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Multiplicity results for systems of asymptotically linear second order equations*

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Abstract

We prove the existence and multiplicity of solutions, with prescribed nodal properties, for a BVP associated with a system of asymptotically linear second order equations. The applicability of an abstract continuation theorem is ensured by upper and lower bounds on the number of zeros of *each component* of a solution.

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1 Introduction and statement of the main result

In this paper we study the existence and multiplicity of solutions of a two-point boundary value problem of the form

$$\begin{cases} u''(t) + G(t, u(t)) = 0 \\ u(0) = 0 = u(\pi), \end{cases} \tag{1.1}$$

where $G : [0, \pi] \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a continuous function which is "asymptotically linear" in a neighbourhood of the origin and at infinity. In the particular case $m = 1$, the literature contains a very large number of contributions. For many authors, as

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it is the case in this paper, the key to multiplicity is some knowledge about the number of zeros of a (possible) solution of (1.1). In this direction, a classical reference is the book by Krasnosel'skii et al.[25]; for more recent results in the spirit of the shooting method, we refer to the papers by Dinca, Sanchez [12], García-Huidobro, Ubilla [19], Klokov, Sadyrbaev [24], Sadyrbaev [34]. When classical phase-plane analysis arguments are combined with topological degree methods, the nonlinearities involved have to satisfy weaker regularity assumptions: in this framework, we refer to the recent contribution of Dambrosio [11] and (as for superlinear problems) to [4],[6],[17]. See also [1],[2],[7],[13],[20],[23],[30] and references therein.

Obviously, in case the function G is of "gradient type" then problem (1.1) can be handled by means of variational methods, even in the more general situation of PDEs. In this direction, we refer, among others, to the work by Fortunato [15] and references therein. See also the book by Mawhin, Willem [32], and [22],[27],[28],[35].

The list of results available in the literature as far as *systems* are concerned is definitely shorter. For a general approach to systems by topological methods we refer to the work of Manásevich, Mawhin [29]; multiplicity results for systems of superlinear ODEs can be found in [5],[10],[21]. See also [3] and [8]. A very recent result for some system of PDEs has been given by Zou [36].

Let us now describe in detail our assumptions on the function G and our main result. In what follows, we set $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_0 = \{x \in \mathbb{R} : x \neq 0\}$. For every $i \in \{1, \dots, m\}$ we assume the following asymptotic conditions:

(H_i^+) there exist $N_i > 0$ and two continuous functions b_i^+, $\beta_i^+ : [0, \pi] \longrightarrow \mathbb{R}^+$ such that

$$x_i > N_i \quad \Longrightarrow \quad b_i^+(t) \leq \frac{G_i(t,x)}{x_i} \leq \beta_i^+(t)$$

for every $t \in [0, \pi]$ and $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \in \mathbb{R}^{m-1}$;

(H_i^-) there exist $M_i > 0$ and two continuous functions b_i^-, $\beta_i^- : [0, \pi] \longrightarrow \mathbb{R}^+$ such that

$$x_i < -M_i \quad \Longrightarrow \quad b_i^-(t) \leq \frac{G_i(t,x)}{x_i} \leq \beta_i^-(t)$$

for every $t \in [0, \pi]$ and $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \in \mathbb{R}^{m-1}$;

(K_i^+) there exist $\delta_i > 0$ and two continuous functions a_i^+, $\alpha_i^+ : [0, \pi] \longrightarrow \mathbb{R}^+$ such that

$$0 < x_i < \delta_i \quad \Longrightarrow \quad a_i^+(t) \leq \frac{G_i(t,x)}{x_i} \leq \alpha_i^+(t)$$

for every $t \in [0, \pi]$ and $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \in \mathbb{R}^{m-1}$;

(K_i^-) there exist $\sigma_i > 0$ and two continuous functions a_i^-, $\alpha_i^- : [0, \pi] \longrightarrow \mathbb{R}^+$ such that

$$-\sigma_i < x_i < 0 \quad \Longrightarrow \quad a_i^-(t) \leq \frac{G_i(t,x)}{x_i} \leq \alpha_i^-(t)$$

for every $t \in [0, \pi]$ and $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m) \in \mathbb{R}^{m-1}$.

Moreover, we assume

(F_i) there exist a continuous function $\widetilde{G}_i : \mathbb{R} \longrightarrow \mathbb{R}$ and a constant $L_i > 0$ such that

$$|G_i(t,x)| \leq L_i \widetilde{G}_i(x_i)$$

for every $t \in [0,\pi]$ and $x \in \mathbb{R}^m$.

We refer to Remark 2.1 and to Corollaries 4.4 and 4.5 for concrete examples of functions G_i satisfying ($\mathrm{H}_i^{\pm}$)-($\mathrm{K}_i^{\pm}$)-(F_i). It is immediate to check that (F_i) is trivial when $m = 1$.

In order to state our main theorem, we introduce the following strictly positive constants (for each fixed i in $\{1,\dots,m\}$)

$$(\mathrm{B}_{\mathrm{i}}^{\pm})^2 := \min_{t\in[0,\pi]} b_i^{\pm}(t)\,, \tag{1.2}$$

$$(\mathcal{B}_i^{\pm})^2 := \max_{t\in[0,\pi]} \beta_i^{\pm}(t)\,, \tag{1.3}$$

$$(\mathrm{A}_{\mathrm{i}}^{\pm})^2 := \min_{t\in[0,\pi]} a_i^{\pm}(t)\,, \tag{1.4}$$

$$(\mathcal{A}_i^{\pm})^2 := \max_{t\in[0,\pi]} \alpha_i^{\pm}(t)\,. \tag{1.5}$$

We also set

$$\begin{aligned} \mathrm{B}_{\mathrm{i}} &:= \frac{2\mathrm{B}_{\mathrm{i}}^{+}\mathrm{B}_{\mathrm{i}}^{-}}{\mathrm{B}_{\mathrm{i}}^{+}+\mathrm{B}_{\mathrm{i}}^{-}}\,, \qquad \mathcal{B}_i := \frac{2\mathcal{B}_i^{+}\mathcal{B}_i^{-}}{\mathcal{B}_i^{+}+\mathcal{B}_i^{-}}\,, \\ \mathrm{A}_{\mathrm{i}} &:= \frac{2\mathrm{A}_{\mathrm{i}}^{+}\mathrm{A}_{\mathrm{i}}^{-}}{\mathrm{A}_{\mathrm{i}}^{+}+\mathrm{A}_{\mathrm{i}}^{-}}\,, \qquad \mathcal{A}_i := \frac{2\mathcal{A}_i^{+}\mathcal{A}_i^{-}}{\mathcal{A}_i^{+}+\mathcal{A}_i^{-}}\,. \end{aligned} \tag{1.6}$$

When the two constants in (1.2), (1.3), (1.4) or (1.5) coincide (cf. Corollary 4.4 and 4.5), we shall omit the superscripts. Note that such a notation is consistent with (1.6). Finally, we set $\tau := \{(s_1,\dots,s_m) : s_i = +1 \text{ or } s_i = -1 \text{ for every } i = 1,\dots,m\}$ and

$$\mathfrak{A}_i = \begin{cases} \mathcal{A}_i + 1 & \text{if } \exists n \in \mathbb{N}\cup\{0\}: \quad \text{either} \quad \mathcal{A}_i < 2n+1 \leq \mathcal{A}_i + 1 - \dfrac{\mathcal{A}_i}{\mathcal{A}_i^{+}} \\ & \qquad\qquad\quad \text{or} \quad \mathcal{A}_i < 2n+1 \leq \mathcal{A}_i + 1 - \dfrac{\mathcal{A}_i}{\mathcal{A}_i^{-}} \\ \mathcal{A}_i & \text{otherwise,} \end{cases} \tag{1.7}$$

$$\mathfrak{B}_i = \begin{cases} \mathcal{B}_i + 1 & \text{if } \exists n \in \mathbb{N} \cup \{0\}: \quad \text{either} \quad \mathcal{B}_i < 2n+1 \leq \mathcal{B}_i + 1 - \dfrac{\mathcal{B}_i}{\mathcal{B}_i^+} \\ & \qquad\qquad\quad \text{or} \quad \mathcal{B}_i < 2n+1 \leq \mathcal{B}_i + 1 - \dfrac{\mathcal{B}_i}{\mathcal{B}_i^-} \\ \mathcal{B}_i & \text{otherwise,} \end{cases} \tag{1.8}$$

$$\mathtt{A}_i = \begin{cases} \mathrm{A}_i - 1 & \text{if } \exists m \in \mathbb{N}: \quad \text{either} \quad \mathrm{A}_i + 1 - \dfrac{\mathrm{A}_i}{\mathrm{A}_i^+} \leq 2m+1 < \mathrm{A}_i \\ & \qquad\qquad\quad \text{or} \quad \mathrm{A}_i + 1 - \dfrac{\mathrm{A}_i}{\mathrm{A}_i^-} \leq 2m+1 < \mathrm{A}_i \\ \mathrm{A}_i & \text{otherwise,} \end{cases} \tag{1.9}$$

$$\mathtt{B}_i = \begin{cases} \mathrm{B}_i - 1 & \text{if } \exists m \in \mathbb{N}: \quad \text{either} \quad \mathrm{B}_i + 1 - \dfrac{\mathrm{B}_i}{\mathrm{B}_i^+} \leq 2m+1 < \mathrm{B}_i \\ & \qquad\qquad\quad \text{or} \quad \mathrm{B}_i + 1 - \dfrac{\mathrm{B}_i}{\mathrm{B}_i^-} \leq 2m+1 < \mathrm{B}_i \\ \mathrm{B}_i & \text{otherwise.} \end{cases} \tag{1.10}$$

Note that if $\mathcal{A}_i^+ = \mathcal{A}_i^-$, then $\mathcal{A}_i = \mathfrak{A}_i$. Similiar remarks are valid for (1.8), (1.9), (1.10).

The above defined constants describe the asymptotic behaviour of G at zero and the one at infinity; a comparison between the former and the latter plays a central role in our multiplicity result. Roughly speaking, we assume that the "gap" between $\mathfrak{A}_i$ and $\mathtt{B}_i$ (or $\mathfrak{B}_i$ and $\mathtt{A}_i$, respectively) is sufficiently large; then, the "bigger" this gap, the greater the number of solutions (with prescribed nodal properties) of (1.1).

We are now in position to state our main result.

Important Remark. In order to simplify the exposition, in what follows we will work, without loss of generality, with $m = 2$.

Theorem. *Assume that G satisfies hypotheses* $(\mathrm{H}_i^{\pm})$-$(\mathrm{K}_i^{\pm})$- (F_i) *and that there exists $m_i \in \mathbb{N}$ such that*

$$\textit{either} \qquad \mathfrak{A}_i < m_i < \mathtt{B}_i \qquad \textit{or} \qquad \mathfrak{B}_i < m_i < \mathtt{A}_i$$

holds for every $i \in \{1,2\}$. Then, there exist h_1, h_2, $\mathcal{H}_1$, $\mathcal{H}_2 \in \mathbb{N}$ such that for every $(n_1, n_2) \in \mathbb{N}^2$ with $h_i \leq n_i \leq \mathcal{H}_i$ $(i = 1,2)$, problem (1.1) has at least four solutions $u^j = (u_1^j, u_2^j)$ $(1 \leq j \leq 4)$ with u_i^j having exactly n_i zeros in $[0,\pi)$ for every $j = 1,2,3,4$. Moreover, for every $(s_1, s_2) \in \tau$ there exists $j \in \{1,...,4\}$ such that $\operatorname{sgn}\Big((u_i^j)'(0)\Big) = s_i$ for $i \in \{1,2\}$.

As a consequence of this main Theorem, we give in Corollary 4.5 a multiplicity result assuming, besides (F_i),

$$\lim_{|x_i|\to+\infty} \frac{G_i(t,x)}{x_i} = C_i^2, \quad \lim_{x_i\to 0} \frac{G_i(t,x)}{x_i} = D_i^2, \tag{1.11}$$

uniformly in $t \in [0,\pi]$ and $x_j \in \mathbb{R}$ $(\forall\, j \neq i)$, where the "gap" between C_i and D_i is sufficiently large. It is then easy to see that the "component-wise" assumptions of the form (1.11) on G in the particular case $C_1 = C_2$ and $D_1 = D_2$ (combined with hypothesis (F_i)) guarantee that the limits $\lim_{x\to 0} \frac{|G(t,x)|}{|x|}$, $\lim_{|x|\to+\infty} \frac{|G(t,x)|}{|x|}$ exist and are finite. Such assumptions of global type would lead to a priori bounds for the possible solutions of (1.1); however, this information would not be sufficient for obtaining estimates on the number of zeros of each component of a (possible) solution. Note that if we consider the particular case of a scalar autonomous equation, then a time-map argument (on the lines of [4]) shows that if $C_i = D_i$ then (1.1) might have only the trivial solution.

To prove our main result, we proceed by using a topological degree approach. The crucial technical result (the so-called "elastic property") is Lemma 2.2 (we refer to Remark 2.3 for more comments on the elastic property); then, we combine some estimates on the number of zeros of each component of a solution to (1.1) with a continuation theorem which carries, by means of a suitable homotopy, the initial problem (1.1) into an uncoupled autonomous one. In this way, the validity of the non-zero degree condition in the continuation theorem is reduced to the computation of the degree for a suitable autonomous problem.

We refer to Remarks 4.7, 4.8, 4.9 for some comparisons between our results and some of the above quoted contributions on asymptotically linear problems associated to *scalar* ODEs. We also point out that the multiplicity results obtained in the framework of PDEs seem (when we particularize to the ODE case) to be less precise. We finally remark that a multiplicity result analogous to our main Theorem can be attained by arguments similar to the ones used in this paper, for a more general system of the form $u_i'' + G_i(t,u(t)) = q_i(t,u(t),u'(t))$, $i \in \{1,\dots,m\}$, where the growth of q_i is subordinate to the constants $\mathtt{B}_i, \mathtt{A}_i$. The details of the proof in this more general case can be found in [9].

In Section 2 we introduce a suitable homotopy; we will study the corresponding autonomous problem in the Appendix. The central part of Section 2 is devoted to the statement and proof of "elastic property" and of another basic Lemma. In Section 3, taking advantage of the Lemmas of the previous section, we obtain the estimates on the number of zeros needed for our main result which is finally proved in Section 4. We end this introductory section with a list of notations.

We denote by $X = C_0^1\left([0,\pi],\mathbb{R}^2\right)$ the space of the functions $u \in C^1\left([0,\pi],\mathbb{R}^2\right)$ such that $u(0) = 0 = u(\pi)$; by $\|u\|_1$ we mean the corrisponding C^1-norm. Moreover, by ∂Y and int Y we denote the boundary and the interior of a set $Y \subset \mathbb{R}^2$. By deg we mean the Leray-Schauder degree. For any real number a and b, we denote by $\overline{ab}$ the open segment whose endpoints (not included) are a and b, i.e. $\overline{ab} = \{x \in$

$\mathbb{R} \mid \exists \lambda \in (0,1) : x = \lambda a + (1-\lambda)b\}$. By $|\mathrm{I}|$ we mean the measure of an interval $\mathrm{I} \subset \mathbb{R}$. For every $x \in \mathbb{R}^2$, we define $\tilde{x}_1 := (x_1, 0)$ and $\tilde{x}_2 = (0, x_2)$.

2 Some preliminary results

In this section, we prove two useful estimates on the norm of each component u_i of a (possible) solution to a parameter-dependent problem associated to (1.1). In the former, for each fixed i, an upper bound on the C^1-norm of u_i is obtained whenever $\min_{t\in[0,\pi]}(|u_i(t)| + |u_i'(t)|)$ is smaller than a given constant; in the latter, a lower bound on $\min_{t\in[0,\pi]}(|u_i(t)| + |u_i'(t)|)$ is obtained whenever the C^1-norm of u_i is greater than a given constant. Let us recall the initial problem

$$\begin{cases} u''(t) \ + \ G(t, u(t)) = 0 \\ u(0) = 0 = u(\pi), \end{cases} \tag{1.1}$$

where $G : [0,\pi] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a continuous function. Let us define

$$\delta := \min\{\delta_1, \delta_2, \sigma_1, \sigma_2\} \quad \text{and} \quad N := \max\{N_1, N_2, M_1, M_2, \delta + 1\}, \tag{2.1}$$

where the constants in (2.1) are defined in $(\mathrm{H}_i^{\pm})$-$(\mathrm{K}_i^{\pm})$. We observe that $(\mathrm{H}_i^{\pm})$ and $(\mathrm{K}_i^{\pm})$ are still valid if we replace the constants N_i, M_i and δ_i, σ_i with N and δ, respectively.

Remark 2.1 It is now important to give some explicit examples of functions G_i satisfying the assumptions of our main result. To this end, we first remark that if G_i satisfies $(\mathrm{H}_i^{\pm})$-$(\mathrm{K}_i^{\pm})$-(F_i) then there exist $f_i : \mathbb{R} \to \mathbb{R}$, $h_i : [0,\pi] \times \mathbb{R}^2 \to \mathbb{R}$ such that $G_i(t, x_1, x_2) = f_i(x_i) - h_i(t, x_1, x_2)$, where the asymptotic behaviour of the ratio $f_i(x_i)/x_i$ is described by inequalities analogous to $(\mathrm{H}_i^{\pm})$-$(\mathrm{K}_i^{\pm})$ and there exist a strictly positive constant l_i such that $|h_i(t, x_1, x_2)| \le l_i |x_i|$ for every $x = (x_1, x_2) \in \mathbb{R}^2$, $t \in [0,\pi]$. Note that the above conditions guarantee that any system of the form $u_i'' + f_i(u_i) = h_i(t, u_1, u_2)$ (with h_i satisfying a sign condition) is of the form (1.1). More concretely, one can think of $f_i(\xi) = c_1 f_i^*(\xi) + c_2 f_i^{**}(\xi)$, where $c_1, c_2 \in \mathbb{R}^+$ and $\lim_{\xi\to 0} \frac{f_i^*(\xi)}{\xi} = 1 = \lim_{|\xi|\to+\infty} \frac{f_i^{**}(\xi)}{\xi}$, $\lim_{|\xi|\to+\infty} \frac{f_i^*(\xi)}{\xi} = 0 = \lim_{\xi\to 0} \frac{f_i^{**}(\xi)}{\xi}$ (cf. Corollary 4.5).

An example in which the dependence of G_i on x_i is not such explicitely stressed is given by

$$G_i(t, x_1, x_2) = \frac{c_1 x_i}{(1 + k^*(t, x_1, x_2))^{x_i^2}} + \frac{c_2 x_i^3 k^{**}(t, x_1, x_2)}{1 + x_i^2 k^{**}(t, x_1, x_2)},$$

where $c_1, c_2 \in \mathbb{R}^+$ and k^*, k^{**} are bounded above and below by strictly positive constants. Finally, in the particular case when $c_1 < c_2$ another example is given by

$$G_i(t, x_1, x_2) = \frac{2 x_i c_2}{\pi} \left\{ \arctan\left(k(t, x_1, x_2) x_i + \tan\left(\frac{\pi c_1}{2 c_2}\right)\right)\right\},$$

being k bounded above and below by strictly positive constants.

In order to apply the continuation theorem we will state in Section 4, we make a homotopy between our initial problem (1.1) and an auxiliary autonomous uncoupled problem of the form

$$\begin{cases} u''(t) + G^0(u(t)) = 0 \\ u(0) = 0 = u(\pi), \end{cases} \tag{2.2}$$

where $G_i^0 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function such that $G_i^0(x) = G_i^0(\tilde{x}_i)$ for every $i \in \{1,2\}$ and for every $x \in \mathbb{R}^2$. Moreover, if we set $g_i^0(x_i) := G_i^0(\tilde{x}_i)$ for every $x_i \in \mathbb{R}$, we also suppose that g_i^0 is a continuous, locally Lipschitz function with the following properties:

$$g_i^0(x)\, x > 0 \quad x \neq 0, \tag{2.3}$$

$$\lim_{x \to +\infty} \frac{g_i^0(x)}{x} = b_{i,+}^2, \qquad \lim_{x \to -\infty} \frac{g_i^0(x)}{x} = b_{i,-}^2, \tag{2.4}$$

$$\lim_{x \to 0^+} \frac{g_i^0(x)}{x} = a_{i,+}^2, \qquad \lim_{x \to 0^-} \frac{g_i^0(x)}{x} = a_{i,-}^2, \tag{2.5}$$

$$x > N \Longrightarrow (\mathrm{B}_i^+)^2 \leq \frac{g_i^0(x)}{x} \leq (\mathcal{B}_i^+)^2, \;\; 0 < x < \delta \Longrightarrow (\mathrm{A}_i^+)^2 \leq \frac{g_i^0(x)}{x} \leq (\mathcal{A}_i^+)^2$$

$$x < -N \Longrightarrow (\mathrm{B}_i^-)^2 \leq \frac{g_i^0(x)}{x} \leq (\mathcal{B}_i^-)^2, \;\; 0 > x > -\delta \Longrightarrow (\mathrm{A}_i^-)^2 \leq \frac{g_i^0(x)}{x} \leq (\mathcal{A}_i^-)^2,$$

where $\mathrm{B}_i^\pm \leq b_{i,\pm} \leq \mathcal{B}_i^\pm$ and $\mathrm{A}_i^\pm \leq a_{i,\pm} \leq \mathcal{A}_i^\pm$ for every $i \in \{1,2\}$.

Now we are able to embed our problem (1.1) into a family of Dirichlet problems of the form

$$\begin{cases} u''(t) + g(t, u(t); \lambda) = 0 \\ u(0) = 0 = u(\pi), \end{cases} \tag{2.6}$$

where $g(t,x;\lambda) := \lambda\, G(t,x) + (1-\lambda)\, G^0(x)$ for every $\lambda \in [0,1], x \in \mathbb{R}^2$ and $t \in [0,\pi]$. From the properties of G^0 and G, we obtain that g satisfies conditions similar to $(\mathrm{H}_i^\pm)$-$(\mathrm{K}_i^\pm)$-(F_i), with $i \in \{1,2\}$. Indeed, for every $i \in \{1,2\}$ the following relations hold:

$(\mathrm{H}_i^+)_\lambda$ for every $t \in [0,\pi]$, $\lambda \in [0,1]$ and $x_j \in \mathbb{R}$ with $j \in \{1,2\}$, $j \neq i$

$$x_i > N \quad \Longrightarrow \quad (\mathrm{B}_i^+)^2 \leq \frac{g_i(t,x;\lambda)}{x_i} \leq (\mathcal{B}_i^+)^2;$$

$(\mathrm{H}_i^-)_\lambda$ for every $t \in [0,\pi]$, $\lambda \in [0,1]$ and in $x_j \in \mathbb{R}$ with $j \in \{1,2\}$, $j \neq i$

$$x_i < -N \quad \Longrightarrow \quad (\mathrm{B}_i^-)^2 \leq \frac{g_i(t,x;\lambda)}{x_i} \leq (\mathcal{B}_i^-)^2;$$

$(\mathrm{K}_i^+)_\lambda$ for every $t \in [0,\pi]$, $\lambda \in [0,1]$ and $x_j \in \mathbb{R}$ with $j \in \{1,2\}$, $j \neq i$

$$0 < x_i < \delta \quad \Longrightarrow \quad (\mathrm{A}_i^+)^2 \leq \frac{g_i(t,x;\lambda)}{x_i} \leq (\mathcal{A}_i^+)^2;$$

$(\mathrm{K}_i^-)_\lambda$ for every $t \in [0,\pi]$, $\lambda \in [0,1]$ and in $x_j \in \mathbb{R}$ with $j \in \{1,2\}$, $j \neq i$

$$-\delta < x_i < 0 \quad \Longrightarrow \quad (\mathrm{A}_i^-)^2 \leq \frac{g_i(t,x;\lambda)}{x_i} \leq (\mathcal{A}_i^-)^2;$$

$(\mathrm{F}_i)_\lambda$ for every $t \in [0,\pi]$ and $x \in \mathbb{R}^m$

$$|g_i(t,x;\lambda)| \leq \widetilde{g}_i(x_i),$$

where $\widetilde{g}_i : \mathbb{R} \longrightarrow \mathbb{R}$ is (continuous, non-negative) defined by $\widetilde{g}_i(y) := L_i \widetilde{G}_i(y) + |g_i^0(y)|$ for every $y \in \mathbb{R}$.

Assume now that the function g satisfies $(\mathrm{H}_\mathrm{i}^\pm)_\lambda$-$(\mathrm{K}_\mathrm{i}^\pm)_\lambda$-$(\mathrm{F}_\mathrm{i})_\lambda$ for every $\lambda \in [0,1]$. In order to simplify the notation, we set $r_i(t) := \sqrt{u_i(t)^2 + u_i'(t)^2}$.

Now we present an important result, the so-called *"Elastic Lemma"*:

Lemma 2.2 *Fix $i \in \{1,2\}$. Then for every $R_1 > 0$ there exists $R_2 \geq R_1^2$ such that for every solution u of (2.6) with*

$$\min_{t\in[0,\pi]} |(\, u_i(t), u_i'(t)\,)| \leq R_1$$

it follows that

$$u_i(t)^2 + u_i'(t)^2 \leq R_2 \qquad \forall t \in [0,\pi]. \tag{2.7}$$

Proof First of all, we introduce the following subsets of $\mathbb{R}^2$:

$$\mathrm{Q} = \{(x,y) \in \mathbb{R}^2 \ : \ |x| \leq R_1 \, , \ |y| \leq R_1\}$$

$$\mathrm{R} = \{(x,y) \in \mathbb{R}^2 \ : \ \text{either } |x| \geq R_1 \ \text{ or } \ |y| \geq R_1\}.$$

Fix $i \in \{1,2\}$ and let u be a solution of (2.6).

- If $(u_i(t), u_i'(t)) \in \mathrm{Q}$ for every $t \in [0,\pi]$, then $r_i^2(t) \leq 2R_1^2$ for all $t \in [0,\pi]$. Whence, choosing $R_2 = 2R_1^2$, we obtain (2.7).

- Assume that there exists $\hat{t} \in [0,\pi]$ such that $(u_i(\hat{t}), u_i'(\hat{t})) \notin \mathrm{Q}$, i.e. $(u_i(\hat{t}), u_i'(\hat{t})) \in \mathrm{int\ R}$. In case $(u_i, u_i')([0,\pi]) \cap \mathrm{int\ Q} \neq \emptyset$, we are able to define

$$t_0 := \inf\{\, t \in [0,\pi] \ : \ (u_i(t), u_i'(t)) \in \mathrm{int\ Q}\}. \tag{2.8}$$

Otherwise, if $(u_i(t), u_i'(t)) \in \mathrm{R}$ for every $t \in [0,\pi]$, we will denote by t_0 an element of $[0,\pi]$ such that $(u_i(t_0), u_i'(t_0)) \in \partial\mathrm{Q}$. We observe that the existence of such t_0 is a consequence of the fact that if $\min_{t\in[0,\pi]} |(\, u_i(t), u_i'(t)\,)| \leq R_1$, then $\mathrm{Q} \cap$

$(u_i, u_i')([0,\pi]) \neq \emptyset$. In order to examine the behaviour of r_i^2 in $[t_0, \pi]$, we need the following:

Claim.

$$[t_0, \pi] = \bigcup_{k \in \mathbb{N}} I_k^1 \cup \bigcup_{h \in \mathbb{N}} I_h^2,$$

where, for every k, $h \in \mathbb{N}$, I_k^1 and I_h^2 are closed intervals with the properties

$$\begin{cases} (u_i, u_i')(I_k^1) \subset \mathrm{Q}, \\ (u_i, u_i')(I_h^2) \subset \mathrm{R}, \\ (u_i(t), u_i'(t)) \in \partial\mathrm{Q} = \partial\mathrm{R}, \end{cases}$$

whenever t is one of the left endpoints of the intervals of the form I_h^2.

Proof of the claim. If $(u_i(t), u_i'(t)) \in \mathrm{R}$ for every $t \in [0, \pi]$, we can choose $I_1^2 = [t_0, \pi]$, while $I_1^1 = \emptyset$ and $I_l^j = \emptyset$ for every $j = 1, 2$, for every $l \neq 1$. In this way, $(u_i, u_i')(I_1^2) \subset \mathrm{R}$ and $(u_i(t_0), u_i'(t_0)) \in \partial\mathrm{Q}$ by definition of t_0. On the other hand, if $(u_i, u_i')([0,\pi]) \cap \operatorname{int} \mathrm{Q} \neq \emptyset$, from the existence of $\hat{t} \in [0, \pi]$ such that $(u_i(\hat{t}), u_i'(\hat{t})) \in \operatorname{int} \mathrm{R}$, we can deduce that

$$\begin{aligned} \exists\, t_1, t_2 \in [0,\pi] : \quad & t_0 < t_1 < t_2 \quad , \quad (u_i(t_1), u_i'(t_1)) \in \partial\,\mathrm{Q} = \partial\,\mathrm{R}, \\ & (u_i(t), u_i'(t)) \in \mathrm{Q} \quad \forall t \in [t_0, t_1], \\ & (u_i(t), u_i'(t)) \in \mathrm{R} \quad \forall t \in [t_1, t_2], \\ & \operatorname{int} \mathrm{R} \cap (u_i, u_i')(]t_1, t_2[) \neq \emptyset. \end{aligned} \tag{2.9}$$

We immediately note that we can choose $I_1^1 = [t_0, t_1]$ and $I_1^2 = [t_1, t_2]$. Moreover, it is not restrictive to suppose that t_2 is the maximum among the numbers t in the interval $[0, \pi]$ satisfying conditions (2.9). If $t_2 = \pi$, the claim is proved if we set $I_l^j = \emptyset$ for every $j = \{1, 2\}$ and $l \neq 1$, $l \in \mathbb{N}$. By the choice of t_2, if $t_2 \neq \pi$ then $(u_i(t_2), u_i'(t_2)) \in \partial\mathrm{Q} = \partial\mathrm{R}$; moreover, there exists $t_3 \in (t_2, \pi]$ such that $(u_i(t), u_i'(t)) \in \mathrm{Q}$ for every $t \in [t_2, t_3]$, with $(u_i(\tilde{t}), u_i'(\tilde{t})) \in \operatorname{int} \mathrm{Q}$ for some $\tilde{t} \in (t_2, t_3)$. Hence, we can define $I_2^1 = [t_2, t_3]$. Furthermore, we can suppose that t_3 is the maximum among the numbers $t \in [0, \pi]$ satisfying the above conditions. If $t_3 = \pi$, the claim is proved if we set $I_2^2 = \emptyset$ and $I_l^j = \emptyset$ for every $j \in \{1, 2\}$ and for every $l \neq 1, 2$. Otherwise, if $t_3 \neq \pi$, it follows that $(u_i(t_3), u_i'(t_3)) \in \partial\mathrm{Q}$ and, arguing as above, we can show that

$$[t_0, \pi] = \bigcup_{k \in \mathbb{N} \cup \{0\}} I_k^1 \cup \bigcup_{h \in \mathbb{N} \cup \{0\}} I_h^2 = \bigcup_{l \in \mathbb{N} \cup \{0\}} [t_l, t_{l+1}],$$

where the intervals that are non-empty are of the form $I_k^1 = [t_{2k}, t_{2k+1}]$ and $I_h^2 = [t_{2h+1}, t_{2h+2}]$, with $k \in \mathbb{N} \cup \{0\}$, and $t_l < t_{l+1} \leq \pi$ for every $l \in \mathbb{N} \cup \{0\}$, and where $(u_i(t_l), u_i'(t_l)) \in \partial\mathrm{Q} = \partial\mathrm{R}$ when $t_l \notin \{0, \pi\}$. Moreover, the intervals I_k^1 and I_h^2 satisfy the following properties:

$$(u_i(t), u_i'(t)) \in \mathrm{Q} \quad \forall t \in I_k^1 \quad , \quad \exists\, \bar{t} \in \operatorname{int} I_k^1 \ : \ (u_i(\bar{t}), u_i'(\bar{t})) \in \operatorname{int} \mathrm{Q} \tag{2.10}$$

$$(u_i(t), u_i'(t)) \in \mathrm{R} \quad \forall t \in I_h^2 \quad , \quad \exists \bar{\bar{t}} \in \text{ int } I_h^2 \ : \ (u_i(\bar{\bar{t}}), u_i'(\bar{\bar{t}})) \in \text{ int } \mathrm{R}\,. \tag{2.11}$$

The proof of the claim is complete. ∎

First, we study the intervals I_k^1, with $k \in \mathbb{N}$. According to (2.10), $(u_i(t), u_i'(t)) \in \mathrm{Q}$ for every $t \in I_k^1$; thus, arguing as before, we obtain

$$r_i^2(t) \le 2R_1^2 \qquad \forall t \in I_k^1\,. \tag{2.12}$$

In order to analyze the intervals I_h^2, let us consider now the equality

$$\frac{\mathrm{d}r_i^2}{\mathrm{d}t}(t) = 2u_i(t)u_i'(t) + 2u_i'(t)u_i''(t) = 2u_i(t)u_i'(t) - 2u_i'(t)g_i(t, u(t); \lambda) \tag{2.13}$$

($t \in [0, \pi]$), which is valid whenever u is a solution of (2.6). It is immediate to check that (2.13) implies

$$\frac{\mathrm{d}r_i^2}{\mathrm{d}t}(t) \le 2|u_i(t)||u_i'(t)| + 2|u_i'(t)||g_i(t, u(t); \lambda)| \le r_i^2(t) + 2|u_i'(t)||g_i(t, u(t); \lambda)|.$$

To simplify the notation, for each fixed $h \in \mathbb{N}$ we introduce the following disjoint subsets of I_h^2

$$S_{1h} = \{t \in I_h^2 \ : \ |u_i(t)| > N\}\,, \qquad S_{2h} = \{t \in I_h^2 \ : \ |u_i(t)| < \delta\}\,,$$

$$S_{3h} = \{t \in I_h^2 \ : \ \delta \le |u_i(t)| \le N\}$$

(cf. (2.1) for the definitions of δ and N). For brevity, in what follows we shall omit, for the above defined sets, the subscript h. Clearly, $I_h^2 = S_1 \cup S_2 \cup S_3$. Note that these sets might be empty.
Now we study separately the sets S_l ($l \in \{1, 2, 3\}$). By the definition of S_1, it follows that $|u_i(t)| > N$, hence from hypotheses $(\mathrm{H}_\mathrm{i}^\pm)_\lambda$ we have $|g_i(t, u(t); \lambda)| \le \widetilde{\mathcal{B}}_i\,|u_i(t)|$ for every $t \in S_1$, where $\widetilde{\mathcal{B}}_i := \max\limits_{i=1,2}\{(\mathcal{B}_i^+)^2, (\mathcal{B}_i^-)^2\}$. Hence,

$$\frac{\mathrm{d}r_i^2}{\mathrm{d}t}(t) \le r_i^2(t) + 2\widetilde{\mathcal{B}}_i\,|u_i(t)||u_i'(t)| \le \left(1 + \widetilde{\mathcal{B}}_i\right) r_i^2(t)\,, \quad \forall t \in S_1\,. \tag{2.14}$$

Similarly, from hypotheses $(\mathrm{K}_\mathrm{i}^\pm)_\lambda$ and by the definition of the set S_2, it follows that $|g_i(t, u(t); \lambda)| \le \widetilde{\mathcal{A}}_i\,|u_i(t)|$ for every $t \in S_2$ (being $\widetilde{\mathcal{A}}_i := \max\limits_{i=1,2}\{(\mathcal{A}_i^+)^2, (\mathcal{A}_i^-)^2\}$). Hence,

$$\frac{\mathrm{d}r_i^2}{\mathrm{d}t}(t) \le r_i^2(t) + 2\widetilde{\mathcal{A}}_i\,|u_i(t)||u_i'(t)| \le \left(1 + \widetilde{\mathcal{A}}_i\right) r_i^2(t)\,, \quad \forall t \in S_2\,. \tag{2.15}$$

As for the behaviour of $\frac{\mathrm{d}r_i^2}{\mathrm{d}t}$ in S_3, we observe that there exists a constant $\widetilde{C}_i > 1$ such that

$$|u_i'(t)| \le \widetilde{C}_i\, r_i^2(t) \qquad \forall t \in S_3. \tag{2.16}$$

Indeed, let us take $t \in S_3$. In particular $|u_i(t)| \geq \delta$. Thus,

$$|u_i'(t)| \leq u_i'(t)^2 + \frac{1}{4} = u_i'(t)^2 + \frac{\delta^2}{4\delta^2} \leq u_i'(t)^2 + \frac{u_i(t)^2}{4\delta^2} \leq \widetilde{C}_i\, r_i^2(t),$$

where $\widetilde{C}_i = \widetilde{C}_i(\delta) := \max\left\{1, \frac{1}{4\delta^2}\right\}$. Setting $C_i^* = C_i^*(N,\delta) := \max_{\delta \leq |\xi| \leq N} \widetilde{g}_i(\xi)$, we obtain from hypothesis $(\mathrm{F}_i)_\lambda$

$$|g_i(t, u(t); \lambda)| \leq \widetilde{g}_i(u_i(t)) \leq C_i^* \qquad \forall\, t \in S_3\,, \tag{2.17}$$

which, combined with (2.16), implies

$$\frac{\mathrm{d}r_i^2}{\mathrm{d}t}(t) \leq r_i^2(t) + 2\,C_i^*\,|u_i'(t)| \leq (1 + 2\,C_i^*\,\widetilde{C}_i)\, r_i^2(t) \qquad \forall\, t \in S_3\,. \tag{2.18}$$

Setting $T := \max\left\{\widetilde{\mathcal{B}}_i,\, \widetilde{\mathcal{A}}_i,\, 2\,C_i^*\,\widetilde{C}_i\right\}$, we are able to define the searched constant $R_2 := 2\,R_1^2\,\mathrm{e}^{(T+1)\pi}$. It is immediate to check that $R_2 \geq R_1^2$; moreover, recalling that I_h^2 is the disjoint union of S_1, S_2 and S_3, from (2.14), (2.15) and (2.18) we obtain

$$\frac{\mathrm{d}r_i^2}{\mathrm{d}t}(t) \leq (T+1)\, r_i^2(t) \tag{2.19}$$

for every $t \in I_h^2$. Being I_h^2 an interval of the form $[a_h, b_h]$ $(\subset [t_0, \pi])$, we easily deduce that

$$r_i^2(t) \leq \mathrm{e}^{(1+T)(t-a_h)}\, r_i^2(a_h) \leq \mathrm{e}^{(1+T)\pi}\, r_i^2(a_h)\,, \tag{2.20}$$

for every $t \in [a_h, b_h]$. Since $(u_i(a_h), u_i'(a_h)) \in \partial\,\mathrm{Q}$, it follows that $r_i^2(a_h) \leq 2R_1^2$. Hence, we can finally conclude that $r_i^2(t) \leq R_2$ for every $t \in I_h^2 = [a_h, b_h]$ and for every $h \in \mathbb{N}$. By the definition of R_2, from (2.12) it follows that $r_i^2(t) \leq 2R_1^2\,\mathrm{e}^{(T+1)\pi} = R_2$ for every $t \in I_k^1$ $(k \in \mathbb{N})$ as well. Hence, recalling that I_k^1 and I_h^2 cover the interval $[t_0, \pi]$, we can conclude that $r_i^2(t) \leq R_2$ for every $t \in [t_0, \pi]$.

For the completion of the proof, it remains to estimate the function r_i^2 in the interval $[0, t_0]$. To this end, we first note that (according to the definition of t_0) when $t_0 > 0$ then $(u_i(t_0), u_i'(t_0)) \in \partial\,\mathrm{Q}$ and $(u_i(t), u_i'(t)) \in \mathrm{R}$ for every $t \in [0, t_0]$. Note that, from (2.13),

$$\frac{\mathrm{d}r_i^2}{\mathrm{d}t}(t) \geq -r_i^2(t) - 2|u_i'(t)||g_i(t, u(t); \lambda)|,$$

for every $t \in [0, \pi]$. Let us now cover the interval $[0, t_0]$ with the following three subsets (analogue to S_1, S_2, S_3)

$$\widetilde{S}_1 = \{t \in [0, t_0] \,:\, |u_i(t)| > N\}\,, \qquad \widetilde{S}_2 = \{t \in [0, t_0] \,:\, |u_i(t)| < \delta\}\,,$$

$$\widetilde{S}_3 = \{t \in [0, t_0] \,:\, \delta \leq |u_i(t)| \leq N\}\,;$$

if we study separately the function $\dfrac{\mathrm{d}r_i^2}{\mathrm{d}t}$ in each of these sets, from $(\mathrm{H}_\mathrm{i}^{\pm})_\lambda$, $(\mathrm{K}_\mathrm{i}^{\pm})_\lambda$, $(\mathrm{F}_\mathrm{i})_\lambda$ we obtain

$$\frac{\mathrm{d}r_i^2}{\mathrm{d}t}(t) \geq -(T+1)\, r_i^2(t) \tag{2.21}$$

for every $t \in [0, t_0]$. Thus, we can easily deduce that for every $t \in [0, t_0]$

$$r_i^2(t) \leq \mathrm{e}^{(1+T)(t_0-t)}\, r_i^2(t_0) \leq 2R_1^2\, \mathrm{e}^{(1+T)\pi} = R_2\,, \tag{2.22}$$

since $(u_i(t_0), u_i'(t_0)) \in \partial\mathrm{Q}$ implies $r_i^2(t_0) \leq 2R_1^2$. We have thus proved that $r_i^2(t) \leq R_2$ for every $t \in [0, \pi]$. This completes the proof.

Remark 2.3 Lemma 2.2 can also be proved with a method similar to the one used in [21]. More precisely, taking into account Remark 2.1 for the definition of f_i and h_i, we can introduce a "guiding function" of the form $V(x,y) = \displaystyle\sum_{i=1}^{2} \left(F_i(x_i) + \frac{1}{2} y_i^2 \right)$, where $F_i(x) = \displaystyle\int_0^x f_i(s)\, ds$. Since it is possible to find constants $K \geq 0$ and $d > 0$ such that

$$|y_i\, h_i(t, x_1, x_2)| \leq d\, V(x,y) \qquad \forall t \in [0,\pi] \qquad \forall (x,y) \in \mathbb{R}^4 : |(x,y)| > K\,,$$

then the thesis of our Lemma 2.2 follows from Lemma 1 in [21]. Nevertheless, in this paper we have given an independent proof. We point out that in our proof of Lemma 2.2 the crucial inequalities are only the ones in $(\mathrm{H}_i^{\pm})_\lambda$ that contain $\mathcal{B}_i^{\pm}$ and $(\mathrm{F}_i)_\lambda$ (cf. Theorem 4.6). However, if we wanted to apply the method used in [21], it would be necessary to prove that $\lim\limits_{|z|\to+\infty} V(z) = +\infty$; for the verification of this condition, the inequalities in $(\mathrm{H}_i^{\pm})_\lambda$ that involve $\mathrm{B}_i^{\pm}$ are to be used as well.

Our second lemma describes, with a statement somehow symmetric to the one of Lemma 2.2, the behaviour of a (possible) solution to (2.6) in a neighbourhood of the origin.

Lemma 2.4 *Fix $i \in \{1,2\}$. Then for every $\rho_1 > 0$ there exists $\rho_2 \in]0, \rho_1^2[$ such that for every solution u of (2.6) with*

$$\max_{t\in[0,\pi]} |(u_i(t), u_i'(t))| > \rho_1 \tag{2.23}$$

it follows that

$$u_i(t)^2 + u_i'(t)^2 > \rho_2 \qquad \forall t \in [0,\pi]\,. \tag{2.24}$$

Proof Let us set $\rho_2 := \dfrac{\rho_1^2}{2}\mathrm{e}^{-(T+1)\pi} \in]0, \rho_1^2[$, where $T = \max\left\{ \widetilde{\mathcal{B}}_i,\, \widetilde{\mathcal{A}}_i,\, 2\, C_i^*\, \widetilde{C}_i \right\}$ as in Lemma 2.2. Let us suppose by contradiction that there exists $\bar{t} \in [0,\pi]$ such that $u_i(\bar{t})^2 + u_i'(\bar{t})^2 \leq \rho_2$; in particular, we have that $\min\limits_{t\in[0,\pi]} r_i(t) \leq \sqrt{\rho_2}$. Then, we

can apply Lemma 2.2 with $R_1 = \sqrt{\rho_2}$. Recall that in Lemma 2.2 the constant R_2 is obtained by means of the explicit expression $R_2 = 2\,R_1^2 e^{(T+1)\pi}$; hence, we can conclude that for every $t \in [0,\pi]$

$$u_i(t)^2 + u_i'(t)^2 \leq 2\,\rho_2\, e^{(T+1)\pi} = 2\,\frac{\rho_1^2}{2} e^{-(T+1)\pi}\, e^{(T+1)\pi} = \rho_1^2;$$

hence, $\max_{t\in[0,\pi]} |(\,u_i(t), u_i'(t)\,)| \leq \rho_1$, a contradiction.

Remark 2.5 We point out that, by means of an independent proof, it is possible to guarantee the validity of Lemma 2.4 only under some assumptions on the behaviour of G in a neighbourhood of the origin; however, due to the fact that for the proof of the main result all the assumptions of Lemma 2.2 are needed, for the sake of brevity we did not develop this alternative proof.

3 Estimates

In this section we develop the estimates on the number of zeros of the components of the possible solutions of the problem

$$\begin{cases} u''(t) \,+\, g\,(t, u(t); \lambda) \,=\, 0 \\ u(0) \,=\, 0 \,=\, u(\pi)\,, \end{cases} \tag{3.1}$$

which will be crucial for the application of the abstract continuation theorem stated in Section 4. To this end, we first remark that the map

$$\begin{array}{rccl} \mathbf{n}: & \mathcal{M} & \longrightarrow & \mathbb{N} \\ & \xi & \longmapsto & \mathbf{n}(\xi) := \text{the number of (simple) zeros of } \xi \text{ in } [0,\pi) \end{array} \tag{3.2}$$

is well defined, where $\mathcal{M} = \{\xi \in C_0^1\,([0,\pi],\mathbb{R}) : \xi(t)^2 + \xi'(t)^2 > 0 \quad \forall t \in [0,\pi]\}$. In what follows we assume, as in the previous sections, that g satisfies $(\mathrm{H}_\mathrm{i}^{\pm})_\lambda$, $(\mathrm{K}_\mathrm{i}^{\pm})_\lambda$, $(\mathrm{F}_\mathrm{i})_\lambda$, with $\lambda \in [0,1]$. We shall first give bounds from below and above, respectively, on the number of zeros of the components of a (possible) solution u of (3.1) for whom $|u_i'(0)|$ is "sufficiently large". Subsequently, bounds for solutions with $|u_i'(0)|$ "sufficiently small" will be developed.

Recalling the definition of $\mathrm{B_i}$ in (1.6), we can state the following

Proposition 3.1 *For every fixed $i \in \{1,2\}$, there exists $d_i > 0$ such that for every $m \in \mathbb{N}$ with $m < \mathrm{B}_i$ and for every solution $u = (u_1, u_2)$ of* (3.1)

$$|u_i'(0)| \geq d_i \qquad \Longrightarrow \qquad \mathbf{n}(u_i) > m\,.$$

Proof Let $u = (u_1, u_2)$ be a solution of (3.1). First, we use Lemma 2.4 in order to write an explicit expression of the number of zeros of u_i in $[0,\pi)$. Indeed, Lemma 2.4 guarantees that whenever $u_i'(0)$ is different from zero then $u_i(t)^2 + u_i(t)'^2 \neq 0$ for every $t \in [0,\pi]$. Hence, according to [14], we can write (for any $\rho > 0$)

$$\begin{aligned} \mathbf{n}(u_i) &= \frac{\rho}{\pi}\int_0^\pi \frac{u_i'(t)^2 - u_i(t)\,u_i''(t)}{\rho^2\,u_i(t)^2 + u_i'(t)^2}\,dt = \\ &= \frac{\rho}{\pi}\int_0^\pi \frac{u_i'(t)^2 + u_i(t)\,g_i(t,u(t);\lambda)}{\rho^2\,u_i(t)^2 + u_i'(t)^2}\,dt\,. \end{aligned} \tag{3.3}$$

We need to distinguish the following three cases:

- $$\frac{\mathbf{n}(u_i)}{2} = \frac{\rho}{\pi}\int_{I_i^\pm} \frac{u_i'(t)^2 - u_i(t)\,u_i''(t)}{\rho^2\,u_i(t)^2 + u_i'(t)^2}\,dt \quad \text{if } \mathbf{n}(u_i) \text{ is even,} \tag{3.4}$$

- $$\frac{\mathbf{n}(u_i)}{2} \pm \frac{1}{2} = \frac{\rho}{\pi}\int_{I_i^\pm} \frac{u_i'(t)^2 - u_i(t)\,u_i''(t)}{\rho^2\,u_i(t)^2 + u_i'(t)^2}\,dt \quad \begin{array}{l}\text{if } \mathbf{n}(u_i) \text{ is odd} \\ \text{and } u_i'(0) > 0\end{array} \tag{3.5}$$

- $$\frac{\mathbf{n}(u_i)}{2} \mp \frac{1}{2} = \frac{\rho}{\pi}\int_{I_i^\pm} \frac{u_i'(t)^2 - u_i(t)\,u_i''(t)}{\rho^2\,u_i(t)^2 + u_i'(t)^2}\,dt \quad \begin{array}{l}\text{if } \mathbf{n}(u_i) \text{ is odd} \\ \text{and } u_i'(0) < 0,\end{array} \tag{3.6}$$

where $I_i^+ := \{t \in [0,\pi] : u_i(t) \geq 0\}$ and $I_i^- := \{t \in [0,\pi] : u_i(t) \leq 0\}$. In what follows, we shall deal with the situation when $\mathbf{n}(u_i)$ is even; the case when $\mathbf{n}(u_i)$ is odd (with the relevant subcases) can be treated analogously and the details will be omitted for brevity. Let us first remark that

$$\begin{aligned} \frac{\mathbf{n}(u_i)}{2} = \frac{\rho}{\pi}\Bigg[&\int_{{}^N I_i^+} \frac{u_i'(t)^2 - u_i(t)u_i''(t)}{\rho^2 u_i(t)^2 + u_i'(t)^2}\,dt + \int_{{}_\delta I_i^+} \frac{u_i'(t)^2 - u_i(t)u_i''(t)}{\rho^2 u_i(t)^2 + u_i'(t)^2}\,dt + \\ &+ \int_{{}_N^\delta I_i^+} \frac{u_i'(t)^2 - u_i(t)u_i''(t)}{\rho^2 u_i(t)^2 + u_i'(t)^2}\,dt\Bigg], \end{aligned} \tag{3.7}$$

where

$${}^N I_i^+ = \{t \in [0,\pi] : u_i(t) > N\}\ ,\ {}_\delta I_i^+ = \{t \in [0,\pi] : 0 \leq u_i(t) < \delta\}\,,$$

$${}_N^\delta I_i^+ = \{t \in [0,\pi] : \delta \leq u_i(t) \leq N\}$$

(the constants $0 < \delta < N$ have been introduced in (2.1)). Let us now set $\rho = \mathrm{B}_\mathrm{i}^+$; from assumption $(\mathrm{H}_\mathrm{i}^+)_\lambda$, it immediately follows that

$$\int_{{}^N I_i^+} \frac{u_i'(t)^2 + u_i(t)g_i(t,u(t);\lambda)}{(\mathrm{B}_\mathrm{i}^+)^2\,u_i(t)^2 + u_i'(t)^2} \geq \int_{{}^N I_i^+} \frac{u_i'(t)^2 + (\mathrm{B}_\mathrm{i}^+)^2\,u_i(t)^2}{(\mathrm{B}_\mathrm{i}^+)^2\,u_i(t)^2 + u_i'(t)^2} = |{}^N I_i^+|\,. \tag{3.8}$$

Notice now that there exists $\eta \in (0,1)$

$$\{m \in \mathbb{N} : m < \mathrm{B}_i\} = \{m \in \mathbb{N} : m < (1-\eta)\,\mathrm{B}_i\}. \tag{3.9}$$

As far as the second addendum in (3.7) is concerned, from $(\mathrm{K}_i^+)_\lambda$ we obtain

$$\int_{\delta I_i^+} \frac{u_i'(t)^2 + u_i(t) g_i(t, u(t); \lambda)}{(\mathrm{B}_i^+)^2 u_i(t)^2 + u_i'(t)^2} dt \geq \int_{\delta I_i^+} \frac{u_i'(t)^2 + (\mathrm{A}_i^+)^2 u_i(t)^2}{(\mathrm{B}_i^+)^2 u_i(t)^2 + u_i'(t)^2} dt ;$$

moreover, there exists $r^+ > 0$ ($r^+ = r^+(\mathrm{B}_i^+, \delta)$) such that for every $t \in {}_\delta I_i^+$ with $u_i'(t)^2 \geq r^+$ the following inequalities hold

$$\frac{u_i'(t)^2 + (\mathrm{A}_i^+)^2 u_i(t)^2}{u_i'(t)^2 + (\mathrm{B}_i^+)^2 u_i(t)^2} \geq \frac{u_i'(t)^2}{u_i'(t)^2 + (\mathrm{B}_i^+)^2 \delta^2} \geq 1 - \eta . \tag{3.10}$$

Now, we look for a lower bound on $u_i'(0)$ which guarantees that $u_i'(t)^2 \geq r^+$ for every $t \in {}_\delta I_i^+$. To this aim let us apply Lemma 2.2 with $R_1 = \sqrt{r^+ + \delta^2}$; then there exists $d_{r^+} > 0$ such that for every solution u of (3.1) with $|u_i'(0)| > d_{r^+}$ it follows that $u_i(t)^2 + u_i'(t)^2 > r^+ + \delta^2$ for every $t \in [0, \pi]$. Thus, for every $t \in {}_\delta I_i^+$

$$|u_i'(0)| > d_{r^+} \qquad \Longrightarrow \qquad u_i'(t)^2 > r^+ + (\delta^2 - u_i(t)^2) \geq r^+ .$$

Therefore, assuming $|u_i'(0)| > d_{r^+}$, inequality (3.10) implies that

$$\int_{\delta I_i^+} \frac{u_i'(t)^2 + u_i(t) g_i(t, u(t); \lambda)}{(\mathrm{B}_i^+)^2 u_i(t)^2 + u_i'(t)^2} dt \geq (1 - \eta) \, |{}_\delta I_i^+| . \tag{3.11}$$

It remains to analyze the third addendum in (3.7). Recalling the definition of C_i^* and the argument in (2.17), from $(\mathrm{F}_i)_\lambda$ we deduce that

$$|u_i(t) g_i(t, u(t); \lambda)| \leq C_i^* \, |u_i(t)| \leq \frac{(C_i^*)^2}{2} + \frac{u_i(t)^2}{2} \qquad \forall t \in {}_N^\delta I_i^+ . \tag{3.12}$$

Arguing as before, we can find $\tilde{r}^+ > 0$ such that for every $t \in {}_N^\delta I_i^+$ with $u_i'(t)^2 \geq \tilde{r}^+$ it follows that

$$\frac{u_i'(t)^2 + u_i(t) g_i(t, u(t); \lambda)}{(\mathrm{B}_i^+)^2 u_i(t)^2 + u_i'(t)^2} \geq \frac{2u_i'(t)^2 - (C_i^*)^2 - N^2}{2(\mathrm{B}_i^+)^2 N^2 + 2u_i'(t)^2} \geq 1 - \eta . \tag{3.13}$$

In order to determine a lower bound on $u_i'(0)$ which leads to $u_i'(t)^2 \geq \tilde{r}^+$ for every $t \in {}_N^\delta I_i^+$, we apply Lemma 2.2 with $R_1 = \sqrt{\tilde{r}^+ + N^2}$. Hence, there exists $d_{\tilde{r}^+} > 0$ such that for every solution u of (3.1) with $|u_i'(0)| > d_{\tilde{r}^+}$ it follows that $u_i(t)^2 + u_i'(t)^2 > \tilde{r}^+ + N^2$ for every $t \in [0, \pi]$. Thus, for every $t \in {}_N^\delta I_i^+$

$$|u_i'(0)| > d_{\tilde{r}^+} \qquad \Longrightarrow \qquad u_i'(t)^2 > \tilde{r}^+ + (N^2 - u_i(t)^2) \geq \tilde{r}^+ .$$

Therefore, assuming $|u_i'(0)| > d_{\tilde{r}^+}$, inequality (3.13) gives

$$\int_{{}_N^\delta I_i^+} \frac{u_i'(t)^2 + u_i(t) g_i(t, u(t); \lambda)}{(\mathrm{B}_i^+)^2 u_i(t)^2 + u_i'(t)^2} dt \geq (1 - \eta) \, |{}_N^\delta I_i^+| . \tag{3.14}$$

Let us now denote by d_i^+ a strictly positive number such that $d_i > \max\{d_{r^+}, d_{\tilde{r}^+}\}$.

Adding (3.8), (3.11) and (3.14), we finally obtain that

$$|u_i'(0)| \geq d_i^+ \quad \Longrightarrow \quad \frac{\mathbf{n}(u_i)}{2} \geq \frac{\mathrm{B}_i^+}{\pi}(1-\eta)\,|I_i^+|\,.$$

With completely similar arguments, from $(\mathrm{K}_i^-)_\lambda$ and $(\mathrm{H}_i^-)_\lambda$, one can also prove that there exists $d_i^- > 0$ such that

$$|u_i'(0)| \geq d_i^- \quad \Longrightarrow \quad \frac{\mathbf{n}(u_i)}{2} \geq \frac{\mathrm{B}_i^-}{\pi}(1-\eta)\,|I_i^-|\,.$$

Thus, setting $d_i = \max\{d_i^+, d_i^-\}$, we infer that if $|u_i'(0)| \geq d_i$, then

$$\pi = |I_i^+| + |I_i^-| \leq \frac{\pi\,\mathbf{n}(u_i)}{2\,(1-\eta)}\left(\frac{1}{\mathrm{B}_i^+} + \frac{1}{\mathrm{B}_i^-}\right).$$

By the definition of B_i in (1.6), we can conclude

$$|u_i'(0)| \geq d_i \quad \Longrightarrow \quad \mathbf{n}(u_i) \geq (1-\eta)\,\mathrm{B}_i\,. \tag{3.15}$$

Let us take an arbitrary $m \in \mathbb{N}$ with $m < \mathrm{B}_i$. From (3.9) it follows that $m < (1-\eta)\,\mathrm{B}_i$. Hence, for every $m < \mathrm{B}_i$ $(m \in \mathbb{N})$

$$|u_i'(0)| \geq d_i \quad \Longrightarrow \quad \mathbf{n}(u_i) > m.$$

Proposition 3.2 *For every fixed $i \in \{1,2\}$, there exists $\tilde{d}_i > 0$ such that for every $M \in \mathbb{N}$ with $M > \mathcal{B}_i$ and for every solution $u = (u_1, u_2)$ of* (3.1)

$$|u_i'(0)| \geq \tilde{d}_i \quad \Longrightarrow \quad \mathbf{n}(u_i) < M.$$

Proof Fix $i \in \{1,2\}$. Let $u = (u_1, u_2)$ be a solution of (3.1). As in the proof of Proposition 3.1, by means of Lemma 2.4 we are able to write, as in (3.3), the explicit expression of the number of zeros of u_i in $[0, \pi)$

$$\mathbf{n}(u_i) \;=\; \frac{\rho}{\pi}\int_0^\pi \frac{u_i'(t)^2 + u_i(t)\,g_i(t, u(t); \lambda)}{\rho^2\,u_i(t)^2 + u_i'(t)^2}\,dt$$

(for any $\rho > 0$), whenever $u_i'(0)$ is different from zero. As before, we will distinguish the case $\mathbf{n}(u_i)$ even, for whom we will use (3.4), and the case $\mathbf{n}(u_i)$ odd, for whom we will apply (3.5) or (3.6).
In what follows, we will sketch the proof assuming that $\mathbf{n}(u_i)$ is even (the other case can be treated in an analogous way). Exactly as in the previous proof, we can write $\frac{\mathbf{n}(u_i)}{2}$ as

$$\frac{\mathbf{n}(u_i)}{2} \;=\; \frac{\rho}{\pi}\left[\int_{N I_i^+} \frac{u_i'(t)^2 - u_i(t)u_i''(t)}{\rho^2 u_i(t)^2 + u_i'(t)^2}\,dt \;+\; \int_{\delta I_i^+} \frac{u_i'(t)^2 - u_i(t)u_i''(t)}{\rho^2 u_i(t)^2 + u_i'(t)^2}\,dt \;+\right.$$

$$\left. +\; \int_{\overset{\delta}{N} I_i^+} \frac{u_i'(t)^2 - u_i(t)u_i''(t)}{\rho^2 u_i(t)^2 + u_i'(t)^2}\,dt\right] \tag{3.16}$$

and set $\rho = \mathcal{B}_i^+$. We first observe that there exists $\tilde{\eta} \in (0,1)$ satisfying

$$\{m \in \mathbb{N} \,:\, m > \mathcal{B}_i\} = \{m \in \mathbb{N} \,:\, m > (1+\tilde{\eta})\,\mathcal{B}_i\}\,. \tag{3.17}$$

From assumption $(\mathrm{H}_i^+)_\lambda$, it immediately follows that

$$\int_{{}^N I_i^+} \frac{u_i'(t)^2 + u_i(t) g_i(t,u(t);\lambda)}{(\mathcal{B}_i^+)^2\, u_i(t)^2 + u_i'(t)^2} dt \le \int_{{}^N I_i^+} \frac{u_i'(t)^2 + (\mathcal{B}_i^+)^2\, u_i(t)^2}{(\mathcal{B}_i^+)^2\, u_i(t)^2 + u_i'(t)^2} dt = |{}^N I_i^+| \tag{3.18}$$

As far as the second addendum in (3.16) is concerned, from $(\mathrm{K}_i^+)_\lambda$, we obtain

$$\int_{{}_\delta I_i^+} \frac{u_i'(t)^2 + u_i(t) g_i(t,u(t);\lambda)}{(\mathcal{B}_i^+)^2\, u_i(t)^2 + u_i'(t)^2} dt \le \int_{{}_\delta I_i^+} \frac{u_i'(t)^2 + (\mathcal{A}_i^+)^2\, u_i(t)^2}{(\mathcal{B}_i^+)^2\, u_i(t)^2 + u_i'(t)^2} dt\,.$$

As before, there exists $\overline{r}^+ > 0$ ($\overline{r}^+ = \overline{r}^+(\mathcal{A}_i^+, \delta)$) such that for every $t \in {}_\delta I_i^+$ with $u_i'(t)^2 \ge \overline{r}^+$

$$\frac{u_i'(t)^2 + (\mathcal{A}_i^+)^2\, u_i(t)^2}{u_i'(t)^2 + (\mathcal{B}_i^+)^2\, u_i(t)^2} \le \frac{u_i'(t)^2 + (\mathcal{A}_i^+)^2\, \delta^2}{u_i'(t)^2} \le 1 + \tilde{\eta}\,.$$

Arguing as in the proof of Proposition 3.1, from Lemma 2.2 we can easily find $d_{\overline{r}^+} > 0$ such that $|u_i'(0)| > d_{\overline{r}^+}$ implies that $u_i'(t)^2 \ge \overline{r}^+$ for every $t \in {}_\delta I_i^+$. Hence, if $|u_i'(0)| > d_{\overline{r}}^+$ then

$$\int_{{}_\delta I_i^+} \frac{u_i'(t)^2 + u_i(t) g_i(t,u(t);\lambda)}{(\mathcal{B}_i^+)^2\, u_i(t)^2 + u_i'(t)^2} dt \le (1+\tilde{\eta})\, |{}_\delta I_i^+|\,. \tag{3.19}$$

It remains to analyze the third addendum. Recalling the argument in (2.17) and exactly as in (3.12), from assumption $(\mathrm{F_i})_\lambda$ we obtain

$$|u_i(t) g_i(t,u(t);\lambda)| \le \frac{(C_i^*)^2}{2} + \frac{u_i(t)^2}{2} \qquad \forall\, t \in {}_N^\delta I_i^+\,.$$

Moreover, there exists $r_+^* > 0$ such that for every $t \in {}_N^\delta I_i^+$ with $u_i'(t)^2 \ge r_+^*$ it follows that

$$\frac{u_i'(t)^2 + u_i(t) g_i(t,u(t);\lambda)}{(\mathcal{B}_i^+)^2\, u_i(t)^2 + u_i'(t)^2} \le \frac{2u_i'(t)^2 + (C_i^*)^2 + N^2}{2(\mathcal{B}_i^+)^2 \delta^2 + 2u_i'(t)^2} \le 1 + \tilde{\eta}.$$

Lemma 2.2 ensures the existence of $d_{r_+^*}$ such that $|u_i'(0)| > d_{r_+^*}$ implies $u_i'(t)^2 \ge r_+^*$ for every $t \in {}_N^\delta I_i^+$. Hence, when $|u_i'(0)| > d_{r_+^*}$, we obtain

$$\int_{{}_N^\delta I_i^+} \frac{u_i'(t)^2 + u_i(t) g_i(t,u(t);\lambda)}{(\mathcal{B}_i^+)^2\, u_i(t)^2 + u_i'(t)^2} dt \le (1+\tilde{\eta})\, |{}_N^\delta I_i^+|. \tag{3.20}$$

Now, denote by $\tilde{d}_i$ a strictly positive number such that $\tilde{d}_i > \max\{d_{\overline{r}^+}, d_{r_+^*}\}$.

Adding (3.18), (3.19) and (3.20), we finally infer that

$$|u_i'(0)| \geq \tilde{d}_i^+ \quad \Longrightarrow \quad \frac{\mathbf{n}(u_i)}{2} \leq \frac{\mathcal{B}_i^+}{\pi}(1+\tilde{\eta})\,|I_i^+|\,.$$

Analogously, from $(\mathrm{K}_i^-)_\lambda$ and $(\mathrm{H}_i^-)_\lambda$, one can also prove that there exists $\tilde{d}_i^- > 0$ such that

$$|u_i'(0)| \geq \tilde{d}_i^- \quad \Longrightarrow \quad \frac{\mathbf{n}(u_i)}{2} \leq \frac{\mathcal{B}_i^-}{\pi}(1+\tilde{\eta})\,|I_i^-|\,.$$

Thus, setting $\tilde{d}_i = \max\{\tilde{d}_i^+, \tilde{d}_i^-\}$, we deduce that $|u_i'(0)| \geq \tilde{d}_i$ implies

$$\pi = |I_i^+| + |I_i^-| \geq \frac{\pi\,\mathbf{n}(u_i)}{2\,(1+\tilde{\eta})}\left(\frac{1}{\mathcal{B}_i^+} + \frac{1}{\mathcal{B}_i^-}\right).$$

By the definition of $\mathcal{B}_i$ in (1.6), we infer

$$|u_i'(0)| \geq \tilde{d}_i \quad \Longrightarrow \quad \mathbf{n}(u_i) \leq (1+\tilde{\eta})\,\mathcal{B}_i\,. \tag{3.21}$$

Let us take an arbitrary $M \in \mathbb{N}$ with $M > \mathcal{B}_i$. From (3.17), it follows that $M > (1+\tilde{\eta})\,\mathcal{B}_i$. Hence, we can conclude that for every $M > \mathcal{B}_i$ $(M \in \mathbb{N})$

$$|u_i'(0)| \geq \tilde{d}_i \quad \Longrightarrow \quad \mathbf{n}(u_i) < M.$$

The two Propositions given below (which conclude the Section) contain lower and upper bounds, respectively, for the number of zeros of the components of the (possible) solutions to (3.1) for whom $|u_i'(0)|$ is (different from zero and) "sufficiently small".

Proposition 3.3 *For every fixed $i \in \{1,2\}$, there exists $d_i^* > 0$ such that for every $m \in \mathbb{N}$ with $m < \mathrm{A}_i$ and for every solution $u = (u_1, u_2)$ of* (3.1)

$$0 < |u_i'(0)| \leq d_i^* \quad \Longrightarrow \quad \mathbf{n}(u_i) > m\,.$$

Proof Let u be a solution of (3.1) and fix $i \in \{1,2\}$. As before, since $|u_i'(0)|$ is different from zero, Lemma 2.4 guarantees that $u \in \mathcal{M}$; hence, we are able to write the explicit expression (3.3) for the number of zeros of u_i in $[0,\pi)$.
Exactly as in the proof of the previous Propositions, we will distinguish the case $\mathbf{n}(u_i)$ even, where we will use (3.4), and the case $\mathbf{n}(u_i)$ odd, where we will apply (3.5) or (3.6). In what follows, we suppose that $\mathbf{n}(u_i)$ is even (the case when $\mathbf{n}(u_i)$ is odd can be treated analogously). Recall that in this situation

$$\frac{\mathbf{n}(u_i)}{2} \;=\; \frac{\rho}{\pi}\int_{I_i^\pm} \frac{u_i'(t)^2 \,+\, u_i(t)\,g_i(t,u(t);\lambda)}{\rho^2\,u_i(t)^2 + u_i'(t)^2}\,dt\,.$$

Let us consider an arbitrary $\tilde{\delta} < \delta$. According to Lemma 2.4 (with $\rho_1 = \tilde{\delta}$), there exists $d_i^* > 0$ such that

$$\max|(u_i(t), u_i'(t))| > \tilde{\delta} \quad \Longrightarrow \quad u_i(t)^2 + u_i'(t)^2 > (d_i^*)^2 \quad \forall\, t \in [0,\pi]\,.$$

In particular, if $|u_i'(0)| \le d_i^*$ then $|(u_i(t), u_i'(t))| \le \tilde{\delta} < \delta$ for every $t \in [0,\pi]$. Assume now $|u_i'(0)| \le d_i^*$; then $|u_i(t)| < \delta$ for every $t \in [0,\pi]$. Thus, from $(\mathrm{K}_i^+)_\lambda$ we obtain

$$\int_{I_i^+} \frac{u_i'(t)^2 + u_i(t) g_i(t, u(t); \lambda)}{(\mathrm{A}_i^+)^2\, u_i(t)^2 + u_i'(t)^2} dt \ge \int_{I_i^+} \frac{u_i'(t)^2 + (\mathrm{A}_i^+)^2\, u_i(t)^2}{(\mathrm{A}_i^+)^2\, u_i(t)^2 + u_i'(t)^2} dt = |I_i^+|\,.$$

Hence, by the choice $\rho = \mathrm{A}_i^+$, it follows that

$$\frac{\mathbf{n}(u_i)}{2} \ge \frac{\mathrm{A}_i^+}{\pi}\, |I_i^+|\,.$$

Analogously, hypothesis $(\mathrm{K}_i^-)_\lambda$ leads to

$$\frac{\mathbf{n}(u_i)}{2} \ge \frac{\mathrm{A}_i^-}{\pi}\, |I_i^-|\,.$$

From the above inequalities we deduce that

$$\pi \le \frac{\pi\, \mathbf{n}(u_i)}{2} \left(\frac{1}{\mathrm{A}_i^+} + \frac{1}{\mathrm{A}_i^-} \right).$$

Hence, by definition of A_i in (1.6), taking an arbitrary $m \in \mathbb{N}$ with $m < \mathrm{A}_i$, we finally conclude that

$$0 < |u_i'(0)| \le d_i^* \quad \Longrightarrow \quad \mathbf{n}(u_i) \ge \mathrm{A}_i > m\,.$$

Remark 3.4 We observe that Proposition 3.3 and Proposition 3.1 are used in the proof of the main results only in those situations when $\mathrm{A}_i > 1$ and $\mathrm{B}_i > 1$, respectively.

Proposition 3.5 *For every fixed $i \in \{1,2\}$, there exists $d_i^* > 0$ such that for every $M \in \mathbb{N}$ with $M > \mathcal{A}_i$ and for every solution $u = (u_1, u_2)$ of* (3.1)

$$0 < |u_i'(0)| \le d_i^* \quad \Longrightarrow \quad \mathbf{n}(u_i) < M\,.$$

Proof Fix $i \in \{1,2\}$. Letting u be a solution of (3.1); as before, thanks to Lemma 2.4, condition $|u_i'(0)| > 0$ enables us to write the explicit expression (3.3) for the number of zeros of u_i in $[0,\pi)$. Suppose now that $\mathbf{n}(u_i)$ is even (the case when $\mathbf{n}(u_i)$ is odd can be treated analogously). Hence,

$$\frac{\mathbf{n}(u_i)}{2} = \frac{\rho}{\pi} \int_{I_i^\pm} \frac{u_i'(t)^2 + u_i(t)\, g_i(t, u(t); \lambda)}{\rho^2\, u_i(t)^2 + u_i'(t)^2}\, dt\,.$$

Exactly as in the proof of Proposition 3.3, we can find $d_i^* > 0$ such that

$$|u_i'(0)| \le d_i^* \quad \Longrightarrow \quad |(u_i(t), u_i'(t))| < \delta \quad \forall\, t \in [0,\pi]\,.$$

In what follows, we will assume $|u_i'(0)| \le d_i^*$, whence, $|u_i(t)| < \delta$ for every $t \in [0,\pi]$. Thus, from $(\mathrm{K}_i^+)_\lambda$ we obtain

$$\int_{I_i^+} \frac{u_i'(t)^2 + u_i(t) g_i(t,u(t);\lambda)}{(\mathcal{A}_i^+)^2\, u_i(t)^2 + u_i'(t)^2}\,dt \le \int_{I_i^+} \frac{u_i'(t)^2 + (\mathcal{A}_i^+)^2\, u_i(t)^2}{(\mathcal{A}_i^+)^2\, u_i(t)^2 + u_i'(t)^2}\,dt = |I_i^+|\,.$$

Hence, choosing $\rho = \mathcal{A}_i^+$ it follows that

$$\frac{\mathbf{n}(u_i)}{2} \le \frac{\mathcal{A}_i^+}{\pi}\,|I_i^+|\,.$$

Analogously, hypothesis $(\mathrm{K}_i^-)_\lambda$ implies that

$$\frac{\mathbf{n}(u_i)}{2} \le \frac{\mathcal{A}_i^-}{\pi}\,|I_i^-|\,.$$

The above inequalities leads to

$$\pi \ge \frac{\pi\,\mathbf{n}(u_i)}{2}\left(\frac{1}{\mathcal{A}_i^+} + \frac{1}{\mathcal{A}_i^-}\right).$$

Let us take an arbitrary $M \in \mathbb{N}$ with $M > \mathcal{A}_i$. From the definition of $\mathcal{A}_i$ in (1.6), we can conclude that

$$0 < |u_i'(0)| \le d_i^* \quad \Longrightarrow \quad \mathbf{n}(u_i) \le \mathcal{A}_i < M\,.$$

Remark 3.6 According to Remark 2.5, Proposition 3.3 and Proposition 3.5 are valid if only assumption $(\mathrm{K}_i^\pm)_\lambda$ is assumed.

Remark 3.7 Our main result (which will be proved in next Section) provides the existence of solutions u to (3.1) having the property $d_i^* < |u_i'(0)| < \max(d_i, \tilde{d}_i)$, for every i. Hence, the two inequalities $|u_i'(0)| \ge d_i$ and $|u_i'(0)| \ge \tilde{d}_i$ (which e.g. in the particular case when $b_i^\pm(t) \equiv \beta_i^\pm(t) \equiv P \notin \mathbb{N}$ would lead, combining (3.15) with (3.9) and (3.21) with (3.17), to the contradiction $P \le \mathbf{n}(u_i) \le P$) never occur simultaneously as far as the solutions we find are concerned. An analogous remark is valid when we compare Proposition 3.3 and Proposition 3.5.

4 The main results

Before proving our main result, we recall the abstract continuation Theorem, on which the proof is based. Indeed, the proof of our main result is concerned with the study of a fixed point equation of the form

$$u = \mathcal{N}(u,\lambda), \tag{4.1}$$

where $\mathcal{N} : \operatorname{dom}\mathcal{N} \subset X \times [0,1] \longrightarrow X$ is a completely continuous operator, X is a Banach space and $(u,\lambda) \in \operatorname{dom}\mathcal{N}$. It is well known (cf. [31]) that problem (3.1) can be written in the form (4.1) with respect to the Banach space $X = C_0^1\left([0,\pi],\mathbb{R}^2\right)$. Let Σ denote the set of the solutions of (4.1), i.e.

$$\Sigma = \{(u,\lambda) \,:\, u = \mathcal{N}(u,\lambda)\}\,.$$

For any subset $D \subset X \times [0,1]$ let us denote by $D_\lambda = \{x \in X \,:\, (x,\lambda) \in D\}$ the section of D at $\lambda \in [0,1]$ and let us set $\mathcal{N}_\lambda := \mathcal{N}(\cdot,\lambda)$. Moreover, let us consider an open set B such that $\overline{B} \subset \operatorname{dom}\mathcal{N}$. Then we can state the following

Theorem 4.1 (Continuation Theorem) [6]. *Let* $\mathbf{k} : \Sigma \cap \overline{B} \longrightarrow \mathbb{N}^2$ *be a continuous function; suppose that there exists* $n \in \mathbb{N}^2$ *satisfying the following conditions*

$$n \notin \mathbf{k}\,(\Sigma \cap \partial\overline{B}\,) \tag{4.2}$$

$$\mathbf{k}^{-1}(n) \quad \textit{is bounded.} \tag{4.3}$$

Then, for every open set U_0^n *such that* $(\mathbf{k}^{-1}(n))_0 \subset U_0^n \subset \overline{U_0^n} \subset \overline{B}_0$ *and* $\Sigma_0 \cap U_0^n = (\mathbf{k}^{-1}(n))_0$, *the Leray-Schauder degree* $\deg\,(\mathrm{Id} - \mathcal{N}_0,\, U_0^n)$ *is defined. If*

$$\deg\,(\mathrm{Id} - \mathcal{N}_0,\, U_0^n) \neq 0\,, \tag{4.4}$$

then there is a continuum $C_n \subset \Sigma$ *with*

$$\{\lambda \in [0,1] \mid \exists u \in X \,:\, (u,\lambda) \in C_n\} = [0,1]$$

and such that

$$(u,\lambda) \in C_n \implies (u,\lambda) \in B \quad \textit{and} \quad \mathbf{k}(u,\lambda) = n\,.$$

In particular, there is at least one solution $\tilde{u} \in B_1$ *of the operator equation*

$$u = \mathcal{N}(u,1)$$

with

$$\mathbf{k}(\tilde{u},1) = n\,.$$

In order to apply Theorem 4.1, we need to introduce a suitable continuous function $\mathbf{k}$, whose domain is contained in $\Sigma \cap \overline{\mathcal{D}}$, being $\mathcal{D}$ a suitable subset of $X \times [0,1]$.
More precisely, we shall set

$$\mathcal{D} = \left\{(u,\lambda) \in X \times [0,1] \,:\, d_i^* < |u_i'(0)| < \max(d_i,\tilde{d}_i) \qquad \forall\, i \in \{1,2\}\right\},$$

where the constants d_i^*, d_i, $\tilde{d}_i$ have been introduced in Section 3. According to Proposition 3.1, 3.2, 3.3, 3.5, it is not restrictive to assume that $d_i^* < \max(d_i,\tilde{d}_i)$; indeed, if we assume that $d_i^* \geq \max(d_i,\tilde{d}_i)$, it will be sufficient to take

$$\mathcal{D} = \left\{(u,\lambda) \in X \times [0,1] \,:\, 0 < \overline{d}_i < |u_i'(0)| < \max(d_i,\tilde{d}_i) \qquad \forall\, i \in \{1,2\}\right\}.$$

We remark that Lemma 3.1 in [17] guarantees that the map $\mathbf{n}$ defined in (3.2) is continuous. This implies that the map

$$\begin{aligned} \mathbf{k}: \ \Sigma \cap \overline{\mathcal{D}} \ &\longrightarrow \quad \mathbb{N}^2 \\ u \ &\longmapsto \ (\mathbf{n}(u_1), \mathbf{n}(u_2)), \end{aligned}$$

is continuous as well.

We are now in position to prove our main result. First, we recall its statement

Theorem 4.2 *Assume that G satisfies hypotheses $(\mathrm{H}_i^{\pm})$, $(\mathrm{K}_i^{\pm})$, (F_i) and that there exists $m_i \in \mathbb{N}$ such that*

$$\textit{either} \qquad \mathfrak{A}_i < m_i < \mathtt{B}_i \qquad \textit{or} \qquad \mathfrak{B}_i < m_i < \mathtt{A}_i$$

holds for every $i \in \{1,2\}$. Then, there exist h_1, h_2, $\mathcal{H}_1$, $\mathcal{H}_2 \in \mathbb{N}$ such that for every $(n_1, n_2) \in \mathbb{N}^2$ with $h_i \le n_i \le \mathcal{H}_i$ $(i = 1,2)$, problem (1.1) has at least four solutions $u^j = (u_1^j, u_2^j)$ $(1 \le j \le 4)$ with u_i^j having exactly n_i zeros in $[0,\pi)$ for every $j = 1,2,3,4$. Moreover, for every $(s_1, s_2) \in \tau$ there exists $j \in \{1,\dots,4\}$ such that $\operatorname{sgn}\big((u_i^j)'(0)\big) = s_i$ for $i \in \{1,2\}$.

Proof For each fixed $i \in \{1,2\}$, the constants h_i and $\mathcal{H}_i$ in the statement of Theorem 4.2 depend on $\mathfrak{A}_i, \mathfrak{B}_i$ and $\mathtt{A}_i, \mathtt{B}_i$ respectively; more precisely, the construction of $h_i, \mathcal{H}_i$ depends on the mutual position of $\mathfrak{A}_i, \mathfrak{B}_i$ and $\mathtt{A}_i, \mathtt{B}_i$ respectively (cf. (4.6), (4.10) and (4.11)). For this reason, we distinguish the following cases:

a. $\mathfrak{A}_i < \mathtt{B}_i$ for every $i \in \{1,2\}$,

b. $\mathfrak{B}_i < \mathtt{A}_i$ for every $i \in \{1,2\}$,

c. $\mathfrak{A}_1 < \mathtt{B}_1$ and $\mathfrak{B}_2 < \mathtt{A}_2$,

d. $\mathfrak{B}_1 < \mathtt{A}_1$ and $\mathfrak{A}_2 < \mathtt{B}_2$.

In each of these four cases, we will apply Theorem 4.1 four times, according to the sign of $u_i'(0)$ $(i \in \{1,2\})$; consequently, in order to find solutions of (1.1) with $\operatorname{sgn}(u_i'(0)) = s_i$ for every $i \in \{1,2\}$ and for $(s_1, s_2) \in \tau$, we will set

$$B_{(s_1,s_2)} = \mathcal{D} \cap \{(u,\lambda) \in X \times [0,1] : \operatorname{sgn}(u_i'(0)) = s_i \quad \forall\, i = 1,2\}. \tag{4.5}$$

We point out that $\mathfrak{A}_i < \mathtt{B}_i$ and $\mathfrak{B}_i < \mathtt{A}_i$ are mutually exclusive: for instance, $\mathfrak{A}_i < \mathtt{B}_i$ implies $\mathtt{A}_i < \mathfrak{B}_i$. Indeed, from (1.6) and by the definition of $\mathfrak{A}_i, \mathfrak{B}_i, \mathtt{A}_i, \mathtt{B}_i$, recalling that $\mathrm{A}_\mathrm{i}^{\pm} \le \mathcal{A}_i^{\pm}$ and $\mathrm{B}_\mathrm{i}^{\pm} \le \mathcal{B}_i^{\pm}$, we easily obtain that $\mathfrak{A}_i < \mathtt{B}_i$ implies

$$\mathtt{A}_i \le \mathrm{A_i} = \frac{2\mathrm{A_i^+A_i^-}}{\mathrm{A_i^+ + A_i^-}} \le \frac{2\mathcal{A}_\mathrm{i}^+\mathcal{A}_\mathrm{i}^-}{\mathcal{A}_\mathrm{i}^+ + \mathcal{A}_\mathrm{i}^-} = \mathcal{A_i} \le \mathfrak{A_i} <$$

$$< \mathtt{B}_i \le \mathrm{B_i} = \frac{2\mathrm{B_i^+B_i^-}}{\mathrm{B_i^+ + B_i^-}} \le \frac{2\mathcal{B}_i^+\mathcal{B}_i^-}{\mathcal{B}_i^+ + \mathcal{B}_i^-} = \mathcal{B}_i \le \mathfrak{B}_i.$$

With analogous arguments, from $\mathfrak{B}_i < \mathtt{A}_i$ it follows that $\mathtt{B}_i < \mathfrak{A}_i$.

First, we analyze case **a**). Let us set

$$h_i := \min\{m \in \mathbb{N} \mid m > \mathfrak{A}_i\}\,, \qquad \mathcal{H}_i := \max\{M \in \mathbb{N} \mid M < \mathtt{B}_i\} \quad \forall\, i = 1,2\,. \tag{4.6}$$

The hypotheses guarantee that $h_i \le \mathcal{H}_i$ for every $i \in \{1,2\}$. Indeed, there exists $m_i \in \mathbb{N}$ such that $m_i \in]\mathfrak{A}_i, \mathtt{B}_i[$ for every $i \in \{1,2\}$. By construction, $h_i \le m_i$ and $\mathcal{H}_i \ge m_i$ for every $i \in \{1,2\}$, whence $h_i \le \mathcal{H}_i$. Thus, let us take $n = (n_1, n_2) \in \mathbb{N}^2$ with $h_i \le n_i \le \mathcal{H}_i$ for every $i \in \{1,2\}$. By the above remark, such an n always exists. Since (by the definition of $\mathcal{H}_i$) $n_i < \mathtt{B}_i \le \mathrm{B}_i$, from Proposition 3.1 it follows that for every $i \in \{1,2\}$, whenever $u = (u_1, u_2)$ is a solution of (3.1) we get

$$|u_i'(0)| \ge \max(d_i, \tilde{d}_i) \ge d_i \qquad \Longrightarrow \qquad \mathbf{n}(u_i) > n_i\,. \tag{4.7}$$

From the definition of h_i, it immediately follows that $n_i > \mathfrak{A}_i \ge \mathcal{A}_i$. Hence, Proposition 3.5 implies that for every solution u of (3.1) and for every $i \in \{1,2\}$

$$0 < |u_i'(0)| \le d_i^* \qquad \Longrightarrow \qquad \mathbf{n}(u_i) < n_i\,. \tag{4.8}$$

We shall now prove the existence of solutions u with $\operatorname{sgn} u_i'(0) = 1$ for every $i = 1,2$ and such that u_i has n_i zeros in $[0,\pi)$. In order to apply Theorem 4.1, let us define the set

$$\begin{aligned} B = B_{(1,1)} &= \mathcal{D} \cap \{(u,\lambda) \in X \times [0,1] : \operatorname{sgn}(u_i'(0)) = 1 \quad \forall\, i = 1,2\} \\ &= \{(u,\lambda) \in X \times [0,1] : d_i^* < u_i'(0) < \max(d_i, \tilde{d}_i) \quad \forall\, i = 1,2\}\,. \end{aligned} \tag{4.9}$$

First, we note that $n \notin \mathbf{k}(\Sigma \cap \partial \overline{B})$. Indeed, suppose by contradiction that there exists $(u,\lambda) \in \partial \overline{B}$ such that $\mathbf{k}(u,\lambda) = n$. This means that there exists $j \in \{1,2\}$ such that $d_j^* = u_j'(0)$ or $u_j'(0) = \max(d_j, \tilde{d}_j)$. In the former case from (4.8) we obtain $\mathbf{n}(u_j) < n_j$, while in the latter from (4.7) we get $\mathbf{n}(u_j) > n_j$. In any case, this contradicts the assumption $\mathbf{k}(u,\lambda) = (\mathbf{n}(u_1), \mathbf{n}(u_2)) = n = (n_1, n_2)$. Thus, (4.2) is fulfilled.

Secondly, we notice that $\mathbf{k}^{-1}(n)$ is bounded. Indeed, let $(u,\lambda) \in \Sigma \cap \overline{B}$ such that $\mathbf{k}(u,\lambda) = n$. From (4.7) we have that $\mathbf{n}(u_i) = n_i$ implies $|u_i'(0)| < \max(d_i, \tilde{d}_i)$ for every $i \in \{1,2\}$. Now we can apply Lemma 2.2 with $R_1 = \max(d_i, \tilde{d}_i)$. Hence, for any $i \in \{1,2\}$ we can find $R_2^i > 0$ such that $u_i(t)^2 + u_i'(t)^2 \le R_2^i \le R_2$ for every $t \in [0,\pi]$ and for every $i = 1,2$, where $R_2 = \max(R_2^1, R_2^2)$. We infer that (4.3) is satisfied.

Finally, we refer to the Appendix (where the autonomous case is studied) to show that there exists an open subset U_0^n such that (4.4) is verified.
Hence, all the assumption of Theorem 4.1 are fulfilled and by Theorem 4.1 it is proved the existence of at least one solution $u^1 = (u_1^1, u_2^1)$ of (1.1) with $0 < d_i^* < (u_i^1)'(0) < \max(d_i, \tilde{d}_i)$ for every $i \in \{1,2\}$ and such that $\mathbf{k}(u^1, 1) = n$, i.e. such that u_i has exactly n_i zeros in $[0,\pi)$ for every $i = 1,2$.

The existence of the other three solutions of (1.1) can be proved analogously and the details are omitted for brevity.

Case **b)** is analogous to the previous one. Thus, we skip the details. We only observe that the role played in case a) by $\mathfrak{A}_i$ and $\mathtt{B}_i$ is now played by $\mathfrak{B}_i$ and $\mathtt{A}_i$ respectively. Consequently, Propositions 3.3 and 3.2 are to be used instead of Propositions 3.1 and 3.5.

We now sketch the proof relative to case **c)**.
First of all, we define the constants h_i and $\mathcal{H}_i$ as follows:

$$h_1 := \min\{m \in \mathbb{N} : \ m > \mathfrak{A}_1\}\,, \qquad \mathcal{H}_1 := \max\{M \in \mathbb{N} : \ M < \mathtt{B}_1\}\,, \tag{4.10}$$

$$h_2 := \min\{m \in \mathbb{N} : \ m > \mathfrak{B}_2\}\,, \qquad \mathcal{H}_2 := \max\{M \in \mathbb{N} : \ M < \mathtt{A}_2\}\,. \tag{4.11}$$

We observe that $h_i \le \mathcal{H}_i$ for every $i \in \{1,2\}$, since in this situation there exist $m_1, m_2 \in \mathbb{N}$ such that $\mathfrak{A}_1 < m_1 < \mathtt{B}_1$ and $\mathfrak{B}_2 < m_2 < \mathtt{A}_2$. Let us take $n = (n_1, n_2) \in \mathbb{N}^2$ with $h_i \le n_i \le \mathcal{H}_i$ for every $i \in \{1,2\}$, and let u be a solution of (3.1). Arguing as before, since $\mathcal{A}_1 \le \mathfrak{A}_1 < n_1 < \mathtt{B}_1 \le \mathrm{B}_1$ from Proposition 3.1 and 3.5, we deduce (respectively) that

$$|u_1'(0)| \ge \max(d_1, \tilde{d}_1) \qquad \Longrightarrow \qquad \mathbf{n}(u_1) > n_1\,,$$

$$0 < |u_1'(0)| \le d_1^* \qquad \Longrightarrow \qquad \mathbf{n}(u_1) < n_1\,.$$

Moreover, since $\mathcal{B}_2 \le \mathfrak{B}_2 < n_2 < \mathtt{A}_2 \le \mathrm{A}_2$, from Proposition 3.3 and 3.2, we get respectively

$$0 < |u_2'(0)| \le d_2^* \qquad \Longrightarrow \qquad \mathbf{n}(u_2) > n_2\,,$$

$$|u_2'(0)| \ge \max(d_1, \tilde{d}_1) \qquad \Longrightarrow \qquad \mathbf{n}(u_2) < n_2\,.$$

We can distinguish four subcases, exactly as before. According to the subcase, we will take a different $(s_1, s_2) \in \tau$ and we will apply Theorem 4.1 with $B = B_{(s_1,s_2)}$ defined as in (4.5) and (4.9). The details, which are analogous to those we already developed, are omitted.

Case **d)** is analogous to the previous one. Indeed, it is sufficient to replace in the previous proof $\mathfrak{A}_1$ and $\mathtt{B}_1$ with $\mathfrak{A}_2$ and $\mathtt{B}_2$, $\mathfrak{B}_2$ and $\mathtt{A}_2$ with $\mathfrak{B}_1$ and $\mathtt{A}_1$.

Remark 4.3 We note that the validity of (4.2) and (4.3) in the abstract continuation theorem, which follows from Proposition 3.1, 3.2, 3.3, 3.5, is ensured whenever the set $\mathcal{D}$ is replaced by any set of the form

$$\widetilde{\mathcal{D}} = \{(u,\lambda) \in X \times [0,1] : \ c_i^* < |u_i'(0)| < \tilde{c}_i \qquad \forall\, i \in \{1,2\}\}\,,$$

where $0 < c_i^* < d_i^*$ and $\tilde{c}_i > \max(d_i, \tilde{d}_i)$.
This remark is useful for the study of the autonomous problem (2.2) that we develop in the Appendix.

In what follows we give some straightforward corollaries of our main result which enable us to perform (in the particular case of a scalar equation) some comparisons with similar results available in the literature (cf. Remark 4.7 and 4.8).

Corollary 4.4 *Assume that for every* $i \in \{1,2\}$ *the function* G *satisfies* $(\mathrm{F_i})$ *and*

$(\mathrm{H_i})$ *there exist* $\widetilde{N}_i > 0$ *and two continuous functions* b_i, $\beta_i : [0,\pi] \longrightarrow \mathbb{R}^+$ *such that*

$$|x_i| > \widetilde{N}_i \quad \Longrightarrow \quad b_i(t) \leq \frac{G_i(t,x)}{x_i} \leq \beta_i(t),$$

for every $t \in [0,\pi]$ *and* $x_j \in \mathbb{R}$ *with* $j \neq i$;

$(\mathrm{K_i})$ *there exist* $\widetilde{M}_i > 0$ *and two continuous functions* a_i, $\alpha_i : [0,\pi] \longrightarrow \mathbb{R}^+$ *such that*

$$|x_i| < \widetilde{M}_i \quad \Longrightarrow \quad a_i(t) \leq \frac{G_i(t,x)}{x_i} \leq \alpha_i(t),$$

for every $t \in [0,\pi]$ *and* $x_j \in \mathbb{R}$ *with* $j \neq i$.

Moreover, suppose that there exists $m_i \in \mathbb{N}$ *such that*

$$\textit{either} \qquad \mathcal{A}_i < m_i < \mathrm{B}_i \qquad \textit{or} \qquad \mathcal{B}_i < m_i < \mathrm{A}_i$$

holds for every $i \in \{1,2\}$. *Then, the same conclusion of Theorem* 4.2 *holds.*

For the applicability of Theorem 4.2, it is sufficient to observe that, according to the remark following (1.10), under the assumptions of Corollary 4.4 (and recalling (1.6)), $\mathfrak{A}_i = \mathcal{A}_i$, $\mathfrak{B}_i = \mathcal{B}_i$, $\mathtt{A}_i = \mathrm{A}_i$ and $\mathtt{B}_i = \mathrm{B}_i$ for every $i \in \{1,2\}$.

Before stating our next result, we recall that for any real number a and b, we denote by $\overline{ab}$ the open segment whose endpoints (not included) are a and b.

Corollary 4.5 *Assume that for every* $i \in \{1,2\}$ *the function* G *satisfies* $(\mathrm{F_i})$ *and that there exist* $C_i, D_i \in \mathbb{R}^+$ *such that*

$$\lim_{|x_i|\to+\infty} \frac{G_i(t,x)}{x_i} = C_i^2 \qquad \textit{and} \qquad \lim_{x_i\to 0} \frac{G_i(t,x)}{x_i} = D_i^2$$

uniformly in $t \in [0,\pi]$ *and* $x_j \in \mathbb{R}$ *with* $j \neq i$. *Suppose that there exists* $m_i \in \mathbb{N}$ *such that* $m_i \in \overline{C_i D_i}$ *for every* $i \in \{1,2\}$. *Then, the same conclusion of Theorem* 4.2 *holds.*

Sketch of the proof. We apply Corollary 4.4 with $b_i(t) \equiv \mathrm{B}_i^2 = C_i^2 - \varepsilon$, $\beta_i(t) \equiv \mathcal{B}_i^2 = C_i^2 + \varepsilon$, $a_i(t) \equiv \mathrm{A}_i^2 = D_i^2 - \varepsilon$, $\alpha_i(t) \equiv \mathcal{A}_i^2 = D_i^2 + \varepsilon$, being $\varepsilon > 0$ (sufficiently small) such that

$$\left\{ m \in \mathbb{N} \,:\, \sqrt{D_i^2+\varepsilon} < m < \sqrt{C_i^2-\varepsilon} \right\} \;=\; \{ m \in \mathbb{N} \,:\, D_i < m < C_i \} \qquad (4.12)$$

and

$$\left\{m \in \mathbb{N} \,:\, \sqrt{C_i^2+\varepsilon} < m < \sqrt{D_i^2-\varepsilon}\right\} \,=\, \{m \in \mathbb{N} \,:\, C_i < m < D_i\}\,. \qquad (4.13)$$

Then, (4.12) and (4.13) guarantee the applicability of Corollary 4.4. ∎

We finally give a Theorem where, besides (F_i), conditions ($\mathrm{H}_i^{\pm}$), ($\mathrm{K}_i^{\pm}$) are replaced by one-sided inequalities. More precisely, we have

Theorem 4.6 *Assume that G satisfies (F_i) with $\widetilde{G}_i(0) = 0$. Moreover, the following properties hold:*

there exist $N_i > 0$ and a continuous function $\beta_i^+ : [0,\pi] \longrightarrow \mathbb{R}^+$ such that

$$x_i > N_i \quad \Longrightarrow \quad 0 \le \frac{G_i(t,x)}{x_i} \le \beta_i^+(t)$$

for every $t \in [0,\pi]$ and $(x_1,\dots,x_i-1,x_i+1,\dots,x_m) \in \mathbb{R}^{m-1}$;

there exist $M_i > 0$ and a continuous function $\beta_i^- : [0,\pi] \longrightarrow \mathbb{R}^+$ such that

$$x_i < -M_i \quad \Longrightarrow \quad 0 \le \frac{G_i(t,x)}{x_i} \le \beta_i^-(t)$$

for every $t \in [0,\pi]$ and $(x_1,\dots,x_i-1,x_i+1,\dots,x_m) \in \mathbb{R}^{m-1}$;

there exist $\delta_i > 0$ and a continuous function $a_i^+ : [0,\pi] \longrightarrow \mathbb{R}^+$ such that

$$0 < x_i < \delta_i \quad \Longrightarrow \quad a_i^+(t) \le \frac{G_i(t,x)}{x_i}$$

for every $t \in [0,\pi]$ and $(x_1,\dots,x_i-1,x_i+1,\dots,x_m) \in \mathbb{R}^{m-1}$;

there exist $\sigma_i > 0$ and a continuous function $a_i^- : [0,\pi] \longrightarrow \mathbb{R}^+$ such that

$$-\sigma_i < x_i < 0 \quad \Longrightarrow \quad a_i^-(t) \le \frac{G_i(t,x)}{x_i}$$

for every $t \in [0,\pi]$ and $(x_1,\dots,x_i-1,x_i+1,\dots,x_m) \in \mathbb{R}^{m-1}$.

Further, suppose that there exists $m_i \in \mathbb{N}$ such that $\mathfrak{B}_i < m_i < \mathtt{A}_i$ for every $i \in \{1,2\}$. Then, there exist h_1, h_2, $\mathcal{H}_1$, $\mathcal{H}_2 \in \mathbb{N}$ such that for every $(n_1,n_2) \in \mathbb{N}^2$ with $h_i \le n_i \le \mathcal{H}_i$ $(i=1,2)$ problem (1.1) has at least four solutions $u^j = (u_1^j, u_2^j)$ $(1 \le j \le 4)$ with u_i^j having exactly n_i zeros in $[0,\pi)$, for every $j = 1,2,3,4$. Moreover, for every $(s_1,s_2) \in \tau$ there exists $j \in \{1,\dots,4\}$ such that $\operatorname{sgn}\left((u_i^j)'(0)\right) = s_i$ for $i \in \{1,2\}$.

For the proof of Theorem 4.6, more refined versions of the lemmas in Section 2 are needed and Proposition 5.1 has to be adapted as well. The details can be found in [9].

Remark 4.7 In [12] there is given a multiplicity result, in the particular case $m = 1$, for a scalar problem of the form (1.1). More precisely, it is assumed that there exist $\varepsilon_0 > 0$, $R > 0$ and $0 < a_0 \leq A_0$, $0 < b_0 \leq B_0$, $0 < a_\infty \leq A_\infty$ and $0 < b_\infty \leq B_\infty$ such that

$$a_0 \leq \frac{G(t,u)}{u} \leq A_0 \qquad \forall (t,u) \in \mathbb{R} \times]\,0, \varepsilon_0\,[,$$

$$b_0 \leq \frac{G(t,u)}{u} \leq B_0 \qquad \forall (t,u) \in \mathbb{R} \times]-\varepsilon_0, 0[,$$

$$a_\infty \leq \frac{G(t,u)}{u} \leq A_\infty \qquad \forall (t,u) \in \mathbb{R} \times]R, +\infty[,$$

$$b_\infty \leq \frac{G(t,u)}{u} \leq B_\infty \qquad \forall (t,u) \in \mathbb{R} \times]-\infty, R[;$$

under a Lipschitz condition on G, there is developed a shooting argument based on the notion of "variation index" (see also [23]). In the particular case when $a_0 = b_0$, $A_0 = B_0$, $a_\infty = b_\infty$, $A_\infty = B_\infty$, then statements (a) and (c) in Theorem 3 in [12] follow from Corollary 4.4. In general, the difference between the number of solutions we find and the number of solutions obtained in [12] is at most one.
Indeed, in this situation, statement (a) in Theorem 3 in [12] ensures the existence of $k_+(a_0, a_0) - k_+(A_\infty, A_\infty)$ solutions positive near $t = 0$, under the further hypothesis $k_+(a_0, a_0) > k_+(A_\infty, A_\infty)$, being $\sqrt{A_\infty} \notin \mathbb{N}$. We point out that in our notation (recalling (4.11))

$$k_+(a_0, a_0) = \max\{m \in \mathbb{N} \ : \ m < \sqrt{a_0}\},$$

$$k_+(A_\infty, A_\infty) = \max\left\{m \in \mathbb{N} \ : \ m < \sqrt{A_\infty}\right\}.$$

Thus, the inequality $k_+(a_0, a_0) > k_+(A_\infty, A_\infty)$ and the fact that $\sqrt{A_\infty} \notin \mathbb{N}$ guarantee the applicability of Corollary 4.4, with $\mathrm{A}_i = \sqrt{a_0}$, $\mathcal{A}_i = \sqrt{A_0}$, $\mathrm{B}_i = \sqrt{a_\infty}$, $\mathcal{B}_i = \sqrt{A_\infty}$, and $m_i = k_+(a_0, a_0) \in]\mathcal{B}_i, \mathrm{A}_i[$. Then, Corollary 4.4 provides the existence of $\mathcal{H} - h + 1$ solutions (with prescribed nodal properties) where, according to (4.11), $\mathcal{H} = \mathcal{H}_2 = k_+(a_0, a_0)$ and $h = h_2 = \min\left\{m \in \mathbb{N} \ : \ m > \sqrt{A_\infty}\right\}$. Finally, using the fact that $\sqrt{A_\infty} \notin \mathbb{N}$, it follows that $h - 1 = k_+(A_\infty, A_\infty)$, exactly as in Statement (a) in Theorem 3 in [12].

Remark 4.8 In [34] Sadyrbaev has given, on the lines of Section 15 in [25], a multiplicity result for a scalar equation of the form

$$u'' + G(t,u) = f(t,u,u'), \tag{4.14}$$

where G is asymptotically linear at infinity and f is sublinear. The argument in [34] is developed through the study of some variation equation (cf. (1.5) in [34]) associated with (4.14), and assumptions are given relatively to the number of zeros of the

solutions of this auxiliary equation. For this approach, it is required that the functions G and f are of class C^1. In our setting (with $f \equiv 0$ and $\lim_{x\to 0} \frac{G(t,x)}{x} = A^2$), the variation equation to be studied is $y'' + \frac{\partial G}{\partial x}(t,0)y = y'' + A^2 y = 0$, and Corollary 4.1 in [34] is, up to minor modifications, the same as Corollary 4.5, Theorem 3 in [12] and Theorem 15.11 in [25].

Remark 4.9 We point out that Amann and Zehnder in [1] have considered the Dirichlet BVP associated to a PDE of the form $-\Delta u = f(u)$; in [1, Theorem 1] is proved the existence of at least one solution, provided that there exists at least one eigenvalue λ of $-\Delta$ such that $\lambda \in \overline{f'(0)\, f'(\infty)}$. Unilateral conditions similar to the assumptions of Theorem 4.6 have been treated, among others, by Esteban in [13, Corollary 11].

5 Appendix

In this Appendix we deal with the autonomous uncoupled problem

$$\begin{cases} u''(t) + G^0(u(t)) = 0 \\ u(0) = 0 = u(\pi), \end{cases} \tag{5.1}$$

where $G^0(x_1, x_2) = \big(g_1^0(x_1), g_2^0(x_2)\big)$ for every $(x_1, x_2) \in \mathbb{R}^2$, being, for instance,

$$g_i^0(\xi) = \begin{cases} \xi\, a_{i,+}^2 & 0 \le \xi \le \delta \\ \xi\, a_{i,-}^2 & -\delta \le \xi < 0 \\ \xi\, b_{i,+}^2 & \xi \ge N \\ \xi\, b_{i,-}^2 & \xi \le -N \\ f_i^+(\xi) & \delta < \xi < N \\ f_i^-(\xi) & -N < \xi < -\delta\,. \end{cases} \tag{5.2}$$

In (5.2), the positive constants $a_{i,\pm}$, $b_{i,\pm}$ are such that

$$\mathrm{B}_i^{\pm} \le b_{i,\pm} \le \mathcal{B}_i^{\pm} \quad \text{and} \quad \mathrm{A}_i^{\pm} \le a_{i,\pm} \le \mathcal{A}_i^{\pm} \tag{5.3}$$

for every $i \in \{1,2\}$, while the piecewise linear functions $f_i^{\pm}$ are suitable for the validity of the following properties

- $$\lim_{x\to+\infty} \frac{g_i^0(x)}{x} = b_{i,+}^2 \quad \text{and} \quad \lim_{x\to-\infty} \frac{g_i^0(x)}{x} = b_{i,-}^2\,, \tag{5.4}$$
- $$\lim_{x\to 0^+} \frac{g_i^0(x)}{x} = a_{i,+}^2 \quad \text{and} \quad \lim_{x\to 0^-} \frac{g_i^0(x)}{x} = a_{i,-}^2\,, \tag{5.5}$$
- $$g_i^0(x)\, x > 0 \qquad \forall\, x \neq 0\,, \tag{5.6}$$
- $$g_i^0 \text{ is continuous and locally Lipschitz.} \tag{5.7}$$

We refer now to the notation introduced for the statement of Theorem 4.1 and to the choice of B in the various steps of the proof of main Theorem 4.2. Consider now $n = (n_1, n_2) \in \mathbb{N}^2$ satisfying conditions (4.2) and (4.3) in Theorem 4.1. We already know that if $\mathfrak{A}_i < n_i < \mathtt{B}_i$ or $\mathfrak{B}_i < n_i < \mathtt{A}_i$, for every $i \in \{1,2\}$, then (4.2)-(4.3) hold. We will exhibit an open set U_0^n such that $\deg(\mathrm{Id} - \mathcal{N}_0, U_0^n) \neq 0$, when $(\mathbf{k}^{-1}(n))_0 \subset U_0^n \subset \overline{U_0^n} \subset \overline{B}_0$ and $\Sigma_0 \cap U_0^n = (\mathbf{k}^{-1}(n))_0$.
To this end, we will study separately

$$\begin{cases} u_i''(t) + g_i^0(u_i(t)) = 0 \\ u_i(0) = 0 = u_i(\pi) \end{cases} \tag{5.8}$$

for fixed $i \in \{1,2\}$. Thus, we fix $i \in \{1,2\}$. From conditions (5.4)-(5.7), time-map techniques (see [33], [4]) lead to the following result:

Proposition 5.1 *Assume that there exists $m \in \mathbb{N}$ such that*

$$\textit{either} \qquad \mathfrak{A}_i < m < \mathtt{B}_i \qquad \textit{or} \qquad \mathfrak{B}_i < m < \mathtt{A}_i$$

holds. Then, there exist $h_i, \mathcal{H}_i \in \mathbb{N}$ such that for every $p \in \mathbb{N}$ with $h_i \leq p \leq \mathcal{H}_i$ problem (5.8) has at least two solutions u_p and v_p having exactly p zeros in $[0,\pi)$, with $u_p'(0) > 0$ and $v_p'(0) < 0$.

It is important to stress that the constants h_i and $\mathcal{H}_i$ obtained in the above Proposition coincide with the ones in the proof of Theorem 4.2.

It is well known that problem (5.8) can be written in the form $u_i = N_0^i(u_i)$ with respect to the Banach space $C_0^1([0,\pi])$, where $N_0^i : \mathrm{dom}\, N_0^i \subset C_0^1([0,\pi]) \longrightarrow C_0^1([0,\pi])$ is a completely continuous operator. Let us denote by Σ_0^i the set of the solutions of (5.8). Let us also define the continuous function $\pi_i : C_0^1([0,\pi],\mathbb{R}^2) \longrightarrow C_0^1([0,\pi])$ setting $\pi(u_1,u_2) = u_i$ for every $(u_1,u_2) \in C_0^1([0,\pi],\mathbb{R}^2)$.

We now use Proposition 5.1 with $m = n_i$; following a continuity argument developed in [16] and taking into accont the concept of "common core" as in [26], one can find (for every fixed $i \in \{1,2\}$) an open set $V_0^{n_i}$ in $C_0^1([0,\pi])$ such that

$$\deg\left(\mathrm{Id} - N_0^i,\, V_0^{n_i}\right) \neq 0, \tag{5.9}$$

where

$$\Sigma_0^i \cap V_0^{n_i} = \mathbf{n}^{-1}(n_i) \cap \Sigma_0^i \cap \pi_i(\overline{B}_0) \subset V_0^{n_i} \subset \overline{V_0^{n_i}} \subset \pi_i(\overline{B}_0). \tag{5.10}$$

We point out that B_0 corresponds to an annulus which (possibly), according to Remark 4.3, strictly contains the annulus B introduced in (4.5); however, with a little abuse of notation, we keep using the letter B.

Since the initial autonomous problem (5.1) is uncoupled and consequently $\mathcal{N}_0(u_1,u_2) = \left(N_0^1(u_1), N_0^2(u_2)\right)$ for every $(u_1,u_2) \in C_0^1([0,\pi],\mathbb{R}^2)$, the following equality for the degree holds

$$\deg\left(\mathrm{Id} - \mathcal{N}_0,\, V_0^{n_1} \times V_0^{n_2}\right) = \deg\left(\mathrm{Id} - N_0^1, V_0^{n_1}\right) \deg\left(\mathrm{Id} - N_0^2, V_0^{n_2}\right). \tag{5.11}$$

Let us define $U_0^n := V_0^{n_1} \times V_0^{n_2}$, where $n = (n_1, n_2) \in \mathbb{N}^2$. From (5.11) and (5.9) (which holds for every $i \in \{1,2\}$), we immediately obtain

$$\deg\left(\mathrm{Id} - \mathcal{N}_0,\, U_0^n\right) \neq 0\,.$$

Moreover, since $\overline{B}_0 = \left(\pi_1(\overline{B}_0), \pi_2(\overline{B}_0)\right)$ by the definition of B and since $\Sigma_0 = \Sigma_0^1 \times \Sigma_0^2$, the validity of (5.10) for every $i \in \{1,2\}$ implies

$$(\mathbf{k}^{-1}(n))_0 \subset U_0^n \subset \overline{U_0^n} \subset \overline{B}_0 \quad \text{and} \quad \Sigma_0 \cap U_0^n = (\mathbf{k}^{-1}(n))_0.$$

References

[1] H. Amann and E. Zehnder, *Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **7** (1980), 539-603.

[2] A. Bahri and P.-L. Lions, *Solutions of superlinear elliptic equations and their Morse indices*, Comm. Pure Appl. Math. **45** (1992), 1205-1215.

[3] R.S. Cantrell, *On solutions to coupled multiparameter nonlinear Sturm-Liouville boundary value problems whose state components have a specified nodal structure*, Results Math. **22** (1992), 470-488.

[4] A. Capietto and W. Dambrosio, *Multiplicity results for some two-point superlinear asymmetric boundary value problem*, Nonlinear Analysis, TMA **38** (1999), 869-896.

[5] A. Capietto and W. Dambrosio, *Multiplicity results for systems of superlinear second order equations*, J. Math. Anal. Appl. **248** (2000), 532- 548.

[6] A. Capietto, J. Mawhin and F. Zanolin, *On the existence of two solutions with a prescribed number of zeros for a superlinear two-point boundary value problem*, Topol. Methods Nonlinear Anal. **6** (1995), 175-188.

[7] A. Castro and A.C. Lazer, *Critical point theory and the number of solutions of a nonlinear Dirichlet problem*, Ann. Mat. Pura Appl. (IV) **120** (1979), 113-137.

[8] A. Castro and A.C. Lazer, *On periodic solutions of weakly coupled systems of differential equations*, Boll. Un. Mat. Ital. B **18** (1981), 733-742.

[9] F. Dalbono, Ph.D. thesis, in preparation.

[10] W. Dambrosio, *Multiple solutions of weakly-coupled systems with p-laplacian operators* Results Math. **36** (1999), 34-54.

[11] W. Dambrosio, *Nodal solutions to semilinear elliptic equations in a ball*, Differential Integral Equations **15** (2002), 945-972.

[12] G. Dinca and L. Sanchez, *Multiple solutions of boundary value problems: an elementary approach via the shooting method*, NoDEA **1** (1994), 163-178.

[13] M. Esteban, *Multiple solutions of semilinear elliptic problems in a ball*, J. Differential Equations **57** (1985), 112-137.

[14] C. Fabry and P. Habets, *Periodic solutions of second order differential equations with superlinear asymmetric nonlinearities*, Arch. Math. **60** (1993), 266-276.

[15] D. Fortunato, *Morse theory and nonlinear elliptic problems* Progress in elliptic and parabolic partial differential equations (Capri, 1994), Pitman Res. Notes Math. Ser. **350**, 163-172, Longman, Harlow, 1996.

[16] M. García-Huidobro, R. Manásevich and F. Zanolin, *Strongly nonlinear second-order ODE's with rapidly growing terms*, J. Math. Analysis Appl. **202** (1996), 1-26.

[17] M. García-Huidobro, R. Manásevich and F. Zanolin, *Infinitely many solutions for a Dirichlet problem with a non homogeneus p-Laplacian like operator in a ball*, Adv. Differential Equations **2** (1997), 203-230.

[18] M. García-Huidobro, R. Manásevich and F. Zanolin, *Splitting the Fučik spectrum to count solutions of a quasilinear ODE*, to appear in Dynam. Systems Appl.

[19] M. García-Huidobro and P. Ubilla, *Multiplicity of solutions for a class of nonlinear second-order equations*, Nonlinear Anal. **28** (1997), 1509-1520.

[20] G. Harris and B. Zinner, *Some remarks concerning exact solution numbers for a class of nonlinear boundary value problems*, J. Math. Anal. Appl. **182** (1994), 571-588.

[21] M. Henrard, *Infinitely many solutions of weakly coupled superlinear systems*, Adv. Differential Equations **2** (1997), 753-778.

[22] N. Hirano, *Multiple nontrivial solutions of semilinear elliptic equations*, Proc. Amer. Math. Soc. **103** (1988), 468-472.

[23] H.G. Kaper and K. Kwong, *On two conjectures concerning the multiplicity of solutions of a Dirichlet problem*, SIAM J. Math. Anal. **23** (1992), 571-578.

[24] Y.A. Klokov and F.Z. Sadyrbaev, *On the number of solutions of second-order boundary value problems with nonlinear asymptotics* (Russian), Differ. Uravn **34** (1998), 471-479. Translation in Differential Equations **34** (1998), 469–477.

[25] M.A. Krasnosel'skiĭ, A.I. Perov, A.I. Povolotskii and P.P. Zabreĭko, Plane vector fields, Academic Press, New York, 1966.

[26] M.A. Krasnosel'skiĭ and P.P. Zabreĭko, Geometric methods of nonlinear analysis, Springer-Verlag, Berlin, 1984.

[27] S. Li and J. Su, *Existence of multiple solutions of a two-point boundary value problem at resonance*, Topol. Methods Nonlinear Anal. **10** (1997), 123-135.

[28] S. Li and M. Willem, *Multiple solutions for asymptotically linear boundary value problems in which the nonlinearity crosses at least one eigenvalue*, NoDEA Nonlinear Differential Equations Appl. **5** (1998), 479- 490.

[29] R. Manásevich and J. Mawhin J, *Boundary value problems for nonlinear perturbations of vector p-Laplacian-like operators*, J. Korean Math. Soc. **37** (2000), 665-685.

[30] A. Margheri, C. Rebelo and F. Zanolin, *Maslov index, Poincaré-Birkhoff theorem and periodic solutions of asymptotically linear planar Hamiltonian systems*, J. Differential Equations **183** (2002), 342-367.

[31] J. Mawhin, Topological degree methods in Nonlinear Boundary Value Problems, CBMS Series, **40**, Amer. Math. Soc., Providence, RI, 1979.

[32] J. Mawhin and M. Willem, Critical point theory and Hamiltonian systems, Applied Mathematical Sciences, **74**, Springer-Verlag, New York, 1989.

[33] Z. Opial, *Sur les périodes des solutions de l'équation différentielle* $x'' + g(x) = 0$, Ann. Polon. Math. **10** (1961), 49-72.

[34] F. Sadyrbaev, *Multiplicity of solutions for two-point boundary value problems with asymptotically asymmetric nonlinearities*, Nonlinear Anal. **27** (1996), 999-1012.

[35] W. Zou, *Multiple solutions results for two-point boundary value problem with resonance*, Discrete Contin. Dynam. Systems **4** (1998), 485-496.

[36] W. Zou, *Multiple solutions for asymptotically linear elliptic systems*, J. Math. Anal. Appl. **255** (2001), 213-229.