > 0$ such that, for n large enough,

$$\|u_n\|_{L^\infty(B^c_R(\tilde{y}_n))} < y. \tag{5.3}$$

Up to a subsequence, we may also assume that for n large enough

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \ge y. \tag{5.4}$$

Indeed, if (5.4) does not hold, up to a subsequence, if necessary, we have $\|u_n\|_\infty < y$. Thus, since $J'_{\epsilon_n}(u_{\epsilon_n}) = 0$, using $(g5)$ and the diamagnetic inequality (2.1) that

$$\int_{\mathbb{R}^3} (|\nabla|u_n||^2 + V_0|u_n|^2)dx \le \int_{\mathbb{R}^3} g(\epsilon_n x, |u_n|^2)|u_n|^2 dx \le \frac{V_0}{K} \int_{\mathbb{R}^3} |u_n|^2 dx$$

and, being $K > 2$, $\|u_n\| = 0$, which is a contradiction.

Taking into account (5.3) and (5.4), we can infer that the global maximum points p_n of $|u_{\epsilon_n}|$ belongs to $B_R(\tilde{y}_n)$, that is $p_n = q_n + \tilde{y}_n$ for some $q_n \in B_R$. Recalling that the associated solution of problem (1.1) is $\hat{u}_n(x) = u_n(x/\epsilon_n)$, we can see that a maximum point η_{ϵ_n} of $|\hat{u}_n|$ is $\eta_{\epsilon_n} = \epsilon_n \tilde{y}_n + \epsilon_n q_n$. Since $q_n \in B_R$, $\epsilon_n \tilde{y}_n \to y_0$ and $V(y_0) = V_0$, the continuity of V allows to conclude that

$$\lim_n V(\eta_{\epsilon_n}) = V_0. \qquad \square$$

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References

[1] A. Ambrosetti, D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem, Commun. Contemp. Math. 10 (2008), 391–404.

[2] V. Ambrosio, Multiplicity and concentration results for a fractional Schrödinger-Poisson type equation with magnetic field, Proc. Roy. Soc. Edinburgh Sect. A 150 (2020), 655-694.

[3] V. Ambrosio, Multiplicity and concentration results for fractional Schrödinger-Poisson equations with magnetic fields and critical growth, Potential Anal. 52 (2020), 565-600.

[4] A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear Schrödinger-Maxwell equations, J. Math. Anal. Appl. 345 (2008), 90–108.

[5] G. Cerami, G. Vaira, Positive solutions for some non-autonomous Schrödinger-Poisson systems, J. Differential Equations 248 (2010), 521–543.

[6] C.O. Alves, G.M. Figueiredo, M.F. Furtado, Multiple solutions for a nonlinear Schrödinger equation with magnetic fields, Comm. Partial Differential Equations 36 (2011), 1565–1586.

[7] G. Arioli, A. Szulkin, A semilinear Schrödinger equation in the presence of a magnetic field, Arch. Rational Mech. Anal. 170 (2003), 277–295.

[8] S. Barile, S. Cingolani, S. Secchi, Single-peaks for a magnetic Schrödinger equation with critical growth, Adv. Differential Equations 11 (2006), 1135–1166.

[9] J. Byeon, L. Jeanjean, M. Maris, Symmetric and monotonicity of least energy solutions, Calc. Var. Partial Differ. Equ. 36 (2009), 481-492.

[10] S. Cingolani, Semiclassical stationary states of nonlinear Schrödinger equations with an external magnetic field, J. Differential Equations 188 (2003), 52–79.

[11] S. Cingolani, L. Jeanjean, S. Secchi, Multi-peak solutions for magnetic NLS equations without non-degeneracy conditions, ESAIM Control Optim. Calc. Var. 15 (2009), 653-675.

[12] S. Cingolani, L. Jeanjean, K. Tanaka, Multiple complex-valued solutions for nonlinear magnetic Schrödinger equations, J. Fixed Point Theory Appl. 19 (2017), no. 1, 37-66.

[13] S. Cingolani, S. Secchi, Semiclassical states for NLS equations with magnetic potentials having polynomial growths, J. Math. Phys. 46 (2005), 053503, 19pp.

[14] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004), 893–906.

[15] T. D'Aprile, D. Mugnai, Non-existence results for the coupled Klein-Gordon-Maxwell equations, Adv. Nonlinear Stud. 4 (2004), 307–322.

[16] P. d'Avenia, C. Ji, Multiplicity and concentration results for a magnetic Schrödinger equation with exponential critical growth in $\mathbb{R}^2$, Int. Math. Res. Not. (2020), DOI: 10.1093/imrn/rnaa074.

[17] M. del Pino, P.L. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, Calc. Var. Partial Differential Equations 4 (1996), 121–137.

[18] I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47 (1974), 324–353.

[19] M.J. Esteban, P.L. Lions, Stationary solutions of nonlinear Schrödinger equations with an external magnetic field, Partial differential equations and the calculus of variations, Vol. I, 401–449, Progr. Nonlinear Differential Equations Appl., 1, Birkhäuser Boston, Boston, 1989.

[20] A. Fiscella, P. Pucci, B.L. Zhang, p-fractional Hardy-Schrödinger-Kirchhoff systems with critical nonlinearities, Adv. Nonlinear Anal. 8 (2019), 1111-1131.

[21] C. Ji, Ground state sign-changing solutions for a class of nonlinear fractional Schrödinger-Poisson system in $\mathbb{R}^3$, Ann. Mat. Pura Appl. (4) 198 (2019), no. 5, 1563-1579.

[22] C. Ji, F. Fang, B.L. Zhang, Least energy sign-changing solutions for the nonlinear Schrödinger-Poisson system, Electron. J. Differential Equations 282 (2017), 1-13.

[23] C. Ji, V.D. Rădulescu, Multi-bump solutions for the nonlinear magnetic Schrödinger equation with exponential critical growth in $\mathbb{R}^2$, Manuscripta Math. (2020), DOI: 10.1007/s00229-020-01195-1.

[24] C. Ji, V.D. Rădulescu, Multiplicity and concentration of solutions to the nonlinear magnetic Schrodinger equation, Calc. Var. Partial Differential Equations, 59 (2020), art 115, pp.28.

[25] X. He, Multiplicity and concentration of positive solutions for the Schrödinger-Poisson equations, Z. Angew. Math. Phys. 62 (2011), 869–889.

[26] X. He, W. Zou, Existence and concentration of ground states for Schrödinger-Poisson equations with critical growth, J. Math. Phys. 53 (2012), 023702, 19pp.

[27] X. He, W. Zou, Multiplicity of concentrating positive solutions for Schrödinger-Poisson equations with critical growth, Nonlinear Anal. 170 (2018), 2150–2164.

[28] E.H. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics 14, American Mathematical Society, Providence, 2001.

[29] Y.L Liu, X. Li, C. Ji, Multiplicity of concentrating solutions for a class of magnetic Schrödinger-Poisson type equation, Adv. Nonlinear Anal. 10 (2021), 131-151.

[30] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Nonlinear analysis-theory and methods, Springer Monographs in Mathematics, Springer, Cham, 2019.

[31] P. H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (1992), 270–291.

[32] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, J. Funct. Anal. 237 (2006), 655–674.

[33] D. Ruiz, S. Gaetano, A note on the Schrödinger-Poisson-Slater equation on bounded domains, Adv. Nonlinear Stud. 8 (2008), 179–190.

[34] A. Szulkin, T. Weth, Ground state solutions for some indefinite variational problems, J. Funct. Anal. 257 (2009), 3802–3822.

[35] A. Szulkin, T. Weth, The method of Nehari manifold, Handbook of Nonconvex Analysis and Applications, pp. 2314–2351, International Press, Boston, 2010.
[36] L.X. Wen, S.T. Chen, V.D. Rădulescu, Axially symmetric solutions of the Schrödinger-Poisson system with zero mass potential in $\mathbb{R}^2$, Appl. Math. Lett. 104 (2020), 106244.
[37] M. Willem, Minimax Theorems, Birkhäuser Boston, Boston, 1996.
[38] A. Xia, Multiplicity and concentration results for magnetic relativistic Schrödinger equations, Adv. Nonlinear Anal. 9 (2020), 1161–1186.
[39] M.Q. Xiang, V.D. Rădulescu, B.L. Zhang, A critical fractional Choquard-Kirchhoff problem with magnetic field. Commun. Contemp. Math. 21 (2019), 1850004, 36 pp.
[40] F. Zhao, L. Zhao, Positive solutions for Schrödinger-Poisson equations with a critical exponent, Nonlinear Anal. 70 (2009) 2150–2164.
[41] A.Q. Zhu, X.M. Sun, Multiple solutions for Schrödinger-Poisson type equation with magnetic field, J. Math. Phys. 56 (2015), 091504, 15pp.