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Absence of Solutions for Some Elliptic Equations and Systems in Half-Space with Nonlinear Neumann Boundary Conditions

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Abstract

This work deals with the nonexistence of solutions of the semilinear elliptic equation $-\Delta u = a(x)|u|^p$ in $R^N_+ = \{x = (x_1, \dots, x_N) \in R^N : x_N > 0\}$, with the boundary condition $-\dfrac{\partial u}{\partial x_N} = b(x')|u|^q$ on $\partial R^N_+ = \{(x', 0) : x' \in R^{N-1}\}$, where $a(x) \geq 0$, $b(x') \geq 0$, $p > 1$ and $q > 1$. We are also concerned with a nonexistence result for the system $-\Delta u_i = a_i(x)|u_{i+1}|^{p_i}$, $x \in R^N_+$, $1 \leq i \leq m$, $u_{m+1} = u_1$, with $\dfrac{\partial u_i}{\partial x_N} \leq 0$ on ∂R^N_+, where $a_i(x) \geq 0$ and $p_i > 1$.

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1 Introduction

Let $\boldsymbol{R}^N_+ = \{x = (x_1, \dots, x_N) = (x', x_N) \in \boldsymbol{R}^N : x_N > 0\}$ and $\partial \boldsymbol{R}^N_+ = \{(x', 0) : x' \in \boldsymbol{R}^{N-1}\}$. In this work, we study the nonexistence of nontrivial solutions of the problem

$$\begin{cases} -\Delta u = a(x)|u|^p & \text{in} \quad \boldsymbol{R}^N_+, \\ -\dfrac{\partial u}{\partial x_N} = b(x')|u|^q & \text{on} \quad \partial \boldsymbol{R}^N_+, \end{cases} \tag{1.1}$$

where $N \geq 2$, $p > 1$ and $q > 1$. The functions $a : \boldsymbol{R}^N_+ \longrightarrow \boldsymbol{R}^+$ and $b : \partial\boldsymbol{R}^N_+ \longrightarrow \boldsymbol{R}^+$ are regular and behave respectively as $|x|^\alpha$ and $|x'|^\beta$ when $|x|$ and $|x'|$ become large. Some conditions on the coefficients α and β will be specified later.

For the particular case where $a(x)$ and $b(x')$ are constant functions, the questions of existence and nonexistence of solutions of this problem have been investigated by many authors. Recently, Chipot *et al* have considered in [1] the problem

$$\begin{cases} -\Delta u = a u^p & \text{in} \quad \boldsymbol{R}^N_+, \\ -\dfrac{\partial u}{\partial x_N} = u^q & \text{on} \quad \partial\boldsymbol{R}^N_+, \end{cases} \tag{1.2}$$

where $a \in \boldsymbol{R}^+$ and $p, q > 1$. Existence of positive solutions of (1.2) has been proved for $p \geq \dfrac{N+2}{N-2}$ and $q \geq \dfrac{N}{N-2}$. Explicit solutions are given for $p = q > \dfrac{N}{N-2}$. It is moreover established in [1] that, for $a > 0$, the problem (1.2) does not admit any solution in each one of the following cases:

(I) $p \leq \dfrac{N+2}{N-2}$ and $q \leq \dfrac{N}{N-2}$, with at least one strict inequality,

(II) $p < \dfrac{N}{N-2}$,

(III) $q < \dfrac{N}{N-1}$.

In the case $N > 2$ with $p = \dfrac{N+2}{N-2}$ and $q = \dfrac{N}{N-2}$, solutions of (1.2) are known to exist (see e.g. [2], [4]). It was shown by Hu [3] that for $a = 0$, positive solutions of (1.2) do not exist if $q < \dfrac{N}{N-2}$.

Our sufficient conditions for the nonexistence of nontrivial solutions of the problem (1.1) are tabulated as follows.

	$\alpha \leq 2\beta$	$\alpha \geq 2\beta$
$2q < p+1$	$1 < p \leq \dfrac{N+2+\alpha}{N-2}$ and $q > 1$	$1 < p \leq \dfrac{N+2+\alpha}{N-2}$ (*) and $1 < q \leq \dfrac{N+\beta}{N-2}$
$2q = p+1$	$1 < p < \min\left(\dfrac{N+2+\alpha}{N-2}, \dfrac{N+2+2\beta}{N-2}\right)$ or $1 < q < \min\left(\dfrac{N+\beta}{N-2}, \dfrac{N+\frac{\alpha}{2}}{N-2}\right)$	
$2q > p+1$	$1 < p \leq \dfrac{N+2+\alpha}{N-2}$ (*) and $1 < q \leq \dfrac{N+\beta}{N-2}$	$p > 1$ and $1 < q \leq \dfrac{N+\beta}{N-2}$

(*) with at least one strict inequality.
This is a summary of conditions $(A.1)$, $(A.2)$, $(B.1)$, $(B.2)$, $(C.1)$ and $(C.2)$ in Theorem 2.1 below.

It can readily be observed that when $a(x) = a$ and $b(x') = b$ are constant functions, then $\alpha = \beta = 0$ and we find again condition (I) while we improve (II) and (III). For this case of constant functions, it is important to point out that our results give the nonexistence of nontrivial solutions of (1.2) when $p \leq \dfrac{N+2}{N-2}$, or $q \leq \dfrac{N}{N-2}$.

In the last part of this work, we are concerned with the study of the nonexistence of solutions for the system:

$$\begin{cases} -\Delta u_i = a_i(x)|u_{i+1}|^{p_i}, \quad x \in \boldsymbol{R}^N_+, \\ \dfrac{\partial u_i}{\partial x_N} \leq 0 \quad \text{on} \quad \partial \boldsymbol{R}^N_+, \\ 1 \leq i \leq m, \quad u_{m+1} = u_1. \end{cases} \tag{1.3}$$

The proofs of our results are based on the choice of a suitable test-function and using the Young's inequality as well as a scaling argument. We are thus concerned with the method called the *test-function method*, developped by Mitidieri & Pohozaev [5], Pohozaev & Tesei [6] and Pohozaev & Veron [7].

2 Nonexistence for the elliptic equation

By weak solutions of (1.1), we mean $u \in L^p_{loc}\left(\boldsymbol{R}^N_+, a(x)dx\right) \cap L^q_{loc}\left(\partial \boldsymbol{R}^N_+, b(x')dx'\right)$ such that, for any positive regular function ψ,

$$\int_{R^N_+} a(x)|u|^p \psi \, dx = -\int_{R^N_+} u \, \Delta\psi \, dx + \int_{\partial R^N_+} \frac{\partial u}{\partial x_N} \psi \, dx' - \int_{\partial R^N_+} \frac{\partial \psi}{\partial x_N} u \, dx'. \tag{2.1}$$

2.1 Main results

Theorem 2.1. *Let $N \geq 2$, $p, q > 1$ and assume that the functions a, b are positive and regular in $\boldsymbol{R}^N_+$ and $\partial \boldsymbol{R}^N_+$ respectively. Let us assume in addition that $a(x)$ and $b(x')$ behave respectively as $|x|^\alpha$ and $|x'|^\beta$, for $|x|$ and $|x'|$ large, with $\alpha > -4$ and $\beta > -2$. Then, problem (1.1) cannot admit nontrivial weak solutions if*
1) $N = 2$;
or
2) $N > 2$ and one of the following conditions holds:

$$(A.1)\quad p \le \frac{N+2+\alpha}{N-2} \quad \text{and} \quad q \le \frac{N+\beta}{N-2} \quad \text{if} \quad 2q < p+1 \quad \text{and} \quad \alpha \ge 2\beta,$$

$$(A.2)\quad p \le \frac{N+2+\alpha}{N-2} \quad \text{and} \quad q \le \frac{N+\beta}{N-2} \quad \text{if} \quad 2q > p+1 \quad \text{and} \quad \alpha \le 2\beta,$$

with at least one strict inequality in the conditions on p and q;

$$(B.1)\qquad p \le \frac{N+2+\alpha}{N-2} \qquad \text{if} \qquad 2q < p+1 \qquad \text{and} \qquad \alpha \le 2\beta,$$

$$(B.2)\qquad q \le \frac{N+\beta}{N-2} \qquad \text{if} \qquad 2q > p+1 \qquad \text{and} \qquad \alpha \ge 2\beta;$$

$$\left.\begin{array}{ll} (C.1) & p < \min\left(\dfrac{N+2+\alpha}{N-2},\ \dfrac{N+2+2\beta}{N-2}\right) \\ & \text{or} \\ (C.2) & q < \min\left(\dfrac{N+\beta}{N-2},\ \dfrac{N+\frac{\alpha}{2}}{N-2}\right) \end{array}\right\} \quad \text{if} \quad 2q = p+1.$$

For the case $a \neq 0$ and $b = 0$, we have the following result.

Theorem 2.2. *Consider the problem*

$$\begin{cases} -\Delta u = a(x)|u|^p & \text{in} \quad \boldsymbol{R}^N_+, \\ \dfrac{\partial u}{\partial x_N} \le 0 & \text{on} \quad \partial \boldsymbol{R}^N_+. \end{cases} \tag{2.2}$$

Let $N \ge 2$, $p > 1$ and assume that the function a is positive and regular in $\boldsymbol{R}^N_+$ and that

$$a(x) \ge a_0 |x|^\alpha, \quad \text{for} \quad |x| \quad \text{large} \quad , \quad \text{with} \quad \alpha > -2 \quad \text{and} \quad a_0 > 0.$$

Let $u \in L^p_{loc}\left(\boldsymbol{R}^N_+, a(x)dx\right)$ a weak solution of problem (2.2). Then, $u \equiv 0$ if
1) $N = 2$;
or
2) $N > 2$ and $p \le \dfrac{N+\alpha}{N-2}$.

2.2 Proofs

Before proving these theorems, we make some remarks on the test-functions that will be used. In a general way we select a test-function under the form:

$$\psi(x) = \Big((x_N + R)^\mu + 1\Big)\,\varphi\left(\frac{|x|}{R}\right),$$

where $\mu \le 0$ and $R > 0$. The function φ defined as

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1, \\ 0 & \text{if } r \ge 2, \end{cases}$$

is considered to be regular, decreasing and positive in $\boldsymbol{R}^+$, with

$$|\varphi'(r)| \le \frac{C}{r} \quad \forall r > 0.$$

We note that

$$\lim_{R\to\infty} \psi(x) = 1 \quad \forall x \in \overline{\boldsymbol{R}^N_+}.$$

Direct computations give

$$\frac{\partial\psi}{\partial x_N} = \mu(x_N + R)^{\mu-1}\,\varphi + \Big((x_N + R)^\mu + 1\Big)\,\frac{\partial\varphi}{\partial x_N},$$

and

$$\Delta\psi = \mu(\mu-1)(x_N + R)^{\mu-2}\,\varphi + 2\mu(x_N + R)^{\mu-1}\,\frac{\partial\varphi}{\partial x_N} + \Big((x_N + R)^\mu + 1\Big)\,\Delta\varphi.$$

If $r = \dfrac{|x|}{R}$, then $\dfrac{\partial\varphi}{\partial x_N} = \dfrac{\partial r}{\partial x_N}\dfrac{\partial\varphi}{\partial r} = \dfrac{x_N}{R^2 r}\dfrac{\partial\varphi}{\partial r}$. Since $\mu \le 0$, we have $\dfrac{\partial\psi}{\partial x_N} = \mu R^{\mu-1}\varphi \le 0$ on $\partial\boldsymbol{R}^N_+$.

Proof of Theorem 2.1. Suppose by contradiction that (1.1) has a nontrivial solution u. From the definition (2.1), we deduce

$$\int_{\boldsymbol{R}^N_+} a(x)\,|u|^p\,\psi\,dx + \int_{\partial\boldsymbol{R}^N_+} b(x')\,|u|^q\,\psi\,dx' \le \int_{\boldsymbol{R}^N_+} |u|\,|\Delta\psi|\,dx + \int_{\partial\boldsymbol{R}^N_+} |u|\left|\frac{\partial\psi}{\partial x_N}\right|dx'.$$

Applying Young's inequality, it follows that

$$\int_{\boldsymbol{R}^N_+} |u|\,|\Delta\psi|\,dx \le \frac{1}{p}\int_{\boldsymbol{R}^N_+} a\,|u|^p\,\psi\,dx + \frac{1}{p'}\int_{\boldsymbol{R}^N_+} |\Delta\psi|^{p'}\,a^{1-p'}\,\psi^{1-p'}\,dx,$$

and therefore

$$\int_{\partial\boldsymbol{R}^N_+} |u|\left|\frac{\partial\psi}{\partial x_N}\right|dx' \le \frac{1}{q}\int_{\partial\boldsymbol{R}^N_+} b\,|u|^q\,\psi\,dx' + \frac{1}{q'}\int_{\partial\boldsymbol{R}^N_+} \left|\frac{\partial\psi}{\partial x_N}\right|^{q'} b^{1-q'}\,\psi^{1-q'}\,dx',$$

where $p+p'=pp'$ and $q+q'=qq'$. Hence,

$$\int_{R_+^N} a(x)\,|u|^p\,\psi\,dx+\int_{\partial R_+^N} b(x')\,|u|^q\,\psi\,dx' \tag{2.3}$$
$$\leq C\left(\int_{R_+^N}|\Delta\psi|^{p'}\,a^{1-p'}\,\psi^{1-p'}\,dx+\int_{\partial R_+^N}\left|\frac{\partial\psi}{\partial x_N}\right|^{q'}\,b^{1-q'}\,\psi^{1-q'}\,dx'\right),$$

where $C=\dfrac{\max(p',q')}{\min(p',q')}$.

Let us now choose the test-function as

$$\psi(x)=\begin{cases}\left((x_N+R)^{-2/(p-1)}+1\right)\varphi\left(\dfrac{|x|}{R}\right), & \text{if}\quad 2q\leq p+1,\\[2ex] \left((x_N+R)^{-1/(q-1)}+1\right)\varphi\left(\dfrac{|x|}{R}\right), & \text{if}\quad 2q\geq p+1,\end{cases} \tag{2.4}$$

with $R>0$. Moreover, let us perform the scaling:

$$y=R^{-1}x,\quad (y_1,y_2,...,y_N)=(R^{-1}x_1,R^{-1}x_2,...,R^{-1}x_N), \tag{2.5}$$

and we notice that $dx=R^N\,dy$ in $\boldsymbol{R}_+^N$ and $dx'=R^{N-1}\,dy'$ on $\partial\boldsymbol{R}_+^N$.

On the one hand, since $1-p'<0$ and $1-q'<0$, it follows that, for R large enough,

$$\psi^{1-p'}\leq R^{\mu(1-p')}\,(1+y_N)^{\mu(1-p')}\,\varphi^{1-p'}(|y|)$$

and

$$\psi^{1-q'}\leq R^{\mu(1-q')}\,(1+y_N)^{\mu(1-q')}\,\varphi^{1-q'}(|y|),$$

where

$$\mu=\begin{cases}-2/(p-1) & \text{if}\quad 2q\leq p+1,\\ -1/(q-1) & \text{if}\quad 2q\geq p+1.\end{cases}$$

When R is still sufficiently large, we get, on the other hand:

$$\left|\frac{\partial\psi}{\partial x_N}\right|\sim R^{\mu-1}\quad\text{on}\quad\partial\boldsymbol{R}_+^N\quad\text{and}\quad|\Delta\psi|\sim R^{\mu-2}\quad\text{in}\quad\boldsymbol{R}_+^N.$$

By considering then $a(x)\sim|x|^\alpha$ and $b(x')\sim|x'|^\beta$, we get the estimate

$$\int_{R_+^N} a(x)\,|u|^p\,\psi\,dx+\int_{\partial R_+^N} b(x')\,|u|^q\,\psi\,dx'\leq C_1\,R^{\lambda_1}+C_2\,R^{\lambda_2}, \tag{2.6}$$

where C_1, $C_2 > 0$,

$$\lambda_1 = (\mu - 2)p' + (\alpha + \mu)(1 - p') + N = N + \mu - \frac{2p + \alpha}{p - 1}$$

and

$$\lambda_2 = (\mu - 1)q' + (\beta + \mu)(1 - q') + N - 1 = N + \mu - \frac{2q + \beta - 1}{q - 1}.$$

We have $\lambda_1 \leq 0$ and $\lambda_2 \leq 0$ if and only if $N \leq \dfrac{2p + \alpha}{p - 1} - \mu$ and $N \leq \dfrac{2q + \beta - 1}{q - 1} - \mu$ respectively.

In order to have $C_1 < \infty$ and $C_2 < \infty$, we take φ^η instead of φ in the definition of ψ for some $\eta \gg 1$.

We inspect (2.6) by distinguishing each one of the following three cases:
- case 1: $\lambda_1 < 0$ and $\lambda_2 < 0$;
- case 2: $\lambda_1 < 0$ and $\lambda_2 = 0$;
- case 3: $\lambda_1 = 0$ and $\lambda_2 < 0$.

In case 1, making $R \to \infty$ in (2.6), we obtain

$$\int_{R_+^N} a(x)\,|u|^p\,dx + \int_{\partial R_+^N} b(x')\,|u|^q\,dx' \leq 0,$$

and hence $u \equiv 0$.

In case 2, we deduce from (2.6) that

$$\int_{\partial R_+^N} b(x')\,|u|^q\,\psi\,dx' < \infty.$$

Let $E_0 = \{R \leq |x'| \leq 2R\}$. We have $\operatorname{supp}\varphi \subset E_0$ and

$$\lim_{R \to \infty} \int_{E_0} b(x')\,|u|^q\,\psi\,dx' = 0.$$

It follows from the definition (2.1) that

$$\int_{R_+^N} a(x)\,|u|^p\,\psi\,dx + \int_{\partial R_+^N} b(x')\,|u|^q\,\psi\,dx' \leq \int_{R_+^N} |u|\,|\Delta\psi|\,dx + \int_{\partial R_+^N} |u| \left|\frac{\partial \psi}{\partial x_N}\right| dx'.$$

Now, on the right-hand side of this inequality, we apply Young's inequality to the first integral and Hölder's inequality to the second integral:

$$\frac{1}{p'}\int_{R_+^N} a(x)\,|u|^p\,\psi\,dx + \int_{\partial R_+^N} b(x')\,|u|^q\,\psi\,dx'$$
$$\leq \frac{1}{p'}\int_{R_+^N} |\Delta\psi|^{p'}\,a^{1-p'}\,\psi^{1-p'}\,dx$$
$$+ \left(\int_{\partial R_+^N} b|u|^q\,\psi\,dx'\right)^{\frac{1}{q}} \left(\int_{\partial R_+^N} \left|\frac{\partial\psi}{\partial x_N}\right|^{q'} b^{1-q'}\,\psi^{1-q'}\,dx'\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{p'}\int_{R_+^N} |\Delta\psi|^{p'}\,a^{1-p'}\,\psi^{1-p'}\,dx$$
$$+ \left(\int_{E_0} b|u|^q\,\psi\,dx'\right)^{\frac{1}{q}} \left(\int_{\partial R_+^N} \left|\frac{\partial\psi}{\partial x_N}\right|^{q'} b^{1-q'}\,\psi^{1-q'}\,dx'\right)^{\frac{1}{q}}.$$

Using again the scaling (2.5), we obtain

$$\frac{1}{p'}\int_{R_+^N} a(x)\,|u|^p\,\psi\,dx + \int_{\partial R_+^N} b(x')\,|u|^q\,\psi\,dx' \leq \tilde{C}_1 R^{\lambda_1} + \tilde{C}_2\left(\int_{E_0} b|u|^q\,\psi\,dx'\right)^{\frac{1}{q}}.$$

Then making $R \to \infty$, it follows that $u \equiv 0$ as in case 1.

In case 3, we have

$$\int_{R_+^N} a(x)\,|u|^p\,\psi\,dx < \infty \quad \text{and} \quad \lim_{R\to\infty}\int_{E_1} a(x)\,|u|^p\,\psi\,dx = 0,$$

where $E_1 = \{R \leq |x| \leq 2R\}$. As in the case 2, we obtain

$$\int_{R_+^N} a(x)\,|u|^p\,\psi\,dx + \int_{\partial R_+^N} b(x')\,|u|^q\,\psi\,dx' \leq \tilde{C}_1\left(\int_{E_1} a(x)\,|u|^p\,\psi\,dx\right)^{\frac{1}{p}} + \tilde{C}_2 R^{\lambda_2}.$$

It follows that $u \equiv 0$ when $R \to \infty$.

By a straightforward computation, we have both $\lambda_1 \leq 0$ and $\lambda_2 < 0$ or $\lambda_1 < 0$ and $\lambda_2 \leq 0$ if one of the conditions $(A.1) - (C.2)$ is satisfied. This achieves the proof.

□

Proof of Theorem 2.2. The proof is by contradiction. Suppose that (2.2) has a nontrivial solution u. We consider

$$\psi(x) = \varphi\left(\frac{|x|}{R}\right),$$

as a test-function. We clearly have $\dfrac{\partial\psi}{\partial x_N} = 0$ on ∂R_+^N. Since $\dfrac{\partial u}{\partial x_N} \leq 0$, we can deduce from the definition (2.1) that

$$\int_{R_+^N} a(x)\,|u|^p\,\psi\,dx \le -\int_{R_+^N} u\,\Delta\psi\,dx.$$

Therefore

$$\int_{R_+^N} a(x)\,|u|^p\,\psi\,dx \le \int_{R_+^N} |u|\,|\Delta\psi|\,dx.$$

Then, applying Young's inequality, we obtain

$$\int_{R_+^N} |u|\,|\Delta\psi|\,dx \le \frac{1}{p}\int_{R_+^N} a\,|u|^p\,\psi_0\,dx + \frac{1}{p'}\int_{R_+^N} |\Delta\psi|^{p'}\,a^{1-p'}\,\psi^{1-p'}\,dx,$$

where $\dfrac{1}{p}+\dfrac{1}{p'}=1$. Hence, it follows that:

$$\int_{R_+^N} a(x)\,|u|^p\,\psi\,dx \le \int_{R_+^N} |\Delta\psi|^{p'}\,a^{1-p'}\,\psi^{1-p'}\,dx. \tag{2.7}$$

Considering the same change of variables as in (2.5), it is immediate to check that, for R large enough, we have:

$$|\Delta\psi| \sim R^{-2}.$$

Then, since $a(x) \sim |x|^\alpha$ for $|x| \to \infty$, it follows from (2.7) that:

$$\int_{R_+^N} a(x)\,|u|^p\,\psi\,dx \le \tilde{C}_0\,R^{\lambda_0}, \tag{2.8}$$

where $\tilde{C}_0 > 0$ and

$$\lambda_0 = -2p' + \alpha(1-p') + N = N + \frac{-2p-\alpha}{p-1}.$$

When $N=2$ and $\alpha > -2$, we obtain $\lambda_0 = \dfrac{-2-\alpha}{p} < 0$.
If $N > 2$, we obtain

$$\lambda_0 \le 0 \Longleftrightarrow N \le \frac{2p+\alpha}{p-1} \Longleftrightarrow p \le \frac{N+\alpha}{N-2}.$$

When $\lambda_0 < 0$, making $R \to \infty$ in (2.8), it follows that $u \equiv 0$. The case $\lambda_0 = 0$ can be treated by using the similar arguments as in the proof of Theorem 2.1.

3 Nonexistence for the system

In this section, we extend the result obtained in Theorem 2.2 to a system.

Theorem 3.1. *Consider the following problem*

$$\begin{cases} -\Delta u_i = a_i(x)|u_{i+1}|^{p_i} \quad in \quad \boldsymbol{R}_+^N, \\ \dfrac{\partial u_i}{\partial x_N} \leq 0 \quad on \quad \partial \boldsymbol{R}_+^N, \\ 1 \leq i \leq m, \quad u_{m+1} = u_1. \end{cases} \tag{3.1}$$

Let $N \geq 2$, $p_i > 1$ *and assume that the functions* a_i *are positive and regular in* $\boldsymbol{R}_+^N$ *and that* $a_i(x) \sim |x|^{\alpha_i}$, *for* $|x|$ *large, with* $\alpha_i > -2$, $1 \leq i \leq m$.
Then, problem (3.1) does not admit any nontrivial weak solution if $N = 2$, *or if*

$$2 < N \leq \min_{1 \leq i \leq m} \left(\frac{2p_i + \alpha_i}{p_i - 1} \right).$$

Proof. By a weak solution of problem (3.1), we mean $(u_1, u_2, ..., u_m)$ such that $u_i \in L^{p_i}_{loc}\left(\boldsymbol{R}_+^N, a_i(x)\,dx\right)$ and for any positive regular function ψ,

$$\int_{R_+^N} a_i(x)\,|u_{i+1}|^{p_i}\,\psi\,dx \leq -\int_{R_+^N} u_i\,\Delta\psi\,dx - \int_{\partial R_+^N} \frac{\partial \psi}{\partial x_N}\,u_i\,dx', \quad 1 \leq i \leq m.$$

Using the same test-function and the same arguments as in the proof of Theorem 2.2, we obtain

$$\sum_{i=1}^{m} \int_{R_+^N} a_i\,|u_{i+1}|^{p_i}\,\psi\,dx \leq \sum_{i=1}^{m} \int_{R_+^N} |\Delta\psi|^{p_i'}\,a_i^{1-p_i'}\,\psi^{1-p_i'}\,dx \leq C\,R^{\lambda_0},$$

where

$$\lambda_0 = N + \max_{1 \leq i \leq m} \left(-2p_i' + \alpha_i(1 - p_i')\right).$$

For $N = 2$ and $\alpha_i > -2$, $1 \leq i \leq m$, we have $\lambda_0 < 0$, and when $N > 2$, we have

$$\lambda_0 \leq 0 \iff N \leq \min_{1 \leq i \leq m} \left(\frac{2p_i + \alpha_i}{p_i - 1} \right).$$

As before, at the limit when $R \to \infty$, we obtain

$$\sum_{i=1}^{m} \int_{R_+^N} a_i\,|u_{i+1}|^{p_i}\,dx \leq 0,$$

leading to $u_{i+1} = 0$ for $1 \leq i \leq m$. This achieves the proof.

□

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